Fall 2012

Construction of the wave operator for non-linear dispersive equations

Kai Erik Tsuruta
University of Iowa

CONSTRUCTION OF THE WAVE OPERATOR FOR NON-LINEAR DISPERSIVE EQUATIONS

by

Kai Erik Tsuruta

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Applied Mathematical and Computational Sciences
in the Graduate College of
The University of Iowa

December 2012

Thesis Supervisors: Associate Professor Xiaoyi Zhang
Assistant Professor Dong Li

ABSTRACT

In this thesis, we will study non-linear dispersive equations. The primary focus will be on the construction of the positive-time wave operator for such equations. The positive-time wave operator problem arises in the study of the asymptotics of a partial differential equation. It is a map from a space of initial data X into itself, and is loosely defined as follows: Suppose that for a solution ψ_{lin} to the dispersive equation with no non-linearity and initial data ψ_+, there exists a unique solution ψ to the non-linear equation with initial data ψ_0 such that ψ behaves as ψ_{lin} as $t \to \infty$. Then the wave operator is the map W_+ that takes ψ_+ to ψ_0.

By its definition, W_+ is injective. An important additional question is whether or not the map is also surjective. If so, then every non-linear solution emanating from X behaves, in some sense, linearly as it evolves (this is known as asymptotic completeness). Thus, there is some justification for treating these solutions as their much simpler linear counterparts.

The main results presented in this thesis revolve around the construction of the wave operator(s) at critical non-linearities. We will study the "semi-relativistic" Schrödinger equation as well as the Klein-Gordon-Schrödinger system on $\mathbb{R}^2$. In both cases, we will impose fairly general quadratic non-linearities for which conservation laws cannot be relied upon. These non-linearities fall below the scaling required to employ such tools as the Strichartz estimates. We instead adapt the"first iteration method" of Jang, Li, and Zhang to our setting which depends crucially on the critical decay of the non-linear interaction of the linear evolution. To see the critical decay in our problem, careful analysis is needed to treat the regime where one has spatial and/or time resonance.

Abstract Approved: ______________________________

Thesis Supervisor

____

Title and Department

____

Date

____

Thesis Supervisor

____

Title and Department

____

Date

CONSTRUCTION OF THE WAVE OPERATOR FOR NON-LINEAR

DISPERSIVE EQUATIONS

by

Kai Erik Tsuruta

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Applied Mathematical and Computational Sciences
in the Graduate College of
The University of Iowa

December 2012

Thesis Supervisors: Associate Professor Xiaoyi Zhang
Assistant Professor Dong Li

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Kai Erik Tsuruta

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Applied Mathematical and Computational Sciences at the December 2012 graduation.

Thesis Committee:

Xiaoyi Zhang, Thesis Supervisor

Dong Li, Thesis Supervisor

Tong Li

Palle Jørgensen

Gerhard Strohmer

ACKNOWLEDGEMENTS

I would like to thank everyone who has helped make this paper possible. In particular, my advisors, Xiaoyi Zhang and Dong Li, who have been unwaveringly enlightening, generous, and patient over the past three years. I cannot overstate how influential and inspiring their insight, expertise, and advice have been to me.

I must also thank Professor Palle Jørgensen, who has been a kind mentor and friend throughout my time in Iowa, and has indulged my many questions during chance meetings in the math lounge, coffee shops, and even hallways.

Throughout my numerous years in school, I have had the good fortune of being educated by many great people. Keith Minkel, Richard Bernatz, Eric Westlund, Lise Kildegaard, Ward Blanton, and Paul Muhly are just a few of the teachers whom I would like to sincerely thank for their wonderful work as educators.

Finally, a great thanks to all of my family members, each of whom have been unconditionally supportive and encouraging.

ABSTRACT

In this thesis, we will study non-linear dispersive equations. The primary focus will be on the construction of the positive-time wave operator for such equations. The positive-time wave operator problem arises in the study of the asymptotics of a partial differential equation. It is a map from a space of initial data X into itself, and is loosely defined as follows: Suppose that for a solution ψ_{lin} to the dispersive equation with no non-linearity and initial data ψ_+, there exists a unique solution ψ to the non-linear equation with initial data ψ_0 such that ψ behaves as ψ_{lin} as $t \to \infty$. Then the wave operator is the map W_+ that takes ψ_+ to ψ_0.

By its definition, W_+ is injective. An important additional question is whether or not the map is also surjective. If so, then every non-linear solution emanating from X behaves, in some sense, linearly as it evolves (this is known as asymptotic completeness). Thus, there is some justification for treating these solutions as their much simpler linear counterparts.

The main results presented in this thesis revolve around the construction of the wave operator(s) at critical non-linearities. We will study the "semi-relativistic" Schrödinger equation as well as the Klein-Gordon-Schrödinger system on $\mathbb{R}^2$. In both cases, we will impose fairly general quadratic non-linearities for which conservation laws cannot be relied upon. These non-linearities fall below the scaling required to employ such tools as the Strichartz estimates. We instead adapt the"first iteration method" of Jang, Li, and Zhang to our setting which depends crucially on the critical decay of the non-linear interaction of the linear evolution. To see the critical decay

in our problem, careful analysis is needed to treat the regime where one has spatial and/or time resonance.

TABLE OF CONTENTS

CHAPTER

CHAPTER 1
INTRODUCTION

The main focus of this thesis is to study the wave operator and its construction for non-linear dispersive equations. Roughly, the positive-time wave operator is defined as follows: Suppose that for a solution ψ_{lin} to the dispersive equation with no non-linearity and initial data ψ_+, there exists a unique solution ψ to the non-linear equation with initial data ψ_0 such that ψ behaves as ψ_{lin} as $t \to \infty$ (this is known as scattering), then the positive time wave operator is the map W_+ that takes ψ_+ to ψ_0. A negative-time wave operator W_-, which considers the the behavior of ψ as $t \to -\infty$ can be defined similarly. Here, we will only handle the construction of W_+, but the construction of W_- can be done analogously.

Before going further, it will be useful to present some notation we will use throughout the paper.

1.1 Notation

• We use the Fourier Transform defined by

$$\mathcal{F}(g)(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} g(x)dx,$$

where n is the spatial dimension. Sometimes we will use the notation $\hat{g}$ to denote $\mathcal{F}(g)$.

• For Schwartz functions, we have the following Fourier Inversion formula

$$g(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{g}(x) d\xi.$$

• We use the operator

$$\Box = \partial_{tt} - \Delta.$$

• The function $\langle \cdot \rangle$, known as the Japanese bracket, is defined for vectors by

$$\langle x \rangle = \sqrt{1 + |x|^2}.$$

• The operators $\langle \nabla \rangle$, $e^{it\Delta}$, and $e^{-it\langle \nabla \rangle}$ (known as Fourier multipliers) are defined by the identities

$$\mathcal{F}(\langle \nabla \rangle f) = \sqrt{1 + |\xi|^2} \hat{f}, \quad \mathcal{F}(e^{it\Delta} f) = e^{-it|\xi|^2} \hat{f}, \text{ and } \mathcal{F}(e^{-it\langle \nabla \rangle} f) = e^{-it\sqrt{1+|\xi|^2}} \hat{f}.$$

• We define the Klein-Gordon linear propagators L and $\dot{L}$ by

$$\mathcal{F}(L(f,g)) = \cos(\langle \xi \rangle t) \hat{f} + \langle \xi \rangle^{-1} \sin(\langle \xi \rangle t) \hat{g}$$

$$\mathcal{F}(\dot{L}(f,g)) = -\langle \xi \rangle^{-1} \sin(\langle \xi \rangle t) \hat{f} + \cos(\langle \xi \rangle t) \hat{g}.$$

• Define $\varphi(r)$ as a smooth cutoff function on $\mathbb{R}^+ \cup \{0\}$ that is congruent to 1 for $r \leq 1$ and supported on $[0, 2]$, the Littlewood-Paley operators P_k are then defined for $k \in \mathbb{Z}$

by

$$\mathcal{F}(P_k f)(\xi) = (\varphi(|\xi|/2^k) - \varphi(|\xi|/2^{k-1}))\hat{f}(\xi).$$

• The space $L^p = L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ is defined by its norm

$$||g||_{L^p} = \left(\int_{\mathbb{R}^n} |g|^p dx \right)^{1/p},$$

while L^∞ is the space of all essentially bounded functions.

• From the L^p spaces, we define the Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ by the norm

$$||g||_{W^{s,p}(\mathbb{R}^n)} = ||\langle \nabla \rangle^s g||_{L^p(\mathbb{R}^n)},$$

and denote $W^{s,2}(\mathbb{R}^n)$ as $H^s(\mathbb{R}^n)$.

• For $1 \leq p, q \leq \infty$, we define the Besov Spaces $B^s_{p,q}(\mathbb{R}^n)$ by the norm

$$||g||_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{k \in \mathbb{Z}} ||\langle 2^k \rangle^s g||^q_{L^p(\mathbb{R}^n)} \right)^{1/q}.$$

• For functions of two variables (t, x), we denote by $||f||_{A_t B_x}$ the mixed norm $|| \, ||f(t, \cdot)||_B||_A$, where A and B are any Banach spaces that appear in this thesis. We will also make the abbreviation

$$||f||_{L^p_t L^p_x(\mathbb{R}^d \times \mathbb{R}^n)} = ||f||_{L^p_{t,x}(\mathbb{R}^d \times \mathbb{R}^n)},$$

where d and n are the dimensions of the variables t and x respectively.

• We introduce the space $Z(\mathbb{R}^n)$, defined by the norm

$$||g||_{Z(\mathbb{R}^n)} = \sum_{|\alpha+\beta|\leq 7} ||x^\alpha \partial^\beta g||_{L^2(\mathbb{R}^n)} + ||g||_{B^{16}_{1,1}(\mathbb{R}^n)} + ||g||_{H^{16}(\mathbb{R}^n)}.$$

The space Z will appear in Chapter 3, where it will be discussed and developed further.

• We also introduce the space $Y(\mathbb{R}^n)$, defined by the norm

$$||g||_{Y(\mathbb{R}^n)} = \sum_{|\alpha+\beta|\leq 12} ||x^\alpha \partial^\beta g||_{L^2(\mathbb{R}^n)} + ||g||_{W^{16,1}(\mathbb{R}^n)} + ||g||_{H^{16}(\mathbb{R}^n)} + ||g||_{B^6_{1,1}(\mathbb{R}^n)}.$$

The space Y appears in Chapter 4. Like the Z space, it will be discussed more in its relevant chapter.

• If A and B are two (usually non-negative) quantities, we will sometimes use the notation $A \lesssim B$ to denote $A \leq CB$, where C is some positive absolute constant. At times, we will be more specific and use $A \lesssim_{a_1,\ldots,a_k} B$ to indicate $A \lesssim C_{a_1,\ldots,a_k} B$ where $C_{a_1,\ldots,a_k}$ is positive and depends only on the parameters $a_1, \ldots a_k$.

1.2 Example of a Wave Operator and Main Results

To help formulate the wave operator problem concretely, we discuss the Schrödinger equation with non-linearity $F(u)$, or more precisely,

$$i\partial_t u + \frac{1}{2}\Delta u = F(u).$$

Solutions to the linear equation have the form $e^{\frac{1}{2}it\Delta}u_+$, where u_+ is the initial data at $t = 0$ and the operator $e^{\frac{1}{2}it\Delta}$ is defined by

$$\mathcal{F}(e^{\frac{1}{2}it\Delta}f)(\xi) = e^{-\frac{1}{2}it|\xi|^2}\hat{f}(\xi).$$

Suppose there is a space X such that if for any $u_+ \in X$, there is a unique global strong X-solution u to the non-linear Schrödinger equation with initial data u_0, such that

$$||u - e^{\frac{1}{2}it\Delta}u_+||_X \to 0 \quad as \quad t \to \infty.$$

Then the positive-time wave operator $W_+ : X \to X$ is defined by $W_+(u_+) = u_0$

One can use the Duhamel formula to show that it is sufficient to find a unique strong solution u satisfying

$$u(t) = e^{\frac{1}{2}it\Delta}u_+ + i\int_t^\infty e^{\frac{1}{2}i(t-s)\Delta}F(u)(s)ds, \tag{1.1}$$

and verify that the integral

$$\int_t^\infty e^{-\frac{1}{2}is\Delta}F(u)(s)ds$$

is conditionally convergent in X and the family of operators

$$e^{\frac{1}{2}it\Delta} \quad t \in [0, \infty)$$

is bounded in X.

The reason one expects a non-linear solution ψ to have this linear behavior as $t \to \infty$ is that if ψ tends to zero over time, then the non-linearity should tend to zero even faster as t increases. Intuitively, this means that it should be easier to establish scattering for a higher degree non-linearity. However, we can only talk about scattering if ψ exists which tends to be harder to establish as the degree non-linearity increases. Hence, the precise degree of non-linearity is very important in scattering theory.

In this thesis, we will present a method for constructing the wave operator(s) for equations with critical degree non-linearities. Specifically, we construct the wave operator(s) for the equation

$$\begin{aligned} \partial_t w + i\langle\nabla\rangle w &= F(w) \\ &= O(w^2 + (\partial_{x_1} w)^2 + (\partial_{x_2} w)^2 + (w_t)^2), \end{aligned} \tag{1.2}$$

and the system

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \pm uv \\ (\Box + 1)v = \pm|\partial_x u|^2 \end{cases} \tag{1.3}$$

on $\mathbb{R}^2$. Our main results are stated in the following two theorems:

Theorem 1.1. Suppose we have that for a sufficiently small $\delta > 0$,

$$||w_+||_Z \leq \delta.$$

Then there is a unique solution w to the equation (1.2) such that

$$||\langle t\rangle \int_t^\infty e^{-i(t-s)\langle\nabla\rangle} F(w)(s)ds||_{L_t^\infty H_x^3} < \infty,$$

and as $t \to \infty$,

$$||e^{-it\langle\nabla\rangle} w_+ - w(t)||_{H_x^3} \to 0.$$

Theorem 1.2. Suppose we have

$$supp(\hat{u}_+(\eta)) \cap \{\eta \mid .75 \leq |\eta| \leq .76\} = \emptyset,$$

and that for a sufficiently small $\delta > 0$,

$$||u_+||_Y + ||v_+||_Y + ||\dot{v}_+||_Y \leq \delta.$$

Then there is a unique solution (u, v) to the system (1.3) such that

$$||\langle t\rangle \int_t^\infty e^{\frac{1}{2}i(t-s)\Delta} u \cdot vds||_{L_t^\infty H_x^3} + ||\langle t\rangle \int_t^\infty e^{-i(t-s)\langle\nabla\rangle} \langle\nabla\rangle^{-1}(\nabla u \cdot \overline{\nabla u})ds||_{L_t^\infty H_x^3} < \infty,$$

and as $t \to \infty$,

$$||e^{\frac{1}{2}it\Delta}u_+ - u(t)||_{H_x^3} + ||L(v_+, \dot{v}_+) - v(t)||_{H_x^3} + ||\dot{L}(v_+, \dot{v}_+) - \partial_t v(t)||_{H_x^2} \to 0.$$

(1.2) is a "semi-relativistic" Schrödinger equation and is closely related to the Klein-Gordon equation

$$(\Box + 1)v = F(v),$$

while (1.3) is a Klein-Gordon-Schrödinger system that describes the interaction of two particle fields. As stated above, a quadratic non-linearity in $\mathbb{R}^2$ is considered a critical degree for both (1.2) and (1.3). Previously, it has been proven that in $\mathbb{R}^2$, one can construct the wave operators for the Schrödinger equation and the Klein-Gordon equation with power type non-linearities of the form $|\phi|^{p-1}\phi$ if $p > 2$, but not if $1 < p \leq 2$ (see, e.g., [3], [6], [15], , [17]).

This criticality is due to the Klein-Gordon, Schrödinger, and "semi-relativistic" Schrödinger equations all seeing a $t^{-d/2}$ decay as $t \to \infty$, where d is the spatial dimension. For an intuition of this critical nature, consider the following energy estimate for the Klein-Gordon equation:

$$||v||_{C_t^0 H_x^1([t_0,\infty)\times\mathbb{R}^d)} + ||\partial_t v||_{C_t^0 L_x^2([t_0,\infty)\times\mathbb{R}^d)} \lesssim$$
$$||\nabla_x v_0||_{L_x^2(R^d)} + ||v_1||_{L_x^2(R^d)} + ||F(v)||_{L_t^1 L_x^2([t_0,\infty)\times\mathbb{R}^d)}.$$

Here, v is a solution to the Klein-Gordon equation with non-linearity F, $v_0 = v(t_0, x)$, and $v_1 = \partial_t v(t_0, x)$. Suppose the non-linearity F is of degree p. If we are to use this energy estimate to control the solution in the energy space, we must bound

$$||F(v)||_{L_t^1 L_x^2([t_0,\infty)\times\mathbb{R}^d)} \sim ||v||_{L_t^p L_x^{2p}([t_0,\infty)\times\mathbb{R}^d)}.$$

Interpolating our decay estimate, we can reduce the problem to bounding

$$||v_0||_{L_x^{\frac{2p-1}{2p}}} \int_{t_0}^{\infty} t^{-d(p-1)/2} dt,$$

which will converge precisely when $d(p-1)/2 > 1$. The critical case comes from when $d(p-1)/2 = 1$.

Another explicit difficulty in dealing with quadratic-type non-linearities in two dimensions is our inability to use the Strichartz estimates and simply treat the non-linearity as a perturbation of the linear evolution. If ϕ is the solution to a dispersive equation, then perturbation methods that make use of the Strichartz estimates require the norm of the non-linearity $F(\phi)$ in some conjugate admissible pair space to be bounded by the norm of ϕ in an admissible pair space. However, since quadratic non-linearities fall below two dimensional Strichartz scaling, we cannot hope to control $F(\phi)$ in this way. Additionally, (1.2) and (1.3) have non-linearities that preclude the use of conservation laws. Typically, this means that global solutions will not exist and thus, solving the "asymptotic problem" (described in Chapter 2) is not enough to guarantee the existence of wave operators.

Instead, to achieve our main results, we directly evolve the solution from $t = +\infty$ to $t = 0$. This is a much harder task than simply solving the asymptotic problem, as we will no longer be able to shrink the time interval to control the size of the linear and/or non-linear term. In solving this harder problem, our work relies on the following, crucial observation: *In a Hilbert-type space, though the linear evolution sees no decay, the non-linear interaction of the linear flow does* and this is, in fact, enough to close the argument. This is a rather surprising statement that distinguishes our work from classical methods. We intend to construct the non-linearity in a space where *the linear evolution does not exist*! This is much different from the classic perturbation methods that work in a space where one may treat the non-linearity as a small perturbation on the linear solution.

Unfortunately, it can be rather difficult to see the decay in the non-linear interaction, and the work involved will take up the bulk of the proofs of our main theorems. To show this decay we must preform involved stationary phase calculations (see [14]), paying special attention to regimes with space and/or time resonance. Based on phase functions in frequency space, we will identify in the integration regime a space resonance set (where the phase function's partial derivative in the integration variable is zero) and a time resonance set (where the phase function is zero). Away from the space resonance set, we will show the decay by integrating by parts in frequency space. Away from the time resonance set, we will integrate by parts in time. The intersection of the space resonance and time resonance sets presents the most difficulty as we cannot integrate by parts in either variable. To overcome this, we must place assumptions on the support of the final data. The idea of this careful analysis comes from [2] and is refined in [7].

CHAPTER 2
CLASSICAL METHODS

Before discussing new results, we first discuss the classical method for which the wave operator is constructed. In doing so, we hope to shed some light on the rich history of dispersive equations, as well as present a standard to which one may compare and contrast our results. Roughly, classical methods take the following steps:

1. Solve the Linear Problem

The first step to understand a non-linear dispersive equation is to solve the equation when the non-linearity is set to zero. This is often achieved by looking at the equation in frequency space. From the linear solution, one may usually derive a *dispersive estimate*, which predicates in what spaces we expect to find solutions.

2. Duhamel Formulation and Local Wellposedness

With the linear problem solved, one can usually use the ODE method of Duhamel's formula to predict the form of the non-linear solution. From there, it is standard to use abstract iteration methods to create a local solution on some small time interval $[0, t]$.

3. Conservation Laws and Global Wellposedness

Once a solution has been constructed on some time interval $[0, t]$, one can sometimes extend this solution to an arbitrarily large time interval $[0, T]$. Usually, this is done by the use of some conservation law satisfied by the particular non-linearity.

4. 'The Asymptotic Problem' and Construction of the Wave Operator

If one has global wellposedness, then evolving the wave operator from $+\infty$ to some arbitrarily large T (this is known as solving the 'asymptotic problem') is sufficient to solve the wave operator problem, as we can then "glue" this evolution to the global solution at T.

Many of these steps are deep, rich subjects in their own right, whose nuances require many years to understand. For the sake of both brevity and clarity, we will focus on one fairly representative equation and setting to demonstrate these steps in action. Specifically, we will follow these four steps to construct the wave operator in $H^1_x(\mathbb{R}^3)$ for the non-linear Schrödinger equation (NLS) with cubic, defocusing power-type non-linearity and initial data at $t = 0$. This initial value problem is given as

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^2 u, \quad u(0,x) = u_0(x), \tag{2.1}$$

where $u \in \mathbb{R}^3$.

It should be noted that most of the author's knowledge on these classical methods comes from [16], and thus, much of the following discussion is taken either directly or indirectly from [16].

2.1 Linear Solution

We seek to solve the initial data problem

$$i\partial_t u = -\frac{1}{2}\Delta u, \quad u(0,x) = u_0(x). \tag{2.2}$$

Assume, for the moment, that u_0 is a Schwartz function and take the spacial Fourier Transform. We have

$$i\partial_t \hat{u}(\xi) = \frac{1}{2}|\xi|^2 \hat{u}(\xi).$$

Hence, $\hat{u} = e^{-\frac{1}{2}it|\xi|^2}\hat{u}_0$. Since u_0 is a Schwartz function, we may use the inversion formula to solve for u,

$$u(t,x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-\frac{1}{2}it|\xi|^2 + ix\cdot\xi}\hat{u}_0(\xi)d\xi.$$

As the Fourier transform is an isometry on L^2, one can extend this definition to L^2, as well as all Hilbert type Sobolev spaces $\dot{H}^s$. To this end, we denote by $e^{\frac{1}{2}it\Delta}$ the linear operator, with the understanding

$$\mathcal{F}(e^{\frac{1}{2}it\Delta}f) = e^{-\frac{1}{2}it|\xi|^2}\hat{f}.$$

Thus, $u = e^{\frac{1}{2}it\Delta}u_0$ is the solution to (2.2).

In light of the form of the solution $u = e^{\frac{1}{2}it\Delta}u_0$, we should also be able to write u as a convolution,

$$u(t,x) = K_t * u_0(x),$$

where K_t is the distributional inverse transform of $e^{-\frac{1}{2}it|\xi|^2}$. By the inversion formula,

$$K_t(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi - i\frac{1}{2}t|\xi|^2}d\xi.$$

It turns out, that by contour integration, we can come up with an explicit formula for this integral. Namely,

$$K_t(x) = \frac{1}{(2\pi it)^{3/2}}e^{i|x|^2/(2t)}$$

for all $t \neq 0$. Hence,

$$u(t,x) = \frac{1}{(2\pi it)^{3/2}}\int_{\mathbb{R}^3} e^{i|x-y|^2/(2t)}u_0(y)dy. \tag{2.3}$$

(2.3) is known as the Schrödinger equation's *fundamental solution*.

Consider the case $u_0(x) = e^{ikx}$, where the initial data is a singular wave with frequency k. Then in the frequency domain, we have $\hat{u}_0(\xi) = \delta(\xi - k)$, and so $u = \frac{1}{8\pi^3}e^{ik(x-\frac{kt}{2})}$. Notice that *the velocity of the wave depends on k*. If instead of a single wave, our initial data is a superposition of n waves with frequency k_j, then u is also a superposition of waves,

$$u(x,t) = \frac{1}{8\pi^3}\sum_{j=1}^{n} e^{ik(x-\frac{k_jt}{2})}.$$

The evolution of these waves with respect to time is *dependent on their frequency k_j*.

Thus, over time, the waves making up the solution will begin to separate from one another. Roughly speaking, it is this property that characterizes dispersive equations.

Along with the linear solution of a dispersive equation comes its *dispersive estimate*. This is an estimate of the decay in time of the L^∞ norm of the linear solution due to dispersion. One common way to obtain this estimate is the method of stationary phase (see [14]). Dispersive estimates for the Klein-Gordon and wave equations are obtained in this way. However, if one is lucky enough to have an explicit fundamental solution (as in the case of the Schrödinger equation), then this formula can be used to achieve a sharp and easily obtained estimate. From (2.3), we have

$$||u||_{L^\infty_x} \leq t^{-3/2}||u_0||_{L^1_x}.$$

One can achieve similar results in any dimension.

2.2 Local Wellposedness

2.2.1 Formulating the Problem

Once the linear solution has been established, one can use Duhamel's method to solve for the form of the non-linear solution. Guessing the solution to (2.1) takes the form

$$u(t,x) = e^{\frac{1}{2}it\Delta}v(t,x),$$

one can solve for $v(t,x)$ and show

$$u(t,x) = e^{\frac{1}{2}it\Delta}u_0 - i\int_0^t e^{\frac{1}{2}i(t-s)\Delta}|u|^2u(s)ds. \tag{2.4}$$

With (2.4) in place, we have arrived at a point where it is important to, at least briefly, address a surprisingly subtle question: what is a solution? We could say that u is only a solution to (2.1) if it solves the equation in the classical sense. However, the presence of the Laplacian operator then immediately requires u to be twice differentiable in the spacial variable. We thus throw away rougher solutions that are, in fact, still of interest. Instead, we could only require u to be a distributional solution to (2.4), and thus minimize any *a priori* assumptions on the regularity of u. This turns out to be too weak of a concept of a solution since the non-linearity forbids us from manipulating these distributions in the same way as classical solutions (this is due to the fact that a distributional solution to (2.4) is only a weak limit of classical solutions).

It turns out that we can make these distributional solutions more useful if we also require them to be locally continuous in time.

Definition 2.1. A *strong H^s_x solution* to (2.4) on a time interval I is a distributional solution which also lies in $C^0_{t,\mathrm{loc}}H^s_x(I \times \mathbb{R}^3)$.

To strengthen the solution further, we also place some addtional assumptions on the solution map $u_0 \mapsto u$. This leads to the following definition of a *wellposed solution*, taken from [16].

Definition 2.2. We say that the Cauchy problem (2.1) is locally wellposed in $H^s_x(\mathbb{R}^d)$ if for any $u_0^* \in H^s_x(\mathbb{R}^d)$ there exists a time $T > 0$ and an open ball B in $H^s_x(\mathbb{R}^d)$ containing u_0^*, and a subset X of $C^0_t H^s_x([0,T] \times \mathbb{R}^d)$, such that for each $u_0 \in B$ there exists a unique strong solution $u \in X$ to the integral equation (2.4) and furthermore the map $u_0 \mapsto u$ is continuous from B (with the H^s_x topology) to X (with the $C^0_t H^s_x([0,T] \times \mathbb{R}^d$ topology). If we can take T arbitrarily large, we say the wellposedness is global rather than local. If the time T depends only on the H^s_x norm of the initial datum we say the wellposedness is in the subcritical sense, otherwise it is in the critical sense.

2.2.2 Preliminary Theorems and Propositions

With a concrete definition of local wellposedness, we can now discuss how one establishes this property in dispersive equations. For (2.1) to be locally wellposed in $H^s_x(\mathbb{R}^3)$, it must admit, for each $u_0 \in H^s_x$, a unique solution to (2.4) in some subspace X of $C^0_t H^s_x([0,T] \times \mathbb{R}^3)$. To put it another way the solution map $\Phi : X \to X$, defined by

$$\Phi(u) = e^{\frac{1}{2}it\Delta}u_0 - i\int_0^t e^{\frac{1}{2}i(t-s)\Delta}|u|^2 u(s)ds,$$

must have a unique fixed point. Recall the Contraction Mapping Theorem, stated as follows:

Theorem 2.1 (Contraction Mapping Theorem)**.** Let (Y, d) be a complete non-empty metric space, and let $\Gamma : Y \to Y$ be a strict contraction. Then Γ has a unique fixed point.

Hence, if we choose X so that $\Phi : X \to X$ is a strict contraction, then we guarantee a unique solution to (2.4) in X. In fact, this solution will be the limit of repeated iterations of the solution map with any initial input. To be more precise, for any $u_1 \in X$, if we define u_{n+1} by

$$u_{n+1} = \Phi(u_n),$$

then the sequence $\{u_n\}$ will converge to the fixed point of Φ. Note that a typical choice for u_1 is the linear solution, in this case $e^{\frac{1}{2}it\Delta}u_0$.

The following proposition, taken directly from [16], compartmentalizes the steps involved in using the Contraction Mapping Theorem to show local wellposedness:

Proposition 2.3 (Abstract Iteration Argument)**.** Let $\mathcal{N}, \mathcal{S}$ be two Banach spaces. Suppose we are given a linear operator $D : \mathcal{N} \to \mathcal{S}$ with the bound

$$||DF||_{\mathcal{S}} \leq C_0 ||F||_{\mathcal{N}}, \tag{2.5}$$

for all $F \in \mathcal{N}$ and some $C_0 > 0$. Suppose further that we are given a non-linear operator $N : \mathcal{S} \to \mathcal{N}$ with $N(0) = 0$, which obeys the Lipschitz bounds

$$||N(u) - N(v)||_{\mathcal{N}} \leq \frac{1}{2C_0} ||u - v||_{\mathcal{S}}, \tag{2.6}$$

for all u, v in the ball $B_\epsilon := \{u \in \mathcal{S} : ||u||_{\mathcal{S}} \leq \epsilon\}$, for some $\epsilon > 0$. Then for all

$u_{lin} \in B_{\epsilon/2}$ there exists a unique solution $u \in B_\epsilon$ to the equation

$$u = u_{lin} + DN(u),$$

with the map $u_{lin} \mapsto u$ Lipschitz with constant at most 2.

Ultimately, the Contraction Mapping Theorem and abstract iteration will be vital in proving local wellposedness, but first we need an idea of how to chose the subspace X of $C_t^0 H_x^s([0,T] \times \mathbb{R}^3)$ as well as a notion of how to estimate the solution map in X. The following space-time bounds will help us greatly in these tasks. They are known as the Strichartz estimates and are derived from the Schrödinger equation's dispersive estimate.

Theorem 2.2 (Strichartz Estimates)**.** Fix the spatial dimension $d \geq 1$ and call a pair (q, r) of exponents admissible if $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $(q, r, d) \neq (2, \infty, 2)$. Then for any admissible exponents (q, r) and $(\tilde{q}, \tilde{r})$ we have the homogeneous Strichartz estimate:

$$||e^{it\Delta} u_0||_{L_t^q L_x^r(I \times \mathbb{R}^d)} \lesssim_{d,q,r} ||u_0||_{L_x^2(\mathbb{R}^d)},$$

the dual homogeneous Strichartz estimate

$$|| \int_R e^{-is\Delta} F(s) ds||_{L_x^2(\mathbb{R}^d)} \lesssim_{d,\tilde{q},\tilde{r}} ||F||_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I \times \mathbb{R}^d)},$$

and the inhomogeneous (or retarded) Strichartz estimate

$$||\int_{t'<t} e^{i(t-t')\Delta}F(t')dt'||_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim_{d,q,r,\tilde{q},\tilde{r}} ||F||_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^d)}.$$

Closely related to the Strichartz estimates is the *Strichartz space*, denoted by $S^0(I\times\mathbb{R}^d)$, and defined as the closure of the Schwartz functions under the norm

$$||u||_{S^0(I\times\mathbb{R}^d)} := \sup_{(q,r)\text{ admissible}} ||u||_{L_t^q L_x^r(I\times\mathbb{R}^d)},$$

where we mean “admissible” in the sense of the Strichartz estimates. Note that the S^0 norm controls the $C_t^0 L_x^2$ norm. Also, with this norm $S^0(I\times\mathbb{R}^d)$ is a Banach space, and has a dual $N^0(I\times\mathbb{R}^d)$ whose norm by construction satisfies

$$||F||_{N^0(I\times\mathbb{R}^d)} \leq ||F||_{L_t^{q'} L_x^{r'}(I\times\mathbb{R}^d)}.$$

We will also make use of the Sobolev Embedding Theorem. Roughly, it states that we may trade some regularity for more integrability. The precise statement is as follows:

Theorem 2.3 (Sobolev Embedding Theorem)**.** Suppose $1<p<q<\infty$ and $s>0$ such that $\frac{1}{p}\leq\frac{1}{q}+\frac{s}{d}$. Then we have

$$||f||_{L_x^q(\mathbb{R}^d)} \lesssim ||f||_{W_x^{s,p}(\mathbb{R}^d)}.$$

In the case $\frac{1}{p} < \frac{s}{d}$, we also have

$$||f||_{L^\infty_x(\mathbb{R}^d)} \lesssim ||f||_{W^{s,p}_x(\mathbb{R}^d)}.$$

2.2.3 Local Wellposedness of the NLS in $H^1_x(\mathbb{R}^3)$

We are finally ready to prove the local wellposedness of (2.1) in H^1_x.

Theorem 2.4 ($H^1_x(\mathbb{R}^3)$ subcritical Schrödinger solutions)**.** The Schrödinger equation (2.1) is locally wellposed in $H^1_x(\mathbb{R}^3)$ in the subcritical sense. More specifically, for any $R > 0$ there exists a $T = T(R) > 0$ such that for all u_0 in the ball $B_R := \{u_0 \in H^1_x(\mathbb{R}^3) \mid ||u_0||_{H^1_x} < R\}$ there exists a unique solution $u \in S^1([0,T] \times \mathbb{R}^3)$ to (2.1). Furthermore, the map $u_0 \mapsto u$ from B_R to $S^1([0,T] \times \mathbb{R}^3)$ is Lipschitz continuous.

Proof. Fix R and choose $T > 0$ later. We will apply Proposition 2.3 with the choices $\mathcal{S} = S^1([0,T] \times \mathbb{R}^3)$ and $\mathcal{N} = N^1([0,T] \times \mathbb{R}^3)$. We will set u_{lin} as the Schrödinger's linear solution

$$u_{lin} = e^{\frac{1}{2}it\Delta}u_0,$$

D as the Duhamel operator

$$DF = -i\int_0^t e^{\frac{1}{2}i(t-s)\Delta}F(u)(s)ds,$$

and the non-linear operator as

$$N(u)(t) = |u|^2u.$$

To place u_{lin} in $B_{\epsilon/2}$, we will use the Strichartz estimates and choose $\epsilon = C_1 R$, where $C_1 > 0$ is some large, independent constant. We may also use the Strichartz estimates to verify the bound in (2.5). Hence, to apply Proposition 2.3, we need only show

$$|| \, |u|^2u - |v|^2v||_{N^1([0,T]\times\mathbb{R}^3)} \leq \frac{1}{2C_0}||u-v||_{S^1([0,T]\times\mathbb{R}^3)},$$

whenever $||u||_{S^1([0,T]\times\mathbb{R}^3)}, ||v||_{S^1([0,T]\times\mathbb{R}^3)} \leq C_1 R$. Using the admissible exponents $(10, 30/13)$ for the S^1 norm and $(2, 6/5)$ for the N^1 norm, it is enough to show that

$$||\nabla^k(|u|^2u - |v|^2v)||_{L_t^2L_x^{6/5}([0,T]\times\mathbb{R}^3)} \leq \frac{1}{2C_0}||u-v||_{L_t^{10}W_x^{1,30/11}([0,T]\times\mathbb{R}^3)},$$

for $k = 0, 1$.

We consider the case $k = 0$. Using the estimate

$$| \, |u|^2u - |v|^2v| \lesssim |u-v|(|u|^2 + |v|^2)$$

and Hölder's inequality, we have

$$\begin{aligned} || \, |u|^2u - |v|^2v||_{L_t^2L_x^{6/5}([0,T]\times\mathbb{R}^3)} &\lesssim ||u-v||_{L_t^4L_x^{30/13}([0,T]\times\mathbb{R}^3)}||w^2||_{L_t^4L_x^{5/2}([0,T]\times\mathbb{R}^3)} \\ &= ||u-v||_{L_t^4L_x^{30/13}([0,T]\times\mathbb{R}^3)}||w||^2_{L_t^8L_x^5([0,T]\times\mathbb{R}^3)}, \end{aligned}$$

where $w = \max\{|u|, |v|\}$. To put the difference $(u - v)$ into an admissible pair space, we may now Hölder in time, extracting some power $\alpha > 0$ of T in the process.

$$||u-v||_{L_t^2 L_x^{30/13}([0,T]\times\mathbb{R}^3)}||w||^2_{L_t^4 L_x^5([0,T]\times\mathbb{R}^3)} \lesssim T^\alpha ||u-v||_{L_t^{10} L_x^{30/13}([0,T]\times\mathbb{R}^3)}||w||^2_{L_t^{10} L_x^5([0,T]\times\mathbb{R}^3)}.$$

Now, by Sobolev Embedding

$$\begin{aligned}
T^\alpha ||u-v||_{L_t^{10} L_x^{30/13}([0,T]\times\mathbb{R}^3)}||w||^2_{L_t^{10} L_x^5([0,T]\times\mathbb{R}^3)} & \\
&\lesssim T^\alpha ||u-v||_{L_t^{10} L_x^{30/13}([0,T]\times\mathbb{R}^3)}||w||^2_{L_t^{10}, W^{1,30/13}([0,T]\times\mathbb{R}^3)} \\
&\lesssim T^\alpha (C_1 R)^2 ||u-v||_{L_t^{10} L_x^{30/13}([0,T]\times\mathbb{R}^3)}.
\end{aligned}$$

Shrinking T now yields the desired bound. The work for the case $k = 1$ is similar, but a bit messier. We omit the details here and refer the reader to [16].

By Proposition 2.3, we may conclude a unique strong solution to (2.1) exists in B_ϵ. By shrinking T, we may use the Lipschitz continuity of the solution map $u_0 \mapsto u$ to strengthen this to the statement that a unique strong solution to (2.1) exists in $S^1([-T,T]\times\mathbb{R}^3) \subset C_t^0 H_x^1$, as desired. □

It is important to note that for a fixed R, the size of T is only a function of $||u_0||_{H_x^1(\mathbb{R}^3)}$. This means that if we are given initial data at t_0 instead at $t = 0$, we can guarantee a solution on $[t_0 - T, t_0 + T]$ as long as $||u(\cdot,t_0)||_{H_x^1(\mathbb{R}^3)} \leq ||u(\cdot,0)||_{H_x^1(\mathbb{R}^3)}$.

2.3 Global Wellposedness

As noted in the previous section, the fact that (2.1) is locally wellposed in the subcritical sense allows us to construct local solutions in a time neighborhood of any initial time t_0. The size of this neighborhood will depend only on $||u(t_0, \cdot)||_{H^1_x}$. The idea for constructing a solution on an arbitrarily large time interval I is to get some *a priori* control on $||u(t, \cdot)||_{H^1_x}$ and then use uniqueness to "glue" several of these local solutions together.

The *a priori* control on $||u(t, \cdot)||_{H^1_x}$ usually comes from conservation laws. In the case of (2.1) we have both mass and energy conservation.

$$M[u(t)] = \int_{\mathbb{R}^3} |u(t, x)|^2 dx = M[u(0)]$$
$$E[u(t)] = \frac{1}{2}||\nabla u(t)||_2^2 + \frac{1}{5}||u(t)||_5^5 = E[u(0)].$$

To show the mass conservation $M[u(t)] = M[u(0)]$ we rearrange (2.1) to get

$$\begin{aligned} 0 &= \operatorname{Im} \int_{\mathbb{R}^3} (iu_t + \Delta u - |u|^2 u)\bar{u} dx \\ &= \operatorname{Im} \int_{\mathbb{R}^3} (iu_t + \Delta u)\bar{u} dx. \end{aligned}$$

Integrating by parts we have $\operatorname{Im} \int_{\mathbb{R}^3} \Delta u \bar{u} = 0$. Hence

$$\operatorname{Im} \int_{\mathbb{R}^3} iu_t \bar{u} dx = \operatorname{Re} \int_{\mathbb{R}^3} u_t \bar{u} dx = 0.$$

Therefore we have

$$\frac{1}{2} \int_{\mathbb{R}^3} u_t \bar{u} + \bar{u}_t u dx = \frac{1}{2}\left(\frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 dx\right) = 0.$$

To show the energy conservation $E[u(t)] = E[u(0)]$, one can start with

$$\mathrm{Re} \int_{\mathbb{R}^3} (iu_t + \Delta u - |u|^2 u)\bar{u}_t dx = 0$$

and proceed in a way similar to that of the proof $M[u(t)] = M[u(0)]$.

Initially, these calculations are only valid for sufficiently smooth solutions. However, using the properties of strong solutions, one may use limiting arguments extend these laws to locally wellposed H^1_x solutions (see [16]).

With the mass and energy conservation laws in place, we can prove that (2.1) has a global $H^1_x(\mathbb{R}^3)$ solution by way of contradiction.

Proposition 2.4 (NLS is Globally Wellposed in $H^1_x(\mathbb{R}^3)$)**.** The Schrödigner equation (2.1) is globally wellposed in $H^1_x(\mathbb{R}^3)$. More specifically, for any $u_0 \in H^1_x$ and $T > 0$, there exists a unique strong solution $u \in S^1([0,T] \times \mathbb{R}^3)$ to (2.1).

Proof. Set

$$R_0 = M[u(0)]^{1/2} + 2E[u(0)]^{1/2}.$$

Then there exists some corresponding τ such that if the initial data $u(t_0, x)$ satisfies $||u(t_0, \cdot)||_{H^1_x} \leq R_0$, a local solution from S^1 exists on $[t_0, t_0 + \tau]$. Suppose there is a "blowup" time $T^* > 0$, such that a local solution $u(t, x) \in S^1$ exists in $[0, T^* - \tau]$ but not on $[0, T^*]$. Taking $t_0 = T^* - \tau$, by energy conservation we have

$$\begin{aligned} ||u(\cdot, t_0)||_{H^1_x} &\leq ||u(\cdot, t_0)||_{L^2_x} + ||\nabla u(\cdot, t_0)||_{L^2_x} \\ &\leq M[u(t_0)]^{1/2} + 2E[u(t_0)]^{1/2} \\ &= M[u(0)]^{1/2} + 2E[u(0)]^{1/2}. \end{aligned}$$

Hence, using $u(t_0, x)$ as initial data, there is a local wellposed solution $u'(t, x) \in S^1([T^* - \tau, T^*] \times \mathbb{R}^3)$. We can then use continuity and uniqueness to "glue" $u'(t, x)$ to $u(t, x)$ to create a wellposed solution on $[0, T^*]$. $\square$

2.4 The "Asymptotic Problem"

If a solution u to (2.1) is to scatter to a free solution u_+ in H^1_x, then we need

$$||u - e^{\frac{1}{2}it\Delta} u_+||_{H^1} \to 0 \quad \text{as} \quad t \to \infty. \tag{2.7}$$

To this end, we will find a unique u satisfying

$$u(t) = e^{\frac{1}{2}it\Delta} u_+ + i \int_t^\infty e^{\frac{1}{2}i(t-s)\Delta} |u|^2 u(s) ds. \tag{2.8}$$

Afterwards, one can easily verify that both the NLS equation and (2.7) are satisfied by $u(t)$. Hence, we may establish the existence of the wave operator W_+ by proving that for any $u_+ \in H^1_x$ there is a unique $u \in C^0_t H^1_x([0, \infty) \times \mathbb{R}^2)$ satisfying (2.8).

Since we have already established the global wellposedness of (2.1) in H^1_x, we need only show that for some large $t > 0$, there is a unique $u(t)$ satisfying (2.8) (this is known as the "asymptotic problem"). After a unique $u(t)$ is shown to exist, we can evolve $u(t)$ to u_0 using (2.1)'s global wellposedness.

Lemma 2.5 (The Asymptotic Problem)**.** Given any $u_+ \in H^1_x$, for some large $T > 0$ and $X \subset C^0_t H^1_x([T,\infty) \times \mathbb{R}^3)$ there is a unique $u(t,x) \in X$ satisfying (2.8) for all $t \in [T,\infty)$.

Proof. As in the case of local wellposedness, we will prove this lemma by an abstract iteration method. However, the asymptotic problem differs from that of local wellposedness in that we are on an infinite time interval and therefore cannot use Hölder in time to extract powers of the interval. Instead, we show the contraction by shrinking the size of B_ϵ.

If we are to shrink B_ϵ, then we must choose an $\mathcal{S}$ where the linear solution can be kept small for some time interval $[T,\infty)$. Because S^1 has an L^∞_t element, no matter what the size of T, we cannot hope to keep u_{lin} small in S^1. Instead, we pass to a smaller norm, controlled by S^1. Define S_0 by

$$||f||_{S_0} = ||f||_{L^5_{t,x}} + ||f||_{L^{10/3}_t W^{1,10/3}_x}.$$

Then for any $u_+ \in H^1_x$, we have

$$\begin{aligned}||e^{\frac{1}{2}it\Delta}u_+||_{S_0(\mathbb{R}\times\mathbb{R}^3)} &\lesssim ||e^{\frac{1}{2}it\Delta}u_+||_{L^5_tW^{1,30/11}_x(\mathbb{R}\times\mathbb{R}^3)} + ||e^{\frac{1}{2}it\Delta}u_+||_{L^{10/3}_tW^{1,10/3}_x(\mathbb{R}\times\mathbb{R}^3)}\\ &\lesssim ||e^{\frac{1}{2}it\Delta}u_+||_{S^1(\mathbb{R}\times\mathbb{R}^3)}\\ &\lesssim 1.\end{aligned}$$

Hence, by the absolute continuity of the integral, we have that for any $\epsilon > 0$ there is a $T = T(u_+, \epsilon)$ such that

$$||e^{\frac{1}{2}it\Delta}u_+||_{S_0([T,\infty)\times\mathbb{R}^3)} \leq \epsilon.$$

With arbitrary control on u_{lin}, we now need only show the non-linearity map $u \mapsto |u|^2u$ from S_0 to its conjugate space (call it N_0) is Lipschitz on some $B_\epsilon = \{u : ||u||_{S_0([T,\infty)\times\mathbb{R}^3)} \leq \epsilon\}$. To accomplish this task, we will pass to the $L^{10/7}_tW^{1,10/7}_x$ norm to control N_0. More precisely, we will show

$$||\nabla^k(|u|^2u - |v|^2v)||_{L^{10/7}_tW^{1,10/7}_x} \leq \frac{1}{2C_0}||u-v||_{S_0},$$

where C_0 is the implicit constant in the inhomogeneous Strichartz estimates and $k = 0, 1$. As in local wellposedness, we will only handle the simpler case of $k = 0$. Using the bound

$$|\ |u|^2u - |v|^2v| \lesssim (|u|^2 + |v|^2)|u-v|$$

and setting $w = \max\{|u|,\ |v|\}$, we have

$$\begin{aligned} ||\ |u|^2u - |v|^2v||_{L_t^{10/7}L_x^{10/7}} &\lesssim ||w^2||_{L_{t,x}^{5/2}}||u-v||_{L_{t,x}^{10/3}} \\ &\lesssim ||w||^2_{L_{t,x}^5}||u-v||_{L_{t,x}^{10/3}}. \end{aligned}$$

Shrinking the size of w in S_0 guarantees the non-linearity is Lipschitz. Thus, there is a unique $u(t,x) \in B_\epsilon$ satisfying (2.8) for all $t \in [T, \infty)$. $\square$

Using global wellposedness, we can uniquely extend $u(t,x)$ to $[0,\infty)$ in the space $S = \{u : u|_{[0,T)\times\mathbb{R}^3} \in S^1([0,T) \times \mathbb{R}^3), u|_{[T,\infty)\times\mathbb{R}^3} \in B_\epsilon\}$. We omit the step of showing u_0 is unique in H_x^1.

CHAPTER 3
CONSTRUCTION OF THE WAVE OPERATOR FOR A SEMI-RELATIVISTIC SCHRÖDINGER EQUATION

In this chapter, we prove the existence of the wave operator for a semi-relativistic Schrödinger equation on $\mathbb{R}^2$ with quadratic non-linearity. This non-linearity type is below Strichartz scaling and therefore classic abstract iteration methods will fail in any Strichartz space. We instead adapt the "first iteration method" of [7] to our setting which depends crucially on the critical decay of the non-linear interaction of the linear evolution. To see the critical decay in our problem, careful stationary phase calculations must be preformed.

3.1 Introduction and Overview

We study the following equation on $\mathbb{R}^2$

$$\begin{aligned} \partial_t u + i\langle\nabla\rangle u &= F(u) \\ &= O(u^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2 + (u_t)^2), \end{aligned} \tag{3.1}$$

and show that wave operators exist under smallness conditions on the "final data". Recall that $\langle\nabla\rangle$ is a Fourier multiplier defined by

$$\mathcal{F}(\langle\nabla\rangle f)(\xi) = \sqrt{1+|\xi|^2}\hat{f}(\xi).$$

(3.1) is referred to as a "semi-relativistic" Schrödinger equation as $\langle\nabla\rangle$ replaces

$\frac{1}{2}\Delta$ in the Schrödinger equation to describe the kinetic and rest energy of a relativistic particle (see [8]) . It is closely related to the Klein-Gordon equation with quadratic-type non-linearity. By the variable assignment $h = v + i\langle\nabla\rangle^{-1}\partial_t v$, one can show that solving

$$\partial_t h + i\langle\nabla\rangle h = \frac{1}{\langle\nabla\rangle} iF(Re(h))$$

is equivalent to solving the Klein-Gordon equation

$$(\Box + 1)v = F(v).$$

Our purpose in this chapter is to construct the wave operator for (3.1). Solutions to (3.1) with no non-linearity have the form $e^{-it\langle\nabla\rangle}u_+$, where u_+ is the initial data at $t = 0$. As discussed before, it suffices to find a unique strong solution for

$$u(t) = e^{-it\langle\nabla\rangle}u_+ - \int_t^\infty e^{-i(t-s)\langle\nabla\rangle} F(u)(s)ds, \tag{3.2}$$

where u_+ is the scattered state of u.

Because (3.1) has a t^{-1} decay as $t \to \infty$, quadratic non-linearities are something of a critical case in two dimensions. These non-linearities fall below Strichartz scaling, and thus the Strichartz estimates cannot be relied upon to close any space-time estimates.

Before stating the main theorem of this chapter, recall that in Chapter 1 we introduced the Z space. The Z space will be where we require the scattered state u_+ to be small. Z is defined by the norm

$$||g||_Z = \sum_{|\alpha+\beta|\leq 7} ||x^\alpha \partial^\beta g||_{L^2} + ||g||_{B^{16}_{1,1}} + ||g||_{H^{16}}.$$

In this chapter, we will use the following dispersive estimate on $\mathbb{R}^2$ for $t > 0$:

$$||e^{\pm i\langle\nabla\rangle t} P_k g||_{L^\infty_x} \lesssim t^{-1}\langle 2^k\rangle^2 ||P_k g||_{L^1_x}.$$

We will also make use of the Coifman-Meyer theorem (see [1], [4]). The version of the theorem we will use comes from [4] and is stated as follows:

Theorem 3.1 (Coifman-Meyer Theorem)**.** Suppose that σ is a function on $(\mathbb{R}^n)^m$ and define T_σ by

$$T_\sigma(f_1, ..., f_m)(x) = \int_{(\mathbb{R}^n)^m} e^{ix\cdot(\xi_1+\cdots\xi_m)} \sigma(\xi_1, ..., \xi_m)\hat{f}_1(\xi_1)\cdots\hat{f}_m(\xi_m)d\xi_1\cdots d\xi_m,$$

where f_i is a Schwartz function in $\mathbb{R}^n$ for $i = 1, ..., m$. Let Ψ be a Schwartz function on $(\mathbb{R}^n)^m$ whose Fourier transform is supported in an annulus of the form $\{\xi :\ c_1 < |\xi| < c_2\}$, is non-vanishing in a smaller annulus $\{\xi :\ c_1' \leq |\xi| \leq c_2'\}$ (for some choice of constants $0 < c_1 < c_1' < c_2' < c_2 < \infty$), and satisfies

$$\sum_{j\in\mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = constant, \qquad \xi \in (\mathbb{R}^n)^m\backslash\{0\}.$$

Suppose that Ψ and σ satisfy for some $\gamma > mn/2$

$$\sup_{k\in\mathbb{Z}} ||\sigma^k\hat{\Psi}||_{H^\gamma_x((\mathbb{R}^n)^m)} = K < \infty,$$

where $\sigma^k(\xi_1, ..., \xi_m) = \sigma(2^k\xi_1, ..., 2^k\xi_m)$. Then the operator T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, whenever

$$\frac{1}{p} = \frac{1}{p_1} + \cdots \frac{1}{p_m},$$

$1 < p_1, ...p_m \leq \infty$, and exactly one of the numbers $p_1, ..., p_m$ is equal to infinity.

In practice, we need only concern ourselves with a very specific instance of this theorem.

Theorem 3.2 (Coifman-Meyer Theorem, Utility-Grade)**.** Let $m(\xi, \eta)$ be a bounded function on $\mathbb{R}^2 \times \mathbb{R}^2$. Suppose that

$$\sup_{k\in\mathbb{Z}} ||m(2^k\cdot)\hat{P}_0||_{H^5_\zeta} < \infty,$$

where $\zeta = (\xi, \eta)$ and P_0 is the $k = 0$ Littlewood-Paley operator. Then the operator $T_m(\xi, \eta)$, defined by

$$\widehat{T_m(f,g)}(\xi) = \int_{\mathbb{R}^2} m(\xi, \eta)\hat{f}(\xi - \eta)\hat{g}(\eta)d\eta,$$

maps $L^p \times L^q \to L^r$ provided $2 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

We may now present the chapter's main result.

Theorem 3.3. Suppose we have that for a sufficiently small $\delta > 0$,

$$||u_+||_Z \leq \delta.$$

Then there is a unique solution u to the equation (3.1) such that

$$||\langle t\rangle \int_t^\infty e^{-i(t-s)\langle\nabla\rangle} F(u)(s)ds||_{L_t^\infty H_x^3} < \infty,$$

and as $t \to \infty$,

$$||e^{-it\langle\nabla\rangle} u_+ - u(t)||_{H^3} \to 0.$$

Our method follows the work of [7]. We briefly explain the proof steps as follows:

Step 1: Duhamel Formulation

We show that constructing the wave operator for (3.1) is equivalent to finding a function u of the form

$$u(t) = e^{-it\langle\nabla\rangle} u_+ - \int_t^\infty e^{-i(t-s)\langle\nabla\rangle} F(u)(s)ds.$$

We will demonstrate how to construct such a solution for the case $F(u) \sim (\partial_x u)^2$, other cases can be treated similarly. For convenience, we will set $F(u) = F_1(u)F_2(u)$ where $F_1, F_2 = \partial_x u$.

Step 2: First Iterate Analysis

With the problem reformulated, we next look to show the natural first iterate of our contraction scheme has a $\langle t\rangle^{-1}$ decay in H^3. More specifically, we show the bilinear operator

$$B(f,g)=\int_t^{\infty}e^{-i(t-s)\langle\nabla\rangle}F_1(f)F_2(g)ds$$

has the following property:

$$||B(e^{-it\langle\nabla\rangle}z_+,e^{-it\langle\nabla\rangle}z_+)||_{H^3_x}\lesssim\frac{1}{\langle t\rangle}.$$

This decay estimate is achieved by going to the frequency domain where we integrate in time and then use the Coifman-Meyer Theorem along with the dispersive estimate to close the argument.

We may only integrate in time provided the phase functions of the exponential are non-zero. Hence, part of our work is to establish some lower bound for these functions. The four phase functions to consider take the form:

$$\phi(\xi,\eta)=\langle\xi\rangle\pm\langle\xi-\eta\rangle\pm\langle\eta\rangle.$$

We will show

$$\phi(\xi,\eta)\gtrsim\frac{1}{\langle\xi\rangle+\langle\eta\rangle}.$$

Step 3: Bilinear Estimates

We introduce the space X, where we construct our solution, by its norm

$$||f||_X = ||\langle t\rangle f||_{L_t^\infty H_x^3}.$$

Through the use of Sobolev embedding, we establish the following estimates on the above defined bilinear operator:

$$\begin{aligned}
||B(f,g)||_X &\lesssim ||f||_X \cdot ||g||_X, \\
||B(f,g)||_X &\lesssim ||\langle t\rangle f||_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)} \cdot ||g||_X, \\
||B(f,g)||_X &\lesssim ||f||_X \cdot ||\langle t\rangle g||_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)}.
\end{aligned}$$

These estimates are used to establish our iterative scheme as a contraction.

Step 4: Contraction in the X-Space

Finally, we define the iteration scheme as follows:

$$u_1 = e^{-it\langle\nabla\rangle}u_+ \quad \text{and} \quad u_{k+1} = u_1 - B(u_k, u_k).$$

We then show this scheme to be a contraction in X. This is done through induction. In the process, we use the dispersive estimates of the linear propagator, the first iterate estimates established in Step 2, and the bilinear estimates from Step 3.

Remark: One should note that because the linear operator is an isometry in H^3, the linear solution does not exist in the space X. In this way, this scheme is very different from any perturbation methods.

We now give a brief overview of the organization of the remainder of this chapter.

In Section 2, we reformulate the wave operator problem using the Duhamel formula. We then introduce the X space on which we will demonstrate the non-linear map is a contraction and establish the bilinear estimates on X. Finally, we prove the Duhamel operator is a contraction on X using induction and the bilinear estimates. Part of this proof relies on the assumption that the first iterate of our scheme is sufficiently small in X.

In Section 3, we prove that the assumption on the smallness of the first iterate is valid if the final data u_+ is sufficiently small in some suitable space. This is done by integrating in time and using the Coifman-Meyer Theorem along with the dispersive estimate.

Note: The following will demonstrate our method for $F(u) = F_1(u)F_2(u)$ where $F_1(u), F_2(u) = \partial_x u$. Adapting the proof for other non-linearities is straightforward.

3.2 Fine Decomposition

3.2.1 Duhamel Formulation

If u is to scatter to a free solution u_+ in H^3 in the positive direction, then we need

$$||u - e^{-it\langle\nabla\rangle}u_+||_{H^3} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.3}$$

To this end, we will find a strong solution to the following:

$$u(t) = e^{-it\langle\nabla\rangle}u_+ - \int_t^\infty e^{-i(t-s)\langle\nabla\rangle}F_1(u)F_2(u)ds. \tag{3.4}$$

Indeed, our main goal here is to construct a solution to (3.4) whose non-linear term lives in X, showing that this solution satisfies (3.3) is easy to verify.

3.2.2 Bilinear Estimates

Define

$$B(f,g) = \int_t^\infty e^{-i\langle\nabla\rangle(t-s)}F_1(f)F_2(g)ds.$$

We introduce the space X defined by the following norm

$$||f||_X = ||\langle t\rangle f(t)||_{L_t^\infty H_x^3([0,\infty)\times\mathbb{R}^2)}.$$

Lemma 3.1. We have

$$\begin{aligned} ||B(f,g)||_X &\lesssim ||f||_X \cdot ||g||_X & (3.5)\\ ||B(f,g)||_X &\lesssim ||\langle t\rangle f||_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)} \cdot ||g||_X & (3.6)\\ ||B(f,g)||_X &\lesssim ||f||_X \cdot ||\langle t\rangle g||_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)}. & (3.7)\end{aligned}$$

Proof. This is a matter of direct calculation. First, by Minkowski

$$||B(f,g)||_X \lesssim \left|\left|\langle t\rangle \int_t^\infty ||F_1(f)F_2(g)(s)||_{H^3}ds\right|\right|_{L_t^\infty([0,\infty))}.$$

Next, by Hölder and Sobolev embedding

$$
\begin{aligned}
||F_1(f)F_2(g)(s)||_{H^3} &\lesssim ||\nabla^3 f||_2||\nabla g||_\infty + ||\nabla^2 f||_4||\nabla^2 g||_4 + ||\nabla f||_\infty||\nabla^3 g||_2 \\
&\lesssim \langle s\rangle^{-2}||f||_X||g||_X.
\end{aligned}
$$

Combining these estimates, we have

$$
\begin{aligned}
||B(f,g)||_X &\lesssim \sup_{t\geq 0}\langle t\rangle\int_t^\infty \langle s\rangle^{-2}ds\cdot||f||_X||g||_X \\
&\lesssim ||f||_X||g||_X.
\end{aligned}
$$

This ends the proof of (3.5). The proofs for (3.6) and (3.7) are similar and we omit the details. □

3.2.3 Contraction in the X-Space

In the following, we will need the estimate

$$||e^{-it\langle\nabla\rangle}f||_{L^\infty} \lesssim \frac{1}{\langle t\rangle}||f||_{B^3_{1,1}}.$$

To establish this, we will use Sobolev embedding, the dispersive estimate, and the following Littlewood-Paley-based inequalities:

$$
\begin{aligned}
||P_k f||_{L^p} \lesssim ||f||_{L^p} \lesssim \textstyle\sum_{k\in\mathbb{Z}}||P_k f||_{L^p} \qquad & 1 < p \leq \infty \\
||f||_{W^{s,p}} \sim ||\langle 2^k\rangle^s P_k f||_{L^p l^2} \qquad & 1 < p < \infty.
\end{aligned}
$$

In the case $t < 1$, we have

$$||e^{-it\langle\nabla\rangle}f||_{L^\infty} \lesssim ||f||_{W^{3,1}} \lesssim ||\langle 2^k\rangle^3 P_k f||_{L^1 l^2} \lesssim ||\langle 2^k\rangle^3 P_k f||_{l^1 L^1}.$$

When $t \geq 1$, we have

$$||e^{-it\langle\nabla\rangle}f||_{L^\infty} \lesssim \sum_{k\in\mathbb{Z}} ||P_k e^{-it\langle\nabla\rangle}f||_{L^\infty} \lesssim \sum_{k\in\mathbb{Z}} t^{-1}\langle 2^k\rangle^2 ||P_k f||_{L^1} \lesssim \frac{1}{\langle t\rangle}||f||_{B^2_{1,1}},$$

and we have established the desired inequality.

Now define

$$u_1 = e^{-i\langle\nabla\rangle t}u_+ \quad \text{and} \quad u_{k+1} = u_1 - B(u_k, u_k).$$

Denote $z_{k+1} = u_{k+1} - u_1$, $k \geq 1$. Then for $k \geq 2$, we have

$$\begin{aligned}
||z_{k+2} - z_{k+1}||_X &= ||B(u_1 + z_{k+1}, u_1 + z_{k+1}) - B(u_1 + z_k, u_1 + z_k)||_X \\
&\lesssim ||B(z_{k+1} - z_k, u_1 + z_{k+1}) + B(u_1 + z_k, z_{k+1} - z_k)||_X \\
&\lesssim ||z_{k+1} - z_k||_X(||z_k||_X + ||z_{k+1}||_X + ||\langle t\rangle e^{-it\langle\nabla\rangle}u_+||_{L^\infty_t W^{3,\infty}([0,\infty)\times\mathbb{R}^2)}) \\
&\lesssim ||z_{k+1} - z_k||_X(||z_k||_X + ||z_{k+1}||_X + ||u_+||_{B^6_{1,1}}). \qquad (3.8)
\end{aligned}$$

Lemma 3.2. Let z_j and the space X be as defined above, and assume

$$||u_+||_{B^6_{1,1}} \leq \delta,$$

and

$$||B(u_1, u_1)||_X \leq \delta$$

for some sufficiently small δ. Then for $n \geq 3$ we have

$$||z_j||_X \leq \delta \cdot \left(\sum_{j=0}^{n} \frac{1}{2^l} \right), \ \forall\ 2 \leq j \leq n. \tag{3.9}$$

Proof. This is a proof by induction on n. For $n = 3$, we have

$$\begin{aligned} ||z_3||_X &= ||B(z_2 + u_1, z_2 + u_1)||_X \\ &\leq C_0||z_2||_X \cdot (||z_2||_X + ||u_+||_{B^6_{1,1}}) + ||B(u_1, u_1)||_X \\ &\leq C_1 \cdot \delta^2 + \delta. \end{aligned}$$

Choosing δ sufficiently small completes the base case. Now consider $j = n + 1$. By (3.8), we have

$$\begin{aligned} ||z_{n+1} - z_n||_X &\leq C \cdot ||z_n - z_{n-1}||_X \cdot (||z_n||_X + ||z_{n-1}||_X + ||u_+||_{B^6_{1,1}}) \\ &\leq (C \cdot 5\delta)^{n-2} \cdot ||z_3 - z_2||_X \\ &\leq (C \cdot 5\delta)^{n-2} \cdot 4\delta. \end{aligned}$$

Therefore, for sufficiently small δ,

$$\begin{aligned} ||z_{n+1}||_X &\leq ||z_n||_X + \delta \cdot \frac{1}{2^{n+1}} \\ &\leq \delta \cdot \left(\sum_{l=0}^{n+1} \frac{1}{2^l} \right). \end{aligned}$$

□

Corollary 3.3. With the hypothesis of Lemma 3.2, z_k is a Cauchy sequence in X and therefore has a strong limit in X.

Using Corollary 3.3, we may show the existence of a solution to (3.4) for positive time values if we assume small u_+ data in $B^6_{1,1}$ and that the first iterate of our scheme is sufficiently small in X. The remainder of the chapter will demonstrate that the later can be achieved if we assume some suitable norm of the data u_+ is small.

3.3 The First Iterate

By the Fourier transform and Plancherel,

$$||B(u_1, u_1)||_X \lesssim \left|\left| \langle \xi \rangle^2 \langle t \rangle \int_{\infty}^{t} \int_{\mathbb{R}^2} e^{is(\langle \xi \rangle \pm \langle \eta \rangle \pm \langle \xi - \eta \rangle)} \hat{f}_+(\xi - \eta) \hat{g}_+(\eta) d\eta ds \right|\right|_{L^2_\xi},$$

where

$$\hat{f}_+(\xi) = O(\xi^\alpha \hat{u}_+(\xi)) \ \text{ or } \ O(\xi^\alpha \bar{\hat{u}}_+(-\xi))$$

$$\hat{g}_+(\xi) = O(\xi^\beta \hat{u}_+(\xi)) \ \text{ or } \ O(\xi^\beta \bar{\hat{u}}_+(-\xi)).$$

Here, α and β are mulit-indices and $0 \leq |\alpha|, |\beta| \leq 1$. Now define

$$\phi_1(\xi, \eta) = \langle \xi \rangle + \langle \xi - \eta \rangle + \langle \eta \rangle$$

$$\phi_2(\xi, \eta) = \langle \xi \rangle + \langle \xi - \eta \rangle - \langle \eta \rangle$$

$$\phi_3(\xi, \eta) = \langle \xi \rangle - \langle \xi - \eta \rangle + \langle \eta \rangle$$

$$\phi_4(\xi, \eta) = \langle \xi \rangle - \langle \xi - \eta \rangle - \langle \eta \rangle.$$

We will start our estimates of $||B(u_1, u_1)||_X$ by first considering the short time case $t \leq 1$. First; however, it will be convenient to introduce the following lemma:

Lemma 3.4. Let $h(\xi, \eta)$ be a bounded function, then for all $n \in \mathbb{N}$ and $f, g \in H^n$, we have

$$\int_{\mathbb{R}^2} |h(\xi, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta)| d\eta \lesssim \frac{1}{\langle \xi \rangle^n} ||f||_{H^n_x} ||g||_{H^n_x}.$$

Proof. First, by the triangle inequality, we have $\langle \xi \rangle \leq \langle \xi - \eta \rangle + \langle \eta \rangle$. Hence, $\langle \xi \rangle^n \lesssim$

$\langle \xi - \eta \rangle^n + \langle \eta \rangle^n$. Thus, since $h(\xi, \eta)$ is assumed to be bounded, we have

$$\begin{aligned}\int_{\mathbb{R}^2} |h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)|d\eta &\lesssim \int_{\mathbb{R}^2} \left| \frac{\langle \xi-\eta\rangle^n + \langle \eta\rangle^n}{\langle \xi \rangle^n} \hat{f}(\xi-\eta)\hat{g}(\eta) \right| d\eta \\ &= \frac{1}{\langle \xi \rangle^n} \int_{\mathbb{R}^2} |\langle \xi-\eta\rangle^n \hat{f}(\xi-\eta)\hat{g}(\eta) + \langle \eta \rangle^n \hat{g}(\eta)\hat{f}(\xi-\eta)| d\eta \\ &\lesssim \frac{1}{\langle \xi \rangle^n} (||\langle \xi - \cdot\rangle^n \hat{f}(\xi-\cdot)\hat{g}(\cdot)||_{L^1_\eta} + ||\langle \cdot \rangle^n \hat{g}(\cdot)\hat{f}(\xi-\cdot)||_{L^1_\eta}).\end{aligned}$$

By Hölder, we then have

$$\begin{aligned}\int_{\mathbb{R}^2} |h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)|d\eta &\lesssim \frac{1}{\langle \xi \rangle^n} (||f||_{H^n_x}||g||_{L^2_x} + ||g||_{H^n_x}||f||_{L^2_x}) \\ &\lesssim \frac{1}{\langle \xi \rangle^n} ||f||_{H^n_x}||g||_{H^n_x}.\end{aligned}$$

$\square$

We may now estimate the short time case $t \leq 1$.

Proposition 3.5 (Short Time Control)**.** Suppose $0 \leq t \leq 1$ and $u_+ \in Z$, then for $j = 1, 2, 3, 4$

$$\left|\left| \langle \xi \rangle^3 \int_t^1 \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)} \hat{f}_+(\xi-\eta)\hat{g}_+(\eta) d\eta ds \right|\right|_{L^2_\xi} \lesssim \frac{1}{\langle t \rangle} ||u_+||^2_Y.$$

Proof. Using Lemma 3.4, we have

$$\left|\int_t^1 \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)} \hat{f}_+(\xi-\eta)\hat{g}_+(\eta)d\eta ds\right| \lesssim \int_t^1 \frac{1}{\langle\xi\rangle^5}||f_+||_{H_x^5}||g_+||_{H_x^5}ds$$
$$\lesssim \frac{1}{\langle\xi\rangle^5}||u_+||_Z^2$$

Hence,

$$\left|\left|\langle\xi\rangle^3 \int_t^1 \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)} \hat{f}_+(\xi-\eta)\hat{g}_+(\eta)d\eta ds\right|\right|_{L_\xi^2} \lesssim ||u_+||_Z^2||\langle\xi\rangle^{-2}||_{L_\xi^2}$$
$$\lesssim \frac{1}{\langle t\rangle}||u_+||_Z^2.$$

□

Using Proposition 3.5, it is now sufficient to estimate the first iterate just for the case $t \geq 1$. When $t \geq 1$, our intent is to integrate in time and then close the estimates using the Coifman-Meyer Theorem. To do this, we must establish a lower bound on the phase functions $\phi_j(\xi,\eta)$.

Lemma 3.6. For any $A, B \in \mathbb{R}$, we have

$$\langle A\rangle + \langle B\rangle - \langle A+B\rangle \gtrsim \frac{1}{\langle A\rangle + \langle B\rangle}. \tag{3.10}$$

Proof. We compute,

$$\begin{aligned}\text{LHS of (3.10)} &= \frac{2 + A^2 + B^2 + 2\langle A\rangle\langle B\rangle - (1 + A^2 + B^2 + 2AB)}{\langle A\rangle + \langle B\rangle + \langle A + B\rangle} \\ &\gtrsim \frac{1}{\langle A\rangle + \langle B\rangle}.\end{aligned}$$

□

Corollary 3.7. For $j = 1, 2, 3, 4$ we have

$$\phi_j(\xi, \eta) \gtrsim \frac{1}{\langle \xi\rangle + \langle \eta\rangle}.$$

With a lower bound on the phase functions, we can now obtain the necessary bound on $||B(u_1, u_1)||_X$ by integrating in time and using the Coifman-Meyer Theorem. We accomplish this in the lemma and proposition that follow.

Lemma 3.8. Let $h(\xi, \eta)$ be a smooth, bounded function. Suppose that for $0 \leq |\gamma_1|, |\gamma_2| \leq 5$ we have

$$|\partial_\xi^{\gamma_1} \partial_\eta^{\gamma_2} h(\xi, \eta)| \lesssim \langle \xi\rangle^5 + \langle \eta\rangle^5.$$

Then for $j = 1, 2, 3, 4$

$$\left|\left|\langle \xi\rangle^3 \int_{\mathbb{R}^2} e^{it\phi_j(\xi,\eta)} h(\xi, \eta)\hat{f}(\xi - \eta)\hat{g}(\eta)d\eta\right|\right|_{L^2_\xi} \lesssim \frac{1}{t}||f||_{B^{15}_{1,1}}||g||_{H^{15}_x}. \tag{3.11}$$

Proof. Let K denote the LHS of (3.11). Then by Plancheral, we see

$$K = ||T_h(e^{itD_m}f, e^{itD_{m'}}g)||_{H^3_x}.$$

Here, the operators D_m, $D_{m'}$ may be $\pm\langle\nabla\rangle$ and T_h is a bilinear operator with symbol $h(\xi, \eta)$. More precisely:

$$\widehat{T_h(f_1, f_2)}(\xi) = \int h(\xi, \eta)\hat{f}_1(\xi - \eta)\hat{f}_2(\eta)d\eta.$$

Rewriting, we have

$$\begin{aligned} T_h(e^{itD_m}f, e^{itD_{m'}}g) &= \int \frac{h(\xi, \eta)}{\langle\eta\rangle^{12}\langle\xi-\eta\rangle^{12}} \widehat{e^{itD_m}\langle\nabla\rangle}^{12} f(\xi - \eta)\widehat{e^{itD_{m'}}\langle\nabla\rangle}^{12} g(\eta)d\eta. \\ &= T_{\tilde{h}}(e^{itD_m}\langle\nabla\rangle^{12}f, e^{itD_{m'}}\langle\nabla\rangle^{12}g), \end{aligned}$$

where

$$\tilde{h}(\xi, \eta) = \frac{h(\xi, \eta)}{\langle\eta\rangle^{12}\langle\xi-\eta\rangle^{12}}.$$

Therefore,

$$\begin{aligned}
K &= ||T_h(e^{itD_m}f, e^{itD_{m'}}g)||_{H^3_x} \\
&\lesssim \sum_{0\leq\alpha\leq 3}\left|\left|\xi^\alpha \int \tilde{h}(\xi,\eta)\widehat{e^{itD_m}\langle\nabla\rangle^{12}f}(\xi-\eta)\widehat{e^{itD_{m'}}\langle\nabla\rangle^{12}g}(\eta)d\eta\right|\right|_{L^2_\xi} \\
&\lesssim \sum_{|\alpha'|\leq 3}\sum_{(\alpha_1,\alpha_2)=\alpha'}\left|\left|(\xi-\eta)^{\alpha_1}\eta^{\alpha_2}\int \tilde{h}(\xi,\eta)\langle\xi-\eta\rangle^{12}\widehat{e^{itD_m}f}(\xi-\eta)\langle\eta\rangle^{12}\widehat{e^{itD_{m'}}g}(\eta)d\eta\right|\right|_{L^2_\xi} \\
&\lesssim \sum_{|\alpha'|\leq 3}\sum_{(\alpha_1,\alpha_2)=\alpha'}\big|\big|T_{\tilde{h}}(e^{itD_m}\langle\nabla\rangle^{12}\partial^{\alpha_1}f, e^{itD_{m'}}\langle\nabla\rangle^{12}\partial^{\alpha_2}g)\big|\big|_{L^2_x}.
\end{aligned}$$

Because we have assumed that, for $0 < \gamma_1, \gamma_2 \leq 5$,

$$|\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}h(\xi,\eta)| \lesssim \langle\xi\rangle^5 + \langle\eta\rangle^5,$$

$\tilde{h}$ is a Coifman-Meyer multiplier. Hence, using the Coifman-Meyer Theorem and the dispersive estimate, we have

$$\begin{aligned}
\sum_{0\leq\alpha_1,\alpha_2\leq 3}\big|\big|T_{\tilde{h}}(e^{itD_m}\langle\nabla\rangle^{12}\partial^{\alpha_1}f, e^{itD_{m'}}\langle\nabla\rangle^{12}\partial^{\alpha_2}g)\big|\big|_{L^2_x} &\lesssim \sum_{0\leq\alpha_1\leq 3}\big|\big|e^{itD_m}\langle\nabla\rangle^{12}\partial^{\alpha_1}f\big|\big|_\infty\big|\big|g\big|\big|_{H^{15}} \\
&\lesssim \frac{1}{t}||f||_{B^{15}_{1,1}}||g||_{H^{15}}.
\end{aligned}$$

$\square$

With Lemma 3.8, we may now close the estimates on $||B(u_1,u_1)||_X$.

Proposition 3.9 (Long Time Control)**.** For $j = 1, 2, 3, 4$, we have

$$\left|\left|\langle\xi\rangle^3 \int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)} \hat{f}_+(\xi-\eta)\hat{g}_+(\eta) d\eta ds\right|\right|_{L^2_\xi} \lesssim \frac{1}{t}||u_+||_Z^2.$$

Proof. Integration in time yields

$$\left|\int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)} \hat{f}_+(\xi-\eta)\hat{g}_+(\eta) d\eta ds\right| \lesssim \lim_{M\to\infty}\left|\int_{\mathbb{R}^2} e^{iM\phi_j(\xi,\eta)} \frac{1}{\phi_j(\xi,\eta)} \hat{f}_+(\xi-\eta)\hat{g}_+(\eta) d\eta\right| \tag{3.12}$$

$$+\left|\int_{\mathbb{R}^2} e^{it\phi_j(\xi,\eta)} \frac{1}{\phi_j(\xi,\eta)} \hat{f}_+(\xi-\eta)\hat{g}_+(\eta) d\eta\right|. \tag{3.13}$$

We wish to control both of these terms by using Lemma 3.8. Hence, we must verify that for $0 \leq \gamma_1, \gamma_2 \leq 5$,

$$\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}\left(\frac{1}{\phi_j(\xi,\eta)}\right) \lesssim \langle\xi\rangle^5 + \langle\eta\rangle^5.$$

One can verify that for any multi-index γ with $|\gamma| \geq 1$ we have $|\partial^\gamma \phi_j(\xi,\eta)| \lesssim 1$. Hence, using Corollary 3.7, we have

$$\begin{aligned} |\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}(\phi_j^{-1}(\xi,\eta))| &\lesssim 1 + |\phi_j(\xi,\eta)|^5 \\ &\lesssim \langle\xi\rangle^5 + \langle\eta\rangle^5. \end{aligned}$$

Thus, we may use Lemma 3.8, to conclude

$$||\langle\xi\rangle^3(3.13)||_{L^2_\xi} \lesssim \frac{1}{t}||u_+||^2_Z.$$

Similarly,

$$||(3.12)||_{L^2_\xi} \lesssim \lim_{M\to\infty}\frac{1}{M}||u_+||^2_Z = 0.$$

Furthermore, one can verify that

$$||(3.12)||_{H^2_\xi} \lesssim \sum_{|\alpha|\le 2}||x^\alpha f_+||_{H^5_x}||g_+||_{H^5_x} \lesssim ||u_+||^2_Z.$$

By Sobloev embedding, (3.12) is then an L^∞_ξ function whose L^2_ξ-norm is zero. Hence, it is identically zero for all ξ.

$\square$

CHAPTER 4
CONSTRUCTION OF THE WAVE OPERATOR FOR THE KLEIN-GORDON-SCHRÖDINGR SYSTEM WITH YUKAWA COUPLING

In this chapter, we prove the existence of the wave operator for the Klein-Gordon-Schrödinger system (KGS) with quadratic non-linearity. Like (3.1), the non-linearity type for this system is below Strichartz scaling and therefore classic perturbation methods will fail in any Strichartz space. The proof will follow the method presented in Chapter 3 up to, but not including, the first iterate analysis.

The work involved in controlling the first iterate for the KGS is notably more difficult than that of (3.1). This is due in large part to the fact that the phase functions $\Phi_i(\xi, \eta)$ for the KGS will have zeros. Hence, we cannot immediately integrate in time everywhere to achieve the necessary decay. Instead, where $\partial_\eta \Phi_i(\xi, \eta)$ is non-zero, we show that one can close the estimates by integration by parts in frequency space. However, to complicate matters further, one phase function will have a non-empty resonance set in $\mathbb{R}^2 \times \mathbb{R}^2$, where both the function and its partial in η are identically zero. On this set we can neither integrate in time nor integrate by parts in frequency space. To account for this set, we must place assumptions on the support of the "final data".

4.1 Introduction and Overview

We study the following system of equations on $\mathbb{R}^2$

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u &= \pm uv \\ (\Box + 1)v &= \pm|\partial_x u|^2 \end{cases}, \tag{4.1}$$

and show that wave operators exist under smallness conditions and a control assumption on a single frequency band of the "final data".

(4.1) is known as the Klein-Gordon-Schrödinger system. A common choice of coupling, known as the Yukawa interaction, is to replace the non-linearity $\pm|\partial_x u|^2$ with $\pm|u|^2$. With the Yukawa coupling, the system describes the interaction of a complex scalar nucleon field u with a real scalar meson field v (see [19]). Our method only relies on the precise form of phase functions in frequency space and not on any conservation laws particular to our choice of non-linearity; therefore the same method can be used to construct the wave operator in the case of Yukawa interaction.

Our purpose in this chapter is to construct the wave operators for this system. Like the "semi-relativistic" Schrödinger equation, both the Klein-Gordon and Schrödinger equations see a $t^{-d/2}$ decay as $t \to \infty$ in the space $\mathbb{R}^d$. Thus, quadratic linearities are a critical case in $\mathbb{R}^2$ for (4.1). Strichartz estimates cannot be used and, because of the choice of coupling, conservation laws do not exist. Hence, classical methods for constructing the wave operator will fail for the system (4.1).

Previous results for (4.1) are mostly restricted to the Yukawa interaction and rely critically on conservation laws. Ozawa and Tsutsumi first studied this problem with the non-linearities uv and $-|u|^2$ in [9]. They proved the existence of wave operators under certain smallness conditions on the scattered states as well as the assumption that the Fourier support of the scattered state of u was outside the unit disc. In [11], [12] and [13], Shimomura improved these results with the same non-linearities. [11] established the existence of wave operators with no smallness condition on v's scattered state, but the Fourier support of u's scattered state was still required to be outside the unit disc. In [12], the support condition on $\hat{u}_+$ was substituted for a smallness condition on v_+ and a controllability assumption on the supports of $\hat{u}_+$ and $\hat{v}_+$ on a single circle. Finally, in [13], wave operators were shown to exist without any smallness condition on v_+ and no support conditions other than the controllability assumption of [12]. All results rely on the energy method and first and second approximations to the asymptotic profiles of u and v to construct solutions on the the interval $[T, \infty)$ for some large T. Global well-posedness results were then used to extend the solution to $[0, \infty)$. Again, these results relied critically on the precise form of the non-linearity.

The Cauchy problem for the KGS system was solved in [10] and [18] for the Yukawa coupling $-uv$ and $|u|^2$. [10] used the Fourier restriction norm method and [18] used the I-method. Both papers used Strichartz estimates on finite intervals and relied on energy and charge conservation to show existence of global solutions. The ability to use these conservation laws depends delicately on the non-linearities, hence, the specific choice of $-uv$ and $|u|^2$ was crucial in the result. In [5], Pecher was able to show local existence without the use of energy conservation, and thus, for a wider variety of Yukawa interactions ($\pm uv$ and $\pm|u|^2$).

We recall some notation presented in Chapter 1:

• The Klein-Gordon linear propagators L and $\dot{L}$ are defined by

$$\mathcal{F}(L(f,g)) = \cos(\langle\xi\rangle t)\hat{f} + \langle\xi\rangle^{-1}\sin(\langle\xi\rangle t)\hat{g}$$
$$\mathcal{F}(\dot{L}(f,g)) = -\langle\xi\rangle^{-1}\sin(\langle\xi\rangle t)\hat{f} + \cos(\langle\xi\rangle t)\hat{g}.$$

• The Y space is defined by the norm

$$||g||_Y = \sum_{|\alpha+\beta|\le 12} ||x^\alpha \partial^\beta g||_{L^2} + ||g||_{W^{16,1}} + ||g||_{H^{16}} + ||g||_{B^6_{1,1}}.$$

We will require the scattered states u_+ and $(v_+, \dot{v}_+)$ to be small in Y.

• For small δ, we introduce the set A_δ, defined as

$$A_\delta = \{(\xi,\eta) \mid r_\xi - \delta \le |\xi| \le r_\xi + \delta,\ r_\eta - \delta \le |\eta| \le r_\eta + \delta\},$$

where the ordered pair (r_ξ, r_η) is the unique solution on $\mathbb{R}^+$ to the system

$$\begin{array}{rcl} \frac{1}{2}r_\xi^2 - \frac{1}{2}(r_\xi - r_\eta)^2 - \langle r_\eta\rangle & = & 0 \\ r_\xi - r_\eta\left(1 + \frac{1}{\langle r_\eta\rangle}\right) & = & 0. \end{array}$$

This system corresponds to the resonance set of a phase function. We will describe the meaning of resonance set and the role A_δ plays in our proof later in the introduction.

Roughly, we have that $(r_\xi, r_\eta) \approx (1.9002, 1.1466)$.

In this chapter, we use the following dispersive estimates on $\mathbb{R}^2$ for $t > 0$:

$$||e^{\frac{1}{2}i\Delta t} f||_{L^\infty_x} \lesssim t^{-1}||f||_{L^1_x}$$
$$||e^{-i\langle\nabla\rangle t} P_k g||_{L^\infty_x} \lesssim t^{-1}\langle 2^k\rangle^2 ||P_k g||_{L^1_x}.$$

We will also make use of the Coifman-Meyer theorem, stated in Chapter 3. The version of the theorem we will use is stated as follows:

Theorem 4.1 (Coifman-Meyer Theorem, Utility-Grade)**.** Let $m(\xi, \eta)$ be a bounded function on $\mathbb{R}^2 \times \mathbb{R}^2$. Suppose that

$$\sup_{k\in\mathbb{Z}} ||m(2^k \cdot)\hat{P}_0||_{H^5_\zeta} < \infty,$$

where $\zeta = (\xi, \eta)$ and P_0 is the $k = 0$ Littlewood-Paley operator. Then the operator $T_m(\xi, \eta)$, defined by

$$\widehat{T_m(f,g)}(\xi) = \int_{\mathbb{R}^2} m(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta,$$

maps $L^p \times L^q \to L^r$ provided $2 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

We may now present our main result.

Theorem 4.2. Suppose we have

$$supp(\hat{u}_+(\eta)) \cap \{\eta \mid .75 \leq |\eta| \leq .76\} = \emptyset,$$

and that for a sufficiently small $\delta > 0$,

$$||u_+||_Y + ||v_+||_Y + ||\dot{v_+}||_Y \leq \delta.$$

Then there is a unique solution (u, v) to the system (4.1) such that

$$||\langle t\rangle \int_t^\infty e^{\frac{1}{2}(t-s)\Delta} u \cdot v ds||_{L_t^\infty H_x^3} + ||\langle t\rangle \int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}(\nabla u \cdot \overline{\nabla u})ds||_{L_t^\infty H_x^3} < \infty,$$

and as $t \to \infty$,

$$||e^{\frac{1}{2}it\Delta}u_+ - u(t)||_{H^3} + ||L(v_+, \dot{v}_+) - v(t)||_{H^3} + ||\dot{L}(v_+, \dot{v}_+) - \partial_t v(t)||_{H^2} \to 0.$$

Remark: In our method of proof, one will see that we may relax the support condition on $\hat{u}_+$ to the assumption

$$\hat{u}_+(\xi - \eta)[v_+(\eta) + i\langle\nabla\rangle^{-1}\dot{v}_+(\eta)] \equiv 0 \text{ on } A_\delta.$$

Remark: One should note that this theorem is established for non-linearities for which it is impossible to use the energy conservation laws on which previous results have relied critically.

Our method follows the work of [7]. We briefly explain the proof steps as follows:

Step 1: Reformulation

We transform the system into one that is first-order in time by the variable assignment $h = v + i\langle\nabla\rangle^{-1}\partial_t v$. In terms of h, the system (4.1) is then transformed to

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u &= \pm u Re(h) \\ -i\partial_t h + \langle\nabla\rangle h &= \pm\langle\nabla\rangle^{-1}(\nabla u \cdot \overline{\nabla u}) \end{cases}.$$

We can then reformulate the problem using Duhamel's formula as:

$$\begin{pmatrix} u \\ h \end{pmatrix}(t) = \begin{pmatrix} e^{i\frac{1}{2}t\Delta}u_+ \pm i\int_t^\infty e^{i\frac{1}{2}(t-s)\Delta}u \cdot Re(h)ds \\ e^{-it\langle\nabla\rangle}h_+ \pm i\int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}(\nabla u \cdot \overline{\nabla u})ds \end{pmatrix}.$$

Step 2: First Iterate Analysis

With the system reformulated, we next look to show the natural first iterate of our contraction scheme has a $\langle t\rangle^{-1}$ decay in H^3. More specifically, we show the bilinear operators

$$\begin{aligned} B_1(f,g) &= \int_t^\infty e^{i\frac{1}{2}(t-s)\Delta}f \cdot Re(g)ds \\ B_2(f,g) &= \int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}(\nabla f \cdot \overline{\nabla g})ds \end{aligned}$$

have the following property:

$$\left\| \begin{matrix} B_1(e^{i\frac{1}{2}\Delta t}u_+, e^{-i\langle\nabla\rangle t}h_+) \\ B_2(e^{i\frac{1}{2}\Delta t}u_+, e^{i\frac{1}{2}\Delta t}u_+) \end{matrix} \right\|_{H^3_x} \lesssim \frac{1}{\langle t\rangle}.$$

Establishing this decay on the first iterate comprises the majority of this chapter. The decay estimates are achieved by going to the frequency domain and carefully analyzing the resonance points for the purpose of using stationary phase methods.

The phase functions to consider are:

$$\begin{aligned}
\phi_0(\xi,\eta) &= \frac{1}{2}|\xi|^2 - \frac{1}{2}|\xi-\eta|^2 + \langle\eta\rangle,\\
\phi_1(\xi,\eta) &= \frac{1}{2}|\xi|^2 - \frac{1}{2}|\xi-\eta|^2 - \langle\eta\rangle,\\
\phi_2(\xi,\eta) &= \langle\xi\rangle - \frac{1}{2}|\xi-\eta|^2 + \frac{1}{2}|\eta|^2.
\end{aligned}$$

We call the sets where $\partial_\eta\phi_i = 0$ and $\phi_i = 0$ the space resonance and time resonance sets of ϕ_i respectively. Roughly, the method is to integrate by parts in frequency space away from the space resonance set and to integrate by parts in time away from the time resonance set. Those points where both $\partial_\eta\phi_i = 0$ and $\phi_i = 0$ comprise the resonance set and cause the greatest difficulty as we cannot integrate by parts in either variable.

When $i = 0, 2$, the resonance set is empty and thus, with carefully chosen cutoff functions, we can always integrate by parts in either frequency space or time. The phase function ϕ_1 is the most problematic and requires the most delicate treatment. This is due to the existence of a resonant set. We have

$$\partial_\eta\phi_1(\xi,\eta) = \xi - \eta\left(1 + \frac{1}{\langle\eta\rangle}\right).$$

Hence, $\partial_\eta \phi_1(\xi, \eta)$ can only be zero if ξ and η are co-linear, and the moduli of ξ and η solve the system

$$\begin{array}{ll} \frac{1}{2}|\xi|^2 - \frac{1}{2}(|\xi| - |\eta|)^2 - \langle|\eta|\rangle & = 0 \\ |\xi| - |\eta|\left(1 + \frac{1}{\langle|\eta|\rangle}\right) & = 0 \end{array}.$$

Hence, the resonance set takes the form

$$\{(\xi, \eta) : \ \xi \parallel \eta, \ |\xi| = r_\xi, \ |\eta| = r_\eta\}.$$

It is because of this set that we must assume

$$\hat{u}_+(\xi - \eta)\hat{h}_+(\eta) \equiv 0 \ \text{ on } \ A_\delta.$$

The method of stationary phase for oscillatory integrals is well known and many classical results can be found in [14]. The idea of carefully analyzing resonance points is presented in [2].

Step 3: Bilinear Estimates

We introduce the space X, where we construct our solutions' non-linear terms, by its norm

$$||f||_X = ||\langle t\rangle f||_{L_t^\infty H_x^3}.$$

Through the use of Sobolev embedding, we establish the following estimates on the above defined bilinear operators:

$$
\begin{aligned}
||B_i(f,g)||_X &\lesssim ||f||_X \cdot ||g||_X, \\
||B_i(f,g)||_X &\lesssim ||\langle t\rangle f||_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)} \cdot ||g||_X, \\
||B_i(f,g)||_X &\lesssim ||f||_X \cdot ||\langle t\rangle g||_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)}.
\end{aligned}
$$

These estimates are used to establish our iterative scheme as a contraction.

Step 4: Contraction in the X-Space

Finally, we define two iteration schemes as follows:

$$
\begin{aligned}
u_1 &= e^{i\frac{1}{2}\Delta t}u_+, \\
h_1 &= e^{-i\langle\nabla\rangle t}h_+,
\end{aligned}
$$

and

$$
\begin{aligned}
u_{k+1} &= u_1 + B_1(u_k, h_k), \\
h_{k+1} &= h_1 + B_2(u_k, u_k).
\end{aligned}
$$

These are then shown to be contractions in X. This is done through induction. In the process, we use the dispersive estimates of the linear propagators, the first iterate estimates established in Step 2, and the bilinear estimates from Step 3.

Remark: One should note that because the linear operators for the Schrődinger and Klein-Gordon equations are isometries in H^3, the linear solutions to both equations do not exist in the space X. In this way, our scheme is very different from any perturbation methods.

We now give a brief overview of the organization of the remainder of this chapter.

In Section 2, we make the system (4.1) a first-order in time system. We then introduce the X space, where we will construct the non-linear terms, and establish the bilinear estimates on X. Finally, we prove the Duhamel operators are contractions on X using induction and the bilinear estimates. Part of this proof relies on the assumption that the first iterate of our scheme is sufficiently small in X.

In Section 3, which accounts for the bulk of the chapter, we prove that the assumption on the smallness of the first iterate is valid if the final data (u_+, h_+) is sufficiently small in some suitable space. This is done by resonance analysis and stationary phase calculations.

4.2 Fine Decomposition

4.2.1 Reformulation

In this section, we will show the non-linear maps are contractions. To this end, we rewrite the system (4.1) into a first-order in time system. To do this, we introduce the variable $h = v + i\langle\nabla\rangle^{-1}\partial_t v$. (4.1) is then transformed to

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u &= \pm u Re(h) \\ -i\partial_t h + \langle\nabla\rangle h &= \pm\langle\nabla\rangle^{-1}(\nabla u \cdot \overline{\nabla u}) \end{cases}. \tag{4.2}$$

If (u, h) are to scatter to free solutions (u_+, h_+) in H^3, then we need

$$||u - e^{\frac{1}{2}it\Delta}u_+||_{H^3} + ||h - e^{-it\langle\nabla\rangle}h_+||_{H^3} \to 0 \quad \text{as} \quad t \to \infty. \tag{4.3}$$

To this end, we will find strong solutions to

$$\begin{pmatrix} u \\ h \end{pmatrix}(t) = \begin{pmatrix} e^{i\frac{1}{2}t\Delta}u_+ \pm i\int_t^\infty e^{i\frac{1}{2}(t-s)\Delta}u \cdot Re(h)ds \\ e^{-it\langle\nabla\rangle}h_+ \pm i\int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}(\nabla u \cdot \overline{\nabla u})ds \end{pmatrix}. \tag{4.4}$$

It is easy to verify that we can establish Theorem 4.2 (and thus, the existence of wave operators for (4.1)) by showing a pair $(u, h)^T$ of the form found in (4.4) exists with the non-linear terms in the X space.

4.2.2 Bilinear Estimates

Based on (4.4), we define the following bilinear operators:

$$B_1(f, g) = \int_t^\infty e^{i\frac{1}{2}(t-s)\Delta}f \cdot Re(g)ds$$
$$B_2(f, g) = \int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}(\nabla f \cdot \overline{\nabla g})ds.$$

Through these operators, we define our two iteration schemes:

$$u_1 = e^{i\frac{1}{2}\Delta t}u_+,$$

$$h_1 = e^{-i\langle\nabla\rangle t}h_+,$$

and

$$u_{k+1} = u_1 + B_1(u_k, h_k),$$

$$h_{k+1} = h_1 + B_2(u_k, u_k).$$

In order to prove these schemes are contractions, we must show the bilinear operators have particular algebra estimates:

Lemma 4.1. For $i = 1, 2$, we have

$$||B_i(f,g)||_X \lesssim ||f||_X \cdot ||g||_X, \tag{4.5}$$

$$||B_i(f,g)||_X \lesssim ||\langle t\rangle f||_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)} \cdot ||g||_X, \tag{4.6}$$

$$||B_i(f,g)||_X \lesssim ||f||_X \cdot ||\langle t\rangle g||_{L_t^\infty W^{3,\infty}([0,\infty)\times\mathbb{R}^2)}. \tag{4.7}$$

Where the space X is defined by the norm

$$||g||_X = ||\langle t\rangle g||_{L_t^\infty H_x^3}.$$

Proof. This is a matter of direct calculation. For $i = 2$, by Minkowski

$$||B_2(f,g)||_X \lesssim \left|\left|\langle t\rangle \int_t^\infty ||\nabla f \cdot \overline{\nabla g}(s)||_{H^2} ds\right|\right|_{L_t^\infty([0,\infty))}.$$

Next, by Hőlder and Sobolev embedding

$$\begin{aligned} ||\nabla f \cdot \overline{\nabla g}(s)||_{H^2} &\lesssim ||\nabla^3 f||_2 ||\nabla g||_\infty + ||\nabla^2 f||_4 ||\nabla^2 g||_4 + ||\nabla f||_\infty ||\nabla^3 g||_2 \\ &\lesssim \langle s\rangle^{-2} ||f||_X ||g||_X. \end{aligned}$$

Combining these estimates, we have

$$\begin{aligned} ||B_2(f,g)||_X &\lesssim \sup_{t\geq 0} \langle t\rangle \int_t^\infty \langle s\rangle^{-2} ds \cdot ||f||_X \cdot ||g||_X \\ &\lesssim ||f||_X ||g||_X. \end{aligned}$$

This ends the proof of (4.5). The proofs for (4.6) and (4.7) and the case $i = 1$ are similar and we omit the details. □

4.2.3 Contraction in the X-Space

With Lemma (4.1), we may now show our iteration schemes are contractions in the X space with fixed points satisfying (4.4). In the following, we will use the estimate

$$||e^{itD} f||_{L^\infty} \lesssim \frac{1}{\langle t\rangle} ||f||_{B^3_{1,1}},$$

where $D = \frac{1}{2}t\Delta$ or $-\langle\nabla\rangle$. For $t > 1$, this bound comes from the dispersive estimates and the inequality

$$||f||_{L^\infty} \lesssim \sum_{k\in\mathbb{Z}} ||P_k f||_{L^\infty}.$$

For $t \leq 1$, we use Sobolev embedding.

Denote $w_{k+1} = u_{k+1} - u_1$, and $z_{k+1} = h_{k+1} - h_1$ for $k \geq 1$. We are interested in the differences between successive values of w_k and z_k as, by construction, they are also the differences of successive values of u_k and h_k respectively. For $k \geq 2$, we have

$$\begin{aligned} ||z_{k+2} - z_{k+1}||_X &= ||B_2(u_1 + w_{k+1}, u_1 + w_{k+1}) - B_2(u_1 + w_k, u_1 + w_k)||_X \\ &= ||B_2(w_{k+1} - w_k, u_1 + w_{k+1}) + B_2(u_1 + w_k, w_{k+1} - w_k)||_X \\ &\lesssim ||w_{k+1} - w_k||_X(||w_k||_X + ||w_{k+1}||_X + ||\langle t\rangle u_1||_{L^\infty_t W^{3,\infty}}) \\ &\lesssim ||w_{k+1} - w_k||_X(||w_k||_X + ||w_{k+1}||_X + ||u_+||_{B^6_{1,1}}). \end{aligned} \tag{4.8}$$

Similarly,

$$||w_{k+2} - w_{k+1}||_X \tag{4.9}$$

$$\begin{aligned} &= ||B_1(u_1 + w_{k+1}, h_1 + z_{k+1}) - B_1(u_1 + w_k, h_1 + z_k)||_X \\ &= ||B_1(w_{k+1} - w_k, h_1 + z_{k+1}) + B_1(u_1 + w_k, z_{k+1} - z_k)||_X \\ &\lesssim ||w_{k+1} - w_k||_X(||z_{k+1}||_X + ||h_+||_{B^6_{1,1}}) + ||z_{k+1} - z_k||_X(||w_k||_X + ||u_+||_{B^6_{1,1}}) \end{aligned} \tag{4.10}$$

With these estimates, we prove by induction that, under suitable conditions on the final data, the iteration scheme is a contraction.

Lemma 4.2. Let w_j, z_j, and the space X be as defined above. Assume

$$||u_+||_{B^6_{1,1}},\ ||h_+||_{B^6_{1,1}} \leq \delta,$$

and

$$||B_1(u_1, h_1)||_X,\ ||B_2(u_1, u_1)||_X \leq \delta,$$

for some sufficiently small δ. Then for $n \geq 3$ we have

$$||w_j||_X,\ ||z_j||_X \leq \delta \cdot \left(\sum_{j=0}^{n} \frac{1}{2^l} \right),\ \forall 2 \leq j \leq n.$$

Proof. This is a proof by induction on n. For $n = 3$, we have

$$\begin{aligned}
||w_3||_X &= ||B_1(w_2 + u_1, z_2 + h_1)||_X \\
&\leq C_0 ||w_2||_X (||z_2||_X + ||h_+||_{B^6_{1,1}}) + ||z_2||_X ||u_+||_{B^6_{1,1}} + ||B_1(u_1, h_1)||_X \\
&\leq C_1 \cdot \delta^2 + \delta.
\end{aligned}$$

$$\begin{aligned}
||z_3||_X &= ||B_2(w_2 + u_1, w_2 + h_1)||_X \\
&\leq C_2 ||w_2||_X (||u_+||_{B^6_{1,1}} + ||h_+||_{B^6_{1,1}}) + ||B_2(u_1, h_1)||_X \\
&\leq C_3 \cdot \delta^2 + \delta.
\end{aligned}$$

Choosing δ sufficiently small completes the base case. Now consider $j = n+1$. By equations (4.8) and (4.10), we have

$$
\begin{aligned}
||w_{n+1} - w_n||_X &\leq C \cdot ||w_n - w_{n-1}||_X \cdot (||z_n||_X + ||h_+||_{B^6_{1,1}}) \\
&+ C \cdot ||z_n - z_{n-1}||_X \cdot (||w_{n-1}||_X + ||u_+||_{B^6_{1,1}}) \\
&\leq (C \cdot 3\delta)^{n-2} \cdot (||w_3 - w_2||_X + ||z_3 - z_2||_X) \\
&\leq (C \cdot 3\delta)^{n-2} \cdot 6\delta.
\end{aligned}
$$

$$
\begin{aligned}
||z_{n+1} - z_n||_X &\leq C' \cdot ||w_n - w_{n-1}||_X (||w_n||_X + ||w_{n-1}||_X + ||u_+||_{B^6_{1,1}}) \\
&\leq (C' \cdot 5\delta)(C \cdot 6\delta)^{n-1} \cdot ||w_3 - w_2||_X \\
&\leq (C' \cdot 5\delta)(C \cdot 6\delta)^{n-1} \cdot 3\delta.
\end{aligned}
$$

Therefore, for sufficiently small δ,

$$
||w_{n+1}||_X \leq ||w_n||_X + \delta \cdot \frac{1}{2^{n+1}} \leq \delta \cdot \left(\sum_{l=0}^{n+1} \frac{1}{2^l} \right),
$$

$$
||z_{n+1}||_X \leq ||z_n||_X + \delta \cdot \frac{1}{2^{n+1}} \leq \delta \cdot \left(\sum_{l=0}^{n+1} \frac{1}{2^l} \right).
$$

□

Corollary 4.3. With the hypothesis of Lemma 4.2, w_k and z_k are Cauchy sequences in X and therefore have strong limits in X.

Using Corollary 4.3, we may show the existence of solutions to (4.4) for positive time values if we assume small u_+ and h_+ data in $B^6_{1,1}$ and the first iterate of our scheme is sufficiently small in the X-space. The remainder of this chapter will demonstrate that the later can be achieved if we assume some suitable norm of the data u_+ and h_+ is small.

4.3 Analysis of the First Iterate

The purpose of this section is to prove that under suitable conditions on (u_+, h_+), $B_1(u_1, h_1)$ and $B_2(u_1, u_1)$ have a decay of $\langle t\rangle^{-1}$ in H^3. By Plancharel's theorem, it is sufficient to measure an L^2 norm in the frequency domain. By definition, we have

$$\begin{pmatrix} B_1(u_1,h_1) \\ \\ B_2(u_1,u_1) \end{pmatrix} = \begin{pmatrix} \int_t^\infty e^{i\frac{1}{2}(t-s)\Delta}\big[\big(e^{i\frac{1}{2}s\Delta}u_+\big)Re\big(e^{-is\langle\nabla\rangle}h_+\big)\big]ds \\ \int_t^\infty e^{-i(t-s)\langle\nabla\rangle}\langle\nabla\rangle^{-1}\big[\nabla\big(e^{i\frac{1}{2}s\Delta}u_+\big)\cdot\overline{\nabla\big(e^{i\frac{1}{2}s\Delta}u_+\big)}\big]ds \end{pmatrix}. \tag{4.11}$$

Hence, in the frequency domain

$$||\begin{pmatrix} B_1(u_1,h_1) \\ \\ B_2(u_1,u_1) \end{pmatrix}||_{H^3_x} \lesssim ||\begin{pmatrix} \langle\xi\rangle^3\mathcal{F}(B_1(u_1,h_1)) \\ \\ \langle\xi\rangle^3\mathcal{F}(B_2(u_1,u_1)) \end{pmatrix}||_{L^2_\xi}$$
$$\lesssim \left\|\begin{pmatrix} \langle\xi\rangle^3\int_t^\infty\int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{g}_j(\eta)d\eta ds \\ \langle\xi\rangle^2\int_t^\infty\int_{\mathbb{R}^2} e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds \end{pmatrix}\right\|_{L^2_\xi}.$$

Where $j = 0, 1$; the phase functions ϕ_0, ϕ_1, and ϕ_2 take the form

$$\begin{aligned}
\phi_0(\xi,\eta) &= \frac{1}{2}|\xi|^2 - \frac{1}{2}|\xi-\eta|^2 + \langle\eta\rangle, \\
\phi_1(\xi,\eta) &= \frac{1}{2}|\xi|^2 - \frac{1}{2}|\xi-\eta|^2 - \langle\eta\rangle, \\
\phi_2(\xi,\eta) &= \langle\xi\rangle - \frac{1}{2}|\xi-\eta|^2 + \frac{1}{2}|\eta|^2;
\end{aligned}$$

and $g_0(\xi) = \bar{\hat{h}}_+(-\xi)$, $g_1(\xi) = \hat{h}_+(\xi)$.

For readability, we will divide the work of establishing the necessary frequency bounds into several lemmas and propositions.

4.3.1 Preliminary Lemmas

Lemma 4.4. Let $h(\xi,\eta)$ be a bounded function, then for all $n \in \mathbb{N}$ and $f, g \in H^n$, we have

$$\int_{\mathbb{R}^2} |h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)|d\eta \lesssim \frac{1}{\langle\xi\rangle^n}||f||_{H^n_x}||g||_{H^n_x}.$$

Proof. First, by the triangle inequality, we have $\langle\xi\rangle \le \langle\xi-\eta\rangle + \langle\eta\rangle$. Hence, $\langle\xi\rangle^n \lesssim \langle\xi-\eta\rangle^n + \langle\eta\rangle^n$. Thus, since $h(\xi,\eta)$ is assumed to be bounded, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} |h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)|d\eta &\lesssim \int_{\mathbb{R}^2} \left|\frac{\langle\xi-\eta\rangle^n + \langle\eta\rangle^n}{\langle\xi\rangle^n}\hat{f}(\xi-\eta)\hat{g}(\eta)\right|d\eta \\
&= \frac{1}{\langle\xi\rangle^n}\int_{\mathbb{R}^2} |\langle\xi-\eta\rangle^n\hat{f}(\xi-\eta)\hat{g}(\eta) + \langle\eta\rangle^n\hat{g}(\eta)\hat{f}(\xi-\eta)|d\eta \\
&\lesssim \frac{1}{\langle\xi\rangle^n}(||\langle\xi-\cdot\rangle^n\hat{f}(\xi-\cdot)\hat{g}(\cdot)||_{L^1_\eta} + ||\langle\cdot\rangle^n\hat{g}(\cdot)\hat{f}(\xi-\cdot)||_{L^1_\eta}).
\end{aligned}$$

By Hőlder, we then have

$$\begin{aligned}\int_{\mathbb{R}^2} |h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)|d\eta &\lesssim \frac{1}{\langle\xi\rangle^n}(||f||_{H^n_x}||g||_{L^2_x} + ||g||_{H^n_x}||f||_{L^2_x}) \\ &\lesssim \frac{1}{\langle\xi\rangle^n}||f||_{H^n_x}||g||_{H^n_x}.\end{aligned}$$

□

Lemma 4.5 (Short Time Control)**.** Suppose $0 \le t \le 1$ and $f, g \in H^5_x$, then for $j = 0, 1, 2$

$$\left|\left|\langle\xi\rangle^3 \int_t^1 \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta ds\right|\right|_{L^2_\xi} \lesssim \frac{1}{\langle t\rangle}||f||_{H^5}||g||_{H^5}.$$

Proof. Using Lemma 4.4, we have

$$\begin{aligned}\left|\int_t^1 \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta ds\right| &\lesssim \int_t^1 \frac{1}{\langle\xi\rangle^5}||f||_{H^5_x}||g||_{H^5_x}ds \\ &\lesssim \frac{1}{\langle\xi\rangle^5}||f||_{H^5_x}||g||_{H^5_x}.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\left|\langle\xi\rangle^3 \int_t^1 \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta ds\right|\right|_{L^2_\xi} &\lesssim ||f||_{H^5_x}||g||_{H^5_x}||\frac{1}{\langle\xi\rangle^2}||_{L^2_\xi} \\ &\lesssim \frac{1}{\langle t\rangle}||f||_{H^5_x}||g||_{H^5_x}.\end{aligned}$$

□

Using Lemma 4.5, it is now sufficient to estimate the first iterate just for the case $t \geq 1$. Therefore; from henceforth, we will assume $t \geq 1$.

In the following lemma, we show how, away from the space resonance set, one may integrate by parts in time to obtain a $\langle t \rangle^{-1}$ decay in H^3.

Lemma 4.6 (Decay Away from Space Resonance)**.** Let $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that $|\partial_\eta \phi(\xi, \eta)| \gtrsim s^{-\alpha_1}$ and $|\partial_\eta^k \phi(\xi, \eta)| \lesssim 1$ for all $k \geq 2$. Further suppose $h(\xi, \eta)$ is a smooth function, with $|\partial_\eta^j h(\xi, \eta)| \lesssim s^{j \cdot \alpha_2}$ for all $j \geq 1$. If $2\alpha_1 + \alpha_2 \leq \frac{2}{3}$, then

$$\left|\left| \langle \xi \rangle^3 \int_t^\infty \int_{\mathbb{R}^2} e^{is\phi(\xi,\eta)} h(\xi, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta ds \right|\right|_{L^2_\xi} \lesssim \frac{1}{t} ||f||_Y ||g||_Y.$$

Proof. Define the operator D_ϕ on sufficiently smooth scalar-valued functions $f(\xi, \eta)$ by

$$D_\phi f(\xi, \eta) = \frac{\nabla_\eta \phi(\xi, \eta)}{is|\nabla_\eta \phi(\xi, \eta)|^2} \cdot \nabla_\eta f(\xi, \eta).$$

By the inequality $2\alpha_1 + \alpha_2 \leq \frac{2}{3}$, we have that there exists a natural number $N \leq 6$ such that $N(1 - 2\alpha_1 - \alpha_2) \geq 2$. Observe that

$$D_\phi^N (e^{is\phi}) = e^{is\phi},$$

so we may integrate by parts N times using D_ϕ. This yields one principal term and N boundary terms. The boundary terms each vanish due to the decay assumptions on f and g and the assumption that $\partial_\eta \phi(\xi, \eta)$ is bounded below. Thus, we are only left to consider the principal term.

$$\left|\int_t^\infty \int_{\mathbb{R}^2} e^{is\phi(\xi,\eta)} h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta ds\right|$$
$$= \left|\int_t^\infty \int_{R^2} e^{is\phi(\xi,\eta)} (D^t)^N [h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)]d\eta ds\right|. \tag{4.12}$$

Inductively, one can establish the bound

$$RHS(4.12) \lesssim$$
$$\int_t^\infty \int_{\mathbb{R}^2} \frac{1}{s^N}\left(\frac{1}{|\nabla_\eta \phi(\xi,\eta)|^{2N}} + \frac{1}{|\nabla_\eta \phi(\xi,\eta)|^N}\right) \sum_{\beta_i \le N} |\partial_\eta^{\beta_1} h(\xi,\eta)\partial_\eta^{\beta_2}\hat{f}(\xi-\eta)\partial^{\beta_3}\hat{g}(\eta)|d\eta ds.$$

Hence

$$\begin{aligned}
RHS(4.12) &\lesssim \int_t^\infty \int_{\mathbb{R}^2} \frac{s^{N\cdot\alpha_2}}{s^{N(1-2\alpha_1)}} \sum_{\beta_2,\beta_3 \le N} |\partial_\eta^{\beta_2}\hat{f}(\xi-\eta)\partial_\eta^{\beta_3}\hat{g}(\eta)|d\eta ds \\
&\lesssim \int_t^\infty \frac{1}{s^2}\frac{1}{\langle\xi\rangle^5} \sum_{\beta_2,\beta_3 \le N} (||\mathcal{F}^{-1}(\partial_\eta^{\beta_2}\hat{f})||_{H_x^5} ||\mathcal{F}^{-1}(\partial_\eta^{\beta_3}\hat{g})||_{H_x^5}) \\
&\lesssim \frac{1}{\langle\xi\rangle^5 t} \sum_{\beta_2,\beta_3 \le N} (||\mathcal{F}^{-1}(\partial_\eta^{\beta_2}\hat{f})||_{H_x^5} ||\mathcal{F}^{-1}(\partial_\eta^{\beta_3}\hat{g})||_{H_x^5}).
\end{aligned}$$

□

We may preform the same calculations with $(\xi-\eta)\hat{f}(\xi-\eta)$ and $\eta\hat{g}(\eta)$ replacing $\hat{f}(\xi-\eta)$ and $\hat{g}(\eta)$ respectively, to obtain the following corollary:

Corollary 4.7. Let $\phi(\xi,\eta)$ and $h(\xi,\eta)$ satisfy the conditions of Lemma 4.6. Then we have the following:

$$\left|\left|\langle\xi\rangle^2\int_t^\infty\int_{\mathbb{R}^2}e^{is\phi(\xi,\eta)}h(\xi,\eta)(\xi-\eta)\hat{f}(\xi-\eta)\eta\hat{g}(\eta)d\eta ds\right|\right|_{L^2_\xi}\lesssim\frac{1}{t}||f||_Y||g||_Y.$$

Now that we have established a $\langle t\rangle^{-1}$ decay away from the frequency space resonance set, we will move on to establishing this decay away from the time resonance set. This will be accomplished by the use of integration in time, the Coifman-Meyer Theorem, and the following lemma:

Lemma 4.8. Let $h(\xi,\eta)$ be a smooth, bounded function. Suppose that for $0\leq\gamma_1,\gamma_2\leq 5$ we have

$$|\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}h(\xi,\eta)|\lesssim\langle\xi\rangle^5+\langle\eta\rangle^5.$$

Then for $j=0,1,2$

$$\left|\left|\langle\xi\rangle^3\int_{\mathbb{R}^2}e^{it\phi_j(\xi,\eta)}h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta\right|\right|_{L^2_\xi}\lesssim\frac{1}{t}||f||_{W_x^{15,1}}||g||_{H_x^{15}}.\tag{4.13}$$

Proof. Let K denote the LHS of (4.13). Then by Plancheral, we see

$$K=||T_h(e^{\frac{1}{2}it\Delta}f,e^{itD_j}g)||_{H_x^3}.$$

Here, T_h is a bilinear operator with symbol $h(\xi,\eta)$ and D_0,D_1, and D_2 are the operators $\langle\nabla\rangle$, $-\langle\nabla\rangle$, and $-\frac{1}{2}\Delta$ respectively. More precisely:

$$\widehat{T_h(f_1, f_2)}(\xi) = \int h(\xi,\eta)\hat{f}_1(\xi-\eta)\hat{f}_2(\eta)d\eta.$$

Rewriting, we have

$$\begin{aligned} T_h(e^{\frac{1}{2}it\Delta}f, e^{itD_j}g) &= \int \frac{h(\xi,\eta)}{\langle\eta\rangle^{12}\langle\xi-\eta\rangle^{12}} \widehat{e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}f}(\xi-\eta)\widehat{e^{itD_j}\langle\nabla\rangle^{12}g}(\eta)d\eta. \\ &= T_{\tilde{h}}(e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}f, e^{itD_j}\langle\nabla\rangle^{12}g), \end{aligned}$$

where

$$\tilde{h}(\xi,\eta) = \frac{h(\xi,\eta)}{\langle\eta\rangle^{12}\langle\xi-\eta\rangle^{12}}.$$

Therefore,

$$\begin{aligned} K &= ||T_h(e^{\frac{1}{2}it\Delta}f, e^{itD_j}g)||_{H^3_x} \\ &\lesssim \sum_{0\le\alpha\le3} \left|\left|\xi^\alpha \int \tilde{h}(\xi,\eta)\widehat{e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}f}(\xi-\eta)\widehat{e^{itD_j}\langle\nabla\rangle^{12}g}(\eta)d\eta\right|\right|_{L^2_\xi} \\ &\lesssim \sum_{|\alpha'|\le3}\sum_{(\alpha_1,\alpha_2)=\alpha'} \left|\left|(\xi-\eta)^{\alpha_1}\eta^{\alpha_2}\int \tilde{h}(\xi,\eta)\langle\xi-\eta\rangle^{12}\widehat{e^{\frac{1}{2}it\Delta}f}(\xi-\eta)\langle\eta\rangle^{12}\widehat{e^{itD_j}g}(\eta)d\eta\right|\right|_{L^2_\xi} \\ &\lesssim \sum_{|\alpha'|\le3}\sum_{(\alpha_1,\alpha_2)=\alpha'} ||T_{\tilde{h}}(e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}\partial^{\alpha_1}f, e^{itD_j}\langle\nabla\rangle^{12}\partial^{\alpha_2}g)||_{L^2_x}. \end{aligned}$$

Because we have assumed that, for $0 < \gamma_1, \gamma_2 \le 5$,

$$|\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}h(\xi,\eta)| \lesssim \langle\xi\rangle^5 + \langle\eta\rangle^5,$$

$\tilde{h}$ is a Coifman-Meyer multiplier. Hence, using the Coifman-Meyer multiplier theorem and the dispersive estimate for the Schrődinger's fundamental solution, we have

$$\sum_{0\le\alpha_1,\alpha_2\le 3}\left|\left|T_{\tilde{h}}(e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}\partial^{\alpha_1}f, e^{itD_j}\langle\nabla\rangle^{12}\partial^{\alpha_2}g)\right|\right|_{L^2_x} \lesssim \sum_{0\le\alpha_1\le 3}\left|\left|e^{\frac{1}{2}it\Delta}\langle\nabla\rangle^{12}\partial^{\alpha_1}f\right|\right|_{\infty}\left|\left|g\right|\right|_{H^{15}}$$
$$\lesssim \frac{1}{t}||\langle\nabla\rangle^{15}f||_{L^1_x}||g||_{H^{15}}.$$

$\square$

With Lemma 4.8, we may now show how away from the time resonance set, one may use integration in time and the Coifman-Meyer Theorem to obtain a $\langle t\rangle^{-1}$ decay in H^3.

Lemma 4.9 (Decay Away Time Resonance)**.** Let $h(\xi,\eta)$ satisfy the bounds

$$|\partial_\xi^\alpha\partial_\eta^\beta h(\xi,\eta)| \lesssim 1,$$

for $0\le\alpha,\beta\le 5$. Furthermore, suppose $\phi_j(\xi,\eta)\gtrsim 1$ for $j=0,1$ or 2. Then for $f,g\in Y$ we have

$$\left|\left|\langle\xi\rangle^3\int_t^\infty\int_{\mathbb{R}^2}e^{is\phi_j(\xi,\eta)}h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta ds\right|\right|_{L^2_\xi} \lesssim \frac{1}{t}||f||_Y||g||_Y.$$

Remark: For our purposes, $h(\xi,\eta)$ will typically be a smooth cutoff function.

Proof. Integration in time yields

$$\left|\int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)} h(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta ds\right| \lesssim \lim_{M\to\infty}\left|\int_{\mathbb{R}^2} e^{iM\phi_j(\xi,\eta)}\frac{h(\xi,\eta)}{\phi_j(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta\right| \tag{4.14}$$

$$+\left|\int_{\mathbb{R}^2} e^{it\phi_j(\xi,\eta)}\frac{h(\xi,\eta)}{\phi_j(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta\right|. \tag{4.15}$$

We wish to control both of these terms by using Lemma 4.8. Hence, we must verify that for $0 \leq \gamma_1, \gamma_2 \leq 5$,

$$\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}\left(\frac{h(\xi,\eta)}{\phi_j(\xi,\eta)}\right) \lesssim \langle\xi\rangle^5 + \langle\eta\rangle^5.$$

By the bounds on $h(\xi,\eta)$ and $\phi_j(\xi,\eta)$, we have

$$|\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}(h(\xi,\eta)\cdot\phi_j^{-1}(\xi,\eta))| \lesssim \sum_{0\leq\gamma_1,\gamma_2\leq 5} |\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}(\phi_j^{-1}(\xi,\eta))|.$$

One can verify that for a mulit-index γ with $|\gamma| \geq 2$, $|\partial^\gamma\phi_j(\xi,\eta)| \lesssim 1$. Hence,

$$\begin{aligned}\sum_{0\leq\gamma_1,\gamma_2\leq 5} |\partial_\xi^{\gamma_1}\partial_\eta^{\gamma_2}(\phi_j^{-1}(\xi,\eta))| &\lesssim 1 + |\partial_\xi\phi_j(\xi,\eta)|^5 + |\partial_\eta\phi_j(\xi,\eta)|^5 \\ &\lesssim 1 + |\eta|^5 + |\xi|^5.\end{aligned}$$

Thus, we may use Lemma 4.8, to conclude

$$||\langle\xi\rangle^3(4.15)||_{L^2_\xi} \lesssim \frac{1}{t}||f||_{W^{15,1}_x}||g||_{H^{15}_x}.$$

Similarly,

$$||(4.14)||_{L^2_\xi} \lesssim \lim_{M\to\infty}\frac{1}{M}||f||_{W^{12,1}_x}||g||_{H^{12}_x} = 0.$$

Furthermore, one can verify that

$$||(4.14)||_{H^2_\xi} \lesssim \sum_{|\alpha|\le 2}||x^\alpha f||_{H^5_x}||g||_{H^5_x} \lesssim ||f||_Y||g||_Y.$$

By Sobloev embedding, (4.14) is then an L^∞_ξ function whose L^2_ξ-norm is zero. Hence, it is identically zero for all ξ. $\square$

With these lemmas in place, we are now ready to establish the necessary decay on $B_1(u_1,h_1)$ and $B_2(u_1,u_1)$. Recall that u_1 and h_1 are the linear evolutions of u and h respectively and that, in frequency space, $B_1(u_1,h_1)$ and $B_2(u_1,u_1)$ have the form

$$B_1(u_1,h_1) = \int_t^\infty\int_{\mathbb{R}^2} e^{is\phi_j(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{g}_j(\eta)d\eta ds$$
$$B_2(u_1,u_1) = \int_t^\infty\int_{\mathbb{R}^2} e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds,$$

where $g_0(\xi)=\bar{\hat{h}}_+(-\xi)$ and $g_1(\xi)=\hat{h}_+(\xi)$.

The principle idea is to identify the space and time resonance sets of each phase function ϕ_j. Away from the space resonance set, we can establish the $\langle t\rangle^{-1}$ decay using

Lemma 4.6; away from the time resonance set, we use Lemma 4.9. Points where both ϕ_j and $\partial_\eta \phi_j$ are identically zero, are in the resonance set. We avoid difficulties at these points by placing proper assumptions on the support of $\hat{u}_+$ and $\hat{h}_+$.

For the purposes of separating the integration regime based on resonance sets, we will make use of some smooth function $\varphi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ with the property that $\varphi(r) = 1$ for $0 \leq r$ and $\varphi(r) = 0$ for $r > 2$. Also, recall that by Lemma 4.5, we need only establish decay for $t \geq 1$. Hence, it is enough to show the H^3 norms of $B_1(u_1, h_1)$ and $B_2(u_1, u_1)$ have a t^{-1} decay.

4.3.2 Analysis of the Phase Function Φ_0

We first estimate $||\langle\xi\rangle^3 \mathcal{F}(B_1(u_1, h_1)||_{L^2_\xi}$. We begin by considering the case when the phase function is ϕ_0. In this case, one can verify, that the set of (ξ, η) for which $\phi_0(\xi, \eta) = 0$ is disjoint from the set for which $\partial_\eta \phi_0(\xi, \eta) = 0$. Thus, with carefully chosen cutoff functions, we can divide the space into several regimes and integrate by parts on every regime in either time or frequency space.

Proposition 4.10. Let

$$\phi_0(\xi, \eta) = \frac{1}{2}|\xi|^2 - \frac{1}{2}|\xi - \eta|^2 + \langle\eta\rangle.$$

Then we have the following:

$$\left|\left|\langle\xi\rangle^3 \int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_0(\xi,\eta)} \hat{u}_+(\xi - \eta)\bar{\hat{h}}_+(-\eta) d\eta ds\right|\right|_{L^2_\xi} \lesssim \frac{1}{t}||u_+||_Y||h_+||_Y.$$

Proof. Dividing the integration regime, we have

$$\int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_0(\xi,\eta)}\hat{u}_+(\xi-\eta)\bar{\hat{h}}_+(-\eta)d\eta ds \tag{4.16}$$

$$= \varphi(A|\xi|)\int_t^\infty \int_{\mathbb{R}^2} \varphi(2|\eta|)e^{is\phi_0(\xi,\eta)}\hat{u}_+(\xi-\eta)\bar{\hat{h}}_+(-\eta)d\eta ds \tag{4.17}$$

$$+ \varphi(A|\xi|)\int_t^\infty \int_{\mathbb{R}^2} [1-\varphi(2|\eta|)]e^{is\phi_0(\xi,\eta)}\hat{u}_+(\xi-\eta)\bar{\hat{h}}_+(-\eta)d\eta ds \tag{4.18}$$

$$+ [1-\varphi(A|\xi|)]\int_t^\infty \int_{\mathbb{R}^2} \varphi(B|\eta-\eta_0|)e^{is\phi_0(\xi,\eta)}\hat{u}_+(\xi-\eta)\bar{\hat{h}}_+(-\eta)d\eta ds \tag{4.19}$$

$$+ [1-\varphi(A|\xi|)]\int_t^\infty \int_{\mathbb{R}^2} [1-\varphi(B|\eta-\eta_0|)]e^{is\phi_0(\xi,\eta)}\hat{u}_+(\xi-\eta)\bar{\hat{h}}_+(-\eta)d\eta ds, \tag{4.20}$$

where A and B are large constants, and for each ξ, the point η_0 is the unique point in $\mathbb{R}^2$ such that

$$\xi = \eta_0\left(1 - \frac{1}{\langle\eta_0\rangle}\right).$$

Note that since

$$\partial_\eta\phi_0(\xi,\eta) = \xi - \eta\left(1 - \frac{1}{\langle\eta\rangle}\right),$$

we have that for a fixed ξ, η_0 is the unique point for which $\partial_\eta\phi_0(\xi,\eta) = 0$.

We begin our estimate with (4.17) where $|\xi| \le 2A^{-1}$. On this regime, we have that $|\eta| \leq 1$. Hence,

$$\begin{aligned}|\phi_0(\xi,\eta)| &\geq \langle \eta \rangle - |\xi||\eta| - \frac{1}{2}|\eta|^2 \\ &\geq 1 - \frac{2}{A} - \frac{1}{2} \\ &\geq \frac{1}{4}.\end{aligned}$$

We therefore apply Lemma 4.9 to obtain

$$||\langle \xi \rangle^3 (4.17)||_{L^2_\xi} \lesssim \frac{1}{t}||u_+||_Y ||h_+||_Y.$$

We next consider (4.18). For this term, we have that $|\xi| \leq 2A^{-1}$ and we are integrating over a regime where $|\eta| \geq 1/2$. Thus

$$\begin{aligned}|\partial_\eta \phi_0(\xi,\eta)| &= \left| \xi - \eta \left(1 - \frac{1}{\langle \eta \rangle}\right) \right| \\ &\geq |\eta| \left(1 - \frac{1}{\langle \eta \rangle}\right) - |\xi| \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\langle \frac{1}{2} \rangle}\right) - \frac{2}{A} \\ &\geq \frac{1}{100}.\end{aligned}$$

Using Lemma 4.6 with $\alpha_1, \alpha_2 = 0$ and $N = 2$ yields

$$||\langle \xi \rangle^3 (4.18)||_{L^2_\xi} \lesssim \frac{1}{t}||u_+||_Y ||h_+||_Y.$$

For (4.19), we seek an absolute lower bound on $|\phi_0(\xi,\eta)|$ for the purpose of using Lemma 4.9. Let θ be the angle between η and η_0. By making B large relative to A, we may assume $0 \leq \cos\theta$. Now, we consider two cases:

Case 1: $|\eta| \leq 2$.

Then we have

$$\begin{aligned}|\phi_0(\xi,\eta)| &\geq \langle\eta\rangle - \frac{1}{2}|\eta|^2 \\ &\geq \langle\eta\rangle - |\eta|.\end{aligned}$$

On the closed ball $B(0,2)$, $\langle\eta\rangle - |\eta|$ is a strictly positive function. Thus, it attains an absolute minimum greater than zero. Hence $|\phi_0(\xi,\eta)| \geq \delta$.

Case 2: $|\eta| > 2$.

Enlarging B we may assume $\eta_0 \geq 1.99$ and $|\eta_0| \geq .99|\eta|$. This gives

$$\begin{aligned}|\phi_0(\xi,\eta)| &\geq |\eta_0|\left(1 - \frac{1}{\langle\eta_0\rangle}\right)|\eta|\cos\theta - \frac{1}{2}|\eta|^2 \\ &\geq |\eta_0|\left(1 - \frac{1}{\langle 1.99\rangle}\right)|\eta|\cos\theta - \frac{1}{2}|\eta|^2 \\ &\geq .53|\eta_0||\eta|\cos\theta - \frac{1}{2}|\eta|^2 \\ &\geq |\eta|(.52\cos\theta - .5).\end{aligned}$$

Now, because $|\eta|$ has a lower bound, if necessary, we may again enlarge B so that $\cos\theta \geq .99$. This gives $|\phi_0(\xi,\eta)| \geq \frac{1}{5}$. With an absolute lower bound on $|\phi_0(\xi,\eta)|$, we may now use Lemma 4.9 to conclude

$$||\langle\xi\rangle^3(4.19)||_{L^2_\xi} \lesssim \frac{1}{t}||u_+||_Y||h_+||_Y.$$

Finally, for (4.20), we wish to show $|\partial_\eta\phi_0(\xi,\eta)| \gtrsim 1$, with the intent of using Lemma 4.6. For this purpose, set θ as the angle between η_0 and η. We consider several cases:

Case1: $|\theta| \geq \pi/2$.

Then

$$|\partial_\eta\phi_0(\xi,\eta)| \geq |\xi| \geq \frac{1}{A}.$$

Case2: $|\eta| \leq \frac{1}{2}|\xi|$.

Then

$$|\partial_\eta\phi_0(\xi,\eta)| \geq |\xi| - |\eta| \geq \frac{1}{2A}.$$

Case 3: $|\eta| \geq \frac{1}{2}|\xi|, \quad 0 \leq \theta < \pi/2$.

By our assumption on the distance between η and η_0, we have

$$B^{-2} \leq |\eta_0|^2 + |\eta|^2 - 2|\eta_0||\eta|\cos\theta.$$

Thus, we can bound $\cos\theta$ by

$$0 \leq \cos\theta \leq \frac{|\eta_0|^2 + |\eta|^2 - \epsilon}{2|\eta_0||\eta|},$$

for $\epsilon = B^{-2}$. Hence

$$
\begin{aligned}
&|\partial_\eta \phi_0(\xi,\eta)|^2 \\
=&|\eta_0|^2\left(1-\frac{1}{\langle\eta_0\rangle}\right)^2 + |\eta|^2\left(1-\frac{1}{\langle\eta\rangle}\right)^2 - 2|\eta_0||\eta|\left(1-\frac{1}{\langle\eta_0\rangle}\right)\left(1-\frac{1}{\langle\eta\rangle}\right)\cos\theta \\
\gtrsim&|\eta_0|^2\left(1-\frac{1}{\langle\eta_0\rangle}\right)^2 + |\eta|^2\left(1-\frac{1}{\langle\eta\rangle}\right)^2 - \left(1-\frac{1}{\langle\eta_0\rangle}\right)\left(1-\frac{1}{\langle\eta\rangle}\right)\left(|\eta_0|^2+|\eta|^2-\epsilon\right) \\
=&\left(\frac{1}{\langle\eta\rangle}-\frac{1}{\langle\eta_0\rangle}\right)\left[|\eta_0|^2\left(1-\frac{1}{\langle\eta_0\rangle}\right) - |\eta|^2\left(1-\frac{1}{\langle\eta\rangle}\right)\right] \qquad (4.21)\\
&+\epsilon\left(1-\frac{1}{\langle\eta_0\rangle}\right)\left(1-\frac{1}{\langle\eta\rangle}\right). \qquad (4.22)
\end{aligned}
$$

Since $|x| \geq |y|$ implies $\frac{1}{\langle y\rangle} \geq \frac{1}{\langle x\rangle}$, the term (4.21) is a product of two numbers with the same sign. Thus,

$$
\begin{aligned}
|\partial_\eta \phi_0(\xi,\eta)| &\gtrsim \epsilon\left(1-\frac{1}{\langle\eta_0\rangle}\right)\left(1-\frac{1}{\langle\eta\rangle}\right) \\
&\gtrsim \epsilon\left(1-\frac{1}{\langle A^{-1}\rangle}\right)\left(1-\frac{1}{\langle\frac{1}{2A}\rangle}\right) \\
&\gtrsim 1.
\end{aligned}
$$

Hence, we may use Lemma 4.6 with $\alpha_1 = \alpha_2 = 0$ and $N = 2$ to obtain

$$
||\langle\xi\rangle^3(4.20)||_{L^2_\xi} \lesssim \frac{1}{t}||u_+||_Y||h_+||_Y.
$$

□

4.3.3 Analysis of the Phase Function Φ_1

We now consider the case where the phase function is ϕ_1. In this case, there is a set $A \subset \mathbb{R}^2 \times \mathbb{R}^2$ on which $\phi_1(\xi, \eta)$ and $\partial_\eta \phi_1(\xi, \eta)$ are both zero. On this set, we can not integrate by parts in either frequency space or time. Instead, we place appropriate assumptions on the final data so that $B_1(u_1, h_1)$ is identically zero around A.

Proposition 4.11. Let

$$\phi_1(\xi, \eta) = \frac{1}{2}|\xi|^2 - \frac{1}{2}|\xi - \eta|^2 - \langle \eta \rangle.$$

Then the resonance set of ϕ_1 is non-empty and contained in the set A_δ defined in the introduction. If

$$\hat{u}_+(\xi - \eta)\hat{h}_+(\eta) \equiv 0 \quad \text{on} \quad A_\delta,$$

then

$$\left|\left| \langle \xi \rangle^3 \int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_1(\xi,\eta)} \hat{u}_+(\xi - \eta)\hat{h}_+(\eta) d\eta ds \right|\right|_{L^2_\xi} \lesssim \frac{1}{t} ||u_+||_Y ||h_+||_Y.$$

Proof. Note that

$$\partial_\eta \phi_1(\xi, \eta) = \xi - \eta \left(1 + \frac{1}{\langle \eta \rangle} \right).$$

Recall that A_δ is defined as

$$A_\delta = \{(\xi, \eta) \mid r_\xi - \delta \le |\xi| \le r_\xi + \delta, \ r_\eta - \delta \le |\eta| \le r_\eta + \delta\},$$

where the ordered pair (r_ξ, r_η) is the unique solution on $\mathbb{R}^+$ to the system

$$\begin{aligned} \tfrac{1}{2}r_\xi^2 - \tfrac{1}{2}(r_\xi - r_\eta)^2 - \langle r_\eta\rangle &= 0 \\ r_\xi - r_\eta\left(1 + \tfrac{1}{\langle r_\eta\rangle}\right) &= 0. \end{aligned}$$

From the form of $\partial_\eta\phi_1(\xi,\eta)$ and the definition of r_ξ and r_η, one can see that if (ξ_R, η_R) is a resonance point of $\phi_1(\xi,\eta)$ then ξ is co-linear with η, $|\xi_R| = r_\xi$, and $|\eta_R| = r_\eta$.

Let ψ_1 be a smooth cutoff function on $\mathbb{R}$ supported on $[r_\xi - \delta/2, r_\xi + \delta/2]$ and congruent to 1 on $[r_\xi - \delta/4, r_\xi + \delta/4]$. Similarly, let ψ_2 be a smooth cutoff function on $\mathbb{R}$ supported on $[r_\eta - \delta/2, r_\eta + \delta/2]$ and congruent to 1 on $[r_\eta - \delta/4, r_\eta + \delta/4]$. For each ξ, we also define η_0 as the unique point in $\mathbb{R}^2$ such that

$$\xi = \eta_0\left(1 + \frac{1}{\langle\eta_0\rangle}\right).$$

Decomposing, we have

$$\int_t^\infty \int_{\mathbb{R}^2} e^{is\phi_1(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{h}_+(\eta)d\eta ds \tag{4.23}$$

$$= \int_t^\infty \int_{\mathbb{R}^2} [1 - \varphi(B|\eta-\eta_0|)]e^{is\phi_1(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{h}_+(\eta)d\eta ds \tag{4.24}$$

$$+ \int_t^\infty \int_{\mathbb{R}^2} \varphi(B|\eta-\eta_0|)[1 - \psi_1(|\xi|)\psi_2(|\eta|)]e^{is\phi_1(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{h}_+(\eta)d\eta ds \tag{4.25}$$

$$+ \int_t^\infty \int_{\mathbb{R}^2} \varphi(B|\eta-\eta_0|)\psi_1(|\xi|)\psi_2(|\eta|)e^{is\phi_1(\xi,\eta)}\hat{u}_+(\xi-\eta)\hat{h}_+(\eta)d\eta ds. \tag{4.26}$$

We begin our estimates with (4.24). On its regime, we have

$$\begin{aligned}|\partial_\eta \phi_1(\xi,\eta)| &= \left|\eta_0\left(1+\frac{1}{\langle\eta_0\rangle}\right) - \eta\left(1+\frac{1}{\langle\eta\rangle}\right)\right| \\ &\geq |\eta-\eta_0| \\ &\geq \frac{1}{B}.\end{aligned}$$

Hence we may use Lemma 4.6 with $\alpha_1, \alpha_2 = 0$ and $N = 2$ to establish

$$||\langle\xi\rangle^3(4.24)||_{L^2_\xi} \lesssim \frac{1}{t}||u_+||_Y||h_+||_Y.$$

We now bound (4.25). To this end, we first consider the values of ϕ_1 at the stationary phase points η_0. More precisely, we will first establish a lower bound on $\phi_1(\xi,\eta_0)$. If $|\xi| > 100$, then we have

$$\begin{aligned}|\phi_1(\xi,\eta_0)| &= |\xi\cdot\eta_0 - \frac{1}{2}|\eta_0| - \langle\eta_0\rangle| \\ &= |\eta_0|^2\left(1+\frac{1}{\langle\eta_0\rangle}\right) - \frac{1}{2}|\eta_0|^2 - \langle\eta_0\rangle \\ &\geq \frac{1}{2}|\eta_0|^2 - \langle\eta_0\rangle \\ &\geq 1.\end{aligned}$$

In the case of $|\xi| \leq 100$, we claim that by enlarging B relative to δ, we may assume

$$0 \leq |\xi| \leq r_\xi - \delta/100,$$

or

$$r_\xi + \delta/100 \leq |\xi| \leq 100.$$

To see this, assume for contradiction that some ξ with

$$|\xi| \in [r_\xi + \delta/100, r_\xi - \delta/100]$$

is in our integration regime. Then we also have that

$$|\eta_0| \in [r_\eta - \delta/20, r_\eta + \delta/20].$$

Hence, by enlarging B, we may use the term $\varphi(B|\eta - \eta_0|)$ to assume that

$$|\eta| \in [r_\eta - \delta/10, r_\eta + \delta/10].$$

But this point must then lie outside our integration regime as $1 - \psi_1(|\xi|)\psi_2(|\eta|) = 0$. Thus, if $|\xi| \leq 100$, then we may also assume that

$$|\xi| \notin [r_\xi - \delta/100, r_\xi + \delta/100].$$

Furthermore, we know that either $\phi_1(\xi, \eta)$ or $\partial_\eta \phi_1(\xi, \eta)$ is non-zero at a given point on the regime of (4.25). Thus, $|\phi_1(\xi, \eta_0)| > 0$ for all relevant ξ. We then have that $\phi_1(\xi, \eta_0)$ is a function in ξ that is continuous and non-zero. This implies that on the compact set

$$[0, r_\xi - \delta/100] \cup [r_\xi + \delta/100, 100],$$

it must attain some absolute lower bound, δ_0. We also have that

$$|\partial_\eta \phi_1(\xi, \eta)| \leq |\eta_0 - \eta| + \left| \frac{\eta_0}{\langle \eta_0 \rangle} - \frac{\eta}{\langle \eta \rangle} \right| \leq 3.$$

Then, by the Fundamental Theorem of Calculus,

$$|\phi_1(\xi, \eta)| \geq \delta_0 - 3|\eta - \eta_0|.$$

Enlarging B, we get that $|\phi_1(\xi, \eta)| \geq \delta_0/2$. Now we may use Lemma 4.9 to obtain

$$||\langle \xi \rangle^3 (4.25)||_{L^2_\xi} \lesssim \frac{1}{t} ||u_+||_Y ||h_+||_Y.$$

Finally, (4.26) is identically zero by the assumption that $\hat{u}_+(\xi - \eta)\hat{h}_+(\eta) \equiv 0$ on A_δ.

□

4.3.4 Analysis of the Phase Function Φ_2

Finally, we estimate $||\langle \xi \rangle^3 \mathcal{F}(B_2(u_1, u_1))||_{L^2_\xi}$.

Proposition 4.12. Let

$$\phi_2(\xi, \eta) = \langle \xi \rangle - \frac{1}{2}|\xi - \eta|^2 + \frac{1}{2}|\eta|^2.$$

Then we have the following:

$$\left|\left|\langle\xi\rangle^2\int_t^\infty\int_{\mathbb{R}^2}e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\right|\right|_{L^2_\xi}\lesssim\frac{1}{t}||u_+||_Y^2.$$

Proof. We decompose as follows:

$$\left|\int_t^\infty\int_{\mathbb{R}^2}e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\right| \tag{4.27}$$

$$\lesssim\left|\int_t^\infty\int_{\mathbb{R}^2}e^{is\phi_2(\xi,\eta)}(1-\varphi(As^{1/3}|\xi|))(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\right| \tag{4.28}$$

$$+\left|\int_t^\infty\varphi(As^{1/3}|\xi|)\int_{\mathbb{R}^2}\varphi\left(\frac{|\eta|}{s^{1/3}}\right)e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\right| \tag{4.29}$$

$$+\left|\int_t^\infty\varphi(As^{1/3}|\xi|)\int_{\mathbb{R}^2}\left(1-\varphi\left(\frac{|\eta|}{s^{1/3}}\right)\right)e^{is\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\right|. \tag{4.30}$$

We first consider (4.28) where we have

$$|\xi|\geq\frac{1}{As^{1/3}}.$$

Since $\partial_\eta\phi_2(\xi,\eta)=\xi$ we may use Corollary 4.7 with $\alpha_1=1/3$, $\alpha_2=0$, and index $N=6$ to obtain

$$||\langle\xi\rangle^2(4.28)||_{L^2_\xi}\lesssim\frac{1}{t}||u_+||_Y^2.$$

We now move on to estimating (4.29), where we have

$$|\xi| \leq \frac{2}{As^{1/3}},$$

and we are integrating over a regime in which

$$|\eta| \leq 2s^{1/3}.$$

Therefore, we have the following bound on $|\phi_2(\xi,\eta)|$:

$$\begin{aligned}|\phi_2(\xi,\eta)| &\geq \langle\xi\rangle - \frac{1}{2}|\xi|^2 - |\xi\cdot\eta| \\ &\geq 1 - \frac{1}{4} - \frac{4}{A} \\ &\geq \frac{1}{2}.\end{aligned}$$

Though $\phi_2(\xi,\eta)$ has an absolute lower bound, we may not directly use Lemma 4.9, as our symbol h is a function of time. Instead, we integrate by parts in time to obtain

$$(4.29) \leq \left|\varphi(At^{1/3}|\xi|)\int_{\mathbb{R}^2} e^{it\phi_2(\xi,\eta)}\frac{\varphi(|\eta|t^{-1/3})}{\phi_2(\xi,\eta)}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta\right| \tag{4.31}$$

$$+\left|\int_{\mathbb{R}^2}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)\int_t^\infty \frac{e^{is\phi_2(\xi,\eta)}}{\phi_2(\xi,\eta)}\partial_s\Big(\varphi(As^{1/3}|\xi|)\varphi(|\eta|s^{-1/3})\Big)dsd\eta\right|. \tag{4.32}$$

Let $\psi(r)$ be a smooth function defined for $r \geq 0$ such that $\psi(r) = 0$ for $r < 1/4$

and $\psi(r) = 1$ for $r \geq 1/2$. Then we have

$$(4.31) \leq \left| \int_{\mathbb{R}^2} e^{it\phi_2(\xi,\eta)} \frac{\psi(|\phi_2(\xi,\eta)|)\varphi(|\eta|t^{-1/3})}{\phi_2(\xi,\eta)} (\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta \right|.$$

Using Lemma 4.8, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^2} e^{it\phi_2(\xi,\eta)} \frac{\psi(|\phi_2(\xi,\eta)|)\varphi(|\eta|t^{-1/3})}{\phi_2(\xi,\eta)} (\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta \right| \\ \lesssim \frac{1}{t}||\nabla u_+||_{W^{15,1}}||\nabla u_+||_{H_x^{15}} \\ \lesssim \frac{1}{t}||u_+||_Y^2. \end{aligned}$$

For (4.32), integrating by parts again yields:

$$\begin{aligned} &(4.32) \\ &\lesssim \left| \int_{\mathbb{R}^2} \frac{e^{it\phi_2(\xi,\eta)}}{(\phi_2(\xi,\eta))^2} \partial_t \Big(\varphi(A|\xi|t^{1/3})\varphi(|\eta|t^{-1/3}) \Big) (\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta \right| \quad (4.33) \\ &+ \left| \int_{\mathbb{R}^2} (\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta) \int_t^\infty \frac{e^{is\phi_2(\xi,\eta)}}{(\phi_2(\xi,\eta))^2} \partial_s^2 \Big(\varphi(As^{1/3}|\xi|)\varphi(|\eta|s^{-1/3}) \Big) ds d\eta \right|. \\ &\qquad (4.34) \end{aligned}$$

Calculating, we have

$$
\begin{aligned}
&(4.33) \\
&\lesssim \left| \int_{\mathbb{R}^2} \frac{e^{it\phi_2(\xi,\eta)}}{(\phi_2(\xi,\eta))^2} |\xi| t^{-2/3} \varphi'(A|\xi|t^{1/3}) \varphi(|\eta|t^{-1/3}))(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta \right| \\
&+ \left| \int_{\mathbb{R}^2} \frac{e^{it\phi_2(\xi,\eta)}}{(\phi_2(\xi,\eta))^2} \varphi(A|\xi|t^{1/3})|\eta|t^{-4/3}\varphi'(|\eta|t^{-1/3}))(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta \right| \\
&\lesssim \int_{\mathbb{R}^2} \left(|\xi| t^{-2/3}\varphi'(A|\xi|t^{1/3}) + |\eta|t^{-4/3}\varphi'(|\eta|t^{-1/3}) \right) |(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)| d\eta \\
&\lesssim \frac{1}{t} \int_{\mathbb{R}^2} |(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)| d\eta \\
&\lesssim \frac{1}{t\langle\xi\rangle^4} ||u_+||_{H_x^5}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(4.34) &\lesssim \int_{\mathbb{R}^2} |(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)| \int_t^\infty \frac{1}{s^2} ds d\eta \\
&\lesssim \frac{1}{t\langle\xi\rangle^4} ||u_+||_{H_x^5}^2.
\end{aligned}
$$

Finally, we bound (4.30) where

$$
|\xi| \leq \frac{2}{As^{1/3}},
$$

and we are integrating over a regime on which

$$
|\eta| \geq s^{1/3}.
$$

For (4.30), we get the decay by assuming high regularity on u_+. More precisely,

$$
\begin{aligned}
&||\langle\xi\rangle^3(4.30)||_{L^2_\xi}\\
&\lesssim \left|\left|\langle\xi\rangle^2\int_t^\infty\int_{\mathbb{R}^2}\left(1-\varphi\left(\frac{|\eta|}{s^{1/3}}\right)\right)e^{is\phi_2(\xi,\eta)}\frac{\langle\eta\rangle^6}{\langle\eta\rangle^6}(\xi-\eta)\hat{u}_+(\xi-\eta)\eta\bar{\hat{u}}_+(-\eta)d\eta ds\right|\right|\\
&\lesssim \left|\left|\langle\xi\rangle^2\int_t^\infty\frac{1}{s^2}\int_{\mathbb{R}^2}\langle\eta\rangle^7|(\xi-\eta)\hat{u}_+(\xi-\eta)\bar{\hat{u}}_+(-\eta)|d\eta ds\right|\right|_{L^2_\xi}\\
&\lesssim \left|\left|\langle\xi\rangle^2\int_t^\infty\frac{1}{s^2\langle\xi\rangle^4}||u_+||_{H^{11}_x}||u_+||_{H^5_x}ds\right|\right|_{L^2_\xi}\\
&\lesssim \frac{1}{t}||u_+||^2_{H^{11}_x}.
\end{aligned}
$$

$\square$

REFERENCES

[1] R. Coifman, Y. Meyer, Au delà des opérateurs pseudo-différentiels. Astérisque, **57** (1978), 1-185

[2] P. Germain, Global existence for coupled Klein-Gordon equations with different velocities. Annales de l'Institut Fourier (to appear), 2011

[3] R. Glassey, On the asymptotic behavior of nonlinear wave equations. Trans. Amer. Math. Soc., **182** (1973), 187-200

[4] L. Grafakos, Z. Si, The Hörmander multiplier theorem for multilinear operators. Crelles Journal (to appear)

[5] A. Grünrock and H. Pecher, Bounds in time for the Klein-Gordon-Schrődinger and the Zakharov system. Hokkaido Math. J. **35** (2006), no. 1, 139-153

[6] N. Hayashi and P. Naumkin, Scattering Operator for nonlinear Klein-Gordon equations. Commun. Contemp. Math, **11** (2009), no. 5, 771-781

[7] J. Jang, D. Li, and X. Zhang, Smooth global solutions for the two dimensional Euler Poisson system. Preprint arXiv:1109.3882v2

[8] E. Lenzmann, Well-posedness for semi-relativistic Hartree equations of critical type. Math. Phys. Anal. Geom. **10** (2007), 43-64.

[9] T. Ozawa and Y. Tsutusmi, Asymptotic behavior of solutions for the coupled Klein-Gordon-Schrődinger equations. Adv. Stud. Pure Math., **23** (1994), 295-305

[10] H. Pecher, Some new well-posedness results for the Klein-Gordon-Schrődinger System. Preprint arXiv: 1106.2116v3

[11] A. Shimomura, Scattering Theory for the coupled Klein-Gordon-Schrődinger equations in two space dimensions. J. Math. Sci. Univ. Tokyo. **10** (2003), no. 4, 661-685

[12] A. Shimomura, Wave operators for the coupled Kein-Gordon-Schrődinger equations in two space dimensions. Funkcial. Ekvac. **47** (2004), no. 1, 63-82

[13] A. Shimomura, Scattering Theory for the coupled Klein-Gordon-Schrődinger equations in two space dimensions II. Hokkaido Math. J. **34** (2005), no. 2, 405-433

[14] E. Stein, Harmonic Analysis. Princeton mathematical series, **43**, Princeton University Press, 1993

[15] W. A. Strauss, Nonlinear Wave equations. CBMS Regional Conference Series in Math., **73**, AMS, 1989

[16] T. Tao, Nonlinear Dispersive Equations: Local and Global Analysis. American Mathematical Society, Providence, Rhode Island (2006)

[17] Y. Tsutsumi and K. Yajima, The asymptotic behavior of nonlinear Schrődinger equations. Bull. Amer. Math. Soc., **11** (1984), no. 1, 186-188

[18] N. Tzirakis, The Cauchy problem for the Klein-Gordon-Schrődinger system in low dimensions below the energy space. Comm. Partial Differential Equations, **30** (2005), no. 4-6, 605-641

[19] H. Yukawa, On the interaction of elementary particles. Proc. Phys.-Math. Soc. Japan, **17** (1935), 48-57