THE ROLE OF THEORETICAL PREDICTION IN FLUID MECHANICS

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INTRODUCTION

It is appropriate that, at the opening session of this conference on hydraulics, the chairman is a representative of the sister field of naval architecture. It symbolizes the parallel efforts between workers in these two fields to introduce into their respective arts rational techniques based upon the science of fluid mechanics.

We live in an era when the boundaries of science are so vast and their rate of growth so rapid that all but the most gifted intellects of those devoted to the extension of the frontiers of science must specialize in a narrow range of a particular field. This is not a new situation. About one hundred years ago, Helmholtz, the great German physiologist and physicist, in an address [1] at the University of Heidelberg, stated:

> "We see scholars and scientific men absorbed in specialties of such vast extent, that the most universal genius cannot hope to master more than a small section of our present range of knowledge. . . . The contemplation of this astounding activity in all branches of science may well make us stand aghast at the audacity of man, and exclaim with the chorus in the Antigone, 'Who can survey the whole field of knowledge; who can grasp the clues and then thread the labyrinth?' One obvious consequence of the vast extension of the limits of science is that every student is forced to choose a narrower and narrower field for his own studies and can only keep up an imperfect acquaintance even with allied fields of research."

If this was the situation one hundred years ago, how much more so is it today? Each worker in science elects to apply himself to the specialty which best suits his interests and talents. In the course of an investigation of a new problem in his specialty, he knows that intense concentration is necessary. The related literature must be gathered and studied to ensure that there is no needless duplication of work already accomplished. He must determine his course of action, be it experimental or theoretical, be concerned about the selection of the experimental or analytical techniques, anticipate that even after great effort has been expended on the investigation, newly discovered facts may make it necessary to reconsider his course.

This need for specialization and concentration is not without its dangers. A student who easily masters a field for which he has both affinity and talent might suffer from conceit and underdevelopment of his weaker faculties if he were not required to participate in other disciplines. A lack of humility might affect his performance even in his own specialty, and give rise to needless antagonisms and misunderstandings among the different fields and sometimes even among parts of the same field. Thus some pure mathematicians are proud of the fact that they work in fields which are uncontaminated by applications and are hardly willing to admit that the applied mathematicians are mathematicians at all. Yet the history of science shows that both kinds of mathematics have been necessary in developing an understanding of the physical world, and alternatively, that the phenomena of nature have been a great stimulus to the development of mathematics. It is clear that for his own sake as well as for the sake of science, a student should be required to study many fields other than his specialty, including especially the history of the evolution of his own and related sciences.

For these reasons I am pleased to observe the breadth of interests on the program of the present and previous Hydraulics Conferences. My own subject also serves as an indication of the increasing awareness of engineers that the science of fluid mechanics can serve to yield practical solutions to many of their problems.

The human mind makes progress in understanding the physical world by many means. Experience and experiment result in intuitions, concepts, ideas, even certain laws relating measurable quantities. In the growth of a science, many men contribute to the accumulation of a large body of facts, concepts and laws, each according to his own talents. These laws, such as those of Galileo for falling bodies, and of Kepler for planetary motion, do not yet constitute a theory.

A theory is born when symbols representing certain basic concepts are assumed to have a certain mathematical relationship to each other from which it is possible to deduce, by purely mathematical operations, known laws relating the same symbols. This is the first role played by theoretical prediction in any science; a new theory must be able to predict known facts and laws. It may happen, however, that several theories may be able to perform this function. To alleviate this embarrassing over-abundance of theories, it has been attempted, at various times in the past, to make a selection on the basis of metaphysical concepts of a Creator and of the nature of the "real" world, or on the basis of simplicity, or authority. The point of view that has won out is that that theory should be selected which can predict new laws, provided these laws are found to be in agreement with experiment. This is the second role of theoretical prediction.

The first successful theory of this kind, Newton's principles of mechanics, was resented and attacked because of the strong support it gave to materialism. Laplace was also chided by Napoleon for having failed to mention the

Creator even once in his monumental work on celestial mechanics, to which the former replied, "Sire, I had no need of that hypothesis." Since Newton's time, successful physical theories have attempted only to understand and predict the relations between natural phenomena. Interpretations and conclusions beyond this have been left to the philosophers.

Dictionaries give many definitions of the word "theory." The one selected above implies that the term should be applied to a science only when the so-called theory is in mathematical form and is capable, by mathematical deduction, of theoretical prediction. Thus the inexorably exact techniques of the infinitesimal calculus play a dominant role in a scientific theory. This characteristic has given rise to two extremes of thought. Mathematicians, finding their problems in the equations of a certain theory, and exulting in their exact techniques and oftentimes esthetically pleasing results, may lose sight of the fact that all theories are at best approximations and that we cannot hope ever to understand the ultimate nature of things. Practical men, on the other hand, noting that the scientists themselves make no claims concerning the absolute validity of their theories and do not hesitate to scrap old theories for new when a theory, even one of long standing, is contradicted by a newly discovered fact of nature, and observing some misapplications of the theory, may consider that theories are but toys of the physicist, whose model of nature is only a mirage, furnishing games for the mathematician. Nevertheless, the fruitfulness, even of theories which are ultimately discarded, in predicting new laws of nature which may lead to practical applications affecting our everyday life or our national defense, has led to an increased respect for theory among those who would naturally distrust it.

THEORY OF INVISCID FLOW

Since the present address is concerned with predictions from the theory of fluid mechanics, it is necessary that we reach an understanding as to the nature of this theory. Although Newton had propounded the principles of mechanics in his "Principia Mathematica Philosophiae Naturalis," published in 1687, it was not until 1755, 68 years later, that a series of papers was published by Euler in which the science of hydrodynamics of an inviscid fluid may be considered to have been established. Like Newton's, Euler's achievement was not an isolated stroke of genius, but was based upon the advances and concepts of his predecessors and contemporaries. His great contribution was the avoidance of unnecessary assumptions as to the inherent structure of a fluid and the precise axiomatic formulation of the theory on the basis of the concepts of internal pressure, the nature of a continuous medium, and the principles of mechanics. Also applied by the new theory was the new technique of elementary abstraction, probably first used by d'Alembert to derive the equation of continuity, in which the equilibrium of an infinitesimal element of the fluid was examined. Thus Euler derived his fundamental equations of motion of an inviscid fluid

$$\mathbf{F} - \frac{1}{\rho} \text{grad } p = \frac{\partial \mathbf{v}}{\partial t} + u \frac{\partial \mathbf{v}}{\partial x} + v \frac{\partial \mathbf{v}}{\partial y} + w \frac{\partial \mathbf{v}}{\partial z}$$

where $\mathbf{F}$ is the external force vector per unit mass of the fluid, ρ is the density of the fluid, p is the internal pressure, and $\mathbf{v}$ is the velocity vector, with components u, v, w, each a function of position (x, y, z) and time t. This, together with the fundamental equation of a continuous medium which expresses that its matter is continuous and indestructible,

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho \mathbf{v}) = 0$$

constitutes a set of 4 partial differential equations in the 4 unknown functions u, v, w and p.

Euler was able to show that the discoveries of the Bernoullis, Clairault, and d'Alembert followed naturally from his equations. Thus, since the predictions from his theory were in accord with existing knowledge, he had little doubt concerning its merit. Among these results was one obtained in 1750 by d'Alembert and entered in a prize competition on the theory of fluid resistance sponsored by the Berlin Academy. When the additional requirement was made that experimental proof must also be submitted, d'Alembert withdrew his paper and subsequently published its contents in his treatise on fluid motion in 1752; this contained his famous paradox that the resistance experienced by a body in steady motion through a fluid is zero. One hardly wonders that the Berlin Academicians balked and demanded experimental confirmation. Since this result is completely contradicted by experience, the new theory of hydrodynamics, almost at its birth, was known to be unable to predict certain fluid phenomena. Thus, although Euler himself applied the theory (with an added force for fluid resistance) to the design of hydraulic machinery, practical men considered the theory which yielded so ridiculous a result as the d'Alembert paradox merely a mathematical curiosity.

Fortunately for the advancement of the science, mathematicians, who in the main were also natural philosophers, actively continued to develop and apply the theory. Lagrange and Laplace found that surface wave phenomena predicted by the theory were in accord with experiment, thereby demonstrating the fruitfulness of the theory. Thus it appeared that the Euler equations were suitable for application to certain classes of phenomena; but the applicability of the equations was limited by the fact that sufficiently powerful mathematical methods for solving the equations had not yet been developed. The difficulty is common to all problems of physics concerned with continuous media. The functions of interest depend upon the space coordinates and time and when one applies physical principles to an element of the medium, the mathematical formulation of the problem is obtained in the form of a set of partial differential equations. General solutions of partial differential equations, when they are obtainable, may contain arbitrary functions and hence are of little practical value. Additional "boundary" conditions, such as the velocity state of the fluid at some initial time, or the constancy of the pressure on a free surface, are required in order to specify a unique solution.

It was stated above that the first role of prediction from a new theory was to show that it encompassed existing laws, and the second role, to discover new laws. The application of the theory to the solution of engineering problems may be considered as the third role of theoretical prediction. One may ask, then, why there was so long a delay before the theory of inviscid fluids began to play this third role. Two reasons can be given; first that the theory of partial differential equations was only being born when Euler published his equations and that about 150 years elapsed before comprehensive treatises on the subject began to appear, and secondly, that the range of physical phenomena to which the inviscid flow theory could be safely applied was not fully understood until the publication of Prandtl's boundary layer theory [2] in 1904, also about 150 years later.

The most rapid and penetrating advances were made in the theory of irrotational flow from which Euler was immediately able to derive the Bernoulli equations. Euler's introduction of the potential function ϕ and its effective use by Laplace in his tome on celestial mechanics were significant advances. Both the velocity potential of hydrodynamics and the gravitational potential satisfied the equation

$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

now known as Laplace's equation, although previously found by Euler in 1752. Instead of four equations in four unknowns, it was necessary to solve only Laplace's equation from which the velocity field could then be obtained as $\mathbf{v} = \text{grad}\ \phi$ and then the pressure field obtained from the Bernoulii equation. Thus the further development of the theory of irrotational, inviscid flow became synonymous with the development of potential theory.

Laplace's "Mécanique céleste," which appeared in 5 volumes between 1799 and 1825 and which culminated the work of the great natural philosophers of the 18th century in applying Newton's principles to the motions of the planets and their satellites, had at first a remarkably ennervating effect upon his contemporaries. It seemed to them that, with the mastery of astronomy, all of nature was understood and that the field of mathematics was exhausted. One recalls that a century later physicists had the same feeling about their subject, just before the discovery of the electron, radioactivity and the publication of the theory of relativity. In both cases the feeling was quickly dissipated as both science and mathematics surged forward to new discoveries and the development of new fields.

PREDICTION OF TWO-DIMENSIONAL FLOWS

For two-dimensional flows the work of the great 19th century mathematicians, Gauss, Cauchy, and Riemann resulted in a powerful technique for attacking such problems. This consisted of the development of the theory of functions of a complex variable, and of its geometrical counterpart, the method of conformal mapping. The theory shows that every possible function of a complex variable is a solution of the two-dimensional Laplace equation and corresponds to a possible irrotational flow, the potential and stream functions of which are the real and imaginary parts of the given function. Thus many such flows have been constructed and collected in handbooks. The method of conformal mapping, on the other hand, often makes it possible to find a direct solution by transforming the given boundaries and boundary conditions for which a complex potential is sought into other boundaries and boundary conditions for which the complex potential is known. The relation between the known and unknown complex potentials given by the mapping transformation then enables the latter to be obtained.

The most common hydrodynamic problem, that of the flow of a fluid past a rigid, impenetrable boundary, is a special case of the Neumann problem of potential theory, in which the normal component of the fluid velocity is prescribed along the boundary. Another type of boundary condition, more common in the field of electrostatics, but also encountered in the free-streamline problems of hydrodynamics, is that of the Dirichlet problem, in which values of the potential are prescribed on the boundaries. It is a curious fact that. although it is easy to prove the uniqueness of the solutions of these problems, the mathematical proof of the existence of solutions was not accomplished until about 1900, by Hilbert.

The history of the evolution of the theory of conformal mapping is, in itself, a fascinating story. Riemann's publication of his geometric function theory in his doctoral dissertation in 1851, in which was included his famous mapping theorem that simply-connected regions can be mapped conformally into a circle, was immediately recognized as an outstanding achievement and a great advance which was applied by other mathematicians in their own researches. Thus the discovery by Weierstrass in 1869 that there was a fundamental flaw in Riemann's work created quite a stir in mathematical circles. Riemann died before he could reply to Weierstrass' criticism, and although many others attempted to salvage the proof, they were unsuccessful until Hilbert succeeded in 1900. As in the case of Fermat's last theorem, which he noted in the margin of a book and of which he did not furnish the proof for lack of space, the attempts to correct the proof of Riemann's mapping theorem stimulated important developments of new methods in the theory of conformal mapping and of the calculus of variations. It should be mentioned that these developments were principally concerned with existence proofs, which mathematicians consider to be one of their primary tasks. Mathematical existence proofs of the solutions of the equations of physics are of significance for two reasons. First, they confirm that the equations are meaningful in a physical sense, and secondly, they often furnish a practical procedure for obtaining particular solutions.

To solve a Dirichlet or Neumann problem for a simply-connected region by conformal mapping, the first and most difficult step is to map the given boundary into a circle. By taking advantage of the mapping properties of the elementary functions, it is possible to transform a great variety of analytical shapes into a circle; but for arbitrary shapes it is necessary to resort to numerical procedures, usually consisting of the solution of a great many linear equations in as many unknowns, in order to obtain an approximate mapping function. The same function also transforms the boundary conditions, so that one now has a boundary-value problem for the circle for which the solution is immediately given in the form of a Fourier series, or as Poisson's integral. Recently, a theorem due to Milne-Thomson, called the circle theorem, gave the solution for a circle immersed in an arbitrary potential flow, and consequently, by appropriate transformation, the solution for a prescribed shape in the corresponding external flow field. Thus, once the transformation of a boundary into a circle is found, one immediately has, for the profile, solutions corresponding to all the boundary conditions for which solutions are known for the circle. It may seem, for example, that the obtainment of a flow at an angle of attack α on a profile from the flow at zero angle is trivial since it is derived so easily by rotating the flow in the circle plane, but when it is considered that there is no such simple correspondence even in the simplest three-dimensional case of axisymmetric flow, it is seen how important and powerful is the transformation theory of two-dimensional irrotational flow.

The impression should not be formed that one is now in a position to solve all such two-dimensional flow problems. The preceding remarks apply only in the important class of problems in which the boundary conditions are linear combinations of the potential and its normal derivative on a known surface. Methods for solving certain problems with free surfaces of unknown shape depend upon transforming them into problems with known boundaries, such as the hodograph method, in which the free streamline is transformed into a circular arc, or linearization techniques in which conditions on an unknown boundary are assumed to prevail on a nearby, known boundary. Although the latter method is suited only for small disturbances, it has proved to be of great practical utility in such varied problems as that of surface gravity waves, the added mass of vibrating bodies, thin airfoil theory, and the cavitation of thin hydrofoils.

Two-dimensional irrotational flow problems of engineering interest, for which general methods of mathematical solution have not yet been evolved, are those in which the exact Bernoulli equation for steady flow,

$$p + \tfrac{1}{2}\rho\ (u^2 + v^2) + gy = C$$

which is non-linear in the velocity components u and v, serves as one of the boundary conditions. An example of this type of problem is the free overfall in which the pressure is constant on two free surfaces. Another is the free-surface gravity-wave problem for which some approximate solutions have been obtained by means of special artifices, but a general technique for obtaining the waves generated by prescribed disturbing mechanisms is not available. When analytical methods are wanting, recourse may be had to numerical procedures in which Laplace's equation is replaced by a difference equation and a solution calculated by the relaxation technique.

PREDICTION OF THREE-DIMENSIONAL, IRROTATIONAL FLOW

It has already been indicated that, for lack of a transformation theory, three-dimensional irrotational flow problems cannot be solved as readily as their two-dimensional counterparts. In many respects, however, the theory for the two types of problems is very similar. One can immediately express solutions of the Dirichlet and Neumann problems for the sphere, and one has a sphere theorem for obtaining the modification of a potential flow field when a sphere is introduced into the flow; these are in exact analogy with the two-dimensional theory for the circle. Futhermore, if a solution of a three-dimensional flow problem has been obtained in terms of the singularities (sources and doublets) which generate the flow, the forces, moments and added masses acting on bodies immersed in it can be expressed in terms of the strengths of these singularities by means of the Lagally and Taylor theorems in which the strengths of the singularities play a role analogous to that of the residues of the functions of a complex variable occurring in the corresponding theorems for two-dimensions, such as that of Blasius.

What mathematical techniques are available for solving three-dimensional irrotational flow problems? A survey of such classical texts as Lamb's

"Hydrodynamics" [3] or Jeans's "Mathematical Theory of Electricity and Magnetism" [4] reveals only two methods, one the separation of variables, the other the method of images. The former leads to solutions which can be expressed either as a series (generally infinite) of products of so-called orthogonal functions or as an integral whose integrand is a separable solution of Laplace's equation. Well known examples are the Fourier series and integrals, series of spherical harmonics, etc. An alternative and equivalent systematic procedure consists of the application of the transform calculus, i.e., of the Fourier, Laplace and other transforms, an approach which was first introduced by Heaviside as an operator calculus for the solution of problems in electrical engineering, but was viewed with distrust until it was placed on a firm mathematical basis with the evolution of the transform calculus. Although in principle separable solutions and corresponding systems of orthogonal functions can be found for general classes of boundaries, from a practical point of view the application of this method is limited to those few types of boundaries which are associated with orthogonal coordinate systems leading to families of orthogonal functions whose properties have been studied.

The other classical mathematical technique is the method of images. It can be applied conveniently to obtain solutions of problems in which the boundaries are plane or spherical, on the basis that the image system of a singularity (source, doublet or vortex) outside (or inside) of a spherical boundary is known. The recently published sphere theorem is the epitome of this principle. If the external singularities are known, their potentials, together with those of the image system, immediately give the complete solution to the problem. When the external singularities are also of the nature of images in spheres (among which planes may be included as spheres of infinite radius), the image system in the particular sphere of interest will disturb the singularity distributions in the external spheres and consequently one is led to the technique of successive images in order to obtain a solution. For example, the flow about a sphere in a channel bounded by parallel planes could be treated in this manner.

The method of images is of interest not only because it is one technique for solving flow problems but also because the forces and moments acting on a body and its added masses can be expressed in terms of the images by applying the Taylor and Lagally theorems. But does a method of images exist for mathematical shapes other than spheres? In two dimensions it was seen that this was essentially the case since, once the image system had been found for the circle, that for the prescribed boundary could be found by conformal transformation. In three dimensions the so-called Kelvin transformation shows that new potential functions can be derived from a given one by inversion; on this basis the sphere theorem can be proved and the method of images justified. It is shown in Kellogg's "Potential Theory," [5] however, that Kelvin's transformation cannot be generalized to yield a transformation theory suitable for other shapes.

There are strong indications, nevertheless, in the papers of Havelock and the newly completed Ph.D. dissertation by Bottaccini [6], that it may be possible to formulate a spheroid theorem which would accomplish the same ends as the sphere theorem, but, undoubtedly, in a more complex manner. Havelock, in a series of papers on a spheroid near a free surface, has made effective use of an image system above the free surface and even computed the secondary image system in the spheroid induced by the first set of images above the surface. Bottaccini has systematized this technique, making use of the orthogonality properties of series of Legendre functions, to obtain these second images within the spheroid which were determinative for evaluating the added masses of a spheroid accelerating near a free surface. The interest in having a spheroid theorem stems from the resemblance of elongated spheroids to bodies of practical interest, such as ships and projectiles.

Fortunately, again at about the turn of the century, the elegant theory of linear integral equations of the Fredholm type was developed and its close relation with the theory of sets of linear algebraic equations and of orthogonal functions was shown. Simultaneously with the development of this theory, it was being applied to prove the existence of solutions of the Dirichlet and Neumann problems by formulating them as integral equations. It has since been discovered that there are many different ways of transforming a flow problem so that its solution depends upon that of an integral equation. Because of its close connection with existence theory, mathematicians have principally considered solutions in terms of the so-called Green's function, the physical interpretation of which, for a body immersed in a potential stream. is that it gives the internal distribution of sources and doublets required, together with the external flow mechanisms, to yield the desired flow in which the body is a stream surface. In the electrostatic case, the Green's function gives the distribution of charge induced on the surface of a conductor when it is introduced into a given electrical field. Thus use of the Green's function may be considered as the extension of the method of images to arbitrary boundaries. In each case the Green's function is to be obtained as the solution of an integral equation.

Other means of representing a flow problem by an integral equation consist of assuming an unknown distribution of singularities, usually on the surface, or, if the body is axisymmetric, on the axis, and then applying the boundary condition to obtain an integral equation with the distribution as the unknown function for which the integral equation is to be solved. This solution is then accomplished either by means of an iteration formula, which yields a solution by successive approximations, or by approximating to the integral equation by a finite set of linear algebraic equations in as many unknowns. Either of these procedures is tedious and time-consuming for manual computation, but easily adapted for processing on a high-speed digital computer.

All of the aforementioned techniques for making theoretical predictions have been mathematical in nature. Another well known method of solving potential flow problems is that of the physical analog, which is set up so as to satisfy the same equations and boundary conditions as the flow system. Measurements in the analog system are then interpreted to predict the behavior of the flow system. Since these predictions are based on measurements it may appear that they should not be considered as theoretical predictions. Actually the results obtained from such an analog are based on the theories of two different fields and consequently should be considered as theoretical in spite of the fact that experimental techniques are involved. Predictions from flow tests with a model, geometrically similar to the prototype, are not theoretical, however, since only the theory of dynamical similarity, and not hydrodynamic theory, is the basis of predicting prototype performance.

VISCOUS FLOW THEORY

The effects of viscosity were not introduced into hydrodynamic theory, in what we now consider to be the proper manner, for almost 100 years after Euler—by Saint Venant in 1843 and by Stokes in 1845—although the Navier-Stokes equations had been published previously by Navier in 1822, but without a proper physical basis. The fundamental law of viscosity, that the stress due to it is proportional to the rate of strain, had already been proposed by Newton, but if it required 68 years after Newton to learn how to apply his laws of motion to a continuous medium subjected to an isotropic stress, it is understandable that another century elapsed before the much more sophisticated concept of the stress tensor and the rate-of-strain tensor and their assumed proportionality was evolved.

It was not clear, at first, that the equations of viscous flow

$$\mathbf{F} - \frac{1}{\rho} \text{grad } p + \mu \nabla^2 \phi = \frac{\partial \mathbf{v}}{\partial t} + u \frac{\partial \mathbf{v}}{\partial x} + v \frac{\partial \mathbf{v}}{\partial y} + w \frac{\partial \mathbf{v}}{\partial z}$$

with the condition of zero slip at the wall, could successfully predict results in agreement with experiment, because of errors in the existing data. Fifteen more years elapsed before the equations were applied to derive the parabolic law of the velocity distribution for laminar flow in a pipe, in agreement with the experimental data of Poiseuille published at about 1840. The second role of the theory, that of predicting new laws, had in this case already preceded the first role of showing agreement with existing facts; for in 1851 Stokes had derived the law of force $F = 6\pi\mu av$ on a sphere of radius a moving through a fluid at a very low velocity v.

The equations of viscous flow were found to be much less susceptible of solution than the Euler equations. By employing special artifices solutions were found for a few cases, but no general techniques for solving the Navier-Stokes equations are available. In general, the mathematical theory for coping with the exact Navier-Stokes equations still has not been developed.

This stalemate was broken with Prandtl's publication of his boundary-layer theory in 1904 [2]. On physical grounds this theory led to a simplification of the Navier-Stokes equations which transformed them from partial differential equations of the elliptic type to equations of the parabolic type which are more amenable to solution. Furthermore the theory indicated clearly what was the proper role of inviscid flow theory in fluid mechanics by delimiting the regions of flow in which viscosity need, and need not, be taken into account. Thus, all at once, the theories of both inviscid and viscous flow became practical and fertile.

LAMINAR FLOW

Only for laminar flows has viscous flow theory proved to be completely successful. Possibly because of its technical importance in aerodynamics, a tremendous literature on boundary-layer theory has evolved, containing not only solutions of particular problems but also general numerical techniques for computing laminar boundary layers on two-dimensional forms and bodies of revolution. For these shapes the viscous flow theory seems to be at least as simple to apply as that of potential flow. In fact, there is even a transformation theory for obtaining an arbitrary number of boundary layers from a given one, although there is no theorem analogous to Riemann's mapping theorem to indicate that arbitrary boundary-layer problems can be reduced to a standard one. On the other hand, axisymmetric boundary-layer problems can be reduced to two-dimensional ones by means of Mangler's transformation, a possibility that certainly does not exist in potential flow. Thus the boundary layer on a cone can be inferred from that on a wedge, for which the exact solution of the boundary-layer equations is known. Although both the Navier-Stokes and the boundary-layer equations are nonlinear, many more exact solutions of the latter equations have been found, a fact attributable to the change in the nature of the partial differential equations of motion from the elliptic (like Laplace's equation) to the parabolic type (diffusion equation). Perhaps of even more significance is the fact that equations of the latter type lend themselves readily to numerical solution by a progressive step-by-step process which is much simpler than the comparable relaxation procedure for elliptic equations.

Considerably less success was had in applying the viscous flow equations to flows about more general three-dimensional forms. These cases are complicated by the possibility of the occurrence of secondary flows within the boundary-layer. The boundary-layer equations for this general case have been formulated only within the past decade. A few applications of them have been made but general techniques for solving the equations are still lacking.

Just as Euler's theory was distrusted because of the d'Alembert paradox, so were the Navier-Stokes equations questioned because of their apparent inability to account for turbulent flow. Experimentally it is observed that a laminar flow, if disturbed at a sufficiently high Reynolds number, may quickly become turbulent. If one has confidence in the validity of the Navier-Stokes equations, then, he would conclude that, for given boundary conditions, although the laminar-flow solution satisfies the equations at all Reynolds numbers, it is not a stable solution beyond some critical limiting value, and that there must be many other solutions of the equations, of greater stability than the laminar flow at the higher Reynolds numbers, whose combined effect is to produce the turbulent-flow solution. Because of this view the extension of the theory of viscous flow has developed in two directions. In one, the stability of the solutions of the equations to various disturbances are being investigated; in the other, attempts are being made to develop a theory of the statistical properties of large families of solutions.

After about 50 years of intensive study, the theory of the stability of laminar flow still cannot be considered to be in a satisfactory state. Although the predictions from two investigations, the Couette flow between rotating concentric cylinders by Taylor [7], and the boundary-layer on a flat plate by Tollmien, Schlichting, and Lin, were confirmed by experiment, the theory of the stability of small perturbations predicts that plane Couette flow (flow between parallel walls in relative motion) will be stable at all Reynolds numbers, contrary to experience. This failure of the "linear" theory of stability indicates that a non-linear theory of the stability of the non-linear Navier-Stokes equations is needed, and, indeed, it has been stated by Neumann [8], that "nothing less than a thorough understanding of the statistical stratifications of the system of all their solutions would seem to be adequate to elucidate the phenomenon of turbulence."

TURBULENT FLOW

It is generally assumed that the Navier-Stokes equations govern turbulent flow also, although this assumption has only the weak justification that certain types of laminar flow have been shown, both theoretically and experimentally, to be unstable. Because of its random nature, the time history of the velocity either at a fixed point or of any particle of the fluid is of little interest and one is concerned only with average values of the velocities and pressures. Recognizing this, the Navier-Stokes equations were averaged by Reynolds to yield what are now called the Reynolds equations, containing the six so-called turbulence stresses in addition to the viscous stresses. This is the essential difficulty, that the averaging process has added six more unknowns, but no more equations. This dilemma could be avoided if it were possible to follow Neumann's advice, to obtain the totality of the solutions of the Navier-Stokes equations first, and then to average them.

The attempts that have been made, by Boussinesq and Prandtl, to complete the theory with the Reynolds equations, by assuming simple additional relations between the turbulence stresses and the properties of the mean flow, have been only partly successful. In present days, a bold and intense effort to grasp the mechanics of turbulence and to utilize it to obtain the equation needed to complete the theory is being made by Max Munk. These theories are sometimes referred to in the literature as phenomenological in contrast with the statistical theories that seek a more fundamental understanding of the phenomena. This designation connotes, to some degree, that even if Prandtl's mixing length theory had been successful it would not have been a desirable theory in the sense that it would not have been based on the ultimate mechanism of turbulence. This is not in accord with our present point of view. Euler's theory of an inviscid fluid (or Newton's principles of mechanics) is not considered any less a theory because it was not based on assumptions concerning the molecular structure of matter, or, in the light of present day physics, on quantum mechanics. One may argue that the random motions in a turbulent flow are visible and measurable, but in reply, it could be pointed out that molecular motion in a fluid is also made essentially visible by the Brownian movement. The analogy could even be pushed a little farther. It is probably no more difficult to solve turbulent flow problems by means of the Navier-Stokes equation than it would be to solve elementary dynamics problems by means of Schrödinger's equation of quantum mechanics.

The modern approach, innovated by Taylor [9] about 20 years ago, is to combine the methods of statistical mechanics with the equations of motion. Although the theory has been developed to the point where it has made certain predictions in conformity with experiment, a completely logical and coherent picture of the mechanics of turbulence is not yet at hand and the theory is not yet suited for practical application to even the simplest of turbulent flows. Instead, certain semi-empirical laws, such as the "law of the wall" and the "outer law," which have a basis in dimensional analysis and are more or less confirmed by experiment, together with the momentum equation are applied to solve boundary-layer problems.

Of course the engineer is not waiting for a completely logical theory of turbulence to be formulated. He has at his disposal a tremendous collection of facts, some laws, and, if he is old enough, the judgment and intuition born of experience to guide him. But if a theory became available, and he became versant in it, he could take advantage of it in either or both of two ways. First, if the mathematics of the theory were sufficiently simple, he could apply its precise predictions to his practical problems and thus relieve his mind of the necessity of remembering a large number of unrelated facts. Secondly, even when a complete mathematical solution of a problem is not feasible, it may be possible to use the concepts and the intimate understanding of the mechanism of turbulence inherent in a successful theory as a guide to proper engineering decisions.

The present state of the theory of turbulence may be compared with that of hydrodynamics in the period between Newton and Euler. Although hydrostatics was mastered, the properties of flowing fluids were a great mystery which, it was believed, could be unravelled by means of Newton's laws. Attempts were made to develop a discrete particle theory of fluids, new basic concepts were introduced, laws were discovered, great scientists quarreled over the merits and priority of their discoveries, until finally Euler succeeded. A comparable statement regarding turbulence is the following: Although laminar flow is mastered, the properties of turbulent flow are a great mystery which, it is believed, can be unravelled by means of the Navier-Stokes equations. Attempts are being made to develop a theory of turbulence by way of three different approaches: through a complete mathematical study of the solutions of the Navier-Stokes equations and their stability, through the discovery of additional relations between the mean flow variables and the Reynolds stresses, and through the application of a theory of statistical mechanics. It could also be said that great scientists are quarreling over the merits of their theories, but, unfortunately, an Euler has not yet appeared.

CONCLUSIONS

The foregoing survey of the theories of fluid mechanics has attempted to indicate those parts of the subject for which a logical and coherent theory exists and for which techniques are available for applying the theory to engineering problems. For turbulent flows which include most flows of interest to hydraulic engineers the theory is not yet in a state so that it can be applied to even the simplest cases.

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DISCUSSION

J. M. Robertson, F. B. Campbell and Philip Hubbard commented on the importance of theory in scientific progress and the increasing acceptance of it as a practical, useful aid.

Captain Wright inquired as to which came first, the mathematics or the physical concept and as to whether a theory of electrohydrodynamics has been developed. The latter is of great interest to those who operate in salt water. The speaker replied that men of many talents and skills must combine to collect facts, develop concepts and even develop laws before a theory can be established. Otherwise, mathematics is just a toy. At present men of great intellect are applying themselves experimentally, theoretically, and analytically to develop concepts about turbulence. The feeling is that we must perform certain basic experiments to understand the mechanism of turbulence before a theory can be developed. About electrohydrodynamics, the electrical theory is well established but the theory of turbulence has yet to be developed and a theory of electrohydrodynamics can not culminate until the turbulence theory is completely formulated.

Alfred Nissan commented that he did not consider either mathematics or experiments essential to development of a scientific theory. He cited the theory of evolution as scientific without requiring either mathematics or experiments and astronomy as a science that could go on for quite a time without experiments. He stated that, in his opinion, observation, prediction, and confirmation are the essentials but they are aided most powerfully by mathematics. Then he referred to G. I. Taylor's work on prediction of instability around rotating cylinders. He developed by mathematics a theory which was numerically exact and which was verified by experiment. For a long time his work was a very interesting example of theoretical hydrodynamics, but recently some of the troubles with paper-making machines were related to his theory and people realize that his theory has practical value.

James Dailey requested comment as to the most effective way to realize the benefits of combining mathematical theory with engineering. Should engineers be trained to do it, or should it be done by specialists, and where should the training be given? The speaker disclaimed being an expert on education but felt that engineers as a group can not be expected to apply the mathematical aspects of theory. They can be taught the concepts and physical ideas of a successful theory. However, some engineers have more scientific curiosity and greater interest in academic pursuits and go on to graduate study. These should be taught the more powerful techniques that can be used to solve engineering problems. A few of them scattered through industry could do much good. For example, the off-shore oil industry has made good use of a few engineers who are also capable analysts and theorists in connection with the hydrodynamics problems involved in erecting drilling rigs and keeping them in place. Some few engineers may be selected who will benefit from current theory and in addition some members of related professions such as applied mathematicians and practical physicists may be attracted to hydrodynamics. The latter may learn the additional theory with less difficulty than engineers who, of necessity, must concentrate on the more practical aspects of the problems.

Maurice Albertson suggested a look at the methods of approaching a problem, that is research in its broad sense, and thinking in terms of (1) theory, (2) laboratory, fundamental, (3) laboratory, applied, and (4) field. No one can be an expert in all fields; hence a team should be organized that can approach the problem from all viewpoints. Thus the fact will be emphasized that the fundamental part of laboratory research can spring either from theory or, as the speaker pointed out, if there is no theory, from a body of fundamental laws and information which must be built up. The fundamental laboratory work blends into the applied and finally goes into the field for confirmation. The usual first incentive for study is a field problem; so by working back and forth the basis for a theory can be established and, although a theory is no better than the assumptions on which it is based, a good theory simplifies the other work appreciably.

Captain Wright closed the discussion with a double tribute to the speaker. First, he said that in his opinion and the opinion of many others he is the outstanding current authority in this country on the mechanics of boundary layers and second, that he had put the laboratory where he had done his under-doctorate work on the basis of using experiment to check theory, and that this is the order of the day there at the present time.