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# Look-back stopping times and their applications to liquidation risk and exotic options

Bin Li

*University of Iowa*

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LOOK-BACK STOPPING TIMES AND THEIR APPLICATIONS TO  
LIQUIDATION RISK AND EXOTIC OPTIONS

by

Bin Li

An Abstract

Of a thesis submitted in partial fulfillment of the  
requirements for the Doctor of Philosophy  
degree in Applied Mathematical and Computational Sciences  
in the Graduate College of  
The University of Iowa

May 2013

Thesis Supervisors: Professor Qihe Tang  
Professor Lihe Wang

## ABSTRACT

In addition to first passage times, many look-back stopping times play a significant role in modeling various risks in insurance and finance as well as in defining financial instruments. Motivated by many recently arisen problems in risk management and exotic options, we study some look-back stopping times including drawdown and drawup, Parisian time and inverse occupation time of some time-homogeneous Markov processes such as diffusion processes and jump-diffusion processes.

Since the structures of these look-back stopping times are much more complex than fundamental stopping times such as first passage times, we aim to develop some general approaches to study these stopping times such as approximation approach and perturbation approach. These approaches can be transformed to a wide class of stochastic processes. Many interesting and explicit formulas for these stopping times are derived and based on which we gain quantitative understandings of these problems in insurance and finance.

In our study, we mainly use the techniques of Laplace transforms and partial differential equations (PDEs). Due to the complex structures, the distributions of these look-back stopping times are usually not explicit even for the simplest linear Brownian motion. However, under Laplace transforms, many important formulas become explicit and it enables us to conduct further derivations and analysis. Besides, PDE methodology provides us an effective and efficient approach in both theoretical investigation and numerical study of these stopping times.

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Graduate College  
The University of Iowa  
Iowa City, Iowa

CERTIFICATE OF APPROVAL

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PH.D. THESIS

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This is to certify that the Ph.D. thesis of

Bin Li

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Applied Mathematical and Computational Sciences at the May 2013 graduation.

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To my parents and grandparents

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In addition to first passage times, many look-back stopping times play a significant role in modeling various risks in insurance and finance as well as in defining financial instruments. Motivated by many recently arisen problems in risk management and exotic options, we study some look-back stopping times including drawdown and drawup, Parisian time and inverse occupation time of some time-homogeneous Markov processes such as diffusion processes and jump-diffusion processes.

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# CHAPTER 1

## INTRODUCTION

### 1.1 Objectives of the Thesis

Stopping times as indicators of a variety of risks play a significant role in modeling insurance contracts, financial instruments, and extreme events in insurance and finance. The objectives of this thesis are as follows:

- To model several types of risks from recently arisen problems in risk management using some stopping times, especially look-back stopping times.
- To develop general approaches to study these look-back stopping times, which can be applied to a wide class of stochastic processes.
- To obtain explicit formulas for many quantities related to these look-back stopping times.
- To gain quantitative understandings of these types of risks based on the explicit formulas.

We first briefly review some stopping times and their applications. First passage times, which do not need us to look back, are the most fundamental and popular stopping times and are extensively used in risk management and financial securities. The law of a first passage time is usually an immediate result of the transition density of a Markov process. Although the transition densities of many Markov processes are not explicit since they are essentially fundamental solutions of partial differential

equations (PDEs), we may still be able to derive explicit formulas for some useful quantities such as the two-sided exit probabilities, which are essentially the solutions of boundary value problems of ordinary differential equations (ODEs). A major approach to study more complex look-back stopping times is to convert them into first passage times.

The first kind of look-back stopping times we will study is drawdown and drawup. Drawdown measures the decline in value from the running maximum (high water mark) and drawup measures the rise in value from the running minimum (low water mark) of an investment, fund or commodity. Empirically, drawdown and drawup are two of the most frequently quoted indicators of extreme financial risks. The study of drawdown and drawup is of great importance for practical applications. In classical studies, major determinants of drawdown and drawup include the expected return rate, the volatility of returns, and the length of track record. However, it is more practical to consider drawdown and drawup jointly with the values of the running maximum and minimum, respectively, because, for example, big drawdowns more likely happen to high-priced stocks. Therefore, we study some generalized drawdowns and drawups by incorporating the corresponding high water mark and low water mark, respectively. In particular, we consider the percentage of a decline with respect to its high water mark.

The second kind of look-back stopping times is inverse occupation times. An occupation time is the time a stochastic process spends within a certain range of space. Note that an occupation time is not a stopping time but its generalized

inverse is. Occupation time, stemming from Lévy arcsine law, is an appealing and challenging topic in stochastic processes and has many applications in insurance and finance. Due to the aggregation feature of occupation times, they are often used to model financial distress by recording the history of depression. Further, motivated by risk management problems, many derivatives related to occupation times are also introduced recently such as corridor options, quantile options and Parisian options. These options can overcome many disadvantages of standard barrier options such as increased volatility around the barrier, discontinuous delta at the barrier, short-term market manipulation, and so on.

The third and the last kind of look-back stopping times considered in this thesis is Parisian times, which are named after Parisian options. A Parisian time is the first time when a stochastic process spends continuously above or below a barrier over a prespecified length of time. Since the first paper on Parisian options published in 1997, many applications of Parisian times have been proposed. One application in quantitative finance is to model the liquidation risk subject to Chapter 11 of the U.S. bankruptcy code. In contrast to straight liquidation, Chapter 11 allows a firm to remain in control of its business operations during a grace period granted by a bankruptcy court. The firm will eventually liquidate if its value stays continuously in a low range during the grace period. Besides, Parisian times are also applied in life insurance to capture the early delayed closure procedure of pension funds. Generally speaking, Parisian times are very useful to model delayed procedures of many insurance and financial events and products.



The challenges of our study are twofold. First, the distributions or joint distributions of these stopping times are usually not explicit. Hence, we mainly apply Laplace transform and PDE techniques in our study. A major advantage of the Laplace transform approach is that we are able to derive explicit formulas for the law of the underlying stopping times. However, the inversion of a Laplace transform is usually very challenging even numerically due to the instability. The PDE approach is an effective alternative since we may express the law of a stopping time as well as other interesting quantities into a corresponding PDE by Itô's formula. Even if explicit solutions of these PDEs may not be tractable, we can still use numerical PDE approaches to determine their values. Numerical PDE approaches are usually more stable than the inverse Laplace transform approach. Moreover, as illustrated in this thesis, we can also investigate analytical properties of these stopping times via the corresponding PDEs. This is sometimes more effective than pure stochastic analysis methods since the theory of differential equations has a longer history and the study is relatively more complete. However, for many quantities of interest, we may fail to derive their corresponding PDEs.

Second, the existing approaches in the literature to study these look-back stopping times, for example, Brownian meander and Brownian excursion, are very limited to their structure models so that they cannot be easily transformed to more general Markov processes. Since the look-back stopping times considered in this thesis only called attention during the last decade, most of the previous works were done in the simplest structures like the Black-Scholes model. Thanks to all kinds of known

formulas of Brownian motion and other fundamental processes, some results of these stopping times were derived under this simple structure model. However, due to the needs of insurance and financial practice, we are required to study the stopping times for a broader class of structure models such as diffusions or Lévy processes. Hence, we also aim to develop some general approaches to study these stopping times and related quantities, such as the approximation argument in Chapter 3, the perturbation approach in Chapters 4 and 6, and the differentiability argument in Chapter 5.

Although the discussions in each chapter of the thesis are driven and motivated by some specific problems in insurance and finance, the ideas and methodologies may be transformed to many other interesting problems which are not discussed in this thesis. We just list a few of such problems below in light of future research.

- Since stopping times are indicators of various extreme risks, the sole study of stopping times is essentially the risk management for the first occurrence of extreme risks over certain sizes. However, from risk management point of view, it is more important to take into account the frequency of such extreme risks as well. A simple example of such problems is the probability of more than five occurrences of big drops over 30% from high water marks in a year.
- Some generalized drawdowns and drawups incorporating the corresponding high and low water marks are studied in Chapter 3 in the framework time-homogeneous diffusion processes. To hedge big drawdowns of underlying assets, we can design some contracts which are triggered if the ratio of the size of a big decline to the high water marks exceeds a certain percentage prior to maturity.

These contracts can recover a part of the “loss” of investors who miss the chance to sell the assets at a relatively high price level. Furthermore, a merit of this type of contracts is that they can be used for an entire portfolio consisting of underlying assets with a wide range of prices because only the percentages of declines are considered.

- Regular occupation times and Parisian times only count the time a process has stayed above or below a level. According to the principle of limited liability, it is more interesting and practical to study occupation times and Parisian times in bounded regions as in Chapters 4–6. From there, we may be able to design various financial instruments such as contingent convertibles (CoCos) to hedge the risks of long depression of underlying assets at a lower cost.

## 1.2 Structure of the Thesis

Chapter 2 serves as a brief introduction of some Markov processes and related concepts. Chapters 3–6 form the main part of this thesis.

In Chapter 3, we study the two-sided exit times of a time-homogeneous diffusion process in the presence of tax payments of loss-carry-forward type. Essentially, the exit times of the firm value process in the presence of taxation are transformed to generalized drawdown and drawup of the firm value process in the absence of taxation. By using an approximation approach, we obtain explicit formulas for the Laplace transforms associated with the two-sided exit times. This argument can be easily transformed to general Markov processes with continuous paths. As corollaries,

we solve the expected present value of tax payments until default, the two-sided exit probabilities, and the non-default probability. An interesting tax identity has been established by researchers in various situations within the Lévy framework. However, we prove that it does not hold in general within the diffusion framework. Further, a sufficient and necessary condition for the tax identity is discovered.

In Chapter 4, we adopt a perturbation approach to solve the joint Laplace transforms of occupation times for time-homogeneous diffusion processes. The occupation time is the amount of time a stochastic process stays within a certain range. Standard methodologies of occupation times include martingale calculus, excursion theory, and Feynman–Kac formulas. Alternatively, an efficient perturbation approach has been developed significantly during the last five years by many researchers. This approach serves to decompose the problems of occupation times into exit problems. This approach significantly simplifies the calculation and can be transformed to general Markov processes as long as the Laplace transforms of the two-sided exit times are explicit. Our results find applications in two recently proposed bankruptcy problems by discovering an intrinsic connection between the new bankruptcy models and the occupation times.

In Chapter 5, we are interested in quantitatively measuring the liquidation risk of a firm subject to both Chapter 7 and Chapter 11 of the U.S. bankruptcy code. A firm liquidates if the firm value stays below a level over a prespecified grace period or it drops below an even lower level. Hence, the liquidation time is essentially the minimum of a first passage time and a Parisian time. First, we model the firm value

by a time-homogeneous diffusion process in which the drift and volatility are level dependent and can be easily adjusted to reflect the state changes of the firm. An explicit formula for the probability of liquidation is established, based on which we gain a quantitative understanding of how the capital structures before and during bankruptcy affect the probability of liquidation. Then we incorporate jumps into the firm value process and derive an analytic formula as well for the probability of liquidation in terms of some well-known probabilities in risk theory. The result involves the regularity of those probabilities, which is a long-standing theoretical issue in the literature. We propose minimum conditions under which these probabilities are classical solutions of associated integro-differential equations.

In Chapter 6, we define some new hybrid Barrier-Parisian options and derive their pricing formulas. The new options are knocked in or out only if the stock price has continuously or cumulatively stayed in a range for a prespecified length of time or has hit some barrier prior to expiration. More generally, we derive the Laplace transforms of hitting times, Parisian times and inverse occupation times with mixed relations for time-homogeneous diffusion processes using a similar perturbation approach as in Chapter 4. Therefore, the structure of the stopping times in this chapter generalizes that in Chapter 5. Moreover, compared with the differentiability argument in Chapter 5, this perturbation approach is simpler and can be extended for general Markov processes with or without jumps.

### 1.3 Notation and Conventions

#### I. General notation

- $e_\lambda$  an independent exponential random variable with rate  $\lambda > 0$
- $E$  expectation
- $\mathcal{F}$   $\{\mathcal{F}_t, t \geq 0\}$ , the natural filtration generated by a stochastic process
- $\mathbb{N}$  the set of all natural numbers
- $\Omega$  probability space
- $P$  probability measure
- $\mathbb{R}$  the set of all real numbers
- $\emptyset$  empty set
- $W$   $\{W_t, t \geq 0\}$ , a standard Brownian motion

#### II. Conventions

- $E^{x_0}[\cdot] = E[\cdot | X_0 = x_0]$
- $E^{x_0}[\cdot; C] = E^{x_0}[\cdot 1_C]$
- $P^{x_0}\{\cdot\} = P\{\cdot | X_0 = x_0\}$
- $\sum_{i \in \emptyset} = 0$

#### III. Stopping times for a general stochastic process $X = \{X_t, t \geq 0\}$

- first passage times  $T_x^\pm = \inf \{t \geq 0 : X_t \gtrless x\}$
- hitting time  $T_x = \inf \{t \geq 0 : X_t = x\}$
- Parisian times  $\tau_{x\pm}(t) = \inf \{u \geq t : u - l_{x\mp}(u) \geq t\}$ ,
- with  $l_{x\mp}(u) = \sup \{s \leq u : X_s \lesseqgtr x\}$
- Inverse occupation times  $\tilde{\tau}_{x\pm}(t) = \inf \left\{ u \geq t : \int_0^u 1_{\{X_s \gtrless x\}} ds \geq t \right\}$

**IV.** Notation for a time-homogeneous diffusion process  $X = \{X_t, t \geq 0\}$  satisfying  $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ ,  $t \geq 0$

$\mu(\cdot)$           the drift of a diffusion

$\sigma(\cdot)$           the volatility of a diffusion

$G(x)$            $\exp \left\{ - \int^x \frac{2\mu(y)}{\sigma^2(y)} dy \right\}$

$g_\lambda^\pm(\cdot)$       fundamental solutions of  $\frac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) = \lambda g(x)$ ,  $\lambda \geq 0$

$f_\lambda(x, y)$        $g_\lambda^-(x)g_\lambda^+(y) - g_\lambda^-(y)g_\lambda^+(x)$

$f_{1,\lambda}(x, y)$      $\frac{\partial}{\partial x} f_\lambda(x, y)$

$f_{2,\lambda}(x, y)$      $\frac{\partial}{\partial y} f_\lambda(x, y)$

$f_{12,\lambda}(x, y)$     $\frac{\partial^2}{\partial x \partial y} f_\lambda(x, y)$

$\psi_\lambda^\pm(x)$          $\pm \frac{(g_\lambda^\pm)'(x)}{g_\lambda^\pm(x)}$ ,

$H_\lambda(x, dy)$      $\frac{1}{\lambda} \mathbb{P}^x \{X_{e_\lambda} \in dy\}$ , where  $e_\lambda$  is an independent exponential random variable with rate  $\lambda > 0$

## CHAPTER 2 PRELIMINARIES

Consider a general stochastic process  $X = \{X_t, t \geq 0\}$  defined on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$ , where, unless stated otherwise,  $\{\mathcal{F}_t, t \geq 0\}$  denotes the natural filtration of  $X$  defined by  $\mathcal{F}_t = \sigma\{X_s : 0 \leq s \leq t\}$ , and it satisfies  $\cup_{t \geq 0} \mathcal{F}_t \subset \mathcal{F}$ .

Martingale property and Markov property are two of the most important and desirable properties of stochastic processes. A uniform integrable stochastic process  $X = \{X_t, t \geq 0\}$  is said to be a martingale if, for every  $0 \leq s \leq t$ , the relation

$$E[X_t | \mathcal{F}_s] = X_s$$

holds almost surely. A stochastic process  $X = \{X_t, t \geq 0\}$  is said to be a Markov process if, for every Borel set  $A \subset \mathbb{R}$  and every  $0 \leq s \leq t$ , the relation

$$P\{X_t \in A | \mathcal{F}_s\} = P\{X_t \in A | X_s\} := P(A, t, X_s, s)$$

holds almost surely. Here  $P(A, t, x, s)$  is called the transition probability of  $X$ . In particular,  $X$  is said to be a time-homogeneous Markov process if  $P(A, t, x, s) = P(A, t - s, x, 0)$  for every  $0 \leq s \leq t$ .

The rest of this chapter is arranged as follows. In Section 2.1, first passage times and some look-back stopping times will be defined. In Sections 2.2 and 2.3, two important classes of Markov processes, diffusion processes and Lévy processes, are introduced briefly.



## 2.1 Stopping Times

A random variable  $\tau : \Omega \rightarrow [0, \infty)$  is called a stopping time if  $\{\tau \leq t\} \in \mathcal{F}_t$  for all  $t \geq 0$ .

First passage times or first hitting times are the most fundamental stopping times. For a level  $x \in \mathbb{R}$ , denote by  $T_x^+$  and  $T_x^-$  the first times when the process  $X$  up-crosses and down-crosses the level  $x$ , respectively; that is,

$$T_x^\pm = \inf \{t \geq 0 : X_t \begin{matrix} \geq \\ \leq \end{matrix} x\}.$$

In particular, when the paths of  $X$  are almost surely continuous, we define the first hitting time

$$T_x = \inf \{t \geq 0 : X_t = x\}. \quad (2.1)$$

It is clear that  $T_x = T_x^+ \vee T_x^-$  in this case. Apparently, in order to decide whether or not  $\{T_x^\pm = t\}$ , one only needs to look forward to check the path of  $X_s$  from  $s = 0$  to  $s = t$ .

However, many stopping times, which we call look-back stopping times, are different from first passage times. Let us look at three kinds of concrete examples of look-back stopping times that will be studied in this thesis. The first kind is called drawdown and drawup. For  $x \geq 0$ , denote by  $DD_x$  (respectively,  $DU_x$ ) the first time when the decline of  $X$  from its running maximum (respectively, the rise of  $X$  from its running minimum) exceeds size  $x$ ; that is,

$$DD_x = \inf \{t \geq 0 : M_t - X_t \geq x\} \quad \text{and} \quad DU_x = \inf \{t \geq 0 : X_t - m_t \geq x\},$$

where the running maximum  $M$  and the running minimum  $m$  of  $X$  are defined by

$M_t = \sup_{0 \leq s \leq t} X_s$  and  $m_t = \inf_{0 \leq s \leq t} X_s$  for  $t \geq 0$ , respectively. In order to determine the drawdown or drawup, one needs to look back at the sample path of  $X$  for its running maxima or minima.

The second kind is called Parisian time, which is named after Parisian option. For an interval  $I \subset \mathbb{R}$  and  $t \geq 0$ , denote by  $\tau_I(t)$  the first time when the process  $X$  has continuously stayed in the interval  $I$  over  $t$  units of time; that is

$$\tau_I(t) = \inf \{u \geq t : u - l_I(u) \geq t\} \quad \text{with} \quad l_I(u) = \sup \{s \leq u : X_s \notin I\}.$$

In order to determine the Parisian time, one needs to look back at the sample path for the last exit time from the interval  $I$ .

The third kind is called inverse occupation time. For a Borel set  $A \subset \mathbb{R}$  and  $t \geq 0$ , denote by  $\tilde{\tau}_A(t)$  the first time when the process  $X$  has cumulatively stayed in the set  $A$  over  $t$  units of time; that is

$$\tilde{\tau}_A(t) = \inf \left\{ u \geq t : \int_0^u 1_{\{X_s \in A\}} ds \geq t \right\},$$

where  $\int_0^u 1_{\{X_s \in A\}} ds$  is called the occupation time of  $X$  in the set  $A$  up to time  $u$ . Note that  $\int_0^u 1_{\{X_s \in A\}} ds$  is not a stopping time. However, it is clear that  $\tilde{\tau}_A(t)$  is a stopping time, which is actually the generalized inverse of the occupation time  $\int_0^t 1_{\{X_s \in A\}} ds$ . In order to determine the inverse occupation time, one needs to look back at the sample path of  $X$  for the cumulative time that  $X$  has stayed within the set  $A$ .

Roughly speaking, a stopping time  $\tau$  is said to be a look-back stopping time if one has to look back when checking the sample path of  $\{X_s, 0 \leq s \leq t\}$  to decide whether or not  $\{\tau = t\}$  occurs.

## 2.2 Diffusion Processes

### 2.2.1 General diffusion processes

**Definition 2.1.** A Markov process  $X = \{X_t, t \geq 0\}$  is said to be a diffusion if the following properties of its transition probability  $P(A, t, x, s)$  hold:

1) for any  $\varepsilon, t > 0$  and  $x \in \mathbb{R}$ ,

$$\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_{|x-y| > \varepsilon} P(dy, t + \delta, x, t) = 0;$$

2) there exist functions  $\mu(x, t)$  and  $\sigma(x, t)$  such that, for all  $\varepsilon, t > 0$  and  $x \in \mathbb{R}$ ,

$$\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_{|x-y| \leq \varepsilon} (y - x) P(dy, t + \delta, x, t) = \mu(x, t),$$

$$\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_{|x-y| \leq \varepsilon} (y - x)^2 P(dy, t + \delta, x, t) = \sigma(x, t).$$

It is usually more convenient to study diffusion processes through the following stochastic differential equation (SDE):

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \quad (2.2)$$

where  $W = \{W_t, t \geq 0\}$  is a standard Brownian motion defined on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$  and the functions  $\mu(x, t)$  and  $\sigma(x, t)$  are measurable for  $x \in \mathbb{R}$  and  $t \geq 0$ .

**Definition 2.2.** A stochastic process  $X = \{X_t, t \geq 0\}$  is said to be a strong solution of (2.2) with initial value  $x_0$  if the following hold:

1) the paths of  $X$  are almost surely continuous;

2)  $P\{X_0 = x_0\} = 1$ ;

3)  $P \left\{ \int_0^t (|\mu(X_s, s)| + \sigma^2(X_s, s)) ds < \infty \right\} = 1$  for every  $t \geq 0$ ;

4)  $X_t = x_0 + \int_0^t \mu(X_s, s) ds + \int_0^t \sigma(X_s, s) dW_s$  almost surely for every  $t > 0$ .

The next theorem (Theorem 2.9 of Karatzas and Shreve (1991) on Page 289) gives sufficient conditions for the existence of a strong solution of SDE (2.2).

**Theorem 2.3.** *Suppose that the coefficients  $\mu(x, t)$  and  $\sigma(x, t)$  satisfy the following:*

1) *there exists a constant  $K$  such that for  $t \geq 0$  and  $x, y \in \mathbb{R}$*

$$|\mu(x, t) - \mu(y, t)| + |\sigma(x, t) - \sigma(y, t)| \leq K |x - y|, \quad (2.3)$$

$$\mu^2(x, t) + \sigma^2(x, t) \leq K^2 (1 + x^2); \quad (2.4)$$

2)  $X_0$  *does not depend on  $W_t$  and  $EX_0^2 < \infty$ .*

*Then there exists a strong solution of (2.2) satisfying  $\sup_{0 \leq t \leq T} EX_t^2 < \infty$  for every  $T > 0$ .*

A strong solution of (2.2) may or may not exist if one of (2.3) and (2.4) fails; see, e.g., Tanaka's SDE  $dX_t = \text{sign}(X_t)dW_t$ ,  $t \geq 0$ , where  $\text{sign}(\cdot)$  is the sign function, and SDE (4.21) of Brownian motion with two-valued drift.

The next theorem (Theorem 1 of Gihman and Skorohod (1972) on Page 67) confirms that the solution of SDE (2.2) is a Markov process and gives its transition probability.

**Theorem 2.4.** *Assume that  $X$  is a strong solution of (2.2). Then  $X$  is a Markov process with transition probability defined by*

$$P(A, t, x, s) = P \{X_t^{x, s} \in A\}$$

for every Borel set  $A \subset \mathbb{R}$  and every  $0 \leq s \leq t$ . Here  $X_t^{x,s} = x + \int_s^t \mu(X_u, u)du + \int_s^t \sigma(X_u, u)dW_u$ .

### 2.2.2 Time-homogeneous diffusion processes

The case of time-independent coefficients in SDE (2.2) corresponds to the so-called time-homogeneous diffusion processes

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t. \quad (2.5)$$

In Chapters 3–6, we will frequently model the firm value processes or the prices of the underlying assets by (2.5). Different from time-inhomogeneous diffusion processes (2.2), time-homogeneous diffusion processes (2.5) permit explicit formulas for many important quantities such as the exit probabilities. This major advantage enables us to conduct further derivations and analysis.

Suppose that  $\mu(x)$  and  $\sigma(x)$  satisfy (2.3) and (2.4) with the argument  $t$  ignored. By Darling and Siegert (1953), we know that the ODE

$$\frac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) = \lambda g(x), \quad \lambda > 0, \quad (2.6)$$

admits two independent, positive and convex fundamental solutions, denoted by  $g_\lambda^\pm(x)$ , where  $g_\lambda^+(x)$  is strictly increasing and  $g_\lambda^-(x)$  is strictly decreasing. Note that the solutions  $g_\lambda^\pm(x)$  play a significant role in subsequent analysis of diffusion processes (2.5). For many specific diffusions of interest, the solutions  $g_\lambda^\pm(x)$  yield explicit expressions. We list a few in the following example and we refer the reader to Borodin and Salminen (2002) for a more complete collection of such results.

**Example 2.5.** 1) For a Brownian motion with drift satisfying  $dX_t = \mu dt + \sigma dW_t$  with  $\mu \in \mathbb{R}$  and  $\sigma > 0$ , we have

$$g_\lambda^\pm(x) = \exp \left\{ \frac{-\mu \pm \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} x \right\}.$$

2) For an Ornstein-Uhlenbeck process satisfying  $dX_t = -\mu X_t dt + \sigma dW_t$  with  $\mu, \sigma > 0$ , we have

$$g_\lambda^\pm(x) = e^{\frac{\mu x^2}{2\sigma^2}} D_{-\lambda/\mu} \left( \pm \sqrt{2\mu} x / \sigma \right),$$

where  $D_{-v}(x)$  is the parabolic cylinder function given by

$$D_{-v}(x) = e^{-x^2/4} 2^{-v/2} \sqrt{\pi} \left\{ \frac{1}{\Gamma((v+1)/2)} \left( 1 + \sum_{k=1}^{\infty} \frac{v(v+2)\cdots(v+2k-2)}{(2k)!} x^{2k} \right) - \frac{\sqrt{2}x}{\Gamma(v/2)} \left( 1 + \sum_{k=1}^{\infty} \frac{(v+1)(v+3)\cdots(v+2k-1)}{(2k+1)!} x^{2k} \right) \right\}.$$

3) For a squared Bessel process satisfying  $dX_t = (2\mu + 2)dt + 2\sqrt{X_t}dW_t$  with  $\mu \in \mathbb{R}$ , we have

$$g_\lambda^-(x) = x^{-\mu/2} I_\mu(\sqrt{2\lambda x}) \quad \text{and} \quad g_\lambda^+(x) = x^{-\mu/2} K_\mu(\sqrt{2\lambda x}),$$

where  $I_\mu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\mu+2k}}{k! \Gamma(\mu+k+1)}$  and  $K_\mu(x) = \frac{\pi}{2 \sin(\mu\pi)} (I_{-\mu}(x) - I_\mu(x))$  are called modified Bessel functions.

Following Darling and Siegert (1953), we are going to introduce several auxiliary functions, which are used in expressing the Laplace transforms associated with the two-sided exit problem. For  $\lambda > 0$  define a function

$$f_\lambda(x, y) = g_\lambda^-(x) g_\lambda^+(y) - g_\lambda^-(y) g_\lambda^+(x). \quad (2.7)$$

Note that the function  $f_\lambda(x, y)$  is strictly decreasing in  $x$  and strictly increasing in  $y$  so that  $f_\lambda(x, y) = 0$  if and only if  $x = y$ . For ease of notation, write

$$f_{1,\lambda}(x, y) = \frac{\partial}{\partial x} f_\lambda(x, y), \quad f_{2,\lambda}(x, y) = \frac{\partial}{\partial y} f_\lambda(x, y) \quad \text{and} \quad f_{12,\lambda}(x, y) = \frac{\partial^2}{\partial x \partial y} f_\lambda(x, y).$$

Denote by  $T_x$  the first hitting time of  $X$  at level  $x$  as in (2.1).

**Theorem 2.6.** *For  $a < x < b$  and  $\lambda > 0$ ,*

$$\mathbb{E}^x [e^{-\lambda T_a}; T_a < T_b] = \frac{f_\lambda(x, b)}{f_\lambda(a, b)} \quad \text{and} \quad \mathbb{E}^x [e^{-\lambda T_b}; T_b < T_a] = \frac{f_\lambda(a, x)}{f_\lambda(a, b)}. \quad (2.8)$$

Letting  $b \rightarrow \infty$  in the first relation of (2.8) and  $a \rightarrow -\infty$  in the second relation of (2.8), respectively, and applying the asymptotic properties  $\lim_{x \rightarrow -\infty} g_\lambda^-(x) = \lim_{x \rightarrow \infty} g_\lambda^+(x) = \infty$  and  $\lim_{x \rightarrow \infty} g_\lambda^-(x) = \lim_{x \rightarrow -\infty} g_\lambda^+(x) = 0$ , we obtain the Laplace transforms of one-sided exit times

$$\mathbb{E}^x e^{-\lambda T_a} = \frac{g_\lambda^-(x)}{g_\lambda^-(a)} \quad \text{and} \quad \mathbb{E}^x e^{-\lambda T_b} = \frac{g_\lambda^+(x)}{g_\lambda^+(b)}. \quad (2.9)$$

From (2.6), when  $\lambda = 0$ , we can choose

$$\begin{cases} g_0^-(x) \equiv 1, & \text{if } \int_x^\infty G(y) dy = \infty, \\ g_0^-(x) = \int_x^\infty G(y) dy, & \text{otherwise,} \end{cases} \quad (2.10)$$

and

$$\begin{cases} g_0^+(x) \equiv 1, & \text{if } \int_{-\infty}^x G(y) dy = \infty, \\ g_0^+(x) = \int_{-\infty}^x G(y) dy, & \text{otherwise,} \end{cases} \quad (2.11)$$

where  $G(x)$  is called the scale function, given by

$$G(x) = \exp \left\{ - \int^x \frac{2\mu(y)}{\sigma^2(y)} dy \right\}, \quad x \in \mathbb{R}. \quad (2.12)$$

In the integral above, the lower bound can be specified to any value in the range of  $X$ . Relying on (2.10) and (2.11), letting  $\lambda \rightarrow 0+$  in (2.8), we obtain the two-sided exit probabilities

$$\mathbb{P}^x\{T_a < T_b\} = \frac{\int_x^b G(y)dy}{\int_a^b G(y)dy} \quad \text{and} \quad \mathbb{P}^x\{T_b < T_a\} = \frac{\int_a^x G(y)dy}{\int_a^b G(y)dy}. \quad (2.13)$$

In particular, letting  $b \rightarrow \infty$  in the first relation of (2.13) and  $a \rightarrow -\infty$  in the second relation of (2.13), or letting  $\lambda \rightarrow 0+$  in (2.9), we obtain the one-sided exit probabilities

$$\mathbb{P}^x\{T_a < \infty\} = \frac{\int_x^\infty G(y)dy}{\int_a^\infty G(y)dy} \quad \text{and} \quad \mathbb{P}^x\{T_b < \infty\} = \frac{\int_{-\infty}^x G(y)dy}{\int_{-\infty}^b G(y)dy}. \quad (2.14)$$

Define a pair of exponents

$$\psi_\lambda^\pm(x) = \pm \frac{(g_\lambda^\pm)'(x)}{g_\lambda^\pm(x)}, \quad \lambda > 0. \quad (2.15)$$

By (2.10) and (2.11), we have

$$\psi_0^-(x) = \frac{G(x)}{\int_x^\infty G(y)dy}, \quad \psi_0^+(x) = \frac{G(x)}{\int_{-\infty}^x G(y)dy} \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \psi_\lambda^\pm(x) = \infty,$$

where the last relation is due to equation (12) of Pitman and Yor (2003).

By Itô and McKean (1974), we denote the Laplace transform of the transition density by

$$\begin{aligned} H_\lambda(x, dy) &= \frac{1}{\lambda} \mathbb{P}^x \{X_{e_\lambda} \in dy\} \\ &= \begin{cases} \frac{g_\lambda^+(x)g_\lambda^-(y)}{f_{2,\lambda}(x,x)} \frac{2}{\sigma^2(y)} \exp \left\{ \int_x^y \frac{2\mu(z)}{\sigma^2(z)} dz \right\} dy, & \text{if } x \leq y, \\ \frac{g_\lambda^-(x)g_\lambda^+(y)}{f_{2,\lambda}(x,x)} \frac{2}{\sigma^2(y)} \exp \left\{ \int_x^y \frac{2\mu(z)}{\sigma^2(z)} dz \right\} dy, & \text{if } x > y, \end{cases} \end{aligned} \quad (2.16)$$

where  $e_\lambda$  is an independent exponential random variable with rate  $\lambda > 0$ .

Finally, we collect some identities to be used later. They can be easily verified by the definitions of  $f_\lambda(x, y)$  and  $\psi_\lambda^\pm(x)$ .



**Theorem 2.7.** For  $x, y, z \in \mathbb{R}$  and  $\lambda > 0$ ,

$$\left\{ \begin{array}{l} f_{2,\lambda}(x, x)f_{1,\lambda}(y, y) - f_{2,\lambda}(x, y)f_{1,\lambda}(x, y) = -f_{12,\lambda}(x, y)f_\lambda(x, y), \\ -f_{1,\lambda}(x, y)\psi_\lambda^-(y) - f_{2,\lambda}(x, y)\psi_\lambda^-(x) = f_{12,\lambda}(x, y) + f_\lambda(x, y)\psi_\lambda^-(x)\psi_\lambda^-(y), \\ f_{1,\lambda}(x, y)\psi_\lambda^+(y) + f_{2,\lambda}(x, y)\psi_\lambda^+(x) = f_{12,\lambda}(x, y) + f_\lambda(x, y)\psi_\lambda^+(x)\psi_\lambda^+(y), \\ f_{1,\lambda}(x, y)g_\lambda^+(x) + f_{2,\lambda}(x, x)g_\lambda^+(y) = f_\lambda(x, y)g_\lambda^+(x)\psi_\lambda^+(x), \\ f_\lambda(x, z)g_\lambda^+(y) + f_\lambda(y, x)g_\lambda^+(z) = f_\lambda(y, z)g_\lambda^+(x). \end{array} \right.$$

### 2.3 Lévy Processes

In Chapter 5, we shall model the firm value process by a jump-diffusion process in the Lévy framework. Besides, some concepts of Lévy processes such as scale functions are also mentioned in other places of the thesis. Let us briefly introduce Lévy processes in this section.

**Definition 2.8.** A stochastic process  $X = \{X_t, t \geq 0\}$  is said to be a Lévy process if the following hold:

- 1) the paths of  $X$  are almost surely right continuous with left limits;
- 2)  $\mathbb{P}\{X_0 = 0\} = 1$ ;
- 3) the increment  $X_t - X_s$ ,  $t > s \geq 0$ , is independent of  $\mathcal{F}_s$ ;
- 4) the increment  $X_t - X_s$ ,  $t > s \geq 0$ , is equal in distribution to  $X_{t-s}$ .

The reader is referred to Kyprianou (2006) and Applebaum (2009) for more discussions on Lévy processes. Suppose that  $a \in \mathbb{R}$ ,  $\sigma \geq 0$  and  $\Pi$  is a measure concentrated on  $\mathbb{R} \setminus \{0\}$  such that  $\int_{\mathbb{R}} (1 \wedge x^2) d\Pi(x) < \infty$ . Based on this triple  $(a, \sigma, \Pi)$ ,

for each  $\theta \in \mathbb{R}$  we define

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x 1_{\{|x|<1\}}) d\Pi(x). \quad (2.17)$$

Then, by Theorem 1.6 of Kyprianou (2006), there exists a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$  on which a Lévy process is defined with the characteristic exponent  $\Psi(\cdot)$  given by (2.17) .

In addition to linear Brownian motions and Poisson processes, another important example of Lévy processes is the compound Poisson process defined by

$$X_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0. \quad (2.18)$$

Here  $N = \{N_t, t \geq 0\}$  is a homogeneous Poisson process with rate  $\lambda > 0$ ,  $\{Y_i, i = 1, 2, \dots\}$  is a sequence of independent and identically distributed (i.i.d.) random variables (independent of  $N$ ) with common distribution function  $F(\cdot)$ . Corresponding to Lévy-Khintchine characterization (2.17), we have  $a = -\lambda \int_{0 < |x| < 1} x dF(x)$ ,  $\sigma = 0$  and  $d\Pi(x) = \lambda dF(x)$ .

In risk theory, the compound Poisson process (2.18) has often been used to model the aggregate amount of losses of an insurance company with  $N_t$  representing the number of claims up to time  $t$  and  $\{Y_i, i = 1, 2, \dots\}$  representing a sequence of claim sizes. See, e.g., Embrechts et al. (1997).

In particular, we call  $X$  a spectrally negative Lévy process if  $\Pi(0, \infty) = 0$ . In the rest of this section, we only consider the case of spectrally negative Lévy processes.

Rather than working with the Lévy exponent  $\Psi(\cdot)$ , we work with the Laplace

exponent

$$\psi(\lambda) = \log \mathbb{E}e^{\lambda X_1} = -\Psi(-i\lambda).$$

The function  $\psi(\cdot)$  is zero at zero and tends to infinity at infinity. Further, it is infinitely differentiable and strictly convex. Define the right inverse of  $\psi(\cdot)$  by

$$\phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}, \quad q \geq 0.$$

For each  $q \geq 0$ , there exists a family of scale functions  $W^{(q)} : \mathbb{R} \rightarrow [0, \infty)$  such that  $W^{(q)}(x) = 0$  for  $x < 0$  and  $W^{(q)}$  is characterized on  $[0, \infty)$  as a strictly increasing and continuous function whose Laplace transform satisfies

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\psi(\lambda) - q} \quad \text{for } \lambda > \phi(q). \quad (2.19)$$

Further, we define

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy, \quad x \in \mathbb{R}.$$

The Laplace transforms associated with the two-sided exit problem of spectrally negative Lévy processes are given in the next theorem (Theorem 8.1 of Kyprianou (2006)).

**Theorem 2.9.** *For any  $q \geq 0$  and  $0 \leq x \leq a$ ,*

$$\begin{aligned} \mathbb{E}^x \left[ e^{-qT_a^+}; T_a^+ < T_0^- \right] &= \frac{W^{(q)}(x)}{W^{(q)}(a)}, \\ \mathbb{E}^x \left[ e^{-qT_0^-}; T_0^- < T_a^+ \right] &= Z^{(q)}(x) - Z^{(q)}(a) \frac{W^{(q)}(x)}{W^{(q)}(a)}. \end{aligned}$$

Letting  $a \rightarrow \infty$  in the second relation of Theorem 2.9, we have, for  $q \geq 0$  and  $x \geq 0$ ,

$$\mathbb{E}^x \left[ e^{-qT_0^-}; T_0^- < \infty \right] = Z^{(q)}(x) - \frac{q}{\phi(q)} W^{(q)}(x).$$

Further, letting  $q \rightarrow 0+$  in the relation above, we have the one-sided exit probability

$$P^x \{T_0^- < \infty\} = \begin{cases} 1 - \psi'(0+)W^{(0)}(x), & \text{if } \psi'(0+) > 0, \\ 1, & \text{if } \psi'(0+) \leq 0. \end{cases}$$

## CHAPTER 3

### TWO-SIDED EXIT TIMES OF TIME-HOMOGENEOUS DIFFUSION PROCESSES WITH TAX

We study the two-sided exit times of a time-homogeneous diffusion process in the presence of tax payments of loss-carry-forward type. Essentially, the exit times of the firm value process in the presence of taxation are transformed to generalized drawdown and drawup of the firm value process in the absence of taxation. By using an approximation approach, we obtain explicit formulas for the Laplace transforms associated with the two-sided exit times. This argument can be easily transformed to general Markov processes with continuous paths. As corollaries, we solve the expected present value of tax payments until default, the two-sided exit probabilities, and the non-default probability. An interesting tax identity has been established by researchers in various situations within the Lévy framework. However, we prove that it does not hold in general within the diffusion framework. Further, a sufficient and necessary condition for the tax identity is discovered. The content of this chapter is mainly based on the paper Li et al. (2013a).

### 3.1 Introduction

Recently, bankruptcy problems with tax have become an appealing research topic. Albrecher and Hipp (2007) first introduced tax payments at a constant rate at profitable times to the compound Poisson risk model and established a charming tax identity for the survival probability. Later on, Albrecher et al. (2009) found a

simple proof using downward excursions and extended the study to a value-dependent tax rate. Further extensions to the Lévy framework were done by Albrecher et al. (2008), Kyprianou and Zhou (2009), Renaud (2009), and Wei (2009), among others. See also Hao and Tang (2009) for the study in the Lévy framework but under periodic taxation. So far there is little study beyond the Lévy framework with difficulty mainly in the two-sided exit problem.

Following this new trend, we study the two-sided exit problem of a time-homogeneous diffusion process with such tax payments. Suppose that the value of a firm in the absence of taxation is modeled by a time-homogeneous diffusion process  $X = \{X_t, t \geq 0\}$ , defined on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$ , with dynamics

$$dX_t = \mu(X_t) dt + \sigma(X_t) dW_t, \quad t \geq 0, \quad (3.1)$$

where  $X_0 = x_0$  is the initial value. The reader is referred to Section 2.2.2 for a brief introduction of time-homogeneous diffusion processes.

Next, we introduce a value-dependent tax rate to the time-homogeneous diffusion model (3.1). More precisely, whenever the process  $X$  coincides with its running maximum  $M^X$  defined by  $M_t^X = \sup_{0 \leq s \leq t} X_s$ ,  $t \geq 0$ , the firm pays tax at rate  $\gamma(M_t^X)$ , where  $\gamma(\cdot) : [x_0, \infty) \rightarrow [0, 1)$  is a measurable function. This is the so-called loss-carry-forward taxation. It is easy to understand that the value process in the presence of taxation satisfies

$$dU_t = dX_t - \gamma(M_t^X) dM_t^X, \quad t \geq 0, \quad (3.2)$$

with  $U_0 = X_0 = x_0$ .

We study the two-sided exit problem of the value process  $U$ . Throughout the chapter, let

$$a < x_0 < b. \tag{3.3}$$

The lower bound  $a$  represents the default threshold, so the firm defaults whenever its value is below  $a$ . In particular, the threshold  $a$  is set to 0 in ruin theory. For a real number  $x$ , introduce the first hitting times of  $X$  and  $U$  as, respectively,

$$T_x^X = \inf \{t \geq 0 : X_t = x\} \quad \text{and} \quad T_x^U = \inf \{t \geq 0 : U_t = x\}.$$

In particular,  $T_a^U$  stands for the time of default in the presence of taxation. Our main goal is to solve the Laplace transforms associated with the two-sided exit problem:

$$E^{x_0} \left[ e^{-\lambda T_b^U}; T_b^U < T_a^U \right] \quad \text{and} \quad E^{x_0} \left[ e^{-\lambda T_a^U}; T_a^U < T_b^U \right].$$

Our idea of the proof of the main result stems from the work of Lehoczky (1977). As corollaries, we study the expected present value of tax payments until default and the two-sided exit probabilities. In particular, we examine the tax identity and prove that it does not hold in general within the current diffusion framework. Furthermore, we discover a sufficient and necessary condition for this tax identity.

The rest of the chapter is arranged as follows. Our main result and its corollaries are presented in Section 3.2. Their proofs are postponed to Section 3.3. A short summary and some remarks are given in Section 3.4.

### 3.2 Main Results

Recall the initial value  $x_0$ , the lower boundary  $a$  and the upper boundary  $b$  as specified by (3.3). Following Kyprianou and Zhou (2009), we define

$$\bar{\gamma}(x) = x - \int_{x_0}^x \gamma(z) dz = x_0 + \int_{x_0}^x (1 - \gamma(z)) dz, \quad x \geq x_0,$$

which is strictly increasing and continuous in  $x$  with  $\bar{\gamma}(x_0) = x_0$ . Thus, its inverse function  $\bar{\gamma}^{-1}(\cdot)$  is well defined on  $[x_0, \bar{\gamma}(\infty))$ . Note that both  $x - \bar{\gamma}(x)$  and  $\bar{\gamma}^{-1}(x) - x$  are non-decreasing and continuous functions. Trivially,  $\bar{\gamma}(\infty) = \infty$  if we assume

$$\int_{x_0}^{\infty} (1 - \gamma(z)) dz = \infty. \quad (3.4)$$

As before, denote by  $M_t^U = \sup_{0 \leq s \leq t} U_s$ ,  $t \geq 0$  the running maximum of  $U$ . In terms of the function  $\bar{\gamma}(\cdot)$ , we can rewrite the process  $U$  in (3.2) as

$$U_t = X_t - M_t^X + \bar{\gamma}(M_t^X), \quad t \geq 0. \quad (3.5)$$

As shown in Lemma 2.1 of Kyprianou and Zhou (2009), we have

$$M_t^U = M_t^X - \int_0^t \gamma(M_s^X) dM_s^X = \bar{\gamma}(M_t^X), \quad t \geq 0, \quad (3.6)$$

and, hence,  $T_x^U = T_{\bar{\gamma}^{-1}(x)}^X$  for  $x \geq x_0$ .

Our main result is the following:

**Theorem 3.1.** *For  $a < x_0 < b$  and  $\lambda > 0$ , we have*

$$\mathbb{E}^{x_0} \left[ e^{-\lambda T_b^U}; T_b^U < T_a^U \right] = \exp \left\{ - \int_{x_0}^{\bar{\gamma}^{-1}(b)} \frac{f_{2,\lambda}(x - \bar{\gamma}(x) + a, x)}{f_\lambda(x - \bar{\gamma}(x) + a, x)} dx \right\} \quad (3.7)$$

and

$$\mathbb{E}^{x_0} \left[ e^{-\lambda T_a^U}; T_a^U < T_b^U \right] = \int_{x_0}^{\bar{\gamma}^{-1}(b)} e^{-\int_{x_0}^y \frac{f_{2,\lambda}(x - \bar{\gamma}(x) + a, x)}{f_\lambda(x - \bar{\gamma}(x) + a, x)} dx} \frac{f_{2,\lambda}(y, y)}{f_\lambda(y - \bar{\gamma}(y) + a, y)} dy. \quad (3.8)$$



The functions  $f_\lambda(x, y)$  is defined in (2.7) and  $f_{2,\lambda}(x, y) = \frac{\partial}{\partial y} f_\lambda(x, y)$ . The complete proof of Theorem 3.1 is deferred to Section 3.3.

One can check that Theorem 3.1 agrees with (2.8) in the case of no taxation, namely,  $\bar{\gamma}(x) \equiv x$ . To check that (3.8) reduces to the first relation of (2.8) when  $\bar{\gamma}(x) \equiv x$ , one uses the identity that, for all  $a < x_0 < b$  and  $\lambda > 0$ ,

$$f_\lambda(a, x_0) f_{2,\lambda}(b, b) = f_{2,\lambda}(x_0, b) f_\lambda(a, b) - f_\lambda(x_0, b) f_{2,\lambda}(a, b),$$

which can be verified by (2.7). In addition, by (3.5) we have

$$T_a^U = \inf \{t \geq 0 : U_t \leq a\} = \inf \{t \geq 0 : M_t^X - X_t \geq \bar{\gamma}(M_t^X) - a\}. \quad (3.9)$$

Therefore, under (3.4), our relation (3.8) with  $b = \infty$  agrees with relation (21) of Lehoczky (1977) with the function  $u(\cdot) = \bar{\gamma}(\cdot) - a$  and  $\alpha = 0$ .

In the example below, we show that, restricted to a Brownian motion and  $a = 0$ , our relation (3.7) coincides with relation (1.5) of Kyprianou and Zhou (2009):

**Example 3.2.** Let  $X_t = \mu t + \sigma W_t$  be a Brownian motion with positive drift  $\mu$  and write  $\mu_\lambda = \sqrt{\mu^2 + 2\lambda}$  for  $\lambda > 0$ . We have

$$g_\lambda^\pm(x) = \exp \left\{ \frac{-\mu \pm \mu_\lambda}{\sigma^2} x \right\}.$$

Then it follows that

$$\frac{f_{2,\lambda}(y, z)}{f_\lambda(y, z)} = \frac{\frac{-\mu + \mu_\lambda}{\sigma^2} \exp \left\{ \frac{\mu_\lambda}{\sigma^2} (z - y) \right\} + \frac{\mu + \mu_\lambda}{\sigma^2} \exp \left\{ -\frac{\mu_\lambda}{\sigma^2} (z - y) \right\}}{\exp \left\{ \frac{\mu_\lambda}{\sigma^2} (z - y) \right\} - \exp \left\{ -\frac{\mu_\lambda}{\sigma^2} (z - y) \right\}}.$$

On the other hand, by inverting the Laplace transform (2.19), the scale function of  $X$  as a spectrally negative Lévy process is

$$W^{(\lambda)}(x) = \frac{\sigma^2}{\mu_\lambda} \left( \exp \left\{ \frac{-\mu + \mu_\lambda}{\sigma^2} x \right\} - \exp \left\{ \frac{-\mu - \mu_\lambda}{\sigma^2} x \right\} \right);$$

It follows that

$$\frac{f_{2,\lambda}(y, z)}{f_\lambda(y, z)} = \frac{W^{(\lambda)'}(y - z)}{W^{(\lambda)}(y - z)}.$$

Then by change of variables, one easily checks that our relation (3.7) with  $a = 0$  agrees with relation (1.5) of Kyprianou and Zhou (2009).

As an application of Theorem 3.1, we derive a formula for the expected present value of tax payments until default. Parallel works in the Lévy framework include Theorem 3.2 of Albrecher et al. (2008), Theorem 1.2 of Kyprianou and Zhou (2009) and Theorem 3.1 of Renaud (2009).

**Corollary 3.3.** *Under (3.4), we have*

$$\mathbb{E}^{x_0} \left[ \int_0^{T_a^U} e^{-\lambda t} \gamma(M_t^X) dM_t^X \right] = \int_{x_0}^{\infty} \gamma(y) \exp \left\{ - \int_{x_0}^y \frac{f_{2,\lambda}(x - \bar{\gamma}(x) + a, x)}{f_\lambda(x - \bar{\gamma}(x) + a, x)} dx \right\} dy.$$

The proof of Corollary 3.3 is deferred to Section 3.3.

Letting  $\lambda \rightarrow 0$  in Theorem 3.1 and using (2.13), we obtain:

**Corollary 3.4.** *It holds that*

$$\mathbb{P}^{x_0} \{T_b^U < T_a^U\} = \exp \left\{ - \int_{x_0}^{\bar{\gamma}^{-1}(b)} \frac{G(x)}{\int_{x - \bar{\gamma}(x) + a}^x G(y) dy} dx \right\} \quad (3.10)$$

and that  $\mathbb{P}^{x_0} \{T_a^U < T_b^U\} = 1 - \mathbb{P}^{x_0} \{T_b^U < T_a^U\}$ .

Here, the scale function  $G(x) = \exp \left\{ - \int^x \frac{2\mu(y)}{\sigma^2(y)} dy \right\}$  is defined in (2.12). A separate proof of Corollary 3.4 can be given by going along the same lines of the proof of Theorem 3.1 with  $\lambda = 0$ . Clearly, in the absence of taxation, namely,  $\bar{\gamma}(x) \equiv x$ , relation (3.10) agrees with the second relation in (2.13). Moreover, we point out that

relation (3.10) is a special case of relation (20) of Lehoczky (1977) with  $u(\cdot) = \bar{\gamma}(\cdot) - a$ .

This is due to the observation that, by (3.6),

$$\mathbb{P}^{x_0} \{T_b^U < T_a^U\} = \mathbb{P}^{x_0} \{M_{T_a^U}^U \geq b\} = \mathbb{P}^{x_0} \{M_{T_a^U}^X \geq \bar{\gamma}^{-1}(b)\}$$

and relation (3.9).

Letting  $b \rightarrow \infty$  in (3.10) yields the non-default probability of  $U$  as follows:

**Corollary 3.5.** *Under (3.4), it holds that*

$$\mathbb{P}^{x_0} \{T_a^U = \infty\} = \exp \left\{ - \int_{x_0}^{\infty} \frac{G(x)}{\int_{x-\bar{\gamma}(x)+a}^x G(y) dy} dx \right\}. \quad (3.11)$$

Tax payments increase default risk, of course, which can be observed by comparing (3.11) with the second relation of (2.14). Thus, relation (3.11) provides us with a quantitative understanding of the impact of tax payments on default risk. In particular, the following example shows that the standard Black-Scholes model without tax has a positive probability to survive forever while any constant tax rate, no matter how small it is, will drive the firm to default eventually:

**Example 3.6.** *Consider the geometric Brownian motion*

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad t \geq 0,$$

where  $X_0 = x_0 > 0$  is the initial value while  $\mu$  and  $\sigma$  are positive constants satisfying  $\rho = 2\mu/\sigma^2 > 1$ . In addition, we assume the default threshold is  $a > 0$ . Then by the second relation of (2.14) with  $G(y) = (a/y)^\rho$  for  $y \geq a$ , the non-default probability without tax is

$$\mathbb{P}^{x_0} \{T_a^X = \infty\} = 1 - \left( \frac{a}{x_0} \right)^{\rho-1} > 0.$$

However, in the presence of a constant tax rate  $0 < \gamma < 1$ , by (3.11) we have

$$\mathbb{P}^{x_0} \{T_a^U = \infty\} = \exp \left\{ - \int_{x_0}^{\infty} \frac{x^{-\rho}}{\int_{\gamma x - \gamma x_0 + a}^x y^{-\rho} dy} dx \right\} = 0.$$

In order to compare our Corollary 3.5 with Theorem 1.1 of Kyprianou and Zhou (2009), we change the variable  $x = \bar{\gamma}^{-1}(s)$  to rewrite relation (3.10). In particular, if  $\gamma(\cdot) \equiv \gamma \in [0, 1)$  is constant, then  $\bar{\gamma}(x) = x - \gamma x + \gamma x_0$  and relation (3.10) reduces to

$$\mathbb{P}^{x_0} \{T_b^U < T_a^U\} = \exp \left\{ - \int_{x_0}^b \frac{G\left(\frac{s - \gamma x_0}{1 - \gamma}\right)}{\int_{\frac{\gamma s - \gamma x_0}{1 - \gamma} + a}^{\frac{s - \gamma x_0}{1 - \gamma}} G(y) dy} ds \right\}^{\frac{1}{1 - \gamma}}. \quad (3.12)$$

As mentioned in Section 3.1, for the case of a constant tax rate  $\gamma$ , the tax identity

$$\mathbb{P}^{x_0} \{T_0^U = \infty\} = \left( \mathbb{P}^{x_0} \{T_0^X = \infty\} \right)^{\frac{1}{1 - \gamma}} \quad (3.13)$$

has been established by researchers in various situations within the Lévy framework. However, relation (3.12) indicates that such an identity does not hold in general within the diffusion framework.

Slightly more generally, we now consider under what condition the identity

$$\mathbb{P}^{x_0} \{T_b^U < T_a^U\} = \left( \mathbb{P}^{x_0} \{T_b^X < T_a^X\} \right)^{\frac{1}{1 - \gamma}} \quad (3.14)$$

holds. Interestingly, the answer is that  $\mu(\cdot)/\sigma^2(\cdot)$  has to be constant.

**Corollary 3.7.** *Consider constant tax rates.*

(1) *For arbitrarily fixed  $x_0$  and  $a$  with  $a < x_0$ , relation (3.14) holds for all  $b > x_0$*

*and  $0 \leq \gamma < 1$  if and only if  $\mu(x)/\sigma^2(x)$  is constant for  $x \geq a$ .*

(2) For arbitrarily fixed  $a$  and  $b$  with  $a < b$ , relation (3.14) holds for all  $a < x_0 < b$  and  $0 \leq \gamma < 1$  if and only if  $\mu(x)/\sigma^2(x)$  is constant for  $a \leq x \leq b$ .

The proof of Corollary 3.7 is deferred to Section 3.3. By letting  $b \rightarrow \infty$  and  $a = 0$  in part (2) of Corollary 3.7 and going along the same lines of its proof, we obtain the following:

**Corollary 3.8.** Consider constant tax rates and assume (3.4) and  $\int^\infty G(y)dy < \infty$ . Then relation (3.13) holds for all  $0 < x_0 < \infty$  and  $0 \leq \gamma < 1$  if and only if  $\mu(x)/\sigma^2(x)$  is constant for  $x \geq 0$ .

The condition  $\int^\infty G(y)dy < \infty$  in Corollary 3.8 is necessary; otherwise, the probability  $P\{T_0^X = \infty\}$  is equal to 0 and relation (3.13) becomes trivial. Note that the square-root process with dynamics

$$dX_t = \mu X_t dt + \sigma \sqrt{X_t} dW_t, \quad t \geq 0,$$

which is widely used in finance, satisfies the condition that  $\mu(\cdot)/\sigma^2(\cdot)$  is constant in Corollaries 3.7 and 3.8.

Corollaries 3.7 and 3.8 confirm that some intrinsic properties of Brownian motions can often be inherited by time-homogenous diffusion processes with constant  $\mu(\cdot)/\sigma^2(\cdot)$ . A similar implication can be found in Lehoczky (1977). Relation (5) therein gives the distribution of the running maximum of a time-homogenous diffusion process at the first time it falls a specified amount below its current maximum. Lehoczky (1977) observed that if  $\mu(\cdot)/\sigma^2(\cdot)$  is constant then this result agrees with that for a Brownian motion.

### 3.3 Proofs

Clearly, in order for  $U$  to hit  $b$  before  $a$ , for every  $s \in [x_0, b)$ , after  $T_s^U$  the process  $U$  must enter  $(s, \infty)$  before it hits  $a$ . By relations (3.5) and (3.6), this fact can be restated in terms of  $X$  as follows. After  $T_{\bar{\gamma}^{-1}(s)}^X$ , the process  $X$  must enter  $(\bar{\gamma}^{-1}(s), \infty)$  before it hits  $\bar{\gamma}^{-1}(s) - s + a$ . Thus, the event  $(T_b^U < T_a^U)$  necessitates a two-sided exit problem of  $X$  for every  $s \in [x_0, b)$ . Based on this intuition, we establish lower and upper discrete approximations for the event  $(T_b^U < T_a^U)$  in the following:

**Lemma 3.9.** *Let  $x_0 = s_0 < s_1 < \dots < s_n = b$  form a partition of the interval  $[x_0, b]$ ,  $n \in \mathbb{N}$ . Then, almost surely,*

$$\bigcap_{i=1}^n A_i \subset (T_b^U < T_a^U) \subset \bigcap_{i=1}^n B_i, \quad (3.15)$$

where each  $A_i$  denotes the event that after  $T_{\bar{\gamma}^{-1}(s_{i-1})}^X$ , the process  $X$  hits  $\bar{\gamma}^{-1}(s_i)$  before  $\bar{\gamma}^{-1}(s_i) - s_i + a$  while each  $B_i$  denotes the event that after  $T_{\bar{\gamma}^{-1}(s_{i-1})}^X$ , the process  $X$  hits  $\bar{\gamma}^{-1}(s_i)$  before  $\bar{\gamma}^{-1}(s_{i-1}) - s_{i-1} + a$ .

*Proof.* To prove the first inclusion in (3.15), assume that the path of  $X$  is continuous such that  $\bigcap_{i=1}^n A_i$  holds. Arbitrarily choose  $t \in [0, T_{\bar{\gamma}^{-1}(b)}^X]$  and suppose that  $t$  falls into the interval  $[T_{\bar{\gamma}^{-1}(s_{i-1})}^X, T_{\bar{\gamma}^{-1}(s_i)}^X]$  for some  $i = 1, \dots, n$ . Then  $M_t^X \leq \bar{\gamma}^{-1}(s_i)$  and, by relation (3.5), the monotonicity of  $s - \bar{\gamma}(s)$  and the description of  $A_i$ , we have

$$U_t = X_t - (M_t^X - \bar{\gamma}(M_t^X)) \geq X_t - (\bar{\gamma}^{-1}(s_i) - \bar{\gamma}(\bar{\gamma}^{-1}(s_i))) > a.$$

In sum,  $U_t > a$  for all  $t \in [0, T_{\bar{\gamma}^{-1}(b)}^X]$ . Hence,  $T_a^U > T_{\bar{\gamma}^{-1}(b)}^X = T_b^U$ .

To prove the second inclusion in (3.15), assume by contradiction that there exists some  $i = 1, \dots, n$  such that after  $T_{\bar{\gamma}^{-1}(s_{i-1})}^X$ , the path of  $X$  hits  $\bar{\gamma}^{-1}(s_{i-1}) - s_{i-1} + a$

before  $\bar{\gamma}^{-1}(s_i)$ . Then at the moment of hitting  $\bar{\gamma}^{-1}(s_{i-1}) - s_{i-1} + a$ , by relation (3.5), the monotonicity of  $s - \bar{\gamma}(s)$  and  $M_t^X \geq \bar{\gamma}^{-1}(s_{i-1})$ , we have

$$U_t = X_t - (M_t^X - \bar{\gamma}(M_t^X)) \leq (\bar{\gamma}^{-1}(s_{i-1}) - s_{i-1} + a) - (\bar{\gamma}^{-1}(s_{i-1}) - \bar{\gamma}(\bar{\gamma}^{-1}(s_{i-1}))) = a,$$

which contradicts to  $T_b^U < T_a^U$ .  $\square$

*Proof of relation (3.7).* Let  $\{s_{n,i}, i = 0, \dots, m_n\}$ ,  $n \in \mathbb{N}$ , constitute a sequence of increasing partitions of the interval  $[x_0, b]$  with  $x_0 = s_{n,0} < s_{n,1} < \dots < s_{n,m_n} = b$  and the maximum length of subintervals  $\Delta_n = \max_{1 \leq i \leq m_n} (s_{n,i} - s_{n,i-1}) \rightarrow 0$  as  $n \rightarrow \infty$ .

By Lemma 3.9, we have

$$\begin{aligned} \mathbb{E}^{x_0} \left[ e^{-\lambda T_b^U}; T_b^U < T_a^U \right] &= \mathbb{E}^{x_0} \left[ \prod_{i=1}^{m_n} e^{-\lambda (T_{s_{n,i}}^U - T_{s_{n,i-1}}^U)}; T_b^U < T_a^U \right] \\ &\leq \mathbb{E}^{x_0} \left[ \prod_{i=1}^{m_n} e^{-\lambda (T_{\bar{\gamma}^{-1}(s_{n,i})}^X - T_{\bar{\gamma}^{-1}(s_{n,i-1})}^X)}; \bigcap_{i=1}^{m_n} B_{n,i} \right] \\ &= E_n, \end{aligned}$$

where each  $B_{n,i}$ , the same as in Lemma 3.9, denotes the event that after  $T_{\bar{\gamma}^{-1}(s_{n,i-1})}^X$ , the process  $X$  hits  $\bar{\gamma}^{-1}(s_{n,i})$  before  $\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a$ . Furthermore, by the strong Markov property of  $X$ ,

$$E_n = \prod_{i=1}^{m_n} \mathbb{E}^{x_0} \left[ e^{-\lambda T_{\bar{\gamma}^{-1}(s_{n,i})}^X}; T_{\bar{\gamma}^{-1}(s_{n,i})}^X < T_{\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a}^X \middle| \mathcal{F}_{T_{\bar{\gamma}^{-1}(s_{n,i-1})}^X} \right].$$

For ease of notation, introduce

$$h(c_1, c_2 | c_0) = 1 - \mathbb{E}^{x_0} \left[ e^{-\lambda T_{c_2}^X}; T_{c_2}^X < T_{c_1}^X \middle| \mathcal{F}_{T_{c_0}^X} \right], \quad c_1 < c_0 < c_2,$$

so that

$$E_n = \exp \left\{ \sum_{i=1}^{m_n} \log \left( 1 - h \left( \bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}) \middle| \bar{\gamma}^{-1}(s_{n,i-1}) \right) \right) \right\}.$$

It can be shown that all the  $h(\cdot, \cdot | \cdot)$  terms in  $E_n$  are uniformly small. In fact, by relation (2.8) and the monotonicity of  $\bar{\gamma}^{-1}(s) - s$ ,

$$\begin{aligned} & h\left(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}) \mid \bar{\gamma}^{-1}(s_{n,i-1})\right) \\ &= \frac{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i})) - f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i-1}))}{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}))} \end{aligned} \quad (3.16)$$

$$\leq K \max_{1 \leq i \leq m_n} (\bar{\gamma}^{-1}(s_{n,i}) - \bar{\gamma}^{-1}(s_{n,i-1})),$$

where the constant  $K$  is defined as

$$K = \frac{\sup_{(y,z) \in D} f_{2,\lambda}(y, z)}{\inf_{(y,z) \in D} f_\lambda(y, z)} < \infty$$

with  $D = \{(y, z) : a \leq y \leq \bar{\gamma}^{-1}(b) - b + a, x_0 \leq z \leq \bar{\gamma}^{-1}(b), z - y \geq x_0 - a\}$ . Notice that, over the closed set  $D$ , the function  $f_\lambda(y, z)$  is strictly positive (hence away from 0) and that the function  $f_{2,\lambda}(y, z)$  is always continuous and strictly positive.

Therefore, by the elementary relation  $\log(1 - h) \sim -h$  as  $h \rightarrow 0$ , it holds for arbitrarily fixed  $0 < \varepsilon < 1$  and all large  $n$  that

$$E_n \leq \exp \left\{ -(1 - \varepsilon) \sum_{i=1}^{m_n} h\left(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}) \mid \bar{\gamma}^{-1}(s_{n,i-1})\right) \right\}.$$

Since for all large  $n$  and  $i = 1, \dots, m_n$ , the numerator of (3.16) is bounded below by

$$(1 - \varepsilon) f_{2,\lambda}\left(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i})\right) (\bar{\gamma}^{-1}(s_{n,i}) - \bar{\gamma}^{-1}(s_{n,i-1})),$$

it follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} E_n &\leq \lim_{n \rightarrow \infty} \exp \left\{ -(1 - \varepsilon)^2 \sum_{i=1}^{m_n} \frac{f_{2,\lambda}(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}))}{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a, \bar{\gamma}^{-1}(s_{n,i}))} \right. \\ &\quad \left. \times (\bar{\gamma}^{-1}(s_{n,i}) - \bar{\gamma}^{-1}(s_{n,i-1})) \right\} \\ &= \exp \left\{ -(1 - \varepsilon)^2 \int_{x_0}^{\bar{\gamma}^{-1}(b)} \frac{f_{2,\lambda}(x - \bar{\gamma}(x) + a, x)}{f_\lambda(x - \bar{\gamma}(x) + a, x)} dx \right\}. \end{aligned} \quad (3.17)$$



The last equality in (3.17) is justified by changing each  $s_{n,i}$  in the second step to  $s_{n,i-1}$ . By the arbitrariness of  $\varepsilon$ , we have

$$\mathbb{E}^{x_0} \left[ e^{-\lambda T_b^U}; T_b^U < T_a^U \right] \leq \limsup_{n \rightarrow \infty} E_n \leq \exp \left\{ - \int_{x_0}^{\bar{\gamma}^{-1}(b)} \frac{f_{2,\lambda}(x - \bar{\gamma}(x) + a, x)}{f_\lambda(x - \bar{\gamma}(x) + a, x)} dx \right\}.$$

The other inequality for the lower bound can be established symmetrically by using the other part of Lemma 3.9.  $\square$

*Proof of relation (3.8).* We employ the same partition of the interval  $[x_0, b]$  as in the proof of relation (3.7). By considering the range of the running maximum of  $U$  before hitting  $a$ , we have

$$\begin{aligned} & \mathbb{E}^{x_0} \left[ e^{-\lambda T_a^U}; T_a^U < T_b^U \right] \\ &= \sum_{i=1}^{m_n} \mathbb{E}^{x_0} \left[ e^{-\lambda T_a^U}; M_{T_a^U}^U \in [s_{n,i-1}, s_{n,i}] \right] \\ &= \sum_{i=1}^{m_n} \mathbb{E}^{x_0} \left[ e^{-\lambda(T_{s_{n,i-1}}^U + T_a^U - T_{s_{n,i-1}}^U)}; T_{s_{n,i-1}}^U < T_a^U < T_{s_{n,i}}^U \right] \\ &= \sum_{i=1}^{m_n} \mathbb{E}^{x_0} \left[ e^{-\lambda T_{s_{n,i-1}}^U} \mathbb{E}^{x_0} \left[ e^{-\lambda T_a^U}; T_a^U < T_{s_{n,i}}^U \mid \mathcal{F}_{T_{s_{n,i-1}}^U} \right]; T_{s_{n,i-1}}^U < T_a^U \right], \end{aligned}$$

where the last step is due to the fact that  $\mathcal{F}_{T_{s_{n,i-1}}^U} = \mathcal{F}_{T_{\bar{\gamma}^{-1}(s_{n,i-1})}^X}$  and the strong Markov property of  $X$ . Clearly, after  $T_{s_{n,i-1}}^U$ , if the process  $U$  hits  $a$  before  $s_{n,i}$ , then the process  $X$  hits  $\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a$  before  $\bar{\gamma}^{-1}(s_{n,i})$  because, otherwise, for  $t \in [T_{s_{n,i-1}}^U, T_{s_{n,i}}^U]$ , by the monotonicity of  $s - \bar{\gamma}(s)$ ,

$$U_t = X_t - (M_t^X - \bar{\gamma}(M_t^X)) \geq X_t - (\bar{\gamma}^{-1}(s_{n,i}) - \bar{\gamma}(\bar{\gamma}^{-1}(s_{n,i}))) > a.$$

Hence, conditional on  $\mathcal{F}_{T_{\bar{\gamma}^{-1}(s_{n,i-1})}^X}$ ,

$$T_{\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a}^X < T_a^U.$$

Therefore, the inner expectation above is dealt with as

$$\begin{aligned}
& \mathbf{E}^{x_0} \left[ e^{-\lambda T_a^U}; T_a^U < T_{s_{n,i}}^U \mid \mathcal{F}_{T_{s_{n,i-1}}^U} \right] \\
& \leq \mathbf{E}^{x_0} \left[ e^{-\lambda T_{\bar{\gamma}^{-1}(s_{n,i})-s_{n,i}+a}^X}; T_{\bar{\gamma}^{-1}(s_{n,i})-s_{n,i}+a}^X < T_{\bar{\gamma}^{-1}(s_{n,i})}^X \mid \mathcal{F}_{T_{\bar{\gamma}^{-1}(s_{n,i-1})}^X} \right] \\
& = \frac{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i}))}{f_\lambda(\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a, \bar{\gamma}^{-1}(s_{n,i}))},
\end{aligned}$$

where the last step is due to relation (2.8). Substituting this into the above and applying relation (3.7), we obtain

$$\begin{aligned}
& \mathbf{E}^{x_0} \left[ e^{-\lambda T_a^U}; T_a^U < T_b^U \right] \\
& \leq \sum_{i=1}^{m_n} \mathbf{E}^{x_0} \left[ e^{-\lambda T_{s_{n,i-1}}^U}; T_{s_{n,i-1}}^U < T_a^U \right] \frac{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i}))}{f_\lambda(\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a, \bar{\gamma}^{-1}(s_{n,i}))} \\
& \leq \sum_{i=1}^{m_n} e^{-\int_{x_0}^{\bar{\gamma}^{-1}(s_{n,i-1})} \frac{f_{2,\lambda}(x-\bar{\gamma}(x)+a,x)}{f_\lambda(x-\bar{\gamma}(x)+a,x)} dx} \frac{f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i}))}{f_\lambda(\bar{\gamma}^{-1}(s_{n,i}) - s_{n,i} + a, \bar{\gamma}^{-1}(s_{n,i}))}. \quad (3.18)
\end{aligned}$$

Since  $f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i-1})) = 0$  and the function  $f_{2,\lambda}(y, z)$  is continuous and strictly positive, for arbitrarily fixed  $0 < \varepsilon < 1$ , it holds that for all large  $n$  and  $i = 1, \dots, m_n$  that

$$\begin{aligned}
& f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i})) \\
& = f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i})) - f_\lambda(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i-1})) \\
& \leq (1 + \varepsilon) f_{2,\lambda}(\bar{\gamma}^{-1}(s_{n,i-1}), \bar{\gamma}^{-1}(s_{n,i-1})) (\bar{\gamma}^{-1}(s_{n,i}) - \bar{\gamma}^{-1}(s_{n,i-1})). \quad (3.19)
\end{aligned}$$

Substituting (3.19) into (3.18), changing each  $s_{n,i}$  in (3.18) to  $s_{n,i-1}$  based on the same reasoning as in deriving (3.17) and letting  $n \rightarrow \infty$ , we obtain

$$\mathbf{E}^{x_0} \left[ e^{-\lambda T_a^U}; T_a^U < T_b^U \right] \leq (1 + \varepsilon) \int_{x_0}^{\bar{\gamma}^{-1}(b)} e^{-\int_{x_0}^y \frac{f_{2,\lambda}(x-\bar{\gamma}(x)+a,x)}{f_\lambda(x-\bar{\gamma}(x)+a,x)} dx} \frac{f_{2,\lambda}(y, y)}{f_\lambda(y - \bar{\gamma}(y) + a, y)} dy.$$

By the arbitrariness of  $\varepsilon$ , the desired upper bound for (3.8) follows.

The corresponding lower bound for (3.8) can be established symmetrically by the fact that, after  $T_{s_{n,i-1}}^U$ , if the process  $X$  hits  $\bar{\gamma}^{-1}(s_{n,i-1}) - s_{n,i-1} + a$  before  $\bar{\gamma}^{-1}(s_{n,i})$ , then the process  $U$  hits  $a$  before  $s_{n,i}$ .  $\square$

*Proof of Corollary 3.3.* Using the change of variables  $y = M_t^X$ , we obtain

$$\begin{aligned} \mathbf{E}^{x_0} \left[ \int_0^{T_a^U} e^{-\lambda t} \gamma(M_t^X) dM_t^X \right] &= \mathbf{E}^{x_0} \left[ \int_{x_0}^{\infty} \gamma(y) e^{-\lambda T_y^X} 1_{\{y \leq M_{T_a^U}^X\}} dy; \right] \\ &= \mathbf{E}^{x_0} \left[ \int_{x_0}^{\infty} \gamma(y) e^{-\lambda T_y^X} 1_{\{T_y^X < T_a^U\}} dy \right] \\ &= \mathbf{E}^{x_0} \left[ \int_{x_0}^{\infty} \gamma(y) e^{-\lambda T_{\bar{\gamma}(y)}^U} 1_{\{T_{\bar{\gamma}(y)}^U < T_a^U\}} dy \right] \\ &= \int_{x_0}^{\infty} \gamma(y) \mathbf{E}^{x_0} \left[ e^{-\lambda T_{\bar{\gamma}(y)}^U}; T_{\bar{\gamma}(y)}^U < T_a^U \right] dy. \end{aligned}$$

Applying relation (3.7) to the right-hand side above yields the desired result.  $\square$

*Proof of Corollary 3.7.* Clearly,  $\mu(x)/\sigma^2(x)$  is constant for  $x \geq a$  if and only if

$$G(x) = c_1 e^{c_2 x}, \quad x \geq a,$$

for some constants  $c_1 > 0$  and  $c_2$ . The sufficiency of both parts can be checked directly. We now prove the necessity separately for both parts.

(1) For arbitrarily fixed  $x_0$  and  $a$  with  $a < x_0$ , we assume that relation (3.14)

holds for all  $b > x_0$  and  $0 \leq \gamma < 1$ . By (3.12), (3.14) and (2.13),

$$\int_{x_0}^b \frac{G\left(\frac{s-\gamma x_0}{1-\gamma}\right)}{\int_{\frac{\gamma s - \gamma x_0}{1-\gamma} + a}^{\frac{s-\gamma x_0}{1-\gamma}} G(y) dy} ds = \int_{x_0}^b \frac{G(s)}{\int_a^s G(y) dy} ds, \quad b > x_0, 0 \leq \gamma < 1.$$

It follows that

$$\frac{G\left(\frac{\gamma s - \gamma x_0}{1-\gamma} + s\right)}{\int_{\frac{\gamma s - \gamma x_0}{1-\gamma} + a}^{\frac{\gamma s - \gamma x_0}{1-\gamma} + s} G(y) dy} = \frac{G(s)}{\int_a^s G(y) dy}, \quad s > x_0, 0 \leq \gamma < 1.$$

Using change of variables  $x = (\gamma s - \gamma x_0)/(1 - \gamma)$  on the left-hand side of above, upon some simple rearrangements we obtain

$$\frac{\int_a^s G(y)dy}{G(s)}G(x+s) = \int_{x+a}^{x+s} G(y)dy, \quad s > x_0, x \geq 0.$$

By the continuity of  $G(\cdot)$ , it follows that

$$\frac{\int_a^s G(y)dy}{G(s)}G(x+s) = \int_{x+a}^{x+s} G(y)dy, \quad s \geq x_0, x \geq 0. \quad (3.20)$$

Taking derivative with respect to  $s$ , upon some simple rearrangements we obtain

$$\frac{G'(x+s)}{G(x+s)} = \frac{G'(s)}{G(s)}, \quad s > x_0, x \geq 0.$$

This means that  $G'(\cdot)/G(\cdot)$  is constant over the interval  $(x_0, \infty)$ . Hence, by the positivity and continuity of  $G(\cdot)$ , it must hold that

$$G(x) = c_1 e^{c_2 x}, \quad x \geq x_0, \quad (3.21)$$

for some constants  $c_1 > 0$  and  $c_2$ . Substituting (3.21) into (3.20) with  $s = x_0$  yields

$$e^{c_2 x} \int_a^{x_0} G(y)dy = \int_{x+a}^{x+x_0} G(y)dy, \quad x \geq 0.$$

Taking derivative with respect to  $x$  and using (3.21) and change of variables, we have

$$G(x) = e^{-c_2 a} \left( c_1 e^{c_2 x_0} - c_2 \int_a^{x_0} G(y)dy \right) e^{c_2 x}, \quad x \geq a.$$

Comparing this with (3.21), we must have  $e^{-c_2 a} (c_1 e^{c_2 x_0} - c_2 \int_a^{x_0} G(y)dy) = c_1$  since  $G(\cdot)$  is continuous at  $x_0$ . One can also easily check this by substitution. Hence,  $G(x) = c_1 e^{c_2 x}$  is valid over  $[a, \infty)$ .

(2) For arbitrarily fixed  $a$  and  $b$  with  $a < b$ , we assume that relation (3.14) holds for all  $x_0 \in (a, b)$  and  $\gamma \in [0, 1)$ . Similarly as in the proof of part (1), by (3.10), (3.14) and (2.13) one sees that

$$\int_{x_0}^{\frac{b-\gamma x_0}{1-\gamma}} \frac{G(x)}{\int_{\gamma x - \gamma x_0 + a}^x G(y) dy} dx = \frac{1}{1-\gamma} \int_{x_0}^b \frac{G(x)}{\int_a^x G(y) dy} dx, \quad x_0 \in (a, b), \gamma \in [0, 1).$$

Taking derivative with respect to  $x_0$  and cancelling  $\gamma$ , we obtain that, over the range  $x_0 \in (a, b)$  and  $\gamma \in (0, 1)$ ,

$$\frac{1}{1-\gamma} \frac{G(x_0)}{\int_a^{x_0} G(y) dy} - \frac{1}{1-\gamma} \frac{G\left(\frac{b-\gamma x_0}{1-\gamma}\right)}{\int_{\frac{\gamma b - \gamma x_0}{1-\gamma} + a}^{\frac{b-\gamma x_0}{1-\gamma}} G(y) dy} = \int_{x_0}^{\frac{b-\gamma x_0}{1-\gamma}} \frac{G(x) G(\gamma x - \gamma x_0 + a)}{\left(\int_{\gamma x - \gamma x_0 + a}^x G(y) dy\right)^2} dx.$$

Letting  $\gamma \rightarrow 0$  yields

$$\frac{G(x_0)}{\int_a^{x_0} G(y) dy} - \frac{G(b)}{\int_a^b G(y) dy} = \int_{x_0}^b \frac{G(x) G(a)}{\left(\int_a^x G(y) dy\right)^2} dx, \quad x_0 \in (a, b).$$

Upon some rearrangements we obtain

$$\frac{G(b) - G(a)}{\int_a^b G(y) dy} \int_a^{x_0} G(y) dy = \int_a^{x_0} G'(y) dy, \quad x_0 \in (a, b),$$

which implies that

$$\frac{G(b) - G(a)}{\int_a^b G(y) dy} G(x) = G'(x), \quad x \in (a, b).$$

Therefore, it must hold that

$$G(x) = c_1 e^{c_2 x}, \quad x \in [a, b],$$

for some constants  $c_1 > 0$  and  $c_2$  by the positivity and continuity of  $G(\cdot)$ .  $\square$

### 3.4 Summary and Some Remarks

Disregarding the background of loss-carry-forward taxation, we essentially solved the Laplace transforms of generalized drawdowns and drawups incorporating the corresponding running maxima and minima for time-homogeneous diffusion processes. The tax identity in the Lévy framework is better to be interpreted as an identity for generalized drawdowns. We obtain a sufficient and necessary condition to this identity for time-homogeneous diffusion processes.

From the proofs of our results, it is clear that the approximation approach in this chapter can be easily transformed to general Markov processes with continuous paths as long as the Laplace transforms of the two-sided exit times or the two-sided exit probabilities are explicit. Further, we hope that this approach can also be extended beyond the Lévy framework such as Markov processes with jumps.

In order to hedge big drawdowns or drawups, we can design some contracts which are triggered if the ratio of the size of a big decline to the high water marks exceeds a certain percentage prior to maturity. Such a contract can be of European type, American type, or be exercised at the moment when the triggering condition is met. These contracts can recover a part of the “loss” of investors who miss the chance to sell the assets at a relatively high price level. Furthermore, a merit of this type of contracts is that they can be used for an entire portfolio consisting of underlying assets with a wide range of prices because only the percentages of declines are considered.

## CHAPTER 4 THE JOINT LAPLACE TRANSFORMS OF DIFFUSION OCCUPATION TIMES

We adopt a perturbation approach to solve the joint Laplace transforms of occupation times for time-homogeneous diffusion processes. Standard methodologies of occupation times include martingale calculus, excursion theory, and Feynman–Kac formulas. Alternatively, an efficient perturbation approach has been developed significantly during the last five years by many researchers. This approach serves to decompose the problems of occupation times into exit problems. This approach significantly simplifies the calculation and can be transformed to general Markov processes as long as the Laplace transforms of the two-sided exit times are explicit. Our results find applications in two recently proposed bankruptcy problems by discovering an intrinsic connection between the new bankruptcy models and the occupation times. The content of this chapter is mainly based on the paper Li and Zhou (2013a).

### 4.1 Introduction

Occupation time, stemming from Lévy arcsine law, is an interesting and challenging topic of stochastic processes and has a wide range of applications in insurance and finance. Many explicit results on the Laplace transforms of occupation times have been obtained for some well known diffusion processes; see, e.g., Borodin and Salminen (2002) for a collection of such results. Some recent results on occupation times of time-homogeneous diffusion processes were obtained by Pitman and Yor (1999,

2003). Besides, the Laplace transforms of occupation times have also been extensively studied for jump processes; see, e.g., dos Reis (1993), Zhang and Wu (2002), Cai et al. (2010), Landriault et al. (2011), Kyprianou et al. (2012), and Loeffen et al. (2012). We consider diffusion processes and aim to solve the joint Laplace transforms of occupation times in multiple regions. Our motivations to treat occupation times in multiple regions include some recent arisen problems in risk management and some exotic options with multiple barriers.

Standard approaches of finding the Laplace transforms of occupation times include excursion theory, martingale calculus, and Feynman–Kac equations; see e.g., Revuz and Yor (1999), Pitman and Yor (1999, 2003). An alternative perturbation approach was recently proposed by Landriault et al. (2011) for spectrally negative Lévy processes. Using this approach, the Laplace transforms of occupation times can be obtained by an approximation argument based on the solutions to the exit problems, which is an excursion theory argument in its spirit. Since the exit problems for time-homogeneous diffusion processes can also be solved explicitly, in this chapter we adopt this perturbation approach to study the joint Laplace transforms of diffusion occupation times in multiple regions. We notice that this idea of perturbation has been developed by many authors in the recent five years to study other excursion related stopping times such as Parisian time; see, e.g., Dassios and Wu (2010), Loeffen et al. (2013) and Chapter 6 of this thesis.

Our results find applications in the study of the risk process with random observation times and the so-called Omega model, where the probabilities of bankruptcy



can be expressed in terms of the occupation times. In addition, the methodology in this chapter will be further exploited to study inverse occupation times in Chapter 6. Moreover, it is anticipated that our results can also be applied to study many exotic options related to occupation times such as corridor options and quantile options.

As introduced in Section 2.2.2, the one-dimensional time-homogeneous diffusion process  $X = \{X_t, t \geq 0\}$ , defined on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$ , is specified by the following stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad (4.1)$$

where  $X_0 = x_0$  is the initial value. For  $x \in \mathbb{R}$ , we define the first hitting time of  $X$  by

$$T_x = \inf \{t \geq 0 : X_t = x\}.$$

The rest of the chapter is arranged as follows. Main results about the Laplace transforms of the occupation times are obtained in Section 4.2. Applications to some recently proposed bankruptcy models are discussed in Section 4.3. Explicit expressions are presented for the cases of Brownian motion with drift and Brownian motion with two-valued drift in Section 4.4. A short summary and some remarks are given in Section 4.5.

## 4.2 Main Results

Throughout the chapter, denote by  $e_\delta$  an independent exponential random variable with rate  $\delta > 0$  and assume that  $\lambda, \lambda_1, \lambda_2 > 0$ . As a warm-up, we first solve the quantity  $\mathbb{E}^0 e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds}$ .

**Theorem 4.1.** *We have*

$$\mathbf{E}^0 e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds} = \frac{\frac{\delta}{\delta+\lambda} \psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)}{\psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)}.$$

*Proof.* We construct an approximation  $L_\varepsilon^\delta$  for the occupation time  $\int_0^{e_\delta} 1_{\{X_s < 0\}} ds$  as follows. Up to time  $e_\delta$ ,  $L_\varepsilon^\delta$  counts both the time for  $X$  to spend below level 0 and the time to move from level 0 to level  $\varepsilon$ , but not from  $\varepsilon$  to 0. To make it rigorous, let  $\theta$  be the shift operator such that  $X_t \circ \theta_s = X_{s+t}$ . Since  $X$  starts with 0, we put  $T_0^1 = 0$ ,  $T_\varepsilon^1 = T_\varepsilon$ ,  $T_0^{i+1} = T_\varepsilon^i + T_0 \circ \theta_{T_\varepsilon^i}$  and  $T_\varepsilon^{i+1} = T_0^{i+1} + T_\varepsilon \circ \theta_{T_0^{i+1}}$  for  $i = 1, 2, \dots$ . Then the approximation

$$L_\varepsilon^\delta = \sum_{i=1}^{\infty} (T_\varepsilon^i \wedge e_\delta - T_0^i \wedge e_\delta).$$

By the memoryless property, we obtain

$$\begin{aligned} \mathbf{E}^0 e^{-\lambda L_\varepsilon^\delta} &= \mathbf{E}^0 [e^{-\lambda L_\varepsilon^\delta}; T_\varepsilon < e_\delta] + \mathbf{E}^0 [e^{-\lambda e_\delta}; e_\delta < T_\varepsilon] \\ &= \mathbf{E}^0 [e^{-\lambda T_\varepsilon}; T_\varepsilon < e_\delta] \mathbf{E}^\varepsilon e^{-\lambda L_\varepsilon^\delta} + \mathbf{E} e^{-\lambda e_\delta} - \mathbf{E}^0 [e^{-\lambda e_\delta}; T_\varepsilon < e_\delta] \\ &= \mathbf{E}^0 e^{-(\delta+\lambda)T_\varepsilon} \mathbf{E}^\varepsilon e^{-\lambda L_\varepsilon^\delta} + \frac{\delta}{\delta+\lambda} (1 - \mathbf{E}^0 e^{-(\delta+\lambda)T_\varepsilon}) \\ &= \frac{g_{\delta+\lambda}^+(0)}{g_{\delta+\lambda}^+(\varepsilon)} \mathbf{E}^\varepsilon e^{-\lambda L_\varepsilon^\delta} + \frac{\delta}{\delta+\lambda} \left( 1 - \frac{g_{\delta+\lambda}^+(0)}{g_{\delta+\lambda}^+(\varepsilon)} \right), \end{aligned} \quad (4.2)$$

where the last step is due to (2.9). Similarly, we have

$$\begin{aligned} \mathbf{E}^\varepsilon e^{-\lambda L_\varepsilon^\delta} &= \mathbf{E}^\varepsilon [e^{-\lambda L_\varepsilon^\delta}; T_0 < e_\delta] + \mathbf{E}^\varepsilon [e^{-\lambda L_\varepsilon^\delta}; e_\delta < T_0] \\ &= \mathbf{P}^\varepsilon \{T_0 < e_\delta\} \mathbf{E}^0 e^{-\lambda L_\varepsilon^\delta} + \mathbf{P}^\varepsilon \{e_\delta < T_0\} \\ &= \frac{g_\delta^-(\varepsilon)}{g_\delta^-(0)} \mathbf{E}^0 e^{-\lambda L_\varepsilon^\delta} + 1 - \frac{g_\delta^-(\varepsilon)}{g_\delta^-(0)}. \end{aligned} \quad (4.3)$$

Substituting (4.3) into (4.2), solving for  $\mathbf{E}^0 e^{-\lambda L_\varepsilon^\delta}$ , and taking limit  $\varepsilon \rightarrow 0+$ , we obtain

$$\mathbf{E}^0 e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds} = \lim_{\varepsilon \rightarrow 0+} \mathbf{E}^0 e^{-\lambda L_\varepsilon^\delta} = \frac{\frac{\delta}{\delta+\lambda} \psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)}{\psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)}.$$

This completes the proof of Theorem 4.1.  $\square$

By letting  $\delta \rightarrow 0+$  in Theorem 4.1, we obtain the following result.

**Corollary 4.2.** *It holds that*

$$\mathbf{E}^0 e^{-\lambda \int_0^\infty 1_{\{X_s < 0\}} ds} = \frac{\psi_0^-(0)}{\psi_\lambda^+(0) + \psi_0^-(0)}.$$

**Corollary 4.3.** *In general, we have for  $x > 0$ ,*

$$\mathbf{E}^x e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds} = \frac{g_\delta^-(x)}{g_\delta^-(0)} \frac{\frac{\delta}{\delta+\lambda} \psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)}{\psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)} + 1 - \frac{g_\delta^-(x)}{g_\delta^-(0)}, \quad (4.4)$$

and for  $x < 0$ ,

$$\mathbf{E}^x e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds} = \frac{g_{\delta+\lambda}^+(x)}{g_{\delta+\lambda}^+(0)} \frac{\frac{\delta}{\delta+\lambda} \psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)}{\psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)} + \frac{\delta}{\delta+\lambda} \left( 1 - \frac{g_{\delta+\lambda}^+(x)}{g_{\delta+\lambda}^+(0)} \right). \quad (4.5)$$

*Proof.* For  $x > 0$ , comparing  $T_0$  with  $e_\delta$  and by the memoryless property we have

$$\mathbf{E}^x e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds} = \mathbf{P}^x \{T_0 < e_\delta\} \mathbf{E}^0 e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds} + \mathbf{P}^x \{e_\delta < T_0\}.$$

Then (4.4) follows from (2.9) and Theorem 4.1. For  $x < 0$ ,

$$\begin{aligned} \mathbf{E}^x e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds} &= \mathbf{E}^x [e^{-\lambda T_0}; T_0 < e_\delta] \mathbf{E}^0 e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds} + \mathbf{E}^x [e^{-\lambda e_\delta}; e_\delta < T_0] \\ &= \mathbf{E}^x e^{-(\delta+\lambda)T_0} \mathbf{E}^0 e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds} + \frac{\delta}{\delta+\lambda} (1 - \mathbf{E}^x e^{-(\delta+\lambda)T_0}). \end{aligned}$$

Then (4.5) also follows from (2.9) and Theorem 4.1.  $\square$

Using the Laplace transforms on exit times of  $X$  in Theorem 2.6, all the results below can be easily generalized for a general initial value  $x \in \mathbb{R}$ . However, we skip this minor generalization for brevity.

The strategy employed in the proof of the single Laplace transform in Theorem 4.1 can be further exploited to find the joint Laplace transforms of occupation times in multiple regions. By the following results of occupation times in two unbounded regions, we can further study Parisian options with double space barriers and double time periods as in Anderluh and van der Weide (2009) and Dassios and Wu (2011) but using occupation times. See Theorem 6.15 below for a special case of such an extension.

**Theorem 4.4.** *For any  $b > 0$ , we have*

$$\begin{aligned}
& \mathbb{E}^0 \exp \left\{ -\lambda_1 \int_0^{e_\delta} 1_{\{X_s < 0\}} ds - \lambda_2 \int_0^{e_\delta} 1_{\{X_s > b\}} ds \right\} \\
&= \frac{\frac{\delta}{\delta + \lambda_1} \left( \frac{f_{2,\delta}(0,b)}{f_\delta(0,b)} + \psi_{\delta + \lambda_2}^-(b) \right) \psi_{\delta + \lambda_1}^+(0) - \left( \frac{\lambda_2}{\delta + \lambda_2} \frac{f_{2,\delta}(0,0)}{f_\delta(0,b)} + \frac{f_{1,\delta}(0,b)}{f_\delta(0,b)} \right) \psi_{\delta + \lambda_2}^-(b) - \frac{f_{12,\delta}(0,b)}{f_\delta(0,b)}}{\left( \frac{f_{2,\delta}(0,b)}{f_\delta(0,b)} + \psi_{\delta + \lambda_2}^-(b) \right) \psi_{\delta + \lambda_1}^+(0) - \frac{f_{1,\delta}(0,b)}{f_\delta(0,b)} \psi_{\delta + \lambda_2}^-(b) - \frac{f_{12,\delta}(0,b)}{f_\delta(0,b)}}, \\
& \mathbb{E}^b \exp \left\{ -\lambda_1 \int_0^{e_\delta} 1_{\{X_s < 0\}} ds - \lambda_2 \int_0^{e_\delta} 1_{\{X_s > b\}} ds \right\} \\
&= \frac{\left( \frac{\lambda_1}{\delta + \lambda_1} \frac{f_{1,\delta}(b,b)}{f_\delta(0,b)} + \frac{f_{2,\delta}(0,b)}{f_\delta(0,b)} \right) \psi_{\delta + \lambda_1}^+(0) + \frac{\delta}{\delta + \lambda_2} \left( -\frac{f_{1,\delta}(0,b)}{f_\delta(0,b)} + \psi_{\delta + \lambda_1}^+(0) \right) \psi_{\delta + \lambda_2}^-(b) - \frac{f_{12,\delta}(0,b)}{f_\delta(0,b)}}{\left( \frac{f_{2,\delta}(0,b)}{f_\delta(0,b)} + \psi_{\delta + \lambda_2}^-(b) \right) \psi_{\delta + \lambda_1}^+(0) - \frac{f_{1,\delta}(0,b)}{f_\delta(0,b)} \psi_{\delta + \lambda_2}^-(b) - \frac{f_{12,\delta}(0,b)}{f_\delta(0,b)}}.
\end{aligned}$$

*Proof.* As in the proof of Theorem 4.1, for arbitrarily small  $\varepsilon > 0$ , we approximate the occupation time  $\int_0^{e_\delta} 1_{\{X_s < 0\}} ds$  by  $L_{\varepsilon,0+}^\delta$ , the sum of durations up to time  $e_\delta$  for all the non-overlapping excursions of the process  $X$  that start with 0 and end with  $\varepsilon$ . Similarly, we approximate  $\int_0^{e_\delta} 1_{\{X_s > b\}} ds$  by  $L_{\varepsilon,b-}^\delta$ , the sum of durations up to time  $e_\delta$  for all the excursions of  $X$  that start with  $b$  and end with  $b - \varepsilon$ .

We define  $I_0(x) = \mathbb{E}^x \exp \left\{ -\lambda_1 \int_0^{e_\delta} 1_{\{X_s < 0\}} ds - \lambda_2 \int_0^{e_\delta} 1_{\{X_s > b\}} ds \right\}$  and its approximation  $I_\varepsilon(x) = \mathbb{E}^x e^{-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta}$  for some  $x \in \mathbb{R}$ . By the strong Markov

property, we have

$$\begin{aligned}
I_\varepsilon(0) &= \mathbb{E}^0[e^{-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta}; T_\varepsilon < e_\delta] + \mathbb{E}^0[e^{-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta}; e_\delta < T_\varepsilon] \\
&= \mathbb{E}^0[e^{-\lambda_1 T_\varepsilon}; T_\varepsilon < e_\delta] I_\varepsilon(\varepsilon) + \mathbb{E}^0[e^{-\lambda_1 e_\delta}; e_\delta < T_\varepsilon] \\
&= \frac{g_{\delta+\lambda_1}^+(0)}{g_{\delta+\lambda_1}^+(\varepsilon)} I_\varepsilon(\varepsilon) + \frac{\delta}{\delta + \lambda_1} \left( 1 - \frac{g_{\delta+\lambda_1}^+(0)}{g_{\delta+\lambda_1}^+(\varepsilon)} \right). \tag{4.6}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I_\varepsilon(\varepsilon) &= \mathbb{E}^\varepsilon[e^{-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta}; T_0 < T_b \wedge e_\delta] + \mathbb{E}^\varepsilon[e^{-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta}; T_b < T_0 \wedge e_\delta] \\
&\quad + \mathbb{E}^\varepsilon[e^{-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta}; e_\delta < T_0 \wedge T_b] \\
&= \mathbb{P}^\varepsilon\{T_0 < T_b \wedge e_\delta\} I_\varepsilon(0) + \mathbb{P}^\varepsilon\{T_b < T_0 \wedge e_\delta\} I_\varepsilon(b) + \mathbb{P}^\varepsilon\{e_\delta < T_0 \wedge T_b\} \\
&= \frac{f_\delta(\varepsilon, b)}{f_\delta(0, b)} I_\varepsilon(0) + \frac{f_\delta(0, \varepsilon)}{f_\delta(0, b)} I_\varepsilon(b) + 1 - \frac{f_\delta(0, \varepsilon) + f_\delta(\varepsilon, b)}{f_\delta(0, b)}. \tag{4.7}
\end{aligned}$$

Substituting (4.7) to (4.6), solving for  $I_\varepsilon(0)$  and taking limit  $\varepsilon \rightarrow 0+$ , we obtain

$$I_0(0) = \frac{\frac{\delta}{\delta+\lambda_1} \psi_{\delta+\lambda_1}^+(0) + \frac{f_{2,\delta}(0,0)}{f_\delta(0,b)} I_0(b) - \frac{f_{1,\delta}(0,b) + f_{2,\delta}(0,0)}{f_\delta(0,b)}}{\psi_{\delta+\lambda_1}^+(0) - \frac{f_{1,\delta}(0,b)}{f_\delta(0,b)}}. \tag{4.8}$$

Here (4.8) is the first equation of a linear system of the desired terms  $I_0(0)$  and  $I_0(b)$ .

To obtain the second equation, we turn to another initial value  $x = b$ . Similarly, we have

$$\begin{aligned}
I_\varepsilon(b) &= \mathbb{E}^b[e^{-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta}; T_{b-\varepsilon} < e_\delta] + \mathbb{E}^b[e^{-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta}; e_\delta < T_{b-\varepsilon}] \\
&= \mathbb{E}^b[e^{-\lambda_2 T_{b-\varepsilon}}; T_{b-\varepsilon} < e_\delta] I_\varepsilon(b - \varepsilon) + \mathbb{E}^b[e^{-\lambda_2 e_\delta}; e_\delta < T_{b-\varepsilon}] \\
&= \frac{g_{\delta+\lambda_2}^-(b)}{g_{\delta+\lambda_2}^-(b - \varepsilon)} I_\varepsilon(b - \varepsilon) + \frac{\delta}{\delta + \lambda_2} \left( 1 - \frac{g_{\delta+\lambda_2}^-(b)}{g_{\delta+\lambda_2}^-(b - \varepsilon)} \right). \tag{4.9}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I_\varepsilon(b - \varepsilon) &= \mathbf{E}^{b-\varepsilon}[e^{-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta}; T_b < T_0 \wedge e_\delta] + \mathbf{E}^{b-\varepsilon}[e^{-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta}; T_0 < T_b \wedge e_\delta] \\
&\quad + \mathbf{E}^{b-\varepsilon}[e^{-\lambda_1 L_{\varepsilon,0+}^\delta - \lambda_2 L_{\varepsilon,b-}^\delta}; e_\delta < T_0 \wedge T_b] \\
&= \mathbf{P}^{b-\varepsilon}\{T_b < T_0 \wedge e_\delta\} I_\varepsilon(b) + \mathbf{P}^{b-\varepsilon}\{T_0 < T_b \wedge e_\delta\} I_\varepsilon(0) + \mathbf{P}^{b-\varepsilon}\{e_\delta < T_0 \wedge T_b\} \\
&= \frac{f_\delta(0, b - \varepsilon)}{f_\delta(0, b)} I_\varepsilon(b) + \frac{f_\delta(b - \varepsilon, b)}{f_\delta(0, b)} I_\varepsilon(0) + 1 - \frac{f_\delta(0, b - \varepsilon) + f_\delta(b - \varepsilon, b)}{f_\delta(0, b)}.
\end{aligned} \tag{4.10}$$

Substituting (4.10) into (4.9), solving for  $I_\varepsilon(b)$  and taking limit  $\varepsilon \rightarrow 0+$ , we obtain

$$I_0(b) = \frac{\frac{\delta}{\delta+\lambda_2} \psi_{\delta+\lambda_2}^-(b) - \frac{f_{1,\delta}(b,b)}{f_\delta(0,b)} I_0(0) + \frac{f_{2,\delta}(0,b) + f_{1,\delta}(b,b)}{f_\delta(0,b)}}{\psi_{\delta+\lambda_2}^-(b) + \frac{f_{2,\delta}(0,b)}{f_\delta(0,b)}}. \tag{4.11}$$

Here (4.11) is the second equation of a linear system of desired terms  $I_0(0)$  and  $I_0(b)$ . Therefore, solving the linear system composed of equations (4.8) and (4.11) and simplifying the result by the first identity in Theorem 2.7 with  $x = 0$ ,  $y = b$  and  $\lambda = \delta$ , we complete the proof of Theorem 4.4.  $\square$

**Remark 4.5.** *In particular, when  $\lambda_2 \rightarrow 0+$ , the first joint Laplace transform in Theorem 4.4 reduces to the one in Theorem 4.1 by the second identity in Theorem 2.7 with  $x = 0$ ,  $y = b$  and  $\lambda = \delta$ .*

Letting  $b \rightarrow 0+$  in Theorem 4.4, and using the facts that  $f_{12,\delta}(0, 0) = 0$  and  $f_{1,\delta}(0, 0) = -f_{2,\delta}(0, 0)$ , we obtain the following compact expression.

**Corollary 4.6.** *We have*

$$\mathbf{E}^0 \exp \left\{ -\lambda_1 \int_0^{e_\delta} 1_{\{X_s < 0\}} ds - \lambda_2 \int_0^{e_\delta} 1_{\{X_s > 0\}} ds \right\} = \frac{\frac{\delta}{\delta+\lambda_1} \psi_{\delta+\lambda_1}^+(0) + \frac{\delta}{\delta+\lambda_2} \psi_{\delta+\lambda_2}^-(0)}{\psi_{\delta+\lambda_1}^+(0) + \psi_{\delta+\lambda_2}^-(0)}.$$

Letting  $\lambda_2 \rightarrow \infty$  and  $\lambda_1 \rightarrow \infty$  in the first and second equation of Theorem 4.4, respectively, and using  $\lim_{\lambda \rightarrow \infty} \psi_\lambda^\pm(x) = \infty$ , we obtain the following result.

**Corollary 4.7.** *For any  $a < 0$ , we have*

$$\begin{aligned} \mathbb{E}^0 \left[ e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds}; e_\delta < T_b \right] &= \frac{\frac{\delta}{\delta+\lambda} f_\delta(0, b) \psi_{\delta+\lambda}^+(0) - f_{2,\delta}(0, 0) - f_{1,\delta}(0, b)}{f_\delta(0, b) \psi_{\delta+\lambda}^+(0) - f_{1,\delta}(0, b)}, \\ \mathbb{E}^b \left[ e^{-\lambda \int_0^{e_\delta} 1_{\{X_s > b\}} ds}; e_\delta < T_0 \right] &= \frac{\frac{\delta}{\delta+\lambda} f_\delta(0, b) \psi_{\delta+\lambda}^-(b) + f_{1,\delta}(b, b) + f_{2,\delta}(0, b)}{f_\delta(0, b) \psi_{\delta+\lambda}^-(b) + f_{2,\delta}(0, b)}. \end{aligned}$$

Next we consider the joint Laplace transforms of occupation times in a bounded region and an unbounded region. Since an unbounded region can be treated as a special case of a bounded region, the results of the following Theorem are more complicated than the ones in Theorem 4.4 as expected. We first introduce some possible applications of the following Theorem 4.8 in risk management. The model in Gauthier (2002) and the liquidation time (5.2) in Chapter 5 can be considered as two applications of Parisian times in risk management with delay constraints. However, we need to replace Parisian times by occupation times in the modeling if there are some cumulative features of these delay constraints. Then, as  $\lambda_2 \rightarrow \infty$ , the following structure essentially generalizes the one in Gauthier (2002) and the liquidation time (5.2) but using occupation times.

**Theorem 4.8.** *For any  $a < 0$ , we have*

$$\begin{aligned} \mathbb{E}^0 \exp \left\{ -\lambda_1 \int_0^{e_\delta} 1_{\{a \leq X_s < 0\}} ds - \lambda_2 \int_0^{e_\delta} 1_{\{X_s < a\}} ds \right\} &= \frac{N_1}{D}, \\ \mathbb{E}^a \exp \left\{ -\lambda_1 \int_0^{e_\delta} 1_{\{a \leq X_s < 0\}} ds - \lambda_2 \int_0^{e_\delta} 1_{\{X_s < a\}} ds \right\} &= \frac{N_2}{D}, \end{aligned}$$

where

$$\begin{aligned}
N_1 &= \frac{\delta(\delta + \lambda_2)f_{2,\delta+\lambda_1}(a, 0) + \delta(\lambda_1 - \lambda_2)f_{2,\delta+\lambda_1}(0, 0)}{(\delta + \lambda_1)(\delta + \lambda_2)f_{\delta+\lambda_1}(a, 0)}\psi_{\delta+\lambda_2}^+(a) \\
&\quad - \frac{f_{1,\delta+\lambda_1}(a, 0)}{f_{\delta+\lambda_1}(a, 0)}\psi_{\delta}^-(0) + \psi_{\delta+\lambda_2}^+(a)\psi_{\delta}^-(0) - \frac{\delta f_{12,\delta+\lambda_1}(a, 0)}{(\delta + \lambda_1)f_{\delta+\lambda_1}(a, 0)}, \\
N_2 &= \frac{\delta f_{2,\delta+\lambda_1}(a, 0)}{(\delta + \lambda_2)f_{\delta+\lambda_1}(a, 0)}\psi_{\delta+\lambda_2}^+(a) - \frac{\delta f_{1,\delta+\lambda_1}(a, 0) - \lambda_1 f_{2,\delta+\lambda_1}(a, a)}{(\delta + \lambda_1)f_{\delta+\lambda_1}(a, 0)}\psi_{\delta}^-(0) \\
&\quad + \frac{\delta}{\delta + \lambda_2}\psi_{\delta+\lambda_2}^+(a)\psi_{\delta}^-(0) - \frac{\delta f_{12,\delta+\lambda_1}(a, 0)}{(\delta + \lambda_1)f_{\delta+\lambda_1}(a, 0)}, \\
D &= \frac{f_{2,\delta+\lambda_1}(a, 0)}{f_{\delta+\lambda_1}(a, 0)}\psi_{\delta+\lambda_2}^+(a) - \frac{f_{1,\delta+\lambda_1}(a, 0)}{f_{\delta+\lambda_1}(a, 0)}\psi_{\delta}^-(0) + \psi_{\delta+\lambda_2}^+(a)\psi_{\delta}^-(0) - \frac{f_{12,\delta+\lambda_1}(a, 0)}{f_{\delta+\lambda_1}(a, 0)}.
\end{aligned}$$

*Proof.* For arbitrarily small  $\varepsilon > 0$ , we approximate the occupation time  $\int_0^{e_\delta} 1_{\{a \leq X_s < 0\}} ds$  by  $L_{\varepsilon, a, 0}^\delta$ , the sum of durations up to time  $e_\delta$  for all the excursions of  $X$  that either start with 0 and end with  $\varepsilon$  avoiding  $a$ , or start with 0 and end with  $a$  avoiding  $\varepsilon$ . Similarly, we approximate  $\int_0^{e_\delta} 1_{\{X_s < a\}} ds$  by  $L_{\varepsilon, a, +}^\delta$ , the sum of durations up to time  $e_\delta$  for all the excursions of  $X$  that start with  $a$  and end with  $a + \varepsilon$ .

We define  $J_0(x) = \mathbb{E}^x \exp \left\{ -\lambda_1 \int_0^{e_\delta} 1_{\{a \leq X_s < 0\}} ds - \lambda_2 \int_0^{e_\delta} 1_{\{X_s < a\}} ds \right\}$  and its approximation  $J_\varepsilon(x) = \mathbb{E}^x e^{-\lambda_1 L_{\varepsilon, a, 0}^\delta - \lambda_2 L_{\varepsilon, a, +}^\delta}$  for some  $x \in \mathbb{R}$ . By the strong Markov property, we have

$$\begin{aligned}
J_\varepsilon(0) &= \mathbb{E}^0[e^{-\lambda_1 L_{\varepsilon, a, 0}^\delta - \lambda_2 L_{\varepsilon, a, +}^\delta}; T_\varepsilon < T_a \wedge e_\delta] + \mathbb{E}^0[e^{-\lambda_1 L_{\varepsilon, a, 0}^\delta - \lambda_2 L_{\varepsilon, a, +}^\delta}; T_a < T_\varepsilon \wedge e_\delta] \\
&\quad + \mathbb{E}^0[e^{-\lambda_1 L_{\varepsilon, a, 0}^\delta - \lambda_2 L_{\varepsilon, a, +}^\delta}; e_\delta < T_\varepsilon \wedge T_a] \\
&= \mathbb{E}^0[e^{-\lambda_1 T_\varepsilon}; T_\varepsilon < T_a \wedge e_\delta] J_\varepsilon(\varepsilon) + \mathbb{E}^0[e^{-\lambda_1 T_a}; T_a < T_\varepsilon \wedge e_\delta] J_\varepsilon(a) \\
&\quad + \mathbb{E}^0[e^{-\lambda_1 e_\delta}; e_\delta < T_\varepsilon \wedge T_a] \\
&= \frac{f_{\delta+\lambda_1}(a, 0)}{f_{\delta+\lambda_1}(a, \varepsilon)} J_\varepsilon(\varepsilon) + \frac{f_{\delta+\lambda_1}(0, \varepsilon)}{f_{\delta+\lambda_1}(a, \varepsilon)} J_\varepsilon(a) + \frac{\delta}{\delta + \lambda_1} \left( 1 - \frac{f_{\delta+\lambda_1}(a, 0) + f_{\delta+\lambda_1}(0, \varepsilon)}{f_{\delta+\lambda_1}(a, \varepsilon)} \right).
\end{aligned} \tag{4.12}$$



Furthermore,

$$\begin{aligned} J_\varepsilon(\varepsilon) &= \mathbb{P}^\varepsilon\{T_0 < e_\delta\}J_\varepsilon(0) + \mathbb{P}^\varepsilon\{e_\delta < T_0\} \\ &= \frac{g_\delta^-(\varepsilon)}{g_\delta^-(0)}J_\varepsilon(0) + 1 - \frac{g_\delta^-(\varepsilon)}{g_\delta^-(0)}. \end{aligned} \quad (4.13)$$

Substituting (4.13) into (4.12), solving for  $J_\varepsilon(0)$  and taking limit  $\varepsilon \rightarrow 0+$ , we obtain

$$J_0(0) = \frac{\psi_\delta^-(0) + \frac{f_{2,\delta+\lambda_1}(0,0)}{f_{\delta+\lambda_1}(a,0)}J_0(a) + \frac{\delta}{\delta+\lambda_1} \frac{f_{2,\delta+\lambda_1}(a,0) - f_{2,\delta+\lambda_1}(0,0)}{f_{\delta+\lambda_1}(a,0)}}{\psi_\delta^-(0) + \frac{f_{2,\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,0)}}. \quad (4.14)$$

Here (4.14) is the first equation of a linear system of desired terms  $J_0(0)$  and  $J_0(a)$ .

Similarly,

$$\begin{aligned} J_\varepsilon(a) &= \mathbb{E}^a[e^{-\lambda_1 L_{\varepsilon,a,0}^\delta - \lambda_2 L_{\varepsilon,a+}^\delta}; T_{a+\varepsilon} < e_\delta] + \mathbb{E}^a[e^{-\lambda_1 L_{\varepsilon,a,0}^\delta - \lambda_2 L_{\varepsilon,a+}^\delta}; e_\delta < T_{a+\varepsilon}] \\ &= \mathbb{E}^a[e^{-\lambda_2 T_{a+\varepsilon}}; T_{a+\varepsilon} < e_\delta]J_\varepsilon(a+\varepsilon) + \mathbb{E}^a[e^{-\lambda_2 e_\delta}; e_\delta < T_{a+\varepsilon}] \\ &= \frac{g_{\delta+\lambda_2}^+(a)}{g_{\delta+\lambda_2}^+(a+\varepsilon)}J_\varepsilon(a+\varepsilon) + \frac{\delta}{\delta+\lambda_2} \left(1 - \frac{g_{\delta+\lambda_2}^+(a)}{g_{\delta+\lambda_2}^+(a+\varepsilon)}\right). \end{aligned} \quad (4.15)$$

Furthermore,

$$\begin{aligned} J_\varepsilon(a+\varepsilon) &= \mathbb{E}^{a+\varepsilon}[e^{-\lambda_1 L_{\varepsilon,a,0}^\delta - \lambda_2 L_{\varepsilon,a+}^\delta}; T_a < T_0 \wedge e_\delta] + \mathbb{E}^{a+\varepsilon}[e^{-\lambda_1 L_{\varepsilon,a,0}^\delta - \lambda_2 L_{\varepsilon,a+}^\delta}; T_0 < T_a \wedge e_\delta] \\ &\quad + \mathbb{E}^{a+\varepsilon}[e^{-\lambda_1 L_{\varepsilon,a,0}^\delta - \lambda_2 L_{\varepsilon,a+}^\delta}; e_\delta < T_a \wedge T_0] \\ &= \mathbb{E}^{a+\varepsilon}[e^{-\lambda_1 T_a}; T_a < T_0 \wedge e_\delta]J_\varepsilon(a) + \mathbb{E}^{a+\varepsilon}[e^{-\lambda_1 T_0}; T_0 < T_a \wedge e_\delta]J_\varepsilon(0) \\ &\quad + \mathbb{E}^{a+\varepsilon}[e^{-\lambda_1 e_\delta}; e_\delta < T_a \wedge T_0] \\ &= \frac{f_{\delta+\lambda_1}(a+\varepsilon, 0)}{f_{\delta+\lambda_1}(a, 0)}J_\varepsilon(a) + \frac{f_{\delta+\lambda_1}(a, a+\varepsilon)}{f_{\delta+\lambda_1}(a, 0)}J_\varepsilon(0) \\ &\quad + \frac{\delta}{\delta+\lambda_1} \left(1 - \frac{f_{\delta+\lambda_1}(a, a+\varepsilon) + f_{\delta+\lambda_1}(a+\varepsilon, 0)}{f_{\delta+\lambda_1}(a, 0)}\right). \end{aligned} \quad (4.16)$$

Substituting (4.16) into (4.15), solving for  $J_\varepsilon(a)$  and taking limit  $\varepsilon \rightarrow 0+$ , we obtain

$$J_0(a) = \frac{\frac{\delta}{\delta+\lambda_1} \frac{-f_{1,\delta+\lambda_1}(a,0) - f_{2,\delta+\lambda_1}(a,a)}{f_{\delta+\lambda_1}(a,0)} + \frac{\delta}{\delta+\lambda_2} \psi_{\delta+\lambda_2}^+(a) + \frac{f_{2,\delta+\lambda_1}(a,a)}{f_{\delta+\lambda_1}(a,0)}J_0(0)}{\psi_{\delta+\lambda_2}^+(a) - \frac{f_{1,\delta+\lambda_1}(a,0)}{f_{\delta+\lambda_1}(a,0)}}. \quad (4.17)$$

Here (4.17) is the second equation of a linear system of desired terms  $J_0(0)$  and  $J_0(a)$ . Finally, solving the linear system composed of equations (4.14) and (4.17), and simplifying the result by the first identity in Theorem 2.7 with  $x = a$ ,  $y = 0$ , and  $\lambda = \delta + \lambda_1$ , we complete the proof of Theorem 4.8.  $\square$

**Remark 4.9.** *In particular, when  $\lambda_1 = \lambda_2 = \lambda$ , the first joint Laplace transform in Theorem 4.8 reduces to the one in Theorem 4.1 by the third identity in Theorem 2.7 with  $x = a$ ,  $y = 0$ , and  $\lambda = \delta + \lambda$ . Further, when  $\lambda_1 \rightarrow 0+$ ,  $\lambda_2 = \lambda$  and  $a = 0$ , the second joint Laplace transform in Theorem 4.8 reduces to the one in Theorem 4.1 by the second identity in Theorem 2.7 with  $x = a$ ,  $y = 0$  and  $\lambda = \delta$ .*

Letting  $\delta \rightarrow 0+$  in Theorem 4.8, we obtain the following Corollary.

**Corollary 4.10.** *For any  $a < 0$ , we have*

$$\begin{aligned} & E^0 \exp \left\{ -\lambda_1 \int_0^\infty 1_{\{a \leq X_s < 0\}} ds - \lambda_2 \int_0^\infty 1_{\{X_s < a\}} ds \right\} \\ &= \frac{-f_{1,\lambda_1}(a, 0)\psi_0^-(0) + f_{\lambda_1}(a, 0)\psi_{\lambda_2}^+(a)\psi_0^-(0)}{f_{2,\lambda_1}(a, 0)\psi_{\lambda_2}^+(a) - f_{1,\lambda_1}(a, 0)\psi_0^-(0) + f_{\lambda_1}(a, 0)\psi_{\lambda_2}^+(a)\psi_0^-(0) - f_{12,\lambda_1}(a, 0)}, \\ & E^a \exp \left\{ -\lambda_1 \int_0^\infty 1_{\{a \leq X_s < 0\}} ds - \lambda_2 \int_0^\infty 1_{\{X_s < a\}} ds \right\} \\ &= \frac{f_{2,\delta+\lambda_1}(a, a)\psi_0^-(0)}{f_{2,\lambda_1}(a, 0)\psi_{\lambda_2}^+(a) - f_{1,\lambda_1}(a, 0)\psi_0^-(0) + f_{\lambda_1}(a, 0)\psi_{\lambda_2}^+(a)\psi_0^-(0) - f_{12,\lambda_1}(a, 0)}. \end{aligned}$$

Letting  $\lambda_2 \rightarrow 0+$  in Theorem 4.8, by the identity  $f_{2,\delta+\lambda}(a, a) = -f_{1,\delta+\lambda}(a, a)$ , we obtain the following result.

**Corollary 4.11.** *For any  $a < 0$ , we have*

$$\begin{aligned} & \mathbb{E}^0 e^{-\lambda \int_0^{\epsilon_\delta} 1_{\{a \leq X_s < 0\}} ds} \\ &= \frac{\frac{\delta f_{2,\delta+\lambda}(a,0) + \lambda f_{2,\delta+\lambda}(0,0)}{(\delta+\lambda)f_{\delta+\lambda}(a,0)} \psi_\delta^+(a) - \frac{f_{1,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^-(0) + \psi_\delta^+(a) \psi_\delta^-(0) - \frac{\delta f_{12,\delta+\lambda}(a,0)}{(\delta+\lambda)f_{\delta+\lambda}(a,0)}}{\frac{f_{2,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^+(a) - \frac{f_{1,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^-(0) + \psi_\delta^+(a) \psi_\delta^-(0) - \frac{f_{12,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)}}, \\ & \mathbb{E}^a e^{-\lambda \int_0^{\epsilon_\delta} 1_{\{a \leq X_s < 0\}} ds} \\ &= \frac{\frac{f_{2,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^+(a) - \frac{\delta f_{1,\delta+\lambda}(a,0) + \lambda f_{1,\delta+\lambda}(a,a)}{(\delta+\lambda)f_{\delta+\lambda}(a,0)} \psi_\delta^-(0) + \psi_\delta^+(a) \psi_\delta^-(0) - \frac{\delta f_{12,\delta+\lambda}(a,0)}{(\delta+\lambda)f_{\delta+\lambda}(a,0)}}{\frac{f_{2,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^+(a) - \frac{f_{1,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)} \psi_\delta^-(0) + \psi_\delta^+(a) \psi_\delta^-(0) - \frac{f_{12,\delta+\lambda}(a,0)}{f_{\delta+\lambda}(a,0)}}. \end{aligned}$$

Letting  $\delta \rightarrow 0+$  in Corollary 4.11, we obtain the following simplified expressions:

**Corollary 4.12.** *For any  $a < 0$ , we have*

$$\begin{aligned} & \mathbb{E}^0 e^{-\lambda \int_0^\infty 1_{\{a \leq X_s < 0\}} ds} \\ &= \frac{f_{2,\lambda}(0,0) \psi_0^+(a) - f_{1,\lambda}(a,0) \psi_0^-(0) + f_\lambda(a,0) \psi_0^+(a) \psi_0^-(0)}{f_{2,\lambda}(a,0) \psi_0^+(a) - f_{1,\lambda}(a,0) \psi_0^-(0) + f_\lambda(a,0) \psi_0^+(a) \psi_0^-(0) - f_{12,\lambda}(a,0)}, \\ & \mathbb{E}^a e^{-\lambda \int_0^\infty 1_{\{a \leq X_s < 0\}} ds} \\ &= \frac{f_{2,\lambda}(a,0) \psi_0^+(a) - f_{1,\lambda}(a,a) \psi_0^-(0) + f_\lambda(a,0) \psi_0^+(a) \psi_0^-(0)}{f_{2,\lambda}(a,0) \psi_0^+(a) - f_{1,\lambda}(a,0) \psi_0^-(0) + f_\lambda(a,0) \psi_0^+(a) \psi_0^-(0) - f_{12,\lambda}(a,0)}. \end{aligned}$$

**Remark 4.13.** *As  $a \rightarrow -\infty$ , the first Laplace transforms in Corollaries 4.11 and 4.12 reduce to the ones in Theorem 4.1 and Corollary 4.2, respectively.*

Letting  $\lambda_2 \rightarrow \infty$  in the first equation of Theorem 4.8, by  $\lim_{\lambda \rightarrow \infty} \psi_\lambda^+(x) = \infty$ , we obtain the following result:

**Corollary 4.14.** *For any  $a < 0$ , we have*

$$\mathbb{E}^0 \left[ e^{-\lambda \int_0^{\epsilon_\delta} 1_{\{a \leq X_s < 0\}} ds}; e_\delta < T_a \right] = \frac{\frac{\delta}{\delta+\lambda} (f_{2,\delta+\lambda}(a,0) - f_{2,\delta+\lambda}(0,0)) + f_{\delta+\lambda}(a,0) \psi_\delta^-(0)}{f_{2,\delta+\lambda}(a,0) + f_{\delta+\lambda}(a,0) \psi_\delta^-(0)}.$$

### 4.3 Applications to Diffusion Risk Processes

#### 4.3.1 The risk process with random observations

Suppose that the firm value process  $X$  follows a time-homogeneous diffusion process defined in (4.1) with initial value  $x_0 > 0$ . Let  $N$  be an independent Poisson process with rate  $\lambda > 0$  of consecutive arrival times  $0 < \tau_1 < \tau_2 < \dots$ . Suppose that the values of the process  $X$  are observed at the random times  $\tau_i$ ,  $i = 1, 2, \dots$ . The bankruptcy time is then defined by

$$\tau_\lambda = \inf_{i=1,2,\dots} \{\tau_i : X_{\tau_i} < 0\}. \quad (4.18)$$

There is a natural connection between the survival probability of this model and the occupation time of the firm value process  $X$ . Actually, for any  $t > 0$ ,

$$\begin{aligned} \mathbb{P}^{x_0} \{\tau_\lambda > t\} &= \mathbb{E}^{x_0} \mathbb{P} \{ \{\tau_i, i = 1, 2, \dots\} \cap \{s \leq t : X_s < 0\} = \emptyset \mid X \} \\ &= \mathbb{E}^{x_0} \mathbb{E} \left[ e^{-\lambda \int_0^t 1_{\{X_s < 0\}} ds} \mid X \right] \\ &= \mathbb{E}^{x_0} e^{-\lambda \int_0^t 1_{\{X_s < 0\}} ds}. \end{aligned}$$

Therefore, it follows that the Laplace transform of the bankruptcy time  $\tau_\lambda$ ,

$$\mathbb{E}^{x_0} e^{-\delta \tau_\lambda} = \mathbb{P}^{x_0} \{\tau_\lambda < e_\delta\} = 1 - \mathbb{E}^{x_0} e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds}.$$

Further, by Corollary 4.3, we have

$$\mathbb{E}^{x_0} e^{-\delta \tau_\lambda} = \frac{\lambda}{\delta + \lambda} \frac{g_\delta^-(x_0)}{g_\delta^-(0)} \frac{\psi_{\delta+\lambda}^+(0)}{\psi_{\delta+\lambda}^+(0) + \psi_\delta^-(0)}. \quad (4.19)$$

Note that when  $\lambda \rightarrow \infty$  and  $\delta \rightarrow 0+$ , by (2.10) and  $\lim_{\lambda \rightarrow \infty} \psi_\lambda^+(\cdot) = \infty$ , relation (4.19) reduces to the traditional probability of bankruptcy.

**Remark 4.15.** *This risk model with random observations was first introduced by Albrecher et al. (2011a, 2013) for the compound Poisson risk process with independent Erlang( $n$ ) inter-observation times. Our model with exponential inter-arrival times is a special case with  $n = 1$ . Unfortunately, it is not apparent that the probability of bankruptcy of this risk model with Erlang( $n$ ) inter-arrival times as in Albrecher et al. (2011a, 2013) can be associated to the occupation times in a nice way due to the lack of memoryless property.*

#### 4.3.2 The Omega risk process

Suppose that the firm value process  $X$  follows a time-homogeneous diffusion process defined in (4.1) with initial value  $x_0 > 0$ . The so-called Omega model was first introduced by Albrecher et al. (2011b). We refer the reader to Gerber et al. (2012) and Albrecher and Loutscham (2013) for recent studies of this model. Actually, bankruptcy in this new model is not immediate even when the firm value is negative. To specify the probability of bankruptcy in this context, a bankruptcy rate function  $\omega(x) \geq 0$ ,  $x \leq 0$ , is introduced. When the firm value is at level  $x \leq 0$ ,  $\omega(x)dt$  is the probability of bankruptcy within  $dt$  time units. To be more precise, we define a Cox process  $N = \{N_t, t \geq 0\}$  with time-dependent intensity  $\omega(X_s)1_{\{X_s < 0\}}$ ,  $s \geq 0$ . We then denote the bankruptcy time by  $\tau_\omega$ , which is the first arrival time of  $N$ .

We can also connect the survival probability of the Omega model with the occupation time of the firm value process  $X$ . For any  $t > 0$ ,

$$\mathbb{P}^{x_0}\{\tau_\omega > t\} = \mathbb{P}^{x_0}\{N_t = 0\} = \mathbb{E}^{x_0}e^{-\int_0^t \omega(X_s)1_{\{X_s < 0\}} ds}.$$

Therefore, the Laplace transform of the bankruptcy time  $\tau_\omega$  is

$$\mathbf{E}^{x_0} e^{-\delta\tau_\omega} = \mathbf{P}^{x_0} \{\tau_\omega < e_\delta\} = 1 - \mathbf{E}^{x_0} e^{-\int_0^{e_\delta} \omega(X_s) 1_{\{X_s < 0\}} ds}. \quad (4.20)$$

For a constant bankruptcy rate  $\omega(x) \equiv \omega$ , the probability of bankruptcy of the Omega model coincides with the one of the random observation model, which was pointed out by Gerber et al. (2012). Therefore, by (4.19) and (4.20),

$$\mathbf{E}^{x_0} e^{-\delta\tau_\omega} = 1 - \mathbf{E}^{x_0} e^{-\omega \int_0^{e_\delta} 1_{\{X_s < 0\}} ds} = \frac{\omega}{\delta + \omega} \frac{g_\delta^-(x_0)}{g_\delta^-(0)} \frac{\psi_{\delta+\omega}^+(0)}{\psi_{\delta+\omega}^+(0) + \psi_\delta^-(0)}.$$

For a piecewise constant bankruptcy rate,

$$\omega(x) = \lambda_1 1_{\{a \leq x < 0\}} + \lambda_2 1_{\{-\infty < x < a\}}, \quad \text{for some } a < 0,$$

the corresponding Laplace transform (4.20) can be obtained by using Theorem 4.8.

## 4.4 Examples

We apply the results in Section 4.2 to two diffusion processes in order to obtain more explicit expressions and to compare them with known results.

### 4.4.1 Brownian motion with drift

Suppose that  $X$  is a Brownian motion with drift satisfying  $dX_t = \mu dt + dW_t$ ,  $t \geq 0$ . Equation (2.6) is reduced to

$$\frac{1}{2} g''(x) + \mu g'(x) = \lambda g(x), \quad \lambda > 0.$$

We have

$$g_\lambda^\pm(x) = e^{\beta_\lambda^\pm x}, \quad \psi_\lambda^\pm(\cdot) = \pm \beta_\lambda^\pm, \quad f_\lambda(x, y) = e^{\beta_\lambda^- x + \beta_\lambda^+ y} - e^{\beta_\lambda^- y + \beta_\lambda^+ x},$$

where  $\beta_\lambda^\pm = -\mu \pm \sqrt{\mu^2 + 2\lambda}$ . By Theorem 4.4, we obtain

$$\begin{aligned} & \mathbb{E}^0 \exp \left\{ -\lambda_1 \int_0^{e\delta} 1_{\{X_s < 0\}} ds - \lambda_2 \int_0^{e\delta} 1_{\{X_s > b\}} ds \right\} \\ &= \frac{C_1 e^{\beta_\delta^+ b} + C_2 e^{\beta_\delta^- b} + \frac{\lambda_2}{\delta + \lambda_2} (\beta_\delta^+ - \beta_\delta^-) \beta_{\delta + \lambda_2}^-}{D_1 e^{\beta_\delta^+ b} + D_2 e^{\beta_\delta^- b}}, \end{aligned}$$

where

$$\left\{ \begin{array}{l} C_1 = \frac{\delta}{\delta + \lambda_1} \beta_\delta^+ \beta_{\delta + \lambda_1}^+ + \beta_\delta^- \beta_{\delta + \lambda_2}^- - \frac{\delta}{\delta + \lambda_1} \beta_{\delta + \lambda_1}^+ \beta_{\delta + \lambda_2}^- + 2\delta, \\ C_2 = -\frac{\delta}{\delta + \lambda_1} \beta_\delta^- \beta_{\delta + \lambda_1}^+ - \beta_\delta^+ \beta_{\delta + \lambda_2}^- + \frac{\delta}{\delta + \lambda_1} \beta_{\delta + \lambda_1}^+ \beta_{\delta + \lambda_2}^- - 2\delta, \\ D_1 = \beta_\delta^+ \beta_{\delta + \lambda_1}^+ + \beta_\delta^- \beta_{\delta + \lambda_2}^- - \beta_{\delta + \lambda_1}^+ \beta_{\delta + \lambda_2}^- + 2\delta, \\ D_2 = -\beta_\delta^- \beta_{\delta + \lambda_1}^+ - \beta_\delta^+ \beta_{\delta + \lambda_2}^- + \beta_{\delta + \lambda_1}^+ \beta_{\delta + \lambda_2}^- - 2\delta. \end{array} \right.$$

By Theorem 4.8, we obtain

$$\begin{aligned} & \mathbb{E}^0 \exp \left\{ -\lambda_1 \int_0^{e\delta} 1_{\{a \leq X_s < 0\}} ds - \lambda_2 \int_0^{e\delta} 1_{\{X_s < a\}} ds \right\} \\ &= \frac{C_3 e^{\beta_{\delta + \lambda_1}^+ a} + C_4 e^{\beta_{\delta + \lambda_1}^- a} + \frac{\delta(\lambda_1 - \lambda_2)}{(\delta + \lambda_1)(\delta + \lambda_2)} (\beta_{\delta + \lambda_1}^+ - \beta_{\delta + \lambda_1}^-) \beta_{\delta + \lambda_2}^+}{D_3 e^{\beta_{\delta + \lambda_1}^+ a} + D_4 e^{\beta_{\delta + \lambda_1}^- a}}, \end{aligned}$$

where

$$\left\{ \begin{array}{l} C_3 = -\beta_\delta^- \beta_{\delta + \lambda_1}^+ + \beta_\delta^- \beta_{\delta + \lambda_2}^+ - \frac{\delta}{\delta + \lambda_1} \beta_{\delta + \lambda_1}^- \beta_{\delta + \lambda_2}^+ - 2\delta, \\ C_4 = \beta_\delta^- \beta_{\delta + \lambda_1}^- - \beta_\delta^- \beta_{\delta + \lambda_2}^+ + \frac{\delta}{\delta + \lambda_1} \beta_{\delta + \lambda_1}^+ \beta_{\delta + \lambda_2}^+ + 2\delta, \\ D_3 = -\beta_\delta^- \beta_{\delta + \lambda_1}^+ + \beta_\delta^- \beta_{\delta + \lambda_2}^+ - \beta_{\delta + \lambda_1}^- \beta_{\delta + \lambda_2}^+ - 2(\delta + \lambda_1), \\ D_4 = \beta_\delta^- \beta_{\delta + \lambda_1}^- - \beta_\delta^- \beta_{\delta + \lambda_2}^+ + \beta_{\delta + \lambda_1}^+ \beta_{\delta + \lambda_2}^+ + 2(\delta + \lambda_1). \end{array} \right.$$

Next we present several results on occupation times of  $X$  with relative simpler expressions. By Corollary 4.2, we have

$$\mathbb{E}^0 e^{-\lambda \int_0^\infty 1_{\{X_s < 0\}} ds} = \frac{2\mu}{\mu + \sqrt{\mu^2 + 2\lambda}}, \quad \text{if } \mu \geq 0$$

which agrees with formula 1.4.3 on Page 255 of Borodin and Salminen (2002). By

Corollary 4.3, we have

$$\mathbb{E}^x e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds} = \begin{cases} 1 - \frac{\lambda}{\delta + \lambda} \frac{\beta_{\delta+\lambda}^+}{\beta_{\delta+\lambda}^+ - \beta_\delta^-} e^{\beta_\delta^- x}, & \text{if } x \geq 0, \\ \frac{\delta}{\delta + \lambda} + \frac{\lambda}{\delta + \lambda} \frac{-\beta_\delta^-}{\beta_{\delta+\lambda}^+ - \beta_\delta^-} e^{\beta_{\delta+\lambda}^+ x}, & \text{if } x \leq 0, \end{cases}$$

which agrees with formula 1.4.1 on Page 255 of Borodin and Salminen (2002). By

Corollary 4.6,

$$\mathbb{E}^0 \exp \left\{ -\lambda_1 \int_0^{e_\delta} 1_{\{X_s < 0\}} ds - \lambda_2 \int_0^{e_\delta} 1_{\{X_s > 0\}} ds \right\} = \frac{2\delta}{-\beta_{\delta+\lambda_1}^- \beta_{\delta+\lambda_2}^+},$$

which agrees with formula 1.6.1 on Page 258 of Borodin and Salminen (2002). By

Corollary 4.7,

$$\begin{aligned} & \mathbb{E}^0 \left[ e^{-\lambda \int_0^{e_\delta} 1_{\{X_s < 0\}} ds}; e_\delta < T_b \right] \\ &= \frac{\left( \frac{\delta}{\delta + \lambda} \beta_{\delta+\lambda}^+ - \beta_\delta^- \right) e^{\beta_\delta^+ b} + \left( \beta_\delta^+ - \frac{\delta}{\delta + \lambda} \beta_{\delta+\lambda}^+ \right) e^{\beta_\delta^- b} - \beta_\delta^+ + \beta_\delta^-}{(\beta_{\delta+\lambda}^+ - \beta_\delta^-) e^{\beta_\delta^+ b} + (\beta_\delta^+ - \beta_{\delta+\lambda}^+) e^{\beta_\delta^- b}}. \end{aligned}$$

By Corollary 4.12,

$$\mathbb{E}^0 e^{-\lambda \int_0^\infty 1_{\{a \leq X_s < 0\}} ds} = \frac{\mu \beta_\lambda^+ e^{\beta_\lambda^+ a} - \mu \beta_\lambda^- e^{\beta_\lambda^- a}}{(\mu \beta_\lambda^+ - \lambda) e^{\beta_\lambda^+ a} - (\mu \beta_\lambda^- - \lambda) e^{\beta_\lambda^- a}}, \quad \text{if } \mu \geq 0,$$

which agrees with formula 1.7.3 on Page 262 of Borodin and Salminen (2002). By

Corollary 4.14,

$$\begin{aligned} & \mathbb{E}^0 \left[ e^{-\lambda \int_0^{e_\delta} 1_{\{a \leq X_s < 0\}} ds}; e_\delta < T_a \right] \\ &= \frac{\left( \beta_\delta^- - \frac{\delta}{\delta + \lambda} \beta_{\delta+\lambda}^- \right) e^{\beta_{\delta+\lambda}^+ a} + \left( \frac{\delta}{\delta + \lambda} \beta_{\delta+\lambda}^+ - \beta_\delta^- \right) e^{\beta_{\delta+\lambda}^- a} - \frac{\delta}{\delta + \lambda} (\beta_{\delta+\lambda}^+ - \beta_{\delta+\lambda}^-)}{(\beta_\delta^- - \beta_{\delta+\lambda}^-) e^{\beta_{\delta+\lambda}^+ a} + (\beta_{\delta+\lambda}^+ - \beta_\delta^-) e^{\beta_{\delta+\lambda}^- a}}, \end{aligned}$$

which agrees with formula 2.7.1(1) on Page 301 of Borodin and Salminen (2002).



## 4.4.2 Brownian motion with two-valued drift

Suppose that  $X$  is a Brownian motion with two-valued drift satisfying

$$dX_t = (\mu_L 1_{(-\infty, 0)}(X_t) - \mu_R 1_{(0, \infty)}(X_t)) dt + dW_t, \quad t \geq 0, \quad (4.21)$$

where  $\mu_L, \mu_R \in \mathbb{R}$ . Although the Lipschitz assumption (2.3) for the diffusion drift  $\mu(\cdot) = \mu_L 1_{(-\infty, 0)}(\cdot) - \mu_R 1_{(0, \infty)}(\cdot)$  fails, equation (4.21) still has a unique strong solution; see Stroock and Varadhan (1969) and Anulova et al. (1998). The diffusion process  $X$  is also of interest to risk theory; see, e.g., Asmussen and Taksar (1997) and Gerber and Shiu (2006). We refer to Simpson (2012) for a recent work on occupation times of  $X$ .

In this two-valued drift model, equation (2.6) reduces to

$$\frac{1}{2}g''(x) + (\mu_L 1_{(-\infty, 0)}(x) - \mu_R 1_{(0, \infty)}(x))g'(x) - \lambda g(x) = 0, \quad \lambda \geq 0.$$

We have

$$g_\lambda^-(x) = e^{(\mu_R - \sqrt{\mu_R^2 + 2\lambda})x} 1_{\{x > 0\}} + [c_- e^{(-\mu_L + \sqrt{\mu_L^2 + 2\lambda})x} + (1 - c_-) e^{(-\mu_L - \sqrt{\mu_L^2 + 2\lambda})x}] 1_{\{x < 0\}},$$

$$g_\lambda^+(x) = [c_+ e^{(\mu_R + \sqrt{\mu_R^2 + 2\lambda})x} + (1 - c_+) e^{(\mu_R - \sqrt{\mu_R^2 + 2\lambda})x}] 1_{\{x > 0\}} + e^{(-\mu_L + \sqrt{\mu_L^2 + 2\lambda})x} 1_{\{x < 0\}},$$

where

$$c_- = \frac{\mu_R - \sqrt{\mu_R^2 + 2\lambda} + \mu_L + \sqrt{\mu_L^2 + 2\lambda}}{2\sqrt{\mu_L^2 + 2\lambda}},$$

$$c_+ = \frac{-\mu_L + \sqrt{\mu_L^2 + 2\lambda} - \mu_R + \sqrt{\mu_R^2 + 2\lambda}}{2\sqrt{\mu_R^2 + 2\lambda}}.$$

It is easy to verify that  $g_\lambda^\pm(\cdot)$  are differentiable at 0. Hence, we have

$$\psi_\lambda^-(0) = -\mu_R + \sqrt{\mu_R^2 + 2\lambda} \quad \text{and} \quad \psi_\lambda^+(0) = -\mu_L + \sqrt{\mu_L^2 + 2\lambda}.$$

By Theorem 4.1, we obtain

$$\begin{aligned} \mathbb{E}^0 e^{-\lambda \int_0^{\varepsilon\delta} 1_{\{X_s < 0\}} ds} &= \frac{\frac{\delta}{\delta+\lambda} \left( -\mu_L + \sqrt{\mu_L^2 + 2\delta + 2\lambda} \right) - \mu_R + \sqrt{\mu_R^2 + 2\delta}}{-\mu_L + \sqrt{\mu_L^2 + 2\delta + 2\lambda} - \mu_R + \sqrt{\mu_R^2 + 2\delta}}, \\ \mathbb{E}^0 e^{-\lambda \int_0^{\varepsilon\delta} 1_{\{X_s > 0\}} ds} &= \frac{\frac{\delta}{\delta+\lambda} \left( -\mu_R + \sqrt{\mu_R^2 + 2\delta + 2\lambda} \right) - \mu_L + \sqrt{\mu_L^2 + 2\delta}}{-\mu_R + \sqrt{\mu_R^2 + 2\delta + 2\lambda} - \mu_L + \sqrt{\mu_L^2 + 2\delta}}. \end{aligned}$$

We have recovered equation (15) of Simpson (2012). Further, we can obtain the probability density functions for the occupation times  $\int_0^t 1_{\{X_s < 0\}} ds$  and  $\int_0^t 1_{\{X_s > 0\}} ds$  by inverting the corresponding Laplace transforms as in Simpson (2012).

In addition, by the explicit expressions for  $g_\lambda^\pm(\cdot)$  given above, we can easily derive all of the joint Laplace transforms in Theorems 4.4 and 4.8 for Brownian motion with two-valued drift.

#### 4.5 Summary and Some Remarks

We adopt a perturbation approach to solve the joint Laplace transforms of occupation times in multiple regions for time-homogeneous diffusion processes. This approach is in the spirit of excursion theory but can serve well as an alternative to other classical approaches. It can significantly simplify the calculation and be transformed to general Markov processes with or without jumps as long as the Laplace transforms of two-sided exit times are explicit.

Our results and approaches may also be applied to study the quantiles of stochastic processes and many other derivatives related to occupation times such as quantile options; see, e.g., Akahori (1995), Dassios (1995), Embrechts et al. (1995), and Linetsky (1999). These options can overcome many disadvantages of standard barrier options such as increased volatility around the barrier, discontinuous delta at

the barrier, short-term market manipulation, and so on.

Another possible extension of the study in this chapter will be the joint distributions of occupation times and other random times such as first passage times, last exit times, local times and Parisian times; see Pitman and Yor (2003). Relations between first passage times, inverse occupation times and Parisian times will be studied more systematically in Chapter 6.

**CHAPTER 5**  
**LIQUIDATION RISK IN THE PRESENCE OF CHAPTER 7 AND**  
**CHAPTER 11 OF THE U.S. BANKRUPTCY CODE**

We are interested in quantitatively measuring the liquidation risk of a firm subject to both Chapter 7 and Chapter 11 of the U.S. bankruptcy code. A firm liquidates if the firm value stays below a level over a prespecified grace period or it drops below an even lower level. Hence, the liquidation time is essentially the minimum of a first passage time and a Parisian time. First, we model the firm value by a time-homogeneous diffusion process in which the drift and volatility are level dependent and can be easily adjusted to reflect the state changes of the firm. An explicit formula for the probability of liquidation is established, based on which we gain a quantitative understanding of how the capital structures before and during bankruptcy affect the probability of liquidation. Then we incorporate jumps into the firm value process and derive an analytic formula as well for the probability of liquidation in terms of some well-known probabilities in risk theory. The result involves the regularity of those probabilities, which is a long-standing theoretical issue in the literature. We propose minimum conditions under which these probabilities are classical solutions of associated integro-differential equations. The content of this chapter is mainly based on the papers Li et al. (2012, 2013b).

## 5.1 Introduction

Stemming from Merton (1974) and Black and Cox (1976), numerous structural models have been proposed in which bankruptcy and liquidation are usually treated as the same event that the firm value reaches an absorbing low barrier.

In the real world, however, the procedures of bankruptcy and liquidation as described in the U.S. bankruptcy code are rather complicated. When a firm is unable to service its debt or pay its creditors but its fiscal situation is not severe, it is usually given the right to declare bankruptcy under Chapter 11 of reorganization rather than Chapter 7 of immediate liquidation. Chapter 11 allows the firm to remain in control of its business operations with a bankruptcy court providing oversight. The court grants the firm a certain observation period during which the firm manager can restructure its debt. The debtor usually proposes a plan of reorganization to keep its business alive and pay creditors over time. The reorganization plan may either succeed or fail. For the latter case, Chapter 11 will be converted to Chapter 7 governed by §1019 of the U.S. bankruptcy code and the firm may be forced to be liquidated. Warren and Westbrook (2009) showed empirical evidences for the conclusion that the Chapter 11 system offers a realistic hope for troubled businesses to turn around their operations and rebuild their capital structures. See also Hotchkiss (1995), Bris et al. (2006), and Denis and Rodgers (2007) for related empirical studies of the role of Chapter 11.

Under these practical considerations, many recent works in the literature of corporate finance have included Chapter 11 reorganization proceedings and made a distinction between bankruptcy and liquidation. In the works of Moraux (2004),

François and Morellec (2004), Galai et al. (2007), and Broadie and Kaya (2007), among others, a firm liquidates when the time its value cumulatively or constantly spending under the bankruptcy barrier exceeds a grace period granted by the bankruptcy court. It is interesting to notice that a similar trend has emerged in risk theory independently. Originating from the study of Parisian options, Parisian ruin was first introduced by Dassios and Wu (2008); see also Czarna and Palmowski (2011). Parisian ruin is essentially a reduced model of our definition of liquidation here. However, in these works only the bankruptcy barrier was considered and, hence, the firm is not necessarily liquidated even when its value is extremely low, which violates the principle of limited liability. Paseka (2003) and Broadie et al. (2007) explicitly incorporated both bankruptcy and liquidation barriers in their models. In this chapter we shall follow Broadie et al. (2007) to describe the intricate procedures of bankruptcy and liquidation.

The probability of liquidation is at the core of evaluation, pricing, and financial risk management. Our goal is to derive explicit formulas for the probability of liquidation in the new framework of Chapter 7 and Chapter 11. We follow Broadie et al. (2007) to describe the procedures of bankruptcy and liquidation by incorporating the Chapter 11 reorganization, the Chapter 7 liquidation, the conversion from Chapter 11 to Chapter 7, and the grace period in Chapter 11. Formally, suppose that the firm value is modeled by a general stochastic process  $X = \{X_t, t \geq 0\}$  starting with  $X_0 = x_0$ .

For a real number  $x$ , denote by  $T_x^+$  and  $T_x^-$  the first time when the process  $X$

up-crosses and down-crosses the level  $x$ , respectively; that is,

$$T_x^\pm = \inf \{t \geq 0 : X_t \gtrless x\}.$$

In particular, when  $X$  is a continuous stochastic process, the first hitting time

$$T_x = T_x^+ \vee T_x^- = \inf \{t \geq 0 : X_t = x\}.$$

Let  $a < b$  and  $c > 0$  be three endogenously determined constants, with  $a$  interpreted as the Chapter 7 liquidation barrier,  $b$  as the Chapter 11 reorganization barrier, and  $c$  as the duration of a grace period in Chapter 11 granted by the bankruptcy court. Let  $\tau_{b-}(c)$  be the first time when the process  $X$  has constantly stayed below level  $b$  for  $c$  units of time, namely,

$$\tau_{b-}(c) = \inf \{t \geq 0 : t - l_{b+}(t) \geq c\} \quad \text{with} \quad l_{b+}(t) = \sup \{s \leq t : X_s \geq b\}. \quad (5.1)$$

As in Section 2.1,  $\tau_{b-}(c)$  is called a Parisian time. Then the liquidation time is defined by

$$T_a^- \wedge \tau_{b-}(c). \quad (5.2)$$

We may also model the liquidation time by reduced-form approaches; see Chapters 8 and 9 of McNeil et al. (2005).

Under this setting three scenarios of liquidation are possible. In the first scenario, if the firm value follows a jump process, a firm may declare Chapter 7 directly if it suffers a catastrophic loss causing its value to drop from a level above the bankruptcy barrier  $b$  to a level below the liquidation barrier  $a$ . In the other two scenarios, the firm first declares Chapter 11 whenever its value drops to a level

between the bankruptcy barrier  $b$  and the liquidation barrier  $a$ . At this moment, an invisible distress clock starts to time. In the second scenario, the firm liquidates when the amount of time continuously spending in bankruptcy exceeds  $c$ , while in the third scenario, the firm liquidates when the firm value continues to drop below the liquidation barrier  $a$  prior to the end of the grace period.

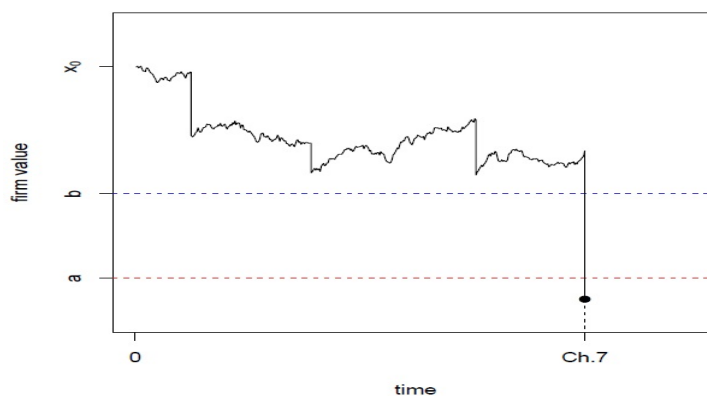


Figure 5.1: The first scenario of liquidation

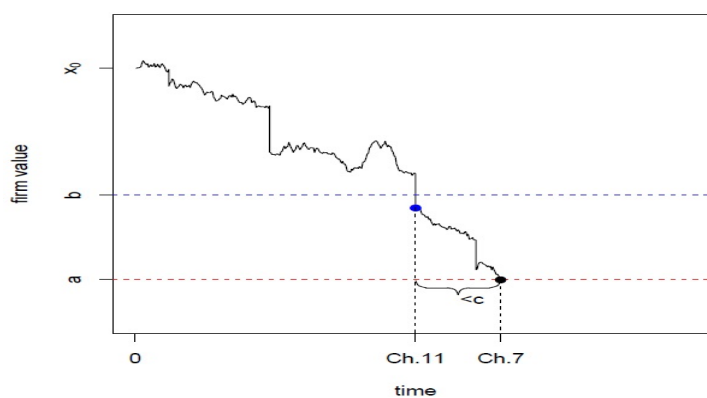


Figure 5.2: The second scenario of liquidation



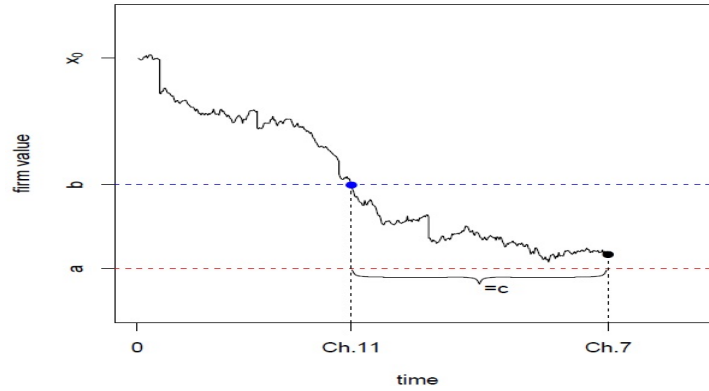


Figure 5.3: The third scenario of liquidation

The probability of liquidation subject to Chapter 7 and Chapter 11 with the liquidation barrier  $a$ , the reorganization barrier  $b$ , and the duration  $c$  of a grace period, or, in short, the probability of liquidation subject to the triple  $(a, b, c)$ , is defined by

$$q(x_0; a, b, c) = P^{x_0} \{T_a^- \wedge \tau_{b-}(c) < \infty\}, \quad a < b \leq x_0, c > 0. \quad (5.3)$$

Accordingly, the survival probability is defined by

$$p(x_0; a, b, c) = 1 - q(x_0; a, b, c) = P^{x_0} \{T_a^- \wedge \tau_{b-}(c) = \infty\}, \quad a < b \leq x_0, c > 0. \quad (5.4)$$

Some remarks on the probability of liquidation follow immediate. First, this probability of liquidation in the infinite-time horizon provides us with a quantitative understanding of the firm's liquidation risk in the long run. Second,  $q(x_0; a, b, c)$  is obviously monotone decreasing in  $c$ . Letting  $c \rightarrow 0$  in (5.3) yields

$$q(x_0; a, b, 0) = P^{x_0} \{T_b^- < \infty\}, \quad (5.5)$$

while letting  $c \rightarrow \infty$  yields

$$q(x_0; a, b, \infty) = \mathbb{P}^{x_0} \{T_a^- < \infty\}. \quad (5.6)$$

Hence, the duration  $c$  serves as a bridge connecting the two traditional probabilities of bankruptcy. Third, letting  $a \rightarrow -\infty$  in (5.3) yields

$$q(x_0; -\infty, b, c) = \mathbb{P}^{x_0} \{\tau_{b-}(c) < \infty\}. \quad (5.7)$$

This is essentially the probability of liquidation introduced by François and Morellec (2004) and Broadie and Kaya (2007) in corporate finance, or the Parisian ruin probability introduced by Dassios and Wu (2008), Czarna and Palmowski (2011), Landriault et al. (2013), and Loeffen et al. (2013) in risk theory.

Compared with the probability of single-barrier liquidation (5.7), the difficulty of the probability of double-barrier liquidation (5.3) exists in that not only the length but also the height of downward excursions need be investigated, causing some classical approaches of excursion theory not immediately applicable.

The rest of the chapter is arranged as follows. We focus on a diffusion structure model and a jump-diffusion structure model in Sections 5.2 and 5.3, respectively. A short summary and some remarks are given in Section 5.4. Hölder Spaces and Sobolev Spaces are briefly introduced as an appendix in Section 5.5.

## 5.2 A Diffusion Structure Model

In this section, we model the firm value by a general time-homogeneous diffusion process  $X = \{X_t, t \geq 0\}$ , defined on the filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq$

$0\}, P)$ , with dynamics

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad (5.8)$$

where  $X_0 = x_0$  is the initial value.

A merit of this way of modeling the firm value is that the drift and volatility are level dependent and can be easily adjusted to reflect the state changes of the firm. Our main result is given by (5.12) below. This formula involves an auxiliary quantity  $A$ , which is the boundary derivative of the solution of a partial differential equation (PDE) and, hence, can be easily determined numerically. Formula (5.12) is completely transparent and explicit except the quantity  $A$ . Through this formula we can gain a quantitative understanding about how the capital structures before and during bankruptcy affect the probability of liquidation.

Note that Gauthier (2002) calculated the Laplace transform of the minimum of a usual first passage time and a Parisian stopping time of a Brownian motion. Gauthier's result implies an analytic formula, nevertheless in a rather involved form, for the probability of liquidation in the current framework for the case with the firm value modeled by a Brownian motion. To the best of our knowledge, our formula (5.12) should be the first analytic formula for the probability of liquidation beyond the Brownian motion case. In addition, our approach, which combines excursion theory with PDE techniques, is very different from Gauthier's.

Solutions to the exit problems for the diffusion process are used intensively in our derivations. This study can be extended to other value processes, such as Lévy-driven models as in the next section or Markov regime-switching models, as long

as the corresponding two-sided exit problems allow mathematically nice solutions. Furthermore, we would like to remark that our result has immediate applications to pricing corporate bonds and other derivatives such as credit default swaps. In particular, in light of this work, it is anticipated that the recent study of Parisian options (see Chesney et al. (1997), Chesney and Gauthier (2006), Anderluh and van der Weide (2009), Dassios and Wu (2009, 2010, 2011), Landriault et al. (2011, 2012), and Albrecher et al. (2012), among others) can be extended to more elaborate options formulated in the current new framework. The reader is referred to Chapter 6 for the study of some new Barrier-Parisian options.

### 5.2.1 Formula for the probability of liquidation

Throughout Section 5.2, to avoid triviality we assume that

$$\int^{\infty} G(y)dy < \infty,$$

where the scale function  $G(y) = \exp\left\{-\int^y \frac{2\mu(z)}{\sigma^2(z)}dz\right\}$ . Thus, for  $u \leq x$ , the second relation in (2.14) yields

$$\mathbb{P}^x \{T_u = \infty\} = \frac{\int_u^x G(y)dy}{\int_u^{\infty} G(y)dy} \in (0, 1). \quad (5.9)$$

We slightly extend the domain of the probability of liquidation  $q(x; a, b, c)$  in (5.3) to  $x \wedge b > a$  and  $c > 0$ . It is sometimes more convenient to carry out our discussions on the corresponding survival probability defined by

$$p(x; a, b, c) = 1 - q(x; a, b, c) = \mathbb{P}^x \{T_a \wedge \tau_{b-}(c) = \infty\}. \quad (5.10)$$

Hereafter, we often drop the arguments  $a, b, c$  from  $p(x; a, b, c)$  and  $q(x; a, b, c)$  when-

ever we proceed with general discussions without emphasis on them. Furthermore, we introduce an auxiliary quantity

$$A(a, b, c) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}^{b-\varepsilon} \{T_b > T_a \wedge c\}}{\varepsilon}. \quad (5.11)$$

Intuitively,  $A(a, b, c)$  represents a mysterious force to prevent the process  $X$  from up-crossing level  $b$ . It will be proved later in Corollary 5.5 that  $A(a, b, c)$  exists, is finite, and equals the boundary derivative of the solution of a PDE. Hence, its value can be easily determined numerically.

Now we are ready to show our main result:

**Theorem 5.1.** *Let  $a < b \leq x_0$  and  $c > 0$ . Then*

$$q(x_0) = \frac{A(a, b, c)}{A(a, b, c) \int_b^\infty G(y) dy + G(b)} \int_{x_0}^\infty G(y) dy. \quad (5.12)$$

*Proof.* We start with the survival probability. For  $x \geq b$ , by the strong Markov property we have

$$\begin{aligned} p(x) &= \mathbb{P}^x \{T_a \wedge \tau_{b-}(c) = \infty, T_b = \infty\} + \mathbb{P}^x \{T_a \wedge \tau_{b-}(c) = \infty, T_b < \infty\} \\ &= \mathbb{P}^x \{T_b = \infty\} + \mathbb{E}^x [\mathbb{P}^x \{T_a \wedge \tau_{b-}(c) = \infty, T_b < \infty \mid \mathcal{F}_{T_b}\}] \\ &= \mathbb{P}^x \{T_b = \infty\} + \mathbb{P}^x \{T_b < \infty\} p(b). \end{aligned} \quad (5.13)$$

It follows that

$$\begin{aligned} p'_+(b) &= \lim_{\varepsilon \rightarrow 0} \frac{p(b + \varepsilon) - p(b)}{\varepsilon} \\ &= (1 - p(b)) \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}^{b+\varepsilon} \{T_b = \infty\}}{\varepsilon} \\ &= (1 - p(b)) \frac{G(b)}{\int_b^\infty G(y) dy}, \end{aligned} \quad (5.14)$$

where the last step is due to (5.9). Similarly as above, for  $x \in (a, b)$  we have

$$p(x) = \mathbb{P}^x \{T_a \wedge \tau_{b-}(c) = \infty\} = \mathbb{P}^x \{T_b \leq T_a \wedge c\} p(b).$$

By Corollary 5.5 below, the limit  $A(a, b, c)$  in (5.11) exists and is finite. It follows that

$$p'_-(b) = \lim_{\varepsilon \rightarrow 0} \frac{p(b) - p(b - \varepsilon)}{\varepsilon} = p(b) \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}^{b-\varepsilon} \{T_b > T_a \wedge c\}}{\varepsilon} = p(b)A(a, b, c). \quad (5.15)$$

By Theorem 5.10 below, the function  $p(\cdot)$  is differentiable at  $b$ . Thus, a combination of (5.14) and (5.15) gives

$$p(b) = \frac{G(b)}{A(a, b, c) \int_b^\infty G(y)dy + G(b)}. \quad (5.16)$$

Substituting (5.16) into (5.13) and using (5.9), we obtain

$$p(x) = \frac{\int_b^x G(y)dy}{\int_b^\infty G(y)dy} + \frac{\int_x^\infty G(y)dy}{\int_b^\infty G(y)dy} \times \frac{G(b)}{A(a, b, c) \int_b^\infty G(y)dy + G(b)}, \quad x \geq b.$$

Hence, relation (5.12) follows from  $q(x_0) = 1 - p(x_0)$ .  $\square$

**Remark 5.2.** *We can rewrite (5.12) slightly more explicitly as*

$$q(x_0) = \frac{A(a, b, c)}{A(a, b, c) \int_b^\infty \exp \left\{ - \int_b^y \frac{2\mu(z)}{\sigma^2(z)} dz \right\} dy + 1} \int_{x_0}^\infty \exp \left\{ - \int_b^y \frac{2\mu(z)}{\sigma^2(z)} dz \right\} dy.$$

*Also recall the definition of the auxiliary quantity  $A(a, b, c)$  in (5.11). We see that the two integrals above involve the values of  $\mu(\cdot)$  and  $\sigma(\cdot)$  over the range  $(b, \infty)$  only while the auxiliary quantity  $A(a, b, c)$  involves the values of  $\mu(\cdot)$  and  $\sigma(\cdot)$  over the range  $[a, b]$  only. This roughly describes how the capital structures before and during bankruptcy affect the probability of liquidation  $q(x_0)$  separately.*

Unfortunately, there seems to exist no simple expression for the auxiliary quantity  $A(a, b, c)$  even for a linear Brownian motion, for which case  $A(a, b, c)$  can be expressed as an infinite series by using formula 2.15.4(1) of Borodin and Salminen (2002). See also Lin (1998), who derived such series expressions for the density functions of hitting times of a (geometric) Brownian motion.

Nevertheless, the quantity

$$A(-\infty, b, c) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}^{b-\varepsilon} \{T_b > c\}}{\varepsilon}$$

has a simple expression for a few diffusion processes such as a Brownian motion with positive drift. More precisely, it is well known that the hitting time  $T_y$  at level  $y > 0$  of the Brownian motion  $X_t = \mu t + \sigma W_t$ ,  $\mu, \sigma, t > 0$ , follows an inverse Gaussian distribution with cumulative distribution function

$$\mathbb{P} \{T_y \leq c\} = \Phi \left( \frac{\mu c - y}{\sigma \sqrt{c}} \right) + \exp \left\{ \frac{2\mu y}{\sigma^2} \right\} \Phi \left( -\frac{\mu c + y}{\sigma \sqrt{c}} \right), \quad (5.17)$$

where  $\Phi(\cdot)$  is the standard normal cumulative distribution function; see, e.g., Chhikara and Folks (1989). By relations (5.12) and (5.17), it is easy to derive the following formulas for the corresponding Parisian ruin probabilities:

**Corollary 5.3.** *Let  $x_0 \geq b$ ,  $c > 0$ , and  $\rho = 2\mu/\sigma^2$ .*

(i) *If  $X$  is a Brownian motion with positive drift, i.e.,  $X$  is given by (5.8) with*

*$\mu(x) \equiv \mu > 0$  and  $\sigma(x) \equiv \sigma > 0$ , then*

$$\mathbb{P}^{x_0} \{\tau_{b-}(c) < \infty\} = \frac{\sqrt{2}e^{-\mu\rho c/4} - \sigma\rho\sqrt{\pi c}\Phi(-\sigma\rho\sqrt{c}/2)}{\sqrt{2}e^{-\mu\rho c/4} + \sigma\rho\sqrt{\pi c}\Phi(\sigma\rho\sqrt{c}/2)} e^{-\rho(x_0-b)}. \quad (5.18)$$

(ii) If  $X$  is a geometric Brownian motion, i.e.,  $X$  is given by (5.8) with  $\mu(x) = \mu x$  and  $\sigma(x) = \sigma x$  for some  $\sigma > 0$ , and if  $b > 0$  and  $\rho > 1$ , then

$$\begin{aligned} & \mathbb{P}^{x_0} \{ \tau_{b-}(c) < \infty \} \\ &= \frac{\sqrt{2}e^{-c\sigma^2(\rho-1)^2/8} - \sigma(\rho-1)\sqrt{\pi c}\Phi(-\sigma(\rho-1)\sqrt{c}/2)}{\sqrt{2}e^{-c\sigma^2(\rho-1)^2/8} + \sigma(\rho-1)\sqrt{\pi c}\Phi(\sigma(\rho-1)\sqrt{c}/2)} \left( \frac{b}{x_0} \right)^{\rho-1}. \end{aligned}$$

Formula (5.18) with  $b = 0$  retrieves equation (33) of Dassios and Wu (2008).

### 5.2.2 Proofs

Define a modified two-sided exit probability as

$$\phi(x; a, b, t) = \mathbb{P}^x \{ T_b \leq T_a \wedge t \}, \quad a < x < b, t \geq 0.$$

In particular,  $\phi(x; 0, \infty, \infty) = \mathbb{P}^x \{ T_0 = \infty \}$  for  $x > 0$  is the survival probability in infinite time. It is well known that, the two-sided exit probability in finite time is the solution of a parabolic PDE; see, e.g., Norberg (1999) for related discussions on this topic of diffusion processes. In the proof of Theorem 5.4 below, in order to give prominence to our main idea we ignore the regularity issue in applying Itô's formula. One can employ a standard approximation approach to pursue a complete proof; see Section 14 of Paulsen (1996) or the proof of Theorem 2.1 of Paulsen and Gjessing (1997) for this approach in the simple infinite-time case.

**Theorem 5.4.** *Suppose that  $h(x, t)$  solves the PDE*

$$h_t(x, t) = \mu(x)h_x(x, t) + \frac{1}{2}\sigma^2(x)h_{xx}(x, t), \quad a < x < b, t > 0, \quad (5.19)$$

*with the boundary conditions  $h(b, t) = 1$  and  $h(a, t) = 0$  for  $t \geq 0$  while  $h(x, 0) = 0$  for  $a < x < b$ . Then  $h(x, t) = \phi(x; a, b, t)$  for  $a \leq x \leq b$  and  $t \geq 0$ .*



*Proof.* For  $a < X_0 = x < b$  and  $0 \leq s \leq T_a \wedge T_b \wedge t$ , applying Itô's formula, equation (5.19) and noticing that  $h_s(X_s, t - s) = -h_t(X_s, t - s)$ , we have

$$\begin{aligned} dh(X_s, t - s) &= -h_t(X_s, t - s)ds + h_x(X_s, t - s)(\mu(X_s)ds + \sigma(X_s)dW_s) + \frac{1}{2}h_{xx}(X_s, t - s)\sigma^2(X_s)ds \\ &= h_x(X_s, t - s)\sigma(X_s)dW_s. \end{aligned}$$

Then  $h(X_s, t - s)$ ,  $0 \leq s \leq T_a \wedge T_b \wedge t$ , is a martingale, which implies that

$$h(x, t) = \mathbb{E}^x [h(X_{T_a \wedge T_b \wedge t}, t - T_a \wedge T_b \wedge t)].$$

By the boundary conditions of  $h(x, t)$ , we obtain

$$\begin{aligned} h(x, t) &= \mathbb{E}^x [h(X_{T_a \wedge T_b \wedge t}, t - T_a \wedge T_b \wedge t)] \\ &= \mathbb{E}^x [h(X_{T_a \wedge T_b}, t - T_a \wedge T_b)1_{\{T_a \wedge T_b \leq t\}}] + \mathbb{E}^x [h(X_t, 0)1_{\{T_a \wedge T_b > t\}}] \\ &= \mathbb{E}^x [h(a, t - T_a)1_{\{T_a = T_a \wedge T_b \leq t\}}] + \mathbb{E}^x [h(b, t - T_b)1_{\{T_b = T_a \wedge T_b \leq t\}}] \\ &= \mathbb{P}^x \{T_b \leq T_a \wedge t\} \\ &= \phi(x; a, b, t). \end{aligned}$$

This ends the proof of Theorem 5.4. □

A merit of Theorem 5.4 is that it not only reduces the existence problem of  $A(a, b, c)$  to that of the boundary derivative of a linear parabolic PDE, but also provides a way to compute  $A(a, b, c)$  numerically. The boundary estimate in Schauder theory of PDE confirms the existence of  $A(a, b, c)$ . Specifically, by using Theorem 4.22 of Lieberman (1996) or Theorem 2.1 of Wang (1992), we immediately have the following:

**Corollary 5.5.** *It holds for every fixed  $t > 0$  that the left derivative  $\phi_{x-}(b; a, b, t)$  is finite and continuous in  $t$ . In particular,*

$$A(a, b, c) = \lim_{\varepsilon \rightarrow 0} \frac{1 - \phi(b - \varepsilon; a, b, c)}{\varepsilon} = \phi_{x-}(b; a, b, c)$$

*exists and is finite. Moreover, we have*

$$\lim_{c \rightarrow 0} A(a, b, c) = \infty \quad \text{and} \quad \lim_{c \rightarrow \infty} A(a, b, c) = \frac{G(b)}{\int_a^b G(y) dy}. \quad (5.20)$$

Substituting (5.20) into (5.12) leads to explicit expressions for the two traditional probabilities of bankruptcy defined by (5.5) and (5.6), which are identical to the expressions obtained by directly applying (5.9).

The following local estimate is the key to the proof of the differentiability of  $p(\cdot)$  at  $b$ ; the same results for a linear Brownian motion can be derived from formulas (3.01) and (3.05) on Page 309 of Borodin and Salminen (2002):

**Lemma 5.6.** *For all  $x \in \mathbb{R}$ , as  $\varepsilon \rightarrow 0$ , we have*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}^x \{T_{x-\varepsilon} < T_{x+\varepsilon}\} = \frac{1}{2} \quad \text{and} \quad \mathbb{E}^x [T_{x+\varepsilon} \wedge T_{x-\varepsilon}] = o(\varepsilon). \quad (5.21)$$

*Proof.* The first relation in (5.21) follows from (2.13) naturally, as

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}^x \{T_{x-\varepsilon} < T_{x+\varepsilon}\} = \lim_{\varepsilon \rightarrow 0} \frac{\int_x^{x+\varepsilon} G(x) dx}{\int_{x-\varepsilon}^{x+\varepsilon} G(x) dx} = \lim_{\varepsilon \rightarrow 0} \frac{G(x + \varepsilon)}{G(x + \varepsilon) + G(x - \varepsilon)} = \frac{1}{2}.$$

Now we turn to the second relation in (5.21). By (2.8), we obtain

$$\begin{aligned} & \mathbb{E}^x [T_{x-\varepsilon}; T_{x-\varepsilon} < T_{x+\varepsilon}] \\ &= - \frac{\partial}{\partial \lambda} \mathbb{E}^x [e^{-\lambda T_{x-\varepsilon}}; T_{x-\varepsilon} < T_{x+\varepsilon}] \Big|_{\lambda=0} \\ &= \frac{f_0(x, x + \varepsilon) \frac{\partial}{\partial \lambda} f_\lambda(x - \varepsilon, x + \varepsilon) \Big|_{\lambda=0} - \frac{\partial}{\partial \lambda} f_\lambda(x, x + \varepsilon) \Big|_{\lambda=0} f_0(x - \varepsilon, x + \varepsilon)}{f_0^2(x - \varepsilon, x + \varepsilon; 0)}. \end{aligned}$$

Symmetrically,

$$\begin{aligned} & \mathbb{E}^x [T_{x+\varepsilon}; T_{x+\varepsilon} < T_{x-\varepsilon}] \\ &= \frac{f_0(x-\varepsilon, x) \frac{\partial}{\partial \lambda} f_\lambda(x-\varepsilon, x+\varepsilon)|_{\lambda=0} - \frac{\partial}{\partial \lambda} f_\lambda(x-\varepsilon, x)|_{\lambda=0} f_0(x-\varepsilon, x+\varepsilon)}{f_0^2(x-\varepsilon, x+\varepsilon)}. \end{aligned}$$

Combining these together and using the identities

$$f_0(x, x+\varepsilon) + f_0(x-\varepsilon, x) = \int_x^{x+\varepsilon} G(y)dy + \int_{x-\varepsilon}^x G(y)dy = f_0(x-\varepsilon, x+\varepsilon),$$

we obtain

$$\begin{aligned} \mathbb{E}^x [T_{x+\varepsilon} \wedge T_{x-\varepsilon}] &= \mathbb{E}^x [T_{x+\varepsilon}; T_{x+\varepsilon} < T_{x-\varepsilon}] + \mathbb{E}^x [T_{x-\varepsilon}; T_{x-\varepsilon} < T_{x+\varepsilon}] \\ &= \frac{\frac{\partial}{\partial \lambda} f_\lambda(x-\varepsilon, x+\varepsilon)|_{\lambda=0} - \frac{\partial}{\partial \lambda} f_\lambda(x-\varepsilon, x)|_{\lambda=0} - \frac{\partial}{\partial \lambda} f_\lambda(x, x+\varepsilon)|_{\lambda=0}}{f_0(x-\varepsilon, x+\varepsilon)}. \end{aligned} \tag{5.22}$$

Trivially, as  $\varepsilon \rightarrow 0$ , the denominator of (5.22) is asymptotically equivalent to  $2G(x)\varepsilon$ .

We use Taylor's expansion to expand each term in the numerator of (5.22) in  $\varepsilon$  up to the  $\varepsilon^2$  term. After cancellations, the numerator becomes

$$\varepsilon^2 \frac{\partial}{\partial \lambda} f_{12,\lambda}(x, x)|_{\lambda=0} + o(\varepsilon^2).$$

Luckily, the symmetric form of (2.7) implies that  $\frac{\partial}{\partial \lambda} f_{12,\lambda}(x, x)|_{\lambda=0} = 0$ . Thus, the second relation in (5.21) follows.  $\square$

**Remark 5.7.** *By the Lipschitz continuity of  $\mu(\cdot)$  and  $\sigma(\cdot)$  in (2.3) and Schauder interior estimate (see, e.g., Theorem 6.2 of Gilbarg and Trudinger (2001)), it can be shown that  $\frac{\partial}{\partial \lambda} g_\lambda^\pm(\cdot)|_{\lambda=0}$  belong to  $C^{2,\alpha}([-K, K])$  for arbitrarily fixed  $0 \leq \alpha < 1$  and  $K > 0$ . For details of Hölder space, the reader is referred to Section 4.1 of Gilbarg and*

Trudinger (2001). Therefore, in the proof of Lemma 5.6 we are allowed to expand each term in the numerator of (5.22) in  $\varepsilon$  up to the  $\varepsilon^2$  term without any stronger regularity assumptions on  $\mu(\cdot)$  and  $\sigma(\cdot)$ . Furthermore, if both  $\mu(\cdot)$  and  $\sigma(\cdot)$  belong to  $C^{1,\alpha}([x-K, x+K])$  for some  $x \in \mathbb{R}$ ,  $0 < \alpha < 1$ , and  $K > 0$ , then the second relation of (5.21) can be improved to

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E}^x [T_{x+\varepsilon} \wedge T_{x-\varepsilon}]}{\varepsilon^2} = C(x),$$

where  $C(x) > 0$  has an explicit expression.

The following two lemmas are in essence restatements of Lemmas 2.2 and 2.3 of Albrecher et al. (2012):

**Lemma 5.8.** *As  $\varepsilon \rightarrow 0$ , we have*

$$\mathbb{P}^b \{ \tau_{b-}(c) < T_{b-\varepsilon} \} = o(\varepsilon).$$

*Proof.* Let  $\theta_t$  denote the shift operator as usual. For  $X$  starting with  $b$ , define the consecutive down-crossing and up-crossing times of levels  $b$  and  $b + \varepsilon$  as follows:

$$\eta_0^- = 0, \quad \eta_0^+ = T_{b+\varepsilon}, \quad \eta_i^- = \eta_{i-1}^+ + T_b \circ \theta_{\eta_{i-1}^+}, \quad \eta_i^+ = \eta_i^- + T_{b+\varepsilon} \circ \theta_{\eta_i^-}, \quad i = 1, 2, \dots$$

Then, by the strong Markov property of  $X$ ,

$$\begin{aligned} \mathbb{P}^b \{ \tau_{b-}(c) < T_{b-\varepsilon} \} &= \sum_{i=0}^{\infty} \mathbb{P}^b \{ \eta_i^- < \tau_{b-}(c) < \eta_i^+ \wedge T_{b-\varepsilon} \} \\ &= \sum_{i=0}^{\infty} \mathbb{E}^b \left[ \mathbb{E}^b \left[ 1_{\{ \eta_i^- < \tau_{b-}(c) < \eta_i^+ \wedge T_{b-\varepsilon} \}} \middle| \mathcal{F}_{\eta_i^-} \right] \right] \\ &= \sum_{i=0}^{\infty} \mathbb{E}^b \left[ 1_{\{ \eta_i^- < \tau_{b-}(c) < T_{b-\varepsilon} \}} \mathbb{E}^b \left[ 1_{\{ \tau_{b-}(c) < \eta_i^+ \wedge T_{b-\varepsilon} \}} \middle| \mathcal{F}_{\eta_i^-} \right] \right] \\ &\leq \mathbb{P}^b \{ \tau_{b-}(c) < T_{b+\varepsilon} \wedge T_{b-\varepsilon} \} \sum_{i=0}^{\infty} \mathbb{P}^b \{ \eta_i^- < T_{b-\varepsilon} \}. \end{aligned}$$

By the strong Markov property and Lemma 5.6, it holds for all small  $\varepsilon > 0$  and all  $i = 0, 1, \dots$  that

$$\mathbb{P}^b \{ \eta_i^- < T_{b-\varepsilon} \} \leq (\mathbb{P}^b \{ T_{b+\varepsilon} < T_{b-\varepsilon} \})^i \leq \left( \frac{2}{3} \right)^i. \quad (5.23)$$

By inequality (5.23) and Lemma 5.6, it follows that, for all small  $\varepsilon > 0$ ,

$$\mathbb{P}^b \{ \tau_{b-}(c) < T_{b-\varepsilon} \} \leq 3\mathbb{P}^b \{ c < T_{b+\varepsilon} \wedge T_{b-\varepsilon} \} \leq \frac{3}{c} \mathbb{E}^b [T_{b+\varepsilon} \wedge T_{b-\varepsilon}] = o(\varepsilon).$$

This completes the proof.  $\square$

**Lemma 5.9.** *As  $\varepsilon \rightarrow 0$ , we have*

$$\mathbb{E}^b [T_{b-\varepsilon} - l_{b+}(T_{b-\varepsilon}); T_{b-\varepsilon} < \infty] = o(\varepsilon),$$

where  $l_{b+}(T_{b-\varepsilon}) = \sup \{ s \leq T_{b-\varepsilon} : X_s \geq b \}$ .

*Proof.* Following the same notation as in the proof of Lemma 5.8, we have

$$\begin{aligned} & \mathbb{E}^b [T_{b-\varepsilon} - l_{b+}(T_{b-\varepsilon}); T_{b-\varepsilon} < \infty] \\ &= \sum_{i=0}^{\infty} \mathbb{E}^b \left[ \mathbb{E}^b \left[ T_{b-\varepsilon} - l_{b+}(T_{b-\varepsilon}); \eta_i^- < T_{b-\varepsilon} < \eta_i^+ \mid \mathcal{F}_{\eta_i^-} \right] \right] \\ &= \sum_{i=0}^{\infty} \mathbb{E}^b \left[ 1_{\{ \eta_i^- < T_{b-\varepsilon} \}} \mathbb{E}^b \left[ T_{b-\varepsilon} - l_{b+}(T_{b-\varepsilon}); T_{b-\varepsilon} < \eta_i^+ \mid \mathcal{F}_{\eta_i^-} \right] \right] \\ &\leq \mathbb{E}^b [T_{b-\varepsilon}; T_{b-\varepsilon} < T_{b+\varepsilon}] \sum_{i=0}^{\infty} \mathbb{P}^b \{ \eta_i^- < T_{b-\varepsilon} \} \\ &\leq \mathbb{E}^b [T_{b-\varepsilon} \wedge T_{b+\varepsilon}] \sum_{i=0}^{\infty} \mathbb{P}^b \{ \eta_i^- < T_{b-\varepsilon} \} \\ &= o(\varepsilon), \end{aligned}$$

where the last step is due to inequality (5.23) and Lemma 5.6.  $\square$

Now we are ready to prove the differentiability eventually.

**Theorem 5.10.** *The function  $p(\cdot)$  defined by (5.10) is differentiable at  $b$ .*

*Proof.* By (5.14), it suffices to show that

$$\lim_{\varepsilon \rightarrow 0} \frac{p(b) - p(b - \varepsilon)}{\varepsilon} = (1 - p(b)) \frac{G(b)}{\int_b^\infty G(y) dy}. \quad (5.24)$$

For all small  $\varepsilon > 0$ , according to whether  $T_{b-\varepsilon}$  is finite or not we split  $p(b)$  into two parts as

$$\begin{aligned} p(b) &= \mathbb{P}^b \{ \tau_{b-}(c) = \infty, T_{b-\varepsilon} = \infty \} + \mathbb{P}^b \{ T_a \wedge \tau_{b-}(c) = \infty, T_{b-\varepsilon} < \infty \} \\ &= I_1 + I_2. \end{aligned} \quad (5.25)$$

By Lemma 5.8,

$$I_1 = \mathbb{P}^b \{ T_{b-\varepsilon} = \infty \} - \mathbb{P}^b \{ \tau_{b-}(c) < \infty, T_{b-\varepsilon} = \infty \} = \mathbb{P}^b \{ T_{b-\varepsilon} = \infty \} + o(\varepsilon). \quad (5.26)$$

On the other hand, it is easy to see that  $I_2$  is equal to

$$\begin{aligned} &I_{21} - I_{22} \\ &= \mathbb{P}^b \{ T_{b-\varepsilon} < \infty, (T_a \wedge \tau_{b-}(c)) \circ \theta_{T_{b-\varepsilon}} = \infty \} \\ &\quad - \mathbb{P}^b \{ T_{b-\varepsilon} < \infty, (T_a \wedge \tau_{b-}(c)) \circ \theta_{T_{b-\varepsilon}} = \infty, \tau_{b-}(c) \leq T_{b-\varepsilon} + T_b \circ \theta_{T_{b-\varepsilon}} \}. \end{aligned} \quad (5.27)$$

By conditioning on  $\mathcal{F}_{T_{b-\varepsilon}}$  and using the strong Markov property of  $X$ , we have

$$I_{21} = \mathbb{P}^b \{ T_{b-\varepsilon} < \infty \} p(b - \varepsilon). \quad (5.28)$$

The estimate for the term  $I_{22}$  is slightly more complex. First,

$$\begin{aligned}
I_{22} &\leq \mathbb{P}^b \{ T_{b-\varepsilon} < \infty, (T_a \wedge \tau_{b-}(c)) \circ \theta_{T_{b-\varepsilon}} = \infty, T_{b-\varepsilon} \leq \tau_{b-}(c) \leq T_{b-\varepsilon} + T_b \circ \theta_{T_{b-\varepsilon}} \} \\
&\quad + \mathbb{P}^b \{ \tau_{b-}(c) < T_{b-\varepsilon} \} \\
&\leq \mathbb{P}^b \{ T_{b-\varepsilon} < \infty, c - (T_{b-\varepsilon} - l_{b+}(T_{b-\varepsilon})) < T_b \circ \theta_{T_{b-\varepsilon}} \leq (T_a \circ \theta_{T_{b-\varepsilon}}) \wedge c \} p(b) + o(\varepsilon),
\end{aligned} \tag{5.29}$$

where in the last step the first term is due to the strong Markov property of  $X$  and the second term due to Lemma 5.8. For arbitrarily fixed  $0 < \delta < c$ , the probability before  $p(b)$  in (5.29) is further bounded by

$$\begin{aligned}
&\mathbb{P}^b \{ T_{b-\varepsilon} < \infty, T_{b-\varepsilon} - l_{b+}(T_{b-\varepsilon}) > \delta \} \\
&\quad + \mathbb{P}^b \{ T_{b-\varepsilon} < \infty, c - \delta < T_b \circ \theta_{T_{b-\varepsilon}} \leq (T_a \circ \theta_{T_{b-\varepsilon}}) \wedge c \} \\
&= o(\varepsilon) + \mathbb{E}^b \left[ \mathbb{P}^b \{ T_{b-\varepsilon} < \infty, T_a \wedge (c - \delta) < T_b \circ \theta_{T_{b-\varepsilon}} \leq (T_a \circ \theta_{T_{b-\varepsilon}}) \wedge c \mid \mathcal{F}_{T_{b-\varepsilon}} \} \right] \\
&= o(\varepsilon) + \mathbb{P}^b \{ T_{b-\varepsilon} < \infty \} \left( \mathbb{P}^{b-\varepsilon} \{ T_b > T_a \wedge (c - \delta) \} - \mathbb{P}^{b-\varepsilon} \{ T_b > T_a \wedge c \} \right), \tag{5.30}
\end{aligned}$$

where we used Lemma 5.9 to deal with the first term and used the strong Markov property of  $X$  to deal with the second term. By Corollary 5.5, we have that

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}^{b-\varepsilon} \{ T_b > T_a \wedge t \}}{\varepsilon} = \phi_{x-}(b; a, b, t) \quad \text{for } t = c \text{ and } c - \delta,$$

and that  $\phi_{x-}(b; a, b, t)$  is continuous at  $t = c$ . It follows that

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}^{b-\varepsilon} \{ T_b > T_a \wedge (c - \delta) \} - \mathbb{P}^{b-\varepsilon} \{ T_b > T_a \wedge c \}}{\varepsilon} = 0.$$

Substituting this into (5.30) and then substituting (5.30) into (5.29), we obtain

$$I_{22} = o(\varepsilon). \tag{5.31}$$

Substituting (5.28) and (5.31) into (5.27) yields

$$I_2 = \mathbb{P}^b \{T_{b-\varepsilon} < \infty\} p(b - \varepsilon) - o(\varepsilon). \quad (5.32)$$

Further substituting (5.26) and (5.32) into (5.25) yields

$$p(b) = \mathbb{P}^b \{T_{b-\varepsilon} = \infty\} + \mathbb{P}^b \{T_{b-\varepsilon} < \infty\} p(b - \varepsilon) - o(\varepsilon).$$

Therefore,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{p(b) - p(b - \varepsilon)}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}^b \{T_{b-\varepsilon} = \infty\} - \mathbb{P}^b \{T_{b-\varepsilon} = \infty\} p(b - \varepsilon)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} (1 - p(b - \varepsilon)) \frac{\mathbb{P}^b \{T_{b-\varepsilon} = \infty\}}{\varepsilon} \\ &= (1 - p(b)) \frac{G(b)}{\int_b^\infty G(y) dy}, \end{aligned}$$

where the last step is due to (5.9). This proves (5.24).  $\square$

### 5.2.3 Numerical examples

As mentioned above, the only implicit part in formula (5.12) is the auxiliary quantity  $A(a, b, c) = \phi_{x-}(b; a, b, c)$ . By Theorem 5.4 and Corollary 5.5,  $A(a, b, c)$  can be computed numerically via a PDE. We use the Crank–Nicolson method (see, e.g., Thomas (1995)) to solve  $A(a, b, c)$ . Usually, the Crank–Nicolson scheme is very accurate for small time steps. This is a second-order implicit finite difference method, which is unconditionally convergent and stable. The local error is of order  $O(\Delta x^2) + O(\Delta t^2)$ , implying that the error for  $A(a, b, c)$  is of order  $O(\Delta x) + O(\Delta t^2)$ .

The numerical experiments are carried out in a computer with an Intel Core i7-2600 CPU 3.40GHz 8GB RAM, and the software programs are run under Matlab R2011b.



First, we assume that the firm value follows a linear Brownian motion and that the capital structure remains unchanged during bankruptcy; that is,

$$dX_t = \mu dt + \sigma dW_t, \quad t \geq 0,$$

where  $X_0 = x_0 \geq b$  and  $\mu, \sigma > 0$ . By Theorem 5.1, the probability of liquidation is given by

$$q(x_0) = \frac{A(a, b, c)}{A(a, b, c) + \rho} e^{-\rho(x_0 - b)} \quad (5.33)$$

with  $\rho = 2\mu/\sigma^2$ . The parameters are set to  $\mu = 0.1$ ,  $\sigma = 0.25$ ,  $a = 0.1$ ,  $b = 0.2$ , and  $c = 1$ .

Table 5.1: The probability of liquidation for a linear Brownian motion without capital restructuring during bankruptcy

mesh	$A(a, b, c)$	$q(x_0)$	elapsed time (s)
$\Delta x = \Delta t = 0.005$	8.5534038	$1.3801420 \times e^{-3.2x_0}$	0.096548
$\Delta x = \Delta t = 0.001$	8.4987776	$1.3777311 \times e^{-3.2x_0}$	3.940972
$\Delta x = \Delta t = 0.0005$	8.4919795	$1.3774294 \times e^{-3.2x_0}$	32.536658
$\Delta x = \Delta t = 0.00025$	8.4885830	$1.3772786 \times e^{-3.2x_0}$	267.074039

By relation (5.9) and Corollary 5.3(i), the two traditional probabilities of bankruptcy and the Parisian ruin probability are equal to, respectively,

$$P^{x_0} \{T_a < \infty\} = 1.3771278 \times e^{-3.2x_0},$$

$$P^{x_0} \{T_b < \infty\} = 1.8964809 \times e^{-3.2x_0},$$

$$P^{x_0} \{\tau_{b-}(c) < \infty\} = 0.6932042 \times e^{-3.2x_0}.$$

The numerical results confirm the following facts:

$$\mathbb{P}^{x_0} \{T_a < \infty\} \vee \mathbb{P}^{x_0} \{\tau_{b-}(c) < \infty\} < q(x_0) < \mathbb{P}^{x_0} \{T_b < \infty\}. \quad (5.34)$$

Noticeably,  $q(x_0)$  is very close to  $\mathbb{P}^{x_0} \{T_a < \infty\}$ , which means that, in the current situation, liquidation happens mainly due to the firm value down-crossing level  $a$  rather than constantly staying below level  $b$ . In other words,  $c = 1$  has been set relatively large. Conversely, for a relatively small  $c > 0$ , it is anticipated that  $q(x_0)$  will be close to  $\mathbb{P}^{x_0} \{T_b < \infty\}$ . These are consistent with analyses in (5.6) and (5.5).

Next, we propose a reorganization plan during bankruptcy. Assume that

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t \geq 0,$$

where

$$\begin{aligned} \mu(x) &= \mu 1_{\{x > b\}} + \mu \left(1 - \frac{b-x}{2(b-a)}\right) 1_{\{a \leq x \leq b\}}, \\ \sigma(x) &= \sigma 1_{\{x > b\}} + \sigma \left(1 - \frac{b-x}{2(b-a)}\right) 1_{\{a \leq x \leq b\}}. \end{aligned}$$

This reorganization plan concerns the priority of the debt holder over the shareholders during bankruptcy by reducing  $\mu(\cdot)$  and  $\sigma(\cdot)$  over the range  $[a, b]$ . However, the Sharpe ratio  $\mu(\cdot)/\sigma(\cdot)$  remains constant. Thus, by Theorem 5.1 and Remark 5.2, formula (5.33) is still valid for  $q(x_0)$  whereas the auxiliary quantity  $A(a, b, c)$  needs to be recalculated according to the values of  $\mu(\cdot)$  and  $\sigma(\cdot)$  over the range  $[a, b]$ .

Table 5.2: The probability of liquidation for a linear Brownian motion with capital restructuring during bankruptcy

mesh	$A(a, b, c)$	$q(x_0)$	elapsed time (s)
$\Delta x = \Delta t = 0.005$	8.2173101	$1.3649425 \times e^{-3.2x_0}$	0.556764
$\Delta x = \Delta t = 0.001$	8.1639584	$1.3624470 \times e^{-3.2x_0}$	4.422223
$\Delta x = \Delta t = 0.0005$	8.1574008	$1.3621387 \times e^{-3.2x_0}$	34.587561
$\Delta x = \Delta t = 0.00025$	8.1541313	$1.3619848 \times e^{-3.2x_0}$	267.168596

Comparing Tables 5.1 and 5.2, we notice that this reorganization plan slightly reduces  $q(x_0)$ . However, we cannot draw a conclusion that capital restructuring according to the priority of the debt holder over the shareholders during bankruptcy always reduces the probability of liquidation. The following example illustrates the contrary.

Similar to the linear Brownian motion case, we first assume that the firm value follows a geometric Brownian motion and that the capital structure remains unchanged during bankruptcy; that is,

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad t > 0,$$

where  $X_0 = x_0 \geq b > 0$  and  $\rho = 2\mu/\sigma^2 > 1$ . By Theorem 5.1, the probability of liquidation is given by

$$q(x_0) = \frac{A(a, b, c)}{A(a, b, c) + (\rho - 1)/b} \left( \frac{b}{x_0} \right)^{\rho-1}. \quad (5.35)$$

The parameters are still set to  $\mu = 0.1$ ,  $\sigma = 0.25$ ,  $a = 0.1$ ,  $b = 0.2$ , and  $c = 1$ .

Table 5.3: The probability of liquidation for a geometric Brownian motion without capital restructuring during bankruptcy

mesh	$A(a, b, c)$	$q(x_0)$	elapsed time (s)
$\Delta x = \Delta t = 0.005$	11.495846	$0.014815100 \times x_0^{-2.2}$	0.460361
$\Delta x = \Delta t = 0.001$	11.145638	$0.014590922 \times x_0^{-2.2}$	4.362864
$\Delta x = \Delta t = 0.0005$	11.101517	$0.014562175 \times x_0^{-2.2}$	31.490827
$\Delta x = \Delta t = 0.00025$	11.079429	$0.014547740 \times x_0^{-2.2}$	276.483609

For this case, the two traditional probabilities of bankruptcy and the Parisian ruin probability are equal to, respectively,

$$P^{x_0} \{T_a < \infty\} = 0.0063095734 \times x_0^{-2.2},$$

$$P^{x_0} \{T_b < \infty\} = 0.028991187 \times x_0^{-2.2},$$

$$P^{x_0} \{\tau_{b-}(c) < \infty\} = 0.014533261 \times x_0^{-2.2}.$$

Again, the numerical results confirm the facts in (5.34). However,  $q(x_0)$  is now very close to  $P^{x_0} \{\tau_{b-}(c) < \infty\}$ , which means that, in the current situation, bankruptcy happens mainly due to the firm value constantly staying below level  $b$  rather than down-crossing level  $a$ .

Next, we propose a reorganization plan during bankruptcy. Assume that

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t > 0,$$

where

$$\begin{aligned}\mu(x) &= \mu x 1_{\{x>b\}} + \mu x \left(1 - \frac{b-x}{2(b-a)}\right) 1_{\{a \leq x \leq b\}}, \\ \sigma(x) &= \sigma x 1_{\{x>b\}} + \sigma x \left(1 - \frac{b-x}{2(b-a)}\right) 1_{\{a \leq x \leq b\}}.\end{aligned}$$

This reorganization plan also concerns the priority of the debt holder over the shareholders during bankruptcy while the Sharpe ratio  $\mu(\cdot)/\sigma(\cdot)$  remains constant. Thus, formula (5.35) is still valid for  $q(x_0)$  whereas the auxiliary quantity  $A(a, b, c)$  needs to be recalculated.

Table 5.4: The probability of liquidation for a geometric Brownian motion with capital restructuring during bankruptcy

mesh	$A(a, b, c)$	$q(x_0)$	elapsed time (s)
$\Delta x = \Delta t = 0.005$	12.348142	$0.015332581 \times x_0^{-2.2}$	0.567849
$\Delta x = \Delta t = 0.001$	11.970123	$0.015107802 \times x_0^{-2.2}$	4.432082
$\Delta x = \Delta t = 0.0005$	11.922681	$0.015079068 \times x_0^{-2.2}$	32.458421
$\Delta x = \Delta t = 0.00025$	11.898945	$0.015064648 \times x_0^{-2.2}$	258.392974

Comparing Tables 5.3 and 5.4, we notice that this reorganization plan slightly increases  $q(x_0)$ . Overall, our numerical results show that the effect of capital restructuring during bankruptcy on the probability of liquidation is ambiguous but intriguing.

### 5.3 A Jump-diffusion Structure Model

In this section, we are concerned with the liquidation risk of a firm whose value follows a jump-diffusion process  $X$ , defined on the filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$ , of the form

$$X_t = x_0 + \mu t - \sum_{i=1}^{N_t} Y_i + \sigma W_t, \quad t \geq 0. \quad (5.36)$$

Here  $N = \{N_t, t \geq 0\}$  is a homogeneous Poisson process with rate  $\lambda > 0$ ,  $\{Y_i, i = 1, 2, \dots\}$  is a sequence of positive, independent and identically distributed (i.i.d.) random variables with common distribution function  $F(\cdot)$  and finite mean  $m$ ,  $W = \{W_t, t \geq 0\}$  is a standard Brownian motion, and  $x_0, \mu > 0$  and  $\sigma > 0$  are constants. As usual, the summation over an empty set produces value zero. Throughout Section 5.3, we assume that  $N, \{Y_i, i = 1, 2, \dots\}$  and  $W$  are mutually independent. We also assume the safety loading condition  $\mu > \lambda m$ .

In risk theory, the stochastic process described by (5.36) has often been used to model the firm value process of an insurance company, where  $N_t$  represents the number of claims up to time  $t$ ,  $\{Y_i, i = 1, 2, \dots\}$  represents a sequence of claim sizes, and  $\mu$  represents the premium rate. See Dufresne and Gerber (1991), Gjessing and Paulsen (1997), and Gerber and Landry (1998), among others. While this application serves as our primary motivation, we carry out our study in a general sense and interpret the stochastic process (5.36) as a general firm value.

We aim to obtain an analytic formula for the probability of liquidation (5.3) in terms of some well-known probabilities in risk theory. The result involves the regularity of these probabilities, which is a long-standing theoretical issue in the

literature. To tackle this issue, we employ regularity theory of PDEs and propose minimum conditions under which these probabilities become classical solutions of associated integro-differential equations. Our approach is effective for a broader class of value processes in insurance and finance.

### 5.3.1 Formula for the probability of liquidation

Our first goal is to derive a formula for the probability of liquidation  $q(x_0)$  for  $x_0 \geq b$ . For this purpose, we introduce

$$\left\{ \begin{array}{ll} \psi(x; b) = \mathbb{P}^x \{T_b^- < \infty\}, & x \geq b, \\ \psi^d(x; b) = \mathbb{P}^x \{T_b^- < \infty, X_{T_b^-} = b\}, & x \geq b, \\ \psi^s(x, y; b) = \mathbb{P}^x \{T_b^- < \infty, X_{T_b^-} < y\}, & y \leq b \leq x, \\ \phi(x; a, b, c) = \mathbb{P}^x \{T_b^+ < T_a^- \wedge c\}, & a \leq x \leq b, c > 0, \end{array} \right. \quad (5.37)$$

where  $\psi(x; b)$  denotes the traditional probability of bankruptcy,  $\psi^d(x; b)$  denotes the probability of bankruptcy due to perturbation,  $\psi^s(x, y; b)$  denotes the distribution function of the deficit at bankruptcy, and  $\phi(x; a, b, c)$  denotes a modified two-sided exit probability. The first three probabilities have been extensively studied; see, e.g., Dufresne and Gerber (1991), Gerber and Shiu (1997), Wang and Wu (2000, 2001), Cai and Yang (2005), and Cai and Xu (2006), among others.

Now we are ready to show the main formula and a sketch of its proof. As usual, we define  $\bar{F}(\cdot) = 1 - F(\cdot)$ .

**Theorem 5.11.** *Suppose that  $F(\cdot) \in C[0, \infty)$ . For  $a < b \leq x_0$  and  $c > 0$ , we have*

$$q(x_0) = \psi(x_0; b) - p(b)\psi^d(x_0; b) - p(b) \int_{(a,b)} \phi(y; a, b, c)\psi^s(x_0, dy; b) \quad (5.38)$$

and

$$p(b) = \frac{2\mu - 2\lambda m}{2\mu + \sigma^2 \phi_{x-}(b; a, b, c) - 2\lambda \int_{(a,b)} \phi(y; a, b, c) \bar{F}(b-y) dy}. \quad (5.39)$$

*Proof.* First we prove relation (5.38). For  $a \leq x \leq b$ , by (5.4) and the strong Markov property of  $X$ ,

$$p(x) = \mathbb{P}^x \{T_a^- \wedge \tau_{b-}(c) = \infty\} = \mathbb{P}^x \{T_b^+ < T_a^- \wedge c\} p(b) = \phi(x; a, b, c) p(b). \quad (5.40)$$

Similarly, for  $x \geq b$  we have

$$\begin{aligned} p(x) &= \mathbb{P}^x \{T_a^- \wedge \tau_{b-}(c) = \infty\} \\ &= \mathbb{P}^x \{T_b^- = \infty\} + \mathbb{P}^x \{T_a^- \wedge \tau_{b-}(c) = \infty, T_b^- < \infty\} \\ &= \mathbb{P}^x \{T_b^- = \infty\} + \mathbb{P}^x \{T_b^- < \infty, X_{T_b^-} = b\} p(b) \\ &\quad + \int_{(a,b)} p(y) \mathbb{P}^x \{T_b^- < \infty, X_{T_b^-} \in dy\} \\ &= 1 - \psi(x; b) + p(b) \psi^d(x; b) + p(b) \int_{(a,b)} \phi(y; a, b, c) \psi^s(x, dy; b), \end{aligned}$$

where the last step is due to (5.40). Then relation (5.38) follows immediately by taking  $x = x_0$  in (5.13).

Next we prove relation (5.39). By (5.40) and Theorem 5.16, the left-hand derivative of  $p(\cdot)$  at  $b$  exists and is equal to

$$\lim_{\varepsilon \rightarrow 0^+} \frac{p(b) - p(b - \varepsilon)}{\varepsilon} = p(b) \lim_{\varepsilon \rightarrow 0^+} \frac{1 - \phi(b - \varepsilon; a, b, c)}{\varepsilon} = p(b) \phi_{x-}(b; a, b, c). \quad (5.41)$$

By (5.13), Theorem 5.16, Lemma 5.19 and Corollary 5.20, the right-hand derivative



of  $p(\cdot)$  at  $b$  exists and is equal to

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0^+} \frac{p(b + \varepsilon) - p(b)}{\varepsilon} &= - \lim_{\varepsilon \rightarrow 0^+} \frac{\psi(b + \varepsilon; b) - 1}{\varepsilon} + p(b) \lim_{\varepsilon \rightarrow 0^+} \frac{\psi^d(b + \varepsilon; b) - 1}{\varepsilon} \\
&\quad + p(b) \int_{(a,b)} \phi(y; a, b, c) \lim_{\varepsilon \rightarrow 0^+} \frac{\psi^s(b + \varepsilon, dy; b)}{\varepsilon} \\
&= \frac{2\mu - 2\lambda m}{\sigma^2} - \frac{2\mu}{\sigma^2} p(b) + \frac{2\lambda}{\sigma^2} p(b) \int_{(a,b)} \phi(y; a, b, c) \bar{F}(b - y) dy.
\end{aligned} \tag{5.42}$$

By Theorem 5.24,  $p(\cdot)$  is differentiable at  $b$ . Thus, simply equating the right-hand sides of (5.41) and (5.42) yields (5.39), as desired.  $\square$

Note that there are two regularity issues involved in the proof of Theorem 5.11. The first issue is the one-sided differentiability of  $p(\cdot)$  at  $b$ , which will be handled in Section 5.3.2. We propose minimum conditions on the distribution function  $F(\cdot)$  under which those traditional probabilities defined in (5.37) are classical solutions of corresponding integro-differential equations. The second issue is the two-sided differentiability of  $p(\cdot)$  at  $b$ , which will be handled in Section 5.3.3.

For three important special cases of the jump-diffusion model (5.36), the compound Poisson model, the Brownian motion with drift and the jump-diffusion with exponential jumps, our main formula in Theorem 5.11 admits more explicit expressions.

**Corollary 5.12.** *In Theorem 5.11, suppose that  $\sigma = 0$ . Then*

$$q(x_0) = \psi(x_0; b) - \frac{\mu - \lambda m}{\mu - \lambda \int_{(a,b)} \phi(y; a, b, c) \bar{F}(b - y) dy} \int_{(a,b)} \phi(y; a, b, c) \psi^s(x_0, dy; b).$$

*Proof.* Since there is no diffusion term,  $\psi^d(\cdot; b) \equiv 0$ . Similar to (5.13), for  $x \geq b$ , we have

$$p(x) = 1 - \psi(x; b) + p(b) \int_{(a,b)} \phi(y; a, b, c) \psi^s(x, dy; b). \quad (5.43)$$

In particular, letting  $x = b$ , we obtain

$$p(b) = \frac{1 - \psi(b; b)}{1 - \int_{(a,b)} \phi(y; a, b, c) \psi^s(b, dy; b)} = \frac{\mu - \lambda m}{\mu - \lambda \int_{(a,b)} \phi(y; a, b, c) \bar{F}(b - y) dy},$$

where the last equality is due to equation (I) of Grandell (1991) and equation (4) of Gerber et al. (1987). The result then follows immediately by substituting the above expression of  $p(b)$  into (5.43).  $\square$

Notice that we can obtain the same result of Corollary 5.12 by plugging  $\sigma = 0$  in (5.38) and (5.39).

**Corollary 5.13.** *In Theorem 5.11, suppose that  $\lambda = 0$ . Then*

$$q(x_0) = \frac{\phi_{x^-}(b; a, b, c)}{2\mu/\sigma^2 + \phi_{x^-}(b; a, b, c)} e^{-2\mu(x_0-b)/\sigma^2}.$$

*Proof.* For this case,  $\psi(x_0; b) = \psi^d(x_0; b) = e^{-2\mu(x_0-b)/\sigma^2}$  and  $\psi^s(\cdot, y; b) \equiv 0$ . Then the result follows immediately by going through the same proof of Theorem 5.11 with  $\lambda = 0$ .  $\square$

Corollary 5.13 agrees with Theorem 5.1 with  $\mu(\cdot) \equiv \mu$  and  $\sigma(\cdot) \equiv \sigma$ .

Now we consider the third special case in which the jump sizes follow an

exponential distribution with mean  $m > 0$ , i.e.,  $F(x) = 1 - e^{-x/m}$ . Define

$$\left\{ \begin{array}{l} \beta_{\pm} = \frac{\frac{\sigma^2}{2m} + \mu \pm \sqrt{\left(\frac{\sigma^2}{2m} - \mu\right)^2 + 2\lambda\sigma^2}}{\sigma^2}, \\ c_1 = \frac{\sigma^2\beta_+^2 - 2\mu\beta_+}{\sigma^2(\beta_+^2 - \beta_-^2) - 2\mu(\beta_+ - \beta_-)}, \\ c_2 = \frac{\sigma^2\beta_+^2 - 2\mu\beta_+ - 2\lambda}{\sigma^2(\beta_+^2 - \beta_-^2) - 2\mu(\beta_+ - \beta_-)}, \\ c_3(y) = \frac{2\lambda e^{-(b-y)/m}}{\sigma^2(\beta_+^2 - \beta_-^2) - 2\mu(\beta_+ - \beta_-)}, \quad y < b. \end{array} \right.$$

**Corollary 5.14.** *In Theorem 5.11, suppose that  $F(\cdot)$  is an exponential distribution with mean  $m > 0$ . Then*

$$q(x_0) = A_1 e^{-\beta_-(x_0-b)} + A_2 e^{-\beta_+(x_0-b)} \quad (5.44)$$

with

$$A_1 = c_1 - \frac{(2\mu - 2\lambda m) \left( c_2 + \frac{1}{m} \int_{(a,b)} \phi(y; a, b, c) c_3(y) dy \right)}{2\mu + \sigma^2 \phi_{x-}(b; a, b, c) - 2\lambda \int_{(a,b)} \phi(y; a, b, c) e^{-(b-y)/m} dy},$$

$$A_2 = 1 - c_1 - \frac{(2\mu - 2\lambda m) \left( 1 - c_2 - \frac{1}{m} \int_{(a,b)} \phi(y; a, b, c) c_3(y) dy \right)}{2\mu + \sigma^2 \phi_{x-}(b; a, b, c) - 2\lambda \int_{(a,b)} \phi(y; a, b, c) e^{-(b-y)/m} dy}.$$

*Proof.* For this case, the probabilities  $\psi(x; b)$ ,  $\psi^d(x; b)$  and  $\psi^s(x, y; b)$  can be solved explicitly from the integro-differential equations (5.46)–(5.48). We have

$$\left\{ \begin{array}{l} \psi(x; b) = c_1 e^{-\beta_-(x-b)} + (1 - c_1) e^{-\beta_+(x-b)}, \quad x > b, \\ \psi^d(x; b) = c_2 e^{-\beta_-(x-b)} + (1 - c_2) e^{-\beta_+(x-b)}, \quad x > b, \\ \psi^s(x, y; b) = c_3(y) e^{-\beta_-(x-b)} - c_3(y) e^{-\beta_+(x-b)}, \quad y < b < x. \end{array} \right.$$

The result follows immediately by plugging these expressions in (5.38).  $\square$

Notice that the only implicit term in (5.44) is  $\phi(y; a, b, c)$ ,  $a \leq y \leq b$ . If we further assume that  $\sigma = 0$  in Corollary 5.14, by equation (II) of Grandell (1991) and

the memoryless property, we have

$$\begin{cases} \psi(x; b) = \frac{\lambda m}{\mu} e^{(\frac{\lambda}{\mu} - \frac{1}{m})(x-b)}, & x > b, \\ \psi^s(x, y; b) = \frac{\lambda m}{\mu} e^{-\frac{b-y}{m} + (\frac{\lambda}{\mu} - \frac{1}{m})(x-b)}, & y < b < x. \end{cases}$$

Hence, by Corollary 5.12, it follows that

$$q(x_0) = \frac{\lambda m - \lambda \int_{(a,b)} \phi(y; a, b, c) e^{-(b-y)/m} dy}{\mu - \lambda \int_{(a,b)} \phi(y; a, b, c) e^{-(b-y)/m} dy} e^{(\frac{\lambda}{\mu} - \frac{1}{m})(x_0-b)}. \quad (5.45)$$

When  $a = -\infty$  and  $b = 0$ , the relation above retrieves Corollary 2.1 of Dassios and Wu (2008), with  $P_{21}(c) = \int_{-\infty}^0 \phi(y; -\infty, 0, c) \frac{1}{m} e^{y/m} dy$  representing the probability that the first excursion of the process  $X$  below level 0 is shorter than  $c$  units of time.

Notice that we can obtain the same formula (5.45) by letting  $\sigma \rightarrow 0$  in (5.44) since

$$\beta_- \rightarrow \frac{1}{m} - \frac{\lambda}{\mu}, \quad \beta_+ \rightarrow \infty, \quad c_1 \rightarrow \frac{\lambda m}{\mu}, \quad c_2 \rightarrow 0, \quad c_3(y) \rightarrow \frac{\lambda m}{\mu} e^{-(b-y)/m} \quad \text{for } y < b.$$

### 5.3.2 Regularity of some well-known bankruptcy related probabilities

In the first part of this subsection, we employ regularity theory of PDE to propose minimum conditions under which  $\psi(x; b)$ ,  $\psi^d(x; b)$ ,  $\psi^s(x, y; b)$  and  $\phi(x; a, b, c)$  are classical solutions of associated integro-differential equations. We would like to point out that this argument is effective for much more general firm value processes such as an insurance process compounded by an investment portfolio. In the second part, we derive the boundary derivatives of  $\psi(\cdot; b)$ ,  $\psi^d(\cdot; b)$  and  $\psi^s(\cdot, y; b)$ , which were used in the proof of Theorem 5.11.

It is well known that the infinitesimal generator of the jump-diffusion process (5.36) is given by

$$\mathcal{L}u(x) = \frac{\sigma^2}{2} u_{xx}(x) + \mu u_x(x) + \lambda \int_0^\infty (u(x-y) - u(x)) dF(y).$$

In the literature of risk theory, the probabilities  $\psi(\cdot; b)$ ,  $\psi^d(\cdot; b)$  and  $\psi^s(\cdot, y; b)$  defined in (5.37) are extensively studied by means of integro-differential equations. Formally, assuming twice continuous differentiability in  $[b, \infty)$ , they satisfy, respectively,

$$\frac{\sigma^2}{2}\psi_{xx}(x; b) + \mu\psi_x(x; b) = \lambda\psi(x; b) - \lambda \int_0^{x-b} \psi(x-y; b)dF(y) - \lambda\bar{F}(x-b) \quad (5.46)$$

with boundary conditions  $\psi(b; b) = 1$  and  $\psi(\infty; b) = 0$ ,

$$\frac{\sigma^2}{2}\psi_{xx}^d(x; b) + \mu\psi_x^d(x; b) = \lambda\psi^d(x; b) - \lambda \int_0^{x-b} \psi^d(x-y; b)dF(y) \quad (5.47)$$

with boundary conditions  $\psi^d(b; b) = 1$  and  $\psi^d(\infty; b) = 0$ , and

$$\frac{\sigma^2}{2}\psi_{xx}^s(x, y; b) + \mu\psi_x^s(x, y; b) = \lambda\psi^s(x, y; b) - \lambda \int_0^{x-b} \psi^s(x-z, y; b)dF(z) - \lambda\bar{F}(x-y) \quad (5.48)$$

with boundary conditions  $\psi^s(b, y; b) = 0$  and  $\psi^s(\infty, y; b) = 0$ . Conversely, if the integro-differential equations (5.46), (5.47) and (5.48) admit twice continuously differentiable solutions in  $[b, \infty)$ , then the solutions equal  $\psi(\cdot; b)$ ,  $\psi^d(\cdot; b)$  and  $\psi^s(\cdot, y; b)$  in the region  $[b, \infty)$ , respectively.

In the following lemma, we derive a similar PDE for the two-sided exit probability  $\phi(x; a, b, c)$  defined in (5.37). To apply Itô's formula in a formal way, we use a standard approximation approach; see a similar idea in detail in Section 14 of Paulsen (1996).

**Lemma 5.15.** *Suppose that  $h(x, t)$  is continuously differentiable in  $t \in (0, \infty)$  and twice continuously differentiable in  $x \in [a, b]$  (where and throughout the chapter, we mean the right-hand derivative at  $x = a$  and the left-hand derivative at  $x = b$ ), and*

that it solves the PDE

$$h_t(x, t) = \frac{\sigma^2}{2} h_{xx}(x, t) + \mu h_x(x, t) - \lambda h(x, t) + \lambda \int_0^{x-a} h(x-y, t) dF(y) \quad (5.49)$$

for  $a < x < b, t > 0$ , with boundary conditions

$$\begin{cases} h(x, t) = 1 & \text{for } x \geq b, t > 0, \\ h(x, t) = 0 & \text{for } x \leq a, t > 0, \\ h(x, 0) = 0 & \text{for } a \leq x \leq b. \end{cases}$$

Then  $h(x, t) = \phi(x; a, b, t)$  for  $(x, t) \in [a, b] \times [0, \infty)$ .

*Proof.* By the boundary conditions, it suffices to prove that  $h(x, t) = \phi(x; a, b, t)$  for  $(x, t) \in (a, b) \times (0, \infty)$ . Fix  $(x, t) \in (a, b) \times (0, \infty)$  and define  $t_n = T_a^- \wedge T_b^+ \wedge (t - 1/n)$  for large  $n \in \mathbb{N}$ . Construct a sequence of approximations  $h^n(\cdot, \cdot) \in C^{2,1}((-\infty, \infty) \times [1/n, t])$ , such that

$$h^n(\cdot, \cdot) = h(\cdot, \cdot) \quad \text{on } ((-\infty, a - 1/n] \cup [a, b] \cup [b + 1/n, \infty)) \times [1/n, t];$$

see Section 5.5.2 below for a strict definition of the  $C^{2,1}$  space. By Itô's formula,  $E^x [h^n(X_{t_n}, t - t_n)] = h^n(x, t)$ . By the bounded convergence theorem, letting  $n \rightarrow \infty$  gives

$$E^x \left[ h(X_{T_a^- \wedge T_b^+ \wedge t}, t - T_a^- \wedge T_b^+ \wedge t) \right] = h(x, t).$$

By the boundary conditions, it follows that

$$\begin{aligned} h(x, t) &= E^x \left[ h(X_{T_a^- \wedge T_b^+}, t - T_a^- \wedge T_b^+); T_a^- \wedge T_b^+ < t \right] + E^x \left[ h(X_t, 0); T_a^- \wedge T_b^+ \geq t \right] \\ &= P^x \{ T_b^+ < T_a^- \wedge t \} \\ &= \phi(x; a, b, t). \end{aligned}$$

This ends the proof.  $\square$

Moreover, it is also easy to show that if  $\phi(x; a, b, t)$  is continuously differentiable in  $t \in (0, \infty)$  and twice continuously differentiable in  $x \in [a, b]$ , then  $\phi(x; a, b, t) = h(x, t)$  for  $(x, t) \in [a, b] \times [0, \infty)$ . Conclusively, in the classical sense, the solutions of the integro-differential equations (5.46)–(5.49) and the probabilities  $\psi(\cdot; b)$ ,  $\psi^d(\cdot; b)$ ,  $\psi^s(\cdot, y; b)$  and  $\phi(\cdot; a, b, \cdot)$  are respectively equivalent to each other.

We are going to summarize conditions for regularities of these four probabilities. The reader is referred to Section 2.2 for a brief introduction of Hölder spaces and Sobolev spaces.

**Theorem 5.16.** *Recall the functions  $\psi(x; b)$ ,  $\psi^d(x; b)$ ,  $\psi^s(x, y; b)$  and  $\phi(x; a, b, t)$  defined by (5.37). We have*

1)  $\phi(\cdot; a, b, \cdot) \in C^{2,1}([a, b] \times [\varepsilon, T])$  for any  $0 < \varepsilon < T$  and any distribution function  $F(\cdot)$ .

2)  $\psi(\cdot; b) \in C^2([b, R])$  for any  $R > b$  and any distribution function  $F(\cdot)$ .

3)  $\psi^d(\cdot; b) \in C^2([b, R])$  for any  $R > b$  if and only if  $F(\cdot) \in C([0, \infty))$ .

4) For fixed  $y \leq b$ ,  $\psi^s(\cdot, y; b) \in C^2([b, R])$  for any  $R > b$  if and only if  $F(\cdot) \in C([b - y, \infty))$ .

*Proof.* For any  $0 < \varepsilon < T$ , define  $U_{\varepsilon, T} = (a, b) \times (\varepsilon, T]$  and its closure  $\bar{U}_{\varepsilon, T} = [a, b] \times [\varepsilon, T]$ .

1) Since  $h(\cdot, \cdot)$  is uniformly bounded by Lemma 5.15, so is  $\int_0^{x-a} h(x-y, t)dF(y)$  as a function of  $(x, t)$ . Hence, both  $h(\cdot, \cdot)$  and  $\int_0^{x-a} h(x-y, t)dF(y)$  belong to  $L^p(U_{\varepsilon, T})$

for any  $1 < p < \infty$ . By the local  $W^{2,p}$  estimates (see, e.g., Theorem 7.30 of Lieberman (1996)), we have

$$\|h_t\|_{L^p(U_{\varepsilon,T})} + \|h_{xx}\|_{L^p(U_{\varepsilon,T})} \leq K_{p,\varepsilon,T}, \quad 1 < p < \infty, \quad (5.50)$$

where  $h_t$  and  $h_{xx}$  are corresponding weak derivatives of  $h(x, t)$  and  $K_{p,\varepsilon,T}$  is a constant depending on  $p, \varepsilon$  and  $T$ . Then by the Sobolev embedding theorem (see, e.g., part II of Theorem 4.12 of Adams (1975)), we have  $h(\cdot, \cdot) \in C^{1+\alpha, (1+\alpha)/2}(\bar{U}_{\varepsilon,T})$  for any  $0 < \alpha < 1$ .

Furthermore, it is easy to check that  $h_t(\cdot, \cdot)$  still satisfies PDE (5.49) in  $(a, b) \times (0, \infty)$  and  $h_t(x, t) = 0$  in  $\mathbb{R} \setminus (a, b) \times (0, \infty)$ . By Young's inequality (see, e.g., Theorem 3.9.4 of Bogachev (2007)) and (5.50), for any  $1 < p < \infty$ ,

$$\begin{aligned} \left\| \int_0^{x-a} h_t(x-y, t) dF(y) \right\|_{L^p(U_{\varepsilon,T})}^p &= \int_{\varepsilon}^T \int_a^b \left| \int_0^{x-a} h_t(x-y, t) dF(y) \right|^p dx dt \\ &\leq \int_{\varepsilon}^T \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} |h_t(x-y, t)| dF(y) \right|^p dx dt \\ &\leq \int_{\varepsilon}^T \int_{-\infty}^{\infty} |h_t(x, t)|^p dx \left( \int_{-\infty}^{\infty} dF(y) \right)^p dt \\ &= \int_{\varepsilon}^T \int_a^b |h_t(x, t)|^p dx dt \\ &\leq (K_{p,\varepsilon,T})^p. \end{aligned}$$

Hence,  $\int_0^{x-a} h_t(x-y, t) dF(y) \in L^p(U_{\varepsilon,T})$  for any  $1 < p < \infty$ . By the same argument as above, it follows that  $h_t(\cdot, \cdot) \in C^{1+\alpha, (1+\alpha)/2}(\bar{U}_{2\varepsilon,T})$  for any  $0 < \alpha < 1$ .

Using integration by parts to PDE (5.49), we obtain

$$h_t(x, t) = \frac{\sigma^2}{2} h_{xx}(x, t) + \mu h_x(x, t) - \lambda \int_0^{x-a} h_x(x-y, t) \bar{F}(y) dy.$$



By  $h(\cdot, \cdot), h_t(\cdot, \cdot) \in C^{1+\alpha, (1+\alpha)/2}(\bar{U}_{2\varepsilon, T})$ , it follows immediately that  $h_{xx}(\cdot, \cdot) \in C(\bar{U}_{2\varepsilon, T})$ .

Finally,  $\phi(\cdot; a, b, \cdot) = h(\cdot, \cdot) \in C^{2,1}(\bar{U}_{2\varepsilon, T})$ .

**2)** By part 1),  $\psi(\cdot; b) \in C^2([b, R])$  for any  $R > b$ .

**3)** As shown in the proof of part 1), we first have  $\psi^d(\cdot; b) \in C^{1,\alpha}([b, R])$  for any  $0 < \alpha < 1, R > b$ . Using integration by parts to (5.47), we obtain

$$\frac{\sigma^2}{2}\psi_{xx}^d(x; b) + \mu\psi_x^d(x; b) = \lambda\bar{F}(x - b) + \lambda \int_0^{x-b} \psi_x^d(x - y; b)\bar{F}(y)dy. \quad (5.51)$$

Hence,  $\psi_{xx}^d(\cdot; b) \in C([b, R])$ , or  $\psi^d(\cdot; b) \in C^2([b, R])$ , for any  $R > b$  if and only if  $\bar{F}(\cdot) \in C([0, \infty))$ .

**4)** As shown in the proof of part 1), we first have  $\psi^s(\cdot, y; b) \in C^{1,\alpha}([b, R])$  for any  $0 < \alpha < 1, R > b$ . Using integration by parts to (5.48), we obtain

$$\frac{\sigma^2}{2}\psi_{xx}^s(x, y; b) + \mu\psi_x^s(x, y; b) = \lambda \int_0^{x-b} \psi_x^s(x - z, y; b)\bar{F}(z)dz - \lambda\bar{F}(x - y). \quad (5.52)$$

Hence,  $\psi_{xx}^s(\cdot, y; b) \in C([b, R])$ , or  $\psi^s(\cdot, y; b) \in C^2([b, R])$ , for any  $R > b$  if and only if  $\bar{F}(\cdot) \in C([b - y, \infty))$ .  $\square$

By Theorem 5.16,  $\psi(\cdot; b)$  is twice differentiable for any  $F(\cdot)$  but  $\psi^d(\cdot; b)$  and  $\psi^s(\cdot, b; b)$  are twice differentiable if and only if  $F(\cdot) \in C([0, \infty))$ . This subtle difference between their regularities is not surprising since, noticing the identity  $\psi(\cdot; b) = \psi^d(\cdot; b) + \psi^s(\cdot, b; b)$ , the sum of two functions may have higher regularity. It is also an exercise to prove the existence, uniqueness and boundedness of solutions of PDEs (5.46)–(5.49) directly by pure PDE techniques.

Part 1) is the core of Theorem 5.16 and the other three parts can be regarded as its corollaries. We point out that Theorem 5.16 rediscovers and extends some

recent results on Doney's conjecture of the regularity of the scale function of  $X$ ; see Theorems 1.3, 3.10 and 3.11 of Kuznetsov et al. (2011) for example.

**Remark 5.17.** *An advantage of the proof of Theorem 5.16 is that it can be easily extended to cover more general risk models outside of the Lévy framework. For example, the drift  $\mu$  and the volatility  $\sigma$  can be generalized to be functions of  $X_t$ . Then some previous regularity results on bankruptcy related quantities in risk theory can be significantly improved. These results include Theorems 2.2 and 3.2 of Wang and Wu (2000), Theorems 2.3, 3.3 and 4.2 of Wang and Wu (2001), Theorem 3.2 of Cai (2004) as well as Theorems 2.1 and 3.1 of Cai and Yang (2005).*

**Lemma 5.18.** *Suppose that  $F(\cdot) \in C([0, \infty))$ . For any  $y \leq b$ , we have*

$$\psi_x(\infty; b) = \psi_x^d(\infty; b) = \psi_x^s(\infty, y; b) = 0.$$

*Proof.* We only prove the last relation as an illustration and we point out that the first two relations can be proven in the same way. For fixed  $y \leq b$ , we rewrite the right-hand side of (5.48) as  $g(x, y)$  so that

$$\frac{\sigma^2}{2} \psi_{xx}^s(x, y; b) + \mu \psi_x^s(x, y; b) = g(x, y).$$

Clearly,  $g(\infty, y) = 0$ . Multiplying both sides of the equation above by  $e^{2\mu x/\sigma^2}$  leads to

$$\frac{\sigma^2}{2} \frac{\partial}{\partial x} \left( e^{\frac{2\mu x}{\sigma^2}} \psi_x^s(x, y; b) \right) = e^{\frac{2\mu x}{\sigma^2}} g(x, y).$$

Integrating this equation with respect to  $dx$  from  $b$  to  $v$  yields

$$\frac{\sigma^2}{2} \psi_x^s(v, y; b) - \frac{\sigma^2}{2} e^{\frac{2\mu(b-v)}{\sigma^2}} \psi_x^s(b+, y; b) = e^{-\frac{2\mu v}{\sigma^2}} \int_b^v e^{\frac{2\mu x}{\sigma^2}} g(x, y) dx, \quad (5.53)$$

where  $\psi_x^s(b+, y; b)$  is the right-hand limit of  $\psi_x^s(x, y; b)$  at  $b$ . By Theorem 5.16,  $\psi_x^s(b+, y; b)$  is bounded. Using the general form of L'Hôpital's rule, we have

$$\lim_{v \rightarrow \infty} e^{-\frac{2\mu v}{\sigma^2}} \int_b^v e^{\frac{2\mu x}{\sigma^2}} g(x, y) dx = \frac{\sigma^2}{2\mu} \lim_{v \rightarrow \infty} g(v, y) = 0.$$

Hence, by taking  $v \rightarrow \infty$  in (5.53) we obtain  $\psi_x^s(\infty, y; b) = 0$ .  $\square$

**Lemma 5.19.** *Suppose that  $F(\cdot) \in C([0, \infty))$ . For any  $y < b$ , we have*

$$\sup_{x \geq b} \psi_y^s(x, y; b) \leq \frac{\lambda}{\mu - \lambda m} \quad \text{and} \quad \sup_{x \geq b} |\psi_{xy}^s(x, y; b)| \leq \frac{4\lambda\mu}{\sigma^2(\mu - \lambda m)}.$$

*Proof.* Integrating (5.52) with respect to  $dx$  from  $u > b$  to  $\infty$  and applying Fubini's theorem and Lemma 5.18, we obtain

$$\begin{aligned} & -\frac{\sigma^2}{2} \psi_x^s(u, y; b) - \mu \psi^s(u, y; b) \\ &= \lambda \int_u^\infty \int_0^{x-b} \psi_x^s(x-z, y; b) \bar{F}(z) dz dx - \lambda \int_u^\infty \bar{F}(x-y) dx \\ &= \lambda \left( \int_b^\infty \int_0^{x-b} - \int_b^u \int_0^{x-b} \right) \psi_x^s(x-z, y; b) \bar{F}(z) dz dx - \lambda \int_{u-y}^\infty \bar{F}(x) dx \\ &= \lambda \left( \int_0^\infty \int_{z+b}^\infty - \int_0^{u-b} \int_{z+b}^u \right) \bar{F}(z) \psi_x^s(x-z, y; b) dx dz - \lambda \int_{u-y}^\infty \bar{F}(x) dx \\ &= -\lambda \int_0^{u-b} \bar{F}(z) \psi^s(u-z, y; b) dz - \lambda \int_{u-y}^\infty \bar{F}(x) dx. \end{aligned}$$

Taking derivative with respect to  $y$  on both sides of the above, we obtain the following equation for  $\psi_y^s(x, y; b)$ : for  $y < b < x$ ,

$$\frac{\sigma^2}{2} \psi_{xy}^s(x, y; b) + \mu \psi_y^s(x, y; b) = \lambda \int_0^{x-b} \bar{F}(z) \psi_y^s(x-z, y; b) dz + \lambda \bar{F}(x-y), \quad (5.54)$$

with boundary conditions  $\psi_y^s(b, y; b) = \psi_y^s(\infty, y; b) = 0$ .

It is clear that  $\psi_y^s(x, y; b) \geq 0$ . We claim that  $\sup_{x \geq b} \psi_y^s(x, y; b)$  is bounded. Otherwise, there exists some  $\tilde{x} > b$  satisfying  $\psi_y^s(\tilde{x}, y; b) \geq \psi_y^s(z, y; b)$  for all  $z \in [b, \tilde{x}]$ ,  $\psi_{xy}^s(\tilde{x}, y; b) \geq 0$ , and  $\psi_y^s(\tilde{x}, y; b) > \lambda/(\mu - \lambda m)$ . Thus,

$$\begin{aligned} \frac{\sigma^2}{2} \psi_{xy}^s(\tilde{x}, y; b) + \mu \psi_y^s(\tilde{x}, y; b) &> \lambda m \psi_y^s(\tilde{x}, y; b) + \lambda \\ &\geq \lambda \int_0^{\tilde{x}-b} \bar{F}(z) \psi_y^s(\tilde{x} - z, y; b) dz + \lambda \bar{F}(\tilde{x} - y), \end{aligned}$$

which contradicts to (5.54). Next, for fixed  $y < b$ , suppose that  $\psi_y^s(x, y; b)$  attains its maximum at  $x = x^*$ , implying that  $\psi_{xy}^s(x^*, y; b) = 0$ . It follows from (5.54) that

$$\mu \psi_y^s(x^*, y; b) = \lambda \int_0^{x^*-b} \bar{F}(z) \psi_y^s(x^* - z, y; b) dz + \lambda \bar{F}(x^* - y) \leq \lambda m \psi_y^s(x^*, y; b) + \lambda,$$

yielding the first inequality. Furthermore, by (5.54) and the first inequality, we obtain

$$\sup_{x \geq b} |\psi_{xy}^s(x, y; b)| \leq \frac{2}{\sigma^2} (\mu + \lambda m) \sup_{x \geq b} \psi_y^s(x, y; b) + \frac{2\lambda}{\sigma^2} \leq \frac{4\lambda\mu}{\sigma^2(\mu - \lambda m)}.$$

This completes the proof.  $\square$

**Corollary 5.20.** *Suppose that  $F(\cdot) \in C([0, \infty))$ . For any  $y < b$ , the right-hand derivatives*

$$\psi_{x+}(b; b) = \frac{2\lambda m - 2\mu}{\sigma^2}, \quad \psi_{x+}^d(b; b) = -\frac{2\mu}{\sigma^2} \quad \text{and} \quad \psi_{x+}^s(b, dy; b) = \frac{2\lambda}{\sigma^2} \bar{F}(b-y) dy.$$

*Proof.* The last relation follows immediately by letting  $x \rightarrow b$  in (5.54). Upon integration by parts to the last term, equation (5.46) is rewritten as

$$\frac{\sigma^2}{2} \psi_{xx}(x; b) + \mu \psi_x(x; b) = \lambda \int_0^{x-b} \psi_x^d(x-y; b) \bar{F}(y) dy.$$

Integrating this equation as well as equation (5.51) with respect to  $dx$  from  $b$  to  $\infty$ , and then going along similar lines as in deriving (5.54), we can prove the first two relations, respectively.  $\square$

## 5.3.3 Proofs

In the proof of the one-sided differentiability, we mainly employed regularity theory of PDE. Differently from that, now we apply stochastic analysis to prove the differentiability of  $p(\cdot)$  at  $b$ . The main idea of the proof stems from the works of Chesney et al. (1997) and Albrecher et al. (2012) on Parisian options.

The following lemma extends Lemma 2.1 of Albrecher et al. (2012) to a general distribution function  $F(\cdot)$  and it is the key to the proof of the differentiability of  $p(\cdot)$  at  $b$ .

**Lemma 5.21.** *For any distribution function  $F(\cdot)$ , we have*

$$\lim_{\varepsilon \rightarrow 0^+} \mathbb{P}^0 \{T_{-\varepsilon}^- < T_{\varepsilon}^+\} = \frac{1}{2} \quad \text{and} \quad \mathbb{E}^0 [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+] = O(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0^+. \quad (5.55)$$

*Proof.* For  $x \in \mathbb{R}$ , we define

$$\tilde{T}_x^+ = \inf \{t \geq 0 : x_0 + \mu t + \sigma W_t > x\} \quad \text{and} \quad \tilde{T}_x^- = \inf \{t \geq 0 : x_0 + \mu t + \sigma W_t < x\}.$$

By formulas 3.01 on Page 309 of Borodin and Salminen (2002) or Lemma 5.2 of Li et al. (2012) with  $\mu(\cdot) \equiv \mu$  and  $\sigma(\cdot) \equiv \sigma$ , we have

$$\lim_{\varepsilon \rightarrow 0^+} \mathbb{P}^0 \{\tilde{T}_{-\varepsilon}^- < \tilde{T}_{\varepsilon}^+\} = \frac{1}{2} \quad \text{and} \quad \mathbb{E}^0 [\tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+] = O(\varepsilon^2). \quad (5.56)$$

In addition, let  $\zeta_1$  be the time when the first jump of  $X$  in (5.36) arrives.

To prove the first relation in (5.55), note that, by the second relation in (5.56),

$$\mathbb{P}^0 \left\{ \zeta_1 < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \right\} = 1 - \mathbb{E}^0 \left[ e^{-\lambda(\tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+)} \right] \leq \lambda \mathbb{E}^0 [\tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+] = O(\varepsilon^2). \quad (5.57)$$

Then by relation (5.57) and the first relation in (5.56),

$$\begin{aligned}
\mathbb{P}^0 \{T_{-\varepsilon}^- < T_{\varepsilon}^+\} &\leq \mathbb{P}^0 \left\{ \zeta_1 < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \right\} + \mathbb{P}^0 \left\{ T_{-\varepsilon}^- < T_{\varepsilon}^+, \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ < \zeta_1 \right\} \\
&= O(\varepsilon^2) + \mathbb{P}^0 \left\{ \tilde{T}_{-\varepsilon}^- < \tilde{T}_{\varepsilon}^+ \wedge \zeta_1 \right\} \\
&\leq O(\varepsilon^2) + \mathbb{P}^0 \left\{ \tilde{T}_{-\varepsilon}^- < \tilde{T}_{\varepsilon}^+ \right\} \\
&\rightarrow \frac{1}{2} \quad \text{as } \varepsilon \rightarrow 0+.
\end{aligned}$$

Symmetrically,  $\limsup_{\varepsilon \rightarrow 0+} \mathbb{P}^0 \{T_{\varepsilon}^+ < T_{-\varepsilon}^-\} \leq 1/2$ . This proves the first relation in (5.55).

Next we prove the second relation in (5.55). Introduce  $e_{\lambda}$  to be an independent exponential random variable with rate  $\lambda > 0$ . For any  $x \in (-\varepsilon, \varepsilon)$ ,

$$\begin{aligned}
&\mathbb{E}^x [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_{\lambda}] \\
&= \mathbb{E}^x [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_{\lambda}; e_{\lambda} < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \wedge \zeta_1] + \mathbb{E}^x [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_{\lambda}; \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ < e_{\lambda} \wedge \zeta_1] \\
&\quad + \mathbb{E}^x [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_{\lambda}; \zeta_1 < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \wedge e_{\lambda}] \\
&= \mathbb{E}^x [e_{\lambda}; e_{\lambda} < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \wedge \zeta_1] + \mathbb{E}^x [\tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+; \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ < e_{\lambda} \wedge \zeta_1] \\
&\quad + \mathbb{E}^x [\zeta_1; \zeta_1 < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \wedge e_{\lambda}, X_{\zeta_1} < -\varepsilon] \\
&\quad + \mathbb{E}^x [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_{\lambda}; \zeta_1 < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \wedge e_{\lambda}, X_{\zeta_1} > -\varepsilon] \\
&= E_1 + E_2 + E_3 + E_4.
\end{aligned} \tag{5.58}$$

By the second relation in (5.56),

$$E_1 + E_2 + E_3 \leq 3\mathbb{E}^x [\tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+] \leq 3\mathbb{E}^0 [\tilde{T}_{-2\varepsilon}^- \wedge \tilde{T}_{2\varepsilon}^+] = O(\varepsilon^2).$$

Further, by the strong Markov property of  $X$  and the memoryless property of expo-

nential distributions,

$$\begin{aligned}
E_4 &= \mathbb{E}^x \left[ \mathbb{E}^x \left[ T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_\lambda; \zeta_1 < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \wedge e_\lambda, X_{\zeta_1} > -\varepsilon \middle| \mathcal{F}_{\zeta_1} \right] \right] \\
&= \mathbb{E}^x \left[ 1_{\{\zeta_1 < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \wedge e_\lambda, X_{\zeta_1} > -\varepsilon\}} \left( \zeta_1 + \mathbb{E}^{X_{\zeta_1}} [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_\lambda] \right) \right] \\
&\leq \mathbb{E}^x \left[ \zeta_1; \zeta_1 < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \wedge e_\lambda, X_{\zeta_1} > -\varepsilon \right] \\
&\quad + \mathbb{P}^x \left\{ \zeta_1 < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \wedge e_\lambda, Y_1 < 2\varepsilon \right\} \sup_{-\varepsilon < y < \varepsilon} \mathbb{E}^y [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_\lambda] \\
&\leq \mathbb{E}^x \left[ \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \right] + \mathbb{P}^x \left\{ \zeta_1 < \tilde{T}_{-\varepsilon}^- \wedge \tilde{T}_{\varepsilon}^+ \right\} \sup_{-\varepsilon < y < \varepsilon} \mathbb{E}^y [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_\lambda] \\
&= O(\varepsilon^2) + O(\varepsilon^2) \sup_{-\varepsilon < y < \varepsilon} \mathbb{E}^y [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_\lambda],
\end{aligned}$$

where the last step is due to (5.56) and (5.57). Substituting these estimates into (5.58) and taking supremum over  $x \in (-\varepsilon, \varepsilon)$ , we obtain

$$\sup_{-\varepsilon < x < \varepsilon} \mathbb{E}^x [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_\lambda] \leq O(\varepsilon^2) + O(\varepsilon^2) \sup_{-\varepsilon < x < \varepsilon} \mathbb{E}^x [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_\lambda].$$

Solving this inequality for  $\sup_{-\varepsilon < x < \varepsilon} \mathbb{E}^x [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+ \wedge e_\lambda]$  and taking limit  $\lambda \rightarrow 0+$ , we obtain  $\sup_{-\varepsilon < x < \varepsilon} \mathbb{E}^x [T_{-\varepsilon}^- \wedge T_{\varepsilon}^+] = O(\varepsilon^2)$ .  $\square$

The following lemma is in essence a restatement of Lemmas 2.2 and 2.3 of Albrecher et al. (2012):

**Lemma 5.22.** *For any distribution function  $F(\cdot)$ , it holds that, as  $\varepsilon \rightarrow 0+$ ,*

$$\mathbb{P}^b \left\{ \tau_{b-}(c) < T_{b-\varepsilon}^- \right\} = O(\varepsilon^2) \quad \text{and} \quad \mathbb{E}^b \left[ T_{b-\varepsilon}^- - l_{b+}(T_{b-\varepsilon}^-); T_{b-\varepsilon}^- < \infty \right] = O(\varepsilon^2),$$

where  $l_{b+}(T_{b-\varepsilon}^-) = \sup \{s \leq T_{b-\varepsilon}^- : X_s \geq b\}$  as defined in (5.1).

The following lemma solves a technical problem for establishing the main result in this subsection:

**Lemma 5.23.** *If  $F(\cdot) \in C([0, \infty))$ , then*

$$\lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \mathbf{P}^b \left\{ \left( T_a^- \circ \theta_{T_{b-\varepsilon}^-} \right) \wedge (c - \delta) \leq T_b^+ \circ \theta_{T_{b-\varepsilon}^-} < \left( T_a^- \circ \theta_{T_{b-\varepsilon}^-} \right) \wedge c, T_{b-\varepsilon}^- < \infty \right\} = 0,$$

where  $\theta_t$  is the usual shift operator.

*Proof.* Let  $0 < \delta < c$ . We consider the cases  $X_{T_{b-\varepsilon}^-} = b - \varepsilon$  and  $X_{T_{b-\varepsilon}^-} < b - \varepsilon$  separately. For the first case, by the strong Markov property of  $X$ ,

$$\begin{aligned} & \mathbf{P}^b \left\{ \left( T_a^- \circ \theta_{T_{b-\varepsilon}^-} \right) \wedge (c - \delta) \leq T_b^+ \circ \theta_{T_{b-\varepsilon}^-} < \left( T_a^- \circ \theta_{T_{b-\varepsilon}^-} \right) \wedge c, T_{b-\varepsilon}^- < \infty, X_{T_{b-\varepsilon}^-} = b - \varepsilon \right\} \\ &= \mathbf{P}^b \left\{ T_{b-\varepsilon}^- < \infty, X_{T_{b-\varepsilon}^-} = b - \varepsilon \right\} \left( \mathbf{P}^{b-\varepsilon} \left\{ T_a^- \wedge (c - \delta) \leq T_b^+ < T_a^- \wedge c \right\} \right) \\ &\leq \mathbf{P}^{b-\varepsilon} \left\{ T_b^+ \geq T_a^- \wedge (c - \delta) \right\} - \mathbf{P}^{b-\varepsilon} \left\{ T_b^+ \geq T_a^- \wedge c \right\}. \end{aligned}$$

By Theorem 5.16, we have that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \mathbf{P}^{b-\varepsilon} \left\{ T_b^+ \geq T_a^- \wedge t \right\} = \phi_{x^-}(b; a, b, t) \quad \text{for } t = c - \delta \text{ and } t = c,$$

and that the left-hand derivative  $\phi_{x^-}(b; a, b, t)$  is continuous at  $t = c > 0$ . It follows that

$$\lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \left[ \mathbf{P}^{b-\varepsilon} \left\{ T_b \geq T_a \wedge (c - \delta) \right\} - \mathbf{P}^{b-\varepsilon} \left\{ T_b \geq T_a \wedge c \right\} \right] = 0.$$

For the second case, by the property of  $X$  having stationary and independent increments,

$$\begin{aligned} & \mathbf{P}^b \left\{ \left( T_a^- \circ \theta_{T_{b-\varepsilon}^-} \right) \wedge (c - \delta) \leq T_b^+ \circ \theta_{T_{b-\varepsilon}^-} < \left( T_a^- \circ \theta_{T_{b-\varepsilon}^-} \right) \wedge c, T_{b-\varepsilon}^- < \infty, X_{T_{b-\varepsilon}^-} < b - \varepsilon \right\} \\ &= \mathbf{P}^{b+\varepsilon} \left\{ \left( T_{a+\varepsilon}^- \circ \theta_{T_b^-} \right) \wedge (c - \delta) \leq T_{b+\varepsilon}^+ \circ \theta_{T_b^-} < \left( T_{a+\varepsilon}^- \circ \theta_{T_b^-} \right) \wedge c, T_b^- < \infty, X_{T_b^-} < b \right\} \\ &= \int_{(a+\varepsilon, b)} \left( \mathbf{P}^y \left\{ T_{b+\varepsilon}^+ \geq T_{a+\varepsilon}^- \wedge (c - \delta) \right\} - \mathbf{P}^y \left\{ T_{b+\varepsilon}^+ \geq T_{a+\varepsilon}^- \wedge c \right\} \right) \psi^s(b + \varepsilon, dy; b) \\ &= \int_{(a+\varepsilon, b)} \left( \phi(y; a + \varepsilon, b + \varepsilon, c - \delta) - \phi(y; a + \varepsilon, b + \varepsilon, c) \right) \psi^s(b + \varepsilon, dy; b). \end{aligned}$$



By Lemma 5.19, Corollary 5.20 and the continuous dependence of the solution of (5.49) on the parameters  $a$  and  $b$ , we obtain

$$\begin{aligned}
& \lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \int_{(a+\varepsilon, b)} (\phi(y; a + \varepsilon, b + \varepsilon, c - \delta) - \phi(y; a + \varepsilon, b + \varepsilon, c)) \frac{\psi^s(b + \varepsilon, dy; b)}{\varepsilon} \\
&= \lim_{\delta \rightarrow 0^+} \frac{2\lambda}{\sigma^2} \int_{(a, b)} (\phi(y; a, b, c - \delta) - \phi(y; a, b, c)) \psi_{x^+}^s(b, dy; b) \\
&= \lim_{\delta \rightarrow 0^+} \frac{2\lambda}{\sigma^2} \int_{(a, b)} (\phi(y; a, b, c - \delta) - \phi(y; a, b, c)) \bar{F}(b - y) dy \\
&= 0.
\end{aligned}$$

This completes the proof of Lemma 5.23.  $\square$

Now we are ready to establish the main result of this subsection:

**Theorem 5.24.** *If  $F(\cdot) \in C([0, \infty))$ , then the survival probability  $p(\cdot)$  defined by (5.4) is differentiable at  $b$ .*

*Proof.* By (5.42), it suffices to show that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{p(b) - p(b - \varepsilon)}{\varepsilon} = \frac{2\mu - 2\lambda m}{\sigma^2} - \frac{2\mu}{\sigma^2} p(b) + \frac{2\lambda}{\sigma^2} p(b) \int_{(a, b)} \phi(y; a, b, c) \bar{F}(b - y) dy.$$

For  $0 < \varepsilon < b - a$ ,

$$p(b) = \mathbb{P}^b \{T_{b-\varepsilon}^- = \infty, \tau_{b-}(c) = \infty\} + \mathbb{P}^b \{T_{b-\varepsilon}^- < \infty, T_a^- \wedge \tau_{b-}(c) = \infty\} = P_1 + P_2.$$

By the first relation in Lemma 5.22 and  $\psi(b; b - \varepsilon) = \psi(b + \varepsilon; b)$ , we have

$$P_1 = \mathbb{P}^b \{T_{b-\varepsilon}^- = \infty\} - \mathbb{P}^b \{T_{b-\varepsilon}^- = \infty, \tau_{b-}(c) < \infty\} = 1 - \psi(b + \varepsilon; b) + O(\varepsilon^2).$$

Now we turn to  $P_2$ . We derive

$$\begin{aligned}
P_2 &= \mathbb{P}^b \left\{ T_{b-\varepsilon}^- < \infty, (T_a^- \wedge \tau_{b-}(c)) \circ \theta_{T_{b-\varepsilon}^-} = \infty \right\} \\
&\quad - \mathbb{P}^b \left\{ T_{b-\varepsilon}^- < \infty, (T_a^- \wedge \tau_{b-}(c)) \circ \theta_{T_{b-\varepsilon}^-} = \infty, \tau_{b-}(c) \leq T_{b-\varepsilon}^- + T_b^+ \circ \theta_{T_{b-\varepsilon}^-} \right\} \\
&= P_{21} - P_{22}.
\end{aligned}$$

By (5.40) and the property of  $X$  having stationary and independent increments,

$$\begin{aligned}
P_{21} &= \mathbb{P}^b \left\{ T_{b-\varepsilon}^- < \infty, X_{T_{b-\varepsilon}^-} = b - \varepsilon, (T_a^- \wedge \tau_{b-}(c)) \circ \theta_{T_{b-\varepsilon}^-} = \infty \right\} \\
&\quad + \mathbb{P}^b \left\{ T_{b-\varepsilon}^- < \infty, X_{T_{b-\varepsilon}^-} < b - \varepsilon, (T_a^- \wedge \tau_{b-}(c)) \circ \theta_{T_{b-\varepsilon}^-} = \infty \right\} \\
&= \mathbb{P}^b \left\{ T_{b-\varepsilon}^- < \infty, X_{T_{b-\varepsilon}^-} = b - \varepsilon \right\} p(b - \varepsilon) + \int_{(a, b-\varepsilon)} p(y) \psi^s(b, dy; b - \varepsilon) \\
&= \psi^d(b; b - \varepsilon) p(b - \varepsilon) + p(b) \int_{(a, b-\varepsilon)} \phi(y; a, b, c) \psi^s(b, dy; b - \varepsilon) \\
&= \psi^d(b + \varepsilon; b) p(b - \varepsilon) + p(b) \int_{(a+\varepsilon, b)} \phi(y; a + \varepsilon, b + \varepsilon, c) \psi^s(b + \varepsilon, dy; b).
\end{aligned}$$

For arbitrarily fixed  $0 < \delta < c$ , by Lemma 5.22, we have

$$\begin{aligned}
&P_{22} \\
&\leq \mathbb{P}^b \left\{ \tau_{b-}(c) < T_{b-\varepsilon}^- \right\} \\
&\quad + \mathbb{P}^b \left\{ T_{b-\varepsilon}^- < \infty, (T_a^- \wedge \tau_{b-}(c)) \circ \theta_{T_{b-\varepsilon}^-} = \infty, T_{b-\varepsilon}^- \leq \tau_{b-}(c) \leq T_{b-\varepsilon}^- + T_b^+ \circ \theta_{T_{b-\varepsilon}^-} \right\} \\
&\leq O(\varepsilon^2) + \mathbb{P}^b \left\{ T_{b-\varepsilon}^- < \infty, c - (T_{b-\varepsilon}^- - l_{b+}(T_{b-\varepsilon}^-)) \leq T_b^+ \circ \theta_{T_{b-\varepsilon}^-} < (T_a^- \circ \theta_{T_{b-\varepsilon}^-}) \wedge c \right\} \\
&\leq O(\varepsilon^2) + \mathbb{P}^b \left\{ T_{b-\varepsilon}^- < \infty, (T_{b-\varepsilon}^- - l_{b+}(T_{b-\varepsilon}^-)) \geq \delta \right\} \\
&\quad + \mathbb{P}^b \left\{ T_{b-\varepsilon}^- < \infty, c - \delta \leq T_b^+ \circ \theta_{T_{b-\varepsilon}^-} < (T_a^- \circ \theta_{T_{b-\varepsilon}^-}) \wedge c \right\} \\
&\leq O(\varepsilon^2) + \mathbb{P}^b \left\{ T_{b-\varepsilon}^- < \infty, c - \delta \leq T_b^+ \circ \theta_{T_{b-\varepsilon}^-} < (T_a^- \circ \theta_{T_{b-\varepsilon}^-}) \wedge c \right\} \\
&= o(\varepsilon),
\end{aligned}$$

where the last step is due to Lemma 5.23. Finally, combining these estimate together we obtain

$$p(b) = 1 - \psi(b+\varepsilon; b) + \psi^d(b+\varepsilon; b)p(b-\varepsilon) + p(b) \int_{(a+\varepsilon, b)} \phi(y; a+\varepsilon, b+\varepsilon, c) \psi^s(b+\varepsilon, dy; b) + o(\varepsilon).$$

It follows from Lemma 5.19, Corollary 5.20 and the continuous dependence of solutions of (5.49) on parameters  $a$  and  $b$ ,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{p(b) - p(b-\varepsilon)}{\varepsilon} = \frac{2\mu - 2\lambda m}{\sigma^2} - \frac{2\mu}{\sigma^2} p(b) + \frac{2\lambda}{\sigma^2} p(b) \int_{(a, b)} \phi(y; a, b, c) \bar{F}(b-y) dy.$$

We complete the proof of Theorem 5.24.  $\square$

#### 5.3.4 Numerical examples

We use the implicit finite difference method to solve the probability of liquidation (5.38) for three types of jump-size distributions. This method is unconditionally convergent and stable. The local error is of order  $O(\Delta x^2) + O(\Delta t)$ . The numerical experiments are carried out on a computer with an Intel Core i7-2600 CPU 3.40GHz 8GB RAM, and the software programs are run under Matlab R2011b. The parameters are set to  $\mu = 0.5$ ,  $\sigma = 0.2$ ,  $\lambda = 0.5$ ,  $a = 0$ ,  $b = 1$  and  $c = 1$ . The expected jump size is  $m = 0.5$  in the following three examples. The step sizes are set to  $\Delta x = 0.025$ ,  $\Delta y = 0.01$  and  $\Delta t = 0.001$ .

In the first example, to test the accuracy of our algorithm, we assume that the jump sizes follow an exponential distribution with probability density function

$$f(x) = 2e^{-2x}, \quad x > 0.$$

We compare the numerical solution by the general algorithm (5.38) with the “analyt-

ic” solution by the simplified formula (5.44). The error of the algorithm is understood as the maximum difference between the numerical and analytic solutions. We obtain the following errors:

$$\max_{x_0 \geq b} |\psi^{\text{num}}(x_0; b) - \psi^{\text{ana}}(x_0; b)| = 5.4012 \times 10^{-3},$$

$$\max_{x_0 \geq b} |\psi^{\text{num}}(x_0; a) - \psi^{\text{ana}}(x_0; a)| = 3.0777 \times 10^{-5},$$

$$\max_{x_0 \geq b} |q^{\text{num}}(x_0) - q^{\text{ana}}(x_0)| = 8.2752 \times 10^{-6}.$$

The elapsed times for the numerical and analytic solutions are 49.29919 and 3.102916 seconds, respectively. It is somewhat surprising that the error for the probability of liquidation  $q(x_0)$  is even smaller than the errors for the two probabilities of bankruptcy,  $\psi(x_0; b)$  and  $\psi(x_0; a)$ . This may be due to the stable structure of  $q(x_0)$ .

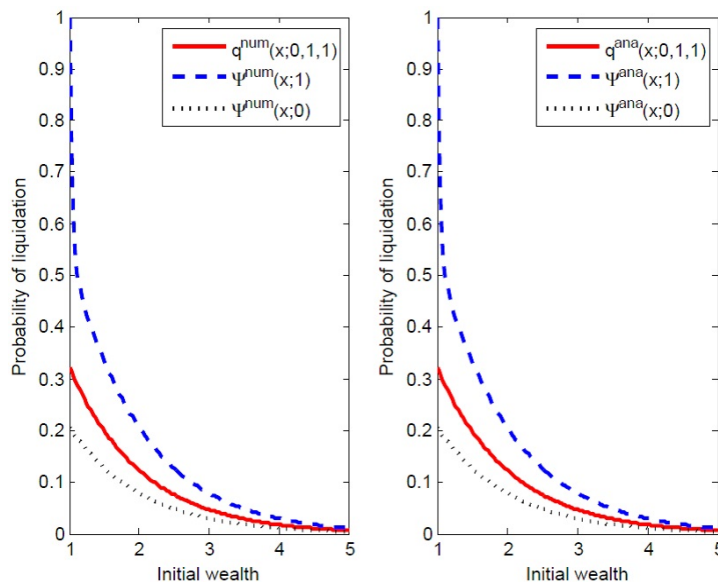


Figure 5.4: Numerical versus analytical solutions of probabilities of liquidation with exponential jumps

In the second example, we assume that the jump sizes follow a gamma distribution with probability density function

$$f(x) = 16xe^{-4x}, \quad x > 0,$$

The elapsed time is 48.912241 seconds.

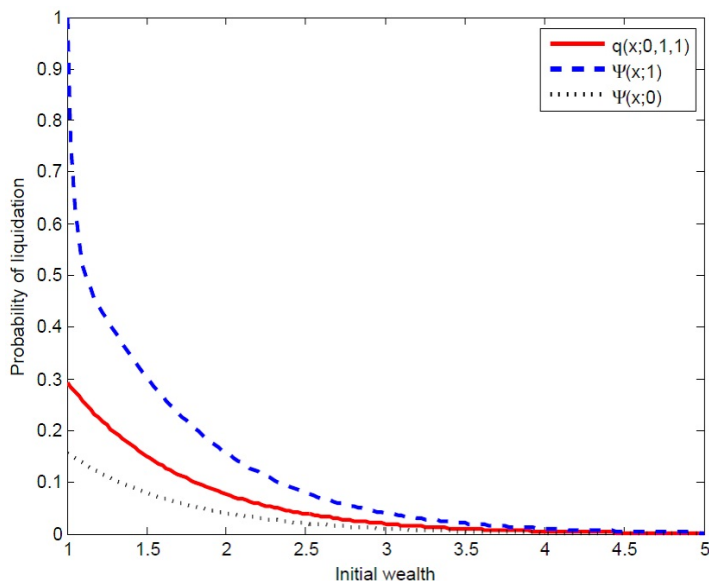


Figure 5.5: Probabilities of liquidation with Gamma jumps

In the third example, we assume that the jump sizes follow a Pareto distribution with probability density function

$$f(x) = \frac{1}{8x^3}, \quad x > 0.25.$$

The elapsed time is 56.898145 seconds.

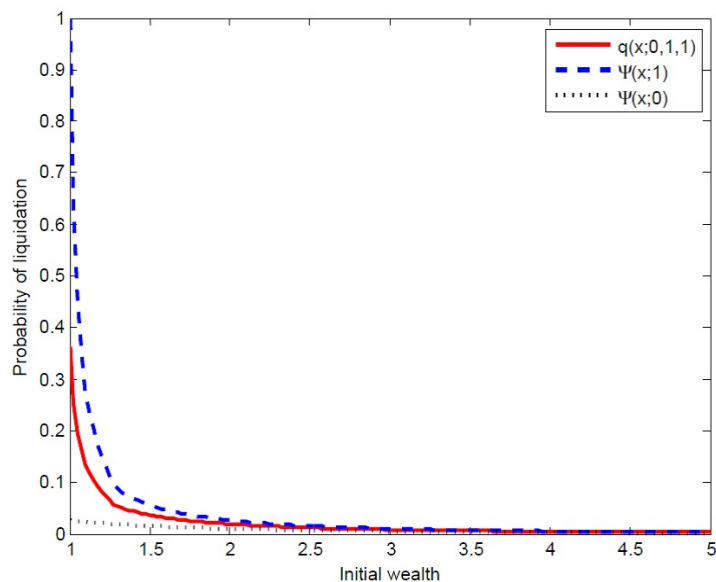


Figure 5.6: Probabilities of liquidation with Pareto jumps

#### 5.4 Summary and Some Remarks

The contributions of this chapter are twofold. First, to quantitatively measure the liquidation risk of a firm subject of Chapter 7 and Chapter 11 of the U.S. bankruptcy code, we solve the probability of liquidation for two structure models by a differentiability argument. Our results indicate that the effect of capital restructuring during Chapter 11 bankruptcy on the probability of liquidation is complex and intriguing. Second, by employing regularity theory of PDEs we obtain minimum conditions under which the probabilities related to first passage times become classical solutions of associated integro-differential equations.

In our model, the liquidation barrier and the reorganization barrier are set to be constants. Broadie et al. (2007) also considered the case that these two barriers are

determined endogenously by maximizing the firm value subject to a limited liability constraint. Then the problem becomes an optimal stopping problem with constraints which was solved in Broadie et al. (2007) by a binomial lattice approach. However, we think that this problem can also be formulated to a free boundary problem of PDEs. This equation should have two time variables as in a similar work of Parisian option by Haber and Schönbucher (1999). Then many efficient numerical PDE approaches can be applied to obtain more precise results than binomial lattice approaches. Another possible extension is to find the optimal reorganization plan to maximize the firm value. As illustrated in our simple numerical examples in Section 5.2.3, the effect of reorganization plans on the probability of liquidation (the firm value may as well) are complex and intriguing. In addition, it is also interesting to study Merton's portfolio problem by replacing the default time (usually a first passage time) by the liquidation time (5.2).

Our Theorem 5.16 rediscovers and extends some recent results on Doney's conjecture. But our results are already beyond the Lévy framework and Doney's conjecture since we considered those probabilities in finite-time horizon and the drift and volatility in our model can be level dependent. One way to completely solve Doney's conjecture might be to approximate Lévy processes by sequences of compound Poisson processes, and meanwhile to prove the corresponding norms are uniformly bounded.

## 5.5 Appendix of Hölder and Sobolev Spaces

We briefly introduce Hölder spaces and Sobolev spaces in one dimension. The reader is referred to Chapter 4 of Gilbarg and Trudinger (2001) and Chapter 4 of Lieberman (1996) for general discussions.

### 5.5.1 Elliptic case

Suppose that  $\bar{U}$  is the closure of an open set  $U \subset \mathbb{R}$ . For  $k \in \mathbb{N}$ ,  $C^k(\bar{U})$  is the space of functions that are continuous in  $\bar{U}$  together with all derivatives  $D^i u$  for  $i \leq k$ , and have the finite norm given by

$$\|u\|_{C^k(\bar{U})} = \sum_{i=0}^k \sup_{x \in \bar{U}} |D^i u(x)|.$$

For  $k \in \mathbb{N}$  and  $0 < \alpha \leq 1$ , Hölder space  $C^{k,\alpha}(\bar{U})$  is the subspace of  $C^k(\bar{U})$  and have the finite norm given by

$$\|u\|_{C^{k,\alpha}(\bar{U})} = \|u\|_{C^k(\bar{U})} + \sup_{x,y \in \bar{U}, x \neq y} \frac{|D^k u(x) - D^k u(y)|}{|x - y|^\alpha}.$$

Assume that  $u$  is locally integrable in  $U$  and  $k \in \mathbb{N}$ . Then a locally integrable function  $v$  is called the  $k^{\text{th}}$  weak derivative of  $u$ , still denoted as  $D^k u = v$ , if it satisfies that

$$\int_U \varphi(x) v(x) dx = (-1)^k \int_U u(x) D^k \varphi(x) dx, \quad \text{for all } \varphi \in C_0^k(\bar{U}),$$

where  $C_0^k(\bar{U})$  is the set of functions in  $C^k(\bar{U})$  with compact support in  $\bar{U}$ . For  $k \in \mathbb{N}$ , denote by  $W^k(U)$  the space of  $k^{\text{th}}$  order weakly differentiable functions.

For  $p \geq 1$ , denote by  $L^p(U)$  the space of measurable functions on  $U$  and have



the finite norm given by

$$\|u\|_{L^p(U)} = \begin{cases} (\int_U |u(x)|^p dx)^{1/p}, & 1 \leq p < \infty, \\ \sup_U |u|, & p = \infty. \end{cases}$$

For  $k \in \mathbb{N}$  and  $p \geq 1$ , Sobolev space  $W^{k,p}(U)$  is the subspace of  $W^k(U)$  and have the finite norm given by

$$\|u\|_{W^{k,p}(U)} = \sum_{i=0}^k \|D^i u(x)\|_{L^p(U)}.$$

### 5.5.2 Parabolic case

Suppose that  $\bar{U}_T$  is the closure of an open set  $U_T \subset \mathbb{R} \times (0, T]$  in the parabolic topology generated by a parabolic distance. Here the parabolic distance between  $Q_1 = (x_1, t_1)$  and  $Q_2 = (x_2, t_2)$  is defined to be  $(|x_1 - x_2|^2 + |t_1 - t_2|)^{1/2}$ .

For  $l \geq 0$ , Hölder space  $C^{l,l/2}(\bar{U}_T)$  is the space of functions that are continuous in  $\bar{U}_T$  together with all derivatives  $D_t^r D_x^s u$  with nonnegative integers  $r, s$  such that  $2r + s \leq l$ , and have the finite norm given by

$$\begin{aligned} \|u\|_{C^{l,l/2}(\bar{U}_T)} &= \sum_{i=0}^{[l]} \sum_{2r+s=i} \sup_{(x,t) \in \bar{U}_T} \|D_t^r D_x^s u(x, t)\| \\ &+ \sum_{2r+s=[l]} \sup_{Q_1, Q_2 \in \bar{U}_T, Q_1 \neq Q_2} \frac{|D_t^r D_x^s u(Q_1) - D_t^r D_x^s u(Q_2)|}{(d(Q_1, Q_2))^{l-[l]}} \\ &+ \sum_{2r+s+1=[l]} \sup_{(x,t_1), (x,t_2) \in \bar{U}_T} \frac{|D_t^r D_x^s u(x, t_1) - D_t^r D_x^s u(x, t_2)|}{|t_1 - t_2|^{(1+l-[l])/2}}, \end{aligned}$$

where  $[l]$  is the greatest integer which is no greater than  $l$ . In particular,

$$\|u\|_{C^{2,1}(\bar{U}_T)} = \sup_{(x,t) \in \bar{U}_T} (|u(x, t)| + |u_x(x, t)| + |u_{xx}(x, t)| + |u_t(x, t)|),$$

and, for  $0 < \alpha < 1$ ,

$$\begin{aligned} \|u\|_{C^{1+\alpha,(1+\alpha)/2}(\bar{U}_T)} &= \sup_{(x,t) \in \bar{U}_T} (|u(x,t)| + |u_x(x,t)|) + \sup_{Q_1, Q_2 \in \bar{U}_T, Q_1 \neq Q_2} \frac{|u_x(Q_1) - u_x(Q_2)|}{(d(Q_1, Q_2))^\alpha} \\ &+ \sup_{(x,t_1), (x,t_2) \in \bar{U}_T} \frac{|u(x,t_1) - u(x,t_2)|}{|t_1 - t_2|^{(1+\alpha)/2}}. \end{aligned}$$

Assume that  $u$  is locally integrable in  $U_T$  and  $k \in \mathbb{N}$ . Then a locally integrable function  $v$  is called a  $k^{\text{th}}$  order weak derivative of  $u$ , still denoted as  $D_t^r D_x^s u = v$  for nonnegative integers  $r, s$  such that  $2r + s = k$ , if it satisfies that

$$\int_{U_T} \varphi(x,t) v(x,t) dx dt = (-1)^{r+s} \int_{U_T} u(x) D_t^r D_x^s \varphi(x) dx dt, \quad \text{for all } \varphi \in C_0^{k,k/2}(\bar{U}_T),$$

where  $C_0^{k,k/2}(\bar{U}_T)$  is the set of functions in  $C^{k,k/2}(\bar{U}_T)$  with compact support in  $\bar{U}_T$ .

For  $k \in \mathbb{N}$ , denote by  $W^{k,k/2}(U_T)$  the space of  $k^{\text{th}}$  order weakly differentiable functions.

For  $p \geq 1$ , denote by  $L^p(U_T)$  the space of measurable functions on  $U_T$  and have the finite norm given by

$$\|u\|_{L^p(U_T)} = \begin{cases} \left( \int_{U_T} |u(x,t)|^p dx dt \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{U_T} |u|, & p = \infty. \end{cases}$$

For  $k \in \mathbb{N}$  and  $p \geq 1$ , Sobolev space  $W^{k,k/2,p}(U_T)$  is the subspace of  $W^{k,k/2}(U_T)$  and

have the finite norm given by

$$\|u\|_{W^{k,k/2,p}(U_T)} = \sum_{2r+s \leq k} \|D_t^r D_x^s u(x,t)\|_{L^p(U_T)}.$$

## CHAPTER 6

### HITTING, PARISIAN AND INVERSE OCCUPATION TIMES OF DIFFUSIONS AND PRICING OF SOME HYBRID BARRIER-PARISIAN OPTIONS

We define some new hybrid Barrier-Parisian options and derive their pricing formulas. The new options are knocked in or out only if the stock price has continuously or cumulatively stayed in a range for a prespecified length of time or has hit some barrier prior to expiration. More generally, we derive the Laplace transforms of hitting times, Parisian times and inverse occupation times with mixed relations for time-homogeneous diffusion processes using a similar perturbation approach as in Chapter 4. Therefore, the structure of the stopping times in this chapter generalizes that in Chapter 5. Moreover, compared with the differentiability argument in Chapter 5, this perturbation approach is simpler and can also be extended for general Markov processes with or without jumps. The content of this chapter is mainly based on the paper Li and Zhou (2013b).

#### 6.1 Introduction

Parisian options, as extensions of Barrier options, were first introduced by Chesney et al. (1997). Parisian options are knocked in or out if the underlying asset price has continuously or cumulatively stayed above or below some level for a pre-specified length of time. Further, Parisian times, the stopping times introduced to define Parisian options, have called much attention recently by many researchers in quantitative finance and insurance. As we discussed in Chapter 4, one application of

Parisian times is to model liquidation risk subject to Chapter 7 and Chapter 11 of the U.S. bankruptcy code. Besides, they are also applied in life insurance to capture the early delayed closure procedure of pension funds; see, e.g., Chen and Suchanecki (2007) and Broeders and Chen (2010). Gauthier (2002) applied Parisian times to study the valuation of investment projects with a delay constraint. Generally speaking, Parisian times are very useful to model delayed procedures of many insurance and financial events and products such as margin call of margin accounts in a stock market.

Since Chesney et al. (1997), some extensions of Parisian options have been made by many authors. Chen and Suchanecki (2008) studied Parisian exchange options. Dassios and Wu (2010) introduced a new type of Parisian options with single space barrier but double time periods. Such options are knocked in or out if the underlying asset price has continuously stayed below the barrier for longer than  $d_1$  units of time or above the barrier for longer than  $d_2$  units of time prior to the maturity of the option. Further, Anderluh and van der Weide (2009) and Dassios and Wu (2011) extended this kind of options to double space barriers and double time periods. Then such options are knocked in or out only if the underlying asset price has continuously stayed below a low barrier  $l_1$  for longer than  $d_1$  units of time or above a high barrier  $l_2$  for longer than  $d_2$  units of time prior to the maturity. Albrecher et al. (2011) studied the valuation of Parisian barrier options in an underlying jump-diffusion model with two-sided exponential jumps. Bernard and Boyle (2011) designed a new type of Parisian options by assuming that the option is exercised immediately

when the barrier condition is met rather than at maturity.

One of the major challenges of valuation of Parisian options is that the pricing formulas of Parisian options (with Parisian times) are not explicit even if in the simplest Black-Scholes framework. Therefore, many numerical approaches are proposed such as Laplace (Fourier) transform, PDE, and Monte Carlo simulation. By doing a Laplace transform with respect to the maturity, explicit pricing formulas can be obtained. Then the value as well as the Greeks of Parisian options can be solved by numerical Laplace inversions; see, e.g., Bernard et al. (2005), Labart and Lelong (2009), and Anderluh and van der Weide (2009). In addition, Haber and Schönbucher (1999) derived a high-dimensional PDE for the price of a Parisian option and solved it by a finite difference method. Parisian options can also be evaluated by Monte Carlo simulation; see Avellaneda and Wu (1999), Costabile (2002), and Bernard and Boyle (2011).

There are three limitations of previous works in the literature. First, there is little work beyond the Black-Scholes framework except Albrecher et al. (2011). Second, all of the theoretical works above except the first paper Chesney et al. (1997) only study the Parisian options with Parisian times. The major challenge of the Parisian options with inverse occupation times is that we may need to examine multiple excursions rather than a single excursion as in Parisian times. Third, the original or generalized knock-in Parisian options mentioned above may become valueless much earlier than the maturity since it always needs to take certain amount of time to trigger knock-in Parisian options. However, in the over-the-counter market, the value of

a knock-in option usually keeps to be positive before its maturity.

Hence, in this chapter, we are motivated to introduce some hybrid Barrier-Parisian options with both Parisian times and inverse occupation times in the context of a general underlying diffusion model. Such new options are knocked in or out only if the underlying asset price has continuously or cumulatively stayed in a range for a certain amount of time or has hit some barrier prior to expiration. Since a knock-in Barrier-Parisian option always has the chance to be triggered before maturity, it will maintain a positive value until maturity.

As in Section 2.2.2, we define a time-homogeneous diffusion process  $X = \{X_t, t \geq 0\}$  on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$  with dynamics

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad (6.1)$$

where  $X_0 = x_0$  is the initial price. See Carr and Nadtochiy (2011) for a recent study of static hedging of Barrier options where the underlying asset is also modeled by (6.1).

Next we define three stopping times that will be studied in this chapter. For  $x \in \mathbb{R}$ , define the first hitting time of  $X$  by

$$T_x = \inf \{t \geq 0 : X_t = x\}.$$

For  $a \in \mathbb{R}$  and  $c > 0$ , denote by  $\tau_{a+}(c)$  and  $\tau_{a-}(c)$  the first times when  $X$  has continuously stayed above and below level  $a$  for  $c$  units of time, respectively. Namely,

$$\tau_{a\pm}(c) = \inf \{t \geq 0 : t - l_{a\mp}(t) > c\} \quad \text{with} \quad l_{a\mp}(t) = \sup \{u \leq t : X_u \stackrel{\leq}{\geq} a\}. \quad (6.2)$$

Further, denote by  $\tilde{\tau}_{a+}(c)$  and  $\tilde{\tau}_{a-}(c)$  the first times when  $X$  has cumulatively stayed above and below level  $a$  for  $c$  units of time, respectively. Namely,

$$\tilde{\tau}_{a\pm}(c) = \inf \left\{ t \geq c : \int_0^t 1_{\{X_s \gtrless a\}} ds \geq c \right\}. \quad (6.3)$$

As in Section 2.1,  $\tau_{a\pm}(c)$  are called Parisian times and  $\tilde{\tau}_{a\pm}(c)$  are called inverse occupation times.

For  $a < b$ ,  $c > 0$ ,  $x \in \mathbb{R}$ , we are interested in valuation of some hybrid Barrier-Parisian options with pricing kernels

$$\mathbb{P}^x \{T_b \wedge \tau_{a\pm}(c) < T, X_T \in dy\}, \quad \mathbb{P}^x \{T_b \wedge \tilde{\tau}_{a\pm}(c) < T, X_T \in dy\} \quad (6.4)$$

for a fixed maturity  $T$ . More generally, we will solve the following Laplace transforms of hitting time, Parisian time and inverse occupation time:

$$\mathbb{P}^x \{T_b < \tau_{a\pm}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\}, \quad \mathbb{P}^x \{\tau_{a\pm}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\}, \quad (6.5)$$

and

$$\mathbb{P}^x \{T_b < \tilde{\tau}_{a\pm}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\}, \quad \mathbb{P}^x \{\tilde{\tau}_{a\pm}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\}, \quad (6.6)$$

where  $e_\lambda$  is an independent exponential random variable with rate  $\lambda > 0$ . Note that we will obtain the Laplace transforms of pricing kernels in (6.4) by adding the two relations in (6.5) and two relations in (6.6), respectively.

Moreover, the structure of stopping times in (6.5) and (6.6) is more general than the one in (5.2). We adopt a similar perturbation approach as in Chapter 4 to study these Laplace transforms. Compared with the differentiability argument in

Chapter 5, the computation by the perturbation approach is simpler and can also be extended to general Markov processes with or without jumps.

The rest of the chapter is arranged as follows. In Section 6.2, we define some auxiliary functions. Section 6.3 focuses on the Laplace transforms of Parisian times and hitting times. And Section 6.4 focuses on the Laplace transforms of inverse occupation times and hitting times. By applying the main results in Sections 6.3 and 6.4, we define some new hybrid Barrier-Parisian options and derive their pricing formulas in Sections 6.5.1 and 6.5.2. The examples under the Black-Scholes framework are solved explicitly in Section 6.5.3. A short summary and some remarks are given in Section 6.6. Some of the proofs are postponed to Section 6.7.

## 6.2 Auxiliary Functions

We first introduce some auxiliary functions that will be used in the rest of this chapter. For  $a \leq x \leq b$  and  $c > 0$ , we define

$$A_\lambda(x; a, b, c) = \mathbb{P}^x \{T_a < T_b \wedge c \wedge e_\lambda\},$$

$$A_\lambda(x; b, a, c) = \mathbb{P}^x \{T_b < T_a \wedge c \wedge e_\lambda\},$$

$$B_\lambda(x, dy; a, b, c) = \mathbb{P}^x \{c < T_a \wedge T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} / \lambda,$$

and for  $x \leq a$ ,

$$A_\lambda(x; a, c) = \mathbb{P}^x \{T_a < c \wedge e_\lambda\},$$

$$B_\lambda(x, dy; a, c) = \mathbb{P}^x \{c < T_a \wedge e_\lambda, X_{e_\lambda} \in dy\} / \lambda.$$



In Section 6.5.3, all of the auxiliary functions defined above are solved explicitly for a linear Brownian motion  $X$ .

In general,  $A_\lambda(x; a, b, c)$  and  $A_\lambda(x; b, a, c)$  solve a standard parabolic PDE (6.7) with eigenvalue  $\lambda > 0$ . The proof of the following Lemma 6.1 is postponed to Section 6.7.

**Lemma 6.1.** *Suppose that  $u_1(x, t)$  and  $u_2(x, t)$  solve the PDE*

$$u_t(x, t) = \mu(x)u_x(x, t) + \frac{1}{2}\sigma^2(x)u_{xx}(x, t) - \lambda u(x, t), \quad a < x < b, t > 0, \quad (6.7)$$

*with the boundary conditions  $u_1(a, t) = 1$ ,  $u_1(b, t) = 0$ ,  $u_2(a, t) = 0$ , and  $u_2(b, t) = 1$  for  $t \geq 0$  while  $u_1(x, 0) = u_2(x, 0) = 0$  for  $a < x < b$ . Then  $u_1(x, t) = A_\lambda(x; a, b, t)$  and  $u_2(x, t) = A_\lambda(x; b, a, t)$  for  $a \leq x \leq b$  and  $t \geq 0$ .*

In addition to PDE techniques, we can also study these auxiliary functions by Laplace transform. In fact, the Laplace transforms of  $A_\lambda(x; a, b, c)$ ,  $A_\lambda(x; b, a, c)$  and  $A_\lambda(x; a, c)$  with respect to  $c$  can be solved explicitly. Suppose that  $e_\delta$  is another exponential random variable with rate  $\delta > 0$  which is independent of  $e_\lambda$  and  $X$ . For  $a \leq x \leq b$ , we have

$$A_{\lambda, \delta}(x; a, b) = \mathbb{P}^x \{T_a < e_\lambda \wedge T_b \wedge e_\delta\} = \frac{f_{\lambda+\delta}(x, b)}{f_{\lambda+\delta}(a, b)}, \quad (6.8)$$

$$A_{\lambda, \delta}(x; b, a) = \mathbb{P}^x \{T_b < e_\lambda \wedge T_a \wedge e_\delta\} = \frac{f_{\lambda+\delta}(a, x)}{f_{\lambda+\delta}(a, b)}, \quad (6.9)$$

and for  $x \leq a$ ,

$$A_{\lambda, \delta}(x; a) = \mathbb{P}^x \{T_a < e_\lambda \wedge e_\delta\} = \frac{g_{\lambda+\delta}^+(x)}{g_{\lambda+\delta}^+(a)}. \quad (6.10)$$

Since the auxiliary functions  $B_\lambda(x, dy; a, b, c)$  and  $B_\lambda(x, dy; a, c)$  have more complex structures, we decompose them into some fundamental quantities that have been extensively studied for time-homogeneous diffusion processes; see, e.g., Borodin and Salminen (2002) for a collection. The proof of the following Lemma 6.2 is postponed to Section 6.7.

**Lemma 6.2.** *For  $a \leq x \leq b$ , we have*

$$\begin{aligned} B_\lambda(x, dy; a, b, c) &= H_\lambda(x, dy) - A_\lambda(x; a, b, c)H_\lambda(a, dy) - A_\lambda(x; b, a, c)H_\lambda(b, dy) \\ &\quad - \int_0^c \mathbb{P}^x \{X_s \in dy\} e^{-\lambda s} ds \\ &\quad + \int_0^c \int_0^s \mathbb{P}^a \{X_{s-t} \in dy\} \mathbb{P}^x \{T_a < T_b, T_a \in dt\} e^{-\lambda s} ds \\ &\quad + \int_0^c \int_0^s \mathbb{P}^a \{X_{s-t} \in dy\} \mathbb{P}^x \{T_b < T_a, T_b \in dt\} e^{-\lambda s} ds, \end{aligned}$$

and for  $x \leq a$ , we have

$$\begin{aligned} B_\lambda(x, dy; a, c) &= H_\lambda(x, dy) - A_\lambda(x; a, c)H_\lambda(a, dy) - \int_0^c \mathbb{P}^x \{X_s \in dy\} e^{-\lambda s} ds \\ &\quad + \int_0^c \int_0^s \mathbb{P}^a \{X_{s-t} \in dy\} \mathbb{P}^x \{T_a \in dt\} e^{-\lambda s} ds. \end{aligned}$$

Since  $e_\delta$  is another exponential random variable with rate  $\delta > 0$  which is independent of  $e_\lambda$  and  $X$ . We denote the Laplace transforms of  $B_\lambda(x, dy; a, b, c)$  and  $B_\lambda(x, dy; a, c)$  with respect to  $c$  by

$$B_{\lambda, \delta}(x, dy; a, b) = \mathbb{P}^x \{e_\delta < T_a \wedge T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} / \lambda,$$

$$B_{\lambda, \delta}(x, dy; a) = \mathbb{P}^x \{e_\delta < T_a \wedge e_\lambda, X_{e_\lambda} \in dy\} / \lambda,$$

respectively. By Lemma 6.2, we obtain the following Corollary and its proof is postponed to Section 6.7.

**Corollary 6.3.** *For  $a \leq x \leq b$ , we have*

$$\begin{aligned}
& B_{\lambda,\delta}(x, dy; a, b) \\
&= H_\lambda(x, dy) - \frac{f_{\lambda+\delta}(x, b)}{f_{\lambda+\delta}(a, b)} H_\lambda(a, dy) - \frac{f_{\lambda+\delta}(a, x)}{f_{\lambda+\delta}(a, b)} H_\lambda(b, dy) \\
&\quad - H_{\lambda+\delta}(x, dy) + \frac{f_{\lambda+\delta}(x, b)}{f_{\lambda+\delta}(a, b)} H_{\lambda+\delta}(a, dy) + \frac{f_{\lambda+\delta}(a, x)}{f_{\lambda+\delta}(a, b)} H_{\lambda+\delta}(b, dy), \quad (6.11)
\end{aligned}$$

and for  $x \leq a$ ,

$$\begin{aligned}
& B_{\lambda,\delta}(x, dy; a) \\
&= H_\lambda(x, dy) - \frac{g_{\lambda+\delta}^+(x)}{g_{\lambda+\delta}^+(a)} H_\lambda(a, dy) - H_{\lambda+\delta}(x, dy) + \frac{g_{\lambda+\delta}^+(x)}{g_{\lambda+\delta}^+(a)} H_{\lambda+\delta}(a, dy). \quad (6.12)
\end{aligned}$$

### 6.3 On Parisian Times and Hitting Times

We first consider the case with an upward hitting time and an upward Parisian time. This structure is more complex than that of regular Parisian options in the sense that we need to examine both the length and the height of upward excursions. The following theorem generalizes the main results in Gauthier (2002) to time-homogeneous diffusion processes. We adopt a perturbation approach which is very different from the classical Brownian meander approach in Gauthier (2002). Compared with Gauthier (2002), our computation is much simpler and can be extended to more general Markov processes with or without jumps.

**Theorem 6.4.** *We have*

$$\mathbb{P}^a \{T_b < \tau_{a+}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} = \frac{\lambda H_\lambda(b, dy) D^+ A_\lambda(a; b, a, c)}{\psi_\lambda^+(a) - D^+ A_\lambda(a; a, b, c)}, \quad (6.13)$$

$$\mathbb{P}^a \{\tau_{a+}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} = \frac{\lambda D^+ B_\lambda(a, dy; a, b, c)}{\psi_\lambda^+(a) - D^+ A_\lambda(a; a, b, c)}, \quad (6.14)$$

where the one-sided derivatives  $D^+A_\lambda(a; a, b, c) = \lim_{\varepsilon \rightarrow 0+} (A_\lambda(a + \varepsilon; a, b, c) - 1) / \varepsilon$ ,  $D^+A_\lambda(a; b, a, c) = \lim_{\varepsilon \rightarrow 0+} A_\lambda(a + \varepsilon; b, a, c) / \varepsilon$  and  $D^+B_\lambda(a, dy; a, b, c) = \lim_{\varepsilon \rightarrow 0+} B_\lambda(a + \varepsilon, dy; a, b, c) / \varepsilon$ .

*Proof.* We define an approximation of  $\tau_{a+}(c)$  by  $\tau_{a+}^\varepsilon(c)$ , which represents the first time when the duration of the excursions starting with level  $a + \varepsilon$  and ending with level  $a$  exceeds  $c$  unites of time. Rigorously, let  $\theta$  be the shift operator such that  $X_t \circ \theta_s = X_{s+t}$ . Since  $X$  starts with  $a$ , we define  $T_a^1 = 0$ ,  $T_{a+\varepsilon}^1 = T_{a+\varepsilon}$ ,  $T_a^{i+1} = T_{a+\varepsilon}^i + T_a \circ \theta_{T_{a+\varepsilon}^i}$ , and  $T_{a+\varepsilon}^{i+1} = T_a^{i+1} + T_{a+\varepsilon} \circ \theta_{T_a^{i+1}}$  for  $i = 1, 2, \dots$ . Define

$$\tau_{a+}^\varepsilon(c) = \inf \{t \in (T_{a+\varepsilon}^i, T_a^{i+1}] : t - T_{a+\varepsilon}^i \geq c \text{ for some } i = 1, 2, \dots\}. \quad (6.15)$$

For ease of notation, let  $I_+^\varepsilon(x) = \mathbb{P}^x \{T_b < \tau_{a+}^\varepsilon(c) \wedge e_\lambda, X_{e_\lambda} \in dy\}$ . By the strong Markov property,

$$I_+^\varepsilon(a) = \mathbb{P}^a \{T_{a+\varepsilon} < e_\lambda\} I_+^\varepsilon(a + \varepsilon) = \frac{g_\lambda^+(a)}{g_\lambda^+(a + \varepsilon)} I_+^\varepsilon(a + \varepsilon), \quad (6.16)$$

and

$$\begin{aligned} I_+^\varepsilon(a + \varepsilon) &= \mathbb{P}^{a+\varepsilon} \{T_a < T_b < \tau_{a+}^\varepsilon(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} \\ &\quad + \mathbb{P}^{a+\varepsilon} \{T_b < T_a \wedge \tau_{a+}^\varepsilon(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} \\ &= \mathbb{P}^{a+\varepsilon} \{T_a < T_b \wedge c \wedge e_\lambda\} I_+^\varepsilon(a) + \mathbb{P}^{a+\varepsilon} \{T_b < T_a \wedge c \wedge e_\lambda\} \mathbb{P}^b \{X_{e_\lambda} \in dy\} \\ &= A_\lambda(a + \varepsilon; a, b, c) I_+^\varepsilon(a) + \lambda A_\lambda(a + \varepsilon; b, a, c) H_\lambda(b, dy). \end{aligned} \quad (6.17)$$

Substituting (6.17) into (6.16), solving for  $I_+^\varepsilon(a)$ , and taking limit  $\varepsilon \rightarrow 0+$ , relation (6.13) follows.

Similarly, let  $J_+^\varepsilon(x) = \mathbb{P}^x \{ \tau_{a+}^\varepsilon(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \}$ . We have,

$$J_+^\varepsilon(a) = \mathbb{P}^a \{ T_{a+\varepsilon} < e_\lambda \} J_+^\varepsilon(a + \varepsilon) = \frac{g_\lambda^+(a)}{g_\lambda^+(a + \varepsilon)} J_+^\varepsilon(a + \varepsilon), \quad (6.18)$$

and

$$\begin{aligned} J_+^\varepsilon(a + \varepsilon) &= \mathbb{P}^{a+\varepsilon} \{ T_a < \tau_{a+}^\varepsilon(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &\quad + \mathbb{P}^{a+\varepsilon} \{ \tau_{a+}^\varepsilon(c) < T_a \wedge T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &= \mathbb{P}^{a+\varepsilon} \{ T_a < T_b \wedge c \wedge e_\lambda \} J_+^\varepsilon(a) + \mathbb{P}^{a+\varepsilon} \{ c < T_a \wedge T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &= A_\lambda(a + \varepsilon; a, b, c) J_+^\varepsilon(a) + \lambda B_\lambda(a + \varepsilon, dy; a, b, c). \end{aligned} \quad (6.19)$$

Substituting (6.19) into (6.18), solving for  $J_+^\varepsilon(a)$ , and taking limit  $\varepsilon \rightarrow 0+$ , relation (6.14) follows.  $\square$

The next corollary generalizes Theorem 6.4 by assuming a general initial value of  $X$  and its proof is postponed to Section 6.7.

**Corollary 6.5.** *For  $x \leq a$ , we have*

$$\mathbb{P}^x \{ T_b < \tau_{a+}(c) \wedge e_\lambda, X_{e_\lambda} \in dy \} = \frac{g_\lambda^+(x) \lambda H_\lambda(b, dy) D^+ A_\lambda(a; b, a, c)}{g_\lambda^+(a) \psi_\lambda^+(a) - D^+ A_\lambda(a; a, b, c)}, \quad (6.20)$$

$$\mathbb{P}^x \{ \tau_{a+}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} = \frac{g_\lambda^+(x) \lambda D^+ B_\lambda(a, dy; a, b, c)}{g_\lambda^+(a) \psi_\lambda^+(a) - D^+ A_\lambda(a; a, b, c)}.$$

and for  $a \leq x \leq b$ ,

$$\begin{aligned} \mathbb{P}^x \{ T_b < \tau_{a+}(c) \wedge e_\lambda, X_{e_\lambda} \in dy \} &= A_\lambda(x; a, b, c) \frac{\lambda H_\lambda(b, dy) D^+ A_\lambda(a; b, a, c)}{\psi_\lambda^+(a) - D^+ A_\lambda(a; a, b, c)} \\ &\quad + A_\lambda(x; b, a, c) \lambda H_\lambda(b, dy), \end{aligned} \quad (6.21)$$

$$\begin{aligned} \mathbb{P}^x \{ \tau_{a+}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} &= A_\lambda(x; a, b, c) \frac{\lambda D^+ B_\lambda(a, dy; a, b, c)}{\psi_\lambda^+(a) - D^+ A_\lambda(a; a, b, c)} \\ &\quad + \lambda B_\lambda(x, dy; a, b, c). \end{aligned}$$

Letting  $c \rightarrow \infty$ , by the fourth identity in Theorem 2.7, we have

$$\lim_{c \rightarrow \infty} \frac{D^+ A_\lambda(a; b, a, c)}{\psi_\lambda^+(a) - D^+ A_\lambda(a; a, b, c)} = \frac{\frac{f_{2,\lambda}(a,a)}{f_\lambda(a,b)}}{\psi_\lambda^+(a) - \frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)}} = \frac{g_\lambda^+(a)}{g_\lambda^+(b)}.$$

Therefore, by the last identity in Theorem 2.7, we can easily verify that relations (6.20) and (6.21) reduce to the same trivial identity

$$\mathbb{P}^x \{T_b < e_\lambda\} \mathbb{P}^b \{X_{e_\lambda} \in dy\} = \frac{g_\lambda^+(x)}{g_\lambda^+(b)} \lambda H_\lambda(b, dy) \quad (6.22)$$

as  $c \rightarrow \infty$ . Integrating the relations in Corollary 6.5 with respect to  $dy$  over  $\mathbb{R}$  and using (2.16), we obtain the following corollary. In fact, an immediate corollary of the following results is that we are able to essentially generalize the results of probability of liquidation (5.3) in Theorem 5.1 to a finite-time horizon by the methodology of inverse Laplace transforms.

**Corollary 6.6.** *For  $x \leq a$ , we have*

$$\begin{aligned} \mathbb{P}^x \{T_b < \tau_{a+}(c) \wedge e_\lambda\} &= \frac{g_\lambda^+(x)}{g_\lambda^+(a)} \frac{D^+ A_\lambda(a; b, a, c)}{\psi_\lambda^+(a) - D^+ A_\lambda(a; a, b, c)}, \\ \mathbb{P}^x \{\tau_{a+}(c) < T_b \wedge e_\lambda\} &= \frac{g_\lambda^+(x)}{g_\lambda^+(a)} \frac{\lambda D^+ B_\lambda(a; a, b, c)}{\psi_\lambda^+(a) - D^+ A_\lambda(a; a, b, c)}. \end{aligned}$$

and for  $a \leq x \leq b$ ,

$$\begin{aligned} \mathbb{P}^x \{T_b < \tau_{a+}(c) \wedge e_\lambda\} &= A_\lambda(x; a, b, c) \frac{D^+ A_\lambda(a; b, a, c)}{\psi_\lambda^+(a) - D^+ A_\lambda(a; a, b, c)} + A_\lambda(x; b, a, c), \\ \mathbb{P}^x \{\tau_{a+}(c) < T_b \wedge e_\lambda\} &= A_\lambda(x; a, b, c) \frac{\lambda D^+ B_\lambda(a; a, b, c)}{\psi_\lambda^+(a) - D^+ A_\lambda(a; a, b, c)} + \lambda B_\lambda(x; a, b, c), \end{aligned}$$

where  $\lambda B_\lambda(x; a, b, c) = \mathbb{P}^x \{c < T_a \wedge T_b \wedge e_\lambda\}$  and  $\lambda D^+ B_\lambda(a; a, b, c) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{P}^{a+\varepsilon} \{c < T_a \wedge T_b \wedge e_\lambda\} / \varepsilon$ .

**Remark 6.7.** *In particular, suppose that  $X$  is a standard Brownian motion and  $x = a$ . Corollary 6.6 retrieves Theorem 1 of Gauthier (2002) with  $f(\cdot) \equiv 1$  by relations (6.63) and (6.64) in Section 6.7.*

Next we consider the case with an upward hitting time and a downward Parisian time. This structure is simpler than the one in Theorem 6.4 since we only need to examine the length of downward excursions. Actually, it is a special case of the structure with double space barriers and double time periods as in Anderluh and van der Weide (2009) and Dassios and Wu (2011). However, we point out that their main results can still be recovered by the this perturbation approach with a little effort.

**Theorem 6.8.** *We have*

$$\mathbb{P}^a \{T_b < \tau_{a-}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} = \frac{-\frac{f_{1,\lambda}(a,a)}{f_\lambda(a,b)} \lambda H_\lambda(b, dy)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + D^- A_\lambda(a; a, c)}, \quad (6.23)$$

$$\mathbb{P}^a \{\tau_{a-}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} = \frac{-\lambda D^- B_\lambda(a, dy; a, c)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + D^- A_\lambda(a; a, c)}, \quad (6.24)$$

where the one-sided derivatives  $D^- A_\lambda(a; a, c) = \lim_{\varepsilon \rightarrow 0^+} (1 - A_\lambda(a - \varepsilon; a, c)) / \varepsilon$  and  $D^- B_\lambda(a, dy; a, c) = -\lim_{\varepsilon \rightarrow 0^+} B_\lambda(a - \varepsilon, dy; a, c) / \varepsilon$ .

*Proof.* We define an approximation of  $\tau_{a-}(t)$  by  $\tau_{a-}^\varepsilon(c)$ , which represents the first time when the duration of the excursions starting with level  $a - \varepsilon$  and ending with level  $a$  exceeds  $c$  units of time. Rigorously, let  $\theta$  be the shift operator such that  $X_t \circ \theta_s = X_{s+t}$ . Since  $X$  starts with  $a$ , we define  $T_a^1 = 0$ ,  $T_{a-\varepsilon}^1 = T_{a-\varepsilon}$ ,  $T_a^{i+1} = T_{a-\varepsilon}^i + T_a \circ \theta_{T_{a-\varepsilon}^i}$ , and  $T_{a-\varepsilon}^{i+1} = T_a^{i+1} + T_{a-\varepsilon} \circ \theta_{T_a^{i+1}}$  for  $i = 1, 2, \dots$ . Define

$$\tau_{a-}^\varepsilon(t) = \inf \{t \in (T_{a-\varepsilon}^i, T_a^{i+1}] : t - T_{a-\varepsilon}^i \geq c \text{ for some } i = 1, 2, \dots\}. \quad (6.25)$$

For ease of notation, let  $I_-^\varepsilon(x) = \mathbb{P}^x \{T_b < \tau_{a-}^\varepsilon(c) \wedge e_\lambda, X_{e_\lambda} \in dy\}$ . By the strong Markov property,

$$\begin{aligned}
I_-^\varepsilon(a) &= \mathbb{P}^a \{T_{a-\varepsilon} < T_b < \tau_{a-}^\varepsilon(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} \\
&\quad + \mathbb{P}^a \{T_b < T_{a-\varepsilon} \wedge \tau_{a-}^\varepsilon(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} \\
&= \mathbb{P}^a \{T_{a-\varepsilon} < T_b \wedge e_\lambda\} I_-^\varepsilon(a - \varepsilon) + \mathbb{P}^a \{T_b < T_{a-\varepsilon} \wedge e_\lambda\} \mathbb{P}^b \{X_{e_\lambda} \in dy\} \\
&= \frac{f_\lambda(a, b)}{f_\lambda(a - \varepsilon, b)} I_-^\varepsilon(a - \varepsilon) + \frac{f_\lambda(a - \varepsilon, a)}{f_\lambda(a - \varepsilon, b)} \lambda H_\lambda(b, dy), \tag{6.26}
\end{aligned}$$

and

$$I_-^\varepsilon(a - \varepsilon) = \mathbb{P}^{a-\varepsilon} \{T_a < e_\lambda \wedge c\} I_-^\varepsilon(a) = A_\lambda(a - \varepsilon; a, c) I_-^\varepsilon(a). \tag{6.27}$$

Substituting (6.27) into (6.26), solving for  $I_-^\varepsilon(a)$ , and taking limit  $\varepsilon \rightarrow 0+$ , relation (6.23) follows.

Similarly, let  $J_-^\varepsilon(x) = \mathbb{P}^x \{\tau_{a-}^\varepsilon(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\}$ . We have,

$$J_-^\varepsilon(a) = \mathbb{P}^a \{T_{a-\varepsilon} < T_b \wedge e_\lambda\} J_-^\varepsilon(a - \varepsilon) = \frac{f_\lambda(a, b)}{f_\lambda(a - \varepsilon, b)} J_-^\varepsilon(a - \varepsilon), \tag{6.28}$$

and

$$\begin{aligned}
J_-^\varepsilon(a - \varepsilon) &= \mathbb{P}^{a-\varepsilon} \{T_a < \tau_{a-}^\varepsilon(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} \\
&\quad + \mathbb{P}^{a-\varepsilon} \{\tau_{a-}^\varepsilon(c) < T_a \wedge T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} \\
&= \mathbb{P}^{a-\varepsilon} \{T_a < e_\lambda \wedge c\} J_-^\varepsilon(a) + \mathbb{P}^{a-\varepsilon} \{c < T_a \wedge e_\lambda, X_{e_\lambda} \in dy\} \\
&= A_\lambda(a - \varepsilon; a, c) J_-^\varepsilon(a) + \lambda B_\lambda(a - \varepsilon, dy; a, c). \tag{6.29}
\end{aligned}$$

Substituting (6.29) into (6.28), solving for  $J_-^\varepsilon(a)$ , and taking limit  $\varepsilon \rightarrow 0+$ , relation (6.24) follows. □



The next corollary generalizes Theorem 6.8 by assuming a general initial value of  $X$ . We skip the proof since it is similar to Corollary 6.5.

**Corollary 6.9.** *For  $x \leq a$ , we have*

$$\begin{aligned} \mathbb{P}^x \{T_b < \tau_{a-}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} &= A_\lambda(x; a, c) \frac{-\frac{f_{1,\lambda}(a,a)}{f_\lambda(a,b)} \lambda H_\lambda(b, dy)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + D^- A_\lambda(a; a, c)}, \\ \mathbb{P}^x \{\tau_{a-}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} &= A_\lambda(x; a, c) \frac{-\lambda D^- B_\lambda(a, dy; a, c)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + D^- A_\lambda(a; a, c)} \\ &\quad + \lambda B_\lambda(x, dy; a, c), \end{aligned} \quad (6.30)$$

and for  $a \leq x \leq b$ ,

$$\begin{aligned} \mathbb{P}^x \{T_b < \tau_{a-}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} &= \frac{f_\lambda(x, b)}{f_\lambda(a, b)} \frac{-\frac{f_{1,\lambda}(a,a)}{f_\lambda(a,b)} \lambda H_\lambda(b, dy)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + D^- A_\lambda(a; a, c)} \\ &\quad + \frac{f_\lambda(a, x)}{f_\lambda(a, b)} \lambda H_\lambda(b, dy), \\ \mathbb{P}^x \{\tau_{a-}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} &= \frac{f_\lambda(x, b)}{f_\lambda(a, b)} \frac{-\lambda D^- B_\lambda(a, dy; a, c)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + D^- A_\lambda(a; a, c)}. \end{aligned} \quad (6.31)$$

Since  $\lim_{c \rightarrow \infty} A_\lambda(x; a, c) = \frac{g_\lambda^+(x)}{g_\lambda^+(a)}$  and  $\lim_{c \rightarrow \infty} D^- A_\lambda(a; a, c) = \psi_\lambda^+(a)$ , by the last two identities in Theorem 2.7, relations (6.30) and (6.31) reduce to the same trivial identity (6.22) as  $c \rightarrow \infty$ . Integrating the relations in Corollary 6.9 with respect to  $dy$  over  $\mathbb{R}$  and using (2.16), we obtain the following results.

**Corollary 6.10.** *For  $x \leq a$ , we have*

$$\begin{aligned} \mathbb{P}^x \{T_b < \tau_{a-}(c) \wedge e_\lambda\} &= A_\lambda(x; a, c) \frac{-\frac{f_{1,\lambda}(a,a)}{f_\lambda(a,b)}}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + D^- A_\lambda(a; a, c)}, \\ \mathbb{P}^x \{\tau_{a-}(c) < T_b \wedge e_\lambda\} &= A_\lambda(x; a, c) \frac{-\lambda D^- B_\lambda(a; a, c)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + D^- A_\lambda(a; a, c)} + \lambda B_\lambda(x; a, c) \end{aligned} \quad (6.32)$$

and for  $a \leq x \leq b$ ,

$$\begin{aligned} \mathbb{P}^x \{T_b < \tau_{a-}(c) \wedge e_\lambda\} &= \frac{f_\lambda(x, b)}{f_\lambda(a, b)} \frac{-\frac{f_{1,\lambda}(a, a)}{f_\lambda(a, b)}}{-\frac{f_{1,\lambda}(a, b)}{f_\lambda(a, b)} + D^- A_\lambda(a; a, c)} + \frac{f_\lambda(a, x)}{f_\lambda(a, b)}, \quad (6.33) \\ \mathbb{P}^x \{\tau_{a-}(c) < T_b \wedge e_\lambda\} &= \frac{f_\lambda(x, b)}{f_\lambda(a, b)} \frac{-\lambda D^- B_\lambda(a; a, c)}{-\frac{f_{1,\lambda}(a, b)}{f_\lambda(a, b)} + D^- A_\lambda(a; a, c)}. \end{aligned}$$

where  $\lambda B_\lambda(x; a, c) = \mathbb{P}^x\{c < T_a \wedge e_\lambda\}$  and  $-\lambda D^- B_\lambda(a; a, c) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{P}^{a-\varepsilon}\{c < T_a \wedge e_\lambda\}/\varepsilon$ .

**Remark 6.11.** *In particular, suppose that  $X$  is a standard Brownian motion, Corollary 6.9 retrieves Theorem 3 of Dassios and Wu (2010) with  $d_1 = 0$ . Further, suppose that  $X$  is a standard Brownian motion, Corollary 6.10 retrieves Theorem 3.2 of Anderluh and van der Weide (2009) with  $D_2 = 0$  and relations (24)–(26) of Theorem 1 of Dassios and Wu (2011) with  $d_1 = 0$ . Please see Section 6.6 for the details.*

#### 6.4 On Inverse Occupation Times and Hitting Times

In this section, we focus on inverse occupation times and hitting times. Our results help to fill in the blank of many previous works of Parisian options or related applications in which only Parisian times are considered. Due to the aggregation feature, the structure of occupation times is more complex than Parisian times. Therefore, we need another Laplace transform with respect to the argument  $c$  representing the required length of related excursions. Essentially, the memoryless property of exponential random variables offsets the obligation to cumulate the length of time.

In the rest of this chapter, denote by  $e_\delta$  another exponential random variable with rate  $\delta > 0$  which is independent of  $e_\lambda$  and  $X$ . We first consider the case with an

upward hitting time and an upward inverse occupation time. The following Theorem 6.12 and Corollaries 6.13 and 6.14 are parallel to Theorem 6.4 and Corollaries 6.5 and 6.6, respectively.

**Theorem 6.12.** *We have*

$$\mathbb{P}^a \{T_b < \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy\} = \frac{\frac{f_{2,\lambda+\delta}(a,a)}{f_{\lambda+\delta}(a,b)} \lambda H_\lambda(b, dy)}{\psi_\lambda^+(a) - \frac{f_{1,\lambda+\delta}(a,b)}{f_{\lambda+\delta}(a,b)}}, \quad (6.34)$$

$$\mathbb{P}^a \{\tilde{\tau}_{a+}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} = \frac{\lambda D^+ B_{\lambda,\delta}(a, dy; a, b)}{\psi_\lambda^+(a) - \frac{f_{1,\lambda+\delta}(a,b)}{f_{\lambda+\delta}(a,b)}}, \quad (6.35)$$

where  $D^+ B_{\lambda,\delta}(a, dy; a, b) = \lim_{\varepsilon \rightarrow 0^+} B_{\lambda,\delta}(a + \varepsilon, dy; a, b)/\varepsilon$  can be explicitly solved by Corollary 6.3.

*Proof.* We define an approximation of  $\tilde{\tau}_{a+}(c)$  by  $\tilde{\tau}_{a+}^\varepsilon(c)$ , which represents the first time when the sum of durations of the excursions starting with level  $a + \varepsilon$  and ending with level  $a$  exceeds  $c$  unites of time. Rigorously, following the notation in the proof of Theorem 6.4, we define

$$\tilde{\tau}_{a+}^\varepsilon(c) = \inf \left\{ t : \sum_{n=1}^{\infty} (T_a^{n+1} \wedge t - T_{a+\varepsilon}^n \wedge t) \geq c \right\}. \quad (6.36)$$

Let  $\tilde{I}_+^\varepsilon(x) = \mathbb{P}^x \{T_b < \tilde{\tau}_{a+}^\varepsilon(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy\}$ , we have

$$\tilde{I}_+^\varepsilon(a) = \mathbb{P}^a \{T_{a+\varepsilon} < e_\lambda\} \tilde{I}_+^\varepsilon(a + \varepsilon) = \frac{g_\lambda^+(a)}{g_\lambda^+(a + \varepsilon)} \tilde{I}_+^\varepsilon(a + \varepsilon), \quad (6.37)$$

and

$$\begin{aligned} & \tilde{I}_+^\varepsilon(a + \varepsilon) \\ &= \mathbb{P}^{a+\varepsilon} \{T_a < T_b < \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy\} + \mathbb{P}^{a+\varepsilon} \{T_b < T_a \wedge \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy\} \\ &= \mathbb{P}^{a+\varepsilon} \{T_a < T_b \wedge e_\delta \wedge e_\lambda\} \tilde{I}_+^\varepsilon(a) + \mathbb{P}^{a+\varepsilon} \{T_b < T_a \wedge e_\delta \wedge e_\lambda, X_{e_\lambda} \in dy\} \\ &= \frac{f_{\lambda+\delta}(a + \varepsilon, b)}{f_{\lambda+\delta}(a, b)} \tilde{I}_+^\varepsilon(a) + \frac{f_{\lambda+\delta}(a, a + \varepsilon)}{f_{\lambda+\delta}(a, b)} \lambda H_\lambda(b, dy). \end{aligned} \quad (6.38)$$

Substituting (6.38) into (6.37), solving for  $\tilde{I}_+^\varepsilon(a)$ , and taking limit  $\varepsilon \rightarrow 0+$ , relation (6.34) follows.

Similarly, let  $\tilde{J}_+^\varepsilon(x) = \mathbb{P}^x \{ \tilde{\tau}_{a+}^\varepsilon(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \}$ . By the strong Markov property,

$$\tilde{J}_+^\varepsilon(a) = \mathbb{P}^a \{ T_{a+\varepsilon} < e_\lambda \} \tilde{J}_+^\varepsilon(a + \varepsilon) = \frac{g_\lambda^+(a)}{g_\lambda^+(a + \varepsilon)} \tilde{J}_+^\varepsilon(a + \varepsilon), \quad (6.39)$$

and

$$\begin{aligned} & \tilde{J}_+^\varepsilon(a + \varepsilon) \\ &= \mathbb{P}^{a+\varepsilon} \{ T_a < \tilde{\tau}_{a+}^\varepsilon(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} + \mathbb{P}^{a+\varepsilon} \{ \tilde{\tau}_{a+}^\varepsilon(e_\delta) < T_a \wedge T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &= \mathbb{P}^{a+\varepsilon} \{ T_a < e_\delta \wedge T_b \wedge e_\lambda \} \tilde{J}_+^\varepsilon(a) + \mathbb{P}^{a+\varepsilon} \{ e_\delta < T_a \wedge T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &= \frac{f_{\lambda+\delta}(a + \varepsilon, b)}{f_{\lambda+\delta}(a, b)} \tilde{J}_+^\varepsilon(a) + \lambda B_{\lambda, \delta}(a + \varepsilon, dy; a, b). \end{aligned} \quad (6.40)$$

Substituting (6.40) into (6.39), solving for  $\tilde{J}_+^\varepsilon(a)$ , and taking limit  $\varepsilon \rightarrow 0+$ , relation (6.35) follows.  $\square$

The next corollary generalizes Theorem 6.12 by assuming a general initial value of  $X$  and its proof is postponed to Section 6.7.

**Corollary 6.13.** *For  $x \leq a$ , we have*

$$\begin{aligned} \mathbb{P}^x \{ T_b < \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy \} &= \frac{g_\lambda^+(x)}{g_\lambda^+(a)} \frac{f_{2, \lambda+\delta}(a, a)}{f_{\lambda+\delta}(a, b)} \lambda H_\lambda(b, dy), \\ \mathbb{P}^x \{ \tilde{\tau}_{a+}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} &= \frac{g_\lambda^+(x)}{g_\lambda^+(a)} \frac{\lambda D^+ B_{\lambda, \delta}(a, dy; a, b)}{\psi_\lambda^+(a) - \frac{f_{1, \lambda+\delta}(a, b)}{f_{\lambda+\delta}(a, b)}}, \end{aligned} \quad (6.41)$$

and for  $a \leq x \leq b$ ,

$$\begin{aligned} \mathbb{P}^x \{T_b < \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy\} &= \frac{f_{\lambda+\delta}(x, b)}{f_{\lambda+\delta}(a, b)} \frac{f_{2, \lambda+\delta}(a, a)}{f_{\lambda+\delta}(a, b)} \lambda H_\lambda(b, dy) \\ &\quad + \frac{f_{\lambda+\delta}(a, x)}{f_{\lambda+\delta}(a, b)} \lambda H_\lambda(b, dy), \quad (6.42) \\ \mathbb{P}^x \{\tilde{\tau}_{a+}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} &= \frac{f_{\lambda+\delta}(x, b)}{f_{\lambda+\delta}(a, b)} \frac{\lambda D^+ B_{\lambda, \delta}(a, dy; a, b)}{\psi_\lambda^+(a) - \frac{f_{1, \lambda+\delta}(a, b)}{f_{\lambda+\delta}(a, b)}} + \lambda B_{\lambda, \delta}(x, dy; a, b). \end{aligned}$$

Letting  $\delta \rightarrow 0+$  and using the last two identities in Theorem 2.7, relations (6.41) and (6.42) reduce to the same trivial identity (6.22). Integrating the two relations in Theorem 6.12 with respect to  $dy$  over  $\mathbb{R}$  and using (2.16) and Corollary 6.3, we obtain the following result.

**Corollary 6.14.** *For  $x \leq a$ , we have*

$$\begin{aligned} \mathbb{P}^x \{T_b < \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda\} &= \frac{g_\lambda^+(x)}{g_\lambda^+(a)} \frac{\frac{f_{2, \lambda+\delta}(a, a)}{f_{\lambda+\delta}(a, b)}}{\psi_\lambda^+(a) - \frac{f_{1, \lambda+\delta}(a, b)}{f_{\lambda+\delta}(a, b)}}, \\ \mathbb{P}^x \{\tilde{\tau}_{a+}(e_\delta) < T_b \wedge e_\lambda\} &= \frac{\delta}{\lambda + \delta} \frac{g_\lambda^+(x) - \frac{f_{1, \lambda+\delta}(a, b)}{f_{\lambda+\delta}(a, b)} - \frac{f_{2, \lambda+\delta}(a, a)}{f_{\lambda+\delta}(a, b)}}{g_\lambda^+(a) - \frac{f_{1, \lambda+\delta}(a, b)}{f_{\lambda+\delta}(a, b)}}, \end{aligned}$$

and for  $a \leq x \leq b$ ,

$$\begin{aligned} \mathbb{P}^x \{T_b < \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda\} &= \frac{f_{\lambda+\delta}(x, b)}{f_{\lambda+\delta}(a, b)} \frac{\frac{f_{2, \lambda+\delta}(a, a)}{f_{\lambda+\delta}(a, b)}}{\psi_\lambda^+(a) - \frac{f_{1, \lambda+\delta}(a, b)}{f_{\lambda+\delta}(a, b)}} + \frac{f_{\lambda+\delta}(a, x)}{f_{\lambda+\delta}(a, b)}, \\ \mathbb{P}^x \{\tilde{\tau}_{a+}(e_\delta) < T_b \wedge e_\lambda\} &= \frac{\delta}{\lambda + \delta} \frac{f_{\lambda+\delta}(x, b) - \frac{f_{1, \lambda+\delta}(a, b)}{f_{\lambda+\delta}(a, b)} - \frac{f_{2, \lambda+\delta}(a, a)}{f_{\lambda+\delta}(a, b)}}{f_{\lambda+\delta}(a, b) - \frac{f_{1, \lambda+\delta}(a, b)}{f_{\lambda+\delta}(a, b)}} \\ &\quad + \frac{\delta}{\lambda + \delta} \left( 1 - \frac{f_{\lambda+\delta}(x, b)}{f_{\lambda+\delta}(a, b)} - \frac{f_{\lambda+\delta}(a, x)}{f_{\lambda+\delta}(a, b)} \right). \end{aligned}$$

Next we consider the case with an upward hitting time and an downward inverse occupation time. The following Theorem 6.15 and Corollaries 6.16 and 6.17 are parallel to Theorem 6.8 and Corollaries 6.9 and 6.10, respectively.

**Theorem 6.15.** *We have*

$$\mathbb{P}^a \{T_b < \tilde{\tau}_{a-}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in \mathrm{d}y\} = \frac{-\frac{f_{1,\lambda}(a,a)}{f_\lambda(a,b)} \lambda H_\lambda(b, \mathrm{d}y)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + \psi_{\lambda+\delta}^+(a)}, \quad (6.43)$$

$$\mathbb{P}^a \{\tilde{\tau}_{a-}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in \mathrm{d}y\} = \frac{-\lambda D^- B_{\lambda,\delta}(a, \mathrm{d}y; a)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + \psi_{\lambda+\delta}^+(a)}, \quad (6.44)$$

where  $-D^- B_{\lambda,\delta}(a, \mathrm{d}y; a) = \lim_{\varepsilon \rightarrow 0+} B_{\lambda,\delta}(a - \varepsilon, \mathrm{d}y; a)/\varepsilon$  can be explicitly solved by Corollary 6.3.

*Proof.* We define an approximation of  $\tilde{\tau}_{a-}(c)$  by  $\tilde{\tau}_{a-}^\varepsilon(c)$ , which represents which represents the first time when the sum of durations of the excursions starting with level  $a - \varepsilon$  and ending with level  $a$  exceeds the time period  $c$ . Rigorously, following the notation in the proof of Theorem 6.8, we define

$$\tilde{\tau}_{a-}^\varepsilon(c) = \inf \left\{ t : \sum_{n=1}^{\infty} (T_a^{n+1} \wedge t - T_{a-\varepsilon}^n \wedge t) \geq c \right\}. \quad (6.45)$$

For ease of notation, let  $\tilde{I}_-^\varepsilon(x) = \mathbb{P}^x \{T_b < \tilde{\tau}_{a-}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in \mathrm{d}y\}$ . By the strong Markov property,

$$\begin{aligned} \tilde{I}_-^\varepsilon(a) &= \mathbb{P}^a \{T_{a-\varepsilon} < T_b < \tilde{\tau}_{a-}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in \mathrm{d}y\} \\ &\quad + \mathbb{P}^a \{T_b < T_{a-\varepsilon} \wedge \tilde{\tau}_{a-}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in \mathrm{d}y\} \\ &= \mathbb{P}^a \{T_{a-\varepsilon} < T_b \wedge e_\lambda\} \tilde{I}_-^\varepsilon(a - \varepsilon) + \mathbb{P}^a \{T_b < T_{a-\varepsilon} \wedge e_\lambda\} \mathbb{P}^b \{X_{e_\lambda} \in \mathrm{d}y\} \\ &= \frac{f_\lambda(a, b)}{f_\lambda(a - \varepsilon, b)} \tilde{I}_-^\varepsilon(a - \varepsilon) + \frac{f_\lambda(a - \varepsilon, a)}{f_\lambda(a - \varepsilon, b)} \lambda H_\lambda(b, \mathrm{d}y), \end{aligned} \quad (6.46)$$

and

$$\tilde{I}_-^\varepsilon(a - \varepsilon) = \mathbb{P}^{a-\varepsilon} \{T_a < e_\delta \wedge e_\lambda\} \tilde{I}_+^\varepsilon(a) = \frac{g_{\lambda+\delta}^+(a - \varepsilon)}{g_{\lambda+\delta}^+(a)} \tilde{I}_+^\varepsilon(a) \quad (6.47)$$

Substituting (6.47) into (6.46), solving for  $\tilde{I}_-^\varepsilon(a)$ , and taking limit  $\varepsilon \rightarrow 0+$ , relation (6.43) follows.

Similarly, let  $\tilde{J}_-^\varepsilon(x) = \mathbb{P}^x \{ \tilde{\tau}_{a-}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \}$ . We have

$$\tilde{J}_-^\varepsilon(a) = \mathbb{P}^a \{ T_{a-\varepsilon} < T_b \wedge e_\lambda \} \tilde{J}_-^\varepsilon(a - \varepsilon) = \frac{f_\lambda(a, b)}{f_\lambda(a - \varepsilon, b)} \tilde{J}_-^\varepsilon(a - \varepsilon), \quad (6.48)$$

and

$$\begin{aligned} \tilde{J}_-^\varepsilon(a - \varepsilon) &= \mathbb{P}^{a-\varepsilon} \{ T_a < \tilde{\tau}_{a-}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &\quad + \mathbb{P}^{a-\varepsilon} \{ \tilde{\tau}_{a-}(e_\delta) < T_a \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &= \mathbb{P}^{a-\varepsilon} \{ T_a < e_\delta \wedge e_\lambda \} \tilde{J}_-^\varepsilon(a) + \mathbb{P}^{a-\varepsilon} \{ e_\delta < T_a \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &= \frac{g_{\lambda+\delta}^+(a - \varepsilon)}{g_{\lambda+\delta}^+(a)} \tilde{J}_-^\varepsilon(a) + \lambda B_\lambda(a - \varepsilon, dy; a, e_\delta). \end{aligned} \quad (6.49)$$

Substituting (6.49) into (6.48), solving for  $\tilde{J}_-^\varepsilon(a)$ , and taking limit  $\varepsilon \rightarrow 0+$ , relation (6.44) follows.  $\square$

The next corollary generalizes Theorem 6.15 by assuming a general initial value of  $X$ . We skip the proof since it is similar to Corollary 6.13.

**Corollary 6.16.** *For  $x \leq a$ , we have*

$$\mathbb{P}^x \{ T_b < \tilde{\tau}_{a-}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy \} = \frac{g_{\lambda+\delta}^+(x)}{g_{\lambda+\delta}^+(a)} \frac{-\frac{f_{1,\lambda}(a,a)}{f_\lambda(a,b)}}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + \psi_{\lambda+\delta}^+(a)} \lambda H_\lambda(b, dy), \quad (6.50)$$

$$\mathbb{P}^x \{ \tilde{\tau}_{a-}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} = \frac{g_{\lambda+\delta}^+(x)}{g_{\lambda+\delta}^+(a)} \frac{-\lambda D^- B_{\lambda,\delta}(a, dy; a)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + \psi_{\lambda+\delta}^+(a)} + \lambda B_{\lambda,\delta}(x, dy; a),$$

and for  $a \leq x \leq b$ ,

$$\begin{aligned} \mathbb{P}^x \{ T_b < \tilde{\tau}_{a-}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy \} &= \frac{f_\lambda(x, b) - \frac{f_{1,\lambda}(a,a)}{f_\lambda(a,b)} \lambda H_\lambda(b, dy)}{f_\lambda(a, b) - \frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + \psi_{\lambda+\delta}^+(a)} \\ &\quad + \frac{f_\lambda(a, x)}{f_\lambda(a, b)} \lambda H_\lambda(b, dy), \end{aligned} \quad (6.51)$$

$$\mathbb{P}^x \{ \tilde{\tau}_{a-}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} = \frac{f_\lambda(x, b) - \lambda D^- B_{\lambda,\delta}(a, dy; a)}{f_\lambda(a, b) - \frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + \psi_{\lambda+\delta}^+(a)}.$$

Letting  $\delta \rightarrow 0+$  and using the last two identities in Theorem 2.7, we have that relations (6.50) and (6.51) reduce to the same trivial identity (6.22). Integrating the two relations in Corollary 6.16 with respect to  $dy$  over  $\mathbb{R}$  using (2.16) and Corollary 6.3, we obtain the following result.

**Corollary 6.17.** *For  $x \leq a$ , we have*

$$\begin{aligned} \mathbb{P}^x \{T_b < \tilde{\tau}_{a-}(e_\delta) \wedge e_\lambda\} &= \frac{g_{\lambda+\delta}^+(x)}{g_{\lambda+\delta}^+(a)} \frac{-\frac{f_{1,\lambda}(a,a)}{f_\lambda(a,b)}}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + \psi_{\lambda+\delta}^+(a)}, \\ \mathbb{P}^x \{\tilde{\tau}_{a-}(e_\delta) < T_b \wedge e_\lambda\} &= \frac{\delta}{\lambda + \delta} \frac{g_{\lambda+\delta}^+(x)}{g_{\lambda+\delta}^+(a)} \frac{\psi_{\lambda+\delta}^+(a)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + \psi_{\lambda+\delta}^+(a)} + \frac{\delta}{\lambda + \delta} \left(1 - \frac{g_{\lambda+\delta}^+(x)}{g_{\lambda+\delta}^+(a)}\right), \end{aligned}$$

and for  $a \leq x \leq b$ ,

$$\begin{aligned} \mathbb{P}^x \{T_b < \tilde{\tau}_{a-}(e_\delta) \wedge e_\lambda\} &= \frac{f_\lambda(x,b)}{f_\lambda(a,b)} \frac{-\frac{f_{1,\lambda}(a,a)}{f_\lambda(a,b)}}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + \psi_{\lambda+\delta}^+(a)} + \frac{f_\lambda(a,x)}{f_\lambda(a,b)}, \\ \mathbb{P}^x \{\tilde{\tau}_{a-}(e_\delta) < T_b \wedge e_\lambda\} &= \frac{\delta}{\lambda + \delta} \frac{f_\lambda(x,b)}{f_\lambda(a,b)} \frac{\psi_{\lambda+\delta}^+(a)}{-\frac{f_{1,\lambda}(a,b)}{f_\lambda(a,b)} + \psi_{\lambda+\delta}^+(a)}. \end{aligned}$$

## 6.5 Barrier-Parisian Options

We assume that the financial prices of all contingent claims are given by discounted expectations of respective payoffs at time of maturity  $T$  under some risk-neutral measure  $\mathbb{Q}$ . We also assume that, under the risk-neutral measure  $\mathbb{Q}$ , the underlying asset  $\{S_t, t \geq 0\} = S = e^X$ , where  $X$  is a time-homogeneous diffusion defined in (2.5); see Carr and Nadtochiy (2011) for a recent study of static hedging of Barrier options where the underlying asset is also modeled by a time-homogeneous diffusion (2.5). The interest rate  $r$  is assumed to be constant. Further, we assume that

$$b > a \vee K \vee S_0 \quad \text{and} \quad 0 < c < T, \quad (6.52)$$



where  $a$  and  $b$  are two space barriers,  $c$  is a time barrier, and  $K$  is the strike price.

For a real number  $x$ , define the first hitting times of the processes  $X$  and  $S$  by

$$T_x^X = \inf \{t \geq 0 : X_t = x\} \quad \text{and} \quad T_x^S = \inf \{t \geq 0 : U_t = x\},$$

respectively. It is clear that  $T_x^S = T_{\log x}^X$ .

### 6.5.1 Barrier-Parisian options with Parisian times

We define two knock-in Barrier-Parisian call options with Parisian times as follows:

$$\begin{aligned} C_{\pm}(S_0, a, b, c, r, K, T) &= e^{-rT} \mathbf{E}_{\mathbb{Q}} [(S_T - K)_+; T_b^S \wedge \tau_{a\pm}^S(c) < T] \\ &= e^{-rT} \mathbf{E}_{\mathbb{Q}}^{\log S_0} [(e^{X_T} - K)_+; T_{\log b}^X \wedge \tau_{(\log a)\pm}^X(c) < T], \end{aligned} \quad (6.53)$$

where the Parisian times  $\tau_{a\pm}^S(c)$  are defined in (6.2). In order to trigger such knock-in Barrier-Parisian options, the underlying asset  $S$  must either hit a higher level  $b$  or constantly stay above/below level  $a$  for  $c$  units of time prior to expiration. In particular, the knock-in Barrier-Parisian options  $C_{\pm}(S_0, a, b, c, r, K, T)$  reduce to original knock-in Parisian options with Parisian times when  $b \rightarrow \infty$ . We can define many other types of knock-out contracts and put contracts in a similar way.

Different from knock-out options, knock-in options in the market usually keep positive values prior to expiration. However, under some circumstances, knock-in Parisian options may become valueless much earlier than expiration since it always takes time to trigger Parisian options. One of the advantages of our knock-in Barrier-Parisian options (6.53) is that they inherit the property of knock-in exotic options to keep positive value prior to expiration.

Note that the price of the knock-in Barrier-Parisian call options can be bounded below and above by the price of some Barrier options and European options. Specifically, for  $S_0 \leq a$ , we have

$$e^{-rT} \mathbf{E}_{\mathbb{Q}} [(S_T - K)_+; T_b^S < T] \leq C_+(S_0, a, b, c, r, K, T) \leq e^{-rT} \mathbf{E}_{\mathbb{Q}} [(S_T - K)_+; T_a^S < T], \quad (6.54)$$

$$e^{-rT} \mathbf{E}_{\mathbb{Q}} [(S_T - K)_+; T_b^S < T] \leq C_-(S_0, a, b, c, r, K, T) \leq e^{-rT} \mathbf{E}_{\mathbb{Q}} [(S_T - K)_+]. \quad (6.55)$$

And for  $a < S_0 < b$ , we have

$$e^{-rT} \mathbf{E}_{\mathbb{Q}} [(S_T - K)_+; T_b^S < T] \leq C_+(S_0, a, b, c, r, K, T) \leq e^{-rT} \mathbf{E}_{\mathbb{Q}} [(S_T - K)_+], \quad (6.56)$$

$$\begin{aligned} e^{-rT} \mathbf{E}_{\mathbb{Q}} [(S_T - K)_+; T_b^S < T] &\leq C_-(S_0, a, b, c, r, K, T) \\ &\leq e^{-rT} \mathbf{E}_{\mathbb{Q}} [(S_T - K)_+; T_a^S \wedge T_b^S < T]. \end{aligned} \quad (6.57)$$

We define the pricing kernel of  $C_{\pm}(S_0, a, b, c, r, K, T)$  by

$$h^{\pm}(\log S_0, dy; T, c) = e^{-rT} \mathbf{P}_{\mathbb{Q}}^{\log S_0} \{T_{\log b}^X \wedge \tau_{(\log a)_{\pm}}^X(c) < T, X_T \in dy\}$$

and its Laplace transform with respect to  $T$  by

$$\begin{aligned} h_{\lambda}^{\pm}(\log S_0, dy; c) &= \frac{1}{\lambda} \mathbf{E}_{\mathbb{Q}}^{\log S_0} [e^{-re\lambda}; T_{\log b}^X \wedge \tau_{(\log a)_{\pm}}^X(c) < e\lambda, X_{e\lambda} \in dy] \\ &= \frac{1}{\lambda + r} \mathbf{P}_{\mathbb{Q}}^{\log S_0} \{T_{\log b}^X \wedge \tau_{(\log a)_{\pm}}^X(c) < e\lambda + r, X_{e\lambda + r} \in dy\}. \end{aligned} \quad (6.58)$$

Trivially, we have

$$C_{\pm}(S_0, a, b, c, r, K, T) = \int_{\log K}^{\infty} (e^y - K) h^{\pm}(\log S_0, dy; T, c).$$

By Corollaries 6.5 and 6.9, we immediately have the following results.

**Corollary 6.18.** *For  $S_0 \leq a$ ,*

$$\begin{aligned} h_\lambda^+(\bar{S}_0, dy; c) &= \frac{g_{\lambda+r}^+(\bar{S}_0)}{g_{\lambda+r}^+(\bar{a})} \frac{H_{\lambda+r}(\bar{b}, dy) D^+ A_{\lambda+r}(\bar{a}; \bar{b}, \bar{a}, c) + D^+ B_{\lambda+r}(\bar{a}, dy; \bar{a}, \bar{b}, c)}{\psi_{\lambda+r}^+(\bar{a}) - D^+ A_{\lambda+r}(\bar{a}; \bar{a}, \bar{b}, c)}, \\ h_\lambda^-(\bar{S}_0, dy; c) &= A_{\lambda+r}(\bar{S}_0; \bar{a}, c) \frac{-\frac{f_{1,\lambda+r}(\bar{a}, \bar{a})}{f_{\lambda+r}(\bar{a}, \bar{b})} H_{\lambda+r}(\bar{b}, dy) - D^- B_{\lambda+r}(\bar{a}, dy; \bar{a}, c)}{-\frac{f_{1,\lambda+r}(\bar{a}, \bar{b})}{f_{\lambda+r}(\bar{a}, \bar{b})} + D^- A_{\lambda+r}(\bar{a}; \bar{a}, c)} \\ &\quad + B_{\lambda+r}(\bar{S}_0, dy; \bar{a}, c), \end{aligned}$$

for  $a \leq S_0 \leq b$ ,

$$\begin{aligned} h_\lambda^+(\bar{S}_0, dy; c) &= A_{\lambda+r}(\bar{S}_0; \bar{a}, \bar{b}, c) \frac{H_{\lambda+r}(\bar{b}, dy) D^+ A_{\lambda+r}(\bar{a}; \bar{b}, \bar{a}, c) + D^+ B_{\lambda+r}(\bar{a}, dy; \bar{a}, \bar{b}, c)}{\psi_{\lambda+r}^+(\bar{a}) - D^+ A_{\lambda+r}(\bar{a}; \bar{a}, \bar{b}, c)} \\ &\quad + A_{\lambda+r}(\bar{S}_0; \bar{b}, \bar{a}, c) H_{\lambda+r}(\bar{b}, dy) + B_{\lambda+r}(\bar{S}_0, dy; \bar{a}, \bar{b}, c), \\ h_\lambda^-(\bar{S}_0, dy; c) &= \frac{f_{\lambda+r}(\bar{S}_0, \bar{b}) - \frac{f_{1,\lambda+r}(\bar{a}, \bar{a})}{f_{\lambda+r}(\bar{a}, \bar{b})} H_{\lambda+r}(\bar{b}, dy) - D^- B_{\lambda+r}(\bar{a}, dy; \bar{a}, c)}{f_{\lambda+r}(\bar{a}, \bar{b}) - \frac{f_{1,\lambda+r}(\bar{a}, \bar{b})}{f_{\lambda+r}(\bar{a}, \bar{b})} + D^- A_{\lambda+r}(\bar{a}; \bar{a}, c)} \\ &\quad + \frac{f_{\lambda+r}(\bar{a}, \bar{S}_0)}{f_{\lambda+r}(\bar{a}, \bar{b})} H_{\lambda+r}(\bar{b}, dy), \end{aligned}$$

where  $\bar{S}_0 = \log S_0$ ,  $\bar{a} = \log a$  and  $\bar{b} = \log b$ .

### 6.5.2 Barrier-Parisian options with inverse occupation times

We define two knock-in Barrier-Parisian call options with inverse occupation times by

$$\begin{aligned} \tilde{C}_\pm(S_0, a, b, c, r, K, T) &= e^{-rT} \mathbf{E}_{\mathbb{Q}} \left[ (S_T - K)_+; T_b^S \wedge \tilde{\tau}_{a\pm}^S(c) < T \right] \\ &= e^{-rT} \mathbf{E}_{\mathbb{Q}}^{\log S_0} \left[ (e^{X_T} - K)_+; T_{\log b}^X \wedge \tilde{\tau}_{(\log a)\pm}^X(c) < T \right], \quad (6.59) \end{aligned}$$

where the inverse occupation times  $\tilde{\tau}_{a\pm}^S(c)$  are defined in (6.3). In order to trigger such knock-in Barrier-Parisian options, the underlying asset  $S$  must either hit a higher

level  $b$  or cumulatively stay above/below level  $a$  for  $c$  units of time prior to expiration. In particular, the knock-in Barrier-Parisian options  $\tilde{C}_{\pm}(S_0, a, b, c, r, K, T)$  reduce to original knock-in Parisian options with inverse occupation time when  $b \rightarrow \infty$ .

Further,  $\tilde{C}_{\pm}(S_0, a, b, c, r, K, T)$  have the same bounds of  $C_{\pm}(S_0, a, b, c, r, K, T)$  as (6.54)–(6.57). Moreover, since  $\tau_{a\pm}^S(c) \leq \tilde{\tau}_{a\pm}^S(c)$ , we have

$$C_{\pm}(S_0, a, b, c, r, K, T) \leq \tilde{C}_{\pm}(S_0, a, b, c, r, K, T).$$

We define the pricing kernel of  $\tilde{C}_{\pm}(S_0, a, b, c, r, K, T)$  by

$$\tilde{h}^{\pm}(\log S_0, dy; T, c) = e^{-rT} \mathbf{P}_{\mathbb{Q}}^{\log S_0} \{T_{\log b}^X \wedge \tilde{\tau}_{(\log a)\pm}^X(c) < T, X_T \in dy\}$$

and its double Laplace transform with respect to  $T$  and  $c$  by

$$\begin{aligned} & \tilde{h}_{\lambda, \delta}^{\pm}(\log S_0, dy) \\ &= \frac{1}{\lambda} \mathbf{E}_{\mathbb{Q}}^{\log S_0} [e^{-re_{\lambda}}; T_{\log b}^X \wedge \tilde{\tau}_{(\log a)\pm}^X(e_{\delta}) < e_{\lambda}, X_{e_{\lambda}} \in dy] \\ &= \frac{1}{\lambda + r} \mathbf{P}_{\mathbb{Q}}^{\log S_0} \{T_{\log b}^X \wedge \tilde{\tau}_{(\log a)\pm}^X(c) < e_{\lambda+r}, X_{e_{\lambda+r}} \in dy\}. \end{aligned} \quad (6.60)$$

Trivially, we have

$$\tilde{C}_{\pm}(S_0, a, b, c, r, K, T) = \int_{\log K}^{\infty} (e^y - K) \tilde{h}^{\pm}(\log S_0, dy; T, c).$$

By Corollaries 6.13 and 6.16, we immediately have the following results.

**Corollary 6.19.** *For  $S_0 \leq a$ ,*

$$\begin{aligned} \tilde{h}_{\lambda, \delta}^+(\bar{S}_0, dy) &= \frac{g_{\lambda+r}^+(\bar{S}_0) \frac{f_{2, \lambda+\delta+r}(\bar{a}, \bar{a})}{f_{\lambda+\delta+r}(\bar{a}, \bar{b})} H_{\lambda+r}(\bar{b}, dy) + D^+ B_{\lambda+r, \delta}(\bar{a}, dy; \bar{a}, \bar{b})}{g_{\lambda+r}^+(\bar{a}) \frac{\psi_{\lambda+r}^+(\bar{a}) - \frac{f_{1, \lambda+\delta+r}(\bar{a}, \bar{b})}{f_{\lambda+\delta+r}(\bar{a}, \bar{b})}}}, \\ \tilde{h}_{\lambda, \delta}^-(\bar{S}_0, dy) &= \frac{g_{\lambda+\delta+r}^+(\bar{S}_0) - \frac{f_{1, \lambda+r}(\bar{a}, \bar{a})}{f_{\lambda+r}(\bar{a}, \bar{b})} H_{\lambda+r}(\bar{b}, dy) - D^- B_{\lambda+r, \delta}(\bar{a}, dy; \bar{a})}{g_{\lambda+\delta+r}^+(\bar{a}) \frac{-\frac{f_{1, \lambda+r}(\bar{a}, \bar{b})}{f_{\lambda+r}(\bar{a}, \bar{b})} + \psi_{\lambda+\delta+r}^+(\bar{a})}} \\ &\quad + B_{\lambda+r, \delta}(\bar{S}_0, dy; \bar{a}), \end{aligned}$$

and for  $a \leq S_0 \leq b$ ,

$$\begin{aligned}\tilde{h}_{\lambda,\delta}^+(\bar{S}_0, dy) &= \frac{f_{\lambda+\delta+r}(\bar{S}_0, \bar{b}) \frac{f_{2,\lambda+\delta+r}(\bar{a}, \bar{a})}{f_{\lambda+\delta+r}(\bar{a}, \bar{b})} H_{\lambda+r}(\bar{b}, dy) + D^+ B_{\lambda+r,\delta}(\bar{a}, dy; \bar{a}, \bar{b})}{f_{\lambda+\delta+r}(\bar{a}, \bar{b}) \psi_{\lambda+r}^+(\bar{a}) - \frac{f_{1,\lambda+\delta+r}(\bar{a}, \bar{b})}{f_{\lambda+\delta+r}(\bar{a}, \bar{b})}} \\ &\quad + \frac{f_{\lambda+\delta+r}(\bar{a}, \bar{S}_0)}{f_{\lambda+\delta+r}(\bar{a}, \bar{b})} H_{\lambda+r}(\bar{b}, dy) + \frac{1}{\delta} B_{\lambda+r,\delta}(x, dy; \bar{a}, \bar{b}), \\ \tilde{h}_{\lambda,\delta}^-(\bar{S}_0, dy) &= \frac{f_{\lambda+r}(\bar{S}_0, \bar{b}) - \frac{f_{1,\lambda+r}(\bar{a}, \bar{a})}{f_{\lambda+r}(\bar{a}, \bar{b})} H_{\lambda+r}(\bar{b}, dy) - D^- B_{\lambda+r,\delta}(\bar{a}, dy; \bar{a})}{f_{\lambda+r}(\bar{a}, \bar{b}) - \frac{f_{1,\lambda+r}(\bar{a}, \bar{b})}{f_{\lambda+r}(\bar{a}, \bar{b})} + \psi_{\lambda+\delta+r}^+(\bar{a})} \\ &\quad + \frac{f_{\lambda+r}(\bar{a}, \bar{S}_0)}{f_{\lambda+r}(\bar{a}, \bar{b})} H_{\lambda+r}(\bar{b}, dy),\end{aligned}$$

where  $\bar{S}_0 = \log S_0$ ,  $\bar{a} = \log a$  and  $\bar{b} = \log b$ .

### 6.5.3 Examples under the Black-Scholes framework

Suppose that  $X$  is a Brownian motion with drift satisfying  $dX_t = \mu dt + dW_t$ ,

$t \geq 0$ . Equation (2.6) is reduced to

$$\frac{1}{2}g''(x) + \mu g'(x) = \lambda g(x), \quad \lambda > 0.$$

Denote by  $\beta_\lambda^\pm = -\mu \pm \mu_\lambda$  where  $\mu_\lambda = \sqrt{\mu^2 + 2\lambda}$ , we have

$$g_\lambda^\pm(x) = e^{\beta_\lambda^\pm x}, \quad \psi_\lambda^\pm(\cdot) = \pm \beta_\lambda^\pm, \quad f_\lambda(x, y) = e^{\beta_\lambda^- x + \beta_\lambda^+ y} - e^{\beta_\lambda^- y + \beta_\lambda^+ x},$$

and

$$H_\lambda(x, dy) = \begin{cases} \frac{1}{\mu_\lambda} e^{-\beta_\lambda^+(y-x)} dy, & x \leq y, \\ \frac{1}{\mu_\lambda} e^{\beta_\lambda^-(x-y)} dy, & x \geq y. \end{cases}$$

For  $x \leq a$ , by (6.10), (6.12) and formula (9) on Page 650 of Borodin and Salminen (2002), we have

$$A_\lambda(x; a, c) = e^{-\beta_\lambda^+(a-x)} \Phi(\mu_\lambda \sqrt{c} - (a-x)/\sqrt{c}) + e^{-\beta_\lambda^-(a-x)} \Phi(-\mu_\lambda \sqrt{c} - (a-x)/\sqrt{c}),$$

and

$$\lambda B_\lambda(x; a, c) = e^{-\lambda c} \left( 1 - \Phi \left( \mu\sqrt{c} - (a-x)/\sqrt{c} \right) - e^{2\mu(a-x)} \Phi \left( -\mu\sqrt{c} - (a-x)/\sqrt{c} \right) \right),$$

where  $\Phi$  is the standard normal distribution. It follows from Corollary 6.10 that

$$\mathbb{P}^a \{T_b < \tau_{a-}(c) \wedge e_\lambda\} = \frac{e^{-\beta_\lambda^+(b-a)}}{e^{-2\mu\lambda(b-a)} + (1 - e^{-2\mu\lambda(b-a)}) \left( \Phi(\mu\lambda\sqrt{c}) + \frac{1}{\mu\lambda\sqrt{2\pi c}} e^{-c\mu_\lambda^2/2} \right)}, \quad (6.61)$$

$$\mathbb{P}^a \{\tau_{a-}(c) < T_b \wedge e_\lambda\} = \frac{\frac{1 - e^{-2\mu\lambda(b-a)}}{2\mu\lambda} e^{-\lambda c} \left( 2\mu\Phi(\mu\sqrt{c}) - 2\mu + \sqrt{\frac{2}{\pi c}} e^{-c\mu^2/2} \right)}{e^{-2\mu\lambda(b-a)} + (1 - e^{-2\mu\lambda(b-a)}) \left( \Phi(\mu\lambda\sqrt{c}) + \frac{1}{\mu\lambda\sqrt{2\pi c}} e^{-c\mu_\lambda^2/2} \right)}. \quad (6.62)$$

The above relations (6.61) and (6.62) retrieve relations (24) and (25) of Dassios and Wu (2011) with  $d_1 = 0$ .

For  $a \leq x \leq b$ , by (6.8), we have

$$A_\lambda(x; a, b, c) = e^{-\mu(x-a)} \int_0^c e^{-\mu_\lambda^2 z/2} ss_z(b-x, b-a) dz$$

$$A_\lambda(x; b, a, c) = e^{\mu(b-x)} \int_0^c e^{-\mu_\lambda^2 z/2} ss_z(x-a, b-a) dz,$$

where  $ss_z(v, t) = \sum_{k=-\infty}^{\infty} \frac{t-v+2kt}{\sqrt{2\pi z^{3/2}}} e^{-(t-v+2kt)^2/2z}$ ,  $v < t$  is defined on Page 641 of Borodin and Salminen (2002). By (6.11), we have

$$\lambda B_\lambda(x; a, b, c) = e^{-\lambda c} \left( 1 - e^{-\mu(x-a)} \int_0^c e^{-\mu^2 z/2} ss_z(b-x, b-a) dz \right) - e^{-\lambda c + \mu(b-x)} \int_0^c e^{-\mu^2 z/2} ss_z(x-a, b-a) dz.$$

It follows from Corollary 6.6 that

$$\mathbb{P}^a \{T_b < \tau_{a+}(c) \wedge e_\lambda\} = \frac{e^{\mu(b-a)} \int_0^c e^{-\mu_\lambda^2 z/2} ss_{1,z}(0, b-a) dz}{\beta_\lambda^+ + \mu + \int_0^c e^{-\mu_\lambda^2 z/2} ss_{1,z}(b-a, b-a) dz}, \quad (6.63)$$

and

$$\begin{aligned} & P^a \{ \tau_{a+}(c) < T_b \wedge e_\lambda \} \\ &= \frac{e^{-\lambda c} \left( \mu + \int_0^c e^{-\mu^2 z/2} s s_{1,z}(b-a, b-a) dz - e^{\mu(b-a)} \int_0^c e^{-\mu^2 z/2} s s_{1,z}(0, b-a) dz \right)}{\beta_\lambda^+ + \mu + \int_0^c e^{-\mu^2 z/2} s s_{1,z}(b-a, b-a) dz}, \end{aligned} \quad (6.64)$$

where  $s s_{1,z}(v, t) = \frac{\partial s s_z(v, t)}{\partial v} = \sum_{k=-\infty}^{\infty} \frac{-z+(t-v+2kt)^2}{\sqrt{2\pi z^{5/2}}} e^{-(t-v+2kt)^2/2z}$ . In particular, when  $\mu = 0$ , relations (6.63) and (6.64) retrieve Theorem 1 of Gauthier (2002) with  $f(\cdot) \equiv 1$ .

By (6.12), we have

$$\begin{aligned} & \frac{1}{dy} B_{\lambda, \delta}(x, dy; a) \\ &= \begin{cases} \frac{e^{-\beta_\lambda^+(y-x)} e^{-\beta_{\lambda+\delta}^+(a-x)} e^{-\beta_\lambda^+(y-a)}}{\mu_\lambda}, & x \leq a \leq y, \\ \frac{e^{-\beta_\lambda^+(y-x)} e^{-\beta_{\lambda+\delta}^+(a-x)} e^{\beta_\lambda^-(a-y)}}{\mu_\lambda} - \frac{e^{-\beta_{\lambda+\delta}^+(y-x)} e^{-\beta_{\lambda+\delta}^+(a-x)} e^{\beta_{\lambda+\delta}^-(a-y)}}{\mu_{\lambda+\delta}}, & x \leq y \leq a, \\ \frac{e^{\beta_\lambda^-(x-y)} e^{-\beta_{\lambda+\delta}^+(a-x)} e^{\beta_\lambda^-(a-y)}}{\mu_\lambda} - \frac{e^{\beta_{\lambda+\delta}^-(x-y)} e^{-\beta_{\lambda+\delta}^+(a-x)} e^{\beta_{\lambda+\delta}^-(a-y)}}{\mu_{\lambda+\delta}}, & y \leq x \leq a. \end{cases} \end{aligned}$$

By partial fraction and formulas (5) and (10) on Page 650 of Borodin and Salminen (2002), we obtain the inversion of  $B_{\lambda, \delta}(x, dy; a)/dy$  in the following corollary. We define  $N_\lambda^\pm(z) = \Phi\left(-\frac{z}{\sqrt{c}} \pm \mu_\lambda \sqrt{c}\right)$ .

**Corollary 6.20.** *For  $x \leq a \leq y$ , we have*

$$\begin{aligned} & \frac{1}{dy} B_\lambda(x, dy; a, c) \\ &= \frac{1}{\mu_\lambda} \left( e^{-\beta_\lambda^+(y-x)} - e^{-\beta_\lambda^+(y-a)} \left( e^{-\beta_\lambda^-(a-x)} N_\lambda^-(a-x) + e^{-\beta_\lambda^+(a-x)} N_\lambda^+(a-x) \right) \right), \end{aligned}$$

for  $x \leq y \leq a$ ,

$$\begin{aligned} & \frac{1}{dy} B_\lambda(x, dy; a, c) \\ &= \frac{1}{\mu_\lambda} \left( e^{-\beta_\lambda^+(y-x)} - e^{\beta_\lambda^-(a-y)} \left( e^{-\beta_\lambda^-(a-x)} N_\lambda^-(a-x) + e^{-\beta_\lambda^+(a-x)} N_\lambda^+(a-x) \right) \right) \\ & \quad - \frac{1}{\mu_\lambda} \left( e^{-\beta_\lambda^+(y-x)} N_\lambda^+(y-x) - e^{-\beta_\lambda^-(y-x)} N_\lambda^-(y-x) \right) \\ & \quad + \frac{e^{\mu(y-x)}}{\mu_\lambda} \left( e^{-\mu_\lambda(2a-x-y)} N_\lambda^+(2a-x-y) - e^{\mu_\lambda(2a-x-y)} N_\lambda^-(2a-x-y) \right), \end{aligned}$$

and for  $y \leq x \leq a$ ,

$$\begin{aligned} & \frac{1}{dy} B_\lambda(x, dy; a, c) \\ &= \frac{1}{\mu_\lambda} \left( e^{\beta_\lambda^-(x-y)} - e^{\beta_\lambda^-(a-y)} \left( e^{-\beta_\lambda^-(a-x)} N_\lambda^-(a-x) + e^{-\beta_\lambda^+(a-x)} N_\lambda^+(a-x) \right) \right) \\ & \quad - \frac{1}{\mu_\lambda} \left( e^{\beta_\lambda^-(x-y)} N_\lambda^+(x-y) - e^{\beta_\lambda^+(x-y)} N_\lambda^-(x-y) \right) \\ & \quad + \frac{e^{\mu(y-x)}}{\mu_\lambda} \left( e^{-\mu_\lambda(2a-x-y)} N_\lambda^+(2a-x-y) - e^{\mu_\lambda(2a-x-y)} N_\lambda^-(2a-x-y) \right). \end{aligned}$$

Similarly, we can also obtain  $B_\lambda(x, dy; a, b, c)$  by inverting  $B_{\lambda,\delta}(x, dy; a, b)$ .

## 6.6 Summary and Some Remarks

We introduced some hybrid Barrier-Parisian options and derived their pricing formulas. More generally, we derived the Laplace transforms of hitting times, Parisian times and inverse occupation times with mixed relations for time-homogeneous diffusion processes by a perturbation approach. As we pointed out, this approach is an efficient alternative to study many other derivatives related to occupation times or Parisian times such as corridor options and quantile options.

Admitting that Parisian style options can overcome many disadvantages of regular Barrier options, an important issue has not been well dealt with for these



options is hedging. Although the computation of price and Greeks of these options is very challenging, it is still possible to design effective and efficient hedging strategies for these options. The reasons are as follows. First, the price of Parisian style options can be bounded below and above by some Barrier options and European options. Since the hedging of Barrier options and European options has been extensively studied in the literature, we may extend some of the ideas and approaches to Parisian style options. Second, since the price of Parisian style options so as the Greeks has a higher regularity than those of Barrier options at the barriers, it is easier for the dealers to provide a good hedge. Further, it may also help to decrease the market volatility around the barriers and the cost of hedging.

## 6.7 Appendix

*Proof of Lemma 6.1.* We only prove the result for  $A_\lambda(x; a, b, t)$ . The other result of  $A_\lambda(x; a, b, t)$  can be proved in the same way. For  $X_0 = x \in (a, b)$  and  $s \in [0, T_a \wedge T_b \wedge t]$ , by Itô's formula and  $u_s(X_s, t - s) = -u_t(X_s, t - s)$ , we have

$$\begin{aligned} & d(e^{-\lambda s} u(X_s, t - s)) \\ &= e^{-\lambda s} (-\lambda u(X_s, t - s) ds - u_t(X_s, t - s) ds + u_x(X_s, t - s) (\mu(X_s) ds + \sigma(X_s) dW_s)) \\ &\quad + e^{-\lambda s} \frac{1}{2} u_{xx}(X_s, t - s) \sigma^2(X_s) ds \\ &= e^{-\lambda s} u_x(X_s, t - s) \sigma(X_s) dW_s, \end{aligned}$$

where the last step is due to equation (5.11). Hence,  $e^{-\lambda s} u(X_s, t - s)$  is a martingale for  $s \in [0, T_a \wedge T_b \wedge t]$ . It follows that

$$u(x, t) = \mathbb{E}^x [e^{-\lambda(T_a \wedge T_b \wedge t)} u(X_{T_a \wedge T_b \wedge t}, t - T_a \wedge T_b \wedge t)].$$

By the boundary conditions of  $u(x, t)$ , we obtain

$$\begin{aligned}
u(x, t) &= \mathbb{E}^x \left[ e^{-\lambda(T_a \wedge T_b \wedge t)} u(X_{T_a \wedge T_b \wedge t}, t - T_a \wedge T_b \wedge t) \right] \\
&= \mathbb{E}^x \left[ e^{-\lambda(T_a \wedge T_b)} u(X_{T_a \wedge T_b}, t - T_a \wedge T_b) 1_{\{T_a \wedge T_b \leq t\}} \right] + \mathbb{E}^x \left[ e^{-\lambda t} u(X_t, 0) 1_{\{T_a \wedge T_b > t\}} \right] \\
&= \mathbb{E}^x \left[ e^{-\lambda T_a} u(a, t - T_a) 1_{\{T_a < T_b \wedge t\}} \right] + \mathbb{E}^x \left[ e^{-\lambda T_b} u(b, t - T_b) 1_{\{T_a < T_b \wedge t\}} \right] \\
&= \mathbb{E}^x \left[ e^{-\lambda T_a}; T_a < T_b \wedge t \right] \\
&= A_\lambda(x; a, b, t).
\end{aligned}$$

This ends the proof. □

*Proof of Lemma 6.2.* We have

$$\begin{aligned}
&B_\lambda(x, dy; a, b, c) \\
&= \frac{1}{\lambda} \mathbb{P}^x \{X_{e_\lambda} \in dy\} - \mathbb{P}^x \{T_a < T_b \wedge c \wedge e_\lambda\} \frac{1}{\lambda} \mathbb{P}^a \{X_{e_\lambda} \in dy\} \\
&\quad - \mathbb{P}^x \{T_b < c \wedge T_a \wedge e_\lambda\} \frac{1}{\lambda} \mathbb{P}^b \{X_{e_\lambda} \in dy\} - \frac{1}{\lambda} \mathbb{P}^x \{e_\lambda < T_a \wedge T_b \wedge c, X_{e_\lambda} \in dy\} \\
&= H_\lambda(x, dy) - A_\lambda(x; a, b, c) H_\lambda(a, dy) - A_\lambda(x; b, a, c) H_\lambda(b, dy) \\
&\quad - \frac{1}{\lambda} \mathbb{P}^x \{e_\lambda < T_a \wedge T_b \wedge c, X_{e_\lambda} \in dy\}. \tag{6.65}
\end{aligned}$$

Further, the last term is equal to

$$\begin{aligned}
&\frac{1}{\lambda} \mathbb{P}^x \{e_\lambda < T_a \wedge T_b \wedge c, X_{e_\lambda} \in dy\} \\
&= \frac{1}{\lambda} \mathbb{P}^x \{e_\lambda < c, X_{e_\lambda} \in dy\} - \frac{1}{\lambda} \mathbb{P}^x \{T_a < e_\lambda < c, T_a < T_b, X_{e_\lambda} \in dy\} \\
&\quad - \frac{1}{\lambda} \mathbb{P}^x \{T_b < e_\lambda < c, T_b < T_a, X_{e_\lambda} \in dy\} \\
&= \int_0^c \mathbb{P}^x \{X_u \in dy\} e^{-\lambda u} du + \int_0^c \int_0^s \mathbb{P}^x \{T_a < T_b \wedge t\} \frac{d\mathbb{P}^a \{X_{s-t} \in dy\}}{dt} e^{-\lambda s} ds \\
&\quad + \int_0^c \int_0^s \mathbb{P}^x \{T_b < T_a \wedge t\} \frac{d\mathbb{P}^b \{X_{s-t} \in dy\}}{dt} e^{-\lambda s} ds
\end{aligned}$$

Relation (6.11) follows immediately by substituting the above equality into (6.65).

For  $x \leq a$ , relation (6.12) can be proved in the same way.  $\square$

*Proof of Corollary 6.3.* By Lemma 6.2 and (6.8)-(6.10), for  $a \leq x \leq b$ , it suffices to compute the following three integrals. First,

$$\begin{aligned} \int_0^\infty \delta e^{-\delta c} dc \int_0^c \mathbb{P}^x \{X_s \in dy\} e^{-\lambda s} ds &= \int_0^\infty \mathbb{P}^x \{X_s \in dy\} e^{-\lambda s} ds \int_s^\infty \delta e^{-\delta c} dc \\ &= H_{\lambda+\delta}(x, dy). \end{aligned}$$

Further,

$$\begin{aligned} &\int_0^\infty \delta e^{-\delta c} dc \int_0^c e^{-\lambda s} ds \int_0^s \mathbb{P}^a \{X_{s-t} \in dy\} \mathbb{P}^x \{T_a < T_b, T_a \in dt\} \\ &= \int_0^\infty \int_0^s \mathbb{P}^a \{X_{s-t} \in dy\} \mathbb{P}^x \{T_a < T_b, T_a \in dt\} e^{-(\lambda+\delta)s} ds \\ &= \int_0^\infty \int_t^\infty \mathbb{P}^a \{X_{s-t} \in dy\} e^{-(\lambda+\delta)s} ds \mathbb{P}^x \{T_a < T_b, T_a \in dt\} \\ &= \int_0^\infty e^{-(\lambda+\delta)t} \mathbb{P}^x \{T_a < T_b, T_a \in dt\} H_{\lambda+\delta}(a, dy) \\ &= \int_0^\infty (\lambda + \delta) e^{-(\lambda+\delta)t} \mathbb{P}^x \{T_a < T_b, T_a < t\} dt H_{\lambda+\delta}(a, dy) \\ &= \frac{f_{\lambda+\delta}(x, b)}{f_{\lambda+\delta}(a, b)} H_{\lambda+\delta}(a, dy). \end{aligned}$$

Similarly,

$$\begin{aligned} &\int_0^\infty \delta e^{-\delta c} dc \int_0^c e^{-\lambda s} ds \int_0^s \mathbb{P}^a \{X_{s-t} \in dy\} \mathbb{P}^x \{T_b < T_a, T_b \in dt\} \\ &= \frac{f_{\lambda+\delta}(a, x)}{f_{\lambda+\delta}(a, b)} H_{\lambda+\delta}(b, dy). \end{aligned}$$

The relation (6.11) then follows immediately. The other relation (6.12) can be proved in the same way.  $\square$

*Proof of Corollary 6.5.* By Theorem 6.4, for  $x \leq a$ , we have

$$\mathbb{P}^x \{T_b < \tau_{a+}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} = \mathbb{P}^x \{T_a < e_\lambda\} \mathbb{P}^a \{T_b < \tau_{a+}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\}$$

and

$$\mathbb{P}^x \{\tau_{a+}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} = \mathbb{P}^x \{T_a < e_\lambda\} \mathbb{P}^a \{\tau_{a+}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\}.$$

While for  $a \leq x \leq b$ ,

$$\begin{aligned} & \mathbb{P}^x \{T_b < \tau_{a+}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} \\ &= \mathbb{P}^x \{T_a < T_b < \tau_{a+}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} + \mathbb{P}^x \{T_b < T_a \wedge \tau_{a+}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} \\ &= \mathbb{P}^x \{T_a < T_b \wedge c \wedge e_\lambda\} \mathbb{P}^a \{T_b < \tau_{a+}(c) \wedge e_\lambda, X_{e_\lambda} \in dy\} \\ &\quad + \mathbb{P}^x \{T_b < T_a \wedge c \wedge e_\lambda\} \mathbb{P}^b \{X_{e_\lambda} \in dy\}, \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P}^x \{\tau_{a+}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} \\ &= \mathbb{P}^x \{T_a < \tau_{a+}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} + \mathbb{P}^x \{\tau_{a+}(c) < T_a \wedge T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} \\ &= \mathbb{P}^x \{T_a < T_b \wedge e_\lambda \wedge c\} \mathbb{P}^a \{\tau_{a+}(c) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy\} \\ &\quad + \mathbb{P}^x \{c < T_a \wedge T_b \wedge e_\lambda, X_{e_\lambda} \in dy\}. \end{aligned}$$

Corollary 6.5 follows immediately by substituting (6.13) and (6.14) into the relations above. □

*Proof of Corollary 6.13.* For  $x \leq a$ , we have

$$\mathbb{P}^x \{T_b < \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy\} = \mathbb{P}^x \{T_a < e_\lambda\} \mathbb{P}^a \{T_b < \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy\},$$

and

$$\mathbb{P}^x \{ \tilde{\tau}_{a+}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} = \mathbb{P}^x \{ T_a < e_\lambda \} \mathbb{P}^a \{ \tilde{\tau}_{a+}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \}.$$

For  $a \leq x \leq b$ ,

$$\begin{aligned} & \mathbb{P}^x \{ T_b < \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &= \mathbb{P}^x \{ T_a < T_b < \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy \} + \mathbb{P}^x \{ T_b < T_a \wedge \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &= \mathbb{P}^x \{ T_a < T_b \wedge e_\delta \wedge e_\lambda \} \mathbb{P}^a \{ T_b < \tilde{\tau}_{a+}(e_\delta) \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ & \quad + \mathbb{P}^x \{ T_b < T_a \wedge e_\delta \wedge e_\lambda \} \mathbb{P}^b \{ X_{e_\lambda} \in dy \}, \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P}^x \{ \tilde{\tau}_{a+}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &= \mathbb{P}^x \{ T_a < \tilde{\tau}_{a+}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} + \mathbb{P}^x \{ \tilde{\tau}_{a+}(e_\delta) < T_a \wedge T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ &= \mathbb{P}^x \{ T_a < e_\delta \wedge T_b \wedge e_\lambda \} \mathbb{P}^a \{ \tilde{\tau}_{a+}(e_\delta) < T_b \wedge e_\lambda, X_{e_\lambda} \in dy \} \\ & \quad + \mathbb{P}^x \{ e_\delta < T_a \wedge T_b \wedge e_\lambda, X_{e_\lambda} \in dy \}. \end{aligned}$$

The results follow immediately by substituting (6.34) and (6.35) of Theorem 6.12 into the above relations. □

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