# THE LOGIC OF MEASUREMENT

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### GUSTAV BERGMANN

Departments of Philosophy and Psychology,
State University of Iowa

The logic of measurement is a chapter in the philosophy of science. One cannot, therefore, discuss it without possessing a few notions that belong to the philosophy of science in general. My first task is thus, inevitably, to acquaint you with some such notions. In doing this I shall for the most part talk about words, sentences, and language in general. Lest that mode of presentation leave you unnecessarily bewildered, I had better, first of all, say a few words about the so-called linguistic turn, which is probably the most important single thing that has happened in analytical philosophy in this century. I shall limit myself to science and its philosophy. The idea is that while the scientist watches the world, the analytical philosopher watches the scientist watching the world. Let me unpack this formula so that we may see exactly where language comes in. Using the word thing very broadly, one may say that the scientist attends to things, observing them, manipulating them, and so on. Having done that for a while he states what he has seen in words. His language is thus about things, or, more generally, about the world. The analytical philosopher watches and eventually speaks about the scientist's watching and eventually speaking about the world. What, then, one may ask, distinguishes the analytical philosopher from either a psychological student of linguistic behavior or a grammarian? The answer is revealing. There is in our tradition a group of questions that have never ceased to challenge intellectual curiosity such as, say, the nature of causation or the peculiar certainty of deductive inference. These are, of course, the classical philosophical questions. The linguistic philosopher believes that they are all linguistic questions, not in the sense that they are mere questions of grammar or of linguistic behavior, but rather in the sense that the only commonsensical and therefore safe way of answering them begins with an investigation of how language reflects what it is about. These, to be sure, are but poor hints. But I fear that I must let it go at that.

I begin with a dichotomy among words, that is, with a division

of all words into two kinds such that each word belongs to one and only one of the two. One of the two classes is exemplified by, say, 'cat', 'color', 'specific heat'; the other, by 'or', 'some', 'is'. The words of the first kind, that exemplified by 'cat' and so on, are called *descriptive*. Those of the second kind, exemplified by 'or' and so on, are called *logical*. The two names, descriptive and logical, allude to the difference between the two kinds. After it has once been pointed out, the difference is, I think, clearly felt. Cats, colors, and specific heats are, to be sure, things very different from each other. Yet they have something in common. They are all "things" or, to put it linguistically, the three words 'cat', 'color', and 'specific heat' each refer to something or name something in a sense in which logical words such as 'or' or 'some' could not by any stretch of the imagination be said to name anything. What the distinction is, is thus clear. To convince one's self that it is worth making one merely has to reflect that in spite of this peculiar feature of logical words, namely, their not referring to anything, no langauge could conceivably get along without them. But let me here add a word of caution that applies equally to all further distinctions. The illustrations I have chosen are clear-cut. Our natural languages, though, English, French, German, and so on, are built for expediency, not for the purpose of exhibiting logical structure. About some words of a natural language there may therefore be doubt whether they belong to the one kind or to the other or, perhaps, depending on how they are used, to both. At this point the analytical philosopher appeals to the schematic languages constructed by the mathematical logician. In such schemata the distinctions are clear-cut. We had better realize, though, that this method raises the question whether one may for the purposes of philosophical analysis without loss or violence replace our natural languages by those schemata. The question must be argued. The answer is affirmative. I shall of course not present the very involved argument on this occasion.

The dichotomy we must consider next is one among sentences. Take the two sentences 'Either it is raining or it is not raining' and 'If everything is green then this is green'. The members of the class they exemplify are called *analytic*. Take next 'Peter is tall' and 'Water if heated boils'. These two sentences belong to the second class. Its members are called *synthetic*. The difference which makes the difference is that while a synthetic sentence says something, an analytical one is tautological or empty. Accordingly, whether a synthetic sentence is true or false depends on what is the case. An analytic sentence, since it says nothing about the

world, could not possibly be false. It fits well with this that whether or not a sentence is analytic does not depend on the meaning of the descriptive words which occur in it. Or, to say the same thing positively, whether or not a sentence is analytic depends only on its form, that is, on the logical words in it and on the order and arrangement of the descriptive words it contains. But then, one may wonder why anyone should bother with analytic sentences at all. At first sight it would appear that, saying nothing, they are merely the pathological limiting case of sentences. To understand why the appearance is deceptive, one only needs to remember the crucial role of deductive inference. Take Euclidean geometry. Every Euclidean theorem, those already known as well as those still to be discovered, is a deductive consequence of the Euclidean axioms. What holds for geometry holds equally for every scientific theory. Deductive inference is thus crucial; and there is only one way to explicate what we mean by deductive inference. The conclusion $C$ can be deduced from the premiss $P$ if and only if the compound sentence 'If $P$ then $C$' is analytic. Presently I shall return to the point; but it will pay if we familiarize ourselves with one more dichotomy among words.

Assume that a foreigner who does not know English very well comes for the first time across the word 'mare'. If he asks us what it means we can do either of two things. If a female horse is at hand we can point at it and say 'This is a mare'. In this case our foreigner learns the meaning of the word directly. Or we could say, without pointing at anything, 'A mare is a female horse'. In this case we have provided the questioner with a *definition*. If he knows what 'horse' and 'female' mean, he will then also know what 'mare' means. Now it is immediately evident that we could not possibly learn the meaning of all words by definition. The meaning of at least some descriptive words, for instance, can be acquired only by becoming acquainted with their referents. On the other hand, it is one of the most important philosophical ideas that we could, in principle, get along with an amazingly small number of undefined words, introducing all others by definitions. Words are thus in principle of two kinds, either *defined* or *undefined* (basic). In a schematic or artificial language the distinction is again clear-cut. To grasp its philosophical significance one merely has to consider that, with one exception, to which I shall attend at the end, even the most abstract descriptive terms of science, with whose referents we are not directly acquainted, can be defined by means of a basic descriptive vocabulary with whose referents we are so acquainted. At the moment, though, I am more interested in the defined logical

words. A defined word, by the way, is descriptive if and only if at least one basic descriptive word occurs essentially in its definition; it is logical if only logical words occur in its definition.

Let me take stock. I have introduced you to three dichotomies. Sentences are either analytic or synthetic. Words are either logical or descriptive and either defined or undefined, which yields, in the familiar fashion, four possibilities. Modest as this apparatus is, it permits one to state intelligently and, I hope, intelligibly, one of the most fundamental results of recent philosophical analysis. All arithmetical words, from the integers 1, 2, 3 . . . . . and the humble '$+$' to the most complicated notions of higher analysis, are logical words. All arithmetical truths, from the simple '$1 + 1 = 2$' to the most esoteric theorem about Hilbert spaces are analytic. The decisive idea is that, rather surprisingly, the integers themselves and the elementary operations among them can be defined in terms of indubitably logical words such as 'and', 'or,' 'all' and 'some'. The great names connected with this discovery are Peano, Frege, and Russell. The classical document, though of course not the last word, is *Principia Mathematica* by Russell and Whitehead.

Measurement is the assignment of numbers to objects or events according to certain rules. That is why it was necessary to begin as I did. Consider the two statements '$5 + 3 = 8$' and '5 feet *and* 3 feet are 8 feet'. One who understands fully the differences and similarities between these two statements has the key to the logic of measurement; but even to state these differences and similarities one needs our little apparatus. At the moment I shall only mention the differences. '$5 + 3 = 8$' is analytic. The '$+$' in it, not naming anything, is a logical word of the kind called an operator, that is, it is a word that makes out of two or several words, in this case out of '3' and '5', an expression, in this case the phrase '$5 + 3$', which functions itself in many respects like a word. The second statement, '5 feet and 3 feet are 8 feet', is a synthetic sentence of the kind called an empirical law. The 'and' in it is a descriptive operator. What it names, very elliptically, is what we do when we put two straight sticks end to end so that they are in a straight line and then, perhaps, either nail or glue them together. But I notice that I just used several words that need explication.

I shall not undertake to explain what a law is beyond mentioning that, as I use them, the word 'law' and the phrases 'empirical law', 'law of nature', and 'synthetic generality' are synonymous. But it will be necessary for us to distinguish laws from what analytical philosophers call *relations*. Take Boyle's law, $pv = c$. Scientists often say that this law is or establishes a relation between the

volume and the pressure of a quantity of gas at a constant temperature. Thus they use 'law' and 'relation' more or less synonymously. This usage blurs a distinction. Strictly speaking, a relation is a character. Being taller, being contiguous, being later are three simple instances of relations. A relation, in other words, is like a property, the only difference being that while a property is exemplified by one thing, a two-term relation is exemplified by two things, a three-term relation, such as betweenness, by three things, and so on. Accordingly, relations are referred to either by words or by phrases which function like words. Laws are expressed by statements. Let 'Peter' be the name of an adult elephant, 'Paul' that of an adult chihuahua. The sentence 'Peter is taller than Paul' says that a certain relation obtains between the individuals Peter and Paul. Accordingly, it contains the relational expression 'is taller than'. But, not being a generality, it is not a law. Take next the sentence 'All crows are black'. It is a generality or law. Yet it does not mention a relation. Accordingly, none of the four words in it is a relation term. 'Crow' and 'black' are the names of descriptive properties; 'all' and 'are' are nonrelational logical words. Take finally 'An adult elephant is taller than an adult chihuahua'. This statement says that every individual of a certain kind stands in a certain relation to every individual of a certain other kind. It is therefore a generality or law. Also, this particular law does mention a relation, as the law 'All crows are black' does not. These examples should go a long way toward convincing anybody that we had better be careful about the way we use 'law' and 'relation'[1].

Relations are either descriptive or logical. Take two straight sticks, put them along side of each other so that one end of the one coincides with one end of the other. If in this position stick $a$ protrudes beyond stick $b$, we say that $a$ is longer than $b$. If the two ends we have not put together coincide, so that neither stick protrudes beyond the other, then we say that $a$ and $b$ are equally long. "Longer" and "equally long" are two descriptive relations. As it happens, they are also defined relations. I have, in fact, just defined them in terms of two other relations, namely, coinciding and protruding; nor would it be difficult to define protruding in terms of coinciding and thus our two relations in terms of coinciding alone. Notice also the following empirical law about straight sticks. Of any two sticks, either the first is longer than the second or the

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[1] It is worth noticing that every empirical law can be construed as a statement to the effect that the *descriptive* characters (up to and including type $n$) mentioned in it satisfy a *logical* relation (of type $n + 1$). In this sense every empirical law has a "logical structure." See also footnotes 2 and 3.

second is longer than the first or they are equally long. Let us next provide ourselves with some instances of logical relations. Those obtaining among numbers are a very important kind of such relations. To be "divisible," for instance, is a relation that may or may not obtain between integers. 6 and 3 exemplify this particular relation, 6 and 5 do not. "Sum" is a three-term relation exemplified by 5, 3, and 8 in this order and in the order 3, 5, 8 but not in any other order nor, say, by 5, 3, and 9. Again, to be "larger" and to be "identical" are logical two-term relations among real numbers and numerical expressions. Notice finally the following analytical truth. Of any two real numbers or numerical expressions either the first is larger than the second or the second is larger than the first or they are identical.

I just called attention to an empirical law for straight sticks, "Of any two (straight) sticks, either the first is longer than the second, and so on," and to an analytic truth, "Of any two (real) numbers, either the first is larger than the second, and so on." Clearly, there is some connection between these two generalities; the one, synthetic, about things; the other, analytic, about numbers. Clearly, this connection is essential for measurement, that is, as I put it, for the rules by which numbers are assigned to things or events. Equally clearly, I think, our task is therefore to state this connection both as precisely and as generally as possible. As to precision, I can of course not offer much on this occasion. As to generality, a few preliminary remarks are necessary.

So far I have spoken as if there were only two kinds of things in the world, individuals, whatever that may mean, and characters, either properties or relations, such that the characters are exemplified by individuals. Now we must rid ourselves of this simplification. Properties and relations do in turn have properties and stand in relations to each other. Or, to say the same thing in Russell's words, there are characters of different types. Consider the sentence 'Green is a color'. If, as on this level of abstractness one must, we take our cue from grammar, the very fact that this sentence makes sense (it is even true) indicates that 'color' names a descriptive character of characters. For our purposes, though, the logical characters of characters are of particular interest. Transitivity, for instance, is such a character, as may be seen from our saying, truly, that the descriptive relation of being longer, which obtains among straight sticks, and the logical relation of being larger, which obtains among real numbers, are both transitive. As everybody knows, a two-term relation is called transitive if and only if, for any three things, it obtains between the first and

the third provided it obtains between the first and the second as well as between the second and the third. Since I just defined it, transitivity obviously is a defined relation[2]. As to its being logical, one merely has to convince one's self that the definiens, that is, the clause following 'if and only if' in the definition I just wrote down, contains only logical words. This is indeed so. The only two words about which one could have any doubts are 'thing' and 'relation'. In a schematism these two words would be represented by what are called variables of unlimited range. That such variables are logical signs is indeed plausible. Even so, the case must be argued; it can of course be argued; but I am sure you will not expect me to expound so subtle a point in the philosophy of logic proper on this occasion. I instead call your attention to the fact that when we say of a descriptive character that it has a certain logical character, e.g., that being longer is transitive, then we state an empirical law, while when we say the same thing of a logical character, then we state an analytic truth.

We notice, then, that the descriptive relation of being longer, whose field is the class of (straight) sticks, shares a logical property, namely, transitivity, with the logical relation of being larger, whose field is the class of (real) numbers. It is easily shown that the same holds for the descriptive relation of being equally long in the field of sticks and the logical relation of identity in the field of numbers. They, too, share some logical properties, e.g., they are both symmetrical, transitive, and reflexive. Similarly, there are certain logical relations exemplified by longer and equally long in the field of sticks as well as by larger and identical in the field of numbers[3]. I shall express this state of affairs by saying that the two fields, that of sticks and that of numbers, share with respect to these two pairs of relations a certain *structure*, or a certain logical structure. This, by the way, is one specific meaning of that desperately vague word, structure. I hurry to add that the word field, in the sense in which I use it here, also can and must be defined precisely. But I trust no harm will be done if, for brevity's sake, I shall continue to use it without further explication.

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[2] There are no undefined logical relations.

[3] Let $R$ be the logical relation between two relations $r$ and $s$ that is defined by
*Df.*

$$R(r, s) = (x, y, z) \ [r(x, y) \cdot s(x, z) \supset s(y, z)].$$

Substitution of 'equally long' and 'longer' for '$r$' and '$s$' respectively yields, for the case of length, one of the axioms of rank order.

I am now ready to generalize from our example. Consider two classes of objects (I use 'object' very broadly, really only to fill the need for a grammatical "object"), $a$, $b$, $c$, ...... and $\alpha$, $\beta$, $\gamma$, ......, such that the first is the field of $n$ relations, $r_1$, $r_2$, ....., $r_n$, the second the field of $n$ other relations, $\rho_1$, $\rho_2$, $\rho_3$, ....., $\rho_n$. Assume that these two groups of relations share a certain logical structure in which $r_1$ corresponds to $\rho_1$, $r_2$ to $\rho_2$, and so on, up to and including $n$, as in our example being longer and equally long correspond to being larger and being identical, respectively. If this is so, then it will often be possible to coordinate to each object $a$ of the first field one and only one object $\alpha$ of the second field so that a certain relation $r$ obtains among objects of the first field if and only if the corresponding relation $\rho$ obtains among those objects of the second field which are coordinated to them. Again I must warn you that all this is not as precise as it can and must be made. But again, it will serve our very limited purpose. In the case of measurement, the $\alpha$ and the $\rho$ are of course the numbers and the arithmetical relations among them; the objects $a$ and the relations $r$ are the things we measure and the descriptive relations among these things. What makes measurement possible is that the two fields share a certain logical structure. But then, as we just saw, to say that a descriptive relation or a group of such has a certain logical structure is the same thing as to say that these relations satisfy certain empirical laws. It follows that measurement can be introduced into a field if and only if the relations which obtain among its things fulfill certain empirical laws.

I am virtually certain that some of you are ready to question the value of all this strained and studied generality. Let me anticipate this sort of criticism. With respect to our example I have, roughly speaking, said no more than this. Since any two straight sticks are either equally long or one is longer than the other and since being longer is transitive, one can to each stick so coordinate a number that (1) two sticks are equally long if and only if the numbers coordinated to them are identical and that (2) one is longer than the other if and only if the number coordinated to the former is larger than that coordinated to the latter. (This, by the way, is merely a rank order and therefore not yet as desirable a measurement as can be established in view of the so-called additivity of length.) I grant cheerfully that you have known this before. Yet I insist that there was some point to our labors. The example is obvious. Naturally; examples ought to be obvious. The point is that the formulation I based on it is so general that it comprehends everything and anything any scientist, either physical or bio-

logical or behavioral, ever called and, I venture to predict, ever will call measurement. If, for instance, some among you wonder whether the formulation also comprehends what physicists call vectors (behavior scientists speak of multiple scores or profiles), let me remind them that what goes for numbers also goes for ordered pairs, triples, and $n$-tuples of numbers, if for no other reason than that these latter entities are themselves defined in terms of numbers. The generality we achieved is thus complete; and to have achieved such generality is the same thing as to have analyzed or explicated the logical nature of measurement. In what follows I shall support this contention by showing that some further comments, which are I think of some interest, all flow from our general formulation.

Measurement, I suggested, is the assignment of numbers to things according to certain rules. The word rule in this formula is ambiguous. Yet I used it deliberately, because I was then not ready to dissolve the ambiguity. Now I am. One meaning of 'rule' is that of convention. Rules in this sense are arbitrary or matters of social agreement. Another meaning of 'rule' is that of law of nature; and there is nothing arbitrary or conventional about laws of nature. As to our formula, I now amend it to read that measurement is the assignment of numbers to things according to both laws and conventions. The laws on which measurement is based are of two kinds. To distinguish among them, it will be convenient to introduce a new term. Instead of always speaking laboriously of a field and the descriptive relations within it, I shall, as one usually does, speak of a *dimension*. Measurement may then be based on no other laws than those within the dimension, that is, on no other laws than those connecting its descriptive relations. Such measurement is called *fundamental*. This is the only case we have so far considered. Or measurement may also utilize laws that connect the characters of one dimension with those of others. Such measurement is called *derived*. I shall attend to it presently. As to the conventions entering measurement, we often find that even if we utilize as many empirical laws as we possibly can, the co-ordination of numbers to things is not yet uniquely determined. Consider two obvious examples. If the empirical laws utilized are those of a rank order and of nothing else as, for instance, in the Mohs hardness scale, then any assignment of numbers that preserves the order is as good as any other. But even in the case of a measurement as desirable as the ordinary measurement of length the choice of the so-called unit is still a matter of convention. The result of the assignment of numbers to the things of a dimension

according to certain laws and conventions is called a *scale*. Thus we are led to make another distinction. We had better not confuse a dimension with the several scales that may be constructed to measure it.

Straight sticks can be ranked. In such a rank order a one-inch stick may receive rank 1, a seven-inch stick rank 2, a 19.5-inch stick rank 3, and so on, quite wildly, provided only that the longer stick always receives the higher rank. But sticks can also be measured in the ordinary way, say, in inches. Everyone agrees that the second measurement is more desirable or better than the first. This agreement sets us the task of stating exactly what it is that we mean when we express such preferences. The answer is not difficult. The essence of measurement is that *some* arithmetical relations among the numbers assigned correspond, by virtue of a shared logical structure, to descriptive relations among the things to which they are assigned. The measurement we prefer to others is so constructed that a *maximum* number of arithmetical relations has such descriptive correlates or, as one also says, empirical meaning. In a mere rank order, for instance, the so-called equality of differences has no empirical meaning. Specifically, it makes no sense to say that the difference in hardness between two minerals of Mohs ranks 2 and 4 is equal to that between two minerals of ranks 7 and 9. What holds for a rank-order hardness also holds for a rank-order length. In ordinary length measurement, on the other hand, the equality of differences has a familiar meaning to which I shall presently attend. First, though, I should like to make another point. The first one who saw it was, as far as I know, the great Helmholtz, in his essay *"Ueber Zaehlen und Messen."*

It follows from our explication that counting is not a species of measuring, or perhaps better, mere counting is not yet measuring. The reason is that numerosity, that is, being of a certain number, is a logical property of classes, not of things, and that the grouping of things in classes, in the sense of 'class' which is here relevant, is arbitrary in a sense in which the grouping of things on the basis of the descriptive characters they exemplify is not. This, however, is not to deny that counting may be an ingredient of measuring. It very often is. If, for instance, we assert that a certain ledge is three inches long, we have counted the layings-off of a unit. But then, these layings-off are the descriptive relational ingredient that is not to be found in mere counting.

The peculiar excellence of ordinary length measurement rests on that feature of the dimension which is known as *additivity*.

Again, I am sure that you know what is involved. To remind your-
self, remember the two sentences with which I started, '5 + 3 = 8'
and '5 feet and 3 feet are 8 feet'. I shall again state the matter as
generally as possible. A dimension is called additive if and only if
it permits of an operation within it that fulfills two conditions.
(1) The operation, called "physical adding," coordinates uniquely
to any two objects of the dimension a third, called their "physical
sum." (2) The descriptive operator has the logical structure of
arithmetical addition. In the case of length the physical operation
consists, schematically speaking, in laying two straight sticks end
to end in a straight line and then either nailing or gluing them
together. Now for two comments. *First:* Notice that I spoke of
operations *within the dimension.* What that excludes is best
shown by examples. Take temperature, which is a linear but not
an additive dimension and assume that a rank-order scale for it
has somehow been constructed. Suppose that someone proposes
to make this rank order additive by defining as the "sum" of ranks
$T_1$ and $T_2$ the rank $T_1 + T_2$. We shall point out to him that he has
done nothing of the sort, since in his definition no physical oper-
ation is mentioned. All he has said, in a rather misleading manner,
is that the number $T_1 + T_2$ is the arithmetical sum of the numbers
$T_1$ and $T_2$. Assume next that someone else proposes as the physical
sum of temperatures $T_1$ and $T_2$ the temperature $T'$ that prevails
in two objects, originally of temperatures $T_1$ and $T_2$, respectively,
after they have been brought into contact and thermic equilibrium
has established itself. This time there is a physical operation. Un-
fortunately, it does not fulfill our first condition. $T'$, as we all
know, is not uniquely determined by $T_1$ and $T_2$. This defect, how-
ever, can be remedied. If we specify that the two bodies are to be
of the same weight and of the same chemical composition, then $T'$
is uniquely determined by $T_1$ and $T_2$. Thus, our first condition
being fulfilled, it would seem that we have at least a candidate for
a physical sum. Again, we all know that however the rank order
may have been scaled, this operation does not fulfill our second
condition, that is, it does not have the logical structure of addition.
This, though, is not the point I want to make. The point is, rather,
that even if the second condition were fulfilled, the operation
would still not be one within the dimension since, in order to secure
the definiteness of $T'$ we had to draw upon extraneous factors,

namely, the dimension of weight and chemical composition[4]. *Second:* Notice that I spoke throughout of the additivity, not of a scale, but of a dimension. To grasp this point, consider again length, this time scaled logarithmically, that is, the way we actually measure it when we use a slide rule. With this scale the number assigned to the physical sum is not, as with the ordinary scale, the arithmetical sum but, rather, the arithmetical product of the numbers assigned to its physical constituents. All one can say, therefore, is this. If a dimension is additive, which is an empirical matter and not one of scales, then it *can* always be so scaled that, as in the case of ordinary length, the number assigned to the physical sum is the arithmetical sum of the numbers assigned to its physical constituents.

We have come upon a new question, namely, whether it is merely a matter of habit that we prefer, as we actually do, except for some very special purposes, the ordinary foot-rule scale to the slide-rule scale of length. The answer, which is again quite general, is this. Other circumstances being equal, we prefer that scale which gives to a maximum number of laws, or, perhaps, to a certain group of laws in which we are specially interested, the simplest mathematical form. In a sense there are thus rational grounds for this kind of preference, too. If I say "in a sense," it is because I, for one, believe that the notion of simplicity itself is by no means simple, or, perhaps better, that it is essentially a psychological notion. This, however, is a long story and a rather controversial one at that. So I shall assume that we know what we mean by mathematical simplicity and show next how this notion operates in the case of derived measurement.

With a few idealizing assumptions it is possible to introduce through fundamental measurement a rank order in the dimension known as density. In the case of nonmixing liquids, for instance, we discover that when we pour any two of them together either one always goes to the top or the one poured last, whichever of the two it may be, stays at the top. On these two empirical relations a rank order can be based. Such a rank order may be quite wild; say, olive oil 1, water 2, concentrated sulfuric acid 17, and so on. Assume

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[4] Some might object that by analogy with this example even mass and weight would not be additive and, perhaps, not even rankable within the dimension since, if we use a balance based on the lever principle, we must specify that its arms are of equal length. I would answer that in this case the extraneous dimension, length, is not, as in my example, a relevant property of the objects to be scaled. Or, to speak metaphorically, the balance plays the role of a parameter, not that of a variable.

next that we have also discovered the law that connects this dimension with two other ones, namely, weight and volume, each of which is additive and already provided with a scale very rich in empirical meaning. The law is, of course, that the rank order we established by fundamental measurement is identical with that produced by the quotient of weight over volume. Under the circumstances we may choose these quotients themselves as the numbers of our original rank order. What we achieve by this choice is that the law connecting the three dimensions takes the simple form $d = \dfrac{W}{V}$. I am sure you see how this example hits two birds with one stone. It shows how arithmetical simplicity enters and it also shows what we mean by derived measurement. For derived measurement, you remember, is measurement that utilizes the laws which connect the characters within a dimension with extraneous ones. Ordinary density is thus a derived scale with weight and volume as the extraneous dimensions. In the case of density, we saw, fundamental measurement within the dimension is, at least in principle, possible. But there is also the possibility of defined magnitudes such that no scale can be based on any relations among them, so that we can in their case no longer strictly speak of a dimension. Even in these cases very satisfactory measurement may be possible. All that is needed is that the defining magnitudes be satisfactorily scaled. The definition then yields automatically a meaningful scaling of the magnitude defined. That is, as it were, the limiting case of derived measurement. The centimeter-gram-second units of theoretical physics are probably the most important application of this idea.

I am ready to sum up. Details and some unavoidable preparations apart, I have asked and answered two questions. The first question was: How is it possible for arithmetical relations among numbers assigned to things to mirror descriptive relations among these things themselves? The answer is: By virtue of a shared logical structure. The second problem with which I dealt is in both question and answer a corollary to the first. The question was: On what rational grounds do we prefer one measurement to another? The answer is: We call that measurement best which manages to endow a maximum of arithmetical relations with empirical meaning. A little reflection will show you that these are the only two problems with which, however sketchily, I have dealt. They are indeed the logical heart of the matter, although they are not the whole of it. In conclusion I shall therefore at least mention four further issues that arise in the logical analysis of measurement.

1. There is an important difference between, say, assigning a number to the momentary strength of an electric current in a wire and assigning numbers to the momentary position or velocity of an electron. The difference is not that in the case of the current we read the number directly from a dial while in the other case, that of the electron, we obtain it by computation from the number or numbers read from one or several dials. The difference is, rather, that the strength of a current can be defined in terms of what we are directly acquainted with while such entities as electrons belong to what is called a model or, more generally, a partially interpreted calculus. It follows that in the case of the electron the numbers assigned depend essentially on the features of the model itself as well as on the way it is fitted to what we are acquainted with. To appreciate the importance of the distinction consider the possibility that by the very way the model is constructed and fitted it yields, even upon the most accurate measurement, not a definite value of, but merely a range for, say, the position of an electron. Clearly, such indeterminacy in assignment must be distinguished from what could reasonably be meant by a lack of accuracy in measurement. As you know, situations whose analysis requires this distinction actually occur in modern physics.

2. No measurement yields or ever will yield a real number. In the ordinary course of events we are satisfied with two or three digits; to ascertain reliably five or six digits is, as every scientist knows, a major effort. All we obtain operationally are thus fractions. In our computations we nevertheless consider those fractions as real numbers or as approximations to such. The advantage of this procedure is essentially computational, that is, logical. That it is advantageous, there is no doubt. The question that arises concerns the legitimacy of the jump from the fractions which we actually obtain through physical operations to the real numbers with which we "operate" verbally. The answer is that whenever we do mention real numbers in synthetic statements we have in effect introduced a partially interpreted calculus.

3. I just spoke several times of accuracy and once of reliability. These words, too, stand in need of elucidation. *Reliability* is a statistical notion, defined by some such terms as, say, the inverse standard deviation of a series of successive measurements. What I mean by *precision* is, simply, the number of digits of a given unit ascertained in a single measurement. The word accuracy is, I believe, often used very inaccurately. The two clear notions are reliability and precision. When we speak of accuracy, we mean

either the one or the other or, perhaps, some ill-defined compound index of both.

4. Imagine a temperature measurement so precise that the heat exchange between the measuring instrument and the system measured cannot be neglected. In this case we do not expect repeated measurements to yield the same result. Quite to the contrary, we would be baffled if they did. This shows that the explication of reliability I just suggested is not yet as general as it could be and, therefore, ought to be. The general issue involved is that of the interaction, or possible interaction, between the object measured and the yardstick with which we measure it. This issue, too, arose in a rather radical form in modern physics. Its analysis leads far beyond the limits of this very elementary discussion.*

## REFERENCE

*See G. Bergmann, "The Logic of Quanta," *Am. J. of Physics*, 15, 1947 (reprinted in H. Feigl and M. Brodbeck, eds., *Readings in the Philosophy of Science*, Appleton-Century-Crofts, 1953).