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TENSOR RENORMALIZATION GROUP METHOD STUDY IN Z5 MODEL

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A thesis submitted in partial fulfillment of the requirements
for graduation with Honors in the Physics

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Thesis Mentor

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All requirements for graduation with Honors in the
Physics have been completed.

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Physics Honors Advisor

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A thesis submitted in partial fulfillment of

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Iowa City, Iowa

CERTIFICATE OF APPROVAL

Honors Thesis

This is to certify that the Honors thesis of

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has been approved by the Department of Physics and Astronomy for the Dec 2017 graduation.

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ABSTRACT

This thesis describes the Tensor Renormalization Group (TRG) method used to solve the Z5 model in lattice gauge theory. Here we consider the Z5 model in two dimensions, we discuss the formulation of the tensor and the process we used to do the TRG blocking, in order to find the phase transition and critical behavior of the model. We give some simple results and plots, and some expectation for further research in the model.

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CHAPTER 1

INTRODUCTION

It has been a great challenge to simulate lattice models like the *O(N)* clock models, and it's unrealistically costly to do blocking on the lattice using traditional renormalization group(RG) method. The Tensor Renormalization Group method is then a renormalization group method developed to solve these statistical models[1]. It provides exact blocking for the models and is much less expansive in terms of computational costs.

Here we use the TRG method to study the Z5 model, which is one of the clock model with N = 5. Unlike the discrete Ising model and continuous *O(2)* model with only one phase transition, the Z5 model is known for having two phase transitions and three phases, arises curiosity.

In this thesis, Chapter 2 will present the Z5 model we are going to research; Chapter 3 will focus on the tensor formulation for both Ising and Z5 models, where the Ising formulation was already solved by predecessors in this area and the Z5 formulation uses a similar technique with only slightly difference in terms of coefficients; Chapter 4 will explain the blocking procedures we use, more specifically the truncation method for reducing the number of states in the tensor, starting from the Ising model and then generalize it to the Z5 model.

CHAPTER 2

THE Z5 MODEL

The Z5 model is a variation of the classic Ising model, which provides a simplified description of the ferromagnetism of materials in statistical mechanics.

Consider a square lattice with total of L x L sites, where each sites are identical spin-1/2 particles with spin $\sigma_i = 1$ or $\sigma_i = -1$, the partition function for the Ising model is

$$Z = \sum_{\{\sigma\}} e^{-\beta H} \tag{2.1}$$

$$= (\prod_i \sum_{\sigma_i}) e^{-\beta H}$$

with

$$H = -\sum_{\langle ij \rangle} \sigma_i \sigma_j \tag{2.2}$$

where $\{\sigma_i\}$ represents the all configurations of the spins and $<ij>$ represents nearest neighbor pairs, and β is the inverse temperature.

For the Z5 model, we have the same lattice but each sites can have 5 spin values around a clock circle. We have the same partition function with only the difference in the Hamiltonian:

$$H = -\sum_{\langle ij \rangle} \cos[(\sigma_i - \sigma_j)\frac{2\pi}{5}] \tag{2.3}$$

with $\sigma_i = 0, 1, 2, 3, 4$

CHAPTER 3

TENSOR RENORMALIZATION GROUP FORMULATION

3.1 Introduction

The Tensor Renormalization Group method is motivated by reformulating the classical partition function in terms of local tensors. We use character expansions and assign degrees of freedom to the links [2]. The model then becomes a square lattice where each link associated to a two sites can have different values, or 'charge'. To make the tensor local, we can simply cut the link in half and assign each half to one site, i.e. mathematically we just take a square root over the coefficient.

We should start by looking at the formulation of 2D Ising model, and then proceed to the Z5 which is very similar to the Ising model.

3.2 TRG formulation for the 2-Dimensional Ising model

For the 2D Ising model, we can rewrite the partition function as

$$Z = (\prod_{i} \sum_{\sigma_i=-1}^{1}) \prod_{\langle i,j \rangle} e^{\beta \sigma_i \sigma_j} \tag{3.1}$$

Using the mathematical property

$$\begin{aligned} e^{\beta \sigma_i \sigma_j} &= \cosh \beta (1 + \tanh \beta \sigma_i \sigma_j) \\ &= \cosh \beta \sum_{n=0}^{1} (\tanh \beta \sigma_i \sigma_j)^n \end{aligned} \tag{3.2}$$

we rewrite the partition function as

$$Z = (\cosh \beta)^{2\mathrm{V}} (\prod_{\mathrm{i}} \sum_{\sigma_i=0}^{4}) \prod_{\langle i,j \rangle} \sum_{\{n\}=0}^{1} (\tanh \beta \sigma_i \sigma_j)^n \tag{3.3}$$

with V being the volume of the lattice.

Notice all the sums and products commute to each other here because they are sum/product over different indices, so we can regroup the four terms involving one single spin σ_i, and we found

$$\sum_{\sigma_i=-1}^{1}(\sigma_i\sqrt{\tanh\beta})^{(n_x+n_y+n_{x'}+n_{y'})}=(\sqrt{\tanh\beta})^{(n_x+n_y-n_{x'}-n_{y'})}2\delta(\mathrm{mod}[n_x+n_y+n_{x'}+n_{y'},2]) \qquad (3.4)$$

where $n_x, n_y, n_{x'}, n_{y'}$ are the n's belong to the four nearest neighbor pairs extended from site i, and the delta function exists because all the odd sums of n's vanish. [3]

Hence if we construct a local tensor for site i,

$$T_{n_x n_y n_{x'} n_{y'}}=\sqrt{F_{n_x}(\beta)F_{n_{x'}}(\beta)F_{n_y}(\beta)F_{n_{y'}}(\beta)}\delta(\mathrm{mod}[n_x+n_y+n_{x'}+n_{y'},2]) \qquad (3.5)$$

with

$$F_n(\beta)=\left(\sqrt{\tanh\beta}\right)^n \qquad (3.6)$$

We can then rewrite the partition function as

$$Z=2^V(\cosh\beta)^{2V}\,Tr\prod_i T^{(i)}_{n_x n_y n_{x'} n_{y'}} \qquad (3.7)$$

the notation 'Tr' means tensor contraction, here it represents sum over 0 and 1.

3.3 TRG formulation for Z5 model

For the Z5 model, we have the partition function

$$Z=(\prod_i\sum_{\sigma_i=0}^{4})e^{\beta\cos[\frac{2\pi}{5}(\sigma_i-\sigma_j)]} \qquad (3.8)$$

The only difference here, besides the sum over 0 to 4, compare to the 2D Ising model, is the exponent, and to rewrite it we need to use Discrete Fourier Transform (DFT), that gives us

$$e^{\beta\cos[\frac{2\pi}{5}(\sigma_i-\sigma_j)]}=\frac{1}{5}\sum_{n=0}^{4}e^{in[\frac{2\pi}{5}(\sigma_i-\sigma_j)]}F_n(\beta) \tag{3.9}$$

With the coefficient function

$$F_n(\beta)=e^{\beta}+2e^{\beta\cos\frac{2\pi}{5}}\cos(\frac{2\pi}{5}n)+2e^{\beta\cos\frac{4\pi}{5}}\cos(\frac{4\pi}{5}n) \tag{3.10}$$

And we have

$$Z=(\frac{1}{5})^{2\mathrm{V}}(\prod_i\sum_{\sigma_i=0}^{4})\sum_{\{n\}}^{4}\prod_{\langle i,j\rangle}e^{in_{\langle i,j\rangle}[\frac{2\pi}{5}(\sigma_i-\sigma_j)]}F_n(\beta) \tag{3.11}$$

If we regroup the four terms involving a given spin σ_i, we then have the constraint

$$\sum_{\sigma_i=0}^{4}e^{i(n_x+n_y-n_{x'}-n_{y'})\frac{2\pi}{5}\sigma_i}=5\delta(\mathrm{mod}[n_x+n_y-n_{x'}-n_{y'},5]) \tag{3.12}$$

Hence, we can construct the tensor by cutting the links in half, which means taking a square root of the coefficient function $F_n(\beta)$, we get

$$T_{n_xn_yn_{x'}n_{y'}}=\sqrt{F_{n_x}(\beta)F_{n_{x'}}(\beta)F_{n_y}(\beta)F_{n_{y'}}(\beta)}\delta(\mathrm{mod}[n_x+n_y-n_{x'}-n_{y'},5]) \tag{3.13}$$

And the partition function becomes

$$Z=Tr\prod_i T^{(i)}_{n_xn_yn_{x'}n_{y'}} \tag{3.14}$$

Here the 'Tr' notation becomes the shortcut for sums over 0 to 4.

CHAPTER 4

BLOCKING PROCEDURES

4.1 Introduction

The tensor formulation provides a lot simpler form of the partition function, with only the tensors that are all local and the coefficient in the front.

We should visualize the tensor $T_{n_x n_y n_{x'} n_{y'}}$ in Eq. (3.5) as a cross, where the four legs represent $n_x, n_y, n_{x'}, n_{y'}$ which can take 5 different values, from 0 to 4. The four legs are half of the links connecting to the neighbor sites.

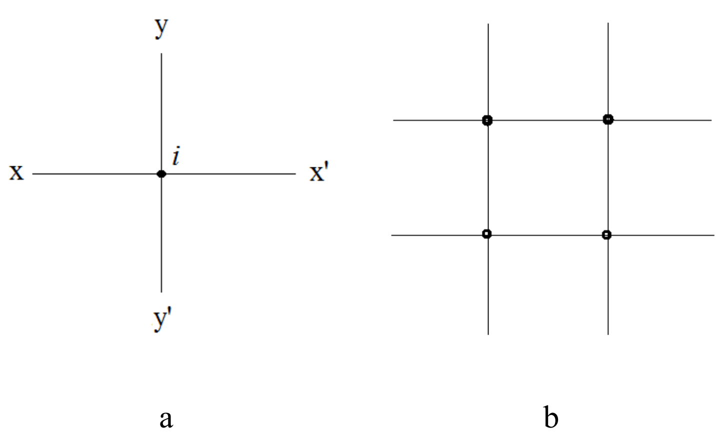


Figure 4.1: a. single tensor visualization. b. four tensors contracted to each other

During the process of calculating the partition function, the object we will make is like Figure 4.1.b, which we have four tensors contracted to each other. This process is called 'blocking'. If we combine the two different indices into one index in each of the four directions, we get a new rank-4 tensor. In 2D Ising model, each '*n*' can take 2 values, thus each leg has 2 x 2 = 4 states; in Z5, each leg has 5 x 5 = 25 states.

The cost is huge and unrealistic if we continue this process contracting four of the new tensors together, because we are squaring the number of states after each step. Some approximation is then necessary. What we really want to achieve is, after each blocking step, we have a new tensor with the same amount of states as the original tensor, i.e. we need to reduce the states to the original amount.

This approximation step was already resolved by mathematicians and it is done by a series of linear transformation. We will again start from the 2D Ising model, and then generalize it to the Z5 model which is a little bit more complicated in terms of the algorithm.

4.2 Blocking procedures for 2D Ising model

We shall start by contracting two tensors together along the y-axis, we can then repeat the same procedure along x-axis to finish one whole step.

Here we give the mathematical solution [4] for the problem along with the geometric representation showing in Figure 4.2:

1. Contract two tensors along the y-axis,

$$M^{(n)}_{xx'yy'} = \sum_i T^{(n)}_{x_1 x_1' yi} T^{(n)}_{x_2 x_2' iy'} \tag{4.1}$$

   where $x = x_1 \otimes x_2$, $x' = x_1' \otimes x_2'$, and the superscript *n* denotes the *n*'th iteration.

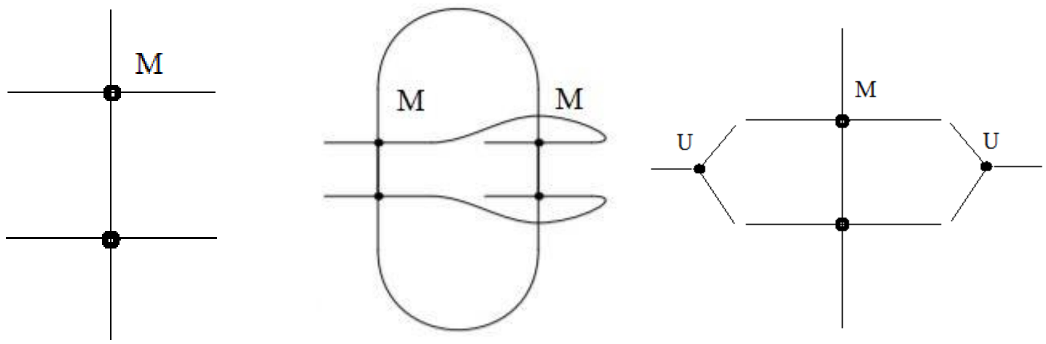
2. Creating the 4x4 matrix A by contracting two M tensors together, with only the left side of the two tensors remains open, i.e.

$$A_{xx_\alpha} = \sum_{x'yy'} M^{(n)}_{xx'yy'} M^{(n)}_{x_\alpha x'yy'} \tag{4.2}$$

3. Find the eigenvectors of matrix A, and construct the 4 x 2 matrix U, whose columns are the two eigenvectors with the largest two eigenvalues, in descending order.
4. Project the U matrix to both left and right sides of the M tensor, reducing the number of states from 4 to 2, as shown in Figure 4.2.c

$$T^{(n+1)}_{xx'yy'} = \sum_{ij} U^{(n+1)}_{ix} M^{(n)}_{ijyy'} U^{(n+1)}_{jx'} \tag{4.3}$$

5. Repeat the same process along the x-axis.



a b c

Figure 4.2: steps for blocking. a. step 1. b. step 2. c. step 4.

This process preserves the initial constraint from the delta function, thus it can be repeated iteratively. It's worth mentioning that we also need to normalize the tensor by T_{0000} after each iteration otherwise the eigenvalues won't converge. We also need to keep track of the T_{0000} value for calculating the partition function at the end. [5]

4.3 Blocking procedures for the Z5 model

For the Z5 model, we need to do a little bit more work to preserve the constraint of the initial tensor, the delta function $\delta(\mathrm{mod}[n_x + n_y - n_{x'} - n_{y'}, 5]$. We want the new tensor after blocking still satisfy this condition. To do this, consider the M tensor in step 1 above:

$$M^{(n)}_{xx'yy'} = \sum_i T^{(n)}_{x_1x_1'yi} T^{(n)}_{x_2x_2'iy'}$$

we have

$$(x_1 - x_1' + y - i) \bmod 5 = 0 \tag{4.4}$$

$$(x_2 - x_2' + i - y') \bmod 5 = 0 \tag{4.5}$$

And we want

$$((x_1, x_2) - (x_1', x_2') + y - y') \bmod 5 = 0 \tag{4.6}$$

where (x_1, x_2) and (x_1', x_2') denotes the charge of the combined state.

Solving the three modulo equations gives us

$$(x_1, x_2) = (x_1 + x_2) \bmod 5$$

$$(x_1', x_2') = (x_1' + x_2') \bmod 5 \qquad \text{(4.70}$$

Thus, we need to define the combined state as the sum of the two original indices modulo 5, which means we need to group the index pairs with their sum modulo 5. For example, we want (0, 1), (1, 0), (2, 4), (3, 3), (4, 2) to be 1, and (0, 3), (1, 2), (2, 1), (3,0) (4, 4) to be 3 in the new tensor. To achieve this, we define (0, 0), (1, 4), (2, 3), (3, 2), (4, 1) to be 0, 1, 2, 3, 4 and (0, 1), (1, 0), (2, 4), (3, 3), (4, 2) to be 5, 6, 7, 8, 9, and so on. This also allows us to have the A matrix in the second step to be a block-diagonal matrix. The A matrix $A_{xx_\alpha} = \sum_{x'yy'} M^{(n)}_{xx'yy'} M^{(n)}_{x_\alpha x'yy'}$

will be 25 x 25, with five 5 x 5 matrices along the diagonal and zeros elsewhere, as shown in Figure 4.3.

$$\begin{pmatrix} (5x5) & & 0 \\ & \ddots & \\ 0 & & (5x5) \end{pmatrix}$$

Figure 4.3: The A matrix, which is block-diagonal, with five 5 x 5 matrices along the diagonal and zeros elsewhere

And in step three, we find the eigenvectors with the largest eigenvalues of these five matrices respectively. In step four, we contract these eigenvectors, which have shapes of 5 x 1, to 5 out of the 25 states of the left side of the *M* tensor with the correct order defined above.

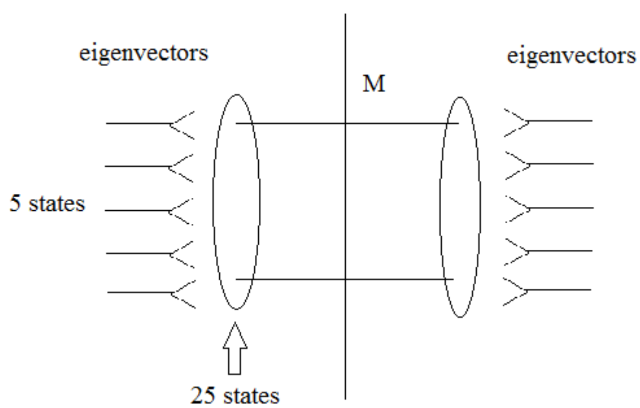


Figure 4.4: Project the eigenvectors to the M tensor to reduce the amount of states from 25 to 5.

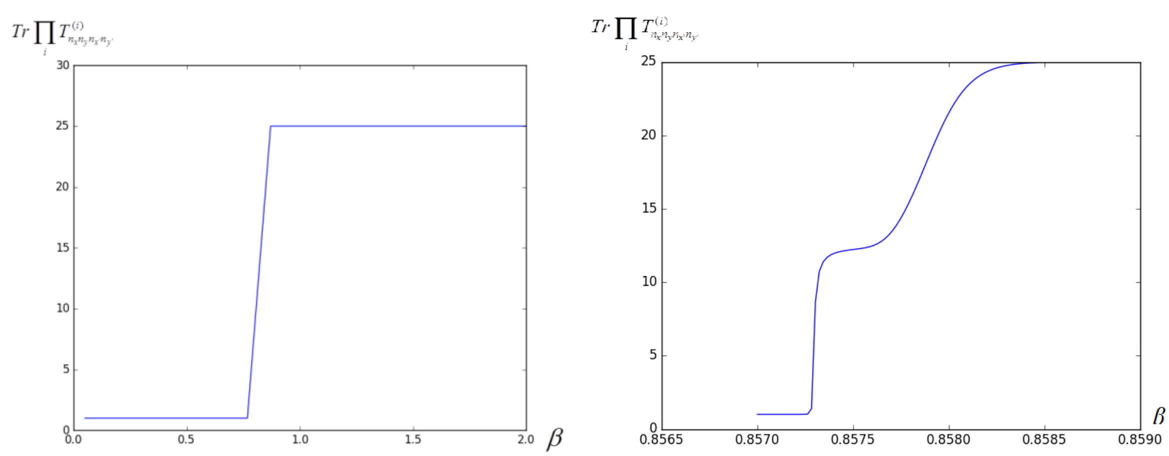
After this step, we will again have a new tensor with 5 states in each of its four legs, and we can repeat this step along the x-axis to complete a full iteration.

CHAPTER 5

RESULTS AND CONCLUSIONS

One of the simplest result we obtained is the critical temperature, where the value of contracted tensor $Tr\prod_i T^{(i)}_{n_x n_y n_{x'} n_{y'}}$ changes dramatically. We can also look at a fixed point[6]. A fixed point could be a site on the lattice. When we do the TRG blocking iteratively over the lattice, the fixed points will eventually become either 0 or 1 depending on the temperature of the system.

We found one of the critical temperature for the Z5 model is at $\beta = 0.858$. It can be seen from Figure 5.1 where the value of the contracted tensor changes from 1 to 25. The other critical temperature still remains to be studied at through other different techniques.



a b

Figure 5.1: a. The value of the contracted tensor $Tr\prod_i T^{(i)}_{n_x n_y n_{x'} n_{y'}}$ after 20 iterations versus β, the transition is at around $\beta = 0.858$. b. Zoom in at $\beta = 0.8565$ *to* $\beta = 0.8585$

We also plotted the natural log of the partition function per site versus temperature, we were supposed to get a smooth line, and surely we did, as shown in Figure 5.2:

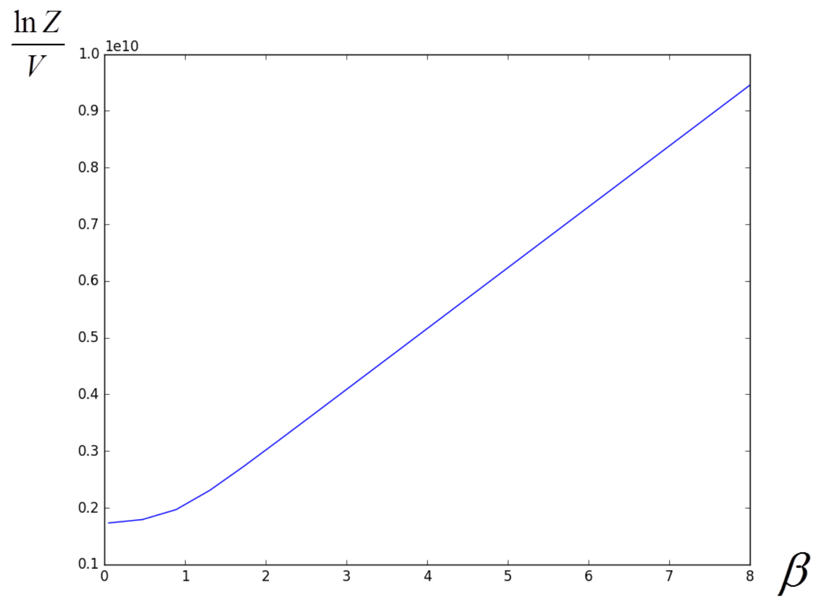


Figure 5.2: plot for *lnZ/V vs β*, which shows a smooth line.

The result we have obtained are only very simple results and there are still a lot more to be looked at, such as the average energy of the system, the average magnetization, the magnetic susceptibility, and so on. But since we already coded the TRG blocking process, these results should be out very soon. And hopefully we will be able to find out what is happening in the third phase of the Z5 model.

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