Sound waves of finite amplitude

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SOUND WAVES OF FINITE AMPLITUDE

by

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SOUND WAVES OF FINITE AMPLITUDE

The problem of sound in a gas, in the most general case, is a problem of the most general motion that a gas may have. We are led to a consideration of the hydrodynamical equations. These are the different equations which a sound wave must satisfy.

A. Hydrodynamical Equations.¹

The hydrodynamical equations are most generally expressed in two systems of representation, that of Euler and that of Lagrange.

I. System of Euler.

In this system of representing the motion of a gas, if we adopt rectangular coordinates, we look upon the gas from a system of fixed rectangular axes and examine what happens at a point whose coordinates are \( x, y, z \). In describing what is happening at a point we are interested in knowing the velocity of flow of the gas, in the density and the pressure magnitude and direction at this point. These quantities will, in general, vary with the time. A knowledge of these quantities also gives us a complete knowledge of the state of the gas, for the temperature is determined by the characteristic equation of the gas as a function of the density and pressure, the gas being assumed homogeneous. Our description of the motion of the gas, expressed in mathematical language would be:-

¹ I have selected the material in this section from Lamb's Hydrodynamics, 1906; Chap. I, Chap. II, Arts. 21-25, Chap. X, Arts. 272-276, 281-282. The dynamical equation for a symmetrical spherical wave is not explicitly given but may be easily obtained by transformation of the general equation to spherical coordinates.
where \( u, v, w \), are the components of the velocity of flow at the point \( x, y, z \) at the time \( t \) in the directions of the \( x, y, \) and \( z \) axes respectively, where \( \rho \) is the density, and where \( p \) is the pressure at the place and time. We can easily see that these functions cannot be perfectly arbitrary. For example, let us imagine any closed surface fixed with respect to the axes of \( x, y, \) and \( z \). The flow of gas outward over this closed surface, determined by the first three functions, evidently has an effect on the density within this surface. This interdependence of the density upon the velocity components is expressed by the so-called equation of continuity:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0
\] (1)

Again, we easily see that the pressure cannot be perfectly arbitrary because the gas is always accelerated so as to move from regions of high pressure to regions of lower pressure. The relations which this involves are called the dynamical equations. These are:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y}
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z}
\] (2)

An important case of the motion of a gas is rectilinear or unidirectional motion. In this case not only is the velocity of flow in one direction but it is the same in magnitude at every point in a plane at right angles to this direction and the density and pressure are also the same at every point in this plane.
Let us take the direction of flow as that of the x axis. Then every quantity involved is a function solely of x and t and v and w disappear from the above equations. The equation of continuity thus becomes:
\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \] (3)
and we have left the one dynamical equation:
\[ \frac{\partial u}{\partial t} + \frac{u \partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \] (4)
If, in addition, at every point, the condition of the gas does not vary with time, we have steady rectilinear motion. This means that u, \( \rho \), and p are functions solely of x. The partial differential equations then become ordinary and we obtain by integration:
\[ \int \rho u = \text{constant} \] (5)
\[ \frac{1}{2} u^2 + \int \frac{dp}{\rho} = \text{constant} \] (6)
The last equation expresses the conservation of energy. The first term is the kinetic energy of unit mass of the gas and the latter its potential energy. This is called Bernoulli's equation.

Besides rectilinear motion, another important case of the motion of a gas is symmetrical spherical motion in which the velocity is everywhere in the direction of the radial lines from a fixed point which is the center of a system of spheres upon each of which the velocity, density, and pressure are constant. If we adopt spherical coordinates and take the center point as the pole, all the quantities involved become functions solely of r and t. Then, if \( u \) denote the value of the radial velocity, the equation of continuity is:
\[ r^2 \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r} (r^2 \rho u) = 0 \] (7)
and the dynamical equation is:
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \] (8)
If the motion is steady \( u, \rho, \) and \( p \) are functions solely of \( r \), the above partial differential equations become ordinary and we obtain by integration:

\[
\frac{r^2 \rho}{2} u = \text{constant} \tag{9}
\]

\[
\frac{1}{2} u^2 + \int \frac{d\rho}{\rho} = \text{constant} \tag{10}
\]

In case the gas is acted upon by a system of force throughout its volume, we must substitute in the dynamical equations for \(-\frac{1}{\rho} \frac{\partial \rho}{\partial n}, F_n = \frac{1}{\rho} \frac{\partial \rho}{\partial n} \) where \( \frac{\partial \rho}{\partial n} \) is the derivative of the pressure in a direction \( n \) and \( F_n \) the component of the resultant of all the forces per unit mass acting at a point \( x, y, z \) in the same direction. For the equations in rectangular coordinates we must substitute in equations (2):

\[
\begin{align*}
\text{for } -\frac{1}{\rho} \frac{\partial \rho}{\partial x}, \quad X &= -\frac{1}{\rho} \frac{\partial \rho}{\partial x} \\
\text{for } -\frac{1}{\rho} \frac{\partial \rho}{\partial y}, \quad Y &= -\frac{1}{\rho} \frac{\partial \rho}{\partial y} \\
\text{for } -\frac{1}{\rho} \frac{\partial \rho}{\partial z}, \quad Z &= -\frac{1}{\rho} \frac{\partial \rho}{\partial z}
\end{align*} \tag{11}
\]

where \( X, Y, Z \) are the coordinates of the resultant of all the forces per unit mass acting at the point \( x, y, z \) at a time \( t \) and where each is supposed to be given as a function of \( x, y, z, t \).

To summarize, I have in this section given the differential equations of motion, which must be satisfied by the velocity, density, and pressure. These three quantities are the determining factors of the motion of a gas in the system of Euler. I have given the reduced equations for the special cases of rectilinear and symmetrical spherical motion and have given the equations of steady motion in these two cases. I have also indicated the changes in the general equations when forces are allowed to act upon the gas throughout its volume.
II System of Lagrange

Let us consider an element of the gas which, in the initial condition was at the point \(a, b, c\) with respect to a fixed system of rectangular axes. At a time \(t\) this element of gas will have moved to a new position \(x, y, z\) with respect to the same system of axes. Then if we determine \(x, y,\) and \(z\) each as a function of \(a, b, c, t\), we know the position of every element of the gas at every instant. If we also know the density and pressure at every point our description of the motion is complete for the temperature is determined by the characteristic equation of the gas as a function of the density and pressure. We may, then, express our description of the motion mathematically in the form:

\[
\begin{align*}
    x &= f_1(a, b, c, t) \\
    y &= f_2(a, b, c, t) \\
    z &= f_3(a, b, c, t) \\
    \rho &= F(a, b, c, t) \\
    p &= \Pi(a, b, c, t)
\end{align*}
\]

We can easily see that these functions cannot be perfectly arbitrary. It is evident that the density in the instantaneous state, compared with that in the original state, varies with the closeness of the packing which the elements of gas undergo and the closeness of the packing depends on the functions \(f_1, f_2,\) and \(f_3\). This interdependence is expressed by the equation of continuity:

\[
\frac{\partial \rho}{\partial \rho} = \left| \begin{array}{ccc}
\frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\
\frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\
\frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c}
\end{array} \right| \tag{12}
\]
Where \( \rho_0 \) is the density in the original state at the point \( a, b, c \). Again, the gas is always accelerated in the direction of the pressure gradient and hence there is an interdependence between the pressure distribution and the motion of the gas as expressed by the functions \( f_1, f_2, \) and \( f_3 \). This interdependence is expressed by the dynamical equation. I shall here give this equation in a form involving both the Eulerian and Lagrangian forms of representation. Let us use the symbol \( \partial \) to denote differentiation in the former system and \( \partial \) for differentiation in the latter system. Then the dynamical equations are:

\[
\frac{\partial^2 x}{\partial t^2} = -\frac{1}{\rho} \frac{\partial}{\partial a} \left( \rho \frac{\partial x}{\partial a} \right) \\
\frac{\partial^2 y}{\partial t^2} = -\frac{1}{\rho} \frac{\partial}{\partial b} \left( \rho \frac{\partial y}{\partial b} \right) \\
\frac{\partial^2 z}{\partial t^2} = -\frac{1}{\rho} \frac{\partial}{\partial c} \left( \rho \frac{\partial z}{\partial c} \right)
\]

Derivatives in the two systems are related by the following formulae:

\[
\frac{\partial}{\partial a} = \frac{\partial}{\partial x} \frac{\partial x}{\partial a} + \frac{\partial}{\partial y} \frac{\partial y}{\partial a} + \frac{\partial}{\partial z} \frac{\partial z}{\partial a} \\
\frac{\partial}{\partial b} = \frac{\partial}{\partial x} \frac{\partial x}{\partial b} + \frac{\partial}{\partial y} \frac{\partial y}{\partial b} + \frac{\partial}{\partial z} \frac{\partial z}{\partial b} \\
\frac{\partial}{\partial c} = \frac{\partial}{\partial x} \frac{\partial x}{\partial c} + \frac{\partial}{\partial y} \frac{\partial y}{\partial c} + \frac{\partial}{\partial z} \frac{\partial z}{\partial c} \\
\frac{\partial}{\partial t} = \frac{\partial}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial}{\partial z} \frac{\partial z}{\partial t}
\]

Let us consider the special case of rectilinear motion and take as the direction of motion that of the \( x \) axis. Since the condition of the gas is the same at all points having the same abscissa \( x \) at the same time, \( x \) is a function solely of \( a \) and \( t \), and \( y \) and \( z \) remain equal to their original values \( b \) and \( c \) respectively. Then the equation of continuity becomes:

\[
\frac{\partial \rho}{\partial a} = \frac{\partial x}{\partial a}
\]

We have left one dynamical equation:

\[
\frac{\partial^2 x}{\partial t^2} = -\frac{1}{\rho} \frac{\partial}{\partial a} \left( \rho \frac{\partial x}{\partial a} \right)
\]

Since a relation between derivatives in the two systems is:
We may substitute in the above equation and we obtain:

\[
\frac{\partial^2 x}{\partial t^2} \cdot \frac{\partial x}{\partial a} = - \frac{1}{\rho} \frac{\partial \rho}{\partial a}
\]

(16)

In case the gas is acted upon by a system of forces throughout its volume, we must substitute in the dynamical equations:

for \(\frac{\partial^3 x}{\partial t^3}, \frac{\partial^2 x}{\partial t^2} - X\)

for \(\frac{\partial^3 y}{\partial t^3}, \frac{\partial^2 y}{\partial t^2} - Y\)

for \(\frac{\partial^3 z}{\partial t^3}, \frac{\partial^2 z}{\partial t^2} - Z\)

(17)

where \(X, Y, Z\), are the components of the resultant of all the forces per unit mass acting on an element of gas whose coordinates are \(a, b, c\) at a time \(t\) and where each is supposed to be given as a function of \(a, b, c, t\).

To summarize, I have in this section given the differential equations of motion, which must be satisfied by the positions of the particles, the density, and the pressure. These three quantities are the determining factors of the motion of a gas in the system of Lagrange. I have given the reduced equations for the case of rectilinear motion. I have also indicated the changes in the general equations when forces are allowed to act upon the gas throughout its volume.
B. Discontinuities

One or more of the quantities used in describing the motion of a gas may at a certain time be discontinuous at certain points or the derivatives of these quantities may be discontinuous at a certain time and at certain points. The discontinuities which I wish to discuss are those that occur at isolated surfaces without singularities. It may appear at first sight as if such discontinuities are exceptional but I shall later show that the occurrence of such discontinuities is general and is of importance in the study of the motion of a gas. A separate analysis must be made as the equations of motion do not apply to the discontinuity.

I. General Properties of Discontinuities and Terminology to be Adopted

Let \( \phi \) be some quantity involving the condition of the gas which is discontinuous at a surface, whose equation in initial coordinates is:

\[
f(a,b,c,t) = 0 \quad (18)
\]

and in terms of instantaneous coordinates is:

\[
f_1(x,y,z,t) = 0 \quad (19)
\]

Let us term the two regions on either side of this surface the regions 1 and 2. Now \( \phi \) is continuous in both these regions and therefore as we approach a point on the surface of discontinuity by any path which lies wholly in region 1, \( \phi \) approaches a definite limit which we shall call \( \phi_1 \). Likewise as we approach a point on the surface of discontinuity...

by any path which lies wholly in region 2, \( \phi \) approaches a definite limit which we shall call \( \phi_2 \). Thus \( \phi \) varies continuously until we reach the surface of discontinuity where it takes a sudden jump of magnitude \( \phi_1 - \phi_2 \), which we shall designate by the symbol \( [\phi] \) after which it again varies continuously.

Now two points, P and Q, on the surface of discontinuity may be connected by paths in regions 1 and 2 which differ by a distance as small as we please from a path lying in the surface of discontinuity. Let the two values of \( \phi \) at P be \( \phi_P, \phi_Q \) and the two values at Q be \( \phi_{1P}, \phi_{1Q} \). Along the path in region 1 \( \phi \) changes continuously from \( \phi_{1P} \) to \( \phi_{1Q} \) and along the path in region 2 \( \phi \) changes continuously from \( \phi_{2P} \) to \( \phi_{2Q} \). We may choose our paths such that as the point Q is made to approach the point P along a path in the surface of discontinuity, the lengths of each of these paths approaches zero. As the lengths of these paths approach zero, since \( \phi \) is continuous upon each of them, \( \phi_P \) must approach \( \phi_{1Q} \) and \( \phi_{2P} \) must approach \( \phi_{2Q} \). Thus \( \phi_1 \), must be a continuous function on the surface of discontinuity and also \( \phi_2 \) must be continuous on this surface.

The Lagrangian system of representation is most useful in the study of discontinuities. The quantities \( x, y, z \) or any of their derivatives in the Lagrangian system of coordinates may be discontinuous when these values are assigned to their positions in a sphere whose coordinates are \( a, b, c \). As a matter of terminology, we shall term the order of the discontinuity the order of the lowest derivative which suffers a discontinuity. Thus \( \frac{\partial^n x}{\partial a \partial b \partial c \partial t^n} \) is of the \( n \)'th order and if this is the lowest order derivative which is discontinuous at the discontinuity, the discontinuity is of the \( n \)'th order.
To summarize, we have proven an important property of a discontinuity, the continuity of the two values there assumed over the surface of discontinuity. This allows us to form a total derivative of either of these values on the surface. We have also indicated the arrangement of discontinuities according to their order.

II. Generality of the Existence of a Discontinuity if the Pressure is a Function wholly of the Density

If in a gas, throughout its motion, heat developed is supposed to distribute itself immediately so that the gas is always at a constant temperature, the pressure of the gas at every point is proportional to the density, in accordance with Boyle's law:

\[ p = c^2 \rho \]  \hspace{1cm} (20)

If, on the other hand, the heat conductivity of the gas may be considered zero, the change of state of each element of gas is adiabatic and the pressure varies as the density to the \( \gamma \) power where \( \gamma = \frac{Sp}{Sv} \) is the ratio of the specific heat at constant pressure to that at constant volume, in accordance with Poisson's adiabatic law, namely:

\[ p = c^2 \rho^{\gamma} \]  \hspace{1cm} (21)

We may write to take care of either case:

\[ p = \phi (\rho) \]  \hspace{1cm} (22)

Now in any mechanical system, the differential equations of motion allow us to calculate the accelerations of its component particles at any instant when we have given the positions of all these particles and their velocities at that instant subject to the condition

* Hadamard, loc. cit. pp. 139, 142.*
that the positions of the different particles and their velocities must satisfy the constraints imposed upon the system. A gas is a mechanical system with an infinite number of degrees of freedom. In the case of the motion of a gas, equations (12), (13) and (22) give upon elimination of \( p \) and \( \rho \) the components of acceleration \( \frac{\partial^2 x}{\partial t^2}, \frac{\partial^2 y}{\partial t^2}, \frac{\partial^2 z}{\partial t^2} \) if we are given \( x, y, \text{and} z \) as functions of \( a, b, c \) at the time at which we are making our observation and also \( \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t} \), as functions of \( a, b, c \) at the same time.

The constraint imposed upon the gas is that at the bounding wall, the elements of gas in contact with the wall must move so as to remain always in contact with this wall. We shall here exempt from discussion at this time cases where the velocity of the bounding wall is one of decompression and of such rapidity that the gas is not able to follow the bounding wall but leaves a vacuum between the surface of the bounding wall and the surface of the gas. Let the equation of the bounding surface be:

\[
f(x, y, z, t) = 0 \tag{23}
\]

Since a particle, whose initial coordinates are \( a, b, c \) and which originally lies on this surface, must continue to lie on this surface, by differentiating the equation of this surface with respect to time, we obtain as a necessary condition upon the given values of the velocity at points on the boundary the equation:

\[
\frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} + \frac{\partial f}{\partial t} = 0 \tag{24}
\]

We suppose that the given values of the velocity at the boundary satisfy this equation. Differentiating once more with respect to time we obtain:

\[
\left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 + \left( \frac{\partial f}{\partial z} \right)^2 + \left( \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial t} + \frac{\partial^2 f}{\partial z^2} \frac{\partial z}{\partial t} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial y}{\partial t} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial x \partial z} \frac{\partial z}{\partial t} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y \partial z} \frac{\partial z}{\partial t} \frac{\partial y}{\partial t} \right) = 0 \tag{25}
\]
Now the components of acceleration are already determined by the equations of motion (13) throughout the volume and hence upon the bounding surface. The components of acceleration thus determined will not, in general, satisfy this equation if the motion of the bounding wall, and hence the function $f$, may be considered as arbitrarily chosen. The contradiction here involved may be resolved by the following conception. We may approach the boundary along a path lying wholly in the enclosed gas. As we approach the boundary along this path by a distance as small as we please, the acceleration is given throughout by the equations of motion (13) but the particle at the terminal point on the boundary has an acceleration which satisfies equation (25). Since \[
\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}
\] are proportional to the direction cosines of the normal to the surface (19) \(^1\) we see that equation (25) affects only the normal component of the acceleration. If the normal component of acceleration obtained as a limit by approach along a path from within the gas does not agree with the value obtained from equation (25), there exists a discontinuity at the bounding surface, $\phi_1$, being the normal component of acceleration, obtained from equation (25) and $\phi_2$ being the value of this component obtained as a limit by approach from within the gas. Thus in this case a second order discontinuity is generated and this will be propagated into the gas according to certain laws which we shall further investigate.

Let us suppose that the values of the acceleration obtained from equations (13) agree with equation (25) on the boundary. We then have further conditions to be fulfilled. When the positions and ve-

\(^1\) Differential and Integral Calculus - Granville, 1911: p. 266
Locities of the particles are given not only the second derivatives of \( x, y, \) and \( z \) may be obtained but every derivative. The second derivatives, 
\[
\frac{\partial^2 x}{\partial a^2}, \frac{\partial^2 x}{\partial a \partial b}, \frac{\partial^2 x}{\partial a \partial c}, \frac{\partial^2 x}{\partial b^2}, \ldots, \frac{\partial^2 z}{\partial c^2}
\]
are known because \( x, y, \) and \( z \) are given as functions of \( a, b, c \). The second derivatives, 
\[
\frac{\partial^2 x}{\partial a \partial t}, \frac{\partial^2 x}{\partial b \partial t}, \ldots, \frac{\partial^2 z}{\partial b \partial t}, \frac{\partial^2 z}{\partial c \partial t}
\]
are known because \( \partial x / \partial t, \partial y / \partial t, \) and \( \partial z / \partial t \) are given as functions of \( a, b, c \). As shown in the preceding paragraph the second derivatives, 
\[
\frac{\partial^2 x}{\partial t^2}, \frac{\partial^2 y}{\partial t^2} \text{ and } \frac{\partial^2 z}{\partial t^2}
\]
are known from the equations of motion (13). Thus all the second derivatives are known as functions of \( a, b, c \). This allows us to calculate by differentiation all the third derivatives except \( \frac{\partial^3 x}{\partial t^3}, \frac{\partial^3 y}{\partial t^3} \) and \( \frac{\partial^3 z}{\partial t^3} \). By differentiating each of equations (13) with respect to \( t \) we obtain these derivatives and hence all the third derivatives are known. Equation (25) may be differentiated with respect to \( t \) for a condition on the boundary and if this relation is not satisfied by the third order time derivatives determined from the given state of the gas, there is a third order discontinuity at the bounding surface. All the fourth order derivatives except \( \frac{\partial^4 x}{\partial t^4}, \frac{\partial^4 y}{\partial t^4} \) and \( \frac{\partial^4 z}{\partial t^4} \) are found by differentiating the third order derivatives with respect to \( a, b, c \) and hence are known, while these derivatives are known by differentiating twice equations (13). This process can obviously be repeated and hence all the derivatives of \( x, y, \) and \( z \) are known. Likewise by differentiating successively equation (25) we obtain added conditions upon these derivatives so that even should the third derivatives at the boundary as determined from the given state agree with the boundary constraint, the fourth order derivatives might not agree and we should have a fourth order discontinuity. Thus, in general, an arbitrary motion of the boundary implies a discontinuity in the gas. In the further progress of the motion beyond the given state should the boundary walls describe a motion discontinuous at a certain
time in the n'th order derivatives, an n'th order discontinuity will be formed in the gas.

To summarize, if the positions and velocities of the particles of gas are given so as to be compatible with the position and velocity of the bounding walls, the acceleration and time derivatives of higher order will not, in general, be compatible with the motion of the bounding walls and discontinuities of the second or higher order will ensue.

II Kinematical Relations Pertaining to a Discontinuity

I wish first to take up the velocity of propagation of a discontinuity in two systems of measurement. The one system of measurement has to do with the equation of the surface of discontinuity in the form:

\[ f(a,b,c,t) = 0 \]  
(26)

and the other in the form:

\[ \phi(x,y,z,t) = 0 \]  
(27)

Let us first consider the former system. In this system we imagine the values \( x, y, z \) assigned to every point in a space whose coordinates are \( a, b, c \). The discontinuity is situated on a moving surface in this space whose equation we may represent by (26). We may imagine a point \( a, b, c \) which moves with a velocity always normal to the surface (26) and so as to remain always in this surface. The velocity of this point we shall call the velocity of propagation of the discontinuity with respect to the initial state at the point considered. We shall designate this velocity by the letter \( \theta \)

1. Hadamard, loc. cit., pp. 101-105. I have elaborated on the proof of the formula for the velocity of propagation which Hadamard only briefly outlines.
The line joining a point \( a', b', c' \) outside the surface of discontinuity to the point \( a, b, c \) on this surface, considered as a vector, has components, \( a-a', b-b', c-c' \). The projection of this line segment upon the normal at \( a, b, c \) is the sum of the projections of these components on the normal. To form the projection of any component we must multiply it by the direction cosine of the normal with respect to the coordinate axis corresponding to this component. Now the direction cosines of the normal are:

\[
\frac{\frac{\partial f}{\partial a}}{\sqrt{\left(\frac{\partial f}{\partial a}\right)^2 + \left(\frac{\partial f}{\partial b}\right)^2 + \left(\frac{\partial f}{\partial c}\right)^2}}, \quad \frac{\frac{\partial f}{\partial b}}{\sqrt{\left(\frac{\partial f}{\partial a}\right)^2 + \left(\frac{\partial f}{\partial b}\right)^2 + \left(\frac{\partial f}{\partial c}\right)^2}}, \quad \frac{\frac{\partial f}{\partial c}}{\sqrt{\left(\frac{\partial f}{\partial a}\right)^2 + \left(\frac{\partial f}{\partial b}\right)^2 + \left(\frac{\partial f}{\partial c}\right)^2}}
\]

since they are proportional to \( \frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial f}{\partial c} \) and since the sum of their squares must equal one. Then the projection of the line segment upon the normal is:

\[
n = \frac{\frac{\partial f}{\partial a} (a-a') + \frac{\partial f}{\partial b} (b-b') + \frac{\partial f}{\partial c} (c-c')}{\sqrt{\left(\frac{\partial f}{\partial a}\right)^2 + \left(\frac{\partial f}{\partial b}\right)^2 + \left(\frac{\partial f}{\partial c}\right)^2}}
\]

(26)

If the point \( a', b', c' \) is fixed and the point \( a, b, c \) is moving with a velocity whose components are \( \frac{da}{dt}, \frac{db}{dt}, \frac{dc}{dt} \), then the rate of change of the normal component of the line segment is obtained by differentiating the above expression for \( n \), namely:

\[
\frac{dn}{dt} = \frac{\frac{\partial f}{\partial a} \frac{da}{dt} + \frac{\partial f}{\partial b} \frac{db}{dt} + \frac{\partial f}{\partial c} \frac{dc}{dt}}{\sqrt{\left(\frac{\partial f}{\partial a}\right)^2 + \left(\frac{\partial f}{\partial b}\right)^2 + \left(\frac{\partial f}{\partial c}\right)^2}}
\]

\[
+ (a-a')\left(\frac{\frac{\partial f}{\partial a}}{\frac{\partial a}{\partial t}} + \frac{\frac{\partial f}{\partial b}}{\frac{\partial b}{\partial t}} + \frac{\frac{\partial f}{\partial c}}{\frac{\partial c}{\partial t}}\right) \frac{\frac{\partial f}{\partial a}}{\sqrt{\left(\frac{\partial f}{\partial a}\right)^2 + \left(\frac{\partial f}{\partial b}\right)^2 + \left(\frac{\partial f}{\partial c}\right)^2}}^2
\]

\[
+ (b-b')\left(\frac{\frac{\partial f}{\partial a}}{\frac{\partial a}{\partial t}} + \frac{\frac{\partial f}{\partial b}}{\frac{\partial b}{\partial t}} + \frac{\frac{\partial f}{\partial c}}{\frac{\partial c}{\partial t}}\right) \frac{\frac{\partial f}{\partial b}}{\sqrt{\left(\frac{\partial f}{\partial a}\right)^2 + \left(\frac{\partial f}{\partial b}\right)^2 + \left(\frac{\partial f}{\partial c}\right)^2}}^2
\]

\[
+ (c-c')\left(\frac{\frac{\partial f}{\partial a}}{\frac{\partial a}{\partial t}} + \frac{\frac{\partial f}{\partial b}}{\frac{\partial b}{\partial t}} + \frac{\frac{\partial f}{\partial c}}{\frac{\partial c}{\partial t}}\right) \frac{\frac{\partial f}{\partial c}}{\sqrt{\left(\frac{\partial f}{\partial a}\right)^2 + \left(\frac{\partial f}{\partial b}\right)^2 + \left(\frac{\partial f}{\partial c}\right)^2}}^2
\]
Now when the point $a', b', c'$ coincides with the point $a, b, c$, it is obvious that $\frac{dn}{dt}$ is the velocity of the point $a, b, c$ such as we have described it and hence is equal to the velocity of propagation $\theta$ at this point. Differentiating equation (25) of the surface, we obtain for the point $a, b, c$, since it lies on this surface, the necessary condition:

$$\frac{\partial f}{\partial \alpha} \frac{da}{dt} + \frac{\partial f}{\partial \beta} \frac{db}{dt} + \frac{\partial f}{\partial \gamma} \frac{dc}{dt} + \frac{\partial f}{\partial t} = 0 \quad (30)$$

Substituting the sum of the first three terms obtained from this equation in the first term of the right hand member of (29) and placing $a = a'$, $b = b'$, $c = c'$, and $\frac{dn}{dt} = \theta$ we obtain:

$$\theta = \frac{-\frac{df}{dt}}{\sqrt{\left(\frac{\partial f}{\partial \alpha}\right)^2 + \left(\frac{\partial f}{\partial \beta}\right)^2 + \left(\frac{\partial f}{\partial \gamma}\right)^2}} \quad (31)$$

We shall adopt the following convention as to sign. The function $f$ in the left member of (26) shall be positive in region 2. The velocity of propagation shall be considered positive or negative according as the surface of discontinuity moves from region 1 into region 2 or in the inverse direction.

By these same methods we may find the velocity of propagation $T$ of the surface of discontinuity with respect to the actual state, merely using equation (27) of the surface and coordinates $x, y, z$ instead of $a, b, c$. The velocity of propagation $T$ is the instantaneous velocity of a point in the surface normal to the surface. For this velocity we have a value analogous to $\theta$ in equation (31), namely

$$T = \frac{-\frac{\partial \phi}{\partial t}}{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2}} \quad (32)$$

The velocities $T$ and $\theta$ are distinct even if the initial state
coincides with the actual state at the instant considered. Now in this
case \( x, y, z \) are respectively equal to \( a, b, c \) at this instant and hence

\( \phi \) and \( f \) are equal at this instant. Therefore \( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \)

are respectively equal to \( \frac{\partial f}{\partial a}, \frac{\partial f}{\partial b}, \frac{\partial f}{\partial c} \). On the other hand we have:

\[
\frac{\partial f}{\partial t} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial t} + \frac{\partial \phi}{\partial t}
\]

(33)

where \( u, v, w \) are components of the velocity. Divide this equation

throughout by \( \sqrt{(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 + (\frac{\partial \phi}{\partial z})^2} \) or its equal \( \sqrt{(\frac{\partial f}{\partial a})^2 + (\frac{\partial f}{\partial b})^2 + (\frac{\partial f}{\partial c})^2} \).

Then by virtue of the statement preceding equation (28), the first
three terms of the right hand member of equation (35) become the normal
component of the velocity at the discontinuity which we shall designate
by \( \nu_n \). Also using equations (31) and (32), equation (33) becomes:

\[-\theta = \nu_n - T\]

or

\[T = \theta + \nu_n\]

(34)

The principal of composition of velocities would also serve to give us
this result for when the initial state is taken coincident with the
actual state, the velocity \( \theta \) is the velocity of propagation of the sur­
face of discontinuity with respect to the medium in its actual state and
the actual velocity of propagation \( T \) may be considered as the result­
ant of the velocity of propagation \( \theta \) with respect to the medium and the
velocity, \( \nu_n \) of the medium itself.

Under this head I wish also to discuss the kinematical relation
existing between the several \( n \)'th order derivatives in an \( n \)'th order
discontinuity. Let \( \phi \) be a quantity which has an \( n \)'th order discon­
tinuity at a surface represented by equation (26). I shall designate
by subscripts 1 and 2 the values of quantities pertaining to the re­
gions 1 and 2 on either side of the surface of discontinuity. Then

\( \phi_1, \phi_2 \)

agree in value and so do corresponding derivatives of order
less than \( n \) of these quantities. Thus we have.
on the surface of discontinuity for \( p, q, r, h \), any positive integers whose sum is \( n - 1 \). Differentiating totally with respect to time, we obtain:

\[
\frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^h} \frac{dx}{dt} + \frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^{h+1}} \frac{dy}{dt} + \frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^h} \frac{dz}{dt} + \frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^{h+1}} = 0
\]

Now according to our assumption of the existence of an \( n \)'th order discontinuity, similarly placed terms in the two members of this equation are not, in general, equal. Bringing all the terms to one side of the equation we may write:

\[
\left[ \frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^h} \right] \frac{dx}{dt} + \left[ \frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^{h+1}} \right] \frac{dy}{dt} + \left[ \frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^h} \right] \frac{dz}{dt} + \left[ \frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^{h+1}} \right] = 0
\]

(35)

where the brackets denote the variations of the quantities they enclose at the surface of discontinuity. Now equation (30) must likewise hold since the total differentiation can only apply on the surface (26).

Dividing the equation (30) throughout by \( \sqrt{\left( \frac{\partial f}{\partial a} \right)^2 + \left( \frac{\partial f}{\partial b} \right)^2 + \left( \frac{\partial f}{\partial c} \right)^2} \) and making use of the statement preceding (28) and of equation (31), we obtain:

\[
\alpha \frac{dx}{dt} + \beta \frac{dy}{dt} + \gamma \frac{dz}{dt} - \theta = 0
\]

(36)

where \( \alpha, \beta, \gamma \) are the direction cosines of the normal at the point considered.

Since \( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \), in equations (35) and (36) still remain arbitrary, the coefficients of these quantities and the non-arbitrary term must be proportional in these two equations and therefore we have:

\[
\frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^h} \alpha = \frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^{h+1}} \beta = \frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^h} \gamma = \frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^{h+1}} \theta
\]

(37)

Since the result is valid for \( p, q, r, h \) any positive integers whose sum is \( n - 1 \), we may write:

\[
\left[ \frac{\partial^n f}{\partial x^p \partial y^q \partial z^r \partial t^m} \right] \alpha \beta \gamma (-\theta)^m = \text{constant} \quad (j + k + l + m = n)
\]

(38)

Placing \( \phi \) successively to \( x, y, z \) and letting the values of the constant in (37) be \( \lambda, \mu, \nu \), respectively, we obtain:

\[
\left[ \frac{\partial^n x}{\partial x^p \partial y^q \partial z^r \partial t^m} \right] = \lambda \alpha \beta \gamma (-\theta)^m \left[ \frac{\partial^n y}{\partial x^p \partial y^q \partial z^r \partial t^m} \right] = \mu \alpha \beta \gamma (-\theta)^m \left[ \frac{\partial^n z}{\partial x^p \partial y^q \partial z^r \partial t^m} \right] = \nu \alpha \beta \gamma (-\theta)^m
\]

As a special case we have :-
The left members of the above equations are components of a vector 
\[ \frac{\partial^n x}{\partial t^n} = \lambda (-\theta)^n \]
\[ \frac{\partial^n y}{\partial t^n} = \mu (-\theta)^n \]
\[ \frac{\partial^n z}{\partial t^n} = \nu (-\theta)^n \]  
(39)

The left members of the above equations are components of a vector and hence \( \lambda, \mu, \nu \) are components of a vector coincident with this vector and called by Hadamard the characteristic segment. Now we see from (38) that as far as the variation in the \( n \)'th derivatives are concerned, the discontinuity depends only upon the characteristic segment and the velocity \( \theta \).

To summarize, I have developed formulae for the velocity of propagation of a discontinuity with respect to the initial and actual state of the gas and have shown the relation between these two velocities when the initial state coincides with the actual state at the instant considered. I have also developed necessary relations between the \( n \)'th order derivatives in an \( n \)'th order discontinuity and have shown that a discontinuity is characterized by a characteristic segment at every point on its surface and a velocity of propagation \( \theta \) at every point.

IV. Stationary Discontinuities

A stationary discontinuity is one which affects always the same particles, that is, a discontinuity for which equation (26) does not involve \( t \) or for which \( \frac{df}{dt} = 0 \). We see from equation (31), therefore, that \( \theta \) is zero. Placing \( \theta = 0 \) in equations (38) we see that
\[ \frac{\partial^n x}{\partial t^n}, \frac{\partial^n y}{\partial t^n}, \frac{\partial^n z}{\partial t^n} \]
are zero unless \( m = 0 \). Thus we see that for a stationary discontinuity, the first derivatives that are discontinuous are those not involving \( t \). We thus see that the \( n \)'th order derivatives
\[ \frac{\partial^n x}{\partial t^n}, \frac{\partial^n y}{\partial t^n}, \frac{\partial^n z}{\partial t^n} \]
can only be discontinuous. 

timous if $x, y, z$ are discontinuous on the surface (26). Then we may take two points $a_1, b_1, c_1$ and $a_2, b_2, c_2$ as close together as we please and separated by the surface of discontinuity, while their actual positions $x_1, y_1, z_1$ and $x_2, y_2, z_2$ are a finite distance apart. Let us consider the equation of the surface in actual coordinates, (27). Then since the discontinuity affects always the same particles, $x_1, y_1, z_1$ and $x_2, y_2, z_2$ must satisfy this equation throughout the motion. Hence a stationary discontinuity represents a motion of the gas on either side of the discontinuity as if the surface of discontinuity was a bounding surface for the two masses of gas on either side. We thus see that the velocity must always be tangential to this surface.

V. Discontinuous Waves

A wave type of discontinuity is one which is propagating from one particle to another, that is, one for which $\delta$ is not zero, and hence this type of discontinuity and the stationary type are mutually exclusive. A discontinuity in $x, y, z$ is therefore eliminated from discussion and we see from equations (36) that a discontinuity of the wave type always affects the time derivatives of $x, y, z$.

1. Derivatives of the Density

The density is defined by the equation of continuity (12)

It is, accordingly, a function of the first derivatives of $x, y, z$ with respect to $a, b, c$. A discontinuity in these first order derivatives produces, in general, a discontinuity in the density. At an $n$'th order discontinuity, the $(n-1)^{th}$ order derivatives of the density are the derivatives first affected. Equation (12) is:

$$\frac{\partial \rho}{\partial t} = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix} = D$$

Let us attempt to form the variation of an \((n-l)\)st order derivative of \(\rho_0/\rho\) at a surface of discontinuity, that is:

\[
\left[ \frac{\partial^m \rho_0}{\partial x^m} \right] = \frac{\partial^{n-l-1} D}{\partial x^{n-l-1} \partial t} - \frac{\partial^{n-l} D}{\partial x^{n-l} \partial t}
\]

(40)

where the first term in the right hand member of this equation is an \((n-l)\)st order derivative by approach from the region 1 and the second term the same derivative by approach from region 2. In forming these two terms by differentiating equation (12), terms containing only derivatives of \(x, y\) or \(z\) of order less than \(n\) may be omitted since the values of these derivatives by approach from regions 1 and 2 agree and hence they disappear from equation (40). The remaining terms are made up of the \((n-l)\)st derivative of each element multiplied by the minor of that element with the algebraic sign corresponding to the position of the element in the determinant. A simplification is introduced if we choose as initial coordinates the actual coordinates in region 2 at the instant considered. Then in region 2, \(x=a, y=b, z=c\) and these equations may be differentiated with respect to \(a, b, c\) for all higher derivatives not involving \(t\). In region 2 we then have \(\frac{\partial x}{\partial a} = \frac{\partial y}{\partial b} = \frac{\partial z}{\partial c} = 1, \frac{\partial x}{\partial b} = \frac{\partial x}{\partial c} = \frac{\partial y}{\partial a} = \frac{\partial y}{\partial c} = \frac{\partial z}{\partial a} = \frac{\partial z}{\partial b} = 0\) and all higher derivatives with respect to \(a, b, c\) zero.

The derivatives of order less than \(n\) on the discontinuity by approach from region 1 agree with the corresponding derivatives by approach from region 2. The evaluation just made for the derivatives in region 2 thus holds for derivatives of order less than \(n\) on the discontinuity by approach from region 1. For a discontinuity of the second or higher order we thus have \(\frac{\partial x}{\partial a} = \frac{\partial y}{\partial b} = \frac{\partial z}{\partial c} = 1\) and \(\frac{\partial x}{\partial b} = \frac{\partial x}{\partial c} = \frac{\partial y}{\partial a} = \frac{\partial y}{\partial c} = \frac{\partial z}{\partial a} = \frac{\partial z}{\partial b} = 0\) and the determinant \(D\) becomes one, all elements except those on the principal diagonal becoming one. The minor of an element not on the principal diagonal is zero while the minor of
a principal element is one. Thus the only terms in the right hand member of (40) remaining are the (n-1)st derivatives of the three principal elements multiplied by their respective minors which are 1. Thus equation (40) becomes:

\[
\left[ \frac{\partial^{n-1}}{\partial x \partial y \partial z \partial x^{h+k}} \right] (\frac{\rho_o}{\rho}) = \left[ \frac{\partial^{n-1}}{\partial x \partial y \partial z \partial x^{h+k}} \right] (\frac{\rho_o}{\rho}) + \left[ \frac{\partial^{n-1}}{\partial x \partial y \partial z \partial x^{h+k}} \right] (\frac{\rho_o}{\rho}) + \left[ \frac{\partial^{n-1}}{\partial x \partial y \partial z \partial x^{h+k}} \right] (\frac{\rho_o}{\rho})
\]

Substituting the values of the variations in the right hand member from equations (38), we have:

\[
\left[ \frac{\partial^{n-1}}{\partial x \partial y \partial z \partial x^{h+k}} \right] (\frac{\rho_o}{\rho}) = \alpha \beta \gamma (\lambda \alpha + \mu \beta + \nu \gamma) (\rho_o) (-\theta)^h \tag{41}
\]

Now any function of \( \rho \) may be expressed in the form \( F(\frac{\rho_o}{\rho}) \) and an (n-1)st derivative of this function is \( F^{n-1}(\frac{\rho_o}{\rho}) \) multiplied by the corresponding derivative of \( \frac{\rho_o}{\rho} \), that is:

\[
\frac{\partial^{n-1}}{\partial x \partial y \partial z \partial x^{h+k}} F \left( \frac{\rho_o}{\rho} \right) = F^{n-1} \left( \frac{\rho_o}{\rho} \right) \frac{\partial^{n-1}}{\partial x \partial y \partial z \partial x^{h+k}} \left( \frac{\rho_o}{\rho} \right)
\]

If we assume an initial state coincident with the actual state in region 2 we have for a second or higher order discontinuity \( \rho = \rho_o \) and we may then take variations at the discontinuity of the above quantity. We thus obtain, making use of equation (41):

\[
\left[ \frac{\partial^{n-1}}{\partial x \partial y \partial z \partial x^{h+k}} \right] F \left( \frac{\rho_o}{\rho} \right) = F^{n-1} \left( \frac{\rho_o}{\rho} \right) \frac{\partial^{n-1}}{\partial x \partial y \partial z \partial x^{h+k}} \left( \frac{\rho_o}{\rho} \right) = F^{n-1} \left( \frac{\rho_o}{\rho} \right) \alpha \beta \gamma (\lambda \alpha + \mu \beta + \nu \gamma) \tag{42}
\]

Let us leave discontinuities of the first order for later discussion. The equations (13) hold valid for the regions 1 and 2. Substituting the value of \( \rho \) from (18), we obtain:

\[
\frac{\partial x}{\partial x} = - \frac{\phi'(\rho_o)}{\delta \rho} \frac{\delta \rho}{\delta x} = \frac{\phi'(\rho)}{\delta \rho} \frac{\delta \rho}{\delta x} \tag{43}
\]

Now for a second or higher order discontinuity, if the initial coordinates are taken as coincidental with the actual instantaneous coordinates in region 2, we see from the relations (14) that the operations, \( \frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z} \) are equivalent respectively to \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \)

the surface of discontinuity. Then for a second order discontinuity equations (43) give, since \( \rho = \rho_0 \),

\[
\begin{align*}
\frac{\partial^2 \chi}{\partial t^2} &= \varphi' (\rho) \frac{\partial}{\partial \rho} (\rho_0) \\
\frac{\partial^2 \eta}{\partial t^2} &= \varphi' (\rho) \frac{\partial}{\partial \rho} (\rho_0) \\
\frac{\partial^2 \zeta}{\partial t^2} &= \varphi' (\rho) \frac{\partial}{\partial \rho} (\rho_0)
\end{align*}
\]  \( (44) \)

Substituting the values of the variations in the left members from equations (38) and the values of those in the right hand members from (41), we obtain:

\[
\begin{align*}
\lambda \theta^2 &= \varphi' (\rho_0) \alpha (\lambda \alpha + \mu \beta + \nu \gamma) \\
\mu \theta^2 &= \varphi' (\rho_0) \beta (\lambda \alpha + \mu \beta + \nu \gamma) \\
\nu \theta^2 &= \varphi' (\rho_0) \gamma (\lambda \alpha + \mu \beta + \nu \gamma)
\end{align*}
\]  \( (45) \)

Since \( \theta \neq 0 \), we may divide one equation by another and we obtain:

\[
\lambda : \mu : \nu = \alpha : \beta : \gamma \]  \( (46) \)

This shows that the characteristic segment is coincident with the normal to the surface and hence we see that the variation of acceleration at the discontinuity is a vector perpendicular to the surface. The quantity \( \lambda \alpha + \mu \beta + \nu \gamma \) by its form designates the projection of the characteristic segment upon the normal but, since the characteristic segment is coincident with the normal, it is the absolute length of the former. If the right hand members of the above equations we have this quantity multiplied by \( \alpha, \beta, \gamma \), which gives the projections of the characteristic segment upon the coordinate axes, that is \( \lambda, \mu, \nu \) respectively. The three equations then reduce to:

\[
\theta^2 = \varphi' (\rho_0) \]  \( (47) \)

The velocity of propagation relative to the medium in its actual state is accordingly that of ordinary sound of infinitesimal amplitude\(^1\). If the pressure \( p \), is not only a function of \( \rho \) but also of \( a, b, c \) in the equations of motion used above we must have in the right members

\(^1\) Lamb, Hydrodynamics, 1911, p 455.
\[
\frac{1}{\rho} \frac{\partial p}{\partial x}, \frac{1}{\rho} \frac{\partial p}{\partial y}, \frac{1}{\rho} \frac{\partial p}{\partial z}
\]
respectively. Since no distinction is involved between the coordinate axes we need only consider one of these expressions and results for the others may be written down by cyclic change of coordinates. Using as initial coordinates the actual instantaneous coordinates in region 2, we have:
\[
\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial p}{\partial a} \frac{\partial a}{\partial x} + \frac{\partial p}{\partial \zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial p}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial p}{\partial \rho} \frac{\partial \rho}{\partial a} + \frac{\partial p}{\partial a}
\]
and
\[
\left[ \frac{\partial p}{\partial a} \right] = \frac{\partial p}{\partial \rho} \left[ \frac{\partial \rho}{\partial a} \right]
\]
since \(\frac{\partial p}{\partial \rho}\) and \(\frac{\partial p}{\partial a}\) are continuous at the discontinuity. On substitution of this result, the only change affected is a displacement of \(\varphi' (\rho)\) by \(\frac{\partial p}{\partial \rho}\). Thus, even under these conditions, the variation in acceleration is normal to the surface and the velocity of propagation is \(\sqrt{\frac{\partial p}{\partial \rho}}\).

We may generalize to a discontinuity of the \(n\)'th order. Differentiating the equation of motion with respect to \(t\) in the initial system of coordinates \(n-2\) times and taking variations we obtain, if the initial coordinates are the instantaneous coordinates in region 2,
\[
\begin{align*}
\frac{\partial y}{\partial t} &= \frac{\partial y}{\partial \rho} \frac{\partial \rho}{\partial t} + \frac{\partial y}{\partial a} \frac{\partial a}{\partial t} + \frac{\partial y}{\partial \zeta} \frac{\partial \zeta}{\partial t} + \frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial t} = \frac{\partial y}{\partial \rho} \frac{\partial \rho}{\partial a} + \frac{\partial y}{\partial a} \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial \rho} \frac{\partial \rho}{\partial t} + \frac{\partial z}{\partial a} \frac{\partial a}{\partial t} + \frac{\partial z}{\partial \zeta} \frac{\partial \zeta}{\partial t} + \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial t} = \frac{\partial z}{\partial \rho} \frac{\partial \rho}{\partial a} + \frac{\partial z}{\partial a}
\end{align*}
\]
Let us consider one of the quantities whose variation occurs in the right member of the above equations. Results for the others may then be deduced by cyclic change of coordinates. Thus
\[
\frac{\partial y}{\partial t} \frac{\partial \rho}{\partial x} = \left( \frac{\partial y}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial y}{\partial a} \frac{\partial a}{\partial x} + \frac{\partial y}{\partial \zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial x} \right) \frac{\partial \rho}{\partial x} = \frac{\partial y}{\partial \rho} \frac{\partial \rho}{\partial a} + \frac{\partial y}{\partial a} + A
\]
Where \(A\) is an expression containing derivatives of \(\frac{\partial \rho}{\partial x}\) of order less than \(n-1\) and derivatives of \(x, y, z\) of order less than \(n\). From the last expression we may transform to the following expression:
\[
\frac{\partial y}{\partial t} \frac{\partial \rho}{\partial x} = \frac{\partial y}{\partial \rho} \frac{\partial \rho}{\partial a} + \frac{\partial y}{\partial a} + B + B \cdot \text{Where } B \text{ is of order less than } n-1 \text{ in } \frac{\partial \rho}{\partial x} \text{ and of order less}
\]
This was suggested by Hadamard, loc. cit., but done here for the first time.
than \( n \) in \( x, y, z \). At the discontinuity \( \frac{\partial}{\partial \nu} \) is equivalent in operation to \( \frac{\partial}{\partial \alpha} \) and in taking variations \( B \) is continuous and hence disappears. Thus we obtain:

\[
\left[ \frac{\partial^{n-2}}{\partial x^{n-2}} \right] \frac{\partial}{\partial y} \left( \frac{\rho_\infty}{\rho} \right) = \left[ \frac{\partial^{n-1}}{\partial x^{n-1}} \right] \frac{\rho_\infty}{\rho}
\]

Similarly, by cyclic change of coordinates we have:

\[
\left[ \frac{\partial^{n-2}}{\partial x^{n-2}} \right] \frac{\partial}{\partial z} \left( \frac{\rho_\infty}{\rho} \right) = \left[ \frac{\partial^{n-1}}{\partial x^{n-1}} \right] \frac{\rho_\infty}{\rho}
\]

Substituting these equations in the equations (48) we obtain:

\[
\frac{\partial x}{\partial t} = \varphi'(\rho_\infty) \left[ \frac{\partial^{n-1}}{\partial x^{n-1}} \right] \frac{\rho_\infty}{\rho}
\]

\[
\frac{\partial y}{\partial t} = \varphi'(\rho_\infty) \left[ \frac{\partial^{n-1}}{\partial x^{n-1}} \right] \frac{\rho_\infty}{\rho}
\]

\[
\frac{\partial z}{\partial t} = \varphi'(\rho_\infty) \left[ \frac{\partial^{n-1}}{\partial x^{n-1}} \right] \frac{\rho_\infty}{\rho}
\]

Substituting the values previously found for the variations in the left (38) and the right (41) members of the equations, we obtain:

\[
\lambda(-\theta)^h = \varphi'(\rho_\infty) \alpha \left( \lambda \alpha + \mu \beta + \nu \gamma \right) (-\theta)^{n-2}
\]

\[
\mu(-\theta)^h = \varphi'(\rho_\infty) \beta \left( \lambda \alpha + \mu \beta + \nu \gamma \right) (-\theta)^{n-2}
\]

\[
\nu(-\theta)^h = \varphi'(\rho_\infty) \gamma \left( \lambda \alpha + \mu \beta + \nu \gamma \right) (-\theta)^{n-2}
\]

For \( \theta \) not equal to zero, we may divide each of these equations by \( (-\theta)^{n-1} \) and the equations become identical with the equations found for a second order discontinuity. The conclusions there drawn, that the characteristic segment is normal to the surface and that the velocity of propagation is \( \sqrt{\varphi'(\rho_\infty)} \) therefore still hold for a \( n \)'th order discontinuity.

3. First Order Discontinuity

a. Continuity Relation

Let us now consider a discontinuity of the first order. Let

Riemann, Über die bortpflanzung ebener Luftwellen von endlicher Schwingungswichte - Göttingen Abhandlung t VIII, 1860
the velocity components, the density and the pressure take on the values $u_1, v_1, w_1, \rho_1, p_1$ and $u_2, v_2, w_2, \rho_2, p_2$, on the surface of discontinuity as we approach this surface from the regions 1 and 2 respectively and let us choose the x-axis coincident with the normal to the surface and reckoned positive in the direction from the region 1 to the region 2. Let $\theta$ be the velocity of propagation with respect to any initial state in which the density has the value $\rho_0$. Then we have from (41) and (38), since $\alpha = 1$, $n = 1$.

$$\begin{align*}
\left[-\frac{\rho_0}{2}\right] &= \left[\frac{\partial x}{\partial \lambda}\right] = \lambda \\
\left[\frac{\partial x}{\partial \lambda}\right] &= \left[\frac{\partial u}{\partial \lambda}\right] = -\lambda \theta
\end{align*}$$

Eliminating $\lambda$,

$$\begin{align*}
[u] + \theta \left[\frac{\rho_0}{2}\right] &= 0
\end{align*}$$

If $\theta_1$ and $\theta_2$ are the velocities of propagation with respect to the regions 1 and 2 reckoned as initial states, we may substitute $\theta_1$ and $\theta_2$ for $\theta$ and $\rho$, and $\rho_1$ and $\rho_2$ respectively for $\rho$. We thus see from the above equation that the following relation must exist between these qualities:

$$\rho_1 \theta_1 = \rho_2 \theta_2 = \rho_0 \theta$$

b. Dynamical Equation and Velocity Formula

Consider a cylinder of infinitesimal cross section area $S$ normal to the surface of discontinuity. In a time $dt$ a mass of gas $\rho_0 S dt$ flows in this cylinder across the surface of discontinuity. This mass of gas may be considered to be enclosed by pistons in this tube. During the passage of this gas across the surface of discontinuity, one piston is on one side of the surface and the other piston on the other side. If $\theta dt$ be taken negligible in comparison with $S$, we may neglect the effect of the pressure on the gas across the lateral wall.
of the cylinder. The force then acting on this gas is $\theta \frac{d}{dt}$ which acts normal to the surface. According to dynamical principles the variation in momentum at the discontinuity must be coincident with the direction of this force and hence the variation of momentum and therefore the variation of velocity, is a vector normal to the surface. We obtain the dynamical equation by equating the impulse of the force to the change in momentum, as follows:

$$(p_1 - p_2) S \frac{d}{dt} = \rho_0 \theta (u_1 - u_2) S \frac{d}{dt}$$

or dropping the factor $S \frac{d}{dt}$,

$$p_1 - p_2 = \rho_0 \theta (u_1 - u_2)$$  \hspace{1cm} (54)$$

Both the continuity equation and the dynamical equation were first given by Riemann. He applied them specifically to plane waves. By eliminating $u_1 - u_2$ between these two equations we obtain the formula for the velocity of propagation:

$$\theta = \sqrt{\frac{p_1 p_2}{\rho_1 \rho_2} \frac{p_1 - p_2}{p_1 + p_2}}$$  \hspace{1cm} (55)$$

As $\rho_1$ approaches $\rho_2$ and $p_1$ approaches $p_2$, this expression approaches the value: $\theta = \frac{p_1}{p_2} \sqrt{\frac{\rho_1}{\rho_2}}$ or from equation (53), $\theta_1 = \frac{dp_1}{d\rho_1} = \theta_2$  \hspace{1cm} (56)$$

and hence the velocity of propagation with respect to the medium on either side of the discontinuity approaches the velocity of propagation of ordinary sound of infinitesimal amplitude. If, however, we assume Poisson's adiabatic law (21) we have:

$$\frac{\rho_1}{\rho_2} \frac{p_1}{p_2}$$

and for given values as large as we please by taking $p_2$ sufficiently large, $\theta$ becomes

c. Equation of energy

Lord Rayleigh \textsuperscript{1} objected to this solution for plane waves because the equation of energy is not satisfied. By impressing upon

\textsuperscript{1} Rayleigh, Theory of Sound - 1896, pp.32-33, 40, 41.
the gas as a whole a velocity equal and opposite to the velocity of propagation of the discontinuity, the latter is brought to rest with respect to our fixed axes. The phenomenon is then one of steady motion and Bernoulli's equation of energy (5) is applicable. This gives:

$$ \int \frac{dp}{\rho} + \frac{1}{2} u^2 = \text{constant} $$

When the discontinuity is brought to rest, \( u_1 = -\theta_1 \), \( u_2 = -\theta_2 \), and from equation (53) we obtain:

$$ \rho_1 u_1 = \rho_2 u_2 \quad \text{then} \quad \rho_1 u_1 = \rho_2 u_2 $$

Bernoulli's equation of energy may be written as:

$$ \int \frac{dp}{\rho} = \frac{1}{2} u_1^2 - \frac{1}{2} u_2^2 = \frac{1}{2} u^2_1 (1 - \frac{\rho_1^2}{\rho_2^2}) $$

Here \( u, \rho, \) and \( p \) may be considered as constant and \( u_2, \rho_2, p_2 \) as variable. Then differentiating we have:

$$ \frac{dp_2}{d\rho_2} = \frac{u_2^2 \rho_1^2}{\rho_2^2} $$

and integrating we obtain:

$$ p_2 = \text{constant} - \frac{u_2^2 \rho_1^2}{\rho_2^2} $$

This is the only law of pressure for which the energy condition and the equation of continuity are satisfied. This is not fulfilled for any actual gas and on this basis Rayleigh concludes that a first order discontinuity is impossible without dissipative forces. A force acting on the gas is a new quantity entering the dynamical equation and may be determined so as to produce a steady state with any law of pressure. Weber\(^1\) considers the energy change in a discontinuity of the first order ab initio and finds for the adiabatic law of Poisson (21) a change of energy in passing the discontinuity. He says that we must in this case consider the law of Carnot, in accordance with which there may be a loss of energy but never a gain. From this criterion he

determines that a discontinuity of condensation is possible but one of rarefaction impossible. Weber says that the loss of energy must be made up by the formes that move the pistons. Beyond this statement he does not go in his explanation. I think a solution to this question must be obtainable by a consideration solely of what happens in the immediate neighborhood of the discontinuity and that, as Rayleigh says, for the adiabatic law of Poisson the discontinuity does not obey the law of energy and that this solution is therefore invalid.

d. Hugoniot's law.

Hugoniot object to the use of the adiabatic law of Poisson (21) as the validity of this law depends on a change of density which is continuous. He said that this law is no more valid for abrupt discontinuous changes of density. He derives a law which is valid for the latter case. He uses the internal energy of a gas which is the energy change not accounted for by the kinetic energy and work done by external forces but which must exist because of the law of conservation of energy. The internal energy of a mass of gas is a function solely of its physical state and since all energy is additive it varies as the mass of gas. Let us consider again the mass of gas, \( \rho_0 \theta S dt \), which flows across the surface of discontinuity through a cylinder of infinitesimal cross-section \( S \) in the time \( dt \). Then the internal energy of this gas suffers a change of magnitude \( \left[ \eta(\rho, p) \right] - \left[ \eta(\rho_2, p_2) \right] \rho_0 \theta S dt \) where \( \eta(\rho, p) \) is the specific internal energy, that is, the internal energy per unit mass.

Since the work done by the pressures in its passage from one side of the surface of discontinuity to the other is \( \left( p_1 u_1 - p_2 u_2 \right) S \) \( dt \) and the change in kinetic energy is \( \frac{1}{2} \rho_0 \theta \left( u_1^2 - u_2^2 \right) S \) \( dt \), the equation of energy becomes, dropping the factor \( S \) \( dt \), \( p_1 u_1 - p_2 u_2 = \left[ \eta(\rho_1, p_1) - \eta(\rho_2, p_2) \right] \rho_0 \theta + \rho_0 \theta \frac{u_1^2 - u_2^2}{2} \) (60)

Hugoniot, Journal de l'ecole Polytechnique, 1887, 1889
This equation should remain valid if any velocity is added to the gas as a whole and the above equation should, therefore, contain only the quantity $u_1 - u_2$. To make this evident, multiply the dynamical equation (54) by $\frac{u_1 + u_2}{2}$ and subtract from the preceding equation and we obtain:

$$\frac{(p_1 + p_2)(u_1 - u_2)}{2} = [\eta(\rho_1, p_1) - \eta(\rho_2, p_2)]\rho_0 \theta \quad (61)$$

Eliminating $u_1 - u_2$ by means of the equation of continuity (52), we obtain

$$\eta(\rho_1, p_1) - \eta(\rho_2, p_2) = \frac{(p_1 + p_2)(\rho_1 - \rho_2)}{2\rho_1^2} \quad (62)$$

For a perfect gas the internal energy has the form $\eta(\rho, p) = \frac{1}{y-1} \frac{\rho}{p}$. and in this case we obtain:

$$\frac{1}{y-1} \left( \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} \right) = \frac{(p_1 + p_2)(\rho_1 - \rho_2)}{2\rho_1^2} \quad (63)$$

or

$$\frac{p_2}{p_1} = \frac{(y+1)\rho_2 - (y-1)\rho_1}{(y+1)\rho_1 - (y-1)\rho_2} \quad (64)$$

This is the adiabatic law of Hugoniot the validity of which is recognized without question by all the French students of this question. Hugoniot called this law the dynamical adiabatic law in contrast with Poisson's relation which he called the static adiabatic law. As $\rho_1$ approaches $\rho_2$ and $p_1$ approaches $p_2$, Hugoniot's law approaches the form:

$$\frac{dp}{d\rho} = \frac{\gamma p}{\rho}$$

and hence agrees with the law of Poisson. By the very method of its derivation the law of Hugoniot satisfies the condition of energy and hence is not open to the objection which we found in the application of Poisson's law to a first order discontinuity.

e. Entropy and Thermodynamic Potential

We shall need the quantities, entropy and thermodynamic potential for later discussion and it is advisable here to define them.

1. Sur le Propagation des Ondes * Hadamard, 1903, p 191
If a system or part of a system at an absolute temperature $T$ receives a quantity of heat $dQ$, the increase of entropy is:

$$d\sigma = \frac{dQ}{T} \quad (65)$$

If the temperature of the system is not constant, we must break it up into elements and sum up the entropy of each element. Now the change in the internal energy in changing from one state to another is made up of the heat gain and of the work done on the gas, when the work necessary to produce the change in kinetic energy as a whole is abstracted from consideration. We shall let $\eta$ be the specific internal energy, and we shall reckon the work, $W$, as positive when it is work done by the gas and negative if it is done on the gas. Then we must have, for a unit mass of gas

$$d\eta = dQ - dW$$

Substituting the value of $d\eta$ from (65) we obtain:

$$d\eta = Td\sigma - dW \quad (66)$$

From the definition of work we have:

$$dW = p\,dv$$

where $v$ is the specific volume of the gas

$$= \frac{-p}{\rho^2} \, d\rho$$

Substituting this value of $dW$ in (66) we obtain:

$$d\eta = Td\sigma + \frac{p}{\rho^2} \, d\rho \quad (67)$$

The thermodynamical potential is defined as:

$$\mathcal{F} = \eta - T\sigma \quad (68)$$

Differentiating we have:

$$d\mathcal{F} = d\eta - Td\sigma - \sigma dT$$

Substituting the value of $d\eta$ from (67) we obtain:

$$d\mathcal{F} = -\sigma dT + \frac{p}{\rho^2} \, d\rho = \frac{\partial \mathcal{F}}{\partial T} \, dT + \frac{\partial \mathcal{F}}{\partial \rho} \, d\rho$$

1. G. H. Bryan, Thermodynamics, 1907, p. 58
2. Bryan, Thermodynamics, 1907, pp. 91, 92.
Hence we have:
\[
\frac{\partial^2 f}{\partial T^2} = -\sigma \quad (69)
\]
\[
\frac{\partial^2 f}{\partial \rho^2} = \frac{\partial p}{\partial \rho^2} \quad (70)
\]
For irreversible processes the entropy of a system must increase.
In fact the increase of entropy may be taken as a measure of the irreversibility of a process. A reversible process is only an ideal and we may consider the increase of entropy a necessary condition in all actual physical changes.

f. Application of the Entropy Condition to a First Order Discontinuity.

Duhem uses as a foundation of his study of first order discontinuities equations (68), (69) (70) and the equations of Riemann (52) and(§4). The condition of increase of entropy, he writes by means of (69):
\[
\frac{\partial^2 f}{\partial T^2} < 0 \quad (71)
\]
He also uses the condition that the pressure must increase with increase of density or that \(\frac{dp}{d\rho}\) is positive always. By means of equation (70) this equation may be written:
\[
2 \frac{\partial f}{\partial \rho} + \rho \frac{\partial^2 f}{\partial \rho^2} > 0 \quad (72)
\]
He also uses Hugoniot's law in the general form (62). All functions are considered functions of \(\rho\) and \(T\). The discontinuity is supposed to have values of the density and temperature \(\rho_1, T_1\) by approach from the region 1 and \(\rho_2, T_2\) by approach from the region 2. At the discontinuity \(\rho_1\) and \(T_1\) are considered as fixed and all functions are then functions of \(\rho_1\). We may thus write:
\[
p = \Pi(\rho_1), \quad T = \Theta(\rho_1), \quad \sigma(\rho_1, T_1) = \Xi(\rho_1)
\]
Substitute in equation (62) the value of \(\eta\) from (68) and we obtain:
\[
\rho_1 \rho_2 \left[f(\rho_1, T_1) - f(\rho_2, T_1) + T_1 \sigma(\rho_1, T_1) - \theta(\rho_1) \xi(\rho_1)\right] = \rho_1 - \rho_2 \left[\rho_1 + \Pi(\rho_1)\right] \quad (73)
\]
Differentiating with respect to \(\rho_2\) and using equations (69) and (70) we obtain:
\[
\frac{\rho_1 \rho_2}{2} \frac{d \Pi(\rho_2)}{d \rho_2} - \frac{\rho_1 - \Pi(\rho_2)}{2} \rho_2 \epsilon(\rho_2) - \frac{\rho_1 \rho_2}{2} \Theta(\rho_2) \frac{d \xi(\rho_2)}{d \rho_2} = 0 \quad (74)
\]
By differentiating the value of \( \sigma \left( \rho_2, T_2 \right) = \Sigma \left( \rho_2 \right) \) from equation (69) and the value of \( p_2 = \mathcal{T} \left( \rho_2 \right) \) from equation (70) with respect to \( \rho_2 \) we obtain:

\[
\frac{d\Sigma \left( \rho_2 \right)}{d\rho_2} = - \frac{\partial^2 \mathcal{T} \left( \rho_2, T_2 \right)}{\partial \rho_2^2} - \frac{\partial^2 \mathcal{S} \left( \rho_2, T_2 \right)}{\partial \rho_2^2} \frac{d\theta \left( \rho_2 \right)}{d\rho_2} \tag{75}
\]

\[
\frac{d\mathcal{T} \left( \rho_2 \right)}{d\rho_2} = \frac{1}{\rho_2} \left[ \rho_2 \frac{d\mathcal{S} \left( \rho_2, T_2 \right)}{d\rho_2} \right] + \frac{\partial^2 \mathcal{T} \left( \rho_2, T_2 \right)}{\partial \rho_2^2} \frac{d\theta \left( \rho_2 \right)}{d\rho_2} \tag{76}
\]

It is of advantage to introduce here the quantity \( \mathcal{V} \left( \rho_2, T_2 \right) = \sqrt{d\mathcal{T} \left( \rho_2 \right)} \) which is the ordinary velocity of sound of infinitesimal amplitude in the region 2 with respect to this medium. Using this quantity and the quantity \( \left( \frac{dp_2}{d\rho_2} \right) (p_2 \text{ constant}) \) we may express the result of the elimination of \( \frac{d\theta \left( \rho_2 \right)}{d\rho_2} \) between the above two equations in the following form:

\[
\frac{d\mathcal{T} \left( \rho_2 \right)}{d\rho_2} = \left[ \mathcal{V} \left( \rho_2, T_2 \right) \right]^2 + \frac{2p_2}{\partial^2 \mathcal{S} \left( \rho_2, T_2 \right)} \frac{d\mathcal{S} \left( \rho_2, T_2 \right)}{d\rho_2} \left( \frac{dp_2}{d\rho_2} \right) \frac{d\Sigma \left( \rho_2 \right)}{d\rho_2} \tag{77}
\]

In equation (74) substitute:

\[
p = \mathcal{T} \left( \rho_2 \right) = \frac{\rho_0 \theta \left( \rho \right)}{T_1} (\rho_1 - \rho_2)
\]

obtained from the velocity equation of Heimann (55). Between the resulting equation and equation (77), eliminate \( \frac{d\mathcal{T} \left( \rho_2 \right)}{d\rho_2} \) and we obtain:

\[
\left[ 2p_2 \frac{d\mathcal{S} \left( \rho_2, T_2 \right)}{d\rho_2} + \rho_2^2 \frac{d^2 \mathcal{S} \left( \rho_2, T_2 \right)}{d\rho_2^2} \left( \frac{dp_2}{d\rho_2} \right) (\rho_1 - \rho_2) \right] \frac{d\Sigma \left( \rho_2 \right)}{d\rho_2} = \left[ \mathcal{V} \left( \rho_2, T_2 \right) \right]^2 \left( \rho_1 - \rho_2 \right)
\]

Duhem \(^1\) writes this equation with the right member of opposite sign but his error is evident on complete investigation. This error led him to advance some wrong conclusions but they are easily righted. This error is evident also from the contradiction in the conclusions which he draws in the general case and in the case of a perfect gas.

From this equation we see that for \( \rho_2 = \rho_1 \),

\[
\frac{d\Sigma \left( \rho_2 \right)}{d\rho_2} = 0
\]

Duhem \(^2\) does not mention that this conclusion might have been obtained from equation (71) since \( \mathcal{T} \left( \rho_2 \right) = p_1 \) when \( \rho_2 = \rho_1 \). Differentiating equation (78) we obtain an equation of the following form:

\(^1\) Loc. cit. p.
\(^2\) Loc. cit. p.
where $A$ and $B$ are finite for $\rho_1 = \rho_2$. Then for $\rho_1 = \rho_2$ we must have from the above equation,

$$\frac{d^2 \xi (\rho_2)}{d \rho_2^2} = 0$$

Thus for a discontinuity with an infinitesimal variation in density, the variation in entropy is an infinitesimal of at least the third order.

According to the convention of sign already used, if $\theta_1$ and $\theta_2$ are positive, the gas passes from the region 2 to the region 1 and $\xi (\rho_2) - \sigma (\rho_1, T)$ must be negative since the entropy must increase; if $\theta_1$ and $\theta_2$ are negative, the gas passes from the region 1 to the region 2 and $\xi (\rho_2) - \sigma (\rho_1, T)$ must be positive. Thus the difference in entropy, $\xi (\rho_2) - \sigma (\rho_1, T)$ is opposite in sign to $\theta_1$ and $\theta_2$. Since the variation of entropy is of the third order $\xi (\rho_2) - \sigma (\rho_1, T)$ has the sign of $(\rho_2 - \rho_1) \frac{d^3 \xi (\rho_2)}{d \rho_2^3}$ as long as $\rho_1 - \rho_2$ does not exceed in absolute value a certain limit.

By differentiating equation (80) and placing $\rho_2 = \rho_1$ we easily obtain:

$$\frac{d^2 \xi (\rho_2)}{d \rho_2^2} = \left[ \frac{l}{2 \rho_1^3} \right]$$

For brevity let us introduce the function

$$H (\rho_1, T_1) = \left[ 2 \frac{d \Pi (\rho_1)}{d \rho_1} + \rho_1 \frac{d^2 \Pi (\rho_1)}{d \rho_1^2} \right]$$

Then $\xi (\rho_2) - \sigma (\rho_1, T_1)$ has the sign of $(\rho_2 - \rho_1) H (\rho_1, T_1)$ and $\theta_1$ and $\theta_2$ must have signs opposite to this quantity. In gases for which $H (\rho_1, T_1)$ is positive a positive discontinuity $(\theta_1, \sigma)$ can only be propagated if $\rho_1 > \rho_2$ or if the wave is one of condensation and for a gas for which $H (\rho_1, T_1)$ is negative, a positive discontinuity can only be propagated if $\rho_1 < \rho_2$ or if the wave is one of rarefaction. In the same way, of course, if the
discontinuity is propagated in the negative direction, condensation or rarefaction are possible according as \( H(\rho_1, T_1) \) is positive or negative respectively. For actual gases \( H(\rho_1, T_1) \) is positive and hence only condensations are possible.

Equation (78) involves the quantity \( V(\rho_1, T_1) - \theta_2^z \). By determining the sign of this quantity, we determine the relative magnitude of the velocity of propagation, with respect to the gas in region 2 compared with the velocity of propagation or ordinary sound in this region. In this equation the coefficient of \( \frac{d\xi(\rho_2)}{d\rho_2} \) is positive if in absolute value is less than a certain limit since \( 2\rho_1 \theta(\rho_2) \) is positive. Also \( \frac{d\xi(\rho_2)}{d\rho_2} \) has the sign of \( H(\rho_1, T_1) \) and therefore is positive or negative according as the wave is one of condensation or rarefaction. We may express these results by saying that \( \frac{d\xi(\rho_2)}{d\rho_2} \) and \( \rho_2 - \rho_1 \) have the same sign if the region 2 is behind the wave and opposite signs if this region is in front of this wave. From the above mentioned equation we thus see that with respect to the region behind the wave, the velocity of propagation is less than that of ordinary sound and with respect to the region in front of the wave it is greater than that of ordinary sound. These results hold, of course, provided \( \rho_2 - \rho_1 \) does not exceed a certain limit in absolute value.

Duhem also examines the sign of the quantity \( \theta_2^z - [V(\rho_1, T_1)]^2 \). In so doing he compares the velocity of propagation with respect to one region with the velocity of ordinary sound in the other medium. For \( \rho_2 = \rho_1 \) this quantity is zero. By differentiating the expression for the velocity as given by Riemann (55) we obtain:

\[
\frac{d\theta_2^z}{d\rho_2} = \frac{\rho_2 \beta(\rho_2)}{\rho_2^z (\rho_2 - \rho_1)} \frac{d\Pi(\rho_2)}{d\rho_2} - (\rho_2 - \rho_1)[\Pi(\rho_2) - \Pi_1]
\]

where \( \beta(\rho_2) = \frac{\rho_2 (\rho_2 - \rho_1)}{d\Pi(\rho_2)} - (\rho_2 - \rho_1)[\Pi(\rho_2) - \Pi_1] \)

For \( \rho_2 = \rho_1 \), \( \beta(\rho_1) = 0 \) therefore we must examine \( \frac{d\theta_2^z}{d\rho_2} \). By differenti-
iating we have:

\[
\frac{d\beta(\rho_2)}{d\rho_2} = \rho_2 (\rho_2 - \rho_1) \frac{d^2 T(\rho_2)}{d\rho_2^2} - 2 \frac{d T(\rho_2)}{d\rho_2} \rho_2 - \frac{d^2 T(\rho_2)}{d\rho_2^2} \tag{85}
\]

For \( \rho_2 = \rho_1 \) this is zero and we must examine the next derivative.

For this we obtain the expression:

\[
\frac{d^2 \beta(\rho_2)}{d\rho_2^2} = (\rho_2 - \rho_1) \left[ \frac{d^3 T(\rho_2)}{d\rho_2^3} + \rho_2 \frac{d^3 T(\rho_2)}{d\rho_2^3} - 2 \frac{d T(\rho_2)}{d\rho_2} \rho_2 \frac{d^2 T(\rho_2)}{d\rho_2^2} \right] \tag{86}
\]

and

\[
\frac{d^2 \beta(\rho_2)}{d\rho_2^2} \rho_2 = \rho_1 = \frac{d^2 T(\rho_2)}{d\rho_2^2} \rho_2 = \rho_1 \tag{87}
\]

Thus we see that for \( \rho_2 - \rho_1 \) less than a certain limit in absolute value, \( \beta(\rho_2) \) has its sign opposite to that of \( \frac{\partial K(\rho_1, T_1)}{\partial \rho_2} \) and \( \Theta - \frac{\sqrt{V(\rho_1, T_1)}}{\rho_1} \) has the sign of \( -(\rho_2 - \rho_1) \frac{\partial K(\rho_1, T_1)}{\partial \rho_2} \).

If \( K(\rho_1, T_1) \) is positive, the velocity of propagation of the discontinuity with respect to the less dense region is greater than the velocity of sound in the more dense region; if \( K(\rho_1, T_1) \) is negative, the velocity of propagation of the discontinuity with respect to the more dense region is less than the velocity of sound in the less dense region.

Duhem\(^1\) then applies these theories to a perfect gas. If \( C \) and \( c \) are the specific heats under constant pressure and volume respectively we have Hugoniot's law (64) in the following form:

\[
\left\{ \frac{\rho_2}{\rho_1} \right\} = \frac{2 c \rho_2 + (C-c)(\rho_2 - \rho_1)}{2 c \rho_1 + (C-c)(\rho_1 - \rho_2)} \tag{88}
\]

Either the numerator or denominator of the fraction in the right member are positive and since the pressures must be positive we must have both numerator and denominator positive.

\[
2 c \rho_2 + (C-c)(\rho_2 - \rho_1) > 0 \tag{89}
\]

\[
2 c \rho_1 + (C-c)(\rho_1 - \rho_2) > 0
\]

We may without loss of generality attribute the subscript 2 to the greater of the two densities. Then the first inequality is fulfilled of itself. The second may be written in the form:

\[
\frac{\rho_2}{\rho_1} < \frac{C + c}{C - c} \tag{90}
\]

Loc. cit. p
For most diatomic gases \( \frac{C}{c} = 1.4 \) approximately and \( \frac{\rho_2}{\rho_1} \) has a limiting value near 6. As the density increases with a fixed \( \rho_1 \), the pressure \( p_2 \) becomes infinite as \( \rho_2 \) approaches \( \frac{C + c}{C - c} \rho_1 \). Eliminating \( p_2 \) between equations (55) and (88) we obtain, when we place \( \Phi = \Phi_1 \) and \( \rho_0 = \rho_1 \)

\[
\frac{\theta_1}{\rho_1} = \frac{p_1}{\rho_1} - \frac{2C\rho_2}{\rho_1} \frac{\rho_2}{\rho_1} + (C - c)(\rho_1 - \rho_2)
\]

Then we have:

\[
\theta_1 - \left[ \frac{\theta}{\rho_1, T_1} \right]^2 = \frac{C}{c} \frac{\rho_1}{\rho_1} \frac{(C + c)(\rho_1 - \rho_2)}{2c \rho_1 + (C - c)(\rho_1 - \rho_2)}
\]  

(91)

This shows that the velocity of propagation of a discontinuity with respect to the region behind the wave is less than the velocity of sound in the same region and the velocity with respect to the region in front of the wave is greater than that of sound in this region. We see that the function of equation (87), \( K(\rho, T) \), has the following form:

\[
K(\rho, T) = \frac{C}{c} \frac{\rho_1}{\rho_1} \frac{C}{C - c}
\]

(92)

This is positive or negative according as \( C \) is less or greater than 3. In all known cases the former is true. In this case \( \theta_1 - \left[ \frac{\theta}{\rho_1, T_1} \right]^2 \) is easily shown to be positive for \( \rho_1 \) less than \( \frac{2C}{C - c} \) and negative if it is greater than this quantity.

M. E. Jouguet\(^1\) obtained some of the above results previous to M. Duhem but by analagous methods. He also used the criterion of entropy and derived the identical conditions to those given above for the possibility of a condensation and rarefaction. He shows that if the medium is of small viscosity, a small stratum of great change in velocity and density will result such that the viscosity may be neglected everywhere save in this stratum. If the thickness of this stratum is so

M. E. Jouguet, Comptes Rendus 138 p 1685, 1904; Comptes Rendus 139, p.786, 1904
ininitely small, the laws of propagation are those given by the form-
ulae of Riemann (equations 52 and 54) and of Hugoniot (equation 64).

3. Motion Following an Initial Plane Wave Discontinuity.

An initial discontinuity of the first order will not in
general, be propagated as such for the equations of Riemann, (52) and
(54), together with (18), giving the law of pressure, will not, in gen-
eral be satisfied for arbitrary values of $u_1, \rho_1, u_2, \rho_2$ in an initial
discontinuity. The initial discontinuity may resolve itself into
two discontinuities of the first order or into discontinuities of the
second order.

1) Riemann's Treatment

Riemann treated of an initial plane wave discontinuity
in an unbounded gas in which the regions 1 and 2 were in a constant
uniform condition of velocity and density such that throughout regions 1 and 2
the velocity was $u_1$ and the density $\rho_1$. Weber gives Riemann's treatment
in a very concise, clear manner. I shall here follow his treatment.

Riemann showed, that, conformably with the equations of motion (3)
and (4) applicable to regions where all the quantities involved are
continuous and with equations (52) and (54) applicable to a first
order discontinuity, assuming the pressure given by (18), an initial
plane wave discontinuity, could be satisfied by one of four cases de-
pending upon the initial values of the velocity and density in regions
1 and 2. These cases involve resolution into discontinuities of
compression and a special type of rarefaction wave. The former have
already been discussed. The latter type of wave involves the appli-
cation to a certain domain of the $x-t$ plane of a solution of the type
\[ u = \pm \int \frac{\phi(\rho)}{\rho} d\rho + c = \mp \sqrt{\phi(\rho)} + \frac{x}{t}. \]

This solution may easily be verified

1. Loc. cit.

2. Weber, Die Partiellen Differential-gleihungen IIpp480-88
by substitution in equations (3) and (4). I shall discuss this solution more fully later but I shall need the result for use in the present case. Riemann's four cases are, then: (1) a resolution into two discontinuities of compression propagated in opposite directions with respect to the gas, (2) a resolution into two rarefaction waves propagated in opposite directions with respect to the gas, (3) a resolution into a positively propagated discontinuity and a negatively propagated rarefaction wave, and (4) a resolution into a negatively propagated discontinuity and a positively propagated rarefaction wave.

We may roughly see the conditions for some of these cases. For the first case, the intermediate region created is a region of greater density than either of the domains 1 and 2 and the velocities must obviously be such that the two masses of gas in regions 1 and 2 impact upon one another. Let us study these cases in detail.

Consider Riemann's first case. The initial discontinuity resolves itself into two discontinuities of compression. In front of the first discontinuity \( u \) and \( \rho \) retain their initial values \( u_1 \) and \( \rho_1 \) and behind the second discontinuity they retain their initial values \( u_2, \rho_2 \) while in the region between the two discontinuities they have the constant values \( u', \rho' \). Then the velocities of propagation of the two discontinuities are constants. We may represent the position of each of the discontinuities at a time \( t \) by a point in a plane in which \( x \) and \( t \) are taken as rectangular coordinates. In this plane the motion of each discontinuity is represented by a line since the velocity of propagation \( \frac{dx}{dt} \) is constant. The angle made by this line with the \( x \)-axis is such that its cotangent is \( \frac{dx}{dt} \), the velocity of propagation. In the accompanying Fig. 1 the discontinuities are propagated along the lines 1 and 2 which make angles \( \alpha_1 \) and \( \alpha_2 \) with the \( x \)-axis. By means of equation (34) we then obtain:
From the above formulae,

\[ T_1 = \cot \alpha_1 = u_1 - \sqrt{\frac{\rho_1}{\rho_1^2 - \rho^2}} \frac{\varphi' - \varphi}{\varphi' - \varphi(\rho)} \]  
(93)

\[ T_2 = \cot \alpha_2 = u_2 + \sqrt{\frac{\rho_2}{\rho_2^2 - \rho^2}} \frac{\varphi' - \varphi}{\varphi' - \varphi(\rho)} \]  
(94)

Adding we have:

\[ u_1 - u_2 = \sqrt{(\rho_1^2 - \rho^2) \frac{\varphi' - \varphi}{\rho_1^2 \rho_2}} + \sqrt{(\rho_2^2 - \rho^2) \frac{\varphi' - \varphi}{\rho_1 \rho_2}} \]  
(97)

The function, \( \frac{(\rho_1^2 - \rho^2) \varphi(\rho') - \varphi(\rho)}{\rho_1^2 \rho_2} = \left( \frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} \right) \varphi(\rho') - \varphi(\rho) \) for \( \rho^2 > \rho_1^2 \). For Riemann's first case, \( \rho' \) must be greater than both \( \rho_1 \) and \( \rho_2 \) and hence the right member of the above equation is an increasing function of \( \rho' \). When \( \rho^2 = \rho_1^2 \) and when \( \rho^2 = \rho_2^2 \), the right member of the above equation becomes \( \frac{(\rho_1^2 - \rho_2^2) \varphi(\rho_1) - \varphi(\rho_2)}{\rho_1 \rho_2} \) and therefore we must have the following inequality:

\[ u_1 - u_2 \geq \sqrt{\frac{(\rho_1^2 - \rho_2^2) \varphi(\rho_1) - \varphi(\rho_2)}{\rho_1 \rho_2}} \]  
(98)

If this inequality is fulfilled, there is a single value of \( \rho' \) which satisfies equation (97) and when \( \rho' \) is found, \( u' \) is obtained from either of equations (95) or (96). The velocities of propagation are found from (93) and (94) and the necessary relation,

\[ \cot \alpha_1 > \cot \alpha_2 \]  
(98)

is, of itself, verified. If the critical inequality (98) above is an equality, we see from the last equation that \( \rho' \) equals either \( \rho_1 \) or \( \rho_2 \) and
we have a single discontinuity propagated backward or forward respectively.

Consider next Riemann's second case. In the accompanying Fig. 2 we take four straight lines, 1, 2, 3, 4 with the angles \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) with the x axis respectively and assume in the sector \((-\infty, 0] \) \( u \) and \( \rho \) constant and equal to their initial values \( u_1, \rho_1 \) and in the sector \((4, 0, +\infty) \) likewise constant and equal to \( u_2, \rho_2 \). In the sector \((2, 0, 3) \) let \( u \) and \( \rho \) be constant and equal to \( u', \rho' \). In the sectors \((1, 0, 2) \) and \((3, 0, 4) \) we assume solutions of the type 

\[
\begin{align*}
  u &= \pm \int \frac{\sqrt{\varphi'(\rho)}}{\rho} \, d\rho + c = \pm \sqrt{\varphi'(\rho)} + \frac{x}{t}.
\end{align*}
\]

taken so that along the lines 1, 2, 3, 4 these solutions shall be continuous with the values in the bordering domains. A solution of this type has the characteristic that for \( \frac{x}{t} \) constant, \( u \) and \( \rho \) are constant, in other words, \( u \) and \( \rho \) have constant values along every line emanating from 0.

Let us for brevity write \( f(\rho) = \int \frac{\varphi'(\rho)}{\rho} \, d\rho \). We may tabulate the above conditions as follows:

\[
\begin{align*}
  (\ -\infty, 0, 1) : & \quad u = u_1, \quad \rho = \rho_1, \text{ constant} \\
  (1, 0, 2) : & \quad u = -f(\rho) + c = \sqrt{\varphi'(\rho)} + \frac{x}{t} \\
  (2, 0, 3) : & \quad u = u', \quad \rho = \rho', \text{ constant} \\
  (3, 0, 4) : & \quad u = f(\rho) + c' = -\sqrt{\varphi'(\rho)} + \frac{x}{t} \\
  (4, 0, +\infty) : & \quad u = u_2, \quad \rho = \rho_2, \text{ constant}
\end{align*}
\]

The condition of continuity between these regions gives the following relations:

\[
\begin{align*}
  u_1' &= -f(\rho') + c = \sqrt{\varphi'(\rho')} + \cot \alpha'
\end{align*}
\]
From these eight equations we must determine the eight quantities, $c, c', u', \rho', \alpha_1, \alpha_2, \alpha_3, \alpha_4$. Eliminating $c$ and $c'$ we obtain:

\begin{align*}
\text{Subtracting we obtain:} \\
\begin{align*}
&u' + f(\rho') = u_1 + f(\rho_1) \\
&u' - f(\rho') = u_2 - f(\rho_2)
\end{align*}
\end{align*}

(101) 

\begin{align*}
\text{Subtracting we obtain:} \\
\begin{align*}
&u_1 - u_2 = 2f(\rho') - f(\rho_1) - f(\rho_2)
\end{align*}
\end{align*}

(103)

Since the waves are of rarefaction, $\rho'$ must be smaller than both $\rho_1$ and $\rho_2$. The derivative function:

\[ f'(\rho) = \sqrt{\phi'(\rho)/\rho} \]

is positive and $f(\rho)$ therefore decreases as $\rho$ decreases. For $\rho_1 > \rho_2$, $u_1 - u_2$, given by the above equation, is less than its value for $\rho' = \rho_2$ and we have:

\[ u_1 - u_2 < f(\rho_2) - f(\rho_1) < 0 \]

(104)

For $\rho_2 > \rho_1$, $u_1 - u_2$ is less than its value for $\rho' = \rho_1$ and we have:

\[ u_1 - u_2 < f(\rho_1) - f(\rho_2) < 0 \]

(105)

In order that a vacuum be not created in the gas, for certain laws of pressure, we must further restrict $u_1 - u_2$ to be in absolute value less than a limit determined by the law of pressure. For example, for Poisson's law, $\phi(\rho) = a\rho^\gamma$ and $f(\rho) = 2a\sqrt{\gamma - 1}/\rho^{\gamma - 1/2}$ which, since $\gamma$ is greater than 1 and since $\rho$ is essentially positive, must be positive. From equation (103) this involves the limitation $u_1 - u_2 < f(\rho_1) - f(\rho_2)$. For Boyle's law no limitation is necessary since $f(\rho) = a\log\rho$ can be any value from $-\infty$ to $+\infty$. If the limitation just mentioned and the inequality (104) or (105) whichever is applicable, is fulfilled, then equation (103) gives a unique value for $\rho'$. Knowing $\rho'$, from either equation (101) or (102) we may obtain $u'$ which lies between
u₁ and u₂, and from equations (100) we may then obtain the cotangents of the angles α₁, α₂, α₃, and α₄. These are in correct order of magnitude without further restrictions.

Consider next Riemann's third case. In this case a discontinuity of condensation is propagated forwards and a wave of rarefaction backward. This involves the assumption that ρ₁ is greater than ρ₂. In the x-t plane (see Fig. 3) we have the wave of rarefaction in the sector (1,0,2) and the positively propagated discontinuity along the line (0,3).

With the terminology indicated in the accompanying diagram, we may tabulate the following conditions:

\[ (-\infty, 0,1) : u = u₁, \quad \rho = \rho₁, \text{ constant} \]
\[ (1,0,2) : \quad u = -f(\rho) + c = \sqrt{\phi'(\rho)} + \frac{\phi'(\rho) - \phi(\rho)}{\rho_1} \]
\[ (2,0,3) : \quad u = u', \quad \rho = \rho', \text{ constant} \]
\[ (3,0,\cdot) : u = u₂, \quad \rho = \rho₂, \text{ constant} \]

We have for the velocity of propagation of the discontinuity:

\[ T₃ = \cot \alpha'_₃ = \frac{u'_₂}{u'_₁ + \sqrt{\frac{\rho₂}{\rho₁}} \frac{\phi'(\rho'') - \phi'(\rho''')}{\rho'' - \rho'''} \quad (107) \]

The condition of continuity along the lines (0,1) and (0,2) demands:

\[ u₁ = -f(\rho₁) + c = \sqrt{\phi'(\rho₁)} + \cot \alpha₁ \quad (108) \]
\[ u' = -f(\rho') + c = \sqrt{\phi'(\rho')} + \cot \alpha₂ \quad (109) \]

From these equations we obtain:
Subtracting we obtain:

$$u_1' - u_2' = f(\rho') - f(\rho_i) + \sqrt{\frac{(\rho' - \rho_i)(\phi - \phi' - \phi_2)}{\rho' \rho_i}}$$  \hspace{1cm} (112)

The right member of the above equation increases with increase of $\rho'$ if $\rho'$ is greater than $\rho_i$ as is necessary. Since $\rho'$ lies between $\rho_1$ and $\rho_2$ in value and hence $u_1' - u_2'$, given by the above equation, must lie between its values for $\rho' = \rho_1$ and $\rho' = \rho_2$. Therefore we have:

$$f(\rho_i) - f(\rho') < u_1' - u_2' < \sqrt{\frac{(\rho' - \rho_i)(\phi - \phi' - \phi_2)}{\rho' \rho_i}}$$  \hspace{1cm} (113)

If this inequality is fulfilled equation (112) gives a unique value for $\rho'$ and when this is known, either equations (110 or (111) determine $u'$ and $\alpha_1, \alpha_2, \alpha_3$ are then found from equations (107), (108) and (109) of this paragraph and are in correct order of magnitude without further restrictions.

Riemann's fourth case is not essentially different from the third and may be obtained from it by changing the direction of the $x$-axis. We may now sum up the preceding results. For brevity, let us write:

$$\sqrt{\frac{(\rho' - \rho_i)(\phi - \phi_1 - \phi_2)}{\rho' \rho_i}} = R$$

$$u_1' - u_2' = [u]$$

$$f(\rho_i) - f(\rho') = \pm \Delta$$ where the sign of $\Delta$ is taken such that $\Delta(\rho_1 - \rho_2) > 0$

Then we have:

**Case I:** $[u] > R$

**Case II:** $-\Delta > [u]$  

**Case III:** $R > [u] > -\Delta, \rho_1 > \rho_2$

**Case IV:** $R > [u] > -\Delta, \rho_1 < \rho_2$

With the exception of the limitation on $u_1' - u_2'$ in Case II
to prevent formation of a vacuum, one of these four gives an acceptable solution for every set of values of the given quantities, \( u_1, u_2, \rho_1, \rho_2 \).

The question still remains open as to whether this solution is unique. Riemann did not discuss this question and Weber also makes no mention of it. Hadamard, however, treats several solutions are sometimes possible which satisfy the equations of motion. Riemann's solution applies specifically to a discontinuity separating two unbounded regions in which \( u \) and \( \rho \) are constant, but the solution also holds for the immediate neighborhood of any discontinuity for the first instant of time. The general problem of discontinuities with given initial conditions has not been solved.


Let us consider the case of discontinuities as treated by Hadamard\(^1\). Each of the waves of rarefaction treated by Riemann are classified two discontinuities of the second order by Hadamard. Hadamard treats of the resolution into discontinuities of the first order. Between the two regions in which \( u \) and \( \rho \) have their original values \( u_1, \rho_1 \), \( u_2, \rho_2 \), we have an intermediate region in which \( u, \rho \) have values \( u', \rho' \). Hadamard considers Poisson's law to hold between these three states, namely \( pw' = \text{constant} \), where \( w = \frac{1}{\rho} \) is introduced for convenience. He visualizes this law as a curve in a plane in which \( w \) and \( p \) are rectangular coordinates. The three points \((w_1, p_1), (w_2, p_2), (w', p')\) lie upon this curve. This assumption makes Hadamard's treatment less general than Riemann's for it implies that only three of the quantities, \( w_1, p_1, w_2, p_2 \) are arbitrary. In effect he assumes that this motion is generated from a gas initially homogeneous such, for example, as by the sudden movement of a piston, in which case the two states would lie on the same

\[ \text{Loc. cit.} \]
\[ \text{Loc. cit.} \]
\[ \text{Hadamard, Sur la Propagation des Undes, 1903, pp.194-200} \]
adiabatic, not taking account of Hugoniot's objection. We have from the equation of Riemann (52) and (54) by elimination of \( \theta \)

\[
\begin{align*}
u' - u_1 &= \pm \sqrt{(p' - p_1)(w_1 - w')}, \\
(114) \\
\frac{u'}{u_2} &= \pm \sqrt{(p' - p_2)(w_2 - w')} = \pm \sqrt{p_2} \\
(115)
\end{align*}
\]

Subtracting we have:

\[
a = u_1 - u_2 = \mp \sqrt{p_1} + (\pm \sqrt{p_2}) \\
(116)
\]

where every combination of upper and lower signs before the two radicals is permissible but for either radical the sign is either upper or lower for all three equations and this rule will be extended to the subsequent equations. By transposition and squaring of both members we obtain for the above:

\[
\begin{align*}
P_1 - P_2 + a^2 &= \mp 2a \sqrt{p_1} \\
(117) \\
P_1 - P_2 - a^2 &= \mp 2a \sqrt{p_2} \\
(118)
\end{align*}
\]

Squaring both of these equations, we amalgamate the two branches expressed by the ambiguous sign into a single curve, as follows:

\[
\begin{align*}
(P_1 - P_2 + a^2)^2 &= 4a^2p_1 \\
(119) \\
(P_1 - P_2 - a^2)^2 &= 4a^2p_2 \\
(120)
\end{align*}
\]

These equations are equivalent. The preceding equations, however, are different as they divide the curve into branches at different points. \( P_1 \) and \( P_2 \) are quadratic expressions but \( P_1 - P_2 \) is linear. Thus the curve is of the second degree and hence, a conic section. For \( P_1 - P_2 + a^2 = 0 \), from equation (119) we must have \( P_1 = (p' - p_1)(w_1 - w') = 0 \) and hence either \( p' = p_1 \) or \( w' = w_1 \). Since the expression \( P_1 - P_2 + a^2 \) occurs twice as a factor in equation (119) the conic is tangent to the lines \( p' = p_1 \) and \( w' = w_1 \), parallel to the coordinate axes at their points of intersection with the straight line \( P_1 - P_2 + a^2 = 0 \). Likewise the line \( P_1 - P_2 - a^2 = 0 \) according to equation (120) intersects the conic at its points of
tangency with the lines $p' = p_1$ and $w' = w_1$. Since the equation of the curve involves the single parameter $a$, the curve is one of a family of conics tangent to the sides of a rectangle composed of the lines $p' = p_1$, $w' = w_1$, $p' = p_2$, $w' = w_2$ drawn with the given points $(w_1, p_1)$ and $(w_2, p_2)$ at opposite corners. We easily see that for $a$ comprised between 0 and $(p_1 - p_2)(w_2 - w_1)$ the conic is an ellipse and if $a$ surpasses this value it is a hyperbola. In the accompanying Fig. 4, $A_1$ and $A_2$ are the given points $(w_1, p_1)$ and $(w_2, p_2)$ respectively and $A_1B_1A_2B_2$ the rectangle to the sides of which the conic is tangent. The conic is represented as an ellipse inscribed in this rectangle. The chord $C_1D_1$ has for its equation $P_1 - P_2 + a^2 = 0$ and the chord $C_2D_2$ has for its equation $P_1 - P_2 - a^2 = 0$ such that $P_1 - P_2 = 0$ represents the diagonal $B_1B_2$.

This conic cuts the adiabatic between $A_1$ and $A_2$ in two points $A'$ and $A''$. Thus we have two values for the intermediate state. There is however a further criterion to distinguish between these two values. Consider the equations of the curve divided into branches. The point $A'$ lies on the arc $C_1D_1$ and the point $A''$ on the arc $C_2D_2$. For the former arc, $P_1 - P_2 + a^2 < 0$ and for the latter, $P_1 - P_2 - a^2 > 0$. We see by comparing the sign before the radicals in equations (117) and (118) with the sign before the same radicals in equations (115) and (116) that for the arc $C_1D_1$, $a(u' - u)^2 < 0$ and for the arc $C_2D_2$, $a(u' - u)^2 > 0$ and $a(u' - u)^2 < 0$ while for the rest of the conic, $a(u' - u) < 0$ and $a(u' - u) > 0$. Let $\theta_1$ and $\theta_2$ be the velocities of propagation of the two
discontinuities with respect to an initial state in which the density is \( w \). Then the equations of continuity (52) become:

\[
P_0 \Theta_1 = \frac{u' - u_1}{w' - w}
\]

\[
P_0 \Theta_2 = \frac{u' - u_2}{w' - w}
\]

Dividing one equation by the other we obtain:

\[
\frac{u' - u_1}{u' - u_2} = \frac{\Theta_1 (w' - u'_1)}{\Theta_2 (w' - u'_2)}
\]

For both the arcs \( C_1D_1 \) and \( C_2D_2 \), \( u' - u_1 \) and \( u' - u_2 \) have the same sign and hence we have:

\[
\frac{u_1 - u'}{u_2 - u'} > 0
\]

Also, if the conic is an ellipse, \( w' \) lies between the values \( w_1 \) and \( w_2 \) and hence \( w_1 - w' \) and \( w_2 - w' \) have opposite signs or

\[
\frac{w_1 - w'}{w_2 - w'} < 0
\]

Therefore from the preceding equation we have:

\[
\theta_1' \theta_2 < 0
\]

Thus on both of these arcs, the discontinuities travel in opposite directions. For the remaining two arcs of our ellipse, \( u' - u_1 \) and \( u' - u_2 \) have opposite signs and hence we have:

\[
\frac{\theta_1'}{\theta_2} > 0
\]

Thus on these arcs, the two discontinuities travel in the same direction. For our points \( A' \) and \( A'' \) on the arcs \( C_1D_1 \) and \( C_2D_2 \) respectively it is necessary to inquire if the condition \( \theta_1' < 0 \) is satisfied for both these points and if for only one, which one. From equation (121), we see that if \( \theta_1 \) is to be negative \( w_1 < w' \) or \( w_1 > w' \) according as \( u' - u_1 = +\sqrt{P_1} \) or \( u' - u_1 = -\sqrt{P_1} \). For \( w_1 < w_2 \), the former alternative must be chosen and this enforces that the intersection which is acceptable must be on the arc \( C_1D_1 \), if the initial discontinuity is one of condensation (See (122)
equation 98) (a > o) and on the arc C₂D₂ if the initial discontinuity is one of rarefaction (a < o) (See equations (104) and (105). For w₁ > w₂ the latter alternative must be chosen and this enforces the choice of the arc C₂D₂ if the initial discontinuity is one of condensation (a > o) and the arc C₁D₁ if the initial discontinuity is one of rarefaction (a < o).

We may put this in a little different form. If the initial discontinuity is one of condensation the point A must be chosen nearer that one of the two points A₁ and A₂ which corresponds to the greater density; if the initial discontinuity is one of rarefaction the point A must be nearer the point corresponding to the lesser density. Thus if the conic is an ellipse and the only intersections occur on the arcs C₁D₁ and C₂D₂ the solution is unique. If the conic is an hyperbola, w' is exterior to the range from w₁ to w₂. We know, however, that if the initial discontinuity is one of condensation, w' is inferior to both w₁ and w₂ and if the discontinuity is one of rarefaction, w' is superior to both w₁ and w₂. Since one intersection will always occur on the side inferior to this range and one on the side superior to this range, this criterion serves to distinguish the correct solution. We may have more than two intersections of the adiabatic with the conic. In the case of the ellipse pictured in the figure, we may have an intersection along the arc C₂D₂. That such points of intersection occur may be seen if the ellipse approximate sufficiently closely its limiting position as a double line along the diagonal A₁A₂. These two intersections, as we have seen by equation (126) represent motions for which θ₁ and θ₂ have the same sign. In these cases the solution is not unique but there are several solutions, all of which are possible. Hadamard does not employ the criterion of entropy, as the application of this principle had not been advanced at the time he
wrote his book. Otherwise, of course, he would reject discontinuities of rarefaction as Zemplen, Jouquet and Duhem have done (See page 25).

h. Objection to Hugoniot's Law

Although the principle of entropy has been extensively employed in connection with first order discontinuities, it is difficult to see how in the absence of any dissipative forces, the gas in passing through a discontinuity can suffer a gain in entropy. Lord Rayleigh declares that it cannot. He shows that we may obtain the equation of Hugoniot by considering a wave of permanent wave form in which the transition between two states need not be abrupt but that dissipative forces must be allowed to maintain the permanency of the wave. A relation identical to Hugoniot's was found fifteen years earlier by Rankine for a wave of permanent wave form maintained by heat conduction. Rayleigh says that the equation of energy is satisfied but not the second law of thermodynamics. The two states on either side of the discontinuity lie, in general, upon two different adiabatics. He says, in effect, that one cannot get from one adiabatic to another without a transfer of heat. In the derivation of Hugoniot's law the internal energy is evaluated from the expression:

$$\int_p \frac{d\nu}{\nu} = \frac{p\nu}{\nu - \mu}$$

The difference of internal energies in the transfer across the discontinuity is taken as:

$$\frac{p_1\nu_1}{\nu - \mu} - \frac{p_2\nu_2}{\nu - \mu}$$

Rayleigh points out that this involves the assumption that nothing is involved in the passage at $w = \infty$ from one adiabatic to another and that

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1. Loc. cit.
2. Loc. cit.
3. Loc. cit.
there is actually required the communication of an infinitesimal quantity of heat. To get from one adiabatic to the other a transfer of heat is required.

Heat can only be gained at the expense of work and hence dissipative forces must enter and an abrupt discontinuity becomes then an impossibility. That dissipative forces could not be allowed in a first order discontinuity is also recognized by Duhem\(^1\) and for this reason he is at a loss for an explanation of the entropy change for this case but insists that, according to the conclusions of Jouguet\(^2\), his results are applicable to a wave with very sudden transition layer maintained by dissipative forces. I think this still remains a disputed point.

4. Summary

It has been shown that discontinuous waves always produce variations in the time derivatives of \(x\), \(y\), and \(z\). The derivatives of the density of order less than the order of discontinuity are the first to become discontinuous. The first order discontinuity is the only case where there is a discontinuity in the density itself. In every discontinuous wave the characteristic segment is normal to the surface and in waves of the second or higher order the velocity of propagation is that of ordinary sound of infinitesimal amplitude with respect to the medium in its actual state on either side of the surface of discontinuity. The continuity and dynamical equations for a first order discontinuity were developed. For the adiabatic law of Poisson the velocity of propagation may be made as great as we please by increasing the pressure on one side of the discontinuity. The energy equation is not satisfied by Poisson's law of pressure and Hugoniot's law was de-

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1. Loc. cit.
2. Loc. cit.
veloped so as to satisfy this equation. The entropy of the gas, however, under this law suffers a change on passing the discontinuity. The condition of increase of entropy applied to actual gases implies that only a discontinuity of condensation is possible. The velocity of propagation under nuguoniou's law is always greater than that of ordinary sound of infinitesimal amplitude when both are measured with respect to the medium in front of the wave and less than the velocity of ordinary sound of infinitesimal amplitude when both are measured with respect to the region behind the wave. The resolution of an initial discontinuity was discussed according to the method of Riemann and of Hadamard. Using Poisson's law it was shown that a resolution into two first order discontinuities satisfying the equation of continuity and the dynamical equation could sometimes be made in more than one way. The objection of Lord Rayleigh to the application of nuguoniou's law to a discontinuity was discussed. This objection, is in substance, that the entropy cannot increase without the entrance of dissipative forces in which case the transition between two states cannot be abrupt.
Before taking up the subject of sound waves of finite amplitude it is proper to indicate the general results attained for infinitesimal amplitude waves as these waves are really the first approximation to the case of finite amplitude waves. For convenience it is customary to change from the variable \( \rho \) to the variable \( S \) defined by

\[
\rho = \rho_0 (1 + S)
\]  

(127)

The quantity \( S \) is called the condensation. The case where \( u, v, w, s, \) and their first derivatives with respect to \( x, y, z, \) and \( t \) are regarded as so small that the second degree terms in these quantities may be dropped from the equations of motion (1) and (2), may be called the case of sound waves of infinitesimal amplitude. The equations of motion then become linear. If the velocity has a potential function, that is, if there exists a function \( \phi \) such that:

\[
u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z}
\]  

(128)

the motion is found to be irrotational. When these values of \( u, v, w, \) are substituted in the dynamical equations (2), we obtain by integration and discarding of second degree terms for the case of infinitesimal amplitude waves the single equation:

\[
c^2 S = \frac{\partial \phi}{\partial t}
\]  

(129)

where \( c^2 = \frac{d\rho}{d\rho} \) is to be considered constant since the density changes are small. With the same substitution the equation of continuity (1) becomes:

\[
\frac{\partial \rho}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}
\]  

(130)

1. The material for this section is selected from Lamb's Hydrodynamics, 1906, pp. 454, 455, 466-172.
The elimination of $S$ between equations (129) and (130) gives:

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right)$$  \hspace{1cm} (131)

If the gas is considered as unbounded and the initial values of the velocity and density are given, the initial value of $\phi$ is everywhere given by integration of any one of equations (128), the constant of integration remaining arbitrary, and the initial value of $\frac{\partial \phi}{\partial t}$ is everywhere given by virtue of equation (129). We may write for these initial conditions:

$$\phi = \psi (x, y, z)$$  \hspace{1cm} (132)

$$\frac{\partial \phi}{\partial t} = \chi (x, y, z)$$  \hspace{1cm} (133)

The value of $\phi$ at any point $P$ at a time $t$ is a function of the original values of $\phi$ and $\frac{\partial \phi}{\partial t}$ on a sphere of radius $ct$ about the point $P$. The effect at $P$, moreover, is unchanged if $\phi$ and $\frac{\partial \phi}{\partial t}$ are constant and equal to their respective mean values on the sphere. Let $\phi_m$ and $\left( \frac{\partial \phi}{\partial t} \right)_m$ designate the mean values of $\phi$ and $\frac{\partial \phi}{\partial t}$ respectively, on a sphere of radius $r$ about the point $P$ in the original condition. If $\omega$ designate a solid angle, we have:

$$\phi_m = \frac{1}{4\pi} \int \int \phi \, d\omega = \frac{1}{4\pi} \int \int \psi (x + lr, y + mr, z + nr) \, d\omega = \Gamma(r)$$  \hspace{1cm} (134)

$$\left( \frac{\partial \phi}{\partial t} \right)_m = \frac{1}{4\pi} \int \int \frac{\partial \phi}{\partial t} \, d\omega = \frac{1}{4\pi} \int \int \chi (x + lr, y + mr, z + nr) \, d\omega = \Lambda(r)$$  \hspace{1cm} (135)

where $P$ is the point $x, y, z$, where $r$ is the radius vector from this point and $l, m, n$ its direction cosines. The effect at $P$ is the same as if we were given a symmetrical spherical motion with initial values of the velocity potential and its time derivative, $\phi_m = \Gamma(r)$ and $\frac{\partial \phi_m}{\partial t} = \Lambda(r)$. The value of $\phi$ as a function of $t$ at the point $P$ in a symmetrical spherical motion about this point with the above initial conditions is:

$$\phi = \frac{d}{dx} \left[ t \Gamma(ct) + t \Lambda(ct) \right]$$  \hspace{1cm} (136)
Substituting the values of $\Gamma$ and $\Lambda$ from equations (134) and (135) in equation (136) and since:

$$
\begin{align*}
    r &= ct, \quad l = \sin \theta \cos \phi, \quad m = \sin \theta \sin \phi, \quad n = \cos \theta, \\
    d\omega &= \sin \theta \, d\theta \, d\phi
\end{align*}
$$

we obtain:

$$
\begin{align*}
\phi &= \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int \psi (x + ct \sin \theta \cos \phi, y + ct \sin \theta \sin \phi, z + ct \cos \theta) \sin \theta \, d\theta \, d\phi \\
&+ \frac{1}{\sqrt{\pi}} \int \chi (x + ct \sin \theta \cos \phi, y + ct \sin \theta \sin \phi, z + ct \cos \theta) \sin \theta \, d\theta \, d\phi 
\end{align*}
$$

(137)

This is a complete solution for an irrotational sound wave of infinitesimal amplitude in an unbounded gas. The solution for a plane wave is:

$$
\phi = f(x - ct) + F'(x + ct)
$$

(138)

The solution for a symmetrical spherical wave is:

$$
\phi = f(r - ct) + F'(r + ct)
$$

(139)

In both cases the functions $f$ and $F'$ are to be determined by the initial conditions.
D. Continuous Finite Amplitude Sound Waves.

I shall now discuss continuous finite amplitude sound waves. In this case no solution has been obtained for the equations in their general form (1) and (2) or (12) and (13). I shall, therefore, consider first the case of plane waves or rectilinear disturbances of finite amplitude.

I. Plane Waves

For plane waves we must obtain a solution of equations (3) and (4) or (15) and (16). In either system of representation there are but two independent variables \( x \) and \( t \) or \( a \) and \( t \). Whichever of the two dependent variables we wish to consider may be considered as a space coordinate with the independent variables the other two coordinates. A solution of the differential equations corresponds then geometrically to a surface in which these are the coordinates. I shall now discuss the properties of the solutions of the equations of motion and obtain solutions for different conditions.

1. General Theory of Characteristics for Plane Waves

Let us consider equations (15) and (16) for plane waves. Instead of expressing \( p \) as a function of \( \rho \) as in equation (22), let us adopt a new variable.

\[
\frac{w}{\rho} = \frac{\partial x}{\partial a}
\]

and express \( p \) as a function of this variable, namely

\[
p = \phi (w)
\]

Eliminating \( \rho \) between equations (15) and (16) and substituting the above value of \( p \), we obtain:

\[
\frac{\partial^2 x}{\partial t^2} = -\frac{1}{\rho} \phi (w) \frac{\partial^2 x}{\partial a^2}
\]

Placing \( -\frac{1}{\rho} \phi (w) = \psi (w) \) we obtain:

\[
\frac{\partial^2 x}{\partial t^2} = \psi (w) \frac{\partial^2 x}{\partial a^2}
\]

1. Hadamard, loc. cit. pp. 154-159, 173-175
This is the equation to be solved to obtain $x$ as a function of $a$ and $t$.

Since the velocity is $u = \frac{\partial x}{\partial t}$ and $w = \frac{\partial x}{\partial a}$, we have:

$$\begin{align*}
\frac{dw}{da} &= \frac{\partial^2 x}{\partial a^2} da + \frac{\partial^2 x}{\partial a \partial t} \, dt \\
\frac{du}{da} &= \frac{\partial^2 x}{\partial a \partial t} da + \frac{\partial^2 x}{\partial t^2} \, dt
\end{align*}$$

(142) 
(143)

Eliminating $\frac{\partial^2 x}{\partial a^2}$ and $\frac{\partial^2 x}{\partial t^2}$ between equations (141), (142) and (143) we obtain:

$$\frac{\partial^2 x}{\partial a \partial t} = \frac{-duda + \psi(w) \, dw \, dt}{-da^2 + \psi(w) \, dt^2}$$

(144)

The equation:

$$-da^2 + \psi(w) \, dt^2 = 0$$

or

$$da = \pm \sqrt{\psi(w)} \, dt$$

(145)

When the surface integral is known, determines two families of curves on this surface, one curve of each family passing through each point on the surface. The curves thus defined are called the characteristics of this surface. If our solution is to be continuous, $\frac{\partial^2 x}{\partial a \partial t}$ must be finite. Along one of the characteristics defined by (145), the denominator of the right member of equation (144) is zero and if $\frac{\partial^2 x}{\partial a \partial t}$ is to be finite, the numerator of the right member of equation (144) must also be zero. Therefore we must have

$$-du \, da + \psi(w) \, dw \, dt = 0$$

(146)

Substituting the value of $da$ from equation (19) in this equation we obtain

$$du = \pm \sqrt{\psi(w)} \, dw$$

(147)

Integrating we obtain:

$$u = \pm \sqrt{\psi(w)} \, dw + c = \pm x(w) + c$$

(148)

This is a differential equation which must be satisfied along the characteristics of our surface integral.

Let us consider the condition that a second surface integral
may exist tangent to the first along a curve $\Gamma$. Now the equation of the tangent plane to the surface integral at the point $x, a, t$, is:

$$x-x_1 = \frac{\partial x}{\partial a}(a - a_1) + \frac{\partial x}{\partial t}(t - t_1)$$

$$= w (a - a_1) + u (t - t_1) \quad (149)$$

Now along the curve $\Gamma$, the tangent planes for the two surfaces coincide and from the above equation we see that $w$ and $u$ must be the same for the two surfaces along this curve. If the contact of the two surfaces is of the first order, $\frac{\partial^2 x}{\partial a \partial t}$ will be different for the two surfaces along the curve and hence its expression from equation (18) must be indeterminate. This is only indeterminate along a characteristic and hence the curve $\Gamma$ must be a characteristic. For any finite order of contact between two surface integrals it is proven that the curve of tangency must be a characteristic. The characteristic is thus the locus of a discontinuity of the second or higher order and equation (145) agrees with equation (47) and shows that the discontinuity is propagated with the ordinary velocity of sound of infinitesimal amplitude with respect to the gas.

2. Single Progressive Waves

A single progressive wave may be defined as one which is propagated into a region at rest. Adopting our geometrical representation, the region at rest is represented by the plane $x = a$. A single progressive wave, then, is one represented by a surface integral tangent to the plane $x=a$.

a. Earnshaw's Condition

Let us consider a surface integral tangent along a characteristic of the first family of equation (145), namely:

$$\frac{da}{dt} = \sqrt{\psi(t)} \quad (150)$$

Since $w = 1$ along this characteristic which pertains to the region at

- Hadamard, loc. cit. pp 174, 195
- Rayleigh, Theory of Sound Vol. II pp. 34, 35.
rest in which \( x = a \). From any point of the surface, draw a characteristic of the family opposite to that of the characteristic of tangency. This characteristic will cut the characteristic of tangency in a point \( P \). Along the former characteristic we must have (equation 148):

\[
u = -X(w) + C
\]

and at the point \( P \), since \( u = 0 \), \( w = 1 \), we have:

\[
o = -X(1) + C
\]

This determines the constant \( C \) and the condition along the former characteristic becomes:

\[
u = X(1) - X(w) \]  \hspace{1cm} (151)

Since this characteristic was arbitrarily chosen and since every point on the surface must have a characteristic of this family passing through it, the above is a condition to be fulfilled over the whole surface of the wave. The characteristic of tangency gives the motion of the front of the wave and from (150) we see the velocity of the front of the wave is positive or that the wave is a positive progressive wave.

Equation (151) is called Earnshaw's condition for a positive progressive wave. If the characteristic of tangency belongs to the other family of equation (145) that is, corresponds to

\[
\frac{da}{dt} = -\sqrt{\psi(1)}
\]

we may similarly show that this surface integral satisfies the relation:

\[
u = X(w) - X(1) \]  \hspace{1cm} (152)

The propagation in the negative direction is not essentially different from that in the positive direction since the one case becomes the other by reversing the directions of the a- and x- axes. This reversal of axes implies a substitution of \( a = -a \), \( x = -x \), and \( u = -u \) and we see from the above formulae that the one case transforms into the other. Thus the two cases differ only as an image in a plane mirror differs from its
object.

Now from equation (148), \( X'(w) = \sqrt{\psi(w)} \) and from the remark preceding equation (141), \( \psi(w) = -\frac{1}{p} \phi'(w) \). If the pressure increases with increase of density we see from equations (15) and (140) that \( \phi'(w) \) is negative and hence \( \psi(w) \) is positive and hence \( X'(w) \) is real and positive. \( X(w) \) is then an increasing function of \( w \). From equations (151) and (152) for positive and negative waves respectively we see that for the former \( u \) decreases with \( w \) and for the latter \( u \) increases with \( w \) or from equation (15) we see that for the former \( u \) increases with \( \phi \) and for the latter \( u \) decreases with \( \phi \). This result necessarily follows if the resistance offered by the gas to the motion of the piston, used to generate a single progressive wave in a cylindrical tube, increases with its speed if this is measured positive in a direction compressing the gas. The resistance to the motion of the piston is \( pS \) where \( S \) is the cross section area of the piston. If the resistance increases with \( u \), \( p \) must increase with \( u \) and since \( p \) is an increasing function of \( \phi \), \( \phi \) must increase with \( u \).

We may obtain Earnshaw's condition by a method due to Lord Rayleigh. We may imagine any point which we are examining to be brought to rest for the instant by impressing upon the gas as a whole a uniform velocity. In the neighborhood of this point the condition is that of a wave of infinitesimal amplitude and the equations given in section C obtain. For a positive progressive infinitesimal wave \( F=0 \) in equation (138) and substituting the value of \( \phi \) from this equation for \( u \), (128) and in the equation for \( S \), (129), we obtain on eliminating \( f' \) between the resulting equations:

\[
\frac{u}{s} = c
\]

or since $c = \sqrt{\phi'(\rho)}$, $u/s = \sqrt{\phi'(\rho)}$ (153)

Now in the wave of finite amplitude we must consider infinitesimal velocities and condensations relative to the point brought to rest. We must, therefore, use $du$ for $u$ and, by equation (127), $dP/\rho$ for $S$ and we must have for the vicinity of every point in the wave:

$$du = \sqrt{\phi'(\rho)} \frac{dP}{\rho}$$

Integrating this expression from a point on the characteristic of tangency where $u = 0$ and $P = P_o$ to any point in the wave having a velocity $u$ and a density $P$ we obtain:

$$u = \int_{P_o}^{P} \sqrt{\phi'(\rho)} \frac{dP}{\rho}$$

which is equivalent to equation (151) and is Earnshaw's condition for a positive progressive wave.

b. General Solution and Properties of a Surface Integral.

Since a positive and negative wave are not essentially different, let us consider a positive wave for which equation (151) holds. Along a characteristic of the same family as the characteristic of tangency, (equation 150) we have equation (145) with the upper sign. To this corresponds equation (148) with the upper sign, that is:

$$u = X(w) + C$$

For the surface of the wave in general we have equation (151) namely:

$$u = X(l) - X(w)$$

Adding and subtracting, we see that $u$ and $w$ are both constant for points on the same characteristic; for which we have equation (145) with the upper sign. This shows that constant values of $u$ and $w$ are transmitted through the gas with velocities equal to $\sqrt{\phi'(w)}$ with respect to the initial state. This is the velocity of infinitesimal sound in a gas of density corresponding to the value of $w$.

For a positive progressive wave we have:
\[ \frac{dx}{dt} = w \cdot d \alpha + u \cdot dt \] since \( w = \frac{\partial x}{\partial \alpha} \) and \( u = \frac{\partial x}{\partial t} \)

Substituting the value of \( u \) from equation (151) we obtain:

\[ \frac{dx}{dt} = w \cdot d \alpha + \left[ X(l) - X(w) \right] dt \quad (156) \]

The surface integral is then found by eliminating \( w \) between the equations:

\[
\begin{align*}
    x &= w \cdot \alpha + \left[ X(l) - X(w) \right] t + \phi(w) \\
    \zeta &= \alpha - X'(w) t + \phi'(w)
\end{align*}
\]

When the arbitrary function \( \phi \) is specified, the elimination may be carried out. This is Earnshaw's method of solution. Equation (157) is the equation of a tangent plane, as we see by comparing with equation (149) to the surface and, since it depends only upon one parameter \( w \), the surface is developable. If \( u \) or \( w \) is taken as dependent variable instead of \( x \), the dependent variable is constant along the characteristics of the family given by equation (145) with the upper sign. The characteristics are then horizontal generatrices of a ruled surface if the \( \alpha t \) plane be taken as horizontal.

Eliminating \( \alpha \) between equations (156) we obtain:

\[
\begin{align*}
    x - \left[ X(l) - X(w) + wX'(w) \right] t &= \phi(w) - W \phi'(w) \\
\end{align*}
\]

from equations (148) and (151) and \( Wx'(w) = \sqrt{\phi'(\rho)} \)

Substituting \( X(l) - X(w) = u \) from (151) and \( wX'(w) = \sqrt{\phi'(\rho)} \), we obtain:

\[
\begin{align*}
    \xi &= \frac{u + \sqrt{\phi'(\rho)}}{t} \\
\end{align*}
\]

Where \( \xi \) is an arbitrary function of \( \alpha \) since by (151) \( u \) is a function of \( w \). Reversing the functional relation, we obtain:

\[
\begin{align*}
    u &= f \left[ x - \left\{ u + \sqrt{\phi'(\rho)} \right\} t \right] \\
\end{align*}
\]

This is known as Poisson's integral although Poisson obtained it only for Boyle's law (21) for which \( \phi'(\rho) = C^2 \) is constant. This equation shows that values of \( u \) are propagated with velocities \( u + \sqrt{\phi'(\rho)} \) with respect to a fixed set of axes. From Lord Rayleigh's method of analysis.

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1. Earnshaw, Proceedings of the Royal Society, Jan 6, 1859.
2. Rayleigh, Theory of Sound Vol II 1896 pp 33, 34
we see that this must be so since any point may be brought to rest by impressing a certain velocity upon the gas as a whole and in the neighborhood of this point the motion must be that of a wave of infinitesimal amplitude and hence values of the velocity are propagated with a velocity \( v \), that of ordinary sound of infinitesimal amplitude. Hence the velocity of propagation in the finite amplitude wave must be the velocity of the gas \( u \) plus the velocity of ordinary sound in the gas which has the velocity \( u \).

Weber \(^1\) takes equation (158A) and changes to a new variable

\[
\eta = u + \sqrt{\phi} = \int_{\phi_f}^{\phi_0} \sqrt{\phi'} \, d\phi + \sqrt{\phi_f} \tag{160}
\]

from equation (154). Then equation (158A) becomes:

\[
x - \eta t = \int (u) = G(\eta) \tag{161}
\]

This shows that \( \eta \) is constant along a straight line in the \( xt \)-plane, the cotangent of whose angle with the \( x \)-axis is \( \eta \).

c. Solution with Given Motion of Piston.

Hadamard \(^2\) treats the problem of the motion of a gas initially at rest in a cylindrical tube when the motion is generated by the motion of a piston. If there are two moving pistons enclosing the gas his solution holds only for such regions and such times as the wave remains a single progressive one, the encounter of the waves issuing from the two pistons producing a new problem. Wherever a reflected wave enters his solution likewise does not apply. Let the initial abscissae of the particles at the surface of the piston considered be \( a=0 \). Let the subscripts zero denote quantities pertaining to the motion of the piston or the motion of the gas at \( a=0 \). Let the motion of the piston be given by:

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\(^1\) Weber, loc. cit. pp. 475-477

\(^2\) Hadamard, loc. cit. pp. 177-178
Then the velocity is given by

\[ u_0 = F'(t_0) \tag{163} \]

where \( F' \) is the derivative function of \( F \). From the first paragraph of section b above we have seen that values of \( u_0 \) and \( w_0 \) are propagated with velocities \( \sqrt{\psi(w_0)} \) with respect to the gas in its initial state. Accordingly, at a time \( t > t_0 \) the velocity \( u_0 \) and corresponding value of \( w_0 \) has been communicated to a particle whose initial abscissa is:

\[ a = \sqrt{\psi(w_0)}(t - t_0) \tag{164} \]

The actual velocity of propagation, according to equation (159) and the succeeding discussion is \( u + \sqrt{\psi'(\rho_0)} \) or from the statement immediately following equation (157) it is \( u_0 + w_0 X'(w_0) \) or from equation (148) \( u_0 + w_0 \sqrt{\psi(w_0)} \). Thus the actual abscissa corresponding to a value of \( u \) and \( w \) is:

\[ x = \mathbf{x}_0 + \left[ u_0 + w_0 \sqrt{\psi(w_0)} \right] (t - t_0) \tag{165} \]

Now \( w_0 \) is expressed in terms of \( u_0 \) by equation (151) and \( u_0 \) is expressed in terms of \( t_0 \) by equation (163). Thus equations (164) and (165) besides \( x, a, \) and \( t \) involve only \( t_0 \) and the elimination of \( t_0 \) between these equations gives the sought for integral.

d. Singularity in the Solution

We may perhaps most easily picture the singularity involved if we adopt Weber's system of representation, equation (161).

Initially we are given \( \eta \) as a function of \( x \), that is, the values of \( \eta \) along the \( x \) axis in the \( x \)t plane. From each point on the \( x \)-axis construct a straight line making an angle with this \( x \) axis whose cotangent is \( \eta \). Then as long as two straight lines do not intersect in a point on the positive side of the \( x \)-axis, the solution is unique, that is \( \eta \) is

uniquely determined for all points for positive values of $t$. In general, however, this family of straight lines will intersect on the positive side of the $x$-axis. These lines will only fail to intersect in the region outside of the envelope of the family of straight lines and it is only there that $\eta$ is uniquely determined. This envelope is given by the elimination of $\eta$ between the equations.

\[
\begin{align*}
    x &= G(\eta) - \eta G'(\eta) \\
    t &= -G'(\eta)
\end{align*}
\]  

(166)

Another interesting method of visualizing the process is that of Lord Rayleigh.\(^2\) At any instant plot a curve with the velocity as ordinate and the actual abscissa as abscissa. Let us see how this curve is deformed with time. A value $u$ is propagated with a velocity $u + \sqrt{\phi'(\rho)}$ (See equation (159) and following discussion). This velocity wave will only retain its original form if $u + \sqrt{\phi'(\rho)}$ is constant.

Substituting the value of $u$ from equation (155), the condition for a permanent wave form becomes:

\[
\int_\rho^\infty \sqrt{\phi'(\rho)} \frac{d\rho}{\sqrt{\rho}} + \sqrt{\phi'(\rho)} = \text{constant}
\]

Differentiating and multiplying by we have:

\[
\sqrt{\phi'(\rho)} \frac{d\rho}{\sqrt{\rho}} + \sqrt{\phi'(\rho)} = 0
\]

Integrating we have:

\[
\sqrt{\phi'(\rho)} = \frac{B}{\rho}
\]

Squaring we have:

\[
\phi'(\rho) = \frac{B^2}{\rho^2}
\]

Integrating again we have:

\[
p = \phi(\rho) = A - \frac{B^2}{\rho}
\]

(167)

It is only under this law of pressure that a velocity wave will retain its original form. This law is not fulfilled for any actual gas and hence the velocity wave is deformed. For Boyle's law (20) and Poisson's

adiabatic law (21), \( u + \sqrt{\phi(p)} \) is an increasing function of \( u \). After a
time \( t \), every point on the velocity curve will have moved forward
parallel to the \( x \)-axis a distance \( u + \sqrt{\phi(p)} \ t \). Along a portion of the
velocity curve which slopes downward in the positive direction, the
velocity of propagation, \( u + \sqrt{\phi(p)} \), is greater the higherup a point is
on the velocity curve such that the slope of the wave becomes ever more
steep until at a certain time it becomes vertical at a certain point
after which a vertical line cuts the curve in more than a single point
and the value of \( u \) is no longer unique at a point. Hence the solution
must be discarded beyond the point where the curve acquires a vertical
tangent. The only type of velocity curve which may escape ultimate
discontinuity is one which has no downward slope but has an upward slope
from \(-\infty\) to \(+\infty\) in the positive direction of the \( x \)-axis. Such a
curve is limited, however, as to the duration of its previous existence
and when projected back into negative time is discontinuous at a
certain time.

a. General Solution.

A more general solution than that for single progressive waves was found by Riemann\(^1\). It occurred to Riemann to adopt as dependent variables instead of \(u\) and \(\rho\) in equations (3) and (4) the quantities \(r\) and \(s\) defined by:

\[
2r = f(\rho) + u
\]
\[
2s = f(\rho) - u
\]

where

\[
f(\rho) = \int \sqrt{\phi'(\rho)} \, d\log\rho \tag{170}
\]

This function \(f(\rho)\) is equivalent to the function \(X(w)\) and we see from equation (148) that \(r\) and \(s\) are constant along the characteristics of the surface integral. Solving the above equations for \(f(\rho)\) and \(u\) we have:

\[
f(\rho) = r + s \tag{171}
\]
\[
u = r - s \tag{172}
\]

Differentiating these expressions, we have by means of equation (170):

\[
\sqrt{\phi'(\rho)} \frac{d \log \rho}{dx} = \frac{du}{dx} + \frac{ds}{dx}
\]
\[
\sqrt{\phi'(\rho)} \frac{d \log \rho}{dt} = \frac{dv}{dt} + \frac{ds}{dt}
\]

\[
\frac{du}{dt} = \frac{dv}{dt} - \frac{ds}{dt}
\]

\[
\frac{dv}{dt} = \frac{du}{dx} - \frac{ds}{dx}
\]

The equations of motion (3) and (4) may be expressed in the form:

\[
\frac{du}{dt} + u \frac{du}{dx} = -\phi'(\rho) \frac{d \log \rho}{dx}
\]

\[
\frac{dv}{dt} + u \frac{dv}{dx} = -\sqrt{\phi'(\rho)} \frac{dv}{dx}
\]

Substituting the expressions evaluated in equations (173) in equations (174) and (175) we obtain:

\[
\frac{dr}{dt} - \frac{ds}{dt} + u \left( \frac{dr}{dx} - \frac{ds}{dx} \right) = -\sqrt{\phi'(\rho)} \left( \frac{dr}{dx} + \frac{ds}{dx} \right)
\]

Riemann, loc. cit.
Adding and subtracting we obtain the equations:

\[ \frac{\partial r}{\partial t} = -(u + \sqrt{\phi'(\rho)}) \frac{\partial r}{\partial x}, \tag{176} \]
\[ \frac{\partial s}{\partial t} = -(u - \sqrt{\phi'(\rho)}) \frac{\partial s}{\partial x}. \tag{177} \]

Since \( u \) and \( \rho \) are given as functions of \( r \) and \( S \) by equations (171) and (172) the above equations are the equations sought for in which \( r \) and \( S \) replace \( u \) and \( \rho \) as dependent variables.

Let us form the total differentials of \( r \) and \( S \). We have by means of equations (176) and (177)

\[ dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial t} dt = \frac{\partial r}{\partial x} [dx - (u + \sqrt{\phi'(\rho)}) dt], \tag{178} \]
\[ ds = \frac{\partial s}{\partial x} dx + \frac{\partial s}{\partial t} dt = \frac{\partial s}{\partial x} [dx - (u - \sqrt{\phi'(\rho)}) dt]. \tag{179} \]

To find the velocities with which \( r \) and \( S \) are propagated we must find the relation between \( dx \) and \( dt \) for \( dr = 0 \) and \( ds = 0 \) respectively.

From equation (178) and (179) we see that

\[ \text{for } dr = 0, \quad \frac{dx}{dt} = u + \sqrt{\phi'(\rho)} \tag{180} \]
\[ \text{and for } ds = 0, \quad \frac{dx}{dt} = u - \sqrt{\phi'(\rho)}. \tag{181} \]

Thus values of \( r \) are propagated in the positive direction with a velocity \( u + \sqrt{\phi'(\rho)} \) and values of \( S \) are propagated in the negative direction with a velocity \( -u + \sqrt{\phi'(\rho)} \). Suppose initially we are given the velocity and density as functions of \( x \). Then by equations (168) and (169) we are also given \( r \) and \( S \) as functions of \( x \). Let us plot two curves, taking \( x \) as abscissae and \( r \) as ordinate in the one case and \( S \) in the other. We are given the curves for \( r \) and \( S \) at \( t = 0 \). After a time \( dt \), a point on the \( r \)-curve has moved a distance \( (u + \sqrt{\phi'(\rho)}) dt \) parallel to the \( x \)-axis and a point on the \( s \)-curve a distance \( (u - \sqrt{\phi'(\rho)}) dt \) parallel to the \( x \)-axis. By taking small enough increments of time and using the above construction successively we may approximate as closely as we please the progress of the wave.
Under certain conditions we may obtain a solution without this tedious graphic method. The principle of this solution obtained by Riemann depends upon a change to linear equations by interchange of the roles of independent and dependent variables. This may always be accomplished if the equations are of the following type, which is the type to which equations (176) and (177) belong:

\[
\begin{align*}
U_1 \frac{\partial u_1}{\partial x_1} + U_2 \frac{\partial u_2}{\partial x_2} + U_3 \frac{\partial u_1}{\partial x_2} + U_4 \frac{\partial u_2}{\partial x_1} &= 0 \\
U_1 \frac{\partial u_1}{\partial x_1} + U_2 \frac{\partial u_2}{\partial x_2} + U_3 \frac{\partial u_1}{\partial x_2} + U_4 \frac{\partial u_2}{\partial x_1} &= 0
\end{align*}
\]

(182) (183)

where the coefficients \(U_1, U_2\) are functions only of \(u_1\) and \(u_2\), not of \(x_1\) and \(x_2\). We have,

\[
\begin{align*}
\frac{du_1}{dx_1} &= \frac{\partial u_1}{\partial x_1} dx_1 + \frac{\partial u_1}{\partial x_2} dx_2 \\
\frac{du_2}{dx_2} &= \frac{\partial u_2}{\partial x_1} dx_1 + \frac{\partial u_2}{\partial x_2} dx_2
\end{align*}
\]

Solving for the differentials \(dx_1\) and \(dx_2\) we obtain:

\[
\begin{align*}
\frac{dx_1}{\partial x_1} &= \frac{\partial x_2}{\partial u_1} du_1 + \frac{\partial x_2}{\partial u_2} du_2 \\
\frac{dx_2}{\partial x_2} &= \frac{\partial x_2}{\partial u_1} du_1 + \frac{\partial x_2}{\partial u_2} du_2
\end{align*}
\]

Where \(\Delta = \frac{\partial x_1}{\partial x_2} \frac{\partial u_2}{\partial u_1} - \frac{\partial x_2}{\partial u_1} \frac{\partial u_2}{\partial u_2}\).

Equating the coefficients of \(du_1\) and \(du_2\) on both sides of these equations we obtain:

\[
\begin{align*}
\frac{\partial x_1}{\partial x_1} &= \Delta \frac{\partial x_2}{\partial u_1} \\
\frac{\partial x_2}{\partial x_2} &= \Delta \frac{\partial x_1}{\partial u_2}
\end{align*}
\]

(184)

Substituting these values in equations (182) and (183), the factor may be dropped and we obtain:

\[
\begin{align*}
U_1 \frac{\partial x_2}{\partial u_2} - U_2 \frac{\partial x_2}{\partial u_1} + U_3 \frac{\partial x_1}{\partial u_1} + U_4 \frac{\partial x_1}{\partial u_2} &= 0 \\
U_1 \frac{\partial x_2}{\partial u_2} - U_2 \frac{\partial x_2}{\partial u_1} + U_3 \frac{\partial x_1}{\partial u_1} + U_4 \frac{\partial x_1}{\partial u_2} &= 0
\end{align*}
\]

(185) (186)

Independent and dependent variables have here been interchanged and since the coefficients are functions of the independent variables only the equations are linear and have the property that the sum of any number of solutions is also a solution. Applying this transformation
to equations (176) and (177) we obtain:

\[
\frac{dx}{ds} = \left( u + \sqrt{\phi'(p)} \right) \frac{dt}{ds} \tag{187}
\]

\[
\frac{dx}{dr} = \left( u - \sqrt{\phi'(p)} \right) \frac{dt}{dr} \tag{188}
\]

These equations may be put in the equivalent form:

\[
\frac{d}{ds} \left[ x - (u + \sqrt{\phi'(p)} t) \right] = -t \frac{d}{ds} \left[ u + \sqrt{\phi'(p)} \right] \tag{189}
\]

\[
\frac{d}{dr} \left[ x - (u - \sqrt{\phi'(p)} t) \right] = -t \frac{d}{dr} \left[ u - \sqrt{\phi'(p)} \right] \tag{190}
\]

From equations (171) and (172) we have:

\[
\frac{d}{ds} \left[ u + \sqrt{\phi'(p)} \right] = -1 + \frac{d}{ds} \log \frac{\sqrt{\phi'(p)}}{p} \tag{191}
\]

\[
\frac{d}{dr} \left[ u - \sqrt{\phi'(p)} \right] = 1 - \frac{d}{dr} \log \frac{\sqrt{\phi'(p)}}{p} \tag{192}
\]

Substituting these quantities in the right members of equations (189) and (190) we have:

\[
\frac{d}{ds} \left[ x - (u + \sqrt{\phi'(p)} t) \right] = t \left[ 1 - \frac{d}{ds} \log \frac{\sqrt{\phi'(p)}}{p} \right] \tag{193}
\]

\[
\frac{d}{dr} \left[ x - (u - \sqrt{\phi'(p)} t) \right] = -t \left[ 1 - \frac{d}{dr} \log \frac{\sqrt{\phi'(p)}}{p} \right] \tag{194}
\]

Since the right members of these equations are equal in magnitude but opposite in sign, the left members are equal in magnitude and opposite in sign. Hence we have:

\[
\frac{d}{ds} \left[ x - (u + \sqrt{\phi'(p)} t) \right] = -\frac{d}{dr} \left[ x - (u - \sqrt{\phi'(p)} t) \right] \tag{195}
\]

This is the necessary and sufficient condition for the existence of a function \( w \) defined by:

\[
x - (u + \sqrt{\phi'(p)} t) = \frac{\partial w}{\partial s} \tag{196}
\]

\[
x - (u - \sqrt{\phi'(p)} t) = -\frac{\partial w}{\partial s} \tag{197}
\]

Subtracting we obtain:

\[
t = -\frac{1}{2(\phi'(p))} \left( \frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} \right) \tag{198}
\]

Substituting the left members of equations (196) and (198) in equation (193) or the left members of equations (197) and (198) in equation (194) we obtain:

\[
\frac{d^2w}{drds} = \frac{1}{2(\phi'(p))} \left( \frac{d}{dr} \log \frac{\sqrt{\phi'(p)}}{p} - 1 \right) \left( \frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} \right) \tag{199}
\]
We may write for brevity,

\[
\frac{\partial^2 \omega}{\partial r \partial s} - m \left( \frac{\partial \omega}{\partial r} + \frac{\partial \omega}{\partial s} \right) = 0
\]  

(200)

where \( m \) by comparison with the above is a function of \( \rho \) and therefore, by equation (171), a function of \( r + s \). The function \( m \) depends on the law of pressure assumed. For Boyle's law, (20) we have:

\[
m = - \frac{1}{2a}
\]

(201)

and for Poisson's adiabatic law, (21) we have:

\[
m = \frac{\gamma + 3}{2(\gamma - 1)(r + s)}
\]

(202)

Thus we have reduced our problem to the solution of a second order linear differential equation (200) with the condition of satisfying a given initial condition. When \( w \) is found as a function of \( r \) and \( S \) as a solution of this equation, equations (196) and (197), when \( r \) and \( S \) have been eliminated by means of equations (168) and (169), give two equations between \( u, \rho, x \) and \( t \) and by solving simultaneously we obtain \( u \) and \( \rho \) as functions of \( x \) \( t \). As to the initial condition to be satisfied by equation (200), we see from equations (196) and (197) that initially we must have:

\[
\frac{\partial \omega}{\partial r} = x, \quad \frac{\partial \omega}{\partial s} = -x
\]

(203)

and

\[
\frac{\partial \omega}{\partial r} \, dr + \frac{\partial \omega}{\partial s} \, ds = x \, (dr-ds)
\]

(204)

Now \( u \) and \( \rho \) and hence, according to equations (168) and (169) \( r \) and \( S \) are initially given as functions of \( X \). Hence, according to (204), \( w \) is determined by integration, except for an additive arbitrary constant which has no effect on the solution, as a function of \( x \) for \( t=0 \).

The equations:

\[
r = \text{constant}
\]

\[
S = \text{constant}
\]

define two families of curves in the \( xt \)-plane. For a curve of the first family we have from equations (178) and (179)
and for a curve of the second family:
\[ \frac{dx}{dt} = \left( u - \sqrt{\phi'(p)} \right) dt, \quad \frac{ds}{dt} = 2 \frac{\partial s}{\partial x} \sqrt{\phi'(p)} dt. \] (206)

Let us confine our attention to a length of $\alpha \beta$ of the x-axis for which $\frac{dr}{dx}$ and $\frac{ds}{dx}$ do not change sign. Then along the curve $r =$ constant $S$ increases or decreases throughout and along the curve $S =$ constant, $r$ increases or decreases throughout according to equations (205) and (206).

If from every point of the line $\alpha \beta$ we draw a curve of each family, these curves form a network and fill a portion of the plane $\alpha \beta \delta$, which is bounded by the x-axis, and the curves $r = r'$, $S = S'$, where $r'$ is the value of $r$ at $\alpha$ and $S'$ the value of $S$ at $\beta$.

The curves $r =$ constant and $S =$ constant may then serve as coordinate in this domain. In a plane in which $r$ and $S$ are taken as rectangular coordinates and in which the curves $r =$ constant and $S =$ constant are straight lines parallel to the coordinate axes, the line $\alpha \beta$ becomes a curve $C$ which has no maximum point between the points $\alpha$ and $\beta$.

In solving equation (200) we use Stokes theorem applied to a plane in the form:
\[ \iint \left( \frac{\partial U}{\partial r} + \frac{\partial V}{\partial s} \right) dr ds = \int (U ds - V dr) \] (207)

where $U$ and $V$ are any continuous functions of $r$ and $s$ and where the double integral is taken over a domain in the $rs$ plane around the boundary of which the line integral is taken. Let $v$ be a function of $r$, $s$ to be later determined and let
\[ U = v \left( \frac{\partial w}{\partial s} - mw \right), \quad V = -w \left( \frac{\partial v}{\partial r} + mw \right) \] (208)

From the above by differentiation we have:
\[ \frac{\partial U}{\partial r} + \frac{\partial V}{\partial s} = v \left[ \frac{\partial^2 w}{\partial r \partial s} - m \left( \frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} \right) \right] - w \left[ \frac{\partial^2 v}{\partial r \partial s} + \frac{\partial mw}{\partial r} + \frac{\partial mw}{\partial s} \right] \] (209)
Let us determine the value of \( v \) so as to satisfy the equation:

\[
\frac{\partial^2 v}{\partial n \partial s} + \frac{\partial (mv)}{\partial n} + \frac{\partial (mv)}{\partial s} = 0
\]  

(210)

From this equation and from equation (200), equation (209) becomes:

\[
\frac{\partial u}{\partial n} + \frac{\partial v}{\partial s} = 0
\]  

(211)

Stoke's theorem, equation (207), then gives:

\[
\int [v\left(\frac{\partial w}{\partial s} - mw\right)ds + w\left(\frac{\partial v}{\partial n} + mv\right)dr] = 0
\]  

(212)

in which the integral is taken around the boundary of a domain in the \( rs \)-plane.

Applying this to the domain \( \partial \beta \xi \), bounded by the curve \( C \) and the straight lines \( r = r', S = S' \), we have:

\[
\int_\alpha^\beta \left[v\left(\frac{\partial w}{\partial s} - mw\right)ds + w\left(\frac{\partial v}{\partial n} + mv\right)dr\right] = 0
\]  

(213)

By partial integration we have:

\[
\int_\alpha^\beta v \frac{\partial w}{\partial s} ds = (vw)_\beta^\xi - (vw)_\alpha^\beta - \int_\alpha^\beta w \frac{\partial v}{\partial s} ds
\]  

(214)

Substituting the value of this integral in (213) we obtain:

\[
(vw)_\beta^\xi - (vw)_\alpha^\beta = \int_\alpha^\beta w\left(\frac{\partial v}{\partial n} + mv\right)ds + \int_\beta^\xi w\left(\frac{\partial v}{\partial n} + mv\right)dr
\]  

(215)

Let us subject \( v \) to the following conditions:

- on \( \partial \beta \xi \), \( \frac{\partial v}{\partial S} + mv \neq 0 \)
- on \( \partial \alpha \xi \), \( \frac{\partial v}{\partial n} + mv = 0 \)

(216)

at the point \( \xi \), \( v = 1 \)

Then equation (215) becomes:

\[
w_\xi = (vw)_\alpha + \int_\alpha^\beta \left[v\left(\frac{\partial w}{\partial s} - mw\right)ds + w\left(\frac{\partial v}{\partial n} + mv\right)dr\right]
\]  

(217)

Thus the problem is changed to one of determining the function \( v \) for when \( v \) is known, \( w_\xi \) is determined by the above formula from the values of \( w \frac{\partial w}{\partial n} \) and \( \frac{\partial w}{\partial s} \) on the curve \( C \).

Let us now consider the determination of the function \( v \).

This function must satisfy equation (210) and boundary conditions (216). The boundary conditions may be simplified by a change of dependent variable. Let us substitute
\[ v = e^{\gamma} V \tag{218} \]

where \( \gamma \) is a function of \( r \) and \( S \) still to be determined. With this substitution the boundary conditions become

\[
\frac{\partial V}{\partial s} + \left( \frac{\partial \gamma}{\partial s} \right)V = 0 \text{ for } r = r' \\
\frac{\partial V}{\partial s} + \left( m + \frac{\partial \gamma}{\partial s} \right) V = 0 \text{ for } S = S' \\
\frac{\partial V}{\partial r} = 0 \text{ for } \sigma = \sigma' \\
V = 1 \text{ for } \sigma = \sigma' \text{ where } \sigma' = r' + S' \\
V = 1 \text{ for } r = r' \text{ and } S = S' , \text{ that is for the sides } \alpha \xi \text{ and } \beta \xi \tag{219}
\]

Since \( m \) is a function of \( \sigma = r + S \), if we take \( \gamma = -\int_{\sigma'}^{\sigma} m d\sigma \) the above conditions become:

\[
\frac{\partial V}{\partial s} = 0 \text{ for } r = r' \\
\frac{\partial V}{\partial s} = 0 \text{ for } S = S' \\
V = 1 \text{ for } \sigma = \sigma' \text{ where } \sigma' = r' + S' \\
\]

and these are obviously equivalent to the condition:

\[ V = 1 \text{ for } r = r' \text{ and } S = S' , \text{ that is for the sides } \alpha \xi \text{ and } \beta \xi \tag{219} \]

By this change of variable equation (210) becomes:

\[
\frac{\partial^2 V}{\partial x \partial s} + \left( \frac{dm}{d\sigma} - m \right) V = 0 \tag{220}
\]

If we suppose that \( V \) is a function of a variable,

\[ z = \mu (S - S') (r - r') \tag{221} \]

where \( \mu \) is a function of \( \sigma \) to be determined, the boundary condition (219) may be fulfilled. If equation (220) allows of this change of variables, this equation becomes an ordinary differential equation. This transformation although not in general possible, is possible for the case of Boyle's law (20) and Poisson's law (21)

For Boyle's law, equation (201) applies, and equation (220) becomes

\[
\frac{\partial^2 V}{\partial x \partial s} - \frac{1}{4a^2} V = 0 \tag{222}
\]
If we let \( z = \frac{(r - r')(s - s')}{4a^2} \) this transforms into
\[
\frac{d^2v}{dz^2} + \frac{1}{z} \frac{dv}{dz} - v = 0
\]
This is reduced to Bessel's equation by the transformation \( 4z = -x^2 \), giving
\[
\frac{d^3v}{dx^2} + \frac{1}{x} \frac{dv}{dx} + v = 0
\]
The solution is thus:
\[
v = 1 + z^2 + \frac{2^2}{(1^2)^2} + \frac{2^2}{(1^3)^2} + \ldots
\]
The solution of our problem is now complete for the case of Boyle's law.

For Poisson's law, by equation (202), we may write
\[
m = \frac{\lambda}{u^2}
\]
where \( \lambda \) is a constant. Equation (220) then becomes:
\[
\frac{\partial^2v}{\partial z^2} - \frac{\lambda + \lambda^2}{\sigma^2} v = 0
\]
If we let \( z = -\frac{(r - r')(s - s')}{(r + s)(r' + s')} \) this transforms to:
\[
z(1-z) \frac{d^2v}{dz^2} + (1 - 2z) \frac{dv}{dz} + \lambda(\lambda + 1) v = 0
\]
The solution may be expressed in terms of a hypergeometric series as follows:
\[
v = F(\lambda + 1, -\lambda, 1, z) = 1 - \lambda(\lambda + 1)z + \frac{(\lambda - 1)\lambda(\lambda + 1)(\lambda + 2)}{(1^2)^2} z^2 + \ldots
\]
The solution of our problem is now complete for the case of Poisson's law.

Previous to the work of Riemann, Ampère succeeded in reducing the differential equations of motion to a linear form when the gas obeyed Boyle's law. He uses quantities \( \alpha, \beta, \eta \) which differ only in sign from the quantities \( r, s, w \) respectively of Riemann. The equation he obtains is
\[
2a \frac{\partial^2\eta}{\partial\alpha\partial\beta} - \frac{\partial\eta}{\partial\alpha} - \frac{\partial\eta}{\partial\beta} = 0
\]

1. Ampère, Journal de l'Ecole Polytechnique 1820 Cahier XVIII, t Xi, p 177
which agrees exactly with Riemann's equation (200) when \( m \) is given by equation (201). The integration of this equation was, however, Riemann's contribution.

Hadamard transforms equation (141) by changing independent variables from \( a, t \) to \( u, w \) and changing the dependent variable from \( x \) to a variable \( z \) defined by:

\[
z = w a + u t - x
\]

(231)

The transformed equation is:

\[
\frac{\partial^2 z}{\partial w^2} = \psi(w) \frac{\partial^2 z}{\partial u^2}
\]

(232)

He then transforms independent variables again from \( u, w \) to \( \xi, \eta \) which are equivalent to the quantities \(-2\gamma, 2\gamma\) of Riemann. The resulting equation is:

\[
4 \frac{\partial^2 z}{\partial \xi \partial \eta} - f(\xi - \eta) \left( \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta} \right) = 0
\]

(233)

where the function \( f \) is defined by:

\[
f \left[ 2 x(w) \right] = \frac{1}{4} \frac{\chi''(\omega)}{\chi'(\omega)^2}
\]

(234)

This equation is of the same form as equation (200) of Riemann and is solved in an analogous manner.

b. Application to Special Cases

1. Riemann's Special Case

A special case treated by Riemann is one in which initially \( r \) and \( S \) have different constant values on either side of a segment \( \alpha \beta \) of the \( x \)-axis and in this segment change without a maximum point from the one constant value to the other. We assume:

\[
r_1, \quad r_2, \quad S_1, \quad S_2
\]

(235)

Hadamard, loc. cit. pp. 162, 163

Riemann, loc. cit.
This case is given by Weber\(^1\) and his treatment is here followed. In the region \(\alpha \beta \xi\) of the \(xt\) plane (See Fig. 5) \(r\) and \(S\) are found as functions of \(x, t\) by the method of the previous section. Outside of this region we assume at least one of the quantities \(r\) and \(S\) to be constant. By comparison of equations \((168)\) and \((169)\) with equation \((155)\) we see that in this case the wave is single progressive. On the curve \(\alpha \xi\), \(r=r_1\) and \(S\) decreases continuously without maximum or minimum points from \(\alpha\) to \(\xi\). On the curve \(\beta \xi\), \(S = S_2\) and \(r\) decreases continuously without maximum or minimum points from \(\beta\) to \(\xi\). From every point \(u\) on the curve \(\alpha \xi\) construct a straight line \(uu'\) making an angle \(\theta\) for which

\[
\cot \theta = u - \sqrt{\phi'(\rho)},
\]

Then, according to the solution for a single progressive wave \(u\) and \(\rho\) are constant along every such line. Also from every point \(v\) on the curve \(\beta \xi\) construct a line \(vv'\) making an angle \(\theta\) for which

\[
\cot \theta = u + \sqrt{\phi'(\rho)}.
\]

Then along every such line \(u\) and \(\rho\) are constant. Construct four straight lines, \(\alpha 1, \xi 2, \xi 3, \beta 4\) with angles \(\theta_1, \theta_2, \theta_3, \theta_4\) such that

\[
\begin{align*}
\cot \theta_1 &= u - \sqrt{\phi'(\rho_1)} \\
\cot \theta_2 &= u' - \sqrt{\phi'(\rho')} \\
\cot \theta_3 &= u' + \sqrt{\phi'(\rho')} \\
\cot \theta_4 &= u + \sqrt{\phi'(\rho_2)}
\end{align*}
\]

Where \(u'\) and \(\rho'\) are the values of \(u\) and \(\rho\) at the point \(\xi\).

Then we assume:

in the sector \((-\infty, \alpha)\), \(u=u_1, \rho = \rho_1\) \(\text{(239)}\)

in the sector \((\alpha, \xi)\), \(u=u_1', \rho = \rho'\)

in the sector \((\xi, \beta)\), \(u=u_2, \rho = \rho_2\)

Weber, loc. cit. pp 515-518
The rest of the x-t-plane is filled by the lines $\mu \mu'$ and $\nu \nu'$. These lines diverge and hence the solution is everywhere unique. Under other assumptions than (235) these lines would not diverge and a discontinuity would set in at a certain time. When $\alpha \beta$ is made to approach zero the above solution becomes Riemann's second case of an initial discontinuity.

2. **Linear Type Waves in $r$ and $s$**

Lord Rayleigh finds a solution for the case where $r$ and $s$ are linear such that we have:

$$r = A x + B, \quad s = C x + D$$

(240)

where $A, B, C, \text{ and } D$ are functions of $t$. Boyle's law, (20) is assumed, for this case equations (176) and (177) become:

$$\frac{\partial r}{\partial t} = - (u + a) \frac{\partial r}{\partial x}$$

(241)

$$\frac{\partial s}{\partial t} = - (u - a) \frac{\partial s}{\partial x}$$

(242)

Substituting the values of $r$ and $s$ from equations (240) we obtain:

$$\left[ \frac{dA}{dt} + A (A - c) x + \frac{dB}{dt} + A (B - D + a) \right] = 0$$

(243)

$$\left[ \frac{dC}{dt} + C (A - c) x + \frac{dD}{dt} + C (B - D - a) \right] = 0$$

(244)

Since these equations are identities in $x$, the coefficients of $x$ in the two equations must be zero and terms without $x$ must be zero. We thus have four equations to determine the four quantities $A, B, C, \text{ and } D$. Four arbitrary constants enter into the expressions for these quantities but two of them merely designate an arbitrary origin of time and arbitrary origin of absises. We are not interested in the transformation of axes and hence we may express these quantities in terms of two arbitrary constants. On solving for these quantities and substituting their values in equations (240) we obtain the following expressions for $r$ and $s$:

\[2r = (H+1) \frac{x}{t} + (H - 1) a \log t + M \quad (245)\]
\[2S = (H-1) \frac{x}{t} + (H - 1) a \log t + N \quad (246)\]

where \(M+N=-2aH\)

Then according to equations (177-2) we obtain for the velocity and density:
\[u = r-S= \frac{x}{t} - aH \quad (247)\]
\[a \log \rho = r + S = H \frac{x}{t} + (H - 1) a \log t - \frac{1}{2}(M+N)\]

or \[\rho = Ct(H^2) - \frac{1}{2e}(H x/a t) \quad \text{where } C=e^H \quad (248)\]

Thus for this case, the velocity remains always linear, remaining constant in magnitude at the origin. The slope of the velocity wave becomes ever more gradual. The density at any instant has an exponential distribution. If \(H=\pm 1\), we see from equations (245) and (246) that either \(r\) or \(S\) is constant and in these cases we fall back upon single progressive waves. If \(H = 0\), \(u=\frac{x}{t}\) and \(\frac{1}{\rho}\) is proportional to \(t\).

The density is constant for a given value of \(t\) throughout the gas but decreases with time in such a way that the specific volume of the gas is proportional to \(t\). The acceleration experienced by a particle of gas is \(\frac{\partial w}{\partial t} + u \frac{\partial u}{\partial x}\) and by substitution of \(u\) from equation (247) we see that this is zero.

3. Limited Initial Disturbance

Riemann's solution does not apply to a region \(\alpha \beta\) in which \(r\) or \(S\) has a maximum. Nevertheless, we can learn something of the propagation of a sound wave from an initial disturbance confined to a region \(\alpha \beta\) outside of which the gas is everywhere at rest and has a constant density \(\rho_0\) by a study of equations (178) and (179) or statements (180) and (181). From these equations we see that \(r\) is propagated in the positive direction with a velocity \(u+\sqrt{\varphi(\rho)}\) and \(S\) is propagated in the negative direction with a velocity \(-u+\sqrt{\varphi(\rho)}\). At a certain time, which is indeterminate until the equations are solved, the hinder limit of the \(r\) varies meets the hinder limit of the region in which \(S\) varies after which the two regions separate and include between them a portion of the fluid in its equilibrium condition.

After the separation of the two regions, S is constant in the gas to the right of the intermediate region and r is constant in the gas to the left of this region and hence there are two singly progressive waves, one propagated in the positive direction and one in the negative direction.
II. Spherical Waves of Finite Amplitude.

Spherical waves of finite amplitude do not seem to have been studied. The case of steady radial flow is given by equations (9) and (10) which determine \( u \) and \( \rho \) as functions of \( r \). From the importance of spherical waves in experiment we have been led to investigate somewhat the case of finite amplitude spherical waves. At any point of a finite amplitude wave we may approximate conditions in the immediate neighborhood of this point at the instant by a steady radial flow with superimposed infinitesimal disturbance. This fact led us to investigate the latter case.

\[ a. \text{Infinitesimal Disturbance Upon a Steady Flow.} \]

We must use equations (7) and (8) where \( p = \phi(\rho) \) [equation (22)].

Since by equation (9) \( r^2 \rho \ u \) is constant for a steady flow, in our case we have:

\[ r^2 \rho \ u = c + s \quad (249) \]

where \( c \) is constant and \( s \) is infinitesimal. Equation (7) then becomes:

\[ r^2 \frac{\partial \rho}{\partial t} + \frac{\partial s}{\partial r} = \phi \quad (250) \]

Now let \( s = \frac{\partial \sigma}{\partial t} \). Substituting this in the above and integrating we obtain:

\[ r^2 \rho + \frac{\partial \sigma}{\partial r} = R \quad (251) \]

or

\[ \rho = \frac{1}{r^2} \left( R - \frac{\partial \sigma}{\partial r} \right) \quad (252) \]

where \( R \) is an arbitrary function of \( r \).

Substituting the values of \( u \) and \( \rho \) from equations (249) and (252) in equation (8) we obtain:

\[ (R - \frac{\partial \sigma}{\partial r}) \frac{\partial^2 \sigma}{\partial t^2} + 2 \frac{\partial^2 \sigma}{\partial r \partial t} = 2 \frac{C^2}{r^2} \left[ (R - \frac{\partial \sigma}{\partial r}) \phi' \left( \frac{R - \frac{\partial \sigma}{\partial r}}{r^2} \right) - \frac{C^2}{r^2} \left[ R - \frac{2 \sigma}{r^2} \rho' \left( \frac{r^2 \rho}{R} \right) - \frac{\partial \sigma}{\partial r} \phi' \left( \frac{R - \frac{\partial \sigma}{\partial r}}{r^2} \right) \right] \right] = 0 \]

Dropping higher powers of infinitesimals than the first, we obtain:-
Expressing \( \phi \left( \frac{R - \frac{2\sigma}{r^2}}{r} \right) \) by the first two terms in the expansion in the powers of \( \frac{\partial \sigma}{\partial r} \) and again dropping terms of higher degree than the first in (253) we have:

\[
\frac{R^2 \sigma}{dt^2} + 2C \frac{\partial \sigma}{\partial r} + \phi \left( \frac{R - \frac{2\sigma}{r^2}}{r} \right) \left\{ \frac{RR'}{R} - \frac{2R}{r} - \frac{2R}{r} \frac{\partial \sigma}{\partial r} - R \frac{\partial \sigma}{\partial r} - R' \frac{\partial \sigma}{\partial r} + \frac{R}{r} \frac{\partial \sigma}{\partial r} \right\}
\]

The terms in derivatives of \( \sigma \) are infinitesimals and hence the terms not containing these derivatives must be placed equal to zero. Thus we have:

\[
- \frac{R^2}{R} \phi \left( \frac{R - \frac{2\sigma}{r^2}}{r} \right) \left( \frac{RR'}{R} - \frac{2R}{r} \right) = 0
\]  
(255)

Thus we have an ordinary differential equation of the first order to determine \( R \). Then equation (254) becomes:

\[
\frac{R^2}{dt^2} + 2CR \frac{\partial \sigma}{\partial r} + \frac{C^2 \partial \sigma}{\partial r^2} - R \frac{\partial \sigma}{\partial r} \left\{ R \phi \left( \frac{R - \frac{2\sigma}{r^2}}{r} \right) \frac{\partial \sigma}{\partial r} \right\} = 0
\]  
(256)

Let \( \sigma = e^P \), where \( P \) is a function of \( r \) to be defined later.

Then equation (256) becomes:

\[
a(R + CP') + aP'' - a^2 R^2 \phi \left( \frac{R}{r^2} \right) P' = a^2 \frac{\partial}{\partial r} \left\{ \frac{R^2}{R} \phi \left( \frac{R}{r^2} \right) P' \right\} = 0
\]  
(257)

This may be put into the form:

\[
\frac{1}{r^4} \left\{ a(R + CP') + aP'' \right\} = (aP + aP') \left( R \phi \left( \frac{R}{r^2} \right) P' \right)
\]  
(258)

Substitute \( S = \frac{R^2}{r^4} \phi \left( \frac{R}{r^2} \right) P' \)

and equation (258) becomes of the first order and takes the form:

\[
a\left\{ r^2 \phi (\eta) + C S \right\} + \eta^2 \phi ' (\eta) \frac{dS}{dr} - S \frac{d}{dr} \left\{ \eta^2 \phi ' (\eta) \right\} = a \eta^2 \phi (\eta) S^2 + \eta \phi (\eta) \frac{dS}{dr}
\]  
(259)

This may be put in the form:

\[
a \left\{ \eta^2 \phi (\eta) + C S \right\} \frac{d}{dr} \left\{ \eta^2 \phi ' (\eta) \right\} = \frac{d}{dr} \left\{ \eta^2 \phi ' (\eta) S \right\}
\]  
(260)
Let $\theta = \xi((\eta^2\phi'(\eta)-1)/(\eta^2\phi'(\eta)))$. Then the above equation becomes:

$$\frac{1}{a} \frac{d}{dr} \left( r^2 \phi'(\eta) \right) = \frac{2cr^2}{\eta^2\phi'(\eta) - 1} \theta + \frac{c^2 - \eta^2\phi''(\eta)}{\eta^2\phi'(\eta) - 1} \theta^2 \tag{261}$$

Now $\eta = \frac{R}{r^2}$ is known when $R$ is determined from equation (255) and hence (261) may be solved for $\theta$ as a function of $r$. Working back through the transformations made, $P$ then is known as a function of $r$. Values of $\sigma$ are then propagated with a velocity $-P$. Since equation (256) is linear, the sum of any number of solutions is a solution. By choosing "a" complex, $\sigma$ becomes a damped periodic function.

b. Finite Amplitude Wave.

Let $r^2 = \frac{\partial F}{\partial r}$ and substitute in equation (7).

Then integrating with respect to $r$ we obtain:

$$\frac{\partial F}{\partial t} + F u = f(t) \tag{262}$$

The function $f(t)$ may be regarded as incorporated in the function $F$. The right member of equation (259) may therefore be regarded as zero. Using these values of $u$ and $\rho$ in equation (8) we obtain:

$$\frac{F_{tt}}{(F_t')^2} - \frac{2F_{tt}}{F_{xx}'} \frac{F_{tt}}{(F_{xx}')^2} = \phi \left( \frac{P_t}{r^2} \right) \left\{ \frac{F_{xx}''}{F_{xx}'} \frac{2}{r} \frac{F_{xx}'}{r} \right\} \tag{263}$$

I have not been able to solve this equation. The solution of the preceding case should be able to be employed to build up a solution for this case similar to Lord Rayleigh's development of Poisson's integral for a plane positive progressive wave as discussed immediately after equation (159).
E. Plane Waves of Permanent Regime With Dissipative Forces.

Waves of permanent regime are those for which the velocity wave does not change its form. We have already seen, by equation (167), that this is only true for a certain law of pressure. This law can only be maintained by conduction of heat or by viscosity or other body forces acting upon the gas. A plane wave of permanent regime can be brought to rest by applying a constant velocity to the gas as a whole. It, therefore, may be treated as a case of steady motion. The equations applicable are (3) and (4) in which according to substitution (11), we substitute for

\[-\frac{1}{\rho} \frac{df}{dx} x - \frac{1}{\rho} \frac{dp}{dx}\].

Assuming steady motion the derivatives with respect to t disappear from these equations and the equations become ordinary. Equations (3) and (4) thus become:

\[\rho u = m\] (264)

\[u \frac{du}{dx} + \frac{1}{\rho} \frac{dp}{dx} = X\] (265)

We shall confine ourselves to a steady wave in which at sufficient distances in the positive and directions, the gas becomes of a uniform state. If the force X is due to viscosity, it vanishes at both terminal states. Also then there is no heat conduction at the terminal state and hence \[\int dq\] is zero where Q is the heat received by an element of gas moving to a point x, and where the integral is taken from one terminal state to the other.

I. Entropy Condition.¹

From equations (264) and (265) we have:

\[x \rho = m \frac{du}{dx} + \frac{dp}{dx}\] (266)

The force operative upon an element of mass is \( Xp \) and the total momentum given to the element is:

\[
\int x^2 \rho \, dx = m (u_2 - u_1) + p_2 - p_1 = \frac{m^2}{\rho_2} - \frac{m^2}{\rho_1} + p_2 - p_1 \tag{267}
\]

According to equations (264) and (266) if the integration is taken from the initial terminal state in which the velocity, density and pressure are \( u_1, \rho_1, \) and \( p_1 \), respectively, to the final terminal state in which these quantities have the values \( u_2, \rho_2, \) and \( p_2 \), respectively. The principle of conservation of momentum applies to the viscosity forces between the particles of the gas and hence \( \int Xp \rho \, dx \) in equation (267) is zero. Therefore we have:

\[
\frac{m^2}{\rho_2} - \frac{m^2}{\rho_1} + p_2 - p_1 = 0 \tag{268}
\]

which is equivalent to equation (167) applied to terminal states.

The energy expended by the viscosity forces is \( \int Xp \rho u \, dx \). From equations (264) and (266) we have:

\[
\frac{1}{m} \int x \rho u \, dx = \frac{1}{2} (u_2^2 - u_1^2) + \int_{p_1}^{p_2} \frac{dp}{\rho} \tag{269}
\]

By equation (264) the right member is a function of \( p \) and \( \rho \).

For Boyle's law, (20), and Poisson's adiabatic law, (21), this quantity is positive or negative according as \( p_2 \) is less or greater than \( p_1 \). The condition of increase of entropy or the condition of dissipation of energy implies that this quantity be negative. Therefore, since the final pressure exceeding the initial pressure denotes a wave of condensation, only waves of condensation are possible.

II. Heat Conduction.

Rankine studies the heat transfers necessary to maintain Earnshaw's law of pressure, (167). He finds for a perfect gas

that to effect a change of pressure \( \frac{dp}{\rho} \), according to this law, the quantity of heat added must be:

\[
dQ = \frac{\frac{dp}{\rho}}{m^2(y-1)} \left\{ p_0 + m^2 v_o - (y+1)p \right\}
\]

(270)

Integrating from the initial to the final state, we have since

\[
\int dQ = 0 \ :
\]

\[
p_0 + m^2 v_o = \frac{1}{2}(y+1)(p_1 + p_2)
\]

(271)

where \( p_0 \) and \( v_o \) are the pressure and specific volume at any point in the wave. If \( p_0 , v_o \), are identified with \( p_1 , v_1 \), we obtain:

\[
m^2 v_1 = \frac{1}{2} (y-1)p_1 + \frac{1}{2}(y+1)p_2
\]

(272)

Since \( v_1 = \frac{1}{\rho_1} \), by equation (264) \( m v_1 \) is equal to \( u_1 \). Multiplying equation (272) by \( v_1 \), we therefore obtain:

\[
u_1^2 = m^2 v_1^2 = v_1 \left\{ \frac{1}{2} (y-1)p_1 + \frac{1}{2} (y+1)p_2 \right\}
\]

(273)

which is the square of the velocity of propagation with respect to region 1. This is greater than velocity of sound of infinitesimal amplitude, for which \( u_1^2 = y p_1 v_1 \), for a condensation and less for a rarefaction, as we see from the above formula.

The absolute temperature is determined for a perfect gas as follows:

\[
\frac{\Theta}{\Theta_0} = \frac{p v}{p_o v_o} = \frac{p}{p_o} \frac{(y+1)(p_1 + p_2)}{(y+1)(p_1 + p_2) - 2p} - 2p
\]

(274)

by the relation inequation (271). The flow of heat is everywhere \( K \frac{d\Theta}{dx} \), where the coefficient of heat conduction \( K \) may be a function of the condition of the gas. If we reckon \( Q \) from the initial condition of constant pressure \( p_1 \),

\[
K \frac{d\Theta}{dx} = m Q
\]

(275)

For the distribution of pressure Rankine finds from the above equation:

\[
dx = -\frac{k}{mC(y+1)} \frac{(y+1)(p_1 + p_2) - 4p}{(p - p_1)(p_2 - p)}
\]

(276)

Under the supposition that \( k \) is constant, Rankine finds integrates equation (276) and obtains:
where \( q = p - (1/2)(p_1 + p_2) \) and \( q_1 = (l/2)(p_2 - p_1) \), \( x \) being measured from the place where \( q \) equals zero. Although mathematically the wave is infinitely long, practically the transition is effected in a distance comparable with \( k/\sqrt{mC(y+1)} \). Since \( \frac{dx}{dp} \) must not change sign, the numerator in (276) must be always positive, and hence \( \frac{p_2}{p_1} \) must not exceed \( \frac{y+1}{3-y} \).

### III. Viscosity

For viscosity we must place in equation (265)

\[
X = \frac{4}{3\rho} \frac{d}{dx}(\mu \frac{dv}{dx})
\]

according to Lamb. In this case also Rankine's condition holds, see equation (268).

Equation (265) integrates into:

\[
p + m^2v - \frac{4}{3}m \mu \frac{dv}{dx} = p_1 + m^2v_1 = p_2 + m^2v_2
\]  

(278)

For Boyle's law we find by integration:

\[
\frac{3m}{4\mu} = \frac{1}{v_1 - v_2} \left\{ v_1 \log(v_1 - v) - v_2 \log(v - v_2) \right\}
\]

Here \( \frac{dx}{dv} \) never changes sign and a permanent wave of condensation is always possible no matter what the value of the ratio \( \frac{p_1}{p_2} \) may be. The velocity of propagation into the rarer medium is:

\[
\frac{v_1}{v_2} = \frac{a}{\frac{v_1}{v_2}}
\]  

(279)

### IV. Viscosity and Heat Conduction

Here the heat change in an element consists of two parts

\( dQ_1 \) and \( dQ_2 \), the former the heat received by conduction and the latter that developed internally under viscosity. For \( \frac{dQ_1}{d\gamma} \) and \( \frac{dQ_2}{dx} \) we find the expressions:

\[
(y - 1)\frac{dQ_1}{d\gamma} = y(p_1 + m^2v)\frac{dv}{dx} - (y + 1)m^2v\frac{dv}{dx} + \frac{4}{3}m\mu\left( \frac{dv}{dx} \right)^2
\]

(280)

Using Rankine's equation (278), equation (280 becomes:
\[
(y - 1) \frac{d\theta}{dx} = \frac{1}{2} (y + 1) m (v_1 - v_2) \frac{dv}{dx} - \frac{3}{2} \frac{d^2\theta}{dx^2} \frac{m}{\mu} \frac{dv}{dx} 
\]
From this equation if we reckon \(Q\) from the terminal state \(v\),
\[
(y - 1) Q = \frac{1}{2} (y + 1) m (v_1 - v)(v - v_2) + \frac{4}{3} \frac{m}{\mu} v \frac{dv}{dx} .
\]
Equation (278) still applies and expressing \(\theta\) as \(\frac{p v}{R}\), we may obtain, by substituting the value of \(p\) from equation (278), \(\theta\) as a function of \(v\).

Then substituting in the equation of conduction, we obtain:
\[
\frac{k}{mc} \left[ m \frac{dv}{dx} \left( \frac{y + 1}{2} (v_1 + v_2) - 2v \right) + \frac{4}{3} \frac{d^2v}{dx^2} \left( \frac{p v}{R} \right) \right] = \frac{1}{2} (y + 1) m (v_1 - v)(v - v_2) \frac{m}{\mu} \frac{dv}{dx} .
\]
Rayleigh assumes Maxwell's theory as connecting the specific heat \(C\), at constant volume, with the coefficient of heat conduction \(K\).

For the theory which assumes a molecular repulsion inversely as the square of the distance,
\[
\frac{C}{K} = \frac{2}{5} .
\]
Calling this ratio \(h\) and writing for \(\frac{\mu}{m}\) equation (282) becomes:
\[
\mu \frac{dv}{dx} (\mu \frac{dv}{dx})^2 + \mu' \frac{d\xi}{dx} \left[ \frac{3}{8} (y + 1) \frac{v_1 + v}{v} - \frac{3}{2} - h \right] = \frac{3}{4} h (y + 1) (v_1 - v)(v - v_2) .
\]
Writing \(\xi\) for \(v^2\) and \(U\) for \(\mu' \frac{d\xi}{dx}\), this equation is reduced to one of first order, namely,
\[
U \frac{d\xi}{dx} + U f (\xi) = F (\xi) .
\]
where \(f (\xi) = \frac{3}{8} (y + 1) \frac{v_1 + v}{v} \left( \sqrt{\xi_1} + \sqrt{\xi_3} \right) - \frac{3}{2} - h\)
and \(F (\xi) = \frac{3}{4} h (y + 1) \left( \sqrt{\xi_1} - \sqrt{\xi_3} \right) \left( \sqrt{\xi_1} + \sqrt{\xi_3} \right) .
\]
When \(U\) is determined as a function of \(\xi\) from this equation, \(x\) is found by simple integration of the equation:
\[
U = \mu' \frac{d\xi}{dx}
\]
Rayleigh solves equation (283) by a series of approximations.
F. Summary and Outlook.

We have treated continuous and discontinuous waves. We were able to treat discontinuous waves in three dimensions. The velocity of propagation of a second or higher order discontinuity with respect to the gas is that of sound of infinitesimal amplitude. Although there is much literature on the subject of first order discontinuities it appears that such waves are probably not possible theoretically. In reality, even if the viscosity and heat conduction were everywhere else negligible, they would not be negligible at the discontinuity and these factors would prevent the formation or the persistence of the discontinuity. Since second or higher order discontinuities are propagated with a velocity of that of ordinary sound with respect to the gas, Poisson's integral for a continuous wave holds if the velocity wave is discontinuous in slope or curvature, or higher derivatives of the velocity with respect to x are discontinuous. If there is a given initial rectilinear disturbance in a limited region, the curves for Riemann's variables r and s are propagated in the positive and negative directions, respectively. After these waves separate we have two single progressive waves, one in the positive and one in the negative direction. The time for complete separation to take place can be approximately determined. The limitation involved in Poisson's integral, by which after a certain time the velocity and density are no longer uniquely determined, is really very serious. The theory of sound waves of finite amplitude should find its application in explosive waves. Now velocities of explosive waves have been measured which are more than twice the velocity.
of ordinary sound. Let us suppose that the velocity of a wave is measured by the velocity of propagation of the maximum point of the velocity curve. The maximum point of the velocity curve under this condition would overtake the front of the wave which moves with the ordinary velocity of sound in a distance less than the wave length of the wave. Hadamard\(^1\) states that the experiments of Vielle\(^2\) on explosive waves, discontinuity would take place in a few centimeters. The introduction of viscosity and heat conduction would seem to be necessary to prevent the wave from becoming discontinuous. The only waves under viscosity and heat conduction that have been successfully treated are waves of permanent regime in which the gas passes from one uniform condition in front of the wave by continuous change without maximum or minimum points to another uniform condition back of the wave. We have seen that a single progressive wave generated from a limited initial disturbance must have a maximum point. To such a wave the solution for the presence of viscosity and heat conduction does not apply. We could infer probably that the forward slope of the wave would attain a state approximating a permanent regime and that for the backward slope of the wave, Poisson's integral would remain approximately correct and that, in accordance therewith, this slope would become more gradual.

As to future investigation, we need a greater theoretical knowledge of waves other than plane waves. A solution for a finite amplitude spherical wave would be valuable for such waves can be more nearly approximated in experiment than plane waves. Waves, supposed to be plane, propagated in tubes, are affected by

the friction of the gas upon the tube. The influence of viscosity and heat conduction on such waves also needs to be studied. We also need a study of the phenomenon of reflection. This can be studied first, as the most simple case, for plane waves. Before we can study refraction we must know more about waves other than plane waves.

As to experimental verification, the velocity of propagation of a wave into a region at rest can be measured, and in so far as the wave approximates a plane wave, the different formulae for the velocity of propagation can be tested. The resistance of the gas to the motion of a solid through it, if we assume the wave generated in front of the solid to be plane, can be used to determine the relation between the pressure and the velocity in a plane wave. Rayleigh assumes for projectiles that the fore part of the wave generated in front of the projectile is one of permanent regime under viscosity and heat conduction and that from the velocity associated with the final state of this wave there is a second transformation under which as the head of the projectile is approached the gas changes according to the adiabatic law and the velocity becomes that of the projectile. These two transformations are supposed to take place in such a small distance that in sound photographs of rifle bullets, etc., they appear as one. The determination of the shape of the density wave does not lend itself readily to experiment. The use of a membrane with a device to record its movements and upon which the wave impinges is of little avail since the free period of the membrane and its inertia affect the results in a manner which cannot be accurately allowed for. If a wave could be maintained steady in a tube by impressing a velocity upon the gas as a whole,

the shape of the wave might be found. Possible experiments on the reflection of finite amplitude waves and their curvature around obstacles, which can be studied by means of sound photographs, await further advances in theory.