

THE FLOW OF SOLID-LIQUID MIXTURES

by

H. E. Babbitt and D. H. Caldwell

University of Illinois

Urbana, Illinois

The purpose of this paper is to discuss the theory of the flow of homogeneous mixtures of solid particles suspended in liquid. By the term "homogeneous mixture" is meant a suspension of solids in liquid with the ratio of solid phase to liquid phase constant throughout the mixture. This rather restricted definition excludes those mixtures of solid particles in liquids in which the rate of settling of the particles is great enough to cause an appreciable solids-concentration gradient in the vertical direction. The types of materials which fit the definition include sewage sludge, sludge from lime-soda water-softening plants, mud from desilting works, rotary drilling mud, ceramic clays, some wet concrete mixtures, and other similar materials.

It has been known for some time that a solid-liquid mixture moving in a pipe exhibits either one of two distinct types of flow depending upon the mean velocity of flow. At low velocities a type of flow similar to laminar flow of true fluids has been found while at high velocities the laminar nature of the flow changed to turbulent. Fig. 1 illustrates the two types of flow as obtained experimentally [1]¹ in a $\frac{3}{8}$ -inch pipe. The figure is a logarithmic plot of head loss against the mean velocity of flow. It will be noticed at the higher velocities—i.e., above 20 feet per second—that the relationships between the friction loss and the velocity of flow for the various sludges tested plot as straight lines which are approximately parallel to the line representing the flow of water. This is in the turbulent region of flow and it is apparent that the same laws govern the flow of a solid-liquid mixture in this region as govern the flow of water, a true fluid. As the velocity of flow is decreased, a point is reached where the plotted line

¹ References appear at the end of the article.

departs from the straight portion representing turbulent flow. The point of departure is called the critical velocity and is analogous to the critical velocity occurring under similar conditions for a true fluid. The lower portions of the curve represent the laminar region of flow, which has been called by various authors "plug flow" and "plastic flow."

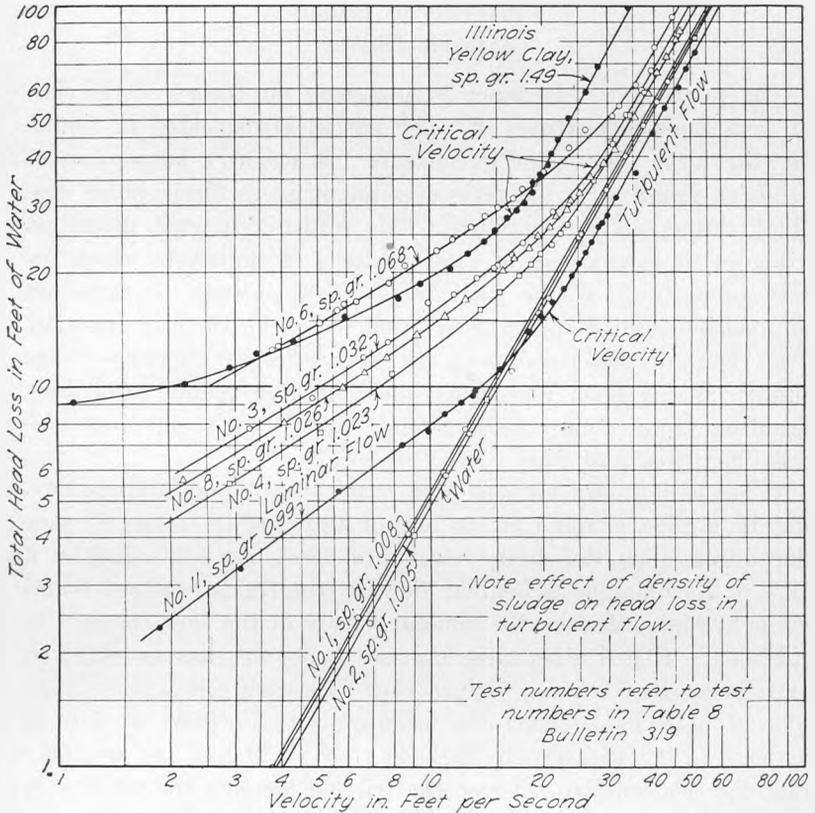


FIG. 1. FRICTION LOSSES FOR VARIOUS SLUDGES FLOWING IN A $\frac{3}{8}$ -INCH PIPE

In the analysis of the phenomenon of plastic flow it is necessary to investigate the various classes of materials which have been classified according to their behavior while flowing. Fig. 2 represents the four recognized classes of materials. Curve I represents the flow of a true fluid, the slope of the line being proportional to

the coefficient of viscosity. The equation for the flow of this class of material in circular pipe is known as Poiseuille's equation. Curves II, III and IV represent the flow of plastic materials. Material following the general form of Curve II is known as a pseudo-plastic material, while material with a stress-flow diagram similar to Curve III is called a true plastic or Bingham plastic. Curve IV represents an inverted plastic or rheopactic material. All classes of materials represented here, with the exception of the true fluids, exhibit variable pseudo-viscosity depending on the rate of shear at which the shearing stress is measured.

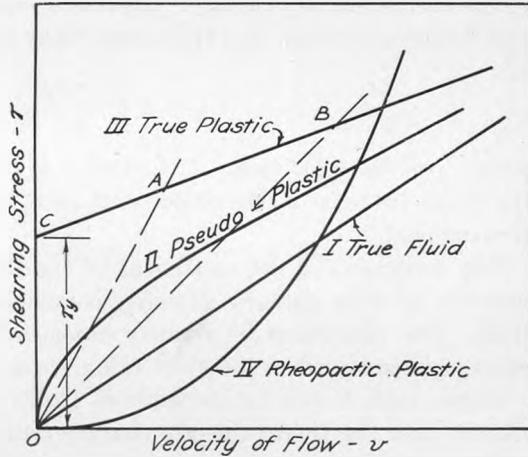


FIG. 2.—MATERIALS CLASSIFIED ACCORDING TO THEIR BEHAVIOR WHILE FLOWING.

For example, the apparent viscosity of the plastic at any point A on Curve III if measured in the usual way for fluids is proportional to the slope of the line OA. It is evident, therefore, why the apparent viscosity is not constant for different velocities or rates of shear. It is seen that two different velocities A and B correspond to entirely different viscosity lines OA and OB, the slopes of which are proportional to the apparent viscosity.

It has been found from the results of tests on different homogeneous solid-liquid mixtures that in the majority of cases they fall into Class III—i.e., many commonly-encountered mixtures are true plastics. The dictionary defines a plastic substance as any substance capable of being continuously and permanently deformed in any direction without rupture under a stress exceeding the yield value. The yield value is defined as that shearing stress under which deformation is impending. Returning to Fig. 2, the point C is the yield value of the true plastic illustrated. After

deformation has started, equal increments of stress produce equal increments in velocity of deformation or flow. The slope of the line for the true plastic is, therefore, a measure of its resistance to deformation and is proportional to the coefficient of rigidity in the same manner that the slope of the true fluid line is proportional to the coefficient of viscosity. Expressed mathematically, the equation defining the flow of a true plastic may be written

$$\eta = k \frac{(\tau - \tau_y)}{v} \quad (1)$$

where η is the coefficient of rigidity, τ is the shearing stress, τ_y is the yield value, v is the velocity of flow, and k is a constant of proportionality.

The reciprocal of the coefficient of rigidity has been termed the mobility by some authors. Mobility is analogous to the fluidity of a fluid. The coefficient of rigidity becomes identical to the coefficient of viscosity when the yield value becomes zero.

It has been shown by the authors [1, 2] that the coefficient of rigidity and the yield value are characteristics of a particular solid-liquid mixture just as the density and solids concentration are characteristics of that mixture, and that all are constants for that mixture.

Using Eq. (1) as a basis and applying it to steady flow in straight circular pipe, the following equation has been developed

$$V = \frac{D}{8\eta} \left(\tau_p - \frac{4}{3} \tau_y + \frac{1}{3} \frac{\tau_y^4}{\tau_p^3} \right) \quad (2)$$

where V is the mean velocity of flow, D the diameter of the pipe, g the acceleration due to gravity, η the coefficient of rigidity, τ_p the shearing stress at the pipe wall, and τ_y the yield value. Neglecting the last term of Eq. (2), it is possible to rewrite it as follows:

$$\tau_p = \eta \frac{8V}{D} + \frac{4}{3} \tau_y \quad (3)$$

When $8V/D$ is plotted as abscissa and τ_p is plotted as ordinate, Eq. (3) is in the form of a straight line with a slope of η and an intercept on the τ_p axis of $4/3 \tau_y$. Since τ_y and η are constant for a particular mixture, the resulting straight line should represent the plastic flow of a particular mixture in a pipe of any diameter

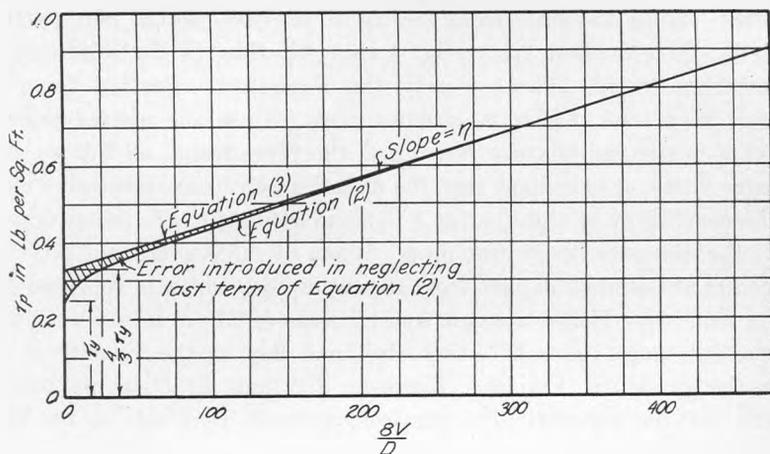


FIG. 3.—PLOT OF EQS. (2) AND (3) FOR PARTICULAR MIXTURE.

except when the ratio τ_y/τ_p approaches unity. Fig. 3 is a plot of both Eq. (2) and Eq. (3) for a particular mixture having a yield value of 0.25 pound per square foot and a rigidity of 0.00093 pound seconds per square foot. The error in neglecting the last term of Eq. (2) is thus shown graphically. The slope of the asymptote is the coefficient of rigidity and the intercept of the asymptote on the τ_p axis is $4/3 \tau_y$, as is readily seen from Eq. (3).

Fig. 4 shows some of the results of experiments made to verify Eqs. (2) and (3). It will be noticed that the experimental pipe sizes varied from 1/2 inch to 3 inches and the sludges tested had

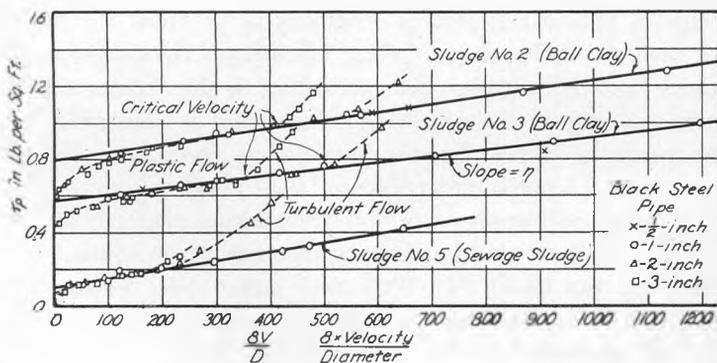


FIG. 4.—COMPARISON OF EXPERIMENTAL RESULTS WITH EQS. (2) AND (3).

yield values ranging from nearly 0 to 0.08 pound per square foot. It is evident from Fig. 4 that the flow of the solid-liquid mixtures tested follows closely the theoretical equation for the flow of a true plastic in circular pipe. Since the plotted points for a particular mixture flowing in the pipes tested all fall on the same curve, it is evident that the magnitudes of the yield value and the coefficient of rigidity for a particular mixture are independent of the diameter of the test pipe. When $8V/D$ is small, the plotted points are seen to depart from the asymptote which is represented by Eq. (3). Hence at these lower values of $8V/D$ in order to determine magnitudes of either the head loss or the velocity it is necessary to use Eq. (2). However, for most practical purposes, and for the common pipe sizes encountered, it is safe to use the simpler form of Eq. (3). This is especially true near the critical velocity and for sludges or mixtures having a large coefficient of rigidity. For small velocities and large pipe the value of the term $8V/D$ becomes small and in these cases Eq. (2) must be used.

Various factors affect the yield value and the coefficient of rigidity. Among these may be listed the concentration of suspended matter, size and shape of the suspended particles, nature of the liquid phase, temperature, agitation, and thixotropy.

The determination of the relation between the yield value and the concentration of suspended particles in a mixture has been the object of many recent investigations. The yield value is generally fairly sharply defined when the particles do not exhibit too large a Brownian movement and are not highly hydrated. Bingham [3] describes a plastic material as a suspension in which the particles touch each other. He states, "Flow necessitates the sliding of these particles the one over the other according to the ordinary laws of friction, so long as their Brownian movement is negligible. It is by no means necessary that the particles be touching at the maximum number of places according to 'close packing'. As a matter of fact, close packing prevents flow from taking place. It is merely necessary that particles touching each other form arches capable of carrying the load." If Bingham's hypothesis is correct, the appearance of a yield value should be noted at a volume concentration of suspended spherical particles of 52.36%. This concentration corresponds to "open packing" or cubic packing of spheres and is theoretically the concentration at which spherical particles

would just begin to touch and form arches capable of carrying stress. However, as observed by the authors and by Bingham [3] and others [4, 5, 6], a measurable yield value is encountered at concentrations of solids as low as 2 percent by volume. Fig. 5 has been prepared from the results of pipe-flow tests by the authors. The material represented on the figure was a colloidal-ball clay. The figure indicates that the yield value varies logarithmically

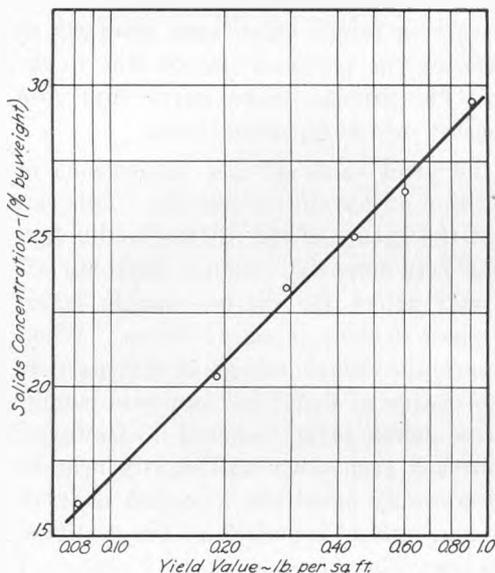


FIG. 5.—PIPE FLOW TESTS ON A COLLOIDAL-BALL CLAY.

with the concentration of suspended matter. These results tend to show that the existence of a yield value might be expected at very low concentrations of suspended matter. In dilute lyophobic systems the existence of a yield value cannot be explained on the basis of mutual touching and interlocking of the particles. Various explanations for this have been given in the literature. These may be grouped as follows: (1) presence of a sheath of adsorbed water or other medium,

called the lysphere, (2) presence of electrical forces between the particles which cause them to assume a quasi-pattern arrangement, and (3) production of networks of particles which are capable of carrying stress by bridge action.

Some lyophobic particles are capable of adsorbing water molecules. Under certain conditions the thickness of the adsorbed sheath, called the lysphere, can be calculated. The effect of the lysphere is to cause the particles to occupy more space than without the lysphere. Norton and Hodgton [7] by experimental investigation on the water-adsorbing power of clay have shown that the thickness of the water layer between the particles was of the

order of 10^3 \AA (one $\text{ \AA} = 0.0001$ micron) when the material exhibited a yield value. In particular they found a value of $3.1 \times 10^3 \text{ \AA}$ for brick clay. Houwink [8] has shown by calculation that the maximum possible thickness of the lysosphere cannot exceed 25 \AA . These apparent contradictions suggest that the hypothesis of the existence of a lysosphere does not adequately explain the existence of a yield value at low concentrations of solids in a solid-liquid mixture.

Electrical forces acting between particles in suspension may either attract or repel. Attracting forces which vary inversely as the cube of the distance between the particles may be due to the van der Waals [9] forces. The diffuse double layer first proposed by Gouy [10] may account for the repelling forces.

It has been found that the yield value of clay suspensions in water is affected by the addition of certain electrolytes. This can be explained by the fact that the nature of the diffuse double layer is greatly affected by the Zeta-potential. Hence anything affecting the Zeta-potential will affect the diffuse double layer. Houwink [8] states, with respect to these repelling forces, "When two particles are very far apart, the electric charge of each particle is largely neutralized by the charge of equal and opposite magnitude of the ionic cloud [diffuse double layer]¹ around it. However, as soon as the particles approach each other sufficiently near in order that these clouds will mutually penetrate, a perfect neutralization is no longer possible and mutual repulsion of the particles, having charges of the same sign (in the case of clay: negative) will become observable." The net result of the attracting and repelling forces between the particles is to create a quasi-structural arrangement which produces a yield value at concentrations far below that concentration at which the particles themselves are in contact. For example, the yield value of clay was found by Wilson and Hall [11] to be reduced by the addition of OH^- .

It is entirely possible that a yield value may be developed at low concentrations of suspended matter as the result of the formation of open nets of the particles capable of transmitting shear [12, 13]. Houwink [8] states, "An important argument in support of such a conception is that suspensions of carbon black, possessing a yield

¹Authors' insert

value, showed electrical conductivity, which was not observed as soon as a yield value was absent."

The size and character of the solid particles also greatly affect the coefficient of rigidity and the yield value. For a given concentration of suspended particles the yield value increases as the size of the particles decreases. Bingham [3] states, "There is abundant evidence that as the diameter of the particles is decreased, the opportunity for the particles touching is increased, which enhances the friction [yield value]¹, but this effect reaches a limit eventually when the particles are so small that their Brownian movement becomes appreciable and strains in the material are not permanent." The result of stress relief by Brownian movement is a stress-flow curve similar to Curve II, Fig. 2—i.e., a pseudo-plastic material. The yield value is not well defined and the application of Eq. (2) under such conditions may lead to serious errors.

Both the coefficient of rigidity and the yield value, as well as other factors, affect the critical velocity at which plastic flow changes to turbulent flow. From the results of a theoretical analysis the authors have proposed the following equation for computing the critical velocity:

$$V_c = \frac{32 \eta + 32 \sqrt{\eta^2 + \frac{f \tau_y \gamma D^2}{96}}}{f \gamma D} \quad (4)$$

where V_c is the critical velocity, γ is the specific weight of the mixture, and f is the friction factor in the Weisbach equation:

$$H = f \frac{L}{D} \frac{V^2}{2g} \quad (5)$$

in which H is the friction loss in head of the flowing material for a pipe of length L and diameter D and V is the velocity of flow.

Table I presents some results of experiments which corroborate Eq. (4). The last three columns show the observed and computed values of the critical velocity together with the percentage error between them. Under field conditions the agreement between computed and observed values of the critical velocities cannot be ex-

¹Authors' insert

TABLE I

OBSERVED AND COMPUTED VALUES OF CRITICAL VELOCITY

Sludge No.	τ_H lb. per sq. ft.	η lb. sec. per sq. ft.	γ lb. per cu. ft.	Diameter of pipe in.	Friction factor f	Critical Vel- ocity, V_c ft. per sec.		Error per cent
						(com- puted)	(ob- served)	
1	0.90	0.00047	75.7	1	0.0216	17.4	17.5	-0.6
				2	0.0196	16.4	16.0	2.5
				3	0.0172	16.6	15.5	7.0
2	0.60	0.00047	75.0	1/2	0.0316	14.0	14.2	-1.4
				1	0.0220	14.9	14.7	1.4
				2	0.0204	13.5	13.5	0.0
				3	0.0180	13.8	13.8	0.0
3	0.44	0.00034	73.8	1/2	0.0324	11.3	12.0	-5.8
				1	0.0224	12.2	12.9	-5.4
				2	0.0208	11.2	11.0	1.8
				3	0.0180	11.6	11.5	0.9
4	0.29	0.00031	72.5	1	0.0236	10.1	10.1	0.0
				2	0.0216	9.2	9.0	2.2
				3	0.0188	9.5	9.5	0.0
5	0.19	0.00025	72.0	1/2	0.0340	7.5	7.7	-2.6
				1	0.0240	8.0	7.9	1.3
				2	0.0208	7.6	7.6	0.0
				3	0.0176	8.0	7.7	3.9
6	0.082	0.00019	70.0	1/2	0.0352	5.2	5.4	-3.7
				1	0.0264	5.2	5.3	-1.9
				2	0.0216	5.1	5.1	0.0
				3	0.0192	5.1	5.1	0.0

pected to be so precise as in these tests, because, in general, the friction factors for pipes in the field are not so precisely known.

In the use of Eqs. (2), (3), and (4) it is necessary to know in addition to other factors the yield value and the coefficient of rigidity of the solid-liquid mixture. These constants can be determined in a modified Stormer viscometer, in an existing pipe line, or by the use of an appropriate formula relating the factors af-

fecting the yield value or the coefficient of rigidity. These methods are described by the authors in Bulletin 319 of the University of Illinois Engineering Experiment Station [1]. A rapid method of determining the yield value of a mixture is to observe the depth to which a thin metallic strip will sink, due to its own weight, into the mixture. The yield value is obtained by equating the downward and upward forces acting on the metal strip. The downward force is the weight of the metal minus the buoyant force of the sludge on the submerged portion, and the upward force is the friction on the sides of the strip due to the yield value of the sludge. At equilibrium these forces are equal and can be equated, and the yield value computed.

The turbulent flow of sludge follows closely the same laws that govern the turbulent flow of true liquids. From the results of about 900 tests on 8 different solid-liquid mixtures [2] it has been found that the viscosity of the dispersion medium can be used in the Reynolds number to obtain a friction factor for use in Eq. (5). The practical result of this observation is that any friction-factor chart designed for use in computing friction losses in the turbulent flow of liquids may be used in computing the head loss due to the *turbulent* flow of a solid-liquid mixture.

In practical applications it has sometimes been found more advantageous to use an approximate exponential type of formula for predicting head loss due to the flow of liquids such as water. An example of an equation of this type is the one devised by Kessler [14] for the flow of water in black steel pipe:

$$V = 80.2 D^{0.678} \left(\frac{H}{L} \right)^{0.543} \quad (6)$$

If the liquid phase of the solid-liquid mixture is water, Eq. (6) can be used to compute the velocity of flow or friction loss of a liquid-solid mixture *when the flow is turbulent*. Table II presents data taken from Bulletin 323 [2]. The data show the degree of precision with which a formula of this type may predict friction losses for the flow of sludge. The yield value of the sludge in this example (sludge No. 2) is of such magnitude that the critical velocity in the 1-inch black steel pipe was computed and observed to be 14.9 feet per second.

It cannot be emphasized too strongly that the foregoing con-

TABLE II

COMPARISON OF HEAD LOSSES COMPUTED BY KESSLER'S FORMULA
WITH OBSERVED HEAD LOSSES FOR SLUDGE FLOWING IN A 1-INCH
BLACK STEEL PIPE

$$\tau_y = 0.60 \text{ lb. per sq. ft.}; \eta = 0.00047 \text{ lb. sec. per sq. ft.};$$

$$\gamma = 75.0 \text{ lb. per cu. ft.}; V_c = 14.9 \text{ ft. per sec.}$$

Velocity ft. per sec.	Head Loss ft. per 100 ft. (computed by Eq. (6))	Head Loss ft. per 100 ft. (observed)	Per Cent Variation
15	86	87	+1.32
18	117	120	+2.65
20	145	145	0.00
22	175	175	0.00
24	200	200	0.00
26	232	232	0.00
28	265	270	+1.88
30	300	310	+3.33
32	339	349	+2.95
34	380	390	+2.65
36	420	430	+2.38
38	465	475	+2.15
40	510	520	+1.96

clusions apply only so long as the flow is turbulent. The critical velocity should in all cases first be computed by Eq. (4). If the actual velocity is below the computed critical velocity, then the plastic-flow relationship of Eq. (3) should be used in computing friction loss. If the actual velocity is larger than the critical velocity then a friction-factor chart and Eq. (5) may be used.

If the settling rate of the particles in suspension is appreciable as compared to the mean velocity of flow, then it is necessary to take account of the energy required to keep them in suspension. Wilson [15] has shown that the following theoretically-derived equation fits observed data when the rate of settling of the particles is appreciable:

$$\frac{H}{L} = \frac{fV^2}{2gD} + \frac{KpV_s}{V} \quad (7)$$

where p is the weight of solids per unit weight of solid-liquid mixture, K is a constant associated with the efficiency of settling, V_s is the rate of settling of the particles, and the other terms have been defined elsewhere in this paper. It will be noted that in this equation the last term may be neglected when the value of V_s is

small or the value of V is large. In either case the magnitude of the first term is much larger than the magnitude of the second term. Here again let it be pointed out that Eq. (7) is valid only when the flow is turbulent.

REFERENCES

- [1] Babbitt, H. E., and Caldwell, D. H., "Laminar Flow of Sludge in Pipes with Special Reference to Sewage Sludge," *Univ. of Illinois Eng. Exp. Station, Bul.* 319, 1939.
- [2] Babbitt, H. E., and Caldwell, D. H., "Turbulent Flow of Sludge in Pipes," *Univ. of Illinois Eng. Exp. Sta., Bul.* 323, 1940.
- [3] Bingham, E. C., *Fluidity and Plasticity*, McGraw-Hill Book Co., New York, 1922.
- [4] Oden, S., "Physico-chemical Properties of Sulphur Hydrosol," *Zeit. physik. Chem.*, Vol. 80, 1911, p. 709-36.
- [5] Bingham, E. C., and Durham, T. C., "Viscosity and Fluidity of Suspensions of Finely Divided Solids in Liquids," *Am. Chem. Journal*, Vol. 46, 1911, p. 278-297.
- [6] Schofield, R. K., and Blair, G. W. S., "The Influence of the Proximity of a Solid Wall on the Consistency of Viscous and Plastic Materials," *J. Phys. Chem.*, Vol. 34, 1930, p. 248-62.
- [7] Norton, F. H., and Hodgdon, F. B., "Some Notes on the Nature of Clay," *J. Am. Cer. Soc.*, Vol. 15, 1932, p. 191.
- [8] Houwink, R., "Second Report on Viscosity and Plasticity," *Academy of Sciences at Amsterdam*, 1939.
- [9] Boer, H. H. de, "Influence of van der Waals' Forces and Primary Bonds on Binding Energy, Strength and Orientation, with Special Reference to some Artificial Resins," *Trans. Faraday Soc.*, Vol. 32, 1936, p. 10-38.
- [10] Gouy, G., "Constitution of the Electric Charge at the Surface of an Electrolyte," *Compt. rend.*, Vol. 149, 1909, p. 654-7.
- [11] Wilson, R. E., and Hall, F. P., "The Measurement of the Plasticity of Clay Slips," *J. Am. Cer. Soc.*, Vol. 5, 1922, p. 916-27.
- [12] Lewis, W. K., Squires, L., and Thompson, W. I., "Structure of Clay Gels," *Trans. Am. Inst. Mining Met. Engrs.*, Vol. 118, 1936, p. 71-80.
- [13] McDowell, C. M., and Usher, F. L., "Viscosity and Rigidity in Suspensions of Fine Particles," I Aqueous Suspensions, *Proc. Royal Soc. (London)*, Vol. A131, 1931, p. 409, II Non-Aqueous Suspensions, *Proc. Royal Soc.*, Vol. A131, 1931, p. 564.
- [14] Kessler, L. H., "Experimental Investigation of Friction Losses in Wrought Iron Pipe When Installed with Couplings," *Univ. of Wis. Exp. Sta., Bul.* 82, 1935.
- [15] Wilson, W. E., "Mechanics of Flow, with Noncolloidal, Inert Solids," *Proc. A. S. C. E.*, Vol. 67, No. 8, Oct. 1941, p. 1434.