Path of a rotating projectile in a resisting medium

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Before Hobins performed his ballistic experiments (1742), the vaguest ideas prevailed as to the real magnitude of the air resistance and consequently of the velocity of cannon shot and musket bullets. It seemed impossible to imagine that a rare medium like air could be the means of producing as great a resistance as it does.

Early in the 17th century, Galileo deduced his famous parabolic theory on the assumption that the horizontal and vertical motion are independent of one another, although he afterward explained that the projectile would not move in the parabolic path, because air resistance had not been taken into account, and gravity did not act in parallel lines.

However it was generally assumed that his theory was correct, and that the air resistance was so slight as to be negligible, with the result that the velocity was very much underestimated. In order to further reconcile theory with practice, they supposed that the first part of the trajectory was a finite straight line, or as they called it the point blank range, through which the shot "flyeth violently."

In 1687, John Wallis published an article on the "Measure of the Air's resistance of Bodies Moved in it," in the Philosophical Transactions, but it is of no value since his treatment is based on the false presumption that the resistance is directly proportional to the "celerity."
"For in a double celerity, there is to be removed in the same time, twice as much air, which is a double impediment; in a triple time, thrice as much, and so in other proportions."

Newton solved the problem of the motion of a projectile when it was moving under gravity with a resistance proportional to the velocity, but believed from experiments that he had performed (with low velocities) that according to the law of nature, the resistance varied as the square of the velocity.

In accordance with a custom prevailing in Newton's time for great Mathematicians, in England and on the continent, to propose difficult problems to one another, Keill, Feb. 1713, sent the following to John Bernoulli, -- "Invenire curvam, quam projectile describit in aere, pro simplicissima suppositions gravitatis atque mediī densitatis uniformis, resistentiae vers in duplicata rationis velocitatis." Bernoulli not only succeeded in solving the problem as it stood, but also the more general one where the resistance varies as the nth power of the velocity and published it, together with a solution by his nephew Nicholas in Acta Erudit Lips 1719, Mai p. 216. His solution as he left it however, is of no practical value.

During the latter half of the 18th century, many eminent mathematicians, some of whom were Euler (1758) Lambert (1765), Borda, (1769), Bezout (1789), Tempelhof 1788-9), D'Ehrenmalm, (1788), Lombard 1796, Legendre, and Poisson worked on the subject but their solutions led to no definite conclusions respecting the law or the magnitude
of the resistance of the air.

No real advance was made in the science of ballistics until Robin's performed his experiments in 1742, with the ballistic pendulum. He was the first to devise means for measuring the velocities of projection satisfactorily. He calculated the amount of resistance offered by the formula \( RS = M V^2 - M V_0^2 \), using data obtained by firing at the ballistic pendulum at different distances, and calculating the loss of speed through the air. He found that the resistance offered by the air to a bullet \( \frac{3}{4} \) inches in diameter, with a weight of \( \frac{1}{2} \) lbs. is about 10 lbs. or 120 times the weight of the bullet at a velocity of 1600 \( \frac{ft}{sec} \).

Afterwards Hutton, about (1775-1783), carried on Robins' Method of experimentation, both with the ballistic pendulum and the whirling machine, making several improvements in them, and using iron projectiles of greater size. Robins came to the conclusion that the resistance is proportional to the square of the velocity, up to a velocity of 1100 feet. Concerning speeds greater than that he says, "The absolute quantity of resistance in these greater velocities is nearly three times as great as it should be, by comparison with smaller velocities." But Hutton writes that Robins' conclusions are not correct. "By more accurate experiment with cannon balls, it appears that the law of resistance begins to increase above the ratio of the square of the velocity from the very slowest motions and thence goes on increasing gradually more and
more above what is assigned by that ratio until we arrive at a velocity of 1600 or 1700, where it is at the greatest amounting in that maximum state to only 2.1 times the quantity resulting from the ratio of the square of the velocity, and at a velocity equal to 1100 ft. instead of answering to the law amounts to 1.86 times the same.

The next experiments of note were those performed at Metz (1839-40) thro' the order of the French Minister of war, by a commission of which M. Frobert, Morin, and Didion were members. Larger spherical projectiles than the bullets of Robins and Hutton, were used, those at Metz being of weights 24, 12 and 8 lbs., with diameters equal to 5.85, 4.66 and 4.06 inches, while those of Robins were but \( \frac{3}{8} \) inches in diameter, and of Hutton, but 1, 3 and 6 lbs. in weight. The results obtained did little more than confirm the conclusions of Hutton's and Robins, and show them applicable to bullets of greater diameter. Didion who made these experiments the basis of his 'Traite de Ballistique' gave a comparison of the results at Metz and those derived from the experiments of Hutton's and Robins, by a careful calculation of their observations. Hutton's experiments gave for the resistance the expression \( R^2 v^2 X 0.027 + 0.0023v \), which compared favorably with that given both the Robin's and the Metz experiments \( R^2 v^2 X 0.027 \).
The next advance was made by F. Bashforth, who points out in his "Treatise on the Motion of a Projectile," the faults of the then-accepted theory of air resistance (1) the assumption that the moving body disturbed only those particles of air, which were in its path, (2) the neglect of air condensation in front of the body as well as of the wave disturbance propagated through the air when the velocity was less than that of sound, and (3) the neglect of the form of the rear of the body which is known to affect the value of the resistance. The series of experiments that he performed in (1864-1880) comprise the only actual determination of air resistance universally employed in work in artillery, and furnish all the data from which his Ballistic tables, contained in all modern naval and foreign treatises are derived. These experiments were carried out with muzzle loading guns, but so carefully that they require only a slight modification in order to serve for our modern artillery.

Bashforth's main purpose was to determine whether the resistance of the air varied exactly as the square of the diameter of the projectile and as the cube of the velocity. It was found to vary as the diameter squared, but was such that when it was expressed under the form \(2b \times v^3\), \(2b\) attained its maximum value for a velocity of about 1200 ft and...
decreased for higher and lower velocities.

Bashforth worked on the assumption that the air resistance followed some law producing a gradual change in the velocities thus making the differences of the times taken by the shot to pass over a succession of equal spaces ultimately zero. He published his results, first in the 'Phil. Trans' 1868 p. 441, then afterwards more in detail in the Government reports in 1870. By means of screens set up at regular intervals of space (1), and an electric chronograph to record the successive instants at which the projectile passed thro' the screens, Bashforth determined and tabulated the times taken to go from one screen to another. Since the second differences of \( t_0, t_2, t_3, t_4, \ldots, t_2, t_3, t_4; \ldots \) being the times of passing each screen, are nearly constant, the equation connecting space and time may be written in the form \( t = as + bs^2 \). If \( v \) is the velocity of the shot at time \( t \) and \( f \) the retarding force,

\[
v = \frac{ds}{dt} = \frac{-1}{a + 2bs}
\]

Let \( v = V_0 \) when \( S = 0 \), \( V_0 = \frac{1}{a} \)

and \( v = \frac{1}{V_0} + \frac{2bs}{V_0} \)

\[
f = \frac{dv}{dt} = -2b \left( \frac{ds}{dt} \right) \left( \frac{1}{V_0} + 2bs \right)^2 = -2bv^3.
\]
If the weight of the shot were $W$ lbs., the resistance of the air acting upon it would be $-2bv^3\frac{w}{g}$ lbs.

In his "Treatise on the Motion of Projectiles" Bachforth gives extensive tables of the coefficient of resistance $K_v$ where $K_v$ is defined by the equation $f = \frac{d^2}{w} K_v \left(\frac{v}{1000}\right)^3$, in which $d$ is the diameter of the projectile in inches and $w$ its weight in pounds. $K_v$ is the same for similar elongated projectiles irrespective of the weight or the diameter. It varies for differently shaped heads, but only slightly for those likely to be used in practice, and remains the same whether the apex of the shot is pointed or rounded off.

The following table given by Bashforth shows the variation of the coefficient of resistance with the shape of the head.

<table>
<thead>
<tr>
<th>Form of head</th>
<th>Mean wt. in lbs.</th>
<th>Value of $2b$ when diam. = $d$ inches.</th>
<th>Value of $2b$ when diam. = $2R$ ft.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Hemispherical</td>
<td>39.34</td>
<td>0.00000077 $\frac{d^2}{w}$</td>
<td>0.000001329 $\frac{d^2}{w}$</td>
</tr>
<tr>
<td>(2) Hemispheroidal</td>
<td>38.70</td>
<td>0.0000060&quot;</td>
<td>0.000001043 &quot;</td>
</tr>
<tr>
<td>(3) Ogival (1 diam.)</td>
<td>39.56</td>
<td>0.0000064&quot;</td>
<td>0.000001114 &quot;</td>
</tr>
<tr>
<td>(4) Ogival (2 diam.)</td>
<td>38.52</td>
<td>0.0000062&quot;</td>
<td>0.000001085 &quot;</td>
</tr>
<tr>
<td>(5) &quot; (1 diam.)</td>
<td>21.81</td>
<td>0.0000063&quot;</td>
<td>0.000001097 &quot;</td>
</tr>
<tr>
<td>(6) &quot; (2 Diam.)</td>
<td>21.94</td>
<td>0.0000060&quot;</td>
<td>0.000001042 &quot;</td>
</tr>
</tbody>
</table>

The values of $K_v$ given in the tables are those given directly by experiment with the exception that those for velocities 1080-1580 are increased by 0.68% to render the density uniform throughout.
The coefficient of resistance $K_v$ in the cubic law of retardation may be easily changed to the coefficient corresponding to laws according to other powers of the velocity. For instance, the cubic law gives as the retardation $-\frac{c^2}{W} K_v \left(\frac{V}{1000}\right)^3$, while the Newtonian law gives $-\frac{c^2}{W} K_v \left(\frac{V}{1000}\right)^2$, so that $K_v = K_v \left(\frac{V}{1000}\right)$. Bashforth found it unnecessary to express the resistance according to other powers of $v$ than the cube, although the value of $K_v$ did not remain constant. Instead, for those velocities when $K_v$ varied, the trajectory could be divided into such small arcs, that for each distance, the average value of $K_v$ could be used without any serious error.

His experimental results showed that for velocities up to 850 $\frac{ft}{sec}$, the resistance varied as the square of the velocity; from 900 to 1100 $\frac{ft}{sec}$, as the sixth power, and for velocities above 1350, as the second power. M. Helie in a work dated 1865, also suggested the cubic law as the law of resistance to elongated projectiles, as a result of experiments that he had made at Gavre 1859, 1860, and 1861.

Corrections of $K_v$ for various reasons are sometimes necessary. In the 'Proc. of R. A. Inst!' 1885, Bashforth introduced certain factors of correction for $K_v$.

Corrections for differences of density are
made by a factor \( \tau \) (relative density as compared with the standard), called the coefficient of tenuity; for unsteadiness of the shot by a factor \( \delta \), and for different forms of head by a factor \( \kappa \) called the coefficient of shape.

\( \tau \) is important in long range, high angle fire. The resistance is reduced by giving the projectile a greater length and a sharper point. \( \kappa \) is not constant for spherical shot, but may be taken as on the average equal to 1.71.

\( \kappa \) for ogival head struck with a radius of two diameters is given as 0.975. The corrected value for \( K_v \) would be

\[
K_v \tau \delta \kappa.
\]

Bashforth’s tables for \( K_v \) were obtained from average results given by 3 to 9 inch shot, so he did not venture to state them applicable to the calculation of the motion of small arm projectiles. But Maj. Re Clintock R. A. Assist. Sup’t. Royal Small Arms factory (about 1883) used them in connection with such projectiles testing the accuracy of the calculation by firing a large number of rounds through paper screens erected at intervals along a 500 yds. and a 1000 yds. range. He stated the result of the experiment as being very satisfactory, the mean heights of the bullet holes in the screens agreeing closely with the heights found by calculation.

Bashforth's experimental work was concluded when he had obtained the values of \( K_v \), but from these experiments, he formed his tables of \( T \) and \( S \) (see Britannica Encyclopaedia, Exterior Ballistics). From \( T \) may be obtained the time
required for the velocity of the projectile to fall from \( V \) to \( V_0 \) and from \( S \), the space traversed during the time \( t \).

Several writers have discussed the effect of rifling on the motion of a projectile. A list of works on this subject are given by Capt. Ingalls in his "Handbook of Problems in Exterior Ballistics." Magnus in "Über die Abweichung der Geschosse," 1860 has shown how to illustrate the motion of a rifled shot with the gyroscope, and later Mallock also uses a demonstration with the gyroscope. Gen. Nayeoski published a mathematical investigation on the subject in the Revue de Technologie Militaire t. v. p. 13, 1866.

+ De l’influence du Mouvement de Rotation sur la Trajectoire des Projectiles oblongs, dans l’air.
treating the spherical projectile, according to the laws of ordinary kinetics of a particle, but neglecting gravity for the time being, we have for our equation of motion \[ S = \frac{-\mathbf{F}}{M} \].

So long as the speed of the projectile is less than that of sound, it may be assumed, as the final result of all experiment from Koblins' time up to Bashforth's that the resistance of the air to the projectile varies approximately as the square of the velocity. Kobins gave as an experimental result the statement that the terminal speed of an iron sphere in ordinary air is equal to that which it would acquire by falling in vacuo through a distance of 300 d yards, where d is the diameter of the sphere in inches. Calculated from this statement, the acceleration of a golf ball due to the resistance of the air should be about \[ \frac{v^2}{400} \] where \( v \) is the velocity of the ball in feet per second and the denominator is given in feet.

\[
-\frac{4\pi}{6} \cdot \frac{d}{A} \cdot \frac{v^2}{v_0} = -2g \left( \frac{900 d}{v_0^2} \right)
\]

Since \( d \) in the golf ball amounts to nearly 1.75 inches, the resistance-acceleration, which in the particular case of the falling body is \(-g\), amounts to \( \frac{v^2}{400} \).

In the "Bashforth chronograph," the acceleration due to the air resistance of an iron shot with a diameter of \( d \) inches and mass of \( w \) pounds is given as \( \frac{-118.3d^2}{1000} \). where \( v \) is in feet per second. This law is true for balls of any material since the resistance depends only
on size and velocity, and the acceleration is inversely as the mass. This formula gives as the acceleration of the average golf-ball, whose weight is about 0.101 lbs. the value \(-v^2\), a result which is 42.5% greater than that deduced from Robins' experiment.

The equation of motion for a golf ball not under the influence of gravity, if \(a\) is written for the 280 will be

\[
\begin{align*}
(1) \quad \ddot{s} &= \frac{\dot{s}^2}{a} \\
(2) \quad \frac{1}{\dot{s}} - \frac{1}{v_0} &= \frac{t}{a}, \quad v_0 \text{ being the initial velocity.} \\
(3) \quad \dot{s} &= \frac{v_0}{1 + \frac{v_0 t}{a}} \\
(4) \quad \left(\frac{1 + \frac{v_0 t}{a}}{e^{s/a}}\right) &= e^{s/a} \\
(5) \quad v &= v_0 e^{-s/a}
\end{align*}
\]

Equation (5) shows that, whatever the speed may be, since \(e^{-0.7} = \frac{1}{2}\), it is reduced to one half when the ball has moved through 196 feet.

From (4), the corresponding time of passage is seen be \(\frac{280}{v_0}\).

Tait and Steele in their "Dynamics of a Particle" derive the approximate equation of a flat trajectory.

\[
y = (\tan \alpha + \frac{q a^2}{2v_0^2})x - \frac{q a^2}{4v_0^2} \left(e^{\frac{x}{a}} - 1\right).
\]
Here the assumption was made that the horizontal component was so nearly equal to the entire velocity that $\frac{dx}{ds}$ could be taken as unity. The equation approximately represents the point, where the angle of inclination is very small, but can hardly be said to hold throughout the latter part where the angle becomes considerable. The error introduced even then is small, so that this equation is as worthy of consideration as the intrinsic equation, which although it can be integrated exactly is always tedious and only to be relied upon when we possess data more accurate than any so far obtained. Bashforth's elaborate and extensive tables would fit our purposes very well, if we had exact information as to the speed at the highest point of the path.

The first data necessary for the use of equation (6) are the values of $a$, $\tan \alpha$, and the initial velocity. $a$ has been assumed as about 280 and $\alpha$ the angle of projection has been carefully determined by means of a clinometer as being on the average about $13^\circ 5$, a value which gives 0.24 for $\tan \alpha$. The horizontal component of the initial velocity is practically equal to the initial velocity itself, and according Tait used the following methods for determining its approximate value. First, a watch ticking four times per second was held to the ear, and a number of rough observations made, with the purpose of ascertaining the distances passed over by the ball in one second. It was found that a well struck ball goes about 100 yards in the first second.
Eliminating \( v \) from formulae (2) and (b),

\[
V_0 = \frac{a}{t} \left( \frac{\frac{g}{v} - 1}{e} \right) = 537 \text{ ft/sec.}
\]

An error of one per cent in the measurement of the distance traversed gives an error of about 1.6\% in the result.

Second: from various data obtained by him, Mr. Tait concluded that the average time of flight for a good drive is about 4.5 sec. and the length of the path, about 600 ft. If these values are substituted in (7), \( V_0 = 463 + \frac{\text{ft}}{\text{sec}} \).

Of course, these estimates are very rough, as the effect of gravity is neglected. If an initial speed of 500 \( \frac{\text{ft}}{\text{sec}} \), a value intermediate between the two calculated values, is taken, the equation of the trajectory becomes

\[
y = 0.258 x - 2.524 \left( e^{\frac{140}{x}} - 1 \right).
\]

The coordinates of the highest point of the trajectory, as given by this equation are

\[
\begin{align*}
x &= 372.4 \text{ ft.} \\
y &= 62.4 \text{ ft.}
\end{align*}
\]

The range \( R \) is also obtained from the same equation with the aid of tables for \( \frac{e^{z} - 1}{e^{z}} \), and is found to be 571 feet. The values of the coordinates of the highest point of the path agree very well with observed values, especially that of \( y \), while that obtained for the range agrees reasonably well, 571 being the calculated and 600 feet, the observed value. The formula gives for ranges corresponding to the following initial velocities respectively, 100, 200, 300—600 \( \frac{\text{ft}}{\text{sec}} \); the values 112, 277, 406, 571, and 631.
From observed values of ranges, it would seem that the majority of golfers have an initial velocity of from 200 to 300 ft.

Later Tait, with the aid of Bashforth's tables, made a more exact determination of the different elements of the trajectory. Using Bashforth's notation, he took

\[ \lambda = 1.9 \quad v_o = 131 \frac{ft}{sec} \quad \text{and} \quad \phi = 13^\circ 5 \] (Bashforth uses \( \lambda \) as \( 2 b n \) (resistance of the air at the vertex of the trajectory, over the weight of shot; \( v_o \), as the velocity at the vertex and \( \phi \), as the initial angle of projection). From the tables, he obtained for the range of carry 542 feet; for the maximum height of the trajectory, 53 feet; the horizontal distance of the highest point from the tee 350 feet; the initial speed 480 feet per second; the terminal speed 48 feet per second and the terminal angle of inclination 38^\circ 5. These numbers agreed remarkably well with the numerical details that had been collected for good drives, but when the time of flight was calculated from Bashforth's tables, a result of 1.51 seconds was obtained for the first part of the path, and 2.13 seconds for the second part. This amounts to 3.6 for the total time of flight, while that observed in a good drive is always over six, and often as much as seven seconds. The initial speed given by this calculation seems to be too large, but if \( v_o \) is decreased,
and Bashforth's value for the coefficient of resistance retained, the angle of projection and the maximum height of the trajectory must be increased in order to keep the same range. If in order to reduce the initial velocity and increase the time of flight, a smaller resistance is assumed, the range, maximum height and the angle of projection agree almost with their observed values, but the vertex of the path is shifted toward the middle. Taking $\lambda = 1.1$, so that $u_0 = 100 \text{ ft.} \over \text{sec.}$ and the initial angle of projection as $23^\circ.25$ Tait obtained for

- Range of Carry 543 feet.
- Maximum height 100 ft.
- Horizontal distance of highest point from the tee 350 ft.
- Initial speed $393 \text{ ft.} \over \text{sec.}$
- Terminal speed $80 \text{ ft.} \over \text{sec.}$
- Terminal inclination $54.6^\circ$

but in this case, the angle of projection is decidedly too great. The calculated time of flight is always too short. The only forces that have been taken into consideration so far are those of resistance and gravity, and these both tend to shorten the time of flight. Besides, for some distance, the path is usually approximately straight and may often be concave upwards. Apparently some additional force not considered must be acting to produce the concavity and to keep the ball in the air for 6 seconds throughout a range of 180 yards,
when the height of no point of its path exceeds 100 feet.

The only force that could possibly do this, is that which would arise from a rotation of the ball due to undercutting. If we allow a rotation to the ball, the longer its path retains its initial inclination to the horizontal, notwithstanding the fact that in doing this, it might lose part of its energy of translation as well as of rotation, and therefore diminish the range, the longer will be its time of flight during the carry.

Several eminent Mathematicians from time to time have tried to determine experimentally the effect of rotation upon the path of a spherical projectile, Newton and Robins being among the first and Magnus one of the most recent. Poisson published a large mathematical treatise on the subject (Recherches sur le Mouvement des Projectiles dans l'air). In spite of all the work attempted, we still know very little more about the entire subject than was known when Newton carried out his experiments on dispersion in 1666. What he thought about the effect of rotation is seen from a paragraph taken from a letter to Oldenburg, then secretary of the Royal Society (1671)- "I remembered that I had often seen a tennis ball, struck with an oblique racket, describe such a curve line. For a circular as well as a progressive motion being communicated to it by that stroke, its parts, on that side where the motions conspire, must press
and beat the contiguous air more violently than on the other; and there excite a reluctancy and reaction of the air proportionably greater." Apparently Newton almost 250 years ago, was aware that the effect of rotation was to produce curvature in the path of a ball and knew also that the effect was great enough to be easily noticed in the short flight of a tennis ball. He gave the direction of the deviation correctly, and in ascribing it to difference of air pressure, for which he, in turn, found a cause, expressed the substance of the present day explanation. Magnus over 150 years later gave practically the same explanation.

Robins, in a paper read before the Royal Society in 1747, spoke of "the hitherto unheeded effects produced by this resistance; for its action is not solely employed in retarding the motions of projectiles, but some part of it exerted in deflecting them from their course, and in twisting them in all kinds of directions from their regular track; this is a doctrine which notwithstanding its prodigious import to the present subject, hath been hitherto entirely unknown, or unattended to; and therefore the experiments, by which I have confirmed it, merit, I conceive, a particular description; as they are themselves of a very singular kind." By means of thin screens placed across the range, Robins accurately measured the deviation either to the right or left of bullets fired in each case from exactly the same point and under exactly the same circumstances. He found that there was
along the path a resultant deviation that increased much more rapidly than the distance from the point of fire. He experimented also and with good success, with a gun, whose barrel was bent a little toward the left near the muzzle, so that a loose fitting bullet would be given a rotation by virtue of its rolling against one side of the bore. Although at first, the bullet deviated a little toward the left, it soon turned away and deviated to the right.

Euler evidently thought differently on the subject for a retranslation of his German version of Robins work* quotes him as saying, "The cause which Mr. Robins assigns for the uncertainty of the shot cannot be the true one, since we have undisputably proved, that it arises from the figure of the ball only. If the ball has a progressive motion, we may, as has been already shown, consider it at rest, and the air flowing against it with the velocity of the ball's motion; for the force with which the particles of air act on the body will be the same in both cases." He followed this statement with an investigation of the motion of the ball, which he summed up thus, --"hence this proposition appears indisputable true: that a spherical body which besides its progressive motion in the air revolves around its centre, will suffer the same resistance, as if it had no such rotation. If therefore, such a ball should receive two such motions in the cannon, yet its progressive motion in the air would be the same as if it had no rotation."

Poisson with Euler does not accept Robins explana-
tion. The effect of rotation on the path of a spherical projectile is given by him as exactly the opposite to that cited by Robins, and the gist of his explanation is this,— Since the density of the air is greater in front of the ball than behind, the friction there is greater than on the back of the ball. This gives rise to a very small lateral force, tending to deflect the ball as if it were rolling upon a cushion of air in front of it. In his own words (Recherches sur le mouvement des projectiles dans l'air, Paris 1839)

"C'est ce que l'on peut aussi regarder comme évident à priori, si l'on considère que cette déviation est due à l'excess de la densité de l'air en avant du projectile, sur sa densité en arrière; excess qui donne lieu à un plus grand frottement du fluide, contre l'hémisphère antérieur, et à un moindre contre l'hémisphère postérieur—il en résultera une force horizontale qui pousera ce point (the centre of inertia) dans le sens du plus grand frottement ou en sens contraire de la rotation à laquelle il répond, c'est-à-dire vers la gauche, quand les points de la partie antérieure du projectile tourneront de gauche à droite, et vers la droite, lorsqu'ils tourneront de droite à gauche."

Several experiments that have been performed show very well, the effect of rotation on a body having translation through the air. Magnus made one in 1852, in which he used a cylinder mounted in such a way that it could be given a
rapid spin about a vertical axis. The cylinder was fastened to one end of a rod, at the opposite end of which was a weight, so adjusted that when the rod was suspended, it would remain horizontal. The rod yielding easily to the action of any horizontal forces showed their effect by motion either in one direction or the other. A blast of air produced by a fan driven by an electric motor was directed through a pipe placed in front of the cylinder, so that it struck the cylinder symmetrically. In this method of procedure, he used the same principle as Euler had previously stated in his reasoning concerning the effects of rotation, namely that, in so far as results are concerned, translation of the ball through air at rest may be replaced by a ball at rest and air flowing past it with a velocity equal to that of the ball. It was found that when the cylinder had no spin about its vertical axis, there was no subsequent motion of the rod; when the cylinder was rotating, the impact of the blast of air against it caused the rod to move off sideways. With one direction of spin, it turned one way; with the reverse direction of spin, the opposite way; and the direction of rotation of the rod was always the same as that of the spin of the cylinder. So we say that if there is no spin, no deflecting force is introduced, but if spin is present there will always be a deflecting force, tending to rotate the rod. We conclude too that a rotating body moving relatively to the
air is acted on by a force, whose direction is at right angles, both to the direction in which the centre of the body is moving and also to the axis about which the body is spinning.

Another way of showing that the rotation of a body introduces an additional force was used by Robins. He suspended a ball by two strings twisted together, so that when the ball was made to swing like the bob of a pendulum, it was also given a rotation about an axis through the point of suspension of the ball. It was found that the plane of vibration did not remain stationary as in the case of a not rotating pendulum, but rotated in the same sense as the ball.

(Illustration) Concerning this experiment, Robins tells us, that "it was always easy to predict before the ball was let go, which way it would deflect, only by considering on which side the whirl would be combined with the progressive motion; for on that side always the deflecting power acted; as the resistance was greater here than on the side where the whirl and progressive motion were opposed to each other." and also, "This experiment is an incontestible proof, that, if any bullet, besides its progressive motion, hath a whirl round its axis, it will be deflected in the manner here described."

Now the question comes up as to how these forces arise. Apparently, there can be little doubt that it is due to a difference in pressure on the two frontsemi-hemispheres of the body. Because of an excess of pressure on one side over that on the other, the ball is pushed away
Plate I.

Illus. I.
in the direction of the least pressure. Robins and Newton both knew this fact, but Poisson's treatment contains no expression from which a formula giving the pressure as different on different parts of the front hemisphere, can be derived, or such a conclusion even deduced.

Another experiment performed by Sir. J. J. Thomson before the Royal Institution of Great Britain shows that the pressures on different sides of a rotating ball are actually different, and that this difference in pressure is dependent on the roughness of the ball. In this experiment, two golf balls, one brambled and one smooth were mounted on a vertical axis which could be set in rapid rotation by an electric motor. An air blast was directed against one or other of the balls, and the difference in pressure measured in the following way. A glass tube was bent as shown in the illustration (illus. 2.) and the ends A and B, so placed that the golf ball would just fit in between them. If the pressure were greatest on the side A of the ball, the liquid in the tube would be depressed at C; if it were greatest at B, the liquid would be depressed at D. The experiment showed that if the balls had no spin, the pressures on both sides were equal. When the sense of spin was as indicated in the illustration, the pressure on the side A was greatest, and the liquid was depressed at C. With a reverse spin, the effect was just the opposite. This experiment furnishes an indisputable
verification of one point at least of the conclusion derived from the experiments previously mentioned, namely—that the pressure is greatest on that part of the ball which the spin carries directly into the air blast, and least at those points which by virtue of the spin are moving in the same direction as the air.

Greater differences of pressure resulted in the case of the brambled ball, than with the smooth one. In fact Thomson found that the difference in level of the two parts of the pressure gauge was more than doubled, and that therefore with the same velocity of spin and speed of the air-current, the difference in pressure on the sides of the rough ball was more than twice that on the smooth ball.

The deflecting force is dependent upon both the speed of rotation, and that of translation, because it does not exist when only one is present, no matter how great in magnitude that one may be. The determination of an expression for that force in terms of their magnitudes has proved no simple matter. Neither Magnus nor Robins gave any suggestion whatever as to its form. Lord Raleigh in his paper, "The Irregular Flight of a Tennis Ball," (Messenger of Mathematics Vol. VI p. 14, 1877) which concerns itself with the mathematical theory of the subject, shows that the difference of pressure gives rise to a deflecting force at right angles both to the direction of motion and to the axis of spin, and
that the magnitude of the force is proportional to the product of the velocity of the ball, by the velocity of spin, multiplied by the sine of the angle between the direction of motion of the ball and the axis of spin.

Several considerations tend to lead to the conclusion that the magnitude is proportional to the product of the speed of translation by that of spin. In the first case, the direction of the deflecting force is reversed, but the magnitude is not changed, when either the direction of translation or rotation is reversed. This could most easily happen if the force were capable of being expressed by the product of the two velocities or proportional to them. Sir. G. G. Stokes wrote to Tait as follows: — "The linear velocity due to rotation even at the surface where it is greatest, being supposed small, or at least tolerably small, compared with the velocity of translation, I think you are right in saying that the force acting laterally upon the ball will vary approximately as \( vw \)." Secondly: the resistance on the non-rotating ball is proportional to the velocity squared. (See illus. 3.) Let the ball be moving with a velocity of translation \( v \) in the sense A-B. Then without rotation, the ball would be carried against the air in front of the ball at a rate \( v \) giving rise to a resisting force proportional to \( v^2 \). Let the ball be given a motion of rotation in addition to its motion of translation. All points of the semi-hemisphere C. will be carried against the air at an additional rate proportional to \( w \), since it is evident that if the ball had an angular velocity twice as great, that part of the rate due to rotation would be twice as great. The vel-
ocities at which points on the semi-hemisphere C are moving against the air is \( (v + ew) \) where \( v \) and \( w \) are velocities of translation and rotation respectively, and \( e \) a constant for any given point. Since the points on the semi-hemisphere D are moving away from the air in front of the ball at the same rate at which those on C are moving into it, the rate at which they are moving against the air is \( (v - ew) \).

These two motions give rise to two different forces \( c_1(v+ew)^2 \) and \( c_1(v-ew)^2 \) acting in opposite senses but both at right angles to the direction of translation and to the axis of rotation. The resultant force is therefore in the sense of the larger force and has a magnitude equal to the difference between the two \( c_1(v+ew)^2 - c_1(v-ew)^2 = c_2vw \). In this case the axis of rotation is at right angles to the direction of translation. If the axis of rotation were at an angle \( \alpha \) with the direction of motion, the component angular velocity about an axis perpendicular to the direction of motion would be \( w \sin \alpha \) and \( w \) in the above formula must be replaced by the value \( w \sin \alpha \). The deflecting force is measured by the expression \( c_2vw \sin \alpha \). Thirdly: however much of assumption there may be in our conclusion concerning the direction and magnitude of the force, it corresponds to a large extent to the results of experiment and observation, which is the ultimate check to which we must refer all mathematical explanation.
Another question arising that might have some bearing on the character of the path of the projectile is whether translation has any appreciable effect on the rate of loss of kinetic energy of rotation by the ball. Tait says that he has often seen "a sliced or heeled ball still rotating rapidly when it reached the ground at the end of its devious course", and that this would be the very thing to be expected, if "the deflecting force were the only result of rotation for this force is perpendicular to the line of flight and does no work." But if the friction on the surface of the ball was dependent on the velocity of translation, the ball while flying would lose its rotation more quickly than if its centre were at rest.

Tait carried on some correspondence with Sir G. G. Stokes on this point, in reply to which Stokes wrote, "As to the decrement of the energy of rotation, I think the second law that you suggested is likely to be approximately true.----- the force laterally on the ball will vary at least approximately as \( vw \). If this acted through the centre, it would have no moment. But I think it will not act through the centre, though probably not far from it, so that it would have a moment varying as \( vw \). Hence the decrement of angular velocity would vary as \( vw \) and the decrement of energy of rotation as \( w \left( \frac{-dw}{dt} \right) \) or as \( w.vw \) or \( vw^2 \) according to your second formula."
Moment of force = $I_c\alpha$

$I \alpha = c.vw$

$\alpha = c.vw$

The decrement of angular velocity varies as $vw$.

$$\alpha = -\frac{dw}{dt} = c.vw$$

$$K.E.\ of\ rotation = \frac{1}{2}Iw^2$$

$$\frac{d}{dt} \left[ \frac{1}{2}Iw^2 \right] = -c_vvw^2$$

He goes on to say, "However, I think the force at any point of the surface, of the nature of that which we have been considering, would act very approximately toward the centre, and therefore would have little moment, so that after all the moment of the force tending to check the rotation may depend rather on the spin directly than on its combination with the velocity of translation. But if this be so, I doubt whether the diminution of rotation during the short time that the ball is flying is sufficient to make it worth while to take it into account."

Another factor of great importance in the study of the trajectory of the projectile, but which it has been found almost impossible to consider with any certainty of results is the effect of the wind. Since the resistance is approximately proportional to the square of the relative speed, a small change in the speed makes a larger change
in the resistance, so that either a light head or following wind might produce a considerable effect.

Even an approximate solution of the simplest case, when the wind is steady and blows in the plane of flight is very difficult. The variation of the speed of the wind with height above the earth introduces further complications. However, the general nature of the problem may be stated. With a head wind, the resistance is greater than in still air. Its direction is not in the line of flight except perhaps at the highest point of the path, but is inclined at a smaller angle to the horizon. Consequently its effect on the horizontal component as compared with that on the vertical is greater than in still air. With a following wind, the resistance is less than in still air. The time of flight is probably lengthened a little by a head wind and shortened by a following wind.

The character of the weather also has its effect, although directly only on the coefficient of resistance. This, if considered proportional to the density may vary as much as 10% from the average. It has been found that a cold dry day gives the greatest value for the coefficient of resistance and the shortest carry; while a moist warm day gives the smallest value for the coefficient of resistance and the longest carry.

The first step toward the complete equation of
motion of a projectile is the consideration of motion taking the resistance of the air into account and disregarding gravity. I have already given (Eq. 5) the equation obtained by taking the component of acceleration along the tangent. Considering the component along the direction of the normal, we have

\[
\begin{align*}
\frac{\dot{s}^2}{c} &= \frac{\dot{s}}{ds} \frac{d\psi}{ds} = \kappa w \frac{\dot{s}}{s} \\
\dot{\psi} &= \kappa w \\
\frac{d\psi}{ds} &= \frac{\kappa w}{s}
\end{align*}
\]

The curvature varies inversely as the velocity.

Substituting for \( \dot{s} \) from (5)

\[
\frac{d\psi}{ds} = \frac{\kappa w}{v_0} \frac{s}{a}
\]

(11) Integrating, \( \psi = \frac{\kappa w a}{v_0} \left( e^{\frac{s}{a}} - 1 \right) \)

where \( \psi \) is measured from the initial direction of projection and \( v_0 \) is the initial speed. This of course implies that \( w \) is practically constant throughout flight. The values for \( x \) and \( y \) coordinates given by this equation when the \( x \) axis taken as the tangent to the curve at the point of projection, are

\[
\begin{align*}
\chi &= a \int_0^\psi \frac{\cos \psi \, d\psi}{\psi - \frac{\kappa w a}{v_0}} \quad \text{and} \quad \eta &= a \int_0^\psi \frac{\sin \psi \, d\psi}{\psi - \frac{\kappa w a}{v_0}}
\end{align*}
\]
more simple forms are obtained if is measured from an initial value \( \left( \frac{-kwa}{v_0} \right) \). In this case, equation (1) becomes
\[
\varphi = \frac{kwa}{v_0} e^{-t}.
\]
If \( \gamma \) is again taken as the angle made by the tangent to the curve with the \( x \) axis, the \( x \) axis has been rotated through an angle \( \left( \frac{kwa}{v_0} \right) \). \( S \) is measured from the point of projection at which point, the tangent makes an angle \( \left( \frac{kwa}{v_0} \right) \) with the new \( x \) axis, which is dependent only on \( a \) and the ratio between the spin and the initial speed.

\[
\frac{S}{a} = \log \left( \frac{v_0 \gamma}{kwa} \right)
\]

\[
\frac{ds}{a} = \frac{dx}{a \cos \gamma} = \frac{dy}{a \sin \gamma} = \frac{dy}{\gamma}.
\]

\[
\left\{ \begin{array}{l}
\frac{\gamma}{a} = \int_{0}^{\gamma} \frac{\cos \gamma \, dy}{\gamma} \\
\frac{y}{a} = \int_{0}^{\gamma} \frac{\sin \gamma \, dy}{\gamma}
\end{array} \right.
\]

These integrals have been tabulated by Mr. J. W. L. Glaisher. In the Phil. transaction 1870, he published a paper containing among other tables, tables of the sine and cosine integrals, \( \int_{0}^{\gamma} \frac{\cos \gamma \, dy}{\gamma} \) and \( \int_{0}^{\gamma} \frac{\sin \gamma \, dy}{\gamma} \) for values of the argument, 0 to 1 at intervals of .01; and 1 to 5 at intervals of .01. From these tables, it may be seen that when \( \gamma = 0 \), \( y = 0 \), and \( x = \infty \) The line \( y = 0 \) is an asymptote to the curve. The curve
has no multiple points and may be plotted from the values given in Ulaher's tables up to $\gamma = 5$. For higher values of $\gamma$, interpolation is necessary as the figures given in the tables are too far apart to insure true drawing. It is an endless spiral whose curvature $\left( \frac{k w}{c S} \right)$ increases continuously from 0 when $s = -\infty$ to $\infty$ when $s = \infty$. Any arc of the curve may be taken as representing the horizontal projection of the path of the sliced golf-ball for the curvature of the trajectory of the projectile increases as the velocity decreases.

If we had accurate values of $'a'$ and $V_0'$, the magnitudes of the initial deflecting and the initial resistance could be obtained. Evidently $'a'$ cannot be found from eye observation alone, for if curves be traced for the path of a golf ball, having a range of 180 yards, and a time of flight equal to 6.5 sec, with the parameter $'a'$ varying from 180 to 360 ft., it will be found that they are strikingly similar. Tait found no means of measuring $'a'$ directly but experimentally determined $V_0'$, first with a ballistic pendulum, and afterwards by measuring the speed of the club head at impact and determining $V_0$ from the known coefficient of restitution of the ball. His results were not satisfactory, but apparently agreed in the fact that $V_0$ was somewhat larger than $300 \, \text{ft./sec.}$ for a good drive. When the carry was taken as 180 yards, with a time of flight equal to 6 sec. $'a'$ as given by the formula

$$t= \frac{a}{V_0} (e-1)$$

was 240 feet. An initial velocity of 350
feet gives an initial resistance $\frac{v^2}{a}$ almost 16 times as great as the weight of the ball. An approximate value of $K_w$ in the flight of a golf ball can be obtained from the formula

$$\varphi = \frac{Kwa}{V_o} (e^{\frac{V_o}{a}} - 1)$$

When the ball is well sliced, its direction of motion when it reaches the ground is often at right angles to the initial direction. If $s$ is 480 feet; $a$, 240 feet; and $V_o$, 350 ft sec,

$$\frac{v^2}{a} = \frac{Kw}{V_o} \left(2\frac{V_o}{3a}\right)(e^{\frac{V_o}{a}} - 1)$$

$$Kw = 0.835 \text{ about}.$$

The acceleration at starting due to rotation is $KwV_o = 125.3$ or nearly four times that of gravity. That is, the deflecting force is four times the weight of the ball.

The problem of motion is much more complicated when wind currents are taken into account. In fact, only in the simplest cases can equations of motion be obtained. Suppose we consider the effect of a steady current of wind in the plane of motion, disregarding gravity.

Let $\varphi$ giving the direction of actual motion be measured from the reverse direction of the wind and let $W$ be the velocity of the wind, and $U$ in the direction $\varphi$ the relative velocity of the ball with respect to the wind. The resistance and the deflecting force both depend on the relative velocity of the ball. See illus. (4).
Plate V.

Illus. 4.
The component of \( U \) in the direction of the wind is

\[ U \cos \psi = W + s \cos \psi \]  

and that perpendicular to the wind is

\[ U \sin \psi = \dot{s} \sin \psi \]

Resolving the acceleration into components along the tangent, and normal respectively,

\[ \dot{s} = -\frac{U^2}{a} \cos(\psi - \psi) + k_w U \sin(\psi - \psi) \]  
\[ \frac{\dot{s}^2}{\psi} = \frac{U^2}{a} \sin(\psi - \psi) + k_w U \cos(\psi - \psi) \]

where \(-\frac{U^2}{a}\) is the acceleration due to the resistance along the direction of relative motion, and \(k_w U\), that due to rotation, along the direction perpendicular to that of relative motion.

Expanding (16) and (17) and substituting to eliminate \(\psi\), the following equation results:

\[ v \frac{d\psi}{ds} = -\frac{U}{a} (W \cos \psi + \dot{s}) + k_w U \sin \psi \]
and
\[ \frac{v^2 d\psi}{ds} = \frac{U W \sin \psi + \kappa}{a}(W \cos \psi + s), \]
where \( \kappa = \kappa^w \) and \( U^2 = W^2 + v^2 + 2Wv \cos \psi. \)

When \( W \) is put equal to 0, and \( U = V \), these equations become (1) and (9) respectively.

If \( \psi \) is small, that is, if the ball is advancing nearly into the wind, the preceding equations may be written approximately as

\[ v \frac{dv}{dw} = -\frac{U}{a} (W + v) + \kappa \cos \psi, \]

\[ = \frac{-(W + v)^2}{a} + \kappa \cos \psi. \]

\[ v^2 \frac{d\psi}{ds} = \frac{U W \cos \psi + \kappa (W + v)}{a} \]

\[ = \frac{(W + v) W \cos \psi + \kappa (W + v)}{a}, \]

for \( \cos \psi = 1 \) and \( \sin \psi = \psi \) approximately and \( U = [U - (-W)] = U + W. \)
Compare these equations with (1) and (9). Equation (20) tells us that due to the action of the wind $\frac{dV}{dt}$, the space rate of diminution of speed is decreased in the ratio $(\frac{W + V}{V})^2$ and that the rotation tends to counteract this effect. Equation (21) tells us that, the space rate of change of direction is increased in the ratio $\frac{W+V}{V}$ due to rotation and also by an additional term entering because of the resistance itself. It can be seen from this that the wind is an important factor in producing upward curvature in the path of the ball. When $\sin \phi$ acquires a finite value by virtue of underspin, resistance enters and further increases it.

Finally, the equations of motion of a golf ball, moving under the action of gravity, in still air, and rotating about an axis perpendicular to the plane of motion, are

\[ (22a) \quad \ddot{S} = -\frac{\dot{S}^2}{a} - g \sin \phi \]

\[ \frac{\dot{S}'}{\rho} = \kappa \dot{\mathbf{w}} \cdot \dot{S} - g \cos \phi \]

or

\[ (22b) \quad \phi = \kappa \dot{\mathbf{w}} - g \frac{\cos \phi}{\dot{S}} \]
When \( \psi \) is small, the term \( g \sin \psi \) in (21) may be neglected and \( \cos \psi \) put equal to unity. Equations (21) and (22) become approximately

\[
\ddot{s} = -\frac{\dot{s}^2}{a}
\]

or by integration,

\[
(23) \quad \dot{s} = v_0 e^{-\frac{s}{a}}.
\]

and

\[
(24) \quad \dot{\psi} = \kappa_1 - \frac{g}{\dot{s}}.
\]

\[
\frac{d\psi}{ds} \dot{s} = \kappa_1 - \frac{g}{\dot{s}}
\]

\[
\frac{d\psi}{ds} = \frac{\kappa_1}{\dot{s}} - \frac{g}{\dot{s}^2}
\]

\[
(25) \quad \frac{d\psi}{ds} = \frac{\kappa_1}{v} e^{\frac{s}{a}} - \frac{g}{\dot{s}^2} e^{\frac{s}{a}}.
\]
In \( y \) and \( y \) coördinates, this becomes nearly

\[
\frac{d^2y}{dx^2} = \frac{K}{v} \frac{\frac{d}{dx} (\frac{y}{v})}{\frac{1}{2} \frac{d^2}{dx^2} \left( \frac{y}{v} \right)}.
\]

There is clearly a point of contrary flexure in the path, for if \( \frac{d^2y}{dx^2} \) is put equal to zero,

\[
\frac{d}{dx} \left( \frac{y}{v} \right) = \frac{K}{v} \quad \text{from which equation, its } x \text{ coördinate may be obtained. The path is concave upward at the point of projection if } Kv \text{ is any larger than } q.
\]

From (26), an expression for \( \frac{dy}{dx} \) is obtained:

\[
\frac{dy}{dx} = \frac{e_0 + \frac{aK}{v} \left( \frac{x}{v} \right) - \frac{aq}{2v^2} \left( \frac{x^2}{v^2} - 1 \right)}{v_0}.
\]

where \( e_0 \) is the initial slope of the tangent.

The \( x \) coördinate of the vertex may be found by putting \( \frac{dy}{dx} = 0 \).
\[ e_0 + a \frac{k}{V_0} \left( e^{\frac{v}{a}} - 1 \right) - \frac{aq}{2V_0} \left( e^{\frac{2v}{a}} - 1 \right) = 0. \]

\[(29) \quad e^{\frac{v}{a}} = \sqrt{\left(1 - \frac{k}{V_0} \right)^2 + \frac{2V_0^2}{g} \frac{e_0}{aq} \frac{V_0}{g} + \frac{k}{V_0}.} \]

Integrating, \( S = V \int e^{-\frac{v}{a}} \), an equation for \( y \) results,

\[ y = a + \frac{k}{a} \left( e^{\frac{2v}{a}} - 1 \right) - \frac{qa}{2V_0^2} \left( e^{\frac{2v}{a}} - 1 \right). \]

This expression for \( y \) might be substituted in (22a), in the term \( g \sin \alpha \), and the result would be a closer approximation to the value of \( S \) than could be had by neglecting this term altogether.

The equation in this case becomes too difficult to handle, so that the first original equations are retained.

Integrating the last equation
again with respect to $s$, we have

$$\int_0^s \varphi \, ds = \left[ \alpha s + \frac{\kappa a^2}{v} \left( e^{2s} - \frac{s^2}{a} \right) - \frac{q a^2}{4 v^2} \left( e^{2\frac{s}{a}} - 2 \frac{s}{a} \right) \right]_0^s.$$  

(30) $$\int_0^s \varphi \, ds = \alpha s + \frac{\kappa a^2}{v} \left( e^{\frac{s}{a}} - 1 - \frac{s^2}{a} \right) - \frac{q a^2}{4 v^2} \left( e^{2\frac{s}{a}} - 1 - 2 \frac{s}{a} \right).$$

If this equation is transformed into rectangular coordinates (using the same origin with $x$ horizontal), we have

$$x = \int_0^s \cos \varphi \, ds = \int_0^s \left( 1 - \frac{\varphi^2}{2} + \cdots \right) \, ds,$$

$$y = \int_0^s \sin \varphi \, ds = \int_0^s \left( \varphi - \frac{\varphi^3}{6} + \cdots \right) \, ds.$$

Discarding the second power terms and over, we have

$$x = \int_0^s \, ds = s.$$

$$y = \int_0^s \varphi \, ds = \alpha s - \frac{\kappa a^2}{v_0} \left( e^{\frac{s}{a}} - 1 - \frac{s}{a} \right) - \frac{q a^2}{4 v_0^2} \left( e^{2\frac{s}{a}} - 1 - 2 \frac{s}{a} \right).$$

$$y = \alpha x - \frac{\kappa a^2}{v_0} \left( e^{\frac{x}{a}} - 1 - \frac{x}{a} \right) - \frac{q a^2}{4 v_0^2} \left( e^{2\frac{x}{a}} - 1 - 2 \frac{x}{a} \right).$$
This equation may be obtained from (28) with a difference only in the first term, which here proves to be no difference since the tangent of the angle of projection is approximately equal to the angle itself.

The most serious fault to be found in the approximation is the omission of $g \sin \alpha$ in the tangential equation of motion (22a). $S$ is too large for the greater part of the path, so that $y$ is too small up to the inflection point, and too large beyond the point of inflection, up to and a little beyond the vertex. In other words, the path rises at first, a little too slowly, and then too rapidly; it rises too high and makes its descent at too small a slope.

When Tait first worked with these equations he chose three values of $\alpha$, one less than, one nearly equal to, and one greater than 280, that given from Bashforth's experiment with iron spheres. Using these values he determined the initial velocity $v_0$ by means of the equation $V(t) = a(e^{\frac{x}{v}} - 1)$ obtaining 528.5, 255, 4 and 192.8 respectively. Then with each pair of values with the successive values .25, .125, and 0.0 for $\alpha$, he found $K$, from the last equation in $x$ and $y$ on the condition that $y = 0$ when $x = 540$ feet. The results showed that as $\alpha$ or $\alpha$ were made less, $K$ became greater.
Since all the constants are found, they may be substituted in the equation of the curve, and the curve traced by a few points. 

The position of the maximum ordinate is found from equation (29) and the equation of the curve. We have for the equation of the trajectory without rotation,

\[ y = (\tan \alpha + \frac{g\alpha}{2V_0^2})x - \frac{g\alpha^2}{4V_0^2} \left( e^{\frac{x}{\alpha}} - 1 \right). \]

The comparison of the two trajectories, that of a non-rotating and a rotating ball is best shown by the curves that I have plotted from the equations, using the same constants as Bashforth used.
Plate VI.

\[ \lambda = 0.25 \quad \lambda = 0.125 \]
\[ \alpha = 180 \quad \alpha = 180 \]
\[ V_o = 528.5 \quad V_o = 528.5 \]
\[ \kappa_1 = 0.236 \quad \kappa_1 = 0.305 \]
The motion of an elongated rotating projectile is more complex than that of the spherical projectile. Different authors seem to disagree in their explanation of its action, but all are unanimous in saying that its major axis remains very nearly tangential to the trajectory and also that it gyrates about the tangent. Bashforth explains it in this way: when an elongated ogival headed shot escapes from a rifled gun, it is moving in the direction of the axis or very nearly so. But the action of gravity soon causes this axis to make a small angle with the direction in which the centre of gravity is moving. The resultant resistance tends to turn the projectile so as to increase the angle between the tangent and the axis of the projectile. If the shot was without rotation it would fly end over end. But if to an observer looking along the shot from behind, it had right handed rotation about its axis, the motion of rotation would be combined with that which the resistance of the air tended to give it and the vertex would be slightly deflected toward the right. The same action is continued and consequently the axis describes a cone about the tangent of the trajectory rotating in the same direction as the shot rotates about its own axis. When the shot has a motion of translation, the air resistance has a resolved component slightly effective in pushing the shot away from the direction of motion. The result will be a sinuous motion.

That the axis of the projectile does remain very nearly tangent to the trajectory is evident from the fact
that the constants in the formula for resistance determined by experiments in which the paths were so nearly horizontal that the projectile must have moved almost point on, can be universally used, with close agreement to actual fact, for calculating the elements, or long range firing, time of flight, angle of elevation, etc.

If the angle between the tangent to the path and the axis of the projectile were as large as $5^\circ$, the surface exposed to the air resist would be increased 25% above that exposed when it moved point on and the range would fall far short of that calculated by the ballistic tables.

It is to be questioned whether the gravity initially causes the axis to make an angle with the trajectory through any action changing the projectile's inclination, but it does so by decreasing the slope of the tangent to the trajectory. Besides, the motion of rotation of the projectile can not be said to be combined with the tendency to upward deflection given by the resistance, thus giving a sidewise deflection. If a rotating projectile lies with its axis in the plane of the trajectory, the resistance acting upon it can have no possible tendency to turn it out of this plane. If there were no atmosphere, the axis of an elongated projectile symmetrical about that axis would remain parallel to its original direction whether the projectile were rotating or not. The air resistance causes the not-rotating projectile to tumble end over while, as experiment
shows, it keeps the axis of one having the proper rotation, along the tangent of the trajectory. The head of the bullet gyrates in a cone about the tangent and at the same time oscillates back and forth through a neutral position in which its axis lies along the line of fire.

By means of paper screens set up at regular intervals along the range it has been found that at one instant or time, the vertex of the bullet lies to one side of the trajectory (the tip of bullet is indicated by the shape of the holes made in the screens); at another instant it is on a different side; and at a third it may exactly coincide with the tangent. Franklin W. Mann represents the combination of the two motions by such diagrams as those given on the following plate.

The centre of the base of the projectile traces out the circles by following along the path in the manner indicated by the order of the numbers given, while the head traces out a similar figure. In the first diagram, the bullet makes three oscillations to one gyration; in the second, five oscillations to one gyration. He points out the similarity of the two motions with that of the gyrating top and the spinning gyroscope. The movement of the point of the top due to the centre of form attempting to describe a circle about the centre of gravity is the same as that which causes the centre of form of the unbalanced bullet (its centre of form
does not coincide with the centre of gravity) to make a small spiral in the air.

If the end of arm of a spinning gyroscope is struck a sharp flow from the right, it does not move directly downward, as it would if pressure were gently applied against its side, but moves downward and to the left and returns again to its initial position, after tracing out a circle. A bullet, so long as it remains in the barrel of the gun, is compelled to spin with its axis along the line of fire and follows this line until its base is released from the gun. Then it abruptly changes its direction of motion to that along the tangent of the bore spiral, thus imparting to the base of the bullet a sudden motion at right angles to its axis. This gives to the axis of the bullet the same action as was given to the arm of the gyroscope.

Granting that the axis of the bullet has a motion of precession about the tangent to the trajectory, we may apparently explain that action by the use of the gyroscope. In Barton's Analytical Mechanics, a rigid development may be found of the torque necessary to maintain the motion of rectangular precession of a body of revolution rotating with a speed of $\Omega$ about its axis while this axis is turning about another axis at right angles to it with a speed $\omega$ and the magnitude of this force is $I \omega \Omega$ where $I$ is the moment of Inertia of the body about the axis of revolution. We may consider the case of the projectile as analogous,
for we have the projectile rotating about an axis of revolution and this axis turning about another through the centre of curvature perpendicular to it. Resistance takes the place of the torque, acting about the direction of the radius of curvature as an axis, and causes the axis of the projectile to describe a cone about the direction of motion.

As a popular way of looking at the subject, we might conceive the action taking place on the bullet as made up of a succession of four different actions gradually shading into one another. Let us take a projectile having left handed rotation, as viewed from the gun. First, when the projectile leaves the muzzle of the gun, it moves along the tangent, direction of motion. The air resistance acts along the tangent to the path and will therefore have a component resistance perpendicular to the axis tending to push the head of the bullet away from the tangent. Due to gyroscopic action, the head of the projectile moves out of the plane of the trajectory from right to left; second, the motion from right to left at the head of the projectile, with the rotation of the bullet sets up another force causing the head of the projectile to move down and the base up; third, the axis of the bullet is again inclined; the air resistance acts downward and the head of the bullet turns from left to right; fourth, the motion from left to right, with the rotation of the bullet sets up a force causing the head to tip up and the base down.

These displacements do not take place to any great extent. Motion begins, but in the beginning sets up
other forces in a different direction, so that a continual succession of such motions produces a motion in a cone about the tangent.

The general shape of the trajectory of the carbine and rifle bullet is shown by the curves on the accompanying plate. The first and third curves correspond to the motion of the rifle bullet, and the second and last, to that of the carbine. They are plotted from tables, giving the ordinates of the trajectory above the line of sight and the horizontal distances from the muzzle. It will be seen that the vertex of the path is always considerably beyond the middle of the range, that of the first and second with a range of 1900 yards being at 1100 yards, and the third and fourth with a range of 2000 being at 1200 yards.

I have been able to give only a very general and narrow treatment of the motion of an elongated rifled projectile, because so little has been done or is known about the subject. An extensive development involves also a consideration of the stream lines and wake formed by the projectile passing through the air, a part of the subject which is attended with the greatest difficulty.
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