FLOW ON A MOVABLE BED

by

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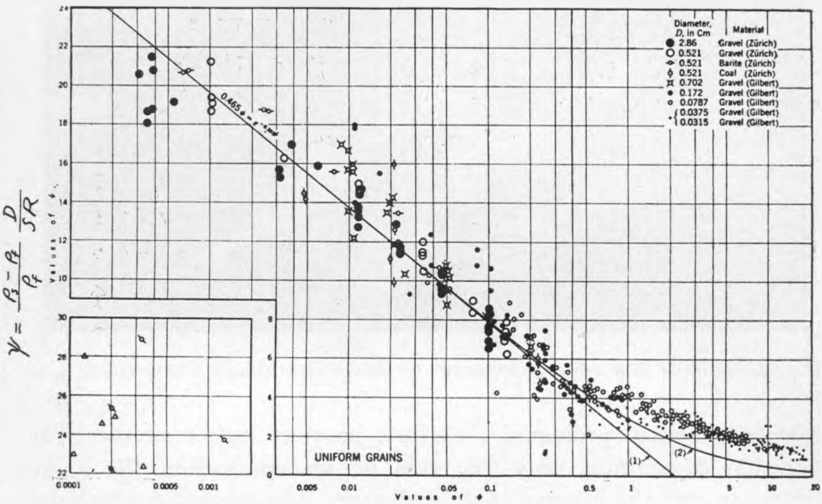
Since the First Hydraulics Conference three years ago, progress in the field of bed-load transportation has consisted mainly of a few revisions of existing formulas. These revisions have been made either by using new measurements or by attempting to introduce into the formulas the latest developments in the theory of turbulence. Although this progress is of theoretical importance to the bed-load problem, it does not seem of sufficient significance to discuss it at this meeting. Instead, the discussion is confined to the practical application of our knowledge of past experience, as expressed by formulas and certain rules, to the calculation of bed-load transportation in a natural stream. The accuracy with which such an application can be made is shown by comparing calculated rates of transportation with actual field measurements.

If our present knowledge of the problem of flow over a movable bed is correct, it is possible to select a reach of a stream and so describe it by cross sections, slope, and physical characteristics of the bed material that the depth of flow and the rate of bed-load transportation can be predicted for any particular discharge. The application of a formula is simplified if a uniform reach of the stream is chosen and if the flow can be assumed as approximately normal.

It can be stated that, in general, the formula with the widest range of applicability affords the best possibility of including the conditions as existing in natural streams. For illustrative purposes, the ψ-ϕ formula [1] [1] represented by the semi-logarithmic plot in Fig. 1 will be used in the following discussion. In connection with the data shown in Fig. 1, it is of interest to note that for the three smallest grain sizes in Gilbert's experiments the slope measurements are questionable. In only a few of these experiments

[1] References appear at the end of the article.

was the slope of the water surface measured, and in those few instances it differed considerably from the bed slope which was used in plotting Fig. 1. Since the energy gradient should be used and because it is better approximated by the slope of the water surface than the bed, the data are very unreliable for these materials and the systematic deviation of the plotted points from the curve in Fig.



$$\phi = \frac{1}{F} \frac{q_s}{(\rho_s - \rho_f) g} \sqrt{\frac{\rho_f}{\rho_s - \rho_f}} \frac{1}{g^{0.5} D^{1.5}}$$

FIG. 1.—BED-LOAD EXPERIMENTS SHOWING THE RELATION BETWEEN ϕ AND ψ.
(Reprinted from "Formulas for the Transportation of Bed-Load" [1]).

1 does not represent necessarily a change of the true curve in that region of the diagram. The truth probably is more closely approached if it is admitted that we do not know the location of the true curve for values of $\phi > 1$ than if any empirical curve is drawn through points plotted from Gilbert's data. This fact is unfortunate because in most rivers in this country the sediment characteristics are such that values of ϕ are greater than 1 at the higher stages. Careful measurements in this range, therefore, are urgently needed.

Although the ψ-ϕ formula appears rather formidable in its complete form, many of the terms are constants and its applica-

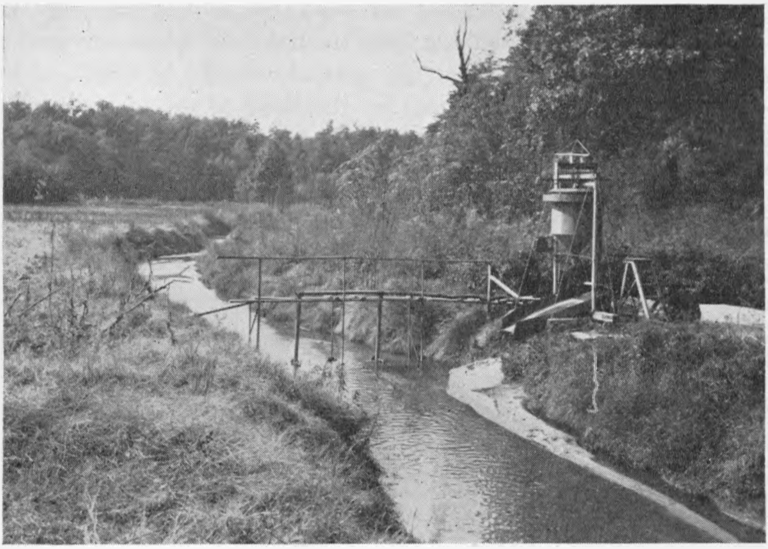


FIG. 2.—SITE OF FIELD MEASUREMENTS ON MOUNTAIN CREEK, GREENVILLE, S. C.

tion to practical problems, therefore, becomes rather simple. The variable ϕ contains only the rate of transportation, the grain diameter, and the density of the sediment. The ψ term also includes the size and density of the sediment, as well as the slope and the hydraulic radius.

The rate of transportation is the rate with which material is moved along the stream bed and excludes all sediment in suspension. The energy gradient should be used for the slope, and the hydraulic radius is that of the bed after the effect of side-wall friction has been eliminated. For the density of the sediment, an average value as determined from bed samples can be used.

Originally the ψ-ϕ formula was derived from experiments on materials of uniform grain sizes, where the diameter of the particles was the mean of the diameters of the two adjacent sieves. In the application of the formula to the movement of a mixture, it is necessary to introduce a representative diameter defined as "that diameter which gives for a certain discharge the same rate of transportation on a bed of uniform material as the mixture does." If close agreement between calculated and observed data over the whole range of measurement is found, the assumption of the existence of a representative diameter is justified. This appears true as illustrated below for observations in Mountain Creek, near Greenville, S. C. For the determination of the representative diameter of a mixture, a rule-of-thumb, for want of something better, is used. This rule designates the representative diameter as the grain size at which 35 percent of the material is finer.

In order to test the applicability of the ψ-ϕ formula to a natural stream, field measurements of rates of bed-load transportation were made in Mountain Creek by means of a portable measuring device. This stream, which is a major tributary of the Enoree River above the Soil Conservation Service sediment-load laboratory [2], has a drainage area of approximately 11.7 square miles. The drainage area is 58 percent woodland, 28 percent cultivated and mostly in cotton, and 14 percent abandoned and urban land. Fig. 2 shows the portable bed-load measuring device in place on the test reach in Mountain Creek. Inasmuch as a full description of the field observations is to be published elsewhere, the method of conducting the experiments is not discussed. Instead, only a comparison between calculated and observed values of depth of flow and rate of bed-load transportation is given (Fig. 3). Because both the portable measuring device and the ψ-ϕ formula exclude sediment in suspension and express the rate of transportation as weight under water per unit of time, the calculated and observed results shown in Fig. 3 are directly comparable.

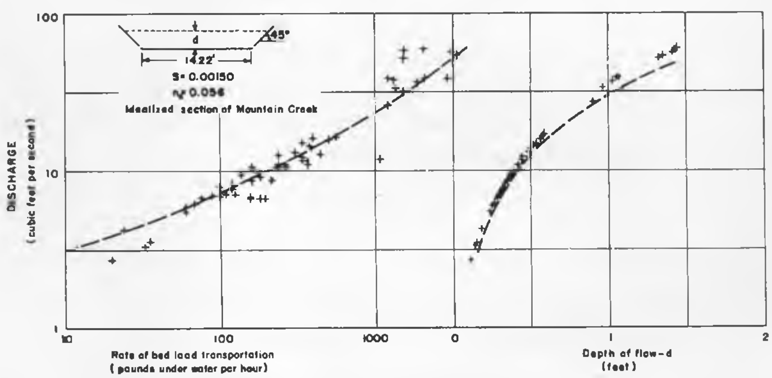


FIG. 3.—MEASURED VALUES OF BED-LOAD TRANSPORTATION AND DEPTH OF FLOW COMPARED WITH VALUES CALCULATED FOR IDEALIZED SECTION ON MOUNTAIN CREEK.

The ψ-ϕ formula is used essentially to determine the relationship between discharge and rate of bed-load transportation. Also of importance in bed-load calculations is a formula to express the roughness of a movable bed, indicating the ability of the bed to convert mechanical energy into turbulence. Past experiments suggest that this friction formula should contain the rate of transportation in one form or another because the bed friction appears to change with the rate of movement. To the writer's knowledge, no previous attempt has been made to develop a general formula for the bed friction in terms of rate of transportation, yet no analytical solution of the bed-load problem appears possible without such a formula. The formula, however, need not necessarily be theoretically correct. Even a fully empirical equation will serve for practical purposes, provided it covers the complete range of conditions involved.

The most obvious approach to the problem is to introduce a special correction for transportation to an established friction formula for flow over a fixed granular bed. As a first approach, von Kármán's equation for flow along a rough wall was used. This formula, which has proved very reliable for surfaces roughened with sand grains, is

$$\frac{V}{\sqrt{\tau_0/\rho}} = y + 5.75 \log \left(\frac{R_b}{D}\right) \tag{1}$$

where V is the average velocity, $\sqrt{\tau_0/\rho}$ is the friction velocity which can also be written as $\sqrt{R_b S g}$, R_b is the hydraulic radius of the bed, and D is the representative grain diameter of the bed. This formula can be adapted even to the transition cases from rough to smooth walls by choosing the proper y-value which is a function of D/δ, where δ is the thickness of the boundary layer. This latter factor is determined from the equation

$$\delta = \frac{11.6\,\nu}{\sqrt{\tau_0/\rho}} \tag{2}$$

where ν is the kinematic viscosity of the fluid. By introducing a dimensionless correction factor, Γ, which is supposed to be a function of the rate of transportation, Eq. (1) may be written

$$\frac{V}{\sqrt{\tau_0/\rho}} = \Gamma \left[y + 5.75 \log \left(\frac{R_b}{D}\right)\right] \tag{3}$$

To obtain information on the change of this correction factor for different rates of transportation, values of Γ are plotted against values of ϕ, which, it is recalled, is a dimensionless expression of the intensity of the transportation. Fig. 4 shows a plot of these values for the Zurich experimental data used in the establishment of the ψ-ϕ relationship shown in Fig. 1. Gilbert's data have not been used in this plot because of the rather wild scattering of points—a scattering which is probably due to unreliable depth and slope measurements. Instead of Gilbert's data, however, the experimental data on United States Waterways Experiment Station sand 2 are used [3]. The smooth curve drawn through the plotted points in Fig. 4 is used in the following discussion for determining the

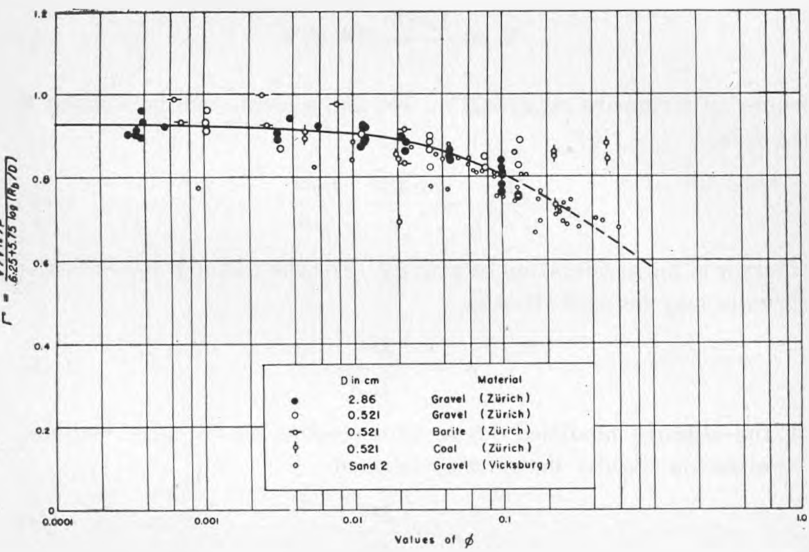


FIG. 4.—CORRECTION Γ OF VON KARMAN'S FRICTION FORMULA IN TERMS OF Φ.

roughness of the bed. The scatter of the points about the curve is random, indicating that scatter is due to experimental errors rather than to the method of representation.

It is of importance to note that the relationship between Γ and ϕ shown in Fig. 4 was obtained from experimental data on materials of uniform grain size. The application of the relationship to mixtures requires another rule-of-thumb to determine the representative diameter of the material, because in describing the relative roughness of the bed a characteristic diameter different from that for the transportation equation must be used. The rule is that for the relative roughness the grain size at which 65 percent of the material is finer describes the mixture with reasonable accuracy. The choice of this diameter, however, is not critical, because its logarithm, as entered in Eq. (3), changes very slowly.

For practical applications the von Kármán equation is rather inconvenient to use. The question, therefore, is whether a more convenient relation, such as the Manning formula, could be used instead. A comparison between the two formulas [4] can be made if the Manning formula is transformed into a slightly different form. Thus the Manning formula,

$$V = \frac{1.486}{n} S^{1/2} R_b^{2/3} \tag{4}$$

where all terms are expressed in feet and seconds, can be written in the form

$$\frac{V}{\sqrt{\tau_0/\rho}} = \frac{1.486}{n} \frac{R_b^{2/3}}{g^{1/2}} \tag{5}$$

where g is the acceleration of gravity. For the factor n the Strickler formula may be used; that is,

$$n = \frac{D^{1/6}}{21.3} \tag{6}$$

or the slightly modified form, which seems to fit most bed-load experiments slightly better, may be used

$$n = \frac{D^{1/6}}{24} \tag{7}$$

where D is the grain diameter. By substituting Eq. (7) into Eq. (5) and expressing all terms in foot and second units, we obtain

$$\frac{V}{\sqrt{\tau_0/\rho}} = 7.66 \left(\frac{R_b}{D}\right)^{1/6} \tag{8}$$

This equation is compared with von Kármán's equation in Fig. 5, and it is seen that for practical purposes either equation can be used for a very wide range of (R_b/D). The Strickler equation, $n = D^{1/6}/21.3$, gives reasonably good results over the wide range

$4 < \frac{R_b}{D} < 2000$, which covers most natural streams of small and medium size. The formula $n = D^{1/6}/24$ gives good results for the range $20 < \frac{R_b}{D} < 200$, which covers most bed-load experiments. Within those limits both the von Kármán and the Manning formula are practically identical and a choice between the two, as to general

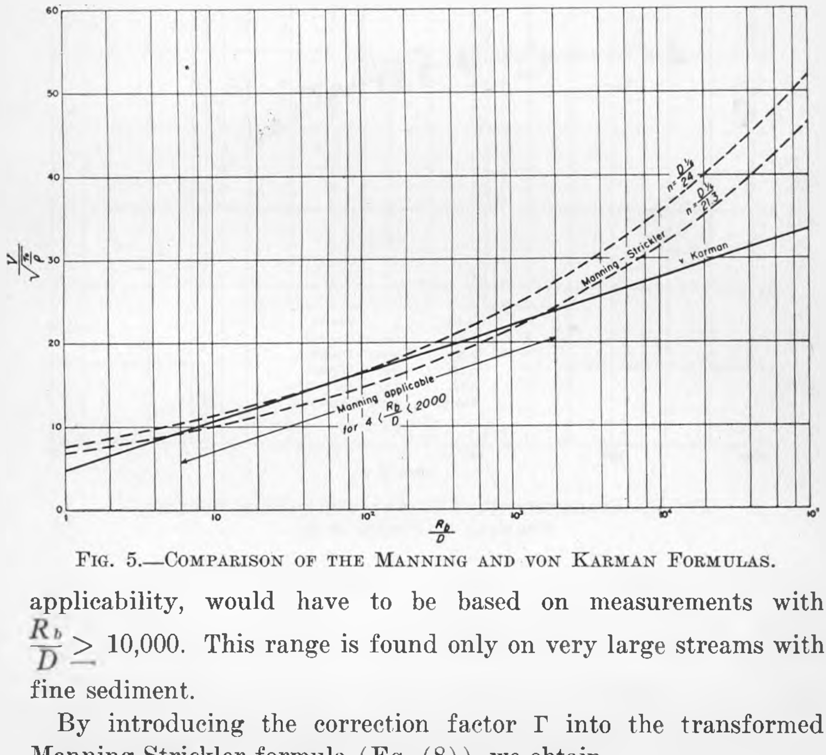


FIG. 5.—COMPARISON OF THE MANNING AND VON KARMAN FORMULAS.

applicability, would have to be based on measurements with $\frac{R_b}{D} \geq 10{,}000$. This range is found only on very large streams with fine sediment.

By introducing the correction factor Γ into the transformed Manning-Strickler formula (Eq. (8)), we obtain

$$\frac{V}{\sqrt{\tau_0/\rho}} = \Gamma \left[7.66 \left(\frac{R_b}{D} \right)^{1/6} \right] \tag{9}$$

Fig. 6 shows a plot of Γ, as obtained by this formula, against ϕ, and it is noticed that the points group around a curve similar to Fig. 4. For practical purposes, both the von Kármán and Manning formulas apply equally well within the above-mentioned limits; however, the Manning formula is generally preferred because of the ease of calculation with a slide rule.

That part of the energy of a stream which is transformed into turbulence along the side walls will not be available for transporting bed load. This was practically emphasized in the analysis of the Mountain Creek data, where the banks were assumed to have a roughness of $n = 0.056$ and were therefore several times more

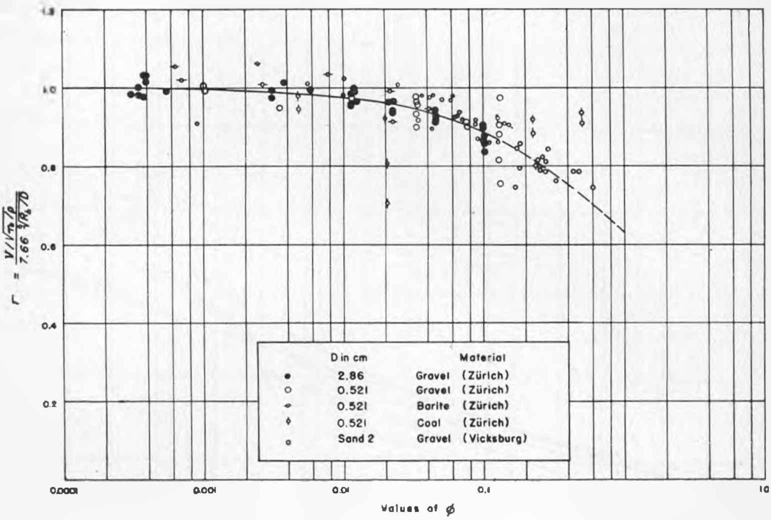


FIG. 6.—CORRECTION Γ OF MANNING-STRICKLER FRICTION FORMULA IN TERMS OF Φ.

effective in dissipating the stream's energy than was the bed, which had a roughness of approximately $n = 0.0134$. A rather narrow strip of relatively rough bank surface, therefore, will be as effective as a much wider strip of the sand bed. In the analysis of the Mountain Creek data, as well as in the preparation of Figs. 4 and 6, the effect of side-wall friction was eliminated by the method employed in determining the ψ-ϕ bed-load formula [1].

The practical application of the foregoing formulas in calculating the curves of discharge and rate of bed-load transportation in a certain reach of a stream, such as that of Mountain Creek (Fig. 2), requires a few simply-observed field data and a few basic assumptions. The field work consists of measuring a set of cross sections in a straight reach and gathering representative composite bed samples. The time required for these observations consists of only 2 or 3 hours for a two-man party. The office work involves only a few simple calculations when the roughness factor for the stream banks is assumed. As was done in preparing the calculated curves in Fig. 3, uniform flow is assumed to prevail. The computation of the curves of rate of bed-load transportation and water depth in terms of discharge requires about two hours' time for an experienced engineer. The whole process, therefore, requires but little more time than the hydraulic computations for a certain cross section where the overall friction factor is assumed. The method is definitely more reliable, though, and gives not only the discharge capacity of a cross section but also the rate of bed-load transportation.

No attempt will be made to elaborate on the importance of a strictly analytical method that permits the stream capacity for both water and bed load to be computed. While other methods may permit the determination of the discharge and rate of bed-load transportation from certain observed factors in an existing stream, the analytical method is the only one that permits a prediction of the consequences of proposed changes in the conditions of a stream. The method makes a systematic planning of river correction works possible. Its full value, however, lies in the planning of soil-conservation and flood-control programs when the watershed as a whole is considered. It also provides a method of tracing sediment down a stream and locating zones where damaging sedimentation may occur. Unfortunately still more field measurements are required to extend the known data into the region for $\phi > 1$. Without such information the application of the method is confined to small rivers and coarse sediment.

REFERENCES

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