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Non uniform thickness and weighted global radius of curvature of smooth curves

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NON UNIFORM THICKNESS AND WEIGHTED GLOBAL RADIUS OF CURVATURE OF SMOOTH CURVES

by

Kimberly Jean Huerter

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy degree
in Mathematics in
the Graduate College of
The University of Iowa

December 2009

Thesis Supervisor: Associate Professor Oguz Durumeric

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Since it is possible for biological structures, such as polymers or DNA strands, to exhibit forces or charges of different strengths the idea of a non-uniform thickness of a knot is explored. In his work, O. Durumeric provides a definition for non-uniform thickness. This thesis will provide an alternative characterization for the non-uniform thickness of a knot, which is more conducive to computer calculations.

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Title and Department

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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To my family. Past, present, and future.

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ABSTRACT

The uniform thickness of knots has been used to investigate knotted polymers including DNA. Even though these structures carry great length, it is unnatural for them to contain knots. However when they do, it can cause gene malfunctions. In fact, scientist have demonstrated that knotting may cause a loss of genetic material by blocking DNA replication and also blocking transcription of a gene into its active protein.

Since it is possible for biological structures, such as polymers or DNA strands, to exhibit forces or charges of different strengths the idea of a non-uniform thickness of a knot is explored. In his work, O. Durumeric provides a definition for non-uniform thickness. This thesis will provide an alternative characterization for the non-uniform thickness of a knot, which is more conducive to computer calculations.

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INTRODUCTION

1.1 Motivation

In the nineteenth century, physicists like Lord Kelvin (Sir William Thomson), proposed that the far reaches of space consisted of three dimensional knotted vorticies [7]. This theory provoked the physicists of the time to employ mathematicians to develop the first tables of knots. However, the theory foundered and even though early research included mathematicians, it failed to capture the interest of the mathematical community.

In the 20th century, knot theory began to gain attention among mathematicians and, today the theory concerns biologists, chemists, and physicists. In particular, the theory of knots has been investigated within the context of DNA and knotted polymers [2]. It is this context that motivates the work for this thesis.

Knot theory is used to study natural biological structures such as polymers including DNA. Even though these structures carry great length, it is unnatural for them to contain knots. However, when they do, it introduces some interesting situations. For instance, when DNA becomes knotted, it can cause gene malfunctions. In fact, in [5] scientist have demonstrated that knotting may cause a loss of genetic material by blocking DNA replication and also blocking transcription of a gene into its active protein. Also in [5] it has been shown that knots cause mutation at rates three to four orders of magnitude higher than those of unknotted DNA.

The uniform thickness of a smooth knot of fixed length can be defined as the radius of the largest uniform tube that can be placed around the knot without self intersection. This definition has been described in different ways. One definition comes from R.A. Litherland, J. Simon, O. Durumeric, and E. Rawdon [8]. They use methods of topology and differential geometry to describe their characterizations.

Later, O. Gonzalez and J. Maddocks [8] came up with their own method of uniform thickness. In their characterization, they use the definition of a circumradius of a circumcircle in order to obtain uniform thickness. The motivation behind this alternative method is to provide a description that would be more conducive to computer calculations of thickness and decreasing thickness.

From here, one main question that arises. What happens if there exists charges along the knot that are not uniform? What if the knot experiences different forces along it? How would the thickness of these types of knots be explained? O. Durumeric generalized his collaborative work with R.A. Litherland, J. Simon, and E. Rawdon on uniform thickness to provide one such explanation. However, it is the goal of this thesis to do the same with the work of O. Gonzalez and J. Maddocks to provide an alternative method. The reason for pursuing this new definition is to provide a method that is more conducive for computer calculations.

The connection between the four methods are highlighted here in the diagram below.

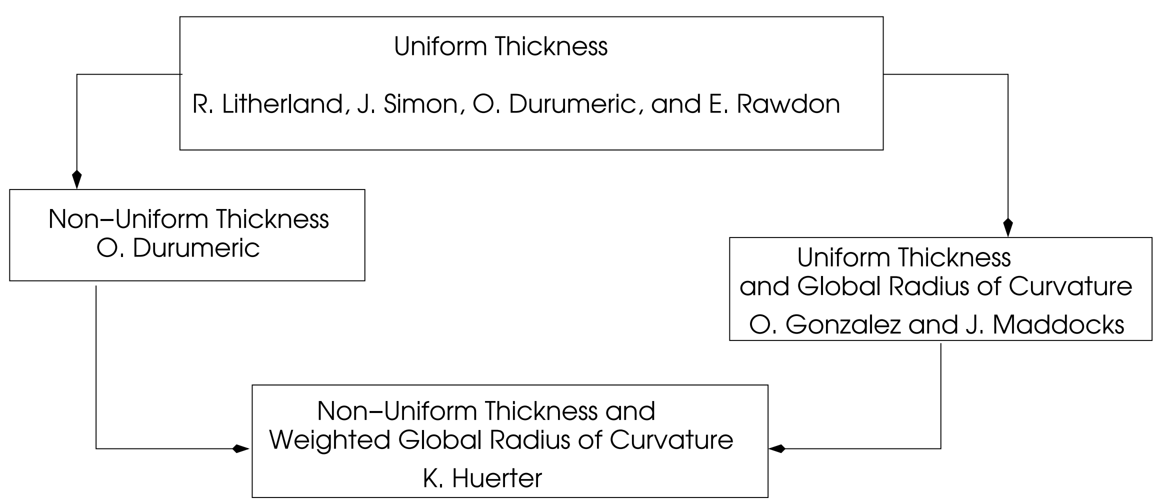

Figure 1.1: Thicknesses

1.2 Overview

Before we can begin the discussion on non-uniform thickness of a knot, some background knowledge and definitions are needed. We begin with the definition of a mathematical knot, move on to what it means for a knot to have thickness, and provide background information on methods used to describe thickness. We then continue with the motivation for finding these structures and introduce non-uniform thickness.

We begin in Chapter 2 by providing some background knowledge of knots. Here we define what it means to be a mathematical knot and what it means for that knot to have thickness. From here, we discuss the significance of equivalent knots and of the ideal shape of a knot. The chapter proceeds by motivating the need for finding the non-uniform thickness of a knot and the reason for the approach used in this thesis.

Chapter 3 provides the main results of this thesis. Chapter 3, section 1 provides some definitions and notations needed throughout the thesis. Section 2 begins the discussion on our non-uniform thickness model. The next few sections provide the ground work for the method. Finally section 3.7 will show the characterization provided in the thesis is equivalent to describe the definition given by O. Durumeric in [4].

CHAPTER 2
BACKGROUND

2.1 Knots

Intuitively, a mathematical knot is similar to a knot that is tied in a piece of rope. However, there is one major difference between a common knot tied in a rope and a mathematical knot. In a mathematical knot, the free ends need to be connected hence forming a closed loop. More formally, a mathematical knot is defined as follows.

Definition 1. *A knot is a proper embedding of a circle into three-dimensional space. In other words, an embedding* $f : [a, b] \to \mathbb{R}^3$ *with* $f(a) = f(b)$*.*

An example of a knot is given in figure 2.1. In particular this example is known as the trefoil knot.

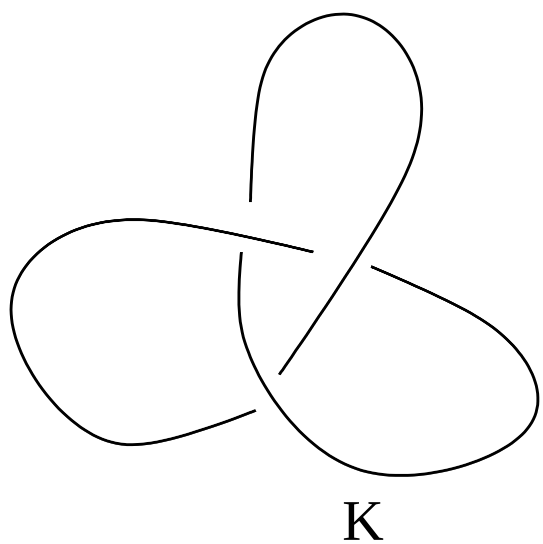

Figure 2.1: Knot

In certain situations it is important to determine when two knots are the same and when they are different. If one knot can be continuously deformed into another knot, then we say that the knots are equivalent and they are in the same knot class. A more formal definition of equivalent knots and knot classes are given here.

Definition 2. *f_0 and f_1 are equivalent knots if there exists an orientation preserving homeomorphism $H : \mathbb{R}^3 \to \mathbb{R}^3$ such that $H \circ f_0 = f_1$.*

Definition 3. *Two knots are said to be in the same knot class if they are equivalent knots.*

The following figure, 2.2 gives examples of two equivalent knots K_0 and K_1 and two knots that are not equivalent, K_1 and K_2. Hence K_0 and K_1 are in the same knot class while K_1 and K_2 will be in separate knot classes.

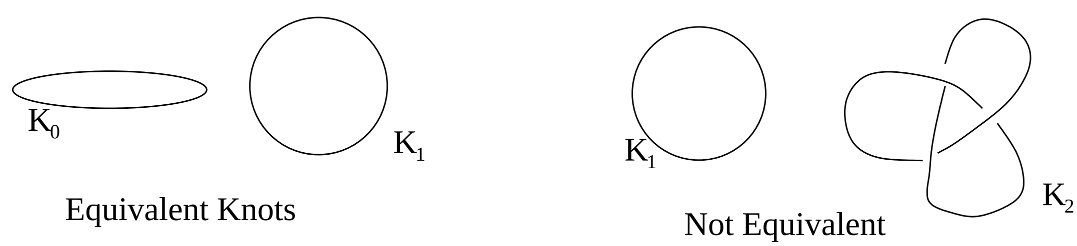

Figure 2.2: Equivalence

It is clear from the definition of a knot that the simplest knot possible will be a circle. Any knot that is equivalent to the to the circle is referred to as the unknot or the trivial knot. If a knot is not in the same class as the unknot, then the knot is referred to as non-trivial.

2.2 Uniform Thickness of Knots

Given a knot embedded in $\mathbb{R}^3$, the uniform thickness of a smooth knot can be defined as the radius of the largest uniform tube that can be placed around the knot without self intersection of normal discs. Over the years, there has been more than one way to describe the uniform thickness of a knot. One of the methods that gives a characterization of thickness is given by O. Gonzalez and J. Maddocks. Since this thesis is a generalization of their ideas, we focus on their characterization and give a brief explanation of it now.

Gonzalez and Maddocks [6] begin with the well-known fact that through any three non-collinear points in $\mathbb{R}^3$ there exists a unique circle called the circumcircle with radius known as the circumradius. Figure 2.3 provides an example of the circumradius.

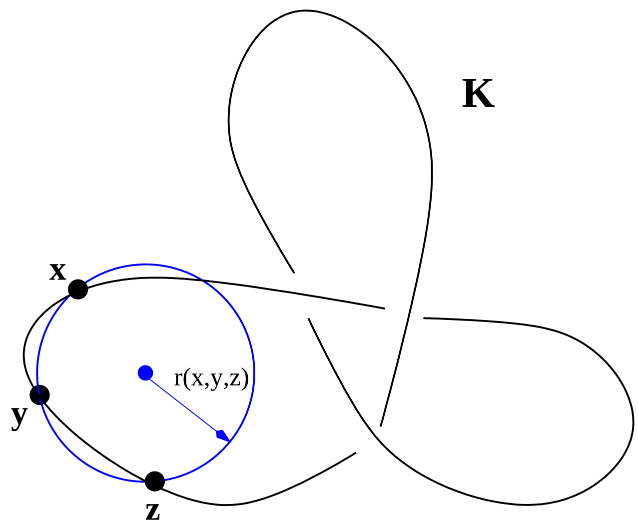

Figure 2.3: Circumradius

From the work in [1], the circumradius can be represented by $r(x, y, z) = \frac{\|x-y\|\|x-z\|\|y-z\|}{4A(x,y,z)}$, where $A(x, y, z)$ is the area of the triangle with vertices at x, y, and z. There are several representations for $A(x, y, z)$, but the one that is most convenient for this process is $A(x, y, z) = \frac{1}{2}\|x - z\|\|y - z\|sin\theta_z$, where θ_z is the angle $\measuredangle xzy$. Therefore, $r(x, y, z) = \frac{\|x-y\|}{2\|sin\theta_z\|}$.

When x, y, and z are points on a simple, smooth curve K, the domain of the function $r(x, y, z)$ can be extended by continuous limits to all triples of points on K [6]. For example, if x, y, and z are three distinct points of K the following limits exist and are defined as follows:

1. $\lim_{z \to y} r(x, y, z) = r(x, x, y)$

2. $\lim_{y,z\to x} r(x,y,z) = r(x,x,x)$

Maddocks and Gonzalez continue by using $r(x, y, z)$ in their definition of global radius of curvature.

Definition 4. *The global radius of curvature at a point x for any simple, smooth curve K is defined to be*

$$\rho_G(x) := \inf_{\substack{y,z\in K\\ x\neq y\neq z\neq x}} r(x,y,z) \tag{2.1}$$

Since K is assumed to be a simple, smooth, closed curve and since the limits $r(x, x, y)$ and $r(x, x, x)$ exist, then the infimum in equation 2.1 can be replaced by a minimum, hence giving the following representation for the global radius of curvature.

$$\rho_G(x) := \min_{y,z\in K} r(x,y,z) \tag{2.2}$$

Maddocks and Gonzalez continue by showing that the minimum global radius of curvature for any point $x \in K$ will either be the standard local radius of curvature at x or the strictly smaller radius of a circle containing x and another distinct point $y \in K$ to which the circle is tangent at y [6]. This of course depends on the point x and the behavior of the curve around x.

We can see which situation will control the thickness by finding the global radius of curvature at each point $x \in K$ and then finding the minimum over all these values. This is known as the Global Radius of Curvature and is defined by [6] to be

$$\Delta[K] := \inf_{x\in K} \rho_G(x).$$

In [6], they observe that when K is simple and smooth, the function $\rho_G(x)$ is a continuous function of x on K and the infimum in the definition above can be replaced with a minimum.

$$\Delta[K] := \min_{x \in K} \rho_G(x).$$

Maddocks and Gonzalez continue their process by showing the thickness will happen at either of the two limit cases $r(x, x, y)$ or $r(x, x, x)$. If the minimum global radius of curvature happens at the first limit, it will do so in a very particular way. It is not enough to have the minimum global radius of curvature at a point x to be of the form $r(x, x, y)$. It must also be the case that the line segment joining the two remaining points must be perpendicular to the curve at both points x and y. An example of this in $\mathbb{R}^2$ can be seen in figure 2.4, below.

The other possibility is the limit $r(x, x, x)$ which is the standard local radius of curvature. It is clear that this quantity will have an effect on the thickness of a knot. In fact, the greater the curvature, the tighter the corners of the knot get resulting in a smaller tube. This can be seen in figure 2.4 as well.

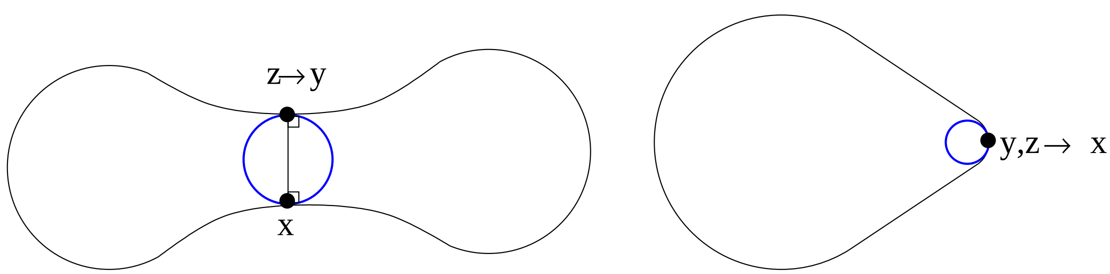

Figure 2.4: Double Critical Self Distance and Focal Distance

Maddocks and Gonzalez have proven that $\Delta[K]$ is equivalent to the definition of thickness given in [8]. An equivalent definition of which we provide here:

Theorem 2.2.1 ([8])**.** *The thickness of a smooth knot is*

$$\eta_*[K] = \min \left\{ \min_{x \in K} \rho(x), \frac{d_*[K]}{2} \right\}$$

Where $\rho(x)$ and $d_*[K]$ are defined as follows.

(i) $\rho(x)$ is the standard local radius of curvature.

(ii) $d_*[K] := \min_{x,y \in \Omega} |x - y|, \Omega$ is defined to be the set of all x, y distinct pairs in K

such that the vector $x - y$ is perpendicular to the tangent lines at both x and y.

In the view of this theorem, we will use

$$\eta_*[K] = \min\left\{\min_{x\in K}\rho(x), \frac{d_*[K]}{2}\right\}$$

as an equivalent definition of uniform thickness.

In their work, [6] show $\min_{x,z\in K x\neq z}(r(x,x,z))$, where $r(x,x,z) = \lim_{y\to x} r(x,y,z)$ is equivalent to $d_*[K]$, and that $\rho(x)$ and $\lim_{y,z\to x} r(x,y,z) = r(x,x,x)$ are the local radius of curvature. Hence they provide a characterization of thickness that is given by [8], but that is more conducive to computer calculations. It is for this reason that we have chosen to generalize this method for thickness given by Maddocks and Gonzalez in this thesis.

Once the thickness of a knot is determined, it is important to find an ideal shape in a knot class. An ideal shape of a knot is a knot within a knot class, with fixed length, that allows for the greatest thickness. In other words,

Definition 5. *The ideal thickness of a knot class K^* is given by,*

$$\Delta[K^*] = \sup_{K\in K^*}\Delta[K] \tag{2.3}$$

where K^* is the class of knots of a given type with a fixed length of one. Any knot K_0 which attains this supremum is called an ideal knot in K^*, and its shape is called an ideal shape.

The figure 2.5 below is an example of a knot in its ideal shape. Notice that the knot at the core of the tube has minimal curvature and maximal distance from itself.

Recall it is the ideal shapes of knots that provide a connection to knotted polymers including DNA in certain equilibrium. Therefore, it is this motivation that makes these definitions so important.

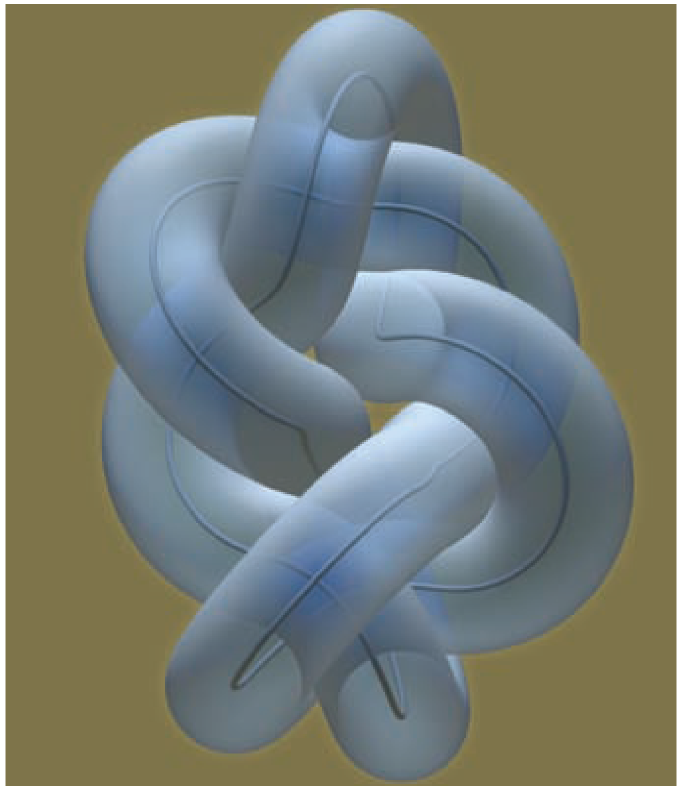

Figure 2.5: Ideal Shape of a Knot: J. Cantarella, M. Piatck, and E. Rawdon.

2.3 Discussion on Computer Calculations

It is the goal of this thesis to find a characterization for the non-uniform thickness of a knot that is more conducive to computer calculations. It will be shown in Chapter 3 that this new characterization gives an algorithmic procedure for finding the non-uniform thickness of a smooth knot. We will see that provided with certain quantities, a step by step procedure for finding the non-uniform thickness will be given. It this step by step approach that makes this characterization better for computer calculations which use discrete data

We can also see that the characterization given in [6] is better for computer calculations in the following discussion.

Recall in [8], the definition for thickness is given by,

$$\eta_*[K] = \min\left\{\min_{x\in K}\rho(x), \frac{d_*[K]}{2}\right\}$$

With this definition, we know that $\min_{x \in K} r(x,x,x)$ is equal to $\min_{x \in K} \rho(x)$ and from [6] we know $\min_{x,y \in K} r(x,x,y)$ is equivalent to $\frac{d_*[K]}{2}$. It is due to this second quantity that we can see why the characterization of thickness given in [6] is better for computer calculations.

Observe that $\frac{d_*[K]}{2} = \min_{x,y \in \Omega} \mid x - y \mid,\ x \neq y$ is a local minimum. Since it would be difficult to tell if there is a local minimum in a local neighborhood, the second quantity would not be ideal to minimize with discrete data. Since the first quantity is global we can see from this perspective when a minimum will exist in this type of situation. Therefore, it will be easier to do computer calculations with the characterization provided by [6].

This explanation will also hold for the non-uniform thickness formula as well.

2.4 Non-Uniform Thickness: O. Durumeric

The non-uniform thickness of a smooth knot can be described as a tube of varying girth at each point along the knot. The changing thickness is due to non-uniform distribution of forces that may be present along the knot. This can be seen in figure 2.6 below.

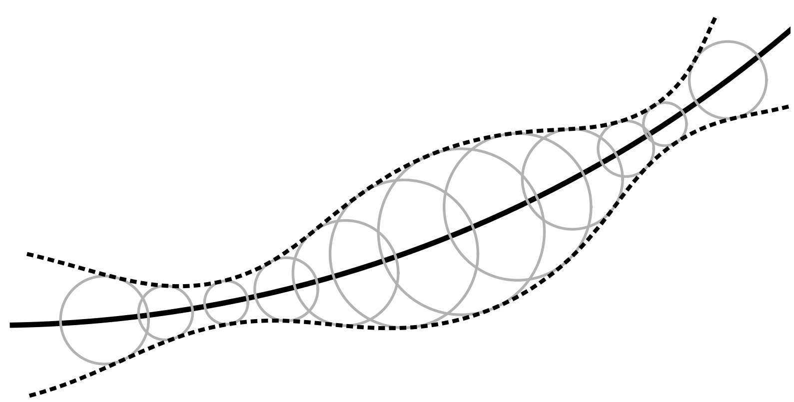

Figure 2.6: Non-Uniform Tubular Neighborhood: O. Durumeric.

Since the forces placed on the knot will have an affect in all directions, Durumeric uses the union of metric balls centered on the knot to define the non-uniform thickness. He uses distance function methods of Riemannian geometry to describe these balls and in turn to define the quantities needed to find the non-uniform thickness. We provide an equivalent definition using definition 1 of [4]. But before we can get to the definition of thickness from [4], we need a preliminary definition.

Definition 6 (Durumeric [4]). *Let K be a union of finitely many disjoint smooth closed curves in $\mathbb{R}^n$, $\mu : K \to (0, \infty)$ be a C^2 function, and $\nabla\mu(q)$ be the gradient of μ. Let NK be the normal bundle of K in $\mathbb{R}^n$. Define $exp^\mu : W \to \mathbb{R}^n$ by*

$$exp^\mu(q, w) = q - \mu(q)\|w\|^2 grad\mu(q) + \mu(q)\sqrt{1 - \|grad\mu(q)\|^2\|w\|^2}w$$

$$where\ W = \left\{w \in NK_q : q \in K \ and\ \|w\| \leq \tfrac{1}{\|grad\mu(q)\|}\ when\ \|grad\mu(q) \neq 0\right\}$$

Now we can give the definition of non-uniform thickness given by [4].

Definition 7 (Durumeric [4]). *Let $D(r) = \{(q, w) \in NK : q \in K and \|w\| < r\}$ Then the differentiable injectivity radius $DIR(K, \mu)$ is*

$$\sup\{r : exp^\mu\ restricted\ to\ D(r)\ is\ a\ diffeomorphism\ onto\ its\ image\ \}$$

From Propositoin 5 part (iii) in [4], O. Durumeric shows that DIR is equivalent to the minimum of the quantities $FocRad^0(K, \mu)$ and $\frac{1}{2}DCSD$.

Proposition 2.4.1 (Durumeric [4]). *Let K be a union of finitely many disjoint simple smooth closed possibly linked or knotted curves in $\mathbb{R}^n$ and $\mu : K \to (0, \infty)$ be given. Then the differentiable non-uniform thickness of a smooth knot can be taken as*

$$DIR = \min\left(\tfrac{1}{2}DCSD\left(K, \mu\right), FocRad^0\left(K, \mu\right)\right).$$

Where,

1. $DCSD\left(K, \mu\right) = \min\left\{\frac{\|q_1 - q_2\|}{\mu(q_1) + \mu(q_2)} : (q_1, q_2), \text{is a double critical pair for} (K, \mu)\right\}$

2. $FocRad^0 =$

$$\left(\max\begin{bmatrix} \max\{\frac{1}{2}(\mu^2)'' + \frac{1}{2}\kappa^2\mu^2 + \kappa\mu\sqrt{\mu(\mu'' + \frac{1}{4}\kappa^2\mu)} & : \mu'' + \frac{1}{4}\kappa^2\mu \geq 0\}, \\ \max\{|\mu'|^2 & : s \in \text{Domain}(\gamma)\} \end{bmatrix}\right)^{-\frac{1}{2}}$$

Where double critical pair is defined as follows.

Definition 8 (Durumeric)**.** *A pair of points* $(q_1, q_2) \in K \times K$ *is called a double critical pair for* (K, μ)*, if* $q_1 \neq q_2$ *and grad*$\Sigma(q_1, q_2) = 0$*, where* $\Sigma : K \times K \to \mathbb{R}$ *is defined by* $\Sigma(q_1, q_2) = \frac{\|q_1 - q_2\|}{\mu(q_1)+\mu(q_2)}$.

This definition is a generalization of the definition for uniform thickness of a knot given in [8]. The quantities $DCSD$ and $FocRad^0$ are generalizations of the ones given in [8]. In fact, if $\mu = 1$ it is easy to show the quantities given here for $FocRad^0$ and $DCSD$ are the same as their counterparts in the definition for uniform thickness.

In Chapter 3, we will see the non-uniform thickness given in this thesis is another characterization of the definition given in [4].

In other words, this work will prove the main theorem

Theorem 2.4.2. $DIR\,[K, \mu] = \Delta\,[K, \mu]$

Where $\Delta\,[K, \mu]$, the weighted global radius of curvature will be the characterization for non-uniform thickness described in this thesis, which is a generalization of $\Delta[K]$ given by O. Gonzales and J. Maddocks in [6] and will be defined in later sections.

CHAPTER 3
NON UNIFORM THICKNESS

The uniform thickness of a smooth knot is described as a uniform tube that encases the particular knot. The structure is constant and uniform. However, if we consider the possibility of a structure having forces placed upon it, the notion of thickness will be altered. This possibility is discussed in [4]. As stated before, [4] is a generalization of [8]. But in this thesis, we give a generalization of the method given in [6]. This is done for the same reasons [6] provided a characterization different from [8], so that a computer program could be built around it.

By introducing a weight function on the knot K, $\mu : K \to (0, \infty)$, the uniform structure of the tube placed around it is altered. Since we are generalizing the work of Gonzalez and Maddocks, we will begin as they do, with three non-collinear points of a given knot. The characterization of non uniform thickness continues with the definition of a weighted global radius of curvature, the consideration of the same limit cases discussed in [6]. It finishes by showing this alternative method is a characterization of thickness defined by [4].

3.1 Notation and Definitions

Throughout this document, we will assume the following notations and definitions.

1. K is a simple, smooth, closed curve in $\mathbb{R}^n$
2. $\gamma : I \to K$ is a C^3 arclength parametrization of K, unless otherwise stated.
3. $\mu : K \to (0, \infty)$ is the weight function on the curve K and must be at least C^3, we will use $\mu(t) = \mu(\gamma(t))$.

Definition 9. *Let $K \subseteq \mathbb{R}^n$ and $\mu : K \to (0, \infty)$ be given and $p_i \in K$ be such that*

$p_i \neq p_j$ when $i \neq j$ unless otherwise stated. Then we define:

1. $R_i(p) = \frac{\|p-p_i\|}{\mu(p_i)}$ *is the weighted distance from* $p_i \in K$ *for* $p \in \mathbb{R}^n$

2. $C_{ij} = \{p \mid R_i(p) = R_j(p)\}$

3. $C_{ijk} = \{p \mid R_i(p) = R_j(p) = R_k(p)\}$

4. $R_{ijk} = \begin{cases} min\{R_i(p) \mid R_i(p) = R_j(p) = R_k(p)\} & if C_{ijk} \neq \varnothing; \\ \infty & if C_{ijk} = \varnothing \end{cases}$

5. $\mathcal{R} = \{R_{ijk} \mid p_i \neq p_j \neq p_k \in K \, and \; C_{ijk} \neq \emptyset\}$

6. *Note: It is common to take* $i = 0$, $j = 1$, *and* $k = 1$ *throughout this paper.*

7. $\Delta[K, \mu] = \inf \mathcal{R} = \min \overline{\mathcal{R}}$

3.2 C_{ijk} and L_{ijk}

In their work describing uniform thickness, Maddocks and Gonzalez use the work of Coxeter to find a suitable representation for the circumradius. Evaluation of Coxeter's work reveals the use of similar triangles in order to achieve his solution for the circumradius. Since the introduction of non-uniform forces, we do not have the luxury of nice geometric figures such as triangles. Therefore, we can not mimic the work of Coxeter and must find an alternate way of finding a representation for the weighted circumradius, R_{ijk}. We can, however, generalize the definition of a circle and its radius to include the weight function and hence arrive at a solution set for the weighted circumradius.

In the uniform case the circumradius $r(p_0, p_1, p_2)$ can also be found by minimizing the distance function $\| p - p_i \|$ on the $n-2$ dimensional hyperplane defined by the following system of equations.

$$\| p - p_i \| = \| p - p_j \| = \| p - p_k \| \tag{3.1}$$

Observe that the point p which minimizes this distance will be the intersection of the line obtained from equation 3.1 and the plane defined by the three points on the curve K, p_i, p_j, and p_k. Once p is found then $r(x, y, z)$ can be determined.

Consider the process for finding $r(x, y, z)$ again, but this time considering weighted distance. Generalizing equation 3.1 produces the following system of equations.

$$\frac{\| p - p_i \|}{\mu(p_i)} = \frac{\| p - p_j \|}{\mu(p_j)} = \frac{\| p - p_k \|}{\mu(p_k)} \tag{3.2}$$

From section 3.1 we know $R_i(p)$, $R_j(p)$ and $R_k(p)$ are defined as follows

1. $R_i(p) = \frac{\|p-p_i\|}{\mu(p_i)}$

2. $R_j(p) = \frac{\|p-p_j\|}{\mu(p_j)}$

3. $R_k(p) = \frac{\|p-p_k\|}{\mu(p_k)}$

Then equation 3.2 becomes:

$$R_i(p) = R_j(p) = R_k(p) \tag{3.3}$$

Solving the system in 3.2 directly for p that would minimize the weighted distance would give a representation for R_{ijk}, but it is long and difficult to work with. Nevertheless, we are able to reduce the situation to a more manageable one and from this we can study the infimum of $\mathcal{R}$ in several steps. We begin by breaking up the solutions set of R_{ijk} in the following way.

Recall from section 3.1 the definition of C_{ij}. Then for points p_i, p_j, and p_k, we have

1. $C_{ij} = \{p | R_i(p) = R_j(p)\}$

2. $C_{ik} = \{p | R_i(p) = R_k(p)\}$

3. $C_{jk} = \{p | R_j(p) = R_k(p)\}$

The next lemma will describe the sets in definition section 3.1. Since the work for each set is similar we focus on one, namely C_{ij}.

Lemma 3.2.1. *The solution set of C_{ij} is either an $n-1$ dimensional sphere of radius $= \frac{\|p_i - p_j\| \mu(p_i)\mu(p_j)}{|\mu^2(p_j) - \mu^2(p_i)|}$ and center $= \frac{p_i \mu^2(p_j) - p_j \mu^2(p_i)}{\mu^2(p_j) - \mu^2(p_i)}$ when $\mu(p_i) \neq \mu(p_j)$, or it is a $n-1$ dimensional plane perpendicular to the line segment $\overline{p_i p_j}$ passing through the point $\frac{p_i + p_j}{2}$, if $\mu(p_i) = \mu(p_j)$. If C_{ij} is a sphere, then its center is on the line through p_i and p_j when $p_i \neq p_j$.*

Proof. For convenience, take $i = 0$, $j = 1$, $k = 2$, $A_0 = \mu(p_0)$, $A_1 = \mu(p_1)$ and $A_2 = \mu(p_2)$.

Case 1:$\mu(p_0) \neq \mu(p_1)$

$$\frac{\| p - p_0 \|}{\mu(p_0)} = \frac{\| p - p_1 \|}{\mu(p_1)}$$

$$\| p - p_0 \|^2 A_1^2 = \| p - p_1 \|^2 A_0^2$$

$$A_1^2(\| p \|^2 - 2p \cdot p_0 + \| p_0 \|^2) = A_0^2(\| p \|^2 - 2p \cdot p_1 + \| p_1 \|^2)$$

$$A_1^2 \| p \|^2 - 2p \cdot p_0 A_1^2 + A_1^2 \| p_0 \|^2 = A_0^2 \| p \|^2 - 2p \cdot p_1 A_0^2 + A_0^2 \| p_1 \|^2$$

$$(A_1^2 - A_0^2) \| p \|^2 - 2p \cdot (A_1^2 p_0 - A_0^2 p_1) = A_0^2 \| p_1 \|^2 - A_1^2 \| p_0 \|^2 \tag{3.4}$$

$$\| p \|^2 - \frac{2p \cdot (A_1^2 p_0 - A_0^2 p_1)}{(A_1^2 - A_0^2)} = \frac{A_0^2 \| p_1 \|^2 - A_1^2 \| p_0 \|^2}{(A_1^2 - A_0^2)} \cdot \frac{A_1^2 - A_0^2}{A_1^2 - A_0^2} \tag{3.5}$$

Consider the numerator on the right hand side of equation 3.5 and simplify:

$$(A_0^2 \| p_1 \|^2 - A_1^2 \| p_0 \|^2)(A_1^2 - A_0^2)$$

$$= A_0^2 A_1^2 \| p_1 \|^2 - A_1^4 \| p_0 \|^2 - A_0^4 \| p_1 \|^2 + A_0^2 A_1^2 \| p_0 \|$$

$$= (A_0^2 A_1^2 \| p_1 \|^2 - 2A_0^2 A_1^2 p_1 \cdot p_0 + A_0^2 A_1^2 \| p_0 \|) - (A_1^4 \| p_0 \|^2$$
$$- 2A_0^2 A_1^2 p_1 \cdot p_0 + A_0^4 \| p_1 \|^2)$$

$$= (A_0^2 A_1^2 \| p_1 - p_0 \|^2) - (\| A_1^2 p_0 - A_0^2 p_1 \|^2) \tag{3.6}$$

Taking equation 3.5 and replacing the numerator on the right hand side with equation 3.6 and adding the second quantity on the right to both sides results in the following expression:

$$\| \ p \ \|^2 - \tfrac{2p\cdot(A_1^2p_0 - A_0^2p_1)}{(A_1^2 - A_0^2)} + \tfrac{\|A_1^2p_0 - A_0^2p_1\|^2}{(A_1^2 - A_0^2)^2} = \tfrac{A_0^2A_1^2\|p_1 - p_0\|^2}{(A_1^2 - A_0^2)^2}$$

$$\left\| p - \tfrac{A_1^2p_0 - A_0^2p_1}{A_1^2 - A_0^2} \right\|^2 = \tfrac{A_0^2A_1^2\|p_1 - p_0\|^2}{(A_1^2 - A_0^2)^2}$$

This gives us the following center and radius for the n-1 dimensional sphere:

$$Center = \frac{A_1^2p_0 - A_0^2p_1}{A_1^2 - A_0^2} = \frac{\mu^2(p_1)p_0 - \mu^2(p_0)p_1}{\mu^2(p_1) - \mu^2(p_0)}$$

$$Radius = \frac{A_0A_1 \ \| \ p_1 - p_0 \ \|}{|A_1^2 - A_0^2|} = \frac{\mu(p_0)\mu(p_1) \ \| \ p_1 - p_0 \ \|}{|\mu^2(p_1) - \mu^2(p_0)|}$$

Remark 1. *From above, we can see that the center of* C_{01} *is a linear combination of the points* p_0 *and* p_1*. Therefore, the center will lie on the line* $\overleftrightarrow{p_0p_1}$*. However, we will show that neither* p_0 *nor* p_1 *can be the center.*

To show this, we will assume that the center is equal to p_0. Observe that not only is the center a linear combination of p_0 and p_1, but also the coefficients of this representation add up to one. Therefore,

$$center = \lambda p_0 + (1 - \lambda)p_1$$

where,

$$\lambda = \tfrac{\mu^2(p_1)}{\mu^2(p_1) - \mu^2(p_0)}$$

$$\lambda - 1 = \tfrac{\mu^2(p_0)}{\mu^2(p_1) - \mu^2(p_0)}$$

is an appropriate representation for the center. With this and our assumption that $p_0 = center$, we get:

$$p_0 = \lambda p_0 + (1 - \lambda)p_1$$

$\Rightarrow \lambda = 1$ and $1 - \lambda = 0 \Rightarrow A_0 = 0$, which would be a contradiction. A similar argument hold if the center is equal to p_1. Therefore, the center of C_{01} cannot be p_0 or p_1.

Case 2: $\mu(p_0) = \mu(p_1)$, that is $A_0 = A_1$

$$\frac{\| p - p_0 \|}{\mu(p_0)} = \frac{\| p - p_1 \|}{\mu(p_1)}$$

$$\Longleftrightarrow \| p - p_0 \|^2 = \| p - p_1 \|^2$$

$$\Longleftrightarrow (\| p \|^2 - 2p \cdot p_0 + \| p_0 \|^2) = (\| p \|^2 - 2p \cdot p_1 + \| p_1 \|^2)$$

$$\Longleftrightarrow -2p \cdot (p_0 - p_1) = \|p_1\|^2 - \|p_0\|^2$$

$$p \cdot (p_1 - p_0) = \tfrac{\|p_1\|^2 - \|p_0\|^2}{2}$$

Which is an n-1 dimensional plane that is perpendicular to the line segment $\overline{p_0 p_1}$ and that passes through the midpoint, $\frac{p_1 + p_0}{2}$. □

From this lemma and the system in equation 3.2, we can see that the three spheres C_{01}, C_{02}, and C_{12}, when they intersect, will do so at a single common point or along an $(n-2)$ dimensional sphere. A 2-dimensional example of which is given in figure 3.1.

Our ultimate goal is to find the minimal generalized circumradius of a curve of non-uniform weight distribution. As we stand so far, the solution to this problem lies on the intersection of three $n-1$ dimensional spheres, C_{ij}. The next set of definitions and lemmas will show this can be reduced to the intersection of an $n-1$ dimensional plane and one of the spheres, C_{ij}.

Definition 10. *Let $p_0, p_1, p_2 \in K$ be distinct, $A_i = \mu(p_i)$, $\forall i$, and let*

$$h_0 = A_0^2 - A_2^2$$

$$h_1 = A_1^2 - A_0^2$$

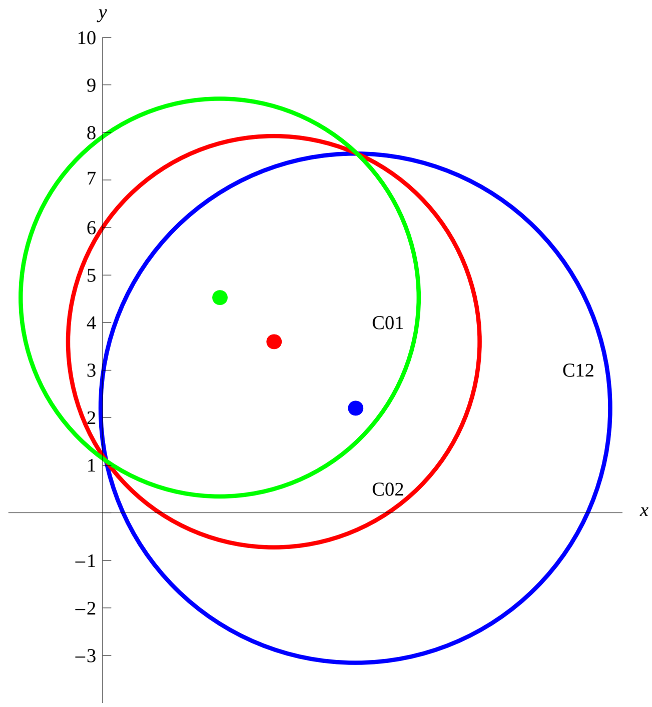

Figure 3.1: Solution Space

$h_2 = A_2^2 - A_1^2$

Then L_{012} *is the plane defined by:*

$$2p \cdot (h_0 p_1 + h_1 p_2 + h_2 p_0) = h_0 \parallel p_1 \parallel^2 + h_1 \parallel p_2 \parallel^2 + h_2 \parallel p_0 \parallel^2$$

provided that $v = h_0 p_1 + h_1 p_2 + h_2 p_0 \neq 0$

Lemma 3.2.2. *Assume that* $v \neq 0$ *and* L_{012} *is defined. Then*

(i) $C_{01} \cap C_{02} \subseteq L_{012}$.

(ii) $C_{012} \subseteq C_{01} \cap L_{012}$

when all are defined and non-empty.

(iii) $C_{01} \cap L_{012} = C_{012}$, *when* $A_0 \neq A_1$.

Proof. (i) Let $A_i = \mu(p_i) > 0$ for $i = 0, 1, 2$, and consider the equations for C_{01} and C_{02} from equation 3.4 in Lemma 3.2.1.

$$(A_1^2 - A_0^2) \parallel p \parallel^2 = A_0^2 \parallel p_1 \parallel^2 - A_1^2 \parallel p_0 \parallel - 2p \cdot (A_0^2 p_1 - A_1^2 p_0) \tag{3.7}$$

$$(A_2^2 - A_0^2) \parallel p \parallel^2 = A_0^2 \parallel p_2 \parallel^2 - A_2^2 \parallel p_0 \parallel - 2p \cdot (A_0^2 p_2 - A_2^2 p_0) \tag{3.8}$$

Multiply equation 3.7 by $(A_2^2 - A_0^2)$ and equation 3.8 by $(A_1^2 - A_0^2)$ and compare.

$$(A_2^2 - A_0^2)(A_1^2 - A_0^2) \parallel p \parallel^2 = (A_2^2 - A_0^2)(A_0^2 \parallel p_1 \parallel^2 - A_1^2 \parallel p_0 \parallel^2 - 2p \cdot (A_0^2 p_1 - A_1^2 p_0)) \tag{3.9}$$

$$(A_1^2 - A_0^2)(A_2^2 - A_0^2) \parallel p \parallel^2 = (A_1^2 - A_0^2)(A_0^2 \parallel p_2 \parallel^2 - A_2^2 \parallel p_0 \parallel^2 - 2p \cdot (A_0^2 p_2 - A_2^2 p_0)) \tag{3.10}$$

Observe that the left-hand side of both 3.9 and 3.10 are equal, hence

$$(A_2^2 - A_0^2)(A_0^2 \parallel p_1 \parallel^2 - A_1^2 \parallel p_0 \parallel^2 - 2p \cdot (A_0^2 p_1 - A_1^2 p_0)) \tag{3.11}$$

$$= (A_1^2 - A_0^2)(A_0^2 \parallel p_2 \parallel^2 - A_2^2 \parallel p_0 \parallel^2 - 2p \cdot (A_0^2 p_2 - A_2^2 p_0))$$

$$\iff 2p \cdot (A_0^2(A_1^2 - A_0^2)p_2 - A_2^2(A_1^2 - A_0^2)p_0 + A_1^2(A_2^2 - A_0^2)p_0 - A_0^2(A_2^2 - A_0^2)p_1)$$

$$= A_0^2(A_1^2 - A_0^2) \parallel p_2 \parallel^2 - A_2^2(A_1^2 - A_0^2) \parallel p_0 \parallel^2 + A_1^2(A_2^2 - A_0^2) \parallel p_0 \parallel^2 - A_0^2(A_2^2 - A_0^2) \parallel p_1 \parallel^2$$

$$\iff 2p \cdot (A_0^2(A_1^2 - A_0^2)p_2 + A_0^2(A_2^2 - A_1^2)p_0 + A_0^2(A_0^2 - A_2^2)p_1)$$

$$= A_0^2(A_1^2 - A_0^2) \parallel p_2 \parallel^2 + A_0^2(A_2^2 - A_1^2) \parallel p_0 \parallel^2 + A_0^2(A_0^2 - A_2^2) \parallel p_1 \parallel^2$$

$$\iff 2p \cdot (h_0 p_1 + h_1 p_2 + h_2 p_0) = h_0 \parallel p_1 \parallel^2 + h_1 \parallel p_2 \parallel^2 + h_2 \parallel p_0 \parallel^2$$

Therefore, $C_{01} \cap C_{02} \subseteq L_{012}$.

(ii) Take $p \in C_{012}$, then $p \in C_{01}$, C_{02}, and C_{12}. This implies that $p \in C_{01} \cap C_{02}$. (or any intersection of two of the spheres). However, from (i) we know that $C_{01} \cap C_{02} \subseteq L_{012}$. Hence $p \in L_{012}$ as well and we have $C_{012} \subseteq C_{01} \cap L_{012}$.

(iii) Take $A_0 \neq A_1$.

Let $p \in L_{012} \cap C_{01}$. Therefore $p \in L_{012}$ will satisfy equation (3.11), provided $v \neq 0$.

Also, since $p \in C_{01}$, p will satisfy equation (3.7) and equation (3.9). Therefore, p will satisfy equation (3.10) by (3.11) and (3.9), and then ultimately equation (3.8) by $A_0 \neq A_1$. Therefore, $p \in C_{02}$. $p \in C_{01} \cap C_{02} \Rightarrow p \in C_{12}$, and consequently, $p \in C_{012}$.

□

Therefore we can reduce our soulution space to be the intersection of L_{012} and C_{01} (or any other sphere for that matter). A two-dimensional example is given in figure 3.2.

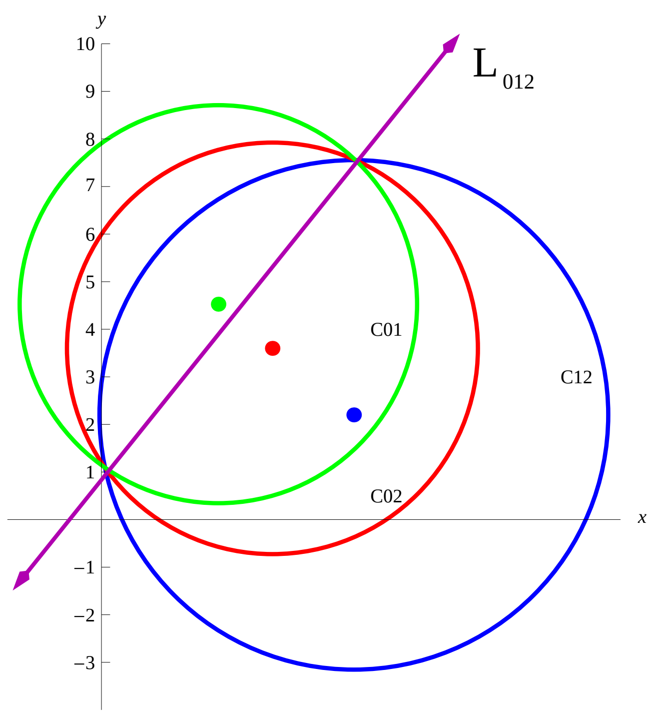

Figure 3.2: Reduced Solution Space

Remark 2. *Recall from Lemma 3.2.1 that if two weights* A_0 *and* A_1 *are equal, then* C_{01} *is an* $n-1$ *dimensional plane. This fact along with the definition for* L_{012} *implies that* $C_{01} = L_{012}$*. This can be seen as follows.*

If $A_0 = A_1$ *and* $p \in C_{01}$*, then equation 3.4 becomes:*

$$0 = A_0^2(\|p_1\|^2 - \|p_0\|^2 - 2 \cdot (p_1 - p_0))$$
$$\iff \|p_1\|^2 - \|p_0\|^2 = 2 \cdot (p_1 - p_0)$$

This is exactly the plane L_{012} *since when* $A_0 = A_1$ *the* h_i *values are as follows,*

$$h_0 = A_0^2 - A_2^2$$

$$h_1 = A_1^2 - A_0^2 = 0$$

$$h_2 = A_2^2 - A_1^2 = A_2^2 - A_0^2 = -h_0$$

This next lemma will show, in our situation, L_{012} will exist for all A_i and non collinear p_i.

Lemma 3.2.3. *Let* $v = h_0p_1 + h_1p_2 + h_2p_0$*. If* p_0*,* p_1*, and* p_2 *are not collinear, then* $v = 0 \Leftrightarrow A_0 = A_1 = A_2$

Proof. ($\Rightarrow$) Let $v = 0$. Observe that $h_0 + h_1 + h_2 = 0$, then $0 = h_0p_1 + h_1p_2 + h_2p_0 = h_0p_1 + h_1p_2 + -h_1p_0 - h_0p_0$, which implies $h_i = 0$ for $i = 0, 1, 2$. By the non-collinearity of p_0, p_1 and p_2. Thus $A_0^2 = A_1^2$, $A_0^2 = A_2^2$, and $A_1^2 = A_2^2$ and hence all the A_i's are equal.

($\Leftarrow$) If $A_0 = A_1 = A_2$, then

$$h_0 = A_0^2 - A_2^2 = 0$$

$$h_1 = A_1^2 - A_0^2 = 0$$

$$h_2 = A_2^2 - A_1^2 = 0$$

and consequently $v = h_0p_1 + h_1p_2 + h_2p_0 = 0$. □

From Lemma 3.2.2 we can see our solution space C_{012} reduces from the intersection of two n-1 dimensional spheres to the intersection of the plane L_{012} with one of those spheres. Since it does not matter which sphere we use, we will take C_{01}.

The next lemma shows the objects in the set are invariant relative to the position of p_i, p_j, and p_k under all isometries of $\mathbb{R}^n$.

Lemma 3.2.4. *Given any points p_i, p_j, and $p_k \in \mathbb{R}^n$ with weights A_i, A_j, and A_k and f an isometry of $\mathbb{R}^n$. Let*

$p_{i'} = f(p_i)$, *with weight* A_i

$p_{j'} = f(p_j)$, *with weight* A_j

$p_{k'} = f(p_k)$, *with weight* A_k

Then C_{ijk} and L_{ijk} translate to the corresponding sets under the rigid motion, f:

1. $f(C_{ij}) = C_{i'j'}$
2. $f(C_{ijk}) = C_{i'j'k'}$
3. $f(L_{ijk}) = L_{i'j'k'}$

Proof. First observe that $R_{i'}(f(p)) = R_i(p)$. By definition of $R_i(p)$ we have,

(i) $R_{i'}(f(p)) = \frac{\|f(p)-p_{i'}\|}{A_i}$

$= \frac{\|f(p)-f(p_i)\|}{A_i}$

$= \frac{\|p-p_i\|}{A_i}$

$= R_i(p)$

(ii) $f(C_{ij}) = \{f(p) \mid R_i(p) = R_j(p)\}$

$= \{f(p) \mid R_{i'}(f(p)) = R_{j'}(f(p))\}$

$= \{q \mid R_{i'}(q) = R_{j'}(q)\}$

$= C_{i'j'}$

It then follows from the definition and of f being an isometry that

$$f(C_{ijk}) = C_{i'j'k'}.$$

Which satisfies (2). For (3), $f(L_{ijk}) = L_{i'j'k'}$, two cases are considered. First, translations and second showing relative invariance under rotation/reflection.

(i) Translation: Let b be a translation vector and $f(p) = p + b$. Recall

$$h_0 + h_1 + h_2 = 0 \text{ and}$$

$$v = h_0 p_1 + h_1 p_2 + h_2 k_0 \neq 0. \text{ Then}$$

$$p + b \in L_{0'1'2'} \Leftrightarrow$$

$$2(p+b)\cdot(h_0(p_1+b)+h_1(p_2+b)+h_2(p_0+b)) = h_0\|p_1+b\|^2+h_1\|p_2+b\|^2+h_2\|p_0+b\|^2$$

$$\Longleftrightarrow 2(p+b)(h_0 p_1 + h_1 p_2 + h_2 p_0 + b(h_0 + h_1 + h_2)$$

$$= h_0(\|p_1\|^2+2p_1\cdot b+\|b\|^2)+h_1(\|p_2\|^2+2p_2\cdot b+\|b\|^2)+h_2(\|p_0\|^2+2p_0\cdot b+\|b\|^2)$$

$$\Longleftrightarrow 2p\cdot(h_0 p_1 + h_1 p_2 + h_2 p_0) + 2b\cdot v =$$
$$h_0\|p_1\|^2 + h_1\|p_2\|^2 + h_2\|p_0\|^2 + \|b\|^2(h_0 + h_1 + h_2) + 2b\cdot v$$

$$\Longleftrightarrow 2p\cdot v = h_0\|p_1\|^2 + h_1\|p_2\|^2 + h_2\|p_0\|^2$$

which is $p \in L_{012}$

(ii) Second, let $A \in O(n)$ and $f(p) = A\cdot p$. Then $f(p) \in L_{0'1'2'}$ becomes

$$2(A\cdot p)\cdot(h_0 A\cdot p_1 + h_1 A\cdot p_2 + h_2 A\cdot p_0) = h_0\|A\cdot p_1\|^2 + h_1\|A\cdot p_2\|^2 + h_2\|A\cdot p_0\|^2$$

$$\Longrightarrow 2(A\cdot p)\cdot A\cdot(h_0 p_1 + h_1 p_2 + h_2 p_0) = (h_0\|p_1\|^2 + h_1\|p_2\|^2 + h_2\|p_0\|^2)$$

$$\Longrightarrow 2(p)\cdot(h_0 p_1 + h_1 p_2 + h_2 p_0) = (h_0\|p_1\|^2 + h_1\|p_2\|^2 + h_2\|p_0\|^2)$$

which is $p \in L_{012}$.

Therefore, L_{ijk} is relatively invariant under rigid motion. $\square$

3.3 Procedure for R_{012} in $\mathbb{R}^2$

We continue our discussion with results in this section given in $\mathbb{R}^2$. An explanation on reducing higher dimensional cases to this one will be given later on. In this section we will outline a procedure used to find the value for R_{ijk} defined in the chapter 3.1. A direct approach in solving for R_{ijk} is not practical so an indirect approach is used. This involves solving for the intersection points of a circle and a line.

Since $C_{01} \cap L_{012}$ is our solution set that we can translate and rotate to any position, we will place the center of C_{01} along the negative x-axis and p_0 at the origin. An example is given in figure 3.3.

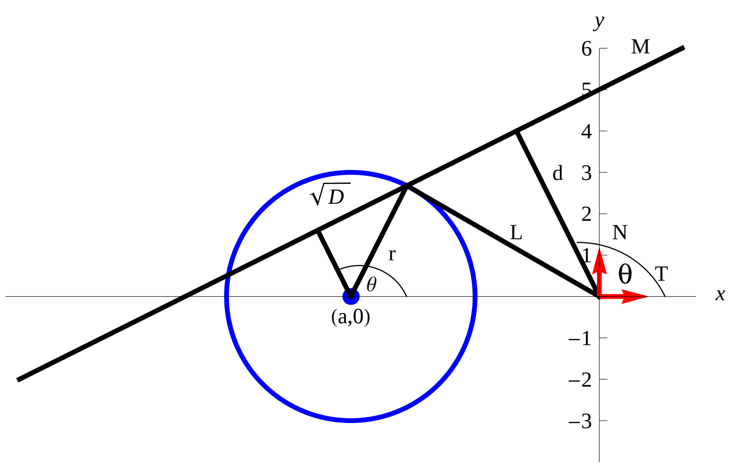

Figure 3.3: R_{ijk}

In this figure the segment from the origin that is perpendicular to the line $M = L_{012}$ has length d. Observe that the point $(d\cos\theta, d\sin\theta)$ is on the line M. By lemma 3.2.4 we may assume $0 \leq \theta \leq \pi$. Therefore, $x\cos\theta + y\sin\theta = d$ is an appropriate representation for the given line. We are also aware that the equation $(x-a)^2 + y^2 = r^2$ describes the circle in figure 3.3.

We wish to find the intersection of this line and circle. Therefore we have the following:

$$Line : x\cos\theta + y\sin\theta = d$$

$$\iff y = \tfrac{d - x\cos\theta}{\sin\theta}$$

$$Circle : (x-a)^2 + y^2 = r^2$$

$$\iff x^2 - 2ax + a^2 + y^2 = r^2$$

$$\iff x^2 - 2ax + a^2 + (\tfrac{d - x\cos\theta}{\sin\theta})^2 = r^2$$

$$\iff x^2\sin^2\theta - 2ax\sin^2\theta + a^2\sin^2\theta + d^2 - 2dx\cos\theta + x^2\cos^2\theta = r^2\sin^2\theta$$

$$\iff x^2 - (2a\sin^2\theta + 2d\cos\theta)x + (d^2 + \sin^2\theta(a^2 - r^2)) = 0$$

Using the quadratic formula to solve for x, we get the following:

$$x = \tfrac{1}{2}\left[(2a\sin^2\theta + 2d\cos\theta) \pm \sqrt{(2a\sin^2\theta + 2d\cos\theta)^2 - 4(d^2 + \sin^2\theta(a^2 - r^2))}\right]$$

To simplify this quantity, let

$$\Delta = (2a\sin^2\theta + 2d\cos\theta)^2 - 4(d^2 + \sin^2\theta(a^2 - r^2))$$

$$= 4a^2\sin^4\theta + 8ad\sin^2\theta\cos\theta + 4d^2\cos^2\theta - 4a^2\sin^2\theta + 4r^2\sin^2\theta - 4d^2$$

$$= 4a^2\sin^2\theta(\sin^2\theta - 1) + 8ad\sin^2\theta\cos\theta + 4d^2(\cos^2\theta - 1) + 4r^2\sin^2\theta$$

$$= 4\sin^2\theta(-a^2\cos^2\theta + 2ad\cos\theta + r^2 - d^2)$$

$$= 4\sin^2\theta\left(r^2 - (a\cos\theta - d)\right)^2$$

$$\Rightarrow x = a\sin^2\theta + d\cos\theta \pm \sin\theta\sqrt{r^2 - (a\cos\theta - d)^2}$$

Recall that we want to find the intersection point that provides a minimum weighted distance to the origin. This will imply that we take

$$x = a\sin^2\theta + d\cos\theta + \sin\theta\sqrt{r^2 - (a\cos\theta - d)^2}$$

.

Therefore, $L^2 = x^2 + y^2 = r^2 - a^2 + 2ax$ from above, becomes

$$L^2 = r^2 - a^2 + 2a(a\sin^2\theta + d\cos\theta + \sin\theta\sqrt{r^2 - (a\cos\theta - d)^2})$$

$$= r^2 - a^2 + 2a^2 \sin^2\theta + 2ad\cos\theta + 2a\sin\theta\sqrt{r^2 - (a\cos\theta - d)^2}$$

$$= r^2 - a^2 + a^2\sin^2\theta + a^2\sin^2\theta + 2ad\cos\theta + 2a\sin\theta\sqrt{r^2 - (a\cos\theta - d)^2}$$

$$= r^2 - a^2\cos^2\theta + a^2\sin^2\theta + 2ad\cos\theta - d^2 + d^2 + 2a\sin\theta\sqrt{r^2 - (a\cos\theta - d)^2}$$

$$= r^2 - (a\cos\theta - d)^2 + d^2 + a^2\sin^2\theta + 2a\sin\theta\sqrt{r^2 - (a\cos\theta - d)^2}$$

Let $D = r^2 - (a\cos\theta - d)^2$. Then L^2 becomes,

$$L^2 = D + d^2 + a^2\sin^2\theta + 2a\sin\theta\sqrt{D}$$

$$= (a\sin\theta + \sqrt{D})^2 + d^2$$

This quantity, L^2 becomes the basis for solving for R_{012} given specific quantities.

Lemma 3.3.1. *Given r, a, θ, and d the value of R_{012} can be determined. In fact, $R_{012}^2 = \frac{L^2}{\mu^2(p_0)}$ where*

$$L^2 = (a\sin\theta + \sqrt{D})^2 + d^2$$

$$D = r^2 - (a\cos\theta - d)^2$$

and if $D < 0$, then there will be no intersection of the spheres described in Lemma 3.2.1

Proof. The work on the previous page provides the information needed to find these quantities. □

Therefore, we have a procedure for finding R_{012} depending on the situation given. A formal explanation is given next.

Procedure 1 (R_{012})**.** *Given points p_0, p_1, and p_2 on K and weights A_0, A_1, and A_2, we can find values for the constants r, a, θ, and d given the following cases.*

Case 1. *p_0, p_1, and p_2 are not collinear, $v \neq 0$, and the weights A_0, A_1, and A_2 are all distinct, then R_{012} can be found as follows. WLOG take $A_0 < A_1$ and $p_0 = 0$. From Lemma 3.2.1 we know the following,*

$a = \frac{-p_1 A_0^2}{A_1^2 - A_0^2}$

$r = \frac{|p_0 - p_1| A_1 A_0}{A_1^2 - A_0^2}$

$\theta = \angle(p_1 - p_0, v)$

$d = \frac{1}{2|v|}\left(h_0\|p_1\|^2 + h_1\|p_2\|^2 + h_2\|p_0\|^2\right)$

Since the points and weights are distinct then these quantities exist and therefore, R_{012} can be determined. A sketch of the situation is given below in figure 3.4.

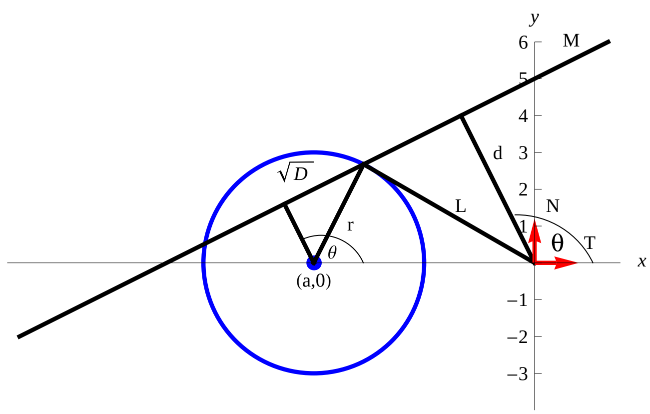

Figure 3.4: R_{ijk}

Case 2. *If the points p_0, p_1, and p_2 are not collinear but the weights are the same then the situation will reduce to the uniform case.*

This can be seen as follows,

$A_0 = A_1 = A_2$ then

$h_0 = A_0^2 - A_2^2 = 0$

$h_1 = A_1^2 - A_0^2$

$h_2 = A_2^2 - A_1^2$

$\Rightarrow \vec{v} = h_0 p_1 + h_1 p_2 + h_2 p_0 = \vec{0}$

Then L_{012} is not defined. The sets C_{01}, C_{02}, and C_{12} are all $n-1$ dimensional planes that intersect at an $n-2$ dimensional plane. It is clear that p value that will minimize the distance from all three points will lie on the intersection of this plane and the plane containing the three points. Therefore, R_{012} is equal to $\frac{\|p-p_0\|}{A_0}$ or equivalently $\frac{r(x,y,z)}{A_0}$ which is the global radius of curvature defined by [6].

Case 3. *p_0, p_1, and p_2 are not collinear and $A_0 = A_2 \neq A_1$.*

Since $A_0 = A_2$ then C_{02} is an $n-1$ dimensional plane in fact, from Lemma 3.2.1 we know $C_{01} = L_{012}$. WLOG take $A_0 > A_1$ and the results follow as in case 1.

Case 4. *p_0, p_1, and p_2 are collinear and $A_0 = A_1 = A_2$.*

Since the weights are the same, the sets C_{01}, C_{02}, and C_{12} are $n-1$ dimensional planes and from Lemma 3.2.1 the planes are perpendicular to the line containing the three points and they are centered at the midpoints of their respective intervals. This results in three parallel planes and therefore there will be no solution in this particular situation.

Case 5. *The points p_0, p_1, p_2 are collinear and A_0, A_1, and A_2 are distinct.*

From Lemma 3.2.1 the spheres C_{01}, C_{02} and C_{12} have centers that lie on the line containing the three points. If D is not negative, then the spheres will intersect and the limit line will be perpendicular to the line containing the three points. Therefore, $\theta = 0$, a and r are easy to find and d is the distance from the origin to the limit line. A two-dimensional example is given in figure 3.5

Case 6. *p_0, p_1, and p_2 are collinear and $A_0 = A_2 \neq A_1$.*

Since $A_0 = A_2$ then C_{02} is an $n-1$ dimensional plane equal to the limit line L_{012}. The other sphere C_{01} will be a sphere with center on the line containing the three points. WLOG take $A_0 < A_1$. The quantities a, r, θ, and d will be reduced similarly to the previous case and R_{012} will be the distance from the origin to the intersection of the sphere C_{01} and the limit line C_{02}.

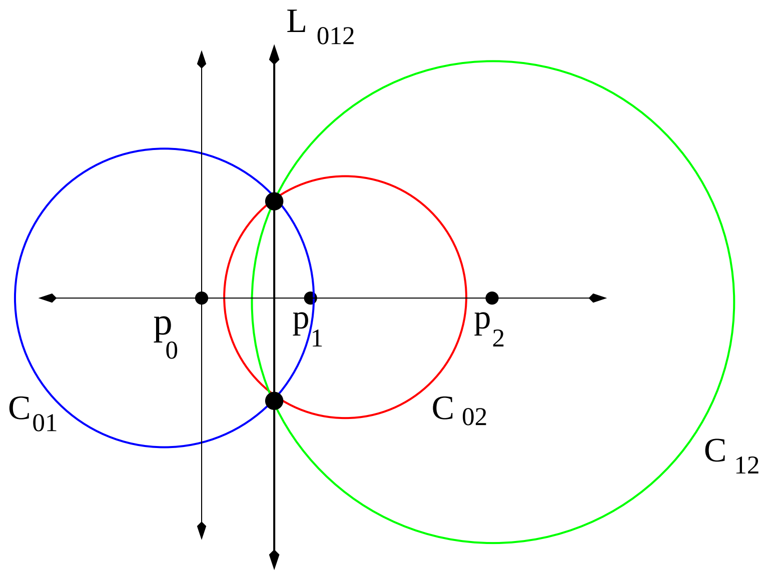

Figure 3.5: Collinear Points, Distinct Weights

3.4 Limit Process

We are now ready to start the limit process. Recall that Maddocks and Gonzalez show that the minimum value of their global radius of curvature will happen at either of the two following limit cases:

1. $\lim_{y \to x} r(x, y, z)$
2. $\lim_{y,z \to x} r(x, y, z)$

We will extend these ideas to include our weight function and show equivalent results. First, we will show the order in which you take the limits does not matter.

Proposition 3.4.1. $\lim_{p_1, p_2 \to p_0} L_{012} = \lim_{p_2 \to p_0} (\lim_{p_1 \to p_0} L_{012})$

Proof. As in definition 10, we have that L_{012} is an $n-1$ dimensional plane defined by:

$$2p \cdot (h_0 p_1 + h_1 p_2 + h_2 p_0) = h_0 \|p_1\|^2 + h_1 \|p_2\|^2 + h_2 \|p_0\|^2$$

For this proof, let $\gamma(t)$ parametrize K, a smooth knot, with respect to arclength such that

$$p_0 = \gamma(0)$$

$$p_1 = \gamma(s)$$

$$p_2 = \gamma(t)$$

and where

$$h_0 = \mu^2(\gamma(0)) - \mu^2(\gamma(t))$$

$$h_1 = \mu^2(\gamma(s)) - \mu^2(\gamma(0))$$

$$h_2 = \mu^2(\gamma(t)) - \mu^2(\gamma(s))$$

Take $f(s,t) = h_0 p_1 + h_1 p_2 + h_2 p_0$

$$= (\mu^2(0) - \mu^2(t))\gamma(s) + (\mu^2(s)) - \mu^2(0))\gamma(t) + (\mu^2(t) - \mu^2(s))\gamma(0)$$

$$f(s,t) = -f(t,s)$$

It is easy to check following results for the partials of f at $(0,0)$.

$$f_s(0,0) = 0$$

$$f_t(0,0) = 0$$

$$f_{st}(0,0) = 0$$

$$f_{ss}(0,0) = 0$$

$$f_{tt}(0,0) = 0$$

$$f_{sss}(0,0) = 0$$

$$f_{ttt}(0,0) = 0$$

$$f_{sst}(0,0) = 2\gamma'(0)\mu(0)\mu''(0) + 2\gamma'(0)\mu'^2(0) - 2\mu(0)\mu'(0)\gamma''(0)$$

$$f_{tts}(0,0) = 2\mu(0)\mu'(0)\gamma''(0) - 2\gamma'(0)\mu(0)\mu''(0) - 2\gamma'(0)\mu'^2(0)$$

$$f_{sst}(0,0) = -f_{tts}(0,0)$$

Therefore the third order Taylor approximation at $(0,0)$ will be

$$\frac{1}{3!}(f_{sst}(0,0)s^2t + f_{sts}(0,0)s^2t + f_{tss}(0,0)s^2t + f_{tts}(0,0)st^2 + f_{tst}(0,0)st^2 + f_{stt}(0,0)st^2)$$

$$= \frac{3}{3!}(f_{sst}(0,0)s^2t + f_{tts}(0,0)t^2s)$$

$$= \frac{1}{2}(f_{sst}(0,0)s^2t + f_{tts}(0,0)t^2s)$$

$$= \frac{1}{2}f_{sst}(0,0)(s-t)st$$

Likewise for the right hand side of L_{ijk}, we will take

$$g(s,t) = h_0\|p_1\|^2 + h_1\|p_2\|^2 + h_2\|p_0\|^2$$

$$= (\mu^2(0) - \mu^2(t))\|\gamma(s)\|^2 + (\mu^2(s) - \mu^2(0))\|\gamma(t)\|^2 + (\mu^2(t) - \mu^2(s))\|\gamma(0)\|^2$$

It is easy to check the following results for the partial derivatives of

$g(s,t)$ at the point $(0,0)$.

$$g_s(0,0) = 0$$

$$g_t(0,0) = 0$$

$$g_{ss}(0,0) = 0$$

$$g_{tt}(0,0) = 0$$

$$g_{st}(0,0) = 0$$

$$g_{sss}(0,0) = 0$$

$$g_{ttt}(0,0) = 0$$

$$g_{sst}(0,0) = 4\left\{(\mu(0)\mu''(0) + \mu'(0)^2)\,\gamma'(0)\cdot\gamma(0) - \mu(0)\mu'(0)\,(\gamma''(0)\cdot\gamma(0) + 1)\right\}$$

$$g_{tts}(0,0) = -g_{sst}(0,0)$$

Therefore, the third order Taylor polynomial of $g(s,t)$ at the point $(0,0)$ will be

$$\frac{1}{2}\left\{g_{sst}(0,0)s^2t + g_{tts}(0,0)t^2s\right\} = \frac{1}{2}g_{sst}(0,0)(s-t)st$$

.

Therefore, the plane L_{012} is defined by

$$2p \cdot f(s,t) = g(s,t)$$

or equivalently

$$2p \cdot \frac{f(s,t)}{st(s-t)} = \frac{g(s,t)}{st(s-t)}$$

for $s \neq 0$, $t \neq 0$ and $s \neq t$. Since both f and g are of class C^∞,

$$\lim_{s\to 0}\left(\lim_{t\to 0}\frac{f(s,t)}{st(s-t)}\right)$$

$$= \lim_{t\to 0}\left(\lim_{s\to 0}\frac{f(s,t)}{st(s-t)}\right)$$

$$= \lim_{s,t\to 0}\frac{f(s,t)}{st(s-t)}$$

$$= \frac{1}{2}f_{sst}(0,0)$$

Similarly for $g(s,t)$.

As long as $f_{sst}(0,0) \neq 0$, the plane L_{012} converges to the plane

$$2p \cdot f_{sst}(0,0) = g_{sst}(0,0) \text{ as } p_1 \to 0, p_2 \to 0$$

Hence, for L_{ijk} the order in which the limits are taken does not matter. □

The next few definitions and lemmas show what happens to the main objects in the solution space after the limits are taken.

Definition 11. *Limit Planes*

1. Let $L_{002} = \lim_{p_1 \to p_0} L_{012}$*, when the limit exists.*

2. Let $L_{000} = \lim_{p_1, p_2 \to p_0} L_{012}$*, when the limit exists.*

Lemma 3.4.2. *Let* $\gamma : (0,1) \to K$ *be a parametrized smooth knot and let* $\mu : K \to (0,\infty)$ *be a weight function. Take* $p_0 = \vec{0}$*,* $p_1 = \gamma(s)$*, and* $p_2 = \gamma(t)$*. Then*

L_{002}*:*

$$p \cdot [(\mu^2(0) - \mu^2(t))\gamma'(0) + 2\mu(0)\mu'(0)\gamma(t)] = \mu(0)\mu'(0)\|\gamma(t)\|^2$$

L_{000}*:*

$$p \cdot [\mu\mu'\kappa N - T(\mu\mu'' + \mu'^2)](0) = \mu\mu'(0)$$

provided that the $\mu\mu'\kappa N - T(\mu\mu'' + \mu'^2) \neq 0$ *and* κ, N, T *denote the curvature, normal and unit tangent of* γ*.*

Proof. Recall from definition 10, L_{012}:

$$2p \cdot (h_0 p_1 + h_1 p_2 + h_2 p_0) = h_0\|p_1\|^2 + h_1\|p_2\|^2 + h_2\|p_0\|^2$$

$h_0 = \mu^2(0) - \mu^2(t)$

$h_1 = \mu^2(s) - \mu^2(0)$

$h_2 = \mu^2(t) - \mu^2(s)$

$$2p \cdot [(\mu^2(0) - \mu^2(t))\gamma(s) + (\mu^2(s) - \mu^2(0))\gamma(t) + (\mu^2(t) - \mu^2(s))\gamma(0)]$$

$$= (\mu^2(0) - \mu^2(t))\|\gamma(s)\|^2 + (\mu^2(s) - \mu^2(0))\|\gamma(t)\|^2 + (\mu^2(t) - \mu^2(s))\|\gamma(0)\|^2$$

Before we take the limit of both sides to find L_{002}, we will divide by the arclength s. Then we will consider the limits for the right and left hand sides separately.

$$2p \cdot \frac{(\mu^2(0)-\mu^2(t))\gamma(s)+(\mu^2(s)-\mu^2(0))\gamma(t)+(\mu^2(t)-\mu^2(s))\gamma(0)}{s}$$

$$= \frac{(\mu^2(0)-\mu^2(t))\|\gamma(s)\|^2+(\mu^2(s)-\mu^2(0))\|\gamma(t)\|^2+(\mu^2(t)-\mu^2(s))\|\gamma(0)\|^2}{s}$$

Considering the left hand side first, aside from the 2p, we get:

$$\lim_{s\to 0}\frac{(\mu^2(0)-\mu^2(t))\gamma(s)+(\mu^2(s)-\mu^2(0))\gamma(t)+(\mu^2(t)-\mu^2(s))\gamma(0)}{s}$$

$$=\lim_{s\to 0}(\mu^2(0)-\mu^2(t))\gamma'(s)+2\mu(s)\mu'(s)\gamma(t)$$

$$=(\mu^2(0)-\mu^2(t))\gamma'(0)+2\mu(0)\mu'(0)\gamma(t)$$

Similarly for the right-hand side, we get:

$$\lim_{s\to 0}\frac{(\mu^2(0)-\mu^2(t))\|\gamma(s)\|^2+(\mu^2(s)-\mu^2(0))\|\gamma(t)\|^2+(\mu^2(t)-\mu^2(s))\|\gamma(0)\|^2}{s}$$

$$=\lim_{s\to 0}\tfrac{(\mu^2(0)-\mu^2(t))\|\gamma(s)\|^2+(\mu^2(s)-\mu^2(0))\|\gamma(t)\|^2}{s}$$

$$=2\mu(0)\mu'(0)\|\gamma(t)\|^2+(\mu^2(0)-\mu^2(t))2\cdot\gamma(0)\cdot\gamma'(0)$$

$$=2\mu(0)\mu'(0)\|\gamma(t)\|^2$$

Therefore, L_{002}:

$$2p\cdot[(\mu^2(0)-\mu^2(t))\gamma'(0)+2\mu(0)\mu'(0)\gamma(t)]=2\mu(0)\mu'(0)\|\gamma(t)\|^2$$

$$\iff p\cdot[(\mu^2(0)-\mu^2(t))\gamma'(0)+2\mu(0)\mu'(0)\gamma(t)]=\mu(0)\mu'(0)\|\gamma(t)\|^2$$

For the second limit, we will again begin with the L_{012} plane. In the previous lemma involving the Taylor polynomial, we have that $f_{sst}=-f_{tts}$. Therefore, in order to find L_{000} we divide the equation of L_{012} by $st(s-t)$ and the order in which s and t tend toward zero does not matter. However, since we have already considered the limit as $s\to 0$ we will begin there.

$$2p\cdot\tfrac{(\mu^2(0)-\mu^2(t))\gamma(s)+(\mu^2(s)-\mu^2(0))\gamma(t)+(\mu^2(t)-\mu^2(s))\gamma(0)}{st(s-t)}$$

$$=\tfrac{(\mu^2(0)-\mu^2(t))\|\gamma(s)\|^2+(\mu^2(s)-\mu^2(0))\|\gamma(t)\|^2+(\mu^2(t)-\mu^2(s))\|\gamma(0)\|^2}{st(s-t)}$$

Considering the two sides separately and aside from the 2p on the right hand side, we get:

$$\lim_{s\to 0}\frac{(\mu^2(0)-\mu^2(t))\gamma(s)+(\mu^2(s)-\mu^2(0))\gamma(t)+(\mu^2(t)-\mu^2(s))\gamma(0)}{st(s-t)}$$

$=\tfrac{(\mu^2(0)-\mu^2(t))\gamma'(0)+2\mu(0)\mu'(0)\gamma(t)}{-t^2}$. Hence,

$$\lim_{t\to 0} \frac{(\mu^2(0)-\mu^2(t))\gamma'(0)+2\mu(0)\mu'(0)\gamma(t)}{-t^2}$$

$$= \lim_{t\to 0} \frac{-2\mu(t)\mu'(t)\gamma'(0)+2\mu(0)\mu'(0)\gamma'(t)}{-2t}$$

$$= \lim_{t\to 0} \frac{-2\gamma'(0)(\mu(t)\mu''(t)+\mu'(t)^2)+2\mu(0)\mu'(0)\gamma''(t)}{-2}$$

$$= \gamma'(0)(\mu(0)\mu''(0)+\mu'(0)^2)-\mu(0)\mu'(0)\gamma''(0)$$

$$= T(\mu(0)\mu''(0)+\mu'(0)^2)-\mu(0)\mu'(0)\kappa N = \tfrac{1}{2} f_{sst}(0,0)$$

where κ is curvature, N is the normal and T is the tangent.

Similarly for the right-hand side:

$$\lim_{s\to 0} \frac{(\mu^2(0)-\mu^2(t))\|\gamma(s)\|^2+(\mu^2(s)-\mu^2(0))\|\gamma(t)\|^2+(\mu^2(t)-\mu^2(s))\|\gamma(0)\|^2}{st(s-t)}$$

$$= \frac{(\mu^2(0)-\mu^2(t))2\gamma(0)\cdot\gamma'(0)+2\mu(0)\mu'(0)\|\gamma(t)\|^2}{-t^2}$$

$$= \frac{2\mu(0)\mu'(0)\|\gamma(t)\|^2}{-t^2}.$$ Hence,

$$\lim_{t\to 0} \frac{(2\mu(0)\mu'(0)\|\gamma(t)\|^2}{-t^2}$$

$$= \lim_{t\to 0} \frac{4\mu(0)\mu'(0)\gamma(t)\cdot\gamma'(t)}{-2t}$$

$$= \lim_{t\to 0} \frac{2\mu(0)\mu'(0)\gamma(t)\cdot\gamma'(t)}{-t}$$

$$= \lim_{t\to 0} \frac{2\mu(0)\mu'(0)(\gamma(t)\cdot\gamma''(t)+1}{-1}$$

$$= -2\mu(0)\mu'(0)+1$$

$$= -2\mu(0)\mu'(0)$$

Therefore, L_{000} is:

$$2p\cdot\big(\mu(0)\mu'(0)\kappa N - T(\mu(0)\mu''(0)+\mu'(0)^2)\big) = 2\mu(0)\mu'(0)$$

□

The next lemma show what happens to the spheres C_{ij} during the limit process. Here, we take $i=0$ and $j=1$.

Lemma 3.4.3. *Let $\gamma(t)$ be a simple, smooth, closed curve in $\mathbb{R}^3$. Take $p_0 = \gamma(0)$ and $p_2 = \gamma(t)$, then the limit sphere C_{00} has the following*

$$Center : \gamma(0) - \tfrac{\mu(0)}{2\mu'(0)}\gamma'(0)$$

$$Radius : \tfrac{\mu(0)}{2|\mu'(0)|},\ provided\ that\ \mu'(0) \neq 0$$

If $\mu'(0) = 0$ then C_{00} is the plane perpendicular to γ at $\gamma(0)$.

Proof. From Lemma 3.2.1, we know that the sphere C_{02} has the following center and radius:

$$\text{Center: } \tfrac{\gamma(0)\mu^2(t)-\gamma(t)\mu^2(0)}{\mu^2(t)-\mu^2(0)}$$

$$\text{Radius: } \tfrac{\|\gamma(0)-\gamma(t)\|\mu(0)\mu(t)}{\|\mu^2(t)-\mu^2(0)\|}$$

First consider the center in this limit process.

$$\lim_{p_2\to p_0} \frac{\gamma(0)\mu^2(t) - \gamma(t)\mu^2(0)}{\mu^2(t) - \mu^2(0)}$$

$$= \tfrac{2\gamma(0)\mu(0)\mu'(0)}{2\mu(0)\mu'(0)} - \tfrac{\mu^2(0)\gamma'(0)}{2\mu(0)\mu'(0)}$$

$$= \gamma(0) - \tfrac{\mu(0)}{2\mu'(0)}\gamma'(0)$$

$$Center = \gamma(0) - \frac{\mu(0)}{2\mu'(0)}\gamma'(0)$$

Now consider the radius.

$$\lim_{p_2\to p_0} \frac{\|\gamma(0) - \gamma(t)\|\mu(0)\mu(t)}{\|\ \mu^2(t) - \mu^2(0)\ \|}$$

$$= \lim_{t\to 0} \frac{|\ \gamma(0) - \gamma(t)\ |}{t} \cdot \frac{t}{|\ \mu^2(t) - \mu^2(0)\ |}\mu(0)$$

$$= 1 \cdot \tfrac{1}{|2\mu'(0)\mu(0)|}\mu^2(0)$$

$$= \tfrac{\mu(0)}{|2\mu'(0)|}$$

□

Corollary 3.4.4. *$C_{00} = \lim_{p_1\to p_0} C_{01}$ is normal to $\gamma(t)$ at $p_0 = \gamma(0)$.*

Proof. From Lemma 2, we know that the center and radius of the sphere C_{00} is as follows:

$$Center : \gamma(0) - \frac{\mu(0)}{2\mu'(0)}\gamma'(0)$$

$$Radius : \frac{\mu(0)}{2 \mid \mu'(0) \mid}$$

From these results, we can see that $\gamma(0)$ lies on C_{00} and that the center lies on the tangent line to γ at $\gamma(0)$. This implies that the tangent to the curve at p_0, $\gamma'(0)$, in the normal to the sphere C_{00}. □

3.5 $\mathbf{R_{iii}}$, The Study of Generalized Focal Points

In this section, we will prove our statements in $\mathbb{R}^2$ and then extend them to higher dimensions.

Now we are ready to discuss finding the values for R_{iii}. In this case, we consider the limit as p_1 travels toward p_0 along the curve. In this situation we have two spheres, C_{00} and C_{02}, and a limit plane L_{002}. From lemma 3.4.3, we have a description for the limit sphere C_{00}. But before we continue the discussion, we will provide an explanation for why our work can be done in $\mathbb{R}^2$. This is the next lemma.

Lemma 3.5.1. *Let P and Q be two perpendicular 2-planes in $\mathbb{R}^3$ and let C be a circle that lies on Q and whose center is on $P \cap Q$. If p is a point on P then $\forall x \in C$, $\|p - x\| \geq \min_{p_i \in C \cap P} \|p - p_i\|$*

Proof. It suffices to show this for $P = xy$-plane and $Q = xz$-plane. Then we can represent C as $(R\cos(t), 0, R\sin(t))$ and $p = (a, b, 0)$. Then the distance from C to p, squared is given by $f(t) = (a - R\cos(t))^2 + b^2 + (R\sin(t))^2 = (a^2 + b^2 + R^2) - 2aR\cos(t)$ which will be minimized when $\cos(t)$ reaches its maximum when $a \geq 0$ (minimum when $a < 0$) and it is clear that this happens at one of the points that lie on $P \cap Q$. □

As we can see, this explanation reduces the three dimensional case involving the intersection of the limit sphere with the limit plane to a two dimensional one. This allows us to reduce the situation and find R_{000}. In fact, in order to find R_{000} it suffices to find the intersection between a circle and a line in a very particular situation. Figure 3.6 gives a 2-dimensional view of this situation. To show this is our next proposition.

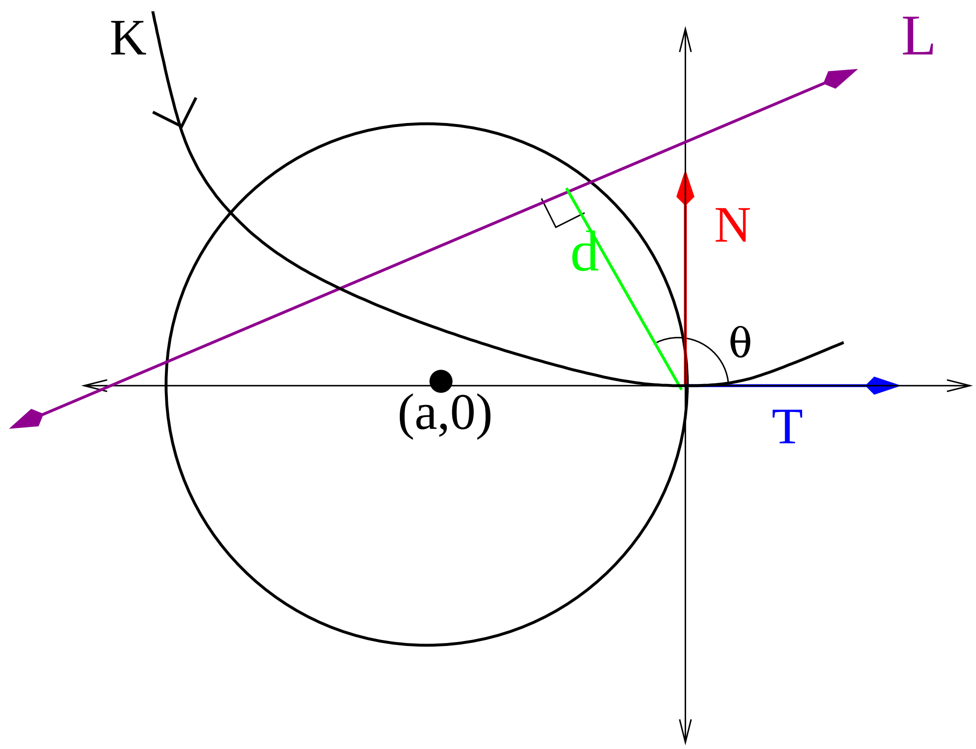

Figure 3.6: R_{iii}

Proposition 3.5.2. *Let K be a simple, smooth, closed curve and take p_0, p_1, and p_2 in $\mathbb{R}^2$, and take $p_0 = \gamma(0)$, p_1, p_2 on K. Consider the $\lim_{p_1,p_2\to p_0} R_{012} = R_{000}$ and assume that*

(i) $\mu'' + \frac{1}{4}\kappa^2\mu^2 \geq 0$ *in a neighborhood of* $t = 0$.

(ii) $[(\mu^2)'\gamma'' - \gamma'(\mu^2)''](0) \neq 0$.

(iii) $(\mu^2)''(0) \neq 0$.

Then R_{000}

$$= \left(\tfrac{1}{2}(\mu^2)'' + \tfrac{1}{2}\kappa^2\mu^2 + \kappa\mu\sqrt{\mu(\mu'' + \tfrac{1}{4}\kappa^2\mu)}(p_0)\right)^{-\frac{1}{2}}$$

If $(\mu'' + \frac{1}{4}\kappa^2\mu^2)(0) < 0$*, then* R_{000} *is not defined.*

Proof. Take $\lambda = \frac{1}{2}\mu^2$ then $\lambda' = \mu\mu'$ and $\lambda'' = (\mu')^2 + \mu\mu'$.

We will first consider the case where

$$\mu'(0) > 0$$

$$\lambda'(0) > 0$$

$$a < 0$$

$$r > 0.$$

By Lemma 3.2.4, we may assume that $\gamma(0) = 0$, $\gamma'(0) = T = e_1$, and $N = e_2$.

The limit line L_{000} reduces to $p \cdot (\lambda' N\kappa - \lambda'' T)(0) = \lambda'(0)$. We would like to use lemma in section 3.3. In the remainder of this proof the (0)'s are omitted. For instance, $\lambda' = \lambda'(0)$. Therefore, $d = \frac{\lambda'}{||\lambda' N\kappa - \lambda'' T||}$. Letting $\Lambda = ||\lambda' N\kappa - \lambda'' T||$, then $d = \frac{\lambda'}{\Lambda}$.

The normal of L_{000} is in the direction of the vector $\lambda' N\kappa - \lambda'' T$. Therefore, if θ is the angle between the positive x-axis and the normal of L_{000}, then we will also have the following:

$$\cos\theta = \frac{-\lambda''}{\Lambda}$$

$$\sin\theta = \frac{\lambda'\kappa}{\Lambda}.$$

$$a = \frac{-\lambda}{\lambda'}$$

$$r = |a| = |\frac{\lambda}{\lambda'}|.$$

The center and radius of C_{00} values come from Lemma 3.4.3 where since $\lambda'(0) = \mu'(0)\mu(0) \neq 0$.

$$Radius : \frac{\mu}{2\mu'}$$

$$Center : \gamma - \frac{\mu}{2\mu'}\gamma'$$

With these quantities we can reduce D and then ultimately L, from section 3.3 which is the quantity that we are trying to minimize.

Recall: $D = r^2 - (a\cos\theta - d)^2$, with the appropriate substitutions, we get the following:

$$D = \left(\tfrac{\lambda}{\lambda'}\right)^2 - \left(\tfrac{-\lambda}{\lambda'}\left(\tfrac{-\lambda''}{\Lambda}\right) - \tfrac{\lambda'}{\Lambda}\right)^2$$

$$= \tfrac{\lambda^2}{\lambda'^2} - \left(\tfrac{\lambda\lambda'' - \lambda'^2}{\lambda'\Lambda}\right)^2$$

$$= \tfrac{\lambda^2\Lambda^2 - (\lambda^2\lambda''^2 - 2\lambda\lambda''\lambda'^2 + \lambda'^4)}{\lambda'^2\Lambda^2}$$

$$= \tfrac{\lambda^2(\lambda'^2\kappa^2 + \lambda''^2) - \lambda^2\lambda''^2 + 2\lambda\lambda''\lambda'^2 - \lambda'^4}{\lambda'^2\Lambda^2}$$

$$= \tfrac{\lambda^2\lambda'^2\kappa^2 + 2\lambda\lambda''\lambda'^2 - \lambda'^4}{\lambda'^2\Lambda^2}$$

$$= \tfrac{\lambda^2\kappa^2 + 2\lambda\lambda'' - \lambda'^2}{\Lambda^2}$$

At this point we observe that

$$\Lambda^2 D = \lambda^2\kappa^2 + 2\lambda\lambda'' - \lambda'^2 \tag{3.12}$$

$$= \mu^2\left(\tfrac{1}{4}\mu^2\kappa^2 + (\mu^2)'' - (\mu')^2\right)$$

$$= \mu^2\left(\tfrac{1}{4}\kappa^2\mu^2 + \mu\mu''\right) \geq 0 \text{ by hypothesis.}$$

Remark 3. *If $\Lambda^2 D < 0$ then $C_{00} \cap L_{000} = \emptyset$, and hence p_1 and p_2 are sufficiently close to p_0 and $R_{012} = \infty$.*

Therefore $L^2 = (a\sin\theta + \sqrt{D})^2 + d^2$

$$= a^2\sin^2\theta + 2a\sqrt{D}\sin\theta + D + d^2$$

$$= a^2\sin^2\theta + 2a\sqrt{D}\sin\theta + (r^2 - a^2\cos^2\theta + 2ad\cos\theta - d^2) + d^2$$

$$= a^2(\sin^2\theta - \cos^2\theta) + r^2 + 2a\sqrt{D}\sin\theta + 2ad\cos\theta$$

$$= \left(\tfrac{-\lambda}{\lambda'}\right)^2\left(\tfrac{\lambda'^2\kappa^2}{\Lambda^2} - \tfrac{\lambda''^2}{\Lambda^2}\right) + \tfrac{\lambda^2}{\lambda'^2} + 2\left(\tfrac{-\lambda}{\lambda'}\right)\left(\tfrac{\lambda'}{\Lambda}\right)\left(\tfrac{-\lambda''}{\Lambda}\right) + 2\left(\tfrac{-\lambda}{\lambda'}\right)\left(\tfrac{\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}}{\Lambda}\right)\left(\tfrac{\lambda'\kappa}{\Lambda}\right)$$

$$= \tfrac{\lambda^2}{\lambda'^2}\left(\tfrac{\lambda'^2\kappa^2-\lambda''^2}{\Lambda^2}\right) + \tfrac{\lambda^2}{\lambda'^2} - \tfrac{2\lambda\lambda'\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}}{\lambda'\Lambda^2} + \tfrac{2\lambda\lambda''}{\Lambda^2}$$

$$= \tfrac{\lambda^2\lambda'^2\kappa^2-\lambda^2\lambda''^2}{\lambda'^2\Lambda^2} + \tfrac{\lambda^2\lambda'^2\kappa^2+\lambda^2\lambda''^2}{\lambda'^2\Lambda^2} - \tfrac{2\lambda\lambda'^2\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}}{\lambda'^2\Lambda^2} + \tfrac{2\lambda\lambda'^2\lambda''}{\lambda'^2\Lambda^2}$$

$$= \tfrac{\lambda^2\kappa^2+\lambda^2\kappa^2-2\lambda\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}+2\lambda\lambda''}{\Lambda^2}$$

$$= \tfrac{2\lambda^2\kappa^2+2\lambda\lambda''-2\lambda\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}}{\Lambda^2}$$

$$= \tfrac{2\lambda\left(\lambda\kappa^2+\lambda''-\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}\right)}{\Lambda^2} = L^2$$

We wish to find R, which in section 3.1 is defined to be $R = \frac{\|p-q\|}{\mu(q)}$. To find R, we consider the reciprocal $\frac{\mu^2(q)}{\|p-q\|^2}$ which is essentially $\frac{2\lambda}{L^2}$.

$$\tfrac{1}{R^2} = \tfrac{2\lambda}{L^2} = \tfrac{2\lambda}{\frac{2\lambda\left(\lambda\kappa^2+\lambda''-\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}\right)}{\Lambda^2}}$$

$$= \tfrac{\Lambda^2}{\lambda\kappa^2+\lambda''-\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}}$$

$$= \tfrac{\Lambda^2\left(\lambda\kappa^2+\lambda''+\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}\right)}{\left(\lambda\kappa^2+\lambda''-\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}\right)\left(\lambda\kappa^2+\lambda''+\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}\right)}$$

$$= \tfrac{\Lambda^2\left(\lambda\kappa^2+\lambda''+\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2}\right)}{(\lambda\kappa^2+\lambda'')^2-\kappa^2(\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2)}$$

Substitution $\Lambda^2 = \lambda'^2\kappa^2 + \lambda''^2$ gives the following:

$$\tfrac{1}{R^2} = \tfrac{(\lambda'^2\kappa^2+\lambda''^2)(\lambda\kappa^2+\lambda''+\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2})}{(\lambda\kappa^2+\lambda'')^2-\kappa^2(\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2)}$$

$$= \tfrac{(\lambda'^2\kappa^2+\lambda''^2)(\lambda\kappa^2+\lambda''+\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2})}{\lambda^2\kappa^4+2\lambda\kappa^2\lambda''+\lambda''^2-\lambda^2\kappa^4-2\lambda\lambda''\kappa^2+\kappa^2\lambda'^2}$$

$$= \tfrac{(\lambda'^2\kappa^2+\lambda''^2)(\lambda\kappa^2+\lambda''+\kappa\sqrt{\lambda^2\kappa^2+2\lambda\lambda''-\lambda'^2})}{\lambda'^2\kappa^2+\lambda''^2}$$

$$\tfrac{1}{R^2} = \lambda\kappa^2 + \lambda'' + \kappa\sqrt{\lambda^2\kappa^2 + 2\lambda\lambda'' - \lambda'^2}$$

$$\Rightarrow R^2 = \left(\lambda\kappa^2 + \lambda'' + \kappa\sqrt{\lambda^2\kappa^2 + 2\lambda\lambda'' - \lambda'^2}\right)^{-1}$$

$$\Rightarrow R = \left(\lambda\kappa^2 + \lambda'' + \kappa\sqrt{\lambda^2\kappa^2 + 2\lambda\lambda'' - \lambda'^2}\right)^{-\frac{1}{2}}$$

$$\text{Hence } R_{000} = \left(\lambda\kappa^2 + \lambda'' + \kappa\sqrt{\lambda^2\kappa^2 + 2\lambda\lambda'' - \lambda'^2}\right)^{-\frac{1}{2}}$$

$$= \tfrac{1}{2}(\mu^2)'' + \tfrac{1}{2}\kappa^2\mu^2 + \kappa\mu\sqrt{\mu\left(\mu'' + \tfrac{1}{4}\kappa^2\mu\right)}(0)^{-\frac{1}{2}}$$

□

Note that this quantitiy, R_{000} is identical to the one given in the definition for the $FocRad^0$ in [4]. From this result and because it is a generaliztation of the uniforom thickness given by O. Gonzalez and J. Maddocks in [6] it must reduce to $r(x, x, x)$ in the case of μ being constant on the knot. This is our next result.

Lemma 3.5.3. *If $\mu(t)$ is constant over an open interval containing 0, then*

$$R_{000} = \left(\tfrac{1}{2}(\mu^2)'' + \tfrac{1}{2}\kappa^2\mu^2 + \kappa\mu\sqrt{\mu\left(\mu'' + \tfrac{1}{4}\kappa^2\mu\right)}\right)(0)^{-\frac{1}{2}}$$

$$= \left(\kappa^2\mu^2\right)(0)^{-\frac{1}{2}} = \tfrac{1}{\kappa\mu}(0)$$

Proof. This is because if μ is constant, then the situation reduces to the case discussed in [6], calculated by using the circumradius and $R = \frac{L}{\mu}$. □

Recall that we wish to show the quantities given in [4] are equivalent to the ones given in this document. For that, it is easier to compare the quantities if we break up the domain of the curve γ. This is our next lemma.

Lemma 3.5.4. *Let γ be a knot in $\mathbb{R}^3$ and $\mu : \gamma \to (0, \infty)$ be a weight function. Then $\overline{D_1} \cup \overline{D_2} \cup \overline{D_3} =$ domain of γ, where*

$$D_1 = \{t \mid \mu \text{ is constant on an open interval containing } t\}.$$

$$D_2 = \{t \mid \left((\mu^2)'(\gamma'') - \gamma'(\mu^2)''\right)(t) \neq 0 \text{ and } \mu'(t) \neq 0\}.$$

$$D_3 = \{t \mid \gamma \text{ has an open line segment containing } \gamma(t)\}.$$

Proof. Let $f = \left((\mu^2)'\right)^2$

$f' = 2(\mu^2)'(\mu^2)'' = 0$ on

$(\text{ domain } \gamma) - \overline{D_2} \subseteq \{t \mid \mu'(t) = 0 \text{ or } \left((\mu^2)'(\gamma'') - \gamma'(\mu^2)''\right)(t) = 0\}$, which is a union of disjoint open intervals.

$f = \text{ constant } = \left((\mu^2)'\right)^2$ on each open disjoint interval, then

$(\mu^2)' = c_1$

$\mu^2 = c_1 t + c_2$

If $c_1 \neq 0$ then $(\mu^2)' = 2\mu\mu' = c_1 \neq 0$

$$\Rightarrow (\mu^2)'(\gamma'') - \gamma'(\mu^2)'' = 0$$
$$\Rightarrow c_1\gamma'' - \gamma' \cdot 0 = 0$$
$$\Rightarrow \gamma'' = 0$$

on that interval. Therefore, γ is a line segment on that interval. If $c_1 = 0$ then μ^2 is constant on that interval. □

We are now ready to compare R_{000} to $FocRad^0$.

Lemma 3.5.5. $FocRad^0\,[K, \mu] \geq \inf R_{000}$.

Proof. Let
$$A = \left\{t \mid \mu'' + \frac{1}{4}\kappa^2\mu \geq 0\right\}$$
and define $F(t)$ as follows.

$$F(t) = \left(\tfrac{1}{2}(\mu^2)'' + \tfrac{1}{2}\kappa^2\mu^2 + \kappa\mu\sqrt{\mu(\mu'' + \tfrac{1}{4}\kappa^2\mu)}\right)^{-\frac{1}{2}} : A \to [0, \infty]$$

Then

F has a finite minimum on the set $A = \left\{t \mid \mu'' + \frac{1}{4}\kappa^2\mu \geq 0\right\}$, from [4].

$$R_{iii}(t) = F(t) \text{ on } D_2 \cap A.$$

$$R_{iii}(t) = F(t) \text{ on } D_1 \cap A.$$

$$\infty = F(t) \text{ on } D_3 \cap A.$$

$$\overline{D_1} \cup \overline{D_2} \cup \overline{D_3} = \text{ Domain of } \gamma$$

$$FocRad^0 = \min_A F = \inf_{(D_1 \cup D_2 \cup D_3) \cap A} F = \inf_{(D_1 \cup D_2) \cap A} R_{iii}(t)$$

$$\geq \inf_{A'} R_{iii}(t),$$

where $A' = \{t \mid R_{iii} \text{ is defined and finite } \} \supseteq (D_1 \cup D_2) \cap A$.

□

Next, we show that the results for this section, up till now, also hold in $\mathbb{R}^3$.

Proposition 3.5.6. *Proposition 3.5.2 and Lemmas 3.5.3, 3.5.4, and 3.5.5 hold for curves in* $\mathbb{R}^3$.

Proof. It suffices to prove Proposition 3.5.2 since the rest does not depend on dimension.

Case 1: $((\mu^2)'\gamma'')(0) \neq 0$

by lemma 3.2.4 we may assume that

$$\gamma(0) = 0$$

$$\mu'(0) > 0$$

$$\lambda'(0) > 0$$

$$a < 0$$

$$r > 0$$

$$\gamma'(0) = e_1$$

$$\gamma''(0) = \kappa \cdot e_2$$

$L_{000} : p \cdot (\lambda' N\kappa - \lambda'' T)(0) = \lambda'(0)$ is a plane perpendicular to the xy-plane, since its normal is the xy-plane. C_{00} is a sphere of radius $\frac{\mu(0)}{2\mu'(0)}$ which has a center on the negative x-axis. One considers the intersection of L_{000} and C_{00} with the xy-plane. By Lemma 3.5.1 the minimum of $\frac{\|p-\gamma(0)\|}{\mu(0)}$ on $L_{000} \cap C_{00}$ which lies on $L_{000} \cap C_{00} \cap xy-$ plane. Then we repeat the proof of prop. 3.5.2. □

Observe that L_{000} intersects C_{00} transversally when $D > 0$, that is $\mu'' + \frac{1}{4}\kappa^2\mu > 0$. This follows from the construction in section 3.3.

Corollary 3.5.7. *Given a simple, smooth, closed, curve K. Then $\exists p_i, p_j, p_k \in K$ such that R_{ijk} is finite.*

For the proof of Corollary 3.5.7, we will use a the following proposition from [4].

Proposition 3.5.8 (Durumeric)**.** *Both $FocRad^0(K,\mu)$ and $FocRad^-(K,\mu) \in \mathbb{R}^+$ are positive (finite) real numbers.*

Proof. (Corollary 3.5.7) Since the definition for $FocRad^0$ from [4] is equivalent to R_{000}, then proposition 3.4.3 [4] will hold for R_{000} as well. Therefore, $\lim\limits_{p_i,p_j \to p_k} R_{ijk} = R_{kkk}$ is finite for some p_k, and R_{ijk} is finite for some p_i, p_j, and p_k. □

3.6 $\mathbf{R_{iij}}$ and Double Critical Points

In this section, we will discuss finding the values for R_{iij}. For this case, we consider the limit as p_1 travels toward p_0. In this situation we have two spheres, C_{00} and C_{02}, and a limit plane L_{002}. From lemma 3.4.3, we have a description for the limit sphere C_{00}.

Before we continue we will need a few definitions. From [4] a critical point is defined as follows.

Definition 12 (Durumeric)**.** *$p_0 = \gamma(0)$ is a critical point to $q (\neq p_0)$ if*

$$\frac{d}{ds}\frac{\|q - \gamma(s)\|^2}{\mu^2(s)} \,|_{s=0} = 0$$

By Lemma 3.4.3 and [4] Proposition 1 part (v), we know that $C_{00} = exp^{\mu}_{p_0}(NK_{p_0})$.

Definition 13 (Durumeric)**.** *(p_0, p_1) is a double critical point for $(K,\mu) \Leftrightarrow \exists R > 0, q$ on the line segment joining p_0 and p_1 such that $\|q - p_i\| = R\mu(p_i)$ for $i = 0, 1$ and both p_i's are critical to q.*

Lemma 3.6.1. *Let γ be a smooth knot in $\mathbb{R}^3$ including planar curves, let μ be a weight function, and take $p_0, p_2 \in K$. If (p_0, p_2) is a double critical pair of (K,μ) and*

$| p_0 - p_2 | < \frac{\mu(p_0)+\mu(p_2)}{\mu'(p_0)}$ *then the point* q *lies on the limit plane* L_{002}*, and consequently* C_{00}.

Note: We are using p_0 and p_2 here because we are considering the case after we have taken $p_1 \rightarrow p_0$.

Remark 4. *Note that in addition to the following proof, we can see that* q *is on* L_{000} *from definition 7 in [4] and Proposition 1 (v) in [4]. From this we have* q *is on* C_{00} *because it is given by the exp map.*

A two dimensional example is give in figure 3.7

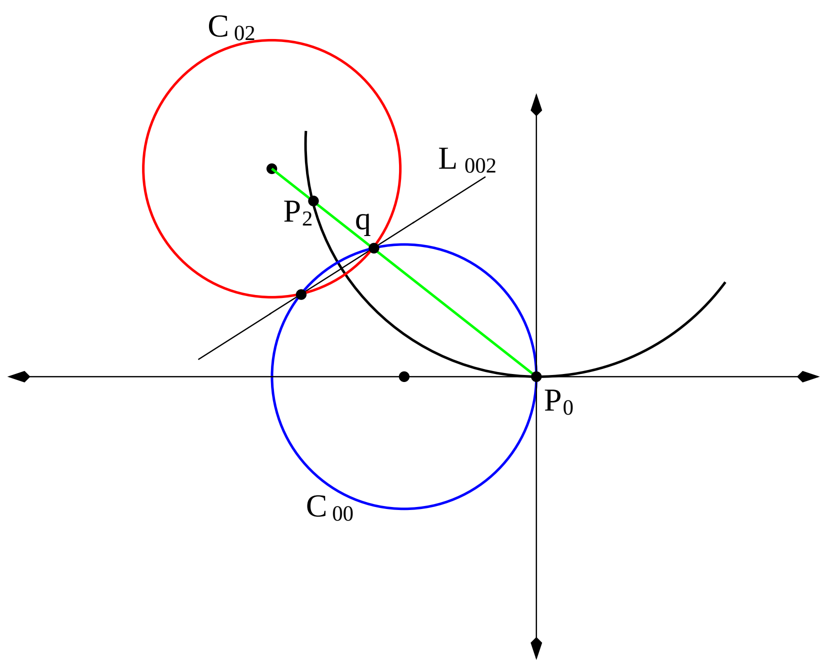

Figure 3.7: Critical Points

Proof. Suppose (p_0, p_2) is a double critical pair of (K, μ). Then $\exists R > 0$ and a point q on the line segment joining p_0 and p_1 such that $\|p_0 - q\| = R\mu(p_0)$ and $\|p_2 - q\| = R\mu(p_2)$. This implies that $R = \frac{\|p_2 - q\|}{\mu(p_2)} = \frac{\|p_0 - q\|}{\mu(p_0)}$ and hence $q \in C_{02}$.

Now, consider the sphere C_{00} which is the limit of the sphere C_{01} as $p_1 \rightarrow p_0$ along the curve. From Lemma 3.4.3, we know the center of this sphere is at

$\gamma(0) - \frac{\mu(0)}{2\mu'(0)}\gamma'(0)$. Let Q be a plane containing this center, p_0, and p_2. Then it is clear that $q \in Q$. Now, since the center of the sphere C_{02} lies on the line segment joining p_0 and p_2, then it will lie on Q as well. By using an isometry, we may assume that Q is the xy-plane and take $p_0 = \gamma(0) = (0,0)$, $p_2 = (a\cos\alpha, a\sin\alpha)$ where α is the angle between the lines $\overline{p_0p_2}$ and $\gamma'(p_0) = e_1$ and reduce the problem to a two-dimensional calculation. From the definition of double critical pair,

$$\begin{aligned} a =\mid p_2 \mid =\mid p_2 - p_0 \mid =\mid p_2 - q \mid + \mid q - p_0 \mid \\ = R\left(\mu(p_2) + \mu(p_0)\right) \\ \mid q \mid =\mid p_0 - q \mid = R\mu(p_0) = \frac{a\mu(p_0)}{\mu(p_0)+\mu(p_2)} \end{aligned}$$

Hence,

$$q = \left(\tfrac{a\mu(p_0)\cos\alpha}{\mu(p_0)+\mu(p_2)}, \tfrac{a\mu(p_0)\sin\alpha}{\mu(p_0)+\mu(p_2)}\right)$$

Case 1:

$$2\mu(p_0)\mu'(p_0)p_2 + \left(\mu^2(p_0) - \mu^2(p_2)\right)\gamma'(p_0) \neq 0$$

.

Then the limit line is

$$p \cdot (2\mu(p_0)\mu'(p_0)p_2 + (\mu^2(p_0) - \mu^2(p_2))\gamma'(p_0)) = \mu(p_0)\mu'(p_0)a^2$$

Therefore, in order to show that q lies on this limit plane, L_{002}, we will show that

$$\begin{pmatrix} \frac{a\mu(p_0)\cos\alpha}{\mu(p_0)+\mu(p_2)} \\ \frac{a\mu(p_0)\sin\alpha}{\mu(p_0)+\mu(p_2)} \end{pmatrix} \cdot [2\mu(p_0)\mu'(p_0)p_2 + (\mu^2(p_0) - \mu^2(p_2))\gamma'(p_0)]$$
$$= \mu(p_0)\mu'(p_0)a^2.$$

Working with the left-hand side of the equation, we get the following:

$$\begin{pmatrix} \frac{a\mu(p_0)\cos\alpha}{\mu(p_0)+\mu(p_2)} \\ \frac{a\mu(p_0)\sin\alpha}{\mu(p_0)+\mu(p_2)} \end{pmatrix} \cdot \left[\begin{pmatrix} 2\mu(p_0)\mu'(p_0)a\cos\alpha \\ 2\mu(p_0)\mu'(p_0)a\sin\alpha \end{pmatrix} + \begin{pmatrix} (\mu^2(p_0) - \mu^2(p_2)) \\ 0 \end{pmatrix}\right]$$
$$= \begin{pmatrix} \frac{a\mu(p_0)\cos\alpha}{\mu(p_0)+\mu(p_2)} \\ \frac{a\mu(p_0)\sin\alpha}{\mu^2(p_0)+\mu^2(p_2)} \end{pmatrix} \cdot \begin{pmatrix} 2\mu(p_0)\mu'(p_0)a\cos\alpha + \mu^2(p_0) - \mu^2(p_2) \\ 2\mu(p_0)\mu'(p_0)a\sin\alpha \end{pmatrix}$$

$$= \frac{2a^2\mu^2(p_0)\mu'(p_0)\cos^2\alpha}{\mu(p_0)+\mu(p_2)} + \frac{a\cos\alpha\mu(p_0)(\mu^2(p_0)-\mu^2(p_2))}{\mu(p_0)+\mu(p_2)} + \frac{2a^2\mu^2(p_0)\mu'(p_0)\sin^2\alpha}{\mu(p_0)+\mu(p_2)}$$

Combining the first and third fractions on the left-hand side of the equation, we get the following reduction:

$$= \tfrac{2a^2\mu^2(p_0)\mu'(p_0)}{\mu(p_0)+\mu(p_2)} + a\mu(p_0)(\mu(p_0)-\mu(p_2))\cos\alpha$$

Since $\gamma(p_0)$ and $\gamma(p_2)$ are a double critical pair, then from Lemma 2 in [4], $\cos\alpha = \frac{-a\mu'(p_0)}{\mu(p_0)+\mu(p_2)}$. Substituting this into the equation above gives the following.

$$= \tfrac{2a^2\mu^2(p_0)\mu'(p_0)}{\mu(p_0)+\mu(p_2)} + a\mu(p_0)(\mu(p_0)-\mu(p_2))\tfrac{-a\mu'(p_0)}{\mu(p_0)+\mu(p_2)}$$

$$= \tfrac{2a^2\mu^2(p_0)\mu'(p_0)-a^2\mu^2(p_0)\mu'(p_0)+a^2\mu(p_0)\mu'(p_0)\mu(p_2)}{\mu(p_0)+\mu(p_2)}$$

$$= \tfrac{a^2\mu(p_0)\mu'(p_0)[2\mu(p_0)-\mu(p_0)+\mu(p_2)]}{\mu(p_0)+\mu(p_2)}$$

$$= a^2\mu(p_0)\mu'(p_0)\tfrac{[\mu(p_0)+\mu(p_2)]}{\mu(p_0)+\mu(p_2)}$$

$$= a^2\mu(p_0)\mu'(p_0)$$

And therefore, p lies on the limit line L_{002} and consequently $q \in C_{00}$.

Case 2:

$$2\mu(p_0)\mu'(p_0)p_2 + \left(\mu^2(p_0)-\mu^2(p_2)\right)\gamma'(0) = 0$$

Subcase 1. *p_2 and $\gamma'(p_0)$ are collinear*

$$\mid p_2 - p_0 \mid < \tfrac{\mu(p_0)+\mu(p_2)}{\mu'(p_0)}$$

$$\Rightarrow \mid p_0 - q \mid < \tfrac{\mu(p_0)}{\mu'(p_0)}$$

$$\Rightarrow \tfrac{|p_0-q|}{\mu(p_0)} < \tfrac{1}{\mu'(p_0)}$$

This case can not occur by [4], Proposition 1(i) #3, since p_0, center of C_{00}, q being collinear requires $R = \frac{1}{\mu'(p_0)}$.

Subcase 2. *p_2 and $\gamma'(p_0)$ are not collinear.*

Then

$$\mu^2(p_2) = \mu^2(p_0)$$

$$\mu(p_2) = \mu(p_0)$$

$$\Rightarrow C_{00} = \text{ line } \perp \overleftrightarrow{p_2 p_0} \text{ at the midpoint } \frac{p_2 + p_0}{2}$$

But $q = \frac{p_2+p_0}{2}$ since $p_0 = 0$, $\mid q \mid = \frac{a}{2} = \frac{|p_2|}{2}$. Hence $q \in C_{02}$. But $A_0 = A_2$

$$\Rightarrow L_{012} = C_{02}$$

$$\Rightarrow L_{002} = C_{02}$$

$$\Rightarrow q \in L_{002}$$

□

Lemma 3.6.2. *If p_0 and p_2 are a double critical pair and*

$$\mid p_0 - p_2 \mid < \tfrac{\mu(p_0)+\mu(p_2)}{\mu'(p_0)}$$

then the limit plane L_{002} and C_{00} intersect transversally (if they intersect).

Proof. We will set this proof up as we did in Lemma 3.6.1.

Case 1: $\alpha \neq \pi$, and

$$2\mu(p_0)\mu'(p_0)p_2 + (\mu^2(p_0) - \mu^2(p_2))\gamma'(p_0) \neq 0 \text{ and } \mu'(p_0) \neq 0$$

Recall from Lemma 3.6.1 that

$$q = \begin{pmatrix} \frac{a\mu(p_0)\cos\alpha}{\mu(p_0)+\mu(p_2)} \\ \frac{a\mu(p_0)\sin\alpha}{\mu(p_0)+\mu(p_2)} \end{pmatrix}$$

and from Lemma 3.4.3 that the center is at

$$\gamma(p_0) - \tfrac{\mu(p_0)}{2\mu'(p_0)}\gamma'(p_0)$$

Therefore $q - \mathit{center}$ is

$$\begin{pmatrix} \frac{a\mu(p_0)\cos\alpha}{\mu(p_0)+\mu(p_2)} \\ \frac{a\mu(p_0)\sin\alpha}{\mu(p_0)+\mu(p_2)} \end{pmatrix} - \begin{pmatrix} \frac{-\mu(p_0)}{2\mu'(p_0)} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a\mu(p_0)\cos\alpha}{\mu(p_0)+\mu(p_2)} + \frac{\mu(p_0)}{2\mu'(p_0)} \\ \frac{a\mu(p_0)\sin\alpha}{\mu(p_0)+\mu(p_2)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a\mu(p_0)\cos\alpha 2\mu'(p_0)+\mu(p_0)(\mu(p_2)+\mu(p_0))}{2\mu'(p_0)(\mu(p_0)+\mu(p_2))} \\ \frac{a\mu(p_0)\sin\alpha}{\mu(p_0)+\mu(p_2)} \end{pmatrix}$$

Therefore, $(q - center)\,(\mu'(p_0))\,(\mu(p_0)+\mu(p_2))\cdot 2$

$$= \begin{pmatrix} 2a\mu(p_0)\cos\alpha\mu'(p_0) + \mu(p_0)\,(\mu(p_2)+\mu(p_0)) \\ 2\mu(p_0)\mu'(p_0)a\sin\alpha \end{pmatrix}$$

The normal of L_{002} is $= \begin{pmatrix} 2a\mu(p_0)\mu'(p_0)\cos\alpha + (\mu^2(p_0) - \mu^2(p_2)) \\ 2\mu(p_0)\mu'(p_0)a\cdot\sin\alpha \end{pmatrix}$

These vectors would be parallel iff $\mu(p_2) = 0$ which is not the case.

Case 2: $\alpha = \pi$ case and zero normal will be excluded as in the proof of Lemma 3.6.1.

Remark 5. *If $\alpha = \pi$ then we get the following situation where $p - center \parallel v$. This situation is dipicted in figure 3.8 below.*

However, in a situation like this, we would have the diameter of C_{00} equal to $R\mu(p_0)$.

$$R\mu(p_0) = 2\frac{\mu(p_0)}{2\mu'(p_0)}$$

$$\Leftrightarrow R = \frac{1}{\mu'(p_0)}$$

From [4], Proposition I part (vi) we have $\frac{1}{\mu'(p_0)} > TIR \geq DIR$. In the case where DIR is equal to double critical self distance, $R = \frac{1}{\mu'(p_0)}$ is too large of a

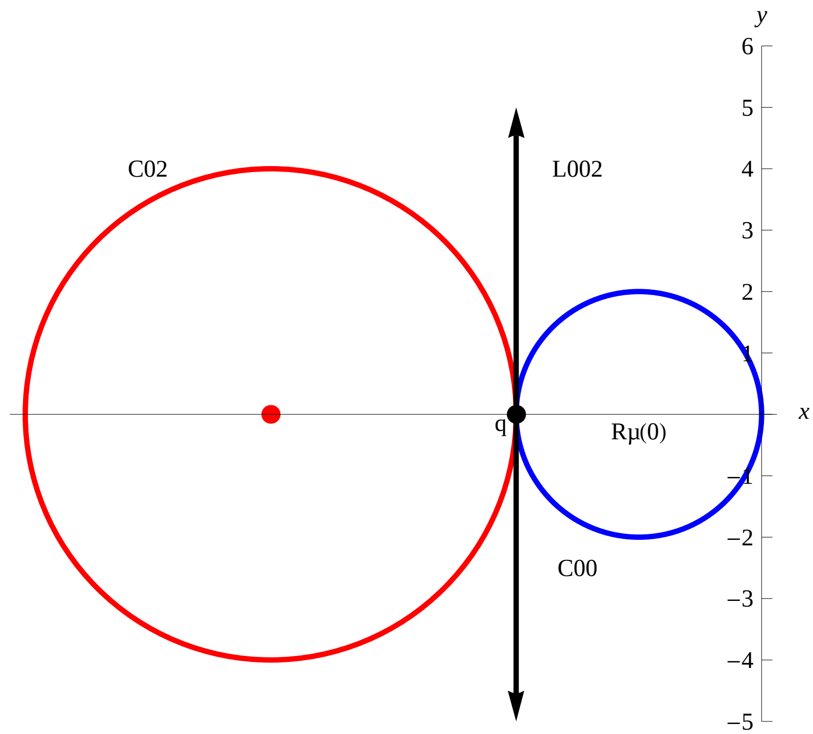

Figure 3.8: Non-Transversality

possibility for the $minR_{ijk}$*. Therefore, this situation will not produce an* R *value that can determine the non-uniform thickness of the curve.*

Hence the intersection of C_{00} and L_{002} are transverse. □

3.7 $\mathbf{DIR}[\mathbf{K}, \mu] = \mathbf{\Delta}[\mathbf{K}, \mu]$

In this section, we show the non uniform thickness described in this paper is a characterization to the definition given by O. Durumeric in [4].

Recall from section 2.4, the $FocRad^0(K, \mu)$ is defined as follows,

Definition 14 (Durumeric)**.** *If* K *is connected, by using a unit speed parametrization* $\gamma(s) : \mathbb{R} \to K$ *such that* $\gamma(s + L) = \gamma(s)$ *where* L *is the length of* K*,* $\mu(s) = \mu(\gamma(s)$*, and the curvature* $\kappa(s)$ *of* $\gamma(s)$*, one defines*

$FocRad^0 =$

$$\left(max \left[\begin{array}{rl} max\{\frac{1}{2}(\mu^2)'' + \frac{1}{2}\kappa^2\mu^2 + \kappa\mu\sqrt{\mu(\mu'' + \frac{1}{4}\kappa^2\mu)} & : \mu'' + \frac{1}{4}\kappa^2\mu \geq 0\}, \\ max\{|\mu'|^2 & : s \in Domain(\gamma)\} \end{array} \right] \right)^{-\frac{1}{2}}.$$

Recall that

$\mathcal{R} = \{R_{ijk} \mid C_{ijk} \neq \emptyset\}$

$DIR(K, \mu) = \min\left\{FocRad^0(K, \mu), \frac{1}{2}DCSD(K, \mu)\right\}$

$\min \overline{\mathcal{R}} = \Delta[K, \mu]$

Proposition 3.7.1. $FocRad^0(K, \mu) \geq \min \overline{\mathcal{R}} = \Delta\,[K, \mu]$

Proof. Since $R_{ijk} \in \mathcal{R}$, $R_{iii} = \lim\limits_{p_j, p_k \to p_i} R_{ijk} \in \overline{\mathcal{R}}$

$$R_{iii} = \left(\tfrac{1}{2}(\mu^2)'' + \tfrac{1}{2}\kappa^2\mu^2 + \kappa\mu\sqrt{\mu(\mu'' + \tfrac{1}{4}\kappa^2\mu)}\right)^{-\frac{1}{2}}(p_i).$$

From Lemma 3.5.5 in section 3.5

$FocRad^0(K, \mu) \geq \inf_{A'} R_{iii} \geq \min \overline{\mathcal{R}}$

$= \Delta\,[K, \mu]$

where $A' = \{t \mid R_{iii}$ is defined and finite $\}$.

It can be easily shown from the definitions that

$R_{iii} = FocRad^0$, for some p_i

$\Rightarrow FocRad^0(K, \mu) \in \overline{\mathcal{R}}$

□

Proposition 3.7.2. $DIR(K, \mu) \geq \Delta\,[K, \mu]$

Proof.

$$DIR(K, \mu) = \min\left(FocRad^0(K, \mu), \frac{1}{2}(K, \mu)\right)$$

Case 1: If $DIR(K,\mu) = FocRad^0(K,\mu)$, then by Proposition 3.7.1 proves the statement.

Case 2: If $DIR(K,\mu) = \frac{1}{2}DCSD(K,\mu)$, then we take a minimal double critical pair p_0 and p_2 on K, with q as in the definition of Double Critical points.

$\frac{\|p_0-p_2\|}{\mu(p_0)+\mu(p_2)} = \frac{1}{2}DCSD(K,\mu) = DIR(K,\mu)$

$\leq TIR(K,\mu) < \frac{1}{\mu'(p_0)}$

By Lemmas 3.6.1 and 3.6.2, L_{002} and C_{00} intersect transversally at q. For p_1 sufficiently close to p_0, L_{012} and C_{01} intersect transversally and $\lim_{p_1 \to p_0} R_{012} = R_{002} \in \overline{\mathcal{R}}$. By the proof of Lemma 3.6.1:

$$\frac{1}{2}DCSD(K.\mu) = DIR(K,\mu) = R_{002} \geq \min \overline{\mathcal{R}} = \Delta[K,\mu]$$

□

Proposition 3.7.3. *$\forall i,j,k, R_{ijk} \geq DIR[K,\mu]$ for p_i, p_j, p_k distinct .*

Remark 6. *Recall from [4], Proposition 1 that $p = exp^{\mu}(q,\omega)$ if and only if q is critical to p and $\mid \omega \mid = \frac{|p-q|}{\mu(q)}$. Critical means that for $F_p(s) = \frac{\|p-\gamma(s)\|^2}{\mu^2(s)}$ and $q = \gamma(s_0)$ one has $F_p'(s_0) = 0$.*

Proof. (Proposition 3.7.3) Suppose $\exists p_0, p_1, p_2$ distinct on K such that $R_{012} < DIR[K,\mu]$. Let $p \in \mathbb{R}^3$ be a weighted center: $\gamma(s_i) = p_i$ and $R_{012} = \frac{\|p-p_i\|}{\mu(s_i)} = F_p(s_i)^{\frac{1}{2}}$, $\forall i = 0,1,2$.

Case 1: $F_p'(s_i) = 0$ for at least two of the values $i = 0,1,2$. WLOG $F_p'(s_0) = F_p'(s_1) = 0$. Then $\gamma(s_0) = p_0$ and $\gamma(s_1) = p_1$ are critical to p. By Proposition 1 of [4]

$$p = exp^{\mu}(p_0,\omega_0) = exp^{\mu}(p_1,\omega_1)$$

for some $\omega_i \in NKp_i$, where $\mid \omega_0 \mid = R_{012} = \mid \omega_1 \mid$, and $p_0 \neq p_1$. By definition of DIR in [4],

$$R_{012} \geq DIR[K,\mu]$$

Case 2: $F_p'(s_i) = 0$ for only one i, say for $i = 0$. $F_p'(s_0) = 0$ and $F_p'(s_1) \neq 0$

By following γ from $\gamma(s_1)$ in the direction where $F_p(s)$ decreases, one can find s' such that

$$F_p(s') < F_p(s_1) = F_p(s_0)$$

and $F_p'(s') = 0$. Then $p = exp^\mu(p_0, \omega_0) = exp^\mu(p', \omega')$ where $p' = \gamma(s')$, $w' \in NK_p'$, by [4] Proposition 1. By Proposition 6 of [4]:

$$R_{012} = F_p(s_0)^{\frac{1}{2}} > AIR[K, \mu] \geq DIR[K, \mu]$$

which contradicts the assumption.

Case 3: All $F_p'(s_i) \neq 0$, $i = 0, 1, 2$. $F_p(s_0) = F_p(s_1) = F_p(s_2) = R_{012}^2$. The domain of γ is a circle. $F_p^{-1}([0, R_{012}^2))$ has at least two components. $\exists s', s''$ such that

$$F_p'(s') = F_p'(s'') = 0, s' \neq s''$$

and both $F_p(s') < R_{012}^2$ and $F_p(s'') < R_{012}^2$. As in the previous case and by Proposition 6 of [4].

$$R_{012} > (\max\{F_p(s'), F_p(s'')\})^{\frac{1}{2}} \geq DIR[K, \mu]$$

Hence none of the cases occur. This proves Proposition 3.7.3. □

Corollary 3.7.4. $\inf\{R_{ijk} \mid p_i, p_j, p_k \text{ } distinct \text{ } C_{ijk} \neq \emptyset\} \geq DIR[K, \mu]$

Theorem 3.7.5. $\Delta[K, \mu] = DIR[K, \mu]$

Proof. This follows from Proposition 3.7.2 and Corollary 3.7.4. □

3.8 Summary

We conclude this document with a brief summary of the connections this document has to uniform and non-uniform thickness definitions and characterizations. Recall that O. Gonzalez and J. Maddocks found another characterization of the definition of uniform thickness given by R.A. Litherland, J. Simon, O. Durumeric, and E. Rawdon. The motivation for this was to find a method that would work well with a computer program. It was also from the [8] document that O. Durumeric found motivation to explain when non-uniform forces are placed on the knot, hence finding a definition for non-uniform thickness. Finally, the motivation for this thesis was equivalent to what O. Gonzalez and J. Maddocks had for uniform thickness, to find an alternative characterization for O. Durumeric's definition that would be more conducive to computer calculations. The following table provides some orginazation to all the thickness quantities given in this document. They are all connected and are interesting to study. It is hopeful, that the discussion on non-uniform thickness of knots can be taken in the same direction as its uniform counterpart. In particular, questions of ideal shapes of knots in the context of non-uniform thickness arise along with connections to biology, chemistry, and other sciences.

	LSDR	Durumeric	Maddocks and Gonzalez	Huerter
2 points	$\frac{d_*[K]}{2}$	DCSD	$r(x, x, y)$	R_{001}
1 point	$\min_{x \in K} \rho(x)$	$FocRad^0$	$r(x, x, x)$	R_{000}

Table 3.1: Thickness Quantities

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