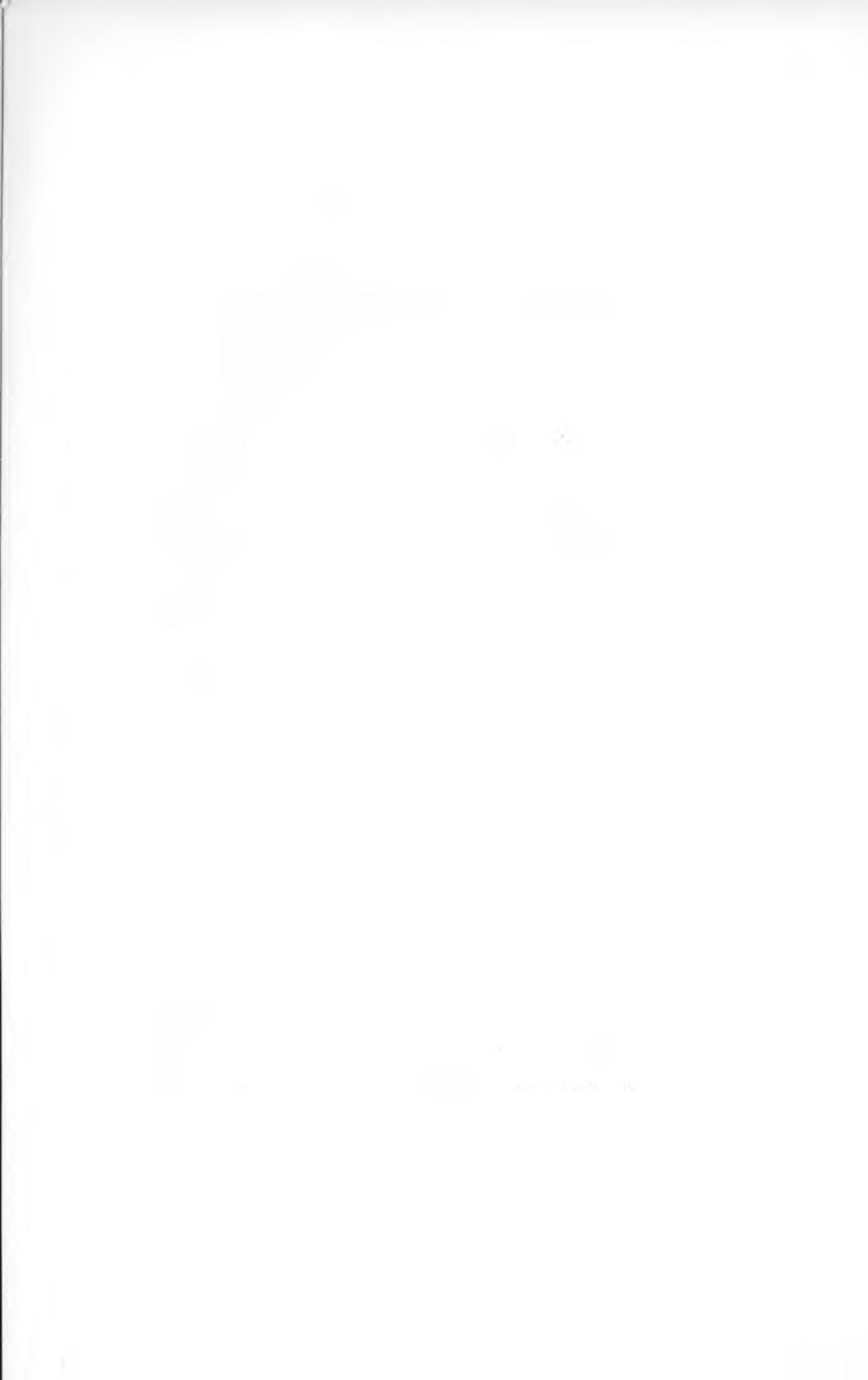


SESSION ON
MODELS AND ANALOGS

Presiding:

RALPH W. POWELL
Associate Professor of Mechanics
Ohio State University

<i>Dimensional Analysis</i>	K. C. REYNOLDS
<i>Use of Analogies in Fluid Mechanics</i>	E. W. LAMB
<i>Demonstration of Fluid Mechanics Phenomena</i>	R. A. DODGE



DIMENSIONAL ANALYSIS

by

Dr. Kenneth C. Reynolds
Associate Professor of Hydraulics
Massachusetts Institute of Technology

DEFINITION

Dimensional analysis is a tool for obtaining a desired physical law by following a systemized procedure. In determining the arrangement and exponents of the universal constants and of the physical quantities, use is made of the fact that a correct physical equation must be homogeneous insofar as the dimensions are concerned.

The various parts of a problem are assigned their dimensional values expressed in terms, not of dimensions such as feet or pounds, but of the assumed primary or fundamental unitary dimensions such as length, L , mass, M , and time, T . By determining the proper arrangement as well as the proper exponents of the various physical quantities, a mathematical formulation of the physical law is obtained such that the dimensional values of the terms on one side of the equation are equal to those on the other, i.e., the equation is homogeneously correct. For example, a velocity on one side of an equality must be equal to a product of terms on the other side having dimensions of a length divided by a time.

Having obtained an equation which is homogeneous, it is necessary to carry on experiments to determine the experimental constants. Dimensional analysis assists one in deciding how to make use of the experimental data in the preparation of a plot from which the experimental constants may be determined.

To be dimensionally correct, an equation must be in its simplest form. Thus, an acceleration would equal a length divided by a time squared while a velocity would be length divided by a time. An acceleration plus a velocity, however, would read dimensionally $L/T^2 + L/T$ which could not be done. Things dissimilar dimensionally cannot be added to or subtracted from one another although they may be multiplied or divided by one another.

Another very important point is that if an analysis is to be complete, *all* the factors which enter the problem must be considered. Failure to observe this fact will result in one's discovering, after an equation has been deduced mathematically, that the observed data, when plotted to obtain the constants of the equation, do not lie on a smooth curve. The omission of a physical quantity prevents the determination of the desired experimental law.

FUNDAMENTAL DIMENSIONS

Dimensional analysis makes use of the dimensions of the various physical quantities, expressed in terms of the adopted primary or fundamental dimensions.

From our present knowledge of physical phenomena, five fundamental dimensions will be sufficient for all problems. They will define geometrical, kinematical, dynamical, thermal, and electrical properties.

Geometrical properties are usually described in terms of the fundamental dimension of length. It is conceivable that if one were studying a number of areas, it might be simpler to use the area of a square as a unit. In general, however, a length dimension, L , has more universal application.

Kinematical properties, as a general rule, are defined in terms of the fundamental dimensions of length, L , and time, T .

Problems in mechanics require length, time, and some other fundamental dimension to describe the dynamical properties. There are several systems which might be used. The system adopted by physicists uses mass as the third dimension, the system being called the absolute system. The gravitational or technical or engineering system uses force, while other systems use energy and the gravitational constant. With a system having the three fundamental dimensions of length, time, and mass, all physical quantities in mechanics may be dimensionally defined. Using these dimensions, Table I has been prepared, giving certain quantities and their dimensional representation.

If the gravitational system with force as a dimension were used, the mass could be replaced by force since Newton's second law of motion states that

$$\text{Force} = \text{mass} \times \text{acceleration}$$

$$F = M \times L/T^2$$

$$\text{from which } M = \frac{F \times T^2}{L} \quad (1)$$

TABLE I
*Symbols and Dimensional Formulae
 in Terms of Mass, M, Length, L, and Time, T,
 for Various Quantities*

QUANTITY	SYMBOL	DIMENSIONAL FORMULA
Length	L	L
Area	A	L^2
Volume	V	L^3
Time	t	T
Revolutions per Time	n	$1/T$
Angular Velocity	ω	$1/T$
Velocity	v	L/T
Acceleration	a	L/T^2
Rate of Discharge	Q	L^3/T
Mass	M	M
Density	ρ	M/L^3
Weight		ML/T^2
Specific Weight (Unit Weight)	w	M/L^2T^2
Force (Total Pressure)	F	ML/T^2
Intensity of Pressure	p	M/LT^2
Pressure Gradient		M/L^2T^2
Work		ML^2/T^2
Energy		ML^2/T^2
Power (Energy per Unit Time)	P	ML^2/T^3
Momentum		ML/T
Impulse		ML/T
Moment of Inertia of Masses		ML^2
Modulus of Elasticity	E	M/LT^2
Coefficient of Viscosity	μ	M/LT
Kinematic Viscosity	ν	L^2/T

For thermodynamic properties, another fundamental dimension must be added to those adopted for the problems in mechanics. The most common system utilizes length, time, mass, and temperature. Temperature could have been added to a length-time-force system or to any other dynamical system. As measured on the ordinary thermometer, whether using the Centigrade or Fahrenheit scale, tempera-

ture is fixed by arbitrarily selecting the interval between the freezing and the boiling point of water. It would be independent of the other three fundamental dimensions and might be selected as the fourth primary dimension for thermodynamical properties. Specific heat could also have been selected.

For electrical properties, a system usually includes length and time. Mass and either electric charge or electrical permeability or the dielectric constant might be included to form the system.

These five dimensions which have been selected—namely: length, time, mass, temperature, and electrical permeability—are sufficient for expressing dimensionally *all* other physical quantities. Thus, density involves mass and length; unit stress involves length, time, mass, and temperature. Dimensions which are thus derived from the primary dimensions are termed secondary or derived dimensions. Most of the physical quantities in Table I are derived quantities.

It is to be noted from this table that some of the quantities have the same dimensional values. From this, it does not follow that these quantities are physically identical.

KINDS OF QUANTITIES

Dimensionally there are two kinds of quantities—dimensional and dimensionless.

A *dimensional* quantity is one that can be expressed in terms of one or more of the arbitrarily chosen primary dimensions. The coefficient of viscosity of a fluid has dimensions of $ML^{-1} T^{-1}$ and is a dimensional quantity.

A *dimensionless* quantity is one without dimensions and is, therefore, a pure number. Slope, which is a vertical distance divided by a horizontal distance, is a length divided by a length or is a length to the zero power and is dimensionless. A well known dimensionless number in the fluid-flow field is the Reynolds number, \mathbf{R} . If \mathbf{R} be the fraction vL/ν where v is a velocity or is L/T ; L is a length such as the diameter of a pipe; and ν , the kinematic viscosity, has dimensions L^2/T , then the Reynolds number reads

$$\mathbf{R} = \frac{vL}{\nu} = L^0 T^0 \quad (2)$$

The units of measure of a dimensionless quantity—consistent within themselves—may be changed without altering the numerical

value of the quantity. This is of great convenience in using data either in English or metric units.

USES OF DIMENSIONAL ANALYSIS

Dimensional analysis serves primarily as a guide in planning the experimental program and, more particularly, in analyzing and correlating the observed data. It has numerous uses, some of which were suggested by Buckingham.¹

(1) It directs attention to the physical quantities which must be measured and indicates the simplifying assumptions that may have to be adopted.

This results in the experimenter first enumerating *all* the factors on which a phenomenon depends, since, unless all the factors are used, the analysis will be incomplete.

(2) Dimensional analysis reduces the number of separate quantities that must be varied and suggests the most economical way of obtaining the desired information.

This leads one to complete as concise an experimental program as will enable him to obtain the desired results.

(3) Dimensional analysis, by showing that certain empirical equations cannot possibly be valid dimensionally, warns against using them without due regard to the units of measurement. Furthermore, it cautions against trusting them too far outside the range of experimental values for which they were deduced and for which they may fit the data. An example will illustrate this use.

In the field of hydraulics, Dubuat's formula for the flow of water over a vertical, rectangular, suppressed weir without velocity of approach is

$$Q = c \frac{2}{3} B \sqrt{2g} H^{3/2} \quad (3)$$

where Q is the rate of discharge, B is the length of the weir crest, g is the acceleration due to gravity, H is the head on the weir.

Dimensionally, this equation reads

$$\frac{L^3}{T} = c L \sqrt{\frac{L}{T^2}} L^{3/2} = \frac{L^3}{T} \quad (4)$$

so that c must be dimensionless.

Professor Theodor Rehbock of Karlsruhe, Germany, studied all reliable experimental data on the flow of water over rectangular,

sharp-crested, fully-aerated, vertical suppressed weirs and proposed for c the value

$$c = 0.605 + \frac{1}{1050 H - 3} + \frac{0.08 H}{Z} \quad (5)$$

where Z is the crest height.

This equation for c is not dimensionally correct, not being dimensionless. Hence, the statement should be added that all quantities are in the meter-second system as the constants were obtained for these units. Since the data for which the formula was derived were for definite ranges of head, crest height, and rate of discharge, the value of c should be used only for conditions within approximately the same ranges. Subsequently an equation was proposed which was dimensionally correct.

It should be observed that *because an equation is dimensionally correct, it does not follow that the equation must be the required physical law*. On the other hand, if an equation is not dimensionally correct, the equation cannot be the correct equation and should be used only under definite restrictions.

(4) Another use of dimensional analysis is to enable one to arrange equations so that the terms involved are dimensionless, making it possible to use English or metric measurements without the bother of conversion.

(5) Dimensional analysis often enables one to dispense with complete experimental investigations and it shows how very incomplete but reliable sets of experiments may give a reliable physical law.

Uses (4) and (5) will be discussed later.

(6) Finally, dimensional analysis simplifies the conversion from one system to another. This transformation may be carried out only if there are the same number of kinds of fundamental dimensions in the two systems. It is based on the fundamental condition for dimensional analysis that the dimensions on the two sides of an equation must be the same.

If a velocity is given as 90 feet per minute and it is to be converted to cm. per sec. (1 foot = 30.48 cm.), one could write dimensionally

$$\text{Velocity} = \frac{\text{length}}{\text{time}} = \frac{1 \text{ foot}}{1 \text{ min.}} \quad (6)$$

where 1 foot = 30.48 cm. and 1 min. = 60 sec.

Then the multiplier for the numerical value of the velocity is

$$\begin{aligned}\frac{1 \text{ foot}}{1 \text{ min.}} &= \frac{30.48 \times 1 \text{ cm.}}{60 \times 1 \text{ sec.}} \\ &= 0.508 \times \frac{1 \text{ cm.}}{1 \text{ sec.}}\end{aligned}$$

$$\text{or a velocity of } 90 \text{ ft. per min.} = 90 \times 0.508 \times \frac{1 \text{ cm.}}{1 \text{ sec.}} \quad (7)$$

$$= 45.72 \text{ cm. per sec.} \quad (8)$$

The above procedure involves changing the units of measure. The following rule may be deduced for converting a numerical value from one system to another system of units: Express the given quantity dimensionally, i.e., in terms of length, time, and the other primary units of the original system of dimensions, each with the proper exponent. A fraction may then be written composed of one unit of each of the terms for the original system of units (each raised to the proper power) in the numerator divided by one unit of each of the terms occurring in the denominator, i.e.,

$$\frac{(1 \text{ foot})}{(1 \text{ min.})}$$

One unit of a physical quantity in the numerator may be replaced by its equivalent value in the desired system (1 foot may be replaced by its equivalent, 30.48 cm.). Similarly, the other units in both the numerator and the denominator must be raised to the indicated power. The resulting fraction gives a multiplier, by which the original numerical quantity is to be multiplied to give the answer in the desired system of units.

THE II-THEOREM

It has been suggested that dimensional analysis may be often used for analyzing what appears to be rather incomplete experimental data and obtaining therefrom a reliable equation or law for the physical phenomenon.

For this purpose Buckingham¹ has derived a mathematical procedure which is very useful. It is first necessary to determine *all* the physical quantities such as a velocity, a length, a rate of discharge, a force, etc. These might be represented mathematically by $Q_1, Q_2, Q_3, \dots, Q_n$, there being n different kinds of physical quantities.

If "F" represents the mathematical term "function of," then the dimensional relationship of these physical quantities to one another may be represented by the equation

$$F(Q_1, Q_2 \dots Q_n) = M^0 L^0 T^0 \quad (9)$$

In considering that the velocity, v , of water flowing from an orifice is dependent on the head on the orifice, H , and on the acceleration due to gravity, g , there are three physical quantities, v , H , and g , or $n=3$.

Each of these physical quantities can be expressed in terms of the adopted fundamental units such as mass, length, and time. Let "k" represent the number of fundamental units. In the problem on the flow from an orifice, k would be two, namely, length and time.

Let Π , represent a product of some or all of these physical quantities, Q_1, Q_2 , etc., each with its proper exponent, x, y , etc., such that Π , is dimensionless. Then

$$\Pi_1 = Q_1^x \cdot Q_2^y \dots Q_n^m = M^0 L^0 T^0 \quad (10)$$

Also, let Π_2 , etc., represent other dimensionless terms obtained by multiplying together some of the other physical quantities, Q_1, Q_2 , etc., with proper exponents, which will be different than for Π_1 .

Thus, for the orifice, v^2/gH consists dimensionally of

$$\frac{\left(\frac{\text{length}}{\text{time}}\right)^2}{\frac{\text{length} \times \text{length}}{\text{time}^2}}$$

and is dimensionless and might be represented by Π_1 . This dimensionless fraction is called the Froude number and is represented by **F**. If the kinematic viscosity, ν , were to be considered, another Π , such as Π_2 , might be the fraction, vH/ν . This dimensionless number is called the Reynolds number, **R**.

Buckingham shows that equation (9) is reducible to the form

$$f(\Pi_1, \Pi_2, \dots \Pi_{n-k}) = 0 \quad (11)$$

in which "f" represents some completely unknown dimensionless function, the form of which is to be found experimentally; n represents the number of physical quantities; and k the number of independent fundamental units—never greater than five. Note that from the

mathematical analysis the relationships between Π_1 and Π_2 or Π_1 and Π_{n-k} are unknown and may only be determined experimentally.

There will be n separate kinds of quantities but some quantity such as length may appear more than once in the equation. Thus, a problem might involve the length of a pipe and its diameter—both of which are represented by L . Rather than represent this ratio by a Π , Buckingham chose to represent the ratio of physical quantities of like dimensions by various r 's such as r' , r'' , etc. All physical quantities having like dimensional values might then be represented by specifying a single one of that kind and letting their relationship to the one specified be represented by ratios r' , r'' , etc. The physical quantity, common to all these ratios, would be used in determining the Π 's and would still be retained as one of the Q 's. These ratios may be deduced by inspection during the initial analysis of the problem.

The original equation (9) then becomes

$$F(Q_1, Q_2, \dots, Q_n, r', r'', \text{etc.}) = M^0 L^0 T^0 \quad (12)$$

Buckingham shows that this equation reduces to his well known Π -equation

$$f(\Pi_1, \Pi_2, \dots, \Pi_{n-k}, r', r'', \text{etc.}) = 0 \quad (13)$$

$$\text{or} \quad \Pi_1 = f(\Pi_2, \dots, \Pi_{n-k}, r', r'', \text{etc.}) \quad (14)$$

This equation is very useful for analyzing problems dimensionally with the objective of determining the underlying physical law.

In using the Π -theorem, the writer has found certain rules to be helpful to the student.

Rule 1. The number of physical quantities whose exponents will be assumed as unknown is the same as the number of fundamental units.

Rule 2. The physical quantities, whose exponents will be assumed as unknown for each of the Π -equations, must be chosen so as to include all fundamental units.

Rule 3. In setting up a Π -equation, the exponents of one term should be assumed as any pure number, usually unity.

Rule 4. If a problem calls for an expression for some physical quantity in terms of the others, consider the exponent of that quantity to be any pure number—such as unity—when setting up one of the Π -equations.

ILLUSTRATIVE PROBLEM

As an illustrative problem, let us use the following data² on the flow of molasses in a 1.61-inch diameter pipe with a gage length of 42.16 feet:

Run No.	Temp. °C.	Friction Head H_f in Ft.	Density ρ slugs/ft. ³	Coef. of Viscosity μ Slugs/ft. sec.	Velocity v Ft./sec.
3	15.0	11.09	2.685	0.01958	0.63
7	18.5	16.79	2.680	0.01353	1.49
10	20.8	18.33	2.677	0.01090	2.04
12	21.3	34.99	2.676	0.01040	4.38

Determine by use of a logarithmic plot the relationship between the various factors.

Equation (9) now reads

$$F(D, L, H_f, \rho, \mu, v, g) = M^0 L^0 T^0 \quad (15)$$

By inspection, the pipe diameter, D , length of pipe, L , and friction head, H_f , have the same fundamental dimension of a length. Our knowledge of pipe flow would indicate that at least the diameter of the pipe is an important physical quantity.

$$\text{Let } r' = \frac{D}{L} \text{ and } r'' = \frac{D}{H_f} \quad (16)$$

$$\text{Then } F'(D, \rho, \mu, v, g, r', r'') = M^0 L^0 T^0 \quad (17)$$

Dimensionally these physical quantities have the following values

$$D = L; \rho = \frac{M}{L^3}; \mu = \frac{M}{LT}; v = \frac{L}{T}; g = \frac{L}{T^2} \quad (18)$$

The number of physical quantities, n , is five and the number of fundamental units, k , is three— L , M , and T . Then $n - k$ equals 2 or there will be 2 Π 's or two dimensionless numbers in addition to the dimensionless ratios r' and r'' .

Then equation (13) reads

$$f(\Pi_1, \Pi_2, r', r'') = 0 \quad (19)$$

Since this is a problem of pipe flow, it is evident that the Reynolds number, \mathbf{R} , should be one of the dimensionless numbers in order to take into account internal fluid friction or the viscous forces.

With this in mind and recalling Rules 1, 2, and 3,

$$\Pi_1 = \rho D^a v^b \mu^c = M^0 L^0 T^0 \tag{20}$$

in which a , b , and c are exponents which must have such values as will make Π_1 dimensionless.

Also,
$$\Pi_2 = g \cdot D^x v^y \mu^z \tag{21}$$

with x , y , and z having values (other than for a , b , and c , respectively) as will make Π_2 dimensionless.

Writing Π_1 dimensionally, we obtain

$$\Pi_1 = \frac{M}{L^3} L^a \left(\frac{L}{T}\right)^b \left(\frac{M}{LT}\right)^c = M^0 L^0 T^0 \tag{22}$$

Collecting the exponents of M , $c = -1$

Collecting the exponents of T , $b = 1$

Collecting the exponents of L , $a = 1$

Hence,
$$\Pi_1 = \rho D v / \mu \tag{23}$$

which is the Reynolds number, since μ/ρ is the kinematic viscosity, ν . This could have been at once set down on the basis of past experience since the only possible combinations of the four physical quantities, D , v , ρ , and μ would have given the Reynolds number with exponents of a positive or negative pure number.

Similarly, if

$$\Pi_2 = g D^x v^y \mu^z, \tag{24}$$

$z = 0; y = -2; x = 1$, and

$$\Pi_2 = \frac{g D}{v^2} \tag{25}$$

which is the reciprocal of the Froude number, F . This could also have been determined by inspection. The only physical quantity with a mass term is μ and hence it must have zero as its exponent. An examination of g , D , and v would have disclosed the Froude number in some form as the only possible dimensionless arrangement of the three quantities.

Thus, there results

$$f\left(\frac{\rho D v}{\mu}, \frac{g D}{v^2}, \frac{D}{L}, \frac{D}{H_f}\right) = 0 \tag{26}$$

or

$$f\left(R, \frac{1}{F}, \frac{D}{L}, \frac{D}{H_f}\right) = 0 \tag{27}$$

Obviously, a plot of $\log D/L$ against \log Reynolds number, $\log R$, yields no information since there is an infinite number of possible values of D/L depending on the particular experimental set-up. This is an important point. Fig. 1 is a plot of $\log 1/F$ and $\log D/H_f$ as ordinates against $\log R$ as abscissae. No definite law can be deduced. If data from the other ten runs had been included, little help would have been obtained.

If we now draw on past experience and if we also recall that a plot of $\log D/L$ against $\log R$ yields no information, we shall have to search further. We might recall that Weisbach-Darcy's coefficient of friction, f , is often used in connection with pipe flow in which

$$H_f = \frac{fL}{D} \frac{v^2}{2g} \tag{28}$$

from which

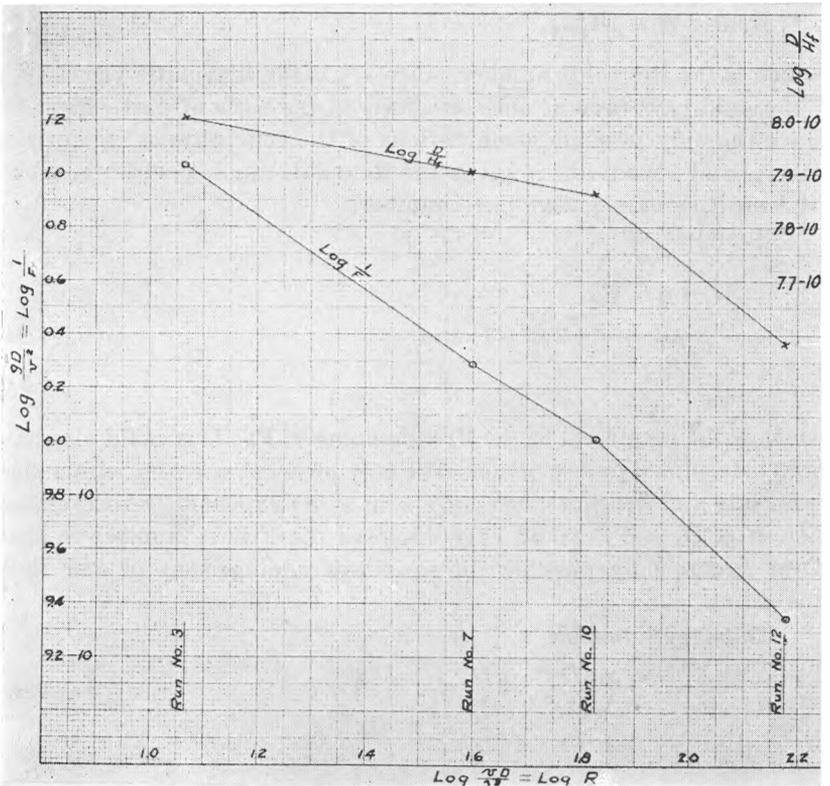


FIG. 1.

$$f = \frac{H_f D 2g}{L v^2} \tag{29}$$

Evidently, this is the reciprocal of the Froude number, Dg/v^2 , multiplied by $2 H_f/L$. If 2 is a dimensionless number, then this fraction is twice the product of the dimensionless ratios r' and the reciprocal of r'' or

$$\frac{r'}{r''} = 2 \frac{D}{L} \times \frac{H_f}{D} = \frac{2H_f}{L} \tag{30}$$

Since D is included in \mathbf{R} and $1/\mathbf{F}$, we may rewrite equation (27) to read

$$f''(\mathbf{R}, f) = 0 \tag{31}$$

It should be pointed out that it is *only* on the basis of past experience that equation (27) is thus simplified.

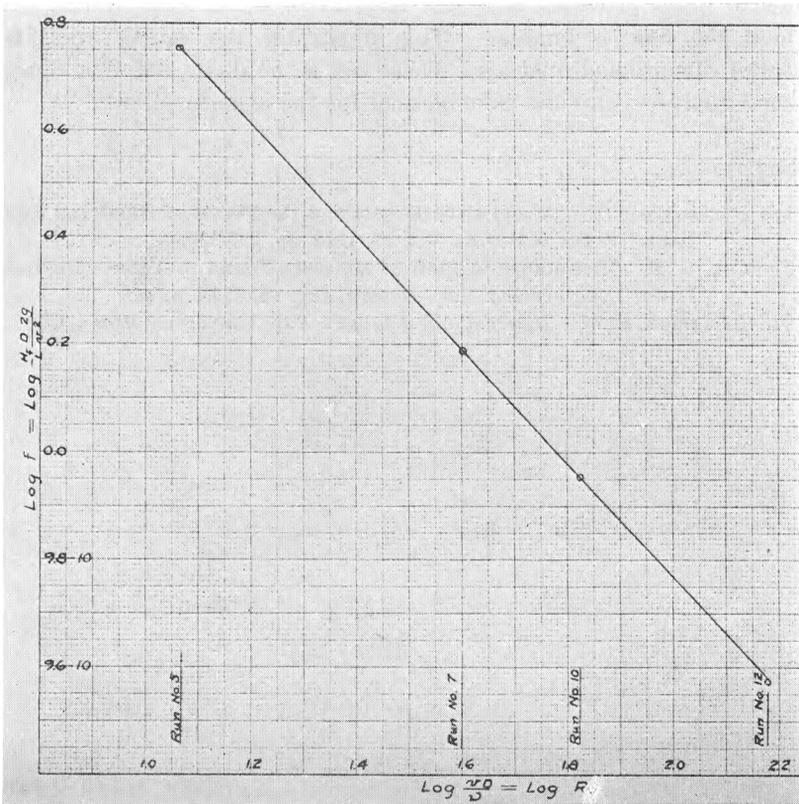


FIG. 2.

A plot of R as abscissae and f as the ordinates is shown in Fig. 2. The observed data now lie on a straight line from which it follows that

$$f = \frac{77.0}{R^{1.057}} \quad (32)$$

Considering that only four sets of data were used, this result compares favorably with the equation often given for the flow of water in pipes with the Reynolds number less than about 2100,

$$f = \frac{64.0}{R} \quad (33)$$

If all available data for molasses, water, and other fluids were plotted on Fig. 2, it would be possible to determine the law for fluid flow in pipes provided R is less than 2100 or, in other words, provided the flow is laminar. This illustrates the fourth and fifth uses of dimensional analysis. There can be no doubt but that dimensional analysis is a most valuable tool for the experimenter.

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