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Resonance for Maass forms in the spectral aspect

Nathan Salazar
University of Iowa

RESONANCE FOR MAASS FORMS IN THE SPECTRAL ASPECT

by

Nathan Salazar

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Thesis Supervisor: Professor Yangbo Ye

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Nathan Salazar

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2016 graduation.

Thesis Committee: ______________________________
Yangbo Ye, Thesis Supervisor

____
Phil Kutzko

____
Muthu Krishnamurthy

____
Victor Camillo

____
Mark McKee

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ABSTRACT

Let f be a Maass cusp form for $\Gamma_0(N)$ with Fourier coefficients $\lambda_f(n)$ and Laplace eigenvalue $1/4+k^2$. For real $\alpha \neq 0$ and $\beta > 0$ consider the sum:

$$\sum_n \lambda_f(n) e(\alpha n^\beta) \phi\left(\frac{n}{X}\right),$$

where ϕ is a smooth function of compact support. We prove bounds on the second spectral moment of this sum, with the eigenvalue tending toward infinity. When the eigenvalue is sufficiently large we obtain an average bound for this sum in terms of X. The method is adopted from proofs of subconvexity bounds for Rankin-Selberg L-functions for $GL(2)\times GL(2)$. It contains in particular the Kuznetsov trace formula and an asymptotic expansion of a well-known oscillatory integral with an enlarged range of $K^\varepsilon \leq L \leq K^{1-\varepsilon}$. The same bounds can be proved in an analogous way for holomorphic cusp forms.

Furthermore, we prove similar bounds for

$$\sum_n \lambda_f(n)\lambda_g(n) e(\alpha n^\beta) \phi\left(\frac{n}{X}\right),$$

where g is a holomorphic cusp form. As a corollary, we obtain a subconvexity bound for the L-function $L(s, f\times g)$. This bound has the significant property of breaking convexity even with a trivial bound toward the Ramanujan Conjecture.

PUBLIC ABSTRACT

Automorphic forms are complex-valued functions which satisfy a number of interesting properties, and are central objects of study in number theory. One way to learn more about these functions is to study their *Fourier coefficients*, which constitute a sequence of complex numbers associated to each automorphic form. *Resonance sums* are a means of investigating the oscillatory behavior of this sequence. For a fixed form, the corresponding resonance sum has been studied extensively, and the information given is with respect to the number of terms in the sum.

For a Maass form (resp. holomorphic form), which is a type of automorphic form, another important number associated to it is its *eigenvalue* (or *level*). Instead of fixing one such form, our approach is to consider the resonance sum for a family of automorphic forms. In this way we allow both the number of terms and its eigenvalue (or level) to vary, thereby gaining insight on the behavior of these forms in a new aspect.

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CHAPTER 1
INTRODUCTION

Through the study of resonance sums one can glean information on the oscillatory behavior of a given sequence of numbers. We typically restrict our consideration to sequences which satisfy additional properties, so as to make them more amenable to analytic techniques. Historically, the first sequence considered was that of the Fourier coefficients for modular forms, and this examination has since been continued to include sequences associated with arithmetic functions and Dirichlet series coefficients.

Recently, bounds for these sums have found interesting applications. For example, in [43] it was shown that knowledge of the asymptotic behavior of these sums can be exploited to determine the level and Laplace eigenvalue of Maass forms. In [4] the behavior of resonance sums corresponding to higher degree L-functions was studied and the results provide evidence for functoriality.

This thesis furthers this inquiry by providing a bound for the second spectral moment of the resonance sum in the $GL(2)$ case, as well as individual bounds for the case of $GL(2) \times GL(2)$. These bounds are unique in that they rely explicitly on the spectral parameter. In this chapter we provide the definitions and notations that will be used throughout the thesis, as well as the statements of our main results. In addition to resonance sums, we also touch upon the history of subconvexity estimates, as these provide the inspiration for the strategy we employ.

In Chapter 2, we present several formulas and lemmas that will be utilized in the proofs of our results. We also derive an asymptotic expansion of a particular exponential integral we need, which improves upon the expansions used to achieve subconvexity bounds for certain degree four L-functions. A detailed proof of our main results for the $GL(2)$ case will be the content of Chapter 3. Chapter 4 contains a complete proof of the results in the $GL(2) \times GL(2)$ case.

1.1 Definitions

We adopt the notation that $e(x) = e^{2\pi ix}$ and we alternate between the Big-O and Vinogradov notation where $f \ll g$ means $f = O(g)$. Moreover, $f \asymp g$ means $g \ll f \ll g$. We write $f \in C^n(D)$ to mean f is a function on some domain D and is n-times continuously differentiable. If $f \in C_c^n(D)$, then we further assume that f has compact support in D.

For a function $f \in L^1(\mathbb{R})$, the Fourier transform of f is defined by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)e(xy)dx. \tag{1.1}$$

We will make use of a version of Parseval's theorem which states that for $f, g \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} f(x)\hat{g}(x)dx = \int_{\mathbb{R}} \hat{f}(x)g(-x)dx$.

By a *Schwartz* function we will mean a function $f \in C^\infty(\mathbb{R})$ such that for all $m \geq 0$, $f^{(m)}(x)$ goes to zero as $|x| \to \infty$ faster than any inverse power of x. That is, for every $n, m \geq 0$ there exists a real constant $C_{n,m}$ such that

$$\sup_{x\in\mathbb{R}} |x^n f^{(m)}(x)| \leq C_{n,m}.$$

The set of all Schwartz functions forms a vector space, and the Fourier transform is an automorphism on this space.

The Kloosterman sum is defined as

$$S(m, n; c) = \sum_{(z,c)=1} e\left(\frac{mz + n\overline{z}}{c}\right)$$

where $z\overline{z} \equiv 1 \mod c$. For the necessary background on modular forms and Maass forms, we refer the reader to [3], but we state the corresponding Fourier expansions here.

If g is a holomorphic cusp form for $\Gamma_0(N)$ of weight l then it can be expressed in the form

$$g(z) = \sum_{n \geq 1} \lambda_g(n) n^{\frac{l-1}{2}} e(nz).$$

Likewise, let f be a Maass cusp form with Laplace eigenvalue $1/4 + k^2$ for $\Gamma_0(N)$, normalized by $\langle f, f \rangle = 1$. Then f has the Fourier-Whittaker expansion

$$f(z) = \sqrt{y} \sum_{n \neq 0} \lambda_f(n) K_{ik}(2\pi|n|y) e(nx).$$

We refer to both $\lambda_g(n)$ and $\lambda_f(n)$ as the n-th Fourier coefficient. Here K_{ik} is the modified Bessel function of the third kind and $z = x + iy$.

To each of these forms f we associate an *automorphic L-function*, $L(s, f)$, defined for $\Re(s) > 1$ as

$$L(s, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s}.$$

We assume f to have trivial nebentypus, and so the Euler product has the form (see [12])

$$\prod_p (1 - \alpha_f(p) p^{-s})^{-1} (1 - \beta_f(p) p^{-s})^{-1} = \prod_p (1 - \lambda_f(p) p^{-s} + p^{-2s})^{-1},$$

which has analytic continuation to all of $\mathbb{C}$.

In general, one must take care when defining the Rankin-Selberg L-function of two automorphic forms f and g. However, when both f and g are of full level and have trivial nebentypus (the case of interest for us), the L-function simplifies considerably and may be neatly expressed as

$$\begin{aligned} L(s, f \times g) &= \prod_p (1 - \alpha_f(p)\alpha_g(p)p^{-s})^{-1}(1 - \alpha_f(p)\beta_g(p)p^{-s})^{-1} \\ &\qquad \times (1 - \beta_f(p)\alpha_g(p)p^{-s})^{-1}(1 - \beta_f(p)\beta_g(p)p^{-s})^{-1}. \\ &= \sum_{n \geq 1} \frac{\lambda_{f\times g}(n)}{n^s}, \end{aligned}$$

where

$$\lambda_{f\times g}(n) = \sum_{a^2 b = n} \lambda_f(b)\lambda_g(b).$$

Throughout this thesis we will be concerned with variations of the following sum, which we call a resonance sum:

$$S_X(f; \alpha, \beta) = \sum_n \lambda_f(n) e(\alpha n^\beta) \phi\Big(\frac{n}{X}\Big). \tag{1.2}$$

Here $\beta \in (0, 1)$ is fixed, $\alpha \in \mathbb{R}\backslash\{0\}$, $\phi \in C_c^\infty((1, 2))$, and X is a parameter tending towards infinity. In the case of the Rankin-Selberg L-function the resonance sum is defined as

$$S_X(f \times g; \alpha, \beta) = \sum_n \lambda_f(n)\lambda_g(n) e(\alpha n^\beta) \phi\Big(\frac{n}{X}\Big). \tag{1.3}$$

Estimating by the Rankin-Selberg theory yields (cf. [34, p. 202])

$$S_X(f; \alpha, \beta) \ll_f k^\varepsilon X^{1+\varepsilon}, \tag{1.4}$$

for both (1.2) and (1.3).

We continue the by now standard convention of using θ to denote the exponent of a bound toward the Ramanujan Conjecture. The trivial bound is $\theta = 1/2$, and currently the best-known bound is $\theta = 7/64$ due to Kim-Sarnak [23].

1.2 History

Resonance sums first arose as objects of interest in a paper by Iwaniec-Luo-Sarnak [19], in which the authors proved various density theorems for the zeros of families of automorphic L-functions. Their results are categorized according to the parity of the corresponding L-function, that is, the sign of ε_f in the functional equation $\Lambda(s,f) = \varepsilon_f \Lambda(1-s,f)$. Moreover, their results depend on the choice of test function ϕ, where the support of $\hat{\phi}$ must be contained in an interval of the form $(-a,a)$. As they note, "in order to be able to distinguish the families of automorphic L-functions of different parity by looking at the distribution of the low lying zeros one must use the test functions ϕ with the support of $\hat{\phi}$ larger than $[-1,1]$". They found that extending the range of these functions was closely linked with what they called *Hypothesis S*.

Hypothesis S. *For any $X \geq 1$, $c \geq 1$, and a with $(a,c) = 1$, we have*

$$\sum_{p \leq X, p \equiv a(c)} e\left(\frac{2\sqrt{p}}{c}\right) \ll_\varepsilon X^{\frac{1}{2}+\varepsilon} \tag{1.5}$$

for any $\varepsilon > 0$.

The authors set about investigating more general sums of the type

$$S_q(X) = \sum_n a_n e(-2\sqrt{nq})\phi\left(\frac{n}{X}\right), \tag{1.6}$$

where ϕ is a smooth function compactly supported on $\mathbb{R}^+$, q, X are positive parameters, and $\{a_n\} \in \mathbb{C}$ is a sequence satisfying $a_n \ll n^\varepsilon \; \forall \varepsilon > 0$. When $a_n = 1$, these sequence elements elements are the Dirichlet series coefficients for $\zeta(s)$. Applying Poisson summation and a stationary phase argument they showed that $S_q(X) \ll (qX)^{1/4}$. When $a_n = \lambda_f(n)$, these are the Dirichlet series coefficients for $L(s,f)$. Applying a Voronoi summation formula and an asymptotic expansion for the J-Bessel function, they found that this sum now has a main term of size $\lambda_f(q)X^{3/4}q^{-1/4}$.

With the sequence $a_n = \lambda_f(n)$, observe that

$$S_q(X) = S_X\left(f; -2\sqrt{q}, \frac{1}{2}\right). \tag{1.7}$$

This sum has since been further explored in Ren-Ye [39] for other values of α and $\beta \leq 1$. They were able to prove that this main term occurs only for these values of α and β. In particular, when $\beta \neq \frac{1}{2}$, (1.2) is bounded by $\max\{X^{\beta}, X^{1/2-\beta/4}\}$. Sun [45] proved similar results, again in the setting where f is holomorphic. It was later shown in Sun-Wu [46] that analagous bounds hold when f is a Maass cusp form. In all these results the form f is assumed to be fixed. In the present work we provide non-trivial bounds for $0 < \beta < 1$ of the spectral second moment, which in turn implies a non-trivial bound for $S_X(f; \alpha, \beta)$ in the X and eigenvalue aspects. Specifically, we have Theorem 1.1.

The strategy we employ is one that has been developed and refined with the aim of obtaining subconvexity estimates for certain L-functions– that is, asymptotic bounds on the value of the L-function along the critical line with respect to various aspects, e.g., the level, $\mathrm{Im}(s)$, etc. The term subconvexity is used to emphasize that it breaks the so-called "convexity" bound, which one acquires by applying the functional equation and the Phragmen-Lindelöf principle.

For the Riemann zeta function, the convexity bound in the t-aspect is

$$\zeta\left(\frac{1}{2} + it\right) \ll |t|^{1/4+\varepsilon}, \tag{1.8}$$

and the exponent was improved to 1/6 by Weyl. This is the classical example for $GL(1)$ functions. When f is a cusp form on $GL(2)/\mathbb{Q}$, subconvexity estimates have been attained in the s-aspect (see [13] and [33]) as well as the other parameters (see the series of papers [5]-[9]). For our purposes we will follow the program designed to estimate certain $GL(4)$ L-functions, specifically, Rankin-Selberg L-functions for $GL(2)$ cusp forms.

As an example, if g were a fixed holomorphic form and f was allowed to range over modular forms of weight k, then the corresponding Rankin-Selberg L-function has $k^{1+\varepsilon}$ as the convexity bound in the k-aspect. In [42] Sarnak was able to reduce this bound to $k^{158/165+\varepsilon}$, the first such subconvexity estimate for this kind of L-function. The initial step of his approach makes use of an approximate functional equation to reduce the problem to that of bounding a sum of the form

$$\sum_{n\geq 1} \lambda_f(n)\lambda_g(n)\phi\left(\frac{n}{X}\right), \tag{1.9}$$

where $\phi \in C_c^\infty((1,2))$. As (1.9) is equal to (1.3) with $\alpha = 0$, it now becomes clear that such a strategy could prove quite useful for bounding resonance sums. Our main result in this case is Theorem 1.2, from which we derive Theorem 1.3. Theorems 1.5 and 1.6 cover the case of $\alpha \neq 0$.

To put Corollary 1.4 into context, consider Sarnak's aforementioned bound of $k^{158/165+\varepsilon}$. The exponent is actually $18/(19-2\theta)+\varepsilon$, with the value of $\theta = \frac{7}{64}$ inserted. We see then that a non-trivial value for θ is crucial for subconvexity. This was a common theme in several results that followed. In [31, 30] Liu and Ye focused on the case where f ranges over Maass cusp forms, and their corresponding subconvexity estimate was $(15 + 2\theta)/16 + \varepsilon$. In [2] Blomer used a different method to obtain $(6-2\theta)/(7-4\theta)+\varepsilon$. The first estimate to break convexity for even the trivial bound of $\theta = 1/2$ was achieved in [28] with an exponent of $1-1/(8+4\theta)+\varepsilon$. Our Corollary 1.4 supersedes these results with an exponent of $1-1/(7+4\theta)+\varepsilon$. Taking the value $\theta = 7/64$, our bound improves upon Blomer's by 11/714. The key to our reduction is the significantly stronger asymptotic expansion of a certain exponential integral (see Theorem 2.9 and the discussion that follows). The exponent $2/3+\varepsilon$ was eventually concluded independently by Lau-Liu-Ye [26] and Jutila-Motohashi [20].

1.3 Main Results

Theorem 1.1. *Suppose $0 < \beta < 1$ and $\alpha \neq 0$ are fixed. Let $\{f_j\}$ be an orthonormal basis of Hecke-Maass eigenforms for $\Gamma_0(N)$ with Laplace eigenvalues $1/4 + k_j^2$. Let K be a parameter tending towards infinity with $K^\varepsilon \leq L \leq K^{1-\varepsilon}$ and $LK \geq X^{\beta+\varepsilon}$. Then*

$$\sum_{K-L \leq k_j \leq K+L} |S_X(f_j;\alpha,\beta)|^2 \ll_{N,\varepsilon} L^{1+\varepsilon}KX + K^\varepsilon X^2. \tag{1.10}$$

A few remarks are in order.

Remark 1.1. The same bound in (1.10) also holds for fixed $\beta > 0$ and varying $\alpha \neq 0$, so long as $\alpha X^\beta \ll X^{1-\varepsilon}$ with $K^\varepsilon \leq L \leq K^{1-\varepsilon}$ and $LK \geq \alpha X^{\beta+\varepsilon}$. These restrictions will be made clear in our proof of Theorem 1.1.

Remark 1.2. We note that (1.10) can be regarded as a large sieve type inequality. That is, if we rewrite (1.10) as

$$\sum_{K-L \leq k_j \leq K+L} \left| \sum_n \lambda_f(n) e(\alpha n^\beta) \phi\Big(\frac{n}{X}\Big) \right|^2 \ll_{N,\varepsilon} (L^{1+\varepsilon}K + K^\varepsilon X)X,$$

then this is very near the best estimate that we could hope to achieve, as explained in Chapter 7.3 of [18].

Remark 1.3. When $X^{\beta+\varepsilon} \ll LK \ll X$, the bound for (1.10) reduces to $K^\varepsilon X^2$. However, when LK is larger with $X \ll L^{1+\varepsilon}K$, (1.10) becomes

$$\sum_{K-L \leq k_j \leq K+L} |S_X(f_j;\alpha,\beta)|^2 \ll L^{1+\varepsilon}KX. \tag{1.11}$$

By Weyl's law, the number of terms in the sum is about LK. Hence, (1.11) can be regarded as $S_X(f_j;\alpha,\beta)$ being bounded by $X^{1/2+\varepsilon}$ on average.

Remark 1.4. The same bounds also hold for holomorphic cusp forms f of weight k with a similar proof, except using Petersson's trace formula instead of Kuznetsov's (see §3.3). We note that similar problems have been studied in [4]-[14], [21], and

[35]-[41] . In particular, Savala [43] proved an asymptotic expansion for $S_X(f;\alpha,\beta)$ explicitly for X, k and N.

For the Rankin-Selberg L-function where g is a fixed holomorphic cusp form we have the following results.

Theorem 1.2. *Let $\{f_j\}$ be an orthonormal basis of Hecke-Maass eigenforms with Laplace eigenvalues $1/4+k_j^2$, and let $3/(7+4\theta)<\mu<1$. Then*

$$\sum_{K-K^\mu\le k_j\le K+K^\mu} |S_X(f_j\times g,0,0)|^2 \ll_{g,\varepsilon} K^{1+\mu}X^{1+\varepsilon} \tag{1.12}$$

for

$$X^{\frac{1+2\theta}{\mu(7+4\theta)-3}}\le K \quad \text{if} \quad \frac{3}{7+4\theta}<\mu<\frac{7}{7+4\theta}, \tag{1.13}$$

and

$$X^{\frac{1+2\theta}{4}}\le K \quad \text{if} \quad \frac{7}{7+4\theta}\le\mu<1. \tag{1.14}$$

Theorem 1.3. *Let f be a Maass cusp form with Laplace eigenvalue $1/4+k^2$. Let g be a fixed holomorphic cusp form for $SL(2,\mathbb{Z})$ or $\Gamma_0(N)$. Then*

$$S_X(f\times g,0,0)\ll_{g,\varepsilon} k^{\frac{5+2\theta}{7+4\theta}}X^{\frac{4+3\theta}{7+4\theta}+\varepsilon} \tag{1.15}$$

if $k>X^{\frac{1+2\theta}{4}}$. Here (1.15) is non-trivial if $k<X^{\frac{3+\theta}{5+2\theta}}$.

By a standard argument (e.g., [26, §4]), the above result implies the following subconvexity bound.

Corollary 1.4. *Let f be a Maass cusp form with Laplace eigenvalue $1/4+k^2$. Let g be a fixed holomorphic cusp form for $SL(2,\mathbb{Z})$ or $\Gamma_0(N)$. Then*

$$L(f\times g,s)\ll_{g,\varepsilon} k^{1-\frac{1}{7+4\theta}+\varepsilon}.$$

Remark 1.5. Again we point out that although our methods are nearly identical to those in [28], our above result is better. For $\theta \neq 0$ it is also stronger than Blomer's result. Our saving in the exponent is a consequence of the improved asymptotic expansion of the function $W_{K,L}(x)$ defined in (2.7).

When $\alpha \neq 0$ the bound on the second spectral moment is given in the following theorem.

Theorem 1.5. *Suppose* $0 < \beta < 1$ *and* $\alpha \neq 0$ *are fixed, with* $\beta \neq 1/2$. *Let* g *be a holomorphic cusp form of weight* l *and let* $\{f_j\}$ *be an orthonormal basis of Hecke-Maass eigenforms with Laplace eigenvalues* $1/4 + k_j^2$. *Let* K *be a parameter tending towards infinity with* $|\alpha| X^\beta = K^{1-\mu+2\varepsilon}$, *where* $\frac{1+2\theta+3\beta}{1+2\theta+7\beta} < \mu < 1$. *Then*

$$\sum_{K-L \leq k_j \leq K+L} |S_X(f_j \times g; \alpha, \beta)|^2 \ll_\varepsilon K^{1+\mu} X^{1+\varepsilon} \ll X^{1+\beta\frac{1+\mu}{1-\mu}+\varepsilon}. \tag{1.16}$$

Observe that in contrast to the $GL(2)$ case, we may obtain a non-trivial individual bound by taking the square root of one term in the sum.

Theorem 1.6. *Fix notation as above and let* f *be a Maass cusp form with Laplace eigenvalue* $1/4 + k^2$ *where* $k \asymp X^{\beta/(1-\mu)}$. *Furthermore, suppose* $\frac{(1+2\theta)(1-\mu)}{7\mu-3} < \beta < \frac{1-\mu}{1+\mu}$. *Then*

$$S_X(f \times g; \alpha, \beta) \ll k^{\frac{1+\mu}{2}+\varepsilon} X^{\frac{1}{2}+\varepsilon} \ll X^{\frac{1}{2}+\frac{\beta}{2}\frac{1+\mu}{1-\mu}+\varepsilon}. \tag{1.17}$$

The implied constant depends at most on ε.

Remark 1.6. The largest allowable β corresponds to the smallest allowable μ, and so we must have $\beta < (1-2\theta)/5$. Moreover, varying the size of μ changes the size of the interval for β, with the largest interval corresponding to

$$\mu = \frac{23 + 4\theta + 10\sqrt{2+4\theta}}{47 - 4\theta}.$$

With this choice of μ and $\theta = 7/64$ we may take any β in

$$\left(\frac{-39+8\sqrt{39}}{160}, \frac{8-\sqrt{39}}{20}\right),$$

which is an interval of length ≈ 0.019.

CHAPTER 2
SUMMATION AND INTEGRAL FORMULAS

To prove our results we will require several summation and integral formulas. In particular, we will need the Poisson and Voronoi summation formulas to convert certain sums of arithmetic terms into sums over integrals. We will then use stationary phase arguments to control these integrals. In this chapter we present the statements of these formulas as well as other analytical tools that we will put to use.

2.1 Poisson, Voronoi, and Trace Formulas

The following theorem is known as the Poisson summation formula (cf. Iwaniec-Kowalski [18, pg. 69]).

Theorem 2.1. *Suppose that both f and $\hat{f}$ are in $L^1(\mathbb{R})$ and have bounded variation. Then*

$$\sum_{m\in\mathbb{Z}} f(m) = \sum_{n\in\mathbb{Z}} \hat{f}(n). \tag{2.1}$$

In [24] Kowalski-Michel-Vanderkam provide a comprehensive Voronoi-type summation formula for both holomorphic and Maass cusp forms with various weight, level, and nebentypus considerations. We will only be concerned with holomorphic cusp forms of full level with trivial nebentypus, and so the Voronoi-type summation formula simplifies considerably.

Theorem 2.2. *Let g be a holomorphic cusp form of weight l with Fourier coefficient $\lambda_g(n)$ as in (1.2). Suppose z, c are positive integers with $(z,c) = 1$. For $F \in C^\infty((0,\infty))$ vanishing in a neighborhood of zero and rapidly decreasing,*

$$\sum_{n\geq 1} \lambda_g(n) e\left(\frac{zn}{c}\right) F(n) \quad = \quad \frac{2\pi i^l}{c} \sum_{r\geq 1} \lambda_g(r) e\left(-\frac{\bar{z}r}{c}\right) \int_0^\infty F(w) J_{l-1}\left(\frac{4\pi\sqrt{rw}}{c}\right) dw.$$

We will also make use of the Petersson and Kuznetsov trace formulas, both of which are established through manipulation of Poincare series in [36] and [25], respectively (see also [16, p. 140]). The upshot of these trace formulas is that they allow us to do away with the enigmatic coefficients $\lambda_f(n)\lambda_f(m)$. We restate them below as given in [30]. Throughout, $\lambda_f(n)$ will denote the n-th Fourier coefficient of f, whether f is a holomorphic or Maass cusp form.

For Petersson's trace formula, let $(\frac{c}{d})$ denote the Kronecker symbol and $\varepsilon_d = 1$ if $d \equiv 1 \mod 4$ and $= i$ if $d \equiv 3 \mod 4$. Then we define a generalized Kloosterman sum by

$$K(m,n;c) = \sum_{\substack{z \mod c \\ (z,c)=1}} \left(\frac{c}{z}\right)^{-2k} \varepsilon_d^{2k} e\left(\frac{m\overline{z}+nz}{c}\right). \tag{2.2}$$

This reduces to the standard Kloosterman sum $S(m,n;c)$ when the weight k is even.

Theorem 2.3. *Let $S_k(\Gamma_0(N))$ denote the space of holomorphic cusp forms of weight k for the Hecke congruence subgroup $\Gamma_0(N)$, and let $\mathcal{F}$ be an orthonormal basis of $S_k(\Gamma_0(N))$. Then*

$$\begin{aligned} &\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathcal{F}} \overline{\lambda}_f(m)\lambda_f(n) \\ &\qquad = \delta_{m,n} + 2\pi i^{-k} \sum_{\substack{c>0 \\ N|c}} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{K(m,n;c)}{c}. \end{aligned} \tag{2.3}$$

For Kuznetsov's trace formula, let $\{f_j\}$ be an orthonormal basis of Hecke-Maass eigenforms with Laplace eigenvalues $1/4 + k_j^2$. Each f_j has the Fourier expansion

$$f_j(z) = (y\cosh \pi k_j)^{1/2} \sum_{n\neq 0} \lambda_j(n) K_{ik_j}(2\pi|n|y)e(nx).$$

Theorem 2.4. *Let $h(r)$ be an even function of complex variable, which is analytic in $-\Delta \leq \text{Im } r \leq \Delta$ for some $\Delta \geq 1/4$. Assume in this region that $h(r) \ll r^{-2-\delta}$ for* *some* $\delta > 0$ *as* $r \to \infty$. *Then for any* n, $m \geq 1$,

$$\sum_{f_j} h(k_j)\lambda_j(n)\overline{\lambda}_j(m) = \frac{\delta_{n,m}}{\pi^2}\int_{\mathbb{R}} \tanh(\pi r)h(r)dr + \frac{2i}{\pi}\sum_{c\geq 1}\frac{S(n,m;c)}{c}\int_{\mathbb{R}} J_{2ir}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{h(r)r}{\cosh(\pi r)}dr - \frac{1}{\pi}\int_{\mathbb{R}} d_{ir}(n)d_{ir}(m)\frac{|\zeta_N(1+2ir)|^2}{|\zeta(1+2ir)|^2}dr, \tag{2.4}$$

where $d_\nu(n) = \sum_{ab=|n|}(a/b)^\nu$, $\zeta_N(s) = \prod_{p|N}(1-p^{-s})^{-1}$, *and* $S(m,n;c)$ *is the classical Kloosterman sum.*

2.2 Weighted Derivative Tests

Throughout this thesis we will often be concerned with *exponential integrals*, that is, integrals of the form

$$\int e(\sigma(t))g(t)dt.$$

The function σ is called the *exponential function*, or *phase function*, while g is the *weight function*, which controls the convergence of the integral. The main idea for bounding the size of these integrals is to apply integration by parts a sufficient number of times, so long as σ' satisfies certain conditions. As an example of this technique, we have the *second derivative test* (see [15, Lemma 5.1.3]).

Lemma 2.5. *Let* $\sigma(t)$ *be real and twice differentiable on the open interval* (a,b) *with* $|\sigma''(t)| \geq \lambda > 0$. *Let* $g(t)$ *be real, and let* V *be the total variation of* $g(t)$ *on the closed interval* $[a,b]$ *plus the maximum modulus of* $g(t)$ *on* $[a,b]$. *Then*

$$\left|\int_a^b e(\sigma(t))g(t)dt\right| \leq \frac{4V}{\sqrt{\pi\lambda}}. \tag{2.5}$$

For computing the total variation the following proposition is useful.

Proposition 2.6. *The total variation of a differentiable function* g, *defined on an* *interval* $[a, b] \subset \mathbb{R}$*, has the following expression if* g' *is Riemann integrable*

$$V = \int_a^b |g'(t)| dt. \tag{2.6}$$

Sometimes it is not satisfactory to merely obtain an upper bound on these exponential integrals. If our functions $\sigma(t)$ or $g(t)$ contain arbitrarily large parameters then we may instead desire an asymptotic expansion. In the case where σ' and σ'' do not change sign over the region of integration such an expansion is given in Lemma 5.5.5 of [15]. However, this theorem requires only that $\sigma \in C^3(\mathbb{R})$. The phase functions that we will work with will be infinitely differentiable, and so it would be nice to have an asymptotic expansion for such integrals with $\sigma \in C^n(\mathbb{R})$, and with error terms that improve for larger n. The following theorem strengthens the weighted first derivative test mentioned in [15] in that it offers more boundary terms and smaller error terms. We will enforce this theorem many times.

Theorem 2.7. *(McKee, Sun and Ye, [32]) Let* $\sigma(t)$ *be a real function,* $n+2$ *times continuously differentiable for* $a \leq t \leq b$*, and let* $g(t)$ *be a real function* $n+1$ *times continuously differentiable for* $a \leq t \leq b$*. Suppose that there are positive parameters* M, N, T, U *with* $M \geq b - a$*, and positive constants* C_r *such that for* $a \leq t \leq b$*,*

$$|\sigma^{(r)}(t)| \leq C_r \frac{T}{M^r}, \quad |g^{(s)}(t)| \leq C_s \frac{U}{N^s},$$

for $r = 2, \ldots, n+2$ *and* $s = 0, \ldots, n+1$*. If* $\sigma'(t)$ *and* $\sigma''(t)$ *do not change signs on* *the interval* $[a, b]$*, then we have*

$$\begin{aligned}\int_a^b e(\sigma(t))g(t)dt &= \Big[e(\sigma(t))\sum_{i=1}^{n} H_i(t)\Big]_a^b \\ &+O\Big(\frac{M}{N}\sum_{j=1}^{[n/2]}\frac{UT^j}{\min|\sigma'|^{n+j+1}M^{2j}}\sum_{t=j}^{n-j}\frac{1}{N^{n-j-t}M^t}\Big) \\ &+O\Big(\Big(\frac{M}{N}+1\Big)\frac{U}{N^n\min|\sigma'|^{n+1}}\Big) \\ &+O\Big(\sum_{j=1}^{n}\frac{UT^j}{\min|\sigma'|^{n+j+1}M^{2j}}\sum_{t=0}^{n-j}\frac{1}{N^{n-j-t}M^t}\Big),\end{aligned}$$

where

$$H_1(t) = \frac{g(t)}{2\pi i\sigma'(t)}, \quad H_i(t) = -\frac{H'_{i-1}(t)}{2\pi i\sigma'(t)}$$

for $i = 2, \ldots, n$*.*

In our applications, the function $g(t)$ will be smooth of compact support contained in (a, b). Hence, all of its derivatives will vanish at a and b so the boundary terms in the above expansion will in fact be zero. Moreover, we will be in a position where we may choose $M = 1 = U$ and $N = K^{-\varepsilon}$, and our function σ will be infinitely differentiable. It follows, then, that the error terms will be negligible provided $\min|\sigma'| \gg T \gg K^{\varepsilon}$.

We will see on multiple occasions that the following lemma will be convenient for computing σ'.

Lemma 2.8. *Suppose*

$$q(t) = B\sqrt{A^2 + p(t)^2} - AB\sinh^{-1}\Big(\frac{A}{p(t)}\Big),$$

where A *and* B *are constants and* $p(t)$ *is differentiable and positive. Then*

$$\frac{dq}{dt} = Bp'(t)\sqrt{\Big(\frac{A}{p(t)}\Big)^2 + 1}.$$

2.3 Asymptotic Expansion for $W_{K,L}(x)$

To prove Theorems 1.1, 1.2 and 1.6 we will eventually require an asymptotic expansion of an oscillatory integral of the form

$$W_{K,L}(\eta x) \quad = \int_{\mathbb{R}} e\left(\tfrac{tK}{L} + \tfrac{\eta x}{2\pi}\cosh\left(\tfrac{\pi t}{L}\right)\right)(h(u)(uL+K))^{\wedge}(t)dt, \tag{2.7}$$

where h is assumed to be a Schwartz function and $\eta = \pm 1$ (note that this is independent of the normalization of f). Asymptotics for such integrals were also required in Lau-Liu-Ye [26] and Li [29], but the technique we use is an application of a new theorem for weighted stationary phase integrals given by Theorem 1.3 of McKee-Sun-Ye [32]. Precisely, with $\varpi_{2\nu}$ as defined in (2.10), we obtain the following

Theorem 2.9. *Suppose $K^{\varepsilon} \le L \le K^{1-\varepsilon}$.*

(i) If $x < LK^{1-\varepsilon}$, then $W_{K,L} \ll_M K^{-M}$ for any $M > 0$.

(ii) If $x \ge LK^{1-\varepsilon}$, then

$$W_{K,L}(\eta x) = \sum_{\nu=0}^{n} \widetilde{W}_{\nu}(\eta x) + O\Big(K\Big(\frac{L}{K^{1-\varepsilon}}\Big)^{n+1}\Big) \tag{2.8}$$

where

$$\begin{aligned} \widetilde{W}_{\nu}(\eta x) \quad &= \quad \frac{i^{\nu}(2\nu-1)!!(1+i)}{\eta^{\nu+1/2}\pi^{2\nu+1/2}} \frac{L^{2\nu+1}}{(4K^2+x^2)^{\nu/2+1/4}}\varpi_{2\nu} \\ &\times \quad e\Big(\frac{\eta}{2\pi}\sqrt{4K^2+x^2} - \eta\frac{K}{\pi}\sinh^{-1}\Big(\frac{2K}{x}\Big)\Big). \end{aligned} \tag{2.9}$$

In [42], [31], [30] and [28] the authors achieved an asymptotic expansion for this integral by expanding $\cosh(\pi t/L)$ into its Taylor series and keeping only the first term, then using the series expansion for e to approximate away the rest. However, to do so required $L \ge K^{1/2+\varepsilon}$. To reduce this restriction, in Lau-Liu-Ye [26], they kept the first two terms, thus only requiring $L \ge K^{1/3+\varepsilon}$. Li [29] followed the same approach. However, it is clear from their computation that keeping more terms will not lift this requirement, as the resulting error term will still have a factor of K/L^3 (for example, see [26, Eq. (13.12)]). Hence, the above theorem is a significant improvement as it allows us to take $L \geq K^{\varepsilon}$, which is crucial in some applications.

As noted before, our technique is to apply Theorem 1.3 from [32] which improves upon the weighted stationary phase result found in Lemma 5.5.6 of Huxley [15], and which allows us to take $L = K^{\varepsilon}$. We restate the theorem here in its entirety.

Theorem 2.10. *(McKee, Sun and Ye, [32]) Let $\sigma(t)$ be a real function $2n+3$ times continuously differentiable for $a \leq t \leq b$, and let $g(t)$ be a real function $2n+1$ times continuously differentiable for $a \leq t \leq b$. Suppose $\sigma'(t)$ changes signs only at $t = \gamma$, from negative to positive, with $a < \gamma < b$. Define $H_k(t)$ by*

$$H_1(t) = \frac{g(t)}{2\pi i \sigma'(t)}, \quad H_i(t) = -\frac{H'_{i-1}(t)}{2\pi i \sigma'(t)}$$

for $i = 2, \ldots, n$. Suppose that there are positivie parameters M, N, T, U with $M > b-a$ and positive constants C_r such that for $a \leq t \leq b$,

$$|\sigma^{(r)}(t)| \leq C_r \frac{T}{M^r}, \quad \text{for } r = 2, 3, \ldots, 2n+3, \quad \sigma''(t) \geq \frac{T}{C_2 M^2}$$

and

$$|g^{(s)}(t)| \leq C_s \frac{U}{N^s}, \quad \text{for } s = 0, 1, 2, \ldots, 2n+1.$$

If T is sufficiently large compared to the constants C_r, we have for $n \geq 2$ that

$$\int_a^b e(\sigma(t))g(t)dt =$$

$$\frac{e\Big(\sigma(\gamma)+\frac{1}{8}\Big)}{\sqrt{\sigma''(\gamma)}}\Big(g(\gamma)+\sum_{\nu=1}^{n}\varpi_{2\nu}\frac{(-1)^\nu(2\nu-1)!!}{(4\pi i\lambda_2)^\nu}\Big)+\Big[e(\sigma(t))\sum_{i=1}^{n+1}H_i(t)\Big]_a^b$$
$$+O\Big(\frac{UM^{2n+5}}{NT^{n+2}}\sum_{j=1}^{[\frac{n+1}{2}]}\Big(\frac{1}{(\gamma-a)^{n+2+j}}+\frac{1}{(b-\gamma)^{n+2+j}}\Big)\sum_{t=j}^{n+1-j}\frac{1}{N^{n+1-j-t}M^t}\Big)$$
$$+O\Big(\frac{UM^{2n+4}}{N^{n+1}T^{n+2}}\Big(\frac{M}{N}+1\Big)\Big(\frac{1}{(\gamma-a)^{n+2}}+\frac{1}{(b-\gamma)^{n+2}}\Big)\Big)$$
$$+O\Big(\frac{UM^{2n+4}}{T^{n+2}}\sum_{j=1}^{n+1}\Big(\frac{1}{(\gamma-a)^{n+2+j}}+\frac{1}{(b-\gamma)^{n+2+j}}\Big)\sum_{t=0}^{n+1-j}\frac{1}{N^{n+1-j-t}M^t}\Big)$$
$$+O\Big(\frac{U}{T^{n+1}}\Big(\frac{M^{2n+2}}{N^{2n+1}}+M\Big)\Big),$$

where

$$\varpi_{2\nu} = \eta_{2\nu}+\sum_{\ell=0}^{2\nu-1}\eta_\ell\sum_{k=1}^{2\nu-\ell}\frac{C_{2\nu,\ell,k}}{\lambda_2^k}\sum_{\substack{3\leq n_1,\dots,n_k\leq 2n+3\\ n_1+\cdots+n_k=2\nu-\ell+2k}}\lambda_{n_1}\cdots\lambda_{n_k}, \tag{2.10}$$

$$\eta_\ell = \frac{g^{(\ell)}(\gamma)}{\ell!} \quad \textit{for } 0\leq \ell\leq 2n,$$

$$\lambda_\ell = \frac{\sigma^{(\ell)}(\gamma)}{\ell!} \quad \textit{for } 0\leq\ell\leq 2n+2, \tag{2.11}$$

and $C_{2\nu,\ell,k}$ is a constant.

Proof of Theorem 2.9. The integral in (3.9) is over all of $\mathbb{R}$, but the above theorem only applies to integrals over a finite range. To that end, we note that $g(t) := (h(u)(uL+K))^\wedge(t)$ is a Schwartz function and thus of rapid decay. We may therefore restrict our range of integration to $(-K^\varepsilon, K^\varepsilon)$, picking up an arbitrarily small error term.

For part (ii), we compute the stationary point of σ.

$$\sigma'(t) = \frac{K}{L}+\frac{\eta x}{2L}\sinh(\frac{\pi t}{L}),$$

and so $\sigma'(\gamma) = 0$ at

$$\gamma := -\eta \frac{L}{\pi} \sinh^{-1}\left(\frac{2K}{x}\right).$$

By part (i) we may suppose $x \geq LK^{1-\varepsilon}$. Since

$$x = \frac{4\pi\sqrt{mn}}{c} \asymp \frac{X}{c},$$

we may henceforth assume $c \leq X/(LK^{1-\varepsilon})$. Thus

$$|\gamma| = \frac{L}{\pi} \sinh^{-1}\left(\frac{2K}{x}\right) < \frac{2LK}{x} \leq \frac{2LKc}{4\pi X} \leq \frac{K^{\varepsilon}}{2\pi},$$

and we see that γ will be contained in our region of integration.

Now, we have

$$\begin{aligned} \sigma(\gamma) &= \frac{\eta x}{2\pi}\sqrt{\left(\frac{2K}{x}\right)^2 + 1} - \eta\frac{K}{\pi}\sinh^{-1}\left(\frac{2K}{x}\right) \\ &= \frac{\eta}{2\pi}\sqrt{4K^2 + x^2} - \eta\frac{K}{\pi}\sinh^{-1}\left(\frac{2K}{x}\right) \end{aligned}$$

and for $r \geq 2$

$$\sigma^{(r)}(t) = \begin{cases} \frac{\eta x}{2\pi}\left(\frac{\pi}{L}\right)^r \cosh\left(\frac{\pi t}{L}\right) & r \text{ is even} \\ \frac{\eta x}{2\pi}\left(\frac{\pi}{L}\right)^r \sinh\left(\frac{\pi t}{L}\right) & r \text{ is odd.} \end{cases} \tag{2.12}$$

Evaluating (2.12) at γ yields

$$\begin{aligned} \sigma^{(r)}(\gamma) &= \begin{cases} \frac{\eta x}{2\pi}\left(\frac{\pi}{L}\right)^r \sqrt{\left(\frac{2K}{x}\right)^2 + 1} & r \text{ is even;} \\ \frac{-K}{\pi}\left(\frac{\pi}{L}\right)^r & r \text{ is odd.} \end{cases} \\ &= \begin{cases} \frac{\eta}{2\pi}\left(\frac{\pi}{L}\right)^r \sqrt{4K^2 + x^2} & r \text{ is even;} \\ \frac{-K}{\pi}\left(\frac{\pi}{L}\right)^r & r \text{ is odd.} \end{cases} \end{aligned} \tag{2.13}$$

We take our parameters to be $T = X/c$, $M = L$, $U = K$ and $N = 1$. Note that N

satisfites

$$\frac{M}{\sqrt{T}} = \frac{L}{\sqrt{X/c}} \leq \sqrt{\frac{L}{K^{1-\varepsilon}}} = O(1) = O(N),$$

which guarantees that our error term will be arbitrarily small. For $|t| \leq K^{\varepsilon}$ from (2.12) we have

$$\begin{aligned} |\sigma^{(r)}(t)| &\leq \frac{x}{2\pi}\left(\frac{\pi}{L}\right)^{r} \cosh\left(\frac{\pi K^{\varepsilon}}{L}\right) \leq 4\pi^{r}\cosh(1)\frac{X/c}{L^{r}} = C_r \frac{T}{M^{r}}, \\ |\sigma^{(2)}(t)| &\geq \frac{x}{2\pi}\left(\frac{\pi}{L}\right)^{2} \geq 2\pi^{2}\frac{X/c}{L^{2}} \geq \frac{X/c}{4\pi^{2}\cosh(1)L^{2}} = \frac{T}{C_2 M^{2}}. \end{aligned}$$

Writing $g(t) = KH(t)$, i.e., letting

$$H(t) = (h(u)(u\frac{L}{K} + 1))^{\wedge}(t), \tag{2.14}$$

we get

$$|g^{(s)}(t)| \leq K \max_{|t|\leq K^{\varepsilon}}\{H^{(s)}(t)\} \leq C_s \frac{U}{N^{s}}.$$

We may now invoke Theorem 2.10 to obtain the following asymptotic expansion of (3.9) by cutting off negligible tails:

$$\begin{aligned} \int_{-K^{\varepsilon}}^{K^{\varepsilon}} e(\sigma(t))g(t)dt &= \frac{e\Big(\sigma(\gamma) + \frac{1}{8}\Big)}{\left|\frac{\eta\pi}{2L^{2}}\sqrt{4K^{2}+x^{2}}\right|^{\frac{1}{2}}}\left[g(\gamma) + \sum_{\nu=1}^{n} \varpi_{2\nu}\frac{(-1)^{\nu}(2\nu-1)!!}{(4\pi i\lambda_2)^{\nu}}\right] \qquad (2.15) \\ &+\text{Boundary terms} + O\Big(\frac{KL^{2n+5}}{(X/c)^{n+2}}\sum_{j=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{1}{(K^{\varepsilon})^{n+2+j}}\sum_{t=j}^{n+1-j}\frac{1}{L^{t}}\Big) \\ &+O\Big(\frac{KL^{2n+4}}{(X/c)^{n+2}}\Big(L+1\Big)\Big(\frac{1}{(K^{\varepsilon})^{n+2}}\Big)\Big) \\ &+O\Big(\frac{KL^{2n+4}}{(X/c)^{n+2}}\sum_{j=1}^{n+1}\Big(\frac{1}{(K^{\varepsilon})^{n+2+j}}\Big)\sum_{t=0}^{n+1-j}\frac{1}{L^{t}}\Big) + O\Big(\frac{KL^{2n+2}}{(X/c)^{n+1}}\Big). \end{aligned}$$

The boundary terms in (2.15) will be negligible since $g(t) = (h(u)(uL+K))^{\wedge}(t)$ is a Schwartz function, and the largest error term in (2.15) will be the last one. Evaluating $e(1/8)$, simplifying the main term, and using that $X/c \geq LK^{1-\varepsilon}$ in the error terms in (2.15) yields

$$\int_{\mathbb{R}} e(\sigma(t))g(t)dt = \tag{2.16}$$
$$\frac{L(1+i)e(\sigma(\gamma))}{(\eta\pi)^{1/2}(4K^2+x^2)^{1/4}}\left[g(\gamma)+\sum_{\nu=1}^{n}\varpi_{2\nu}\frac{(-1)^{\nu}(2\nu-1)!!}{(4\pi i\lambda_2)^{\nu}}\right]+O\Big(\frac{KL^{2n+2}}{(LK^{1-\varepsilon})^{n+1}}\Big)$$
$$=\sum_{\nu=0}^{n}\widetilde{W}_{\nu}(x)+O\Big(K\Big(\frac{L}{K^{1-\varepsilon}}\Big)^{n+1}\Big).$$

Since σ and g are both infinitely differentiable, we may take n to be as large as we need in (2.16) to ensure that the error term is negligible. This proves Theorem 2.9 part (ii).

Now, for part (i), we know that this integral will be negligible for $x < LK^{1-\varepsilon}$ as has already been proven in [26, §13]. Alternatively, we can show this by considering the stationary point of the integral, which occurs at $\frac{-\eta L}{\pi}\sinh^{-1}(\frac{2K}{x})$ (see above). If this value is too large, say $\frac{L}{\pi}\sinh^{-1}(\frac{2K}{x}) > K^{\varepsilon}$, then it will lie outside the effective range of integration (due to the rapid decay of $g(t)$). Solving for x we get the inequality

$$x < \frac{2K}{\sinh\left(\pi\frac{K^{\varepsilon}}{L}\right)} < \frac{4}{\pi}LK^{1-\varepsilon}.$$

Hence, our integral will be negligible for large values of K if x is too small.

□

We rewrite $\varpi_{2\nu}$ in (2.10) as $KH_{2\nu}(\gamma)$, where

$$H_{2\nu}(\gamma) = \frac{H^{(2\nu)}(\gamma)}{(2\nu)!} + \sum_{\ell=0}^{2\nu-1}\frac{H^{(\ell)}(\gamma)}{\ell!}\sum_{k=1}^{2\nu-\ell}\frac{2^k C_{2\nu,\ell,k}}{\left(\frac{\eta\pi}{2L^2}\sqrt{4K^2+x^2}\right)^k}\sum_{\substack{3\le n_1,\dots,n_k\le 2n+3\\ n_1+\cdots+n_k=2\nu-\ell+2k}}\lambda_{n_1}\cdots\lambda_{n_k} \tag{2.17}$$

with λ_{n_i} being given in (2.11) and $H(t)$ in (2.14).

CHAPTER 3
RESONANCE FOR $GL(2)$

We now present the proof of Theorem 1.1.

3.1 Averaging and the Kuznetsov Trace Formula

As in [31], our strategy is to take a smooth averaging of $|S_X(f;\alpha,\beta)|^2$ and then use the Kuznetsov trace formula (Theorem 2.4). Let $\{f_j\}$ be an orthonormal basis of Hecke-Maass eigenforms with Laplace eigenvalues $1/4+k_j^2$.

Let h be a positive Schwartz function and $K^\varepsilon \le L \le K^{1-\varepsilon}$, where K is a parameter tending to infinity. We will bound the sum

$$\sum_{f_j}\Big[h\Big(\frac{k_j-K}{L}\Big)+h\Big(-\frac{k_j+K}{L}\Big)\Big]|S_X(f_j;\alpha,\beta)|^2 \tag{3.1}$$
$$=\sum_{n,m} e\Big(\alpha(n^\beta-m^\beta)\Big)\phi\Big(\frac{n}{X}\Big)\bar{\phi}\Big(\frac{m}{X}\Big)\sum_{f_j}\Big[h\Big(\frac{k_j-K}{L}\Big)+h\Big(-\frac{k_j+K}{L}\Big)\Big]\lambda_{f_j}(n)\bar{\lambda}_{f_j}(m).$$

We take the sum of two h terms in the brackets to ensure that our function is even with respect to the variable k_j. Since h is a Schwartz function, it is negligible outside the interval $(-L^\varepsilon, L^\varepsilon)$. Hence, the h-terms will only contribute if

$$-L^\varepsilon \le \frac{k_j-K}{L}\le L^\varepsilon, \text{ or } K-L^{1+\varepsilon}\le k_j\le K+L^{1+\varepsilon}.$$

By Weyl's law, the number of such terms is $\asymp KL^{1+\varepsilon}$, and so using the Rankin-Selberg bound (1.4) yields

$$\sum_{f_j}\Big[h\Big(\frac{k_j-K}{L}\Big)+h\Big(-\frac{k_j+K}{L}\Big)\Big]|S_X(f_j;\alpha,\beta)|^2 \ll L^{1+\varepsilon}KX^{2+2\varepsilon}.$$

We seek a square-root saving on $S_X(f;\alpha,\beta)$ on average, i.e., to prove (3.1) is $\ll L^{1+\epsilon}KX^{1+\epsilon}$.

Employing Kuznetsov's trace formula (Theorem 2.4), the inner sum of (3.1) becomes

$$\sum_{f_j}\Big[h\Big(\frac{k_j-K}{L}\Big)+h\Big(-\frac{k_j+K}{L}\Big)\Big]\lambda_{f_j}(n)\bar{\lambda}_{f_j}(m)$$

$$= \frac{\delta_{n,m}}{\pi^2}\int_{\mathbb{R}}\tanh(\pi r)\Big(h\Big(\frac{r-K}{L}\Big)+h\Big(-\frac{r+K}{L}\Big)\Big)rdr \tag{3.2}$$

$$+\frac{2i}{\pi}\sum_{N|c\geq 1}\frac{S(n,m;c)}{c}\int_{\mathbb{R}}J_{2ir}\Big(\frac{4\pi\sqrt{nm}}{c}\Big)\Big(h\Big(\frac{r-K}{L}\Big)+h\Big(-\frac{r+K}{L}\Big)\Big)\frac{rdr}{\cosh(\pi r)} \tag{3.3}$$

$$-\frac{1}{\pi}\int_{\mathbb{R}}d_{ir}(n)d_{ir}(m)\Big(h\Big(\frac{r-K}{L}\Big)+h\Big(-\frac{r+K}{L}\Big)\Big)\frac{|\zeta_N(1+2ir)|^2}{|\zeta(1+2ir)|^2}dr \tag{3.4}$$

where

$$d_{ir}(n)=\sum_{ab=n}(a/b)^{ir},\quad \zeta_N(s)=\prod_{p|N}(1-p^{-s})^{-1}.$$

For (3.4), first note that the integral is convergent since

$$|\zeta(1+2ir)|\geq c\log^{-2/3}(2+|r|)$$

for some $c>0$. We split the integral as

$$\int_{\mathbb{R}}d_{ir}(n)d_{ir}(m)h\Big(\frac{r-K}{L}\Big)\frac{|\zeta_N(1+2ir)|^2}{|\zeta(1+2ir)|^2}dr + \int_{\mathbb{R}}d_{ir}(n)d_{ir}(m)h\Big(-\frac{r+K}{L}\Big)\frac{|\zeta_N(1+2ir)|^2}{|\zeta(1+2ir)|^2}dr$$

and then change variables $u=(r-K)/L$ in the first integral and $u=-(r+K)/L$ in the second to get

$$L\int_{\mathbb{R}}d_{i(uL+K)}(n)d_{i(uL+K)}(m)h(u)\frac{|\zeta_N(1+2i(uL+K))|^2}{|\zeta(1+2i(uL+K))|^2}du + L\int_{\mathbb{R}}d_{-i(uL+K)}(n)d_{-i(uL+K)}(m)h(u)\frac{|\zeta_N(1-2i(uL+K))|^2}{|\zeta(1-2i(uL+K))|^2}du.$$

For the second integral, note that $d_{-ir}=d_{ir}$, and $|\zeta(1-2ir)|=|\zeta(1+2ir)|$, as can be verified by comparing the modulus of each local factor individually. The same is true for $|\zeta_N(1-2ir)|$, so it follows that these two integrals are equal. Plugging this back into (3.4) and then taking the sum over n and m yields

$$-\frac{2L}{\pi}\sum_{n,m} e(\alpha(n^\beta - m^\beta))\phi\left(\frac{n}{X}\right)\bar{\phi}\left(\frac{m}{X}\right) \tag{3.5}$$
$$\times \int_{\mathbb{R}} d_{i(uL+K)}(n) d_{i(uL+K)}(m) \frac{\left|\zeta_N(1+2i(uL+K))\right|^2}{\left|\zeta(1+2i(uL+K))\right|^2} h(u)du$$
$$= -\frac{2L}{\pi}\int_{\mathbb{R}}\left|\sum_n e(\alpha n^\beta)\phi\Big(\frac{n}{X}\Big) d_{i(uL+K)}(n)\right|^2 \frac{\left|\zeta_N(1+2i(uL+K))\right|^2}{\left|\zeta(1+2i(uL+K))\right|^2} h(u)du$$

since $d_{ir} = \bar{d}_{ir}$. Moreover, since $h(u)$ is always positive, we can move this whole term (3.4) to the other side of (3.1), which is also positive. Thus we need only bound (3.2) and (3.3), which we refer to as the diagonal and off-diagonal terms, respectively.

3.1.1 Diagonal Terms

The term (3.2) is zero except when $n = m$, so substituting this into (3.1) yields

$$\sum_n \left|\phi\Big(\frac{n}{X}\Big)\right|^2 \Big[\frac{1}{\pi^2}\int_{\mathbb{R}} \tanh(\pi r)\Big(h\Big(\frac{r-K}{L}\Big) + h\Big(-\frac{r+K}{L}\Big)\Big) r dr\Big] \tag{3.6}$$
$$\ll \sum_n \left|\phi\Big(\frac{n}{X}\Big)\right|^2 \Big[\int_{K-L^{1+\varepsilon}}^{K+L^{1+\varepsilon}} \tanh(\pi r) r dr\Big].$$

Here we've used that $h = O(1)$ and that the rapid decay of h isolates $|(r-K)/L|$ to L^ε. Since $\tanh(\pi r) = sgn(r) + O(e^{-\pi|r|})$, the integral on the right hand side of (3.6) will be

$$\ll \int_{K-L^{1+\varepsilon}}^{K+L^{1+\varepsilon}} |r| \left(1 + O(e^{-\pi|r|})\right) dr \ll L^{1+\varepsilon} K.$$

Combined with the fact that $\phi(\frac{n}{X}) \ll 1$, this shows that the diagonal terms are

$$\frac{1}{\pi^2}\sum_n \left|\phi\Big(\frac{n}{X}\Big)\right|^2 \int_{\mathbb{R}} \tanh(\pi r)\Big(h\Big(\frac{r-K}{L}\Big) + h\Big(-\frac{r+K}{L}\Big)\Big) r dr \ll L^{1+\varepsilon} K X. \tag{3.7}$$

3.1.2 Off-Diagonal Terms

In the rest of this proof we will compute the off-diagonal terms (3.3) where we may have to impose relations between X and K. Following the method in [31] closely, we write

$$V_{K,L}(x) \;=\; \int_{\mathbb{R}} J_{2ir}(x) \left(h\Big(\frac{r-K}{L}\Big) + h\Big(-\frac{r+K}{L}\Big)\right) \frac{rdr}{\cosh(\pi r)}.$$

If we change variables $r \mapsto -r$ we get

$$= \;-\int_{\mathbb{R}} J_{-2ir}(x) \left(h\Big(\frac{r-K}{L}\Big) + h\Big(-\frac{r+K}{L}\Big)\right) \frac{rdr}{\cosh(\pi r)},$$

so

$$\begin{aligned} V_{K,L}(x) \;&=\; \frac{1}{2}\Big[\int_{\mathbb{R}} J_{2ir}(x) \left(h\Big(\frac{r-K}{L}\Big) + h\Big(-\frac{r+K}{L}\Big)\right) \frac{rdr}{\cosh(\pi r)} \\ &\quad -\int_{\mathbb{R}} J_{-2ir}(x) \left(h\Big(\frac{r-K}{L}\Big) + h\Big(-\frac{r+K}{L}\Big)\right) \frac{rdr}{\cosh(\pi r)}\Big] \\ &=\; \frac{1}{2}\int_{\mathbb{R}} \frac{J_{2ir}(x) - J_{-2ir}(x)}{\sinh(\pi r)} \left(h\Big(\frac{r-K}{L}\Big) + h\Big(-\frac{r+K}{L}\Big)\right) r\tanh(\pi r)dr. \end{aligned}$$

Since $\tanh(\pi r) = sgn(r) + O(e^{-\pi|r|})$, we can write this as

$$V_{K,L}(x) = \frac{1}{2}\int \frac{J_{2ir}(x) - J_{-2ir}(x)}{\sinh(\pi r)} \left(h\Big(\frac{r-K}{L}\Big) + h\Big(-\frac{r+K}{L}\Big)\right) |r|dr + O\big(K^{-M}\big)$$

for $M > 0$ sufficiently large. Now we use Parseval's theorem on this integral to transform it into

$$\begin{aligned} &\frac{1}{2}\int \Big(\frac{J_{2ir}(x) - J_{-2ir}(x)}{\sinh(\pi r)}\Big)^{\wedge}(y)\Big(h\Big(\frac{r-K}{L}\Big)|r| + h\Big(-\frac{r+K}{L}\Big)|r|\Big)^{\wedge}(-y)dy + O\big(K^{-M}\big) \\ &=\; \frac{1}{2i}\int \cos(x\cosh(\pi y))\Big(h\Big(\frac{r-K}{L}\Big)|r| + h\Big(-\frac{r+K}{L}\Big)|r|\Big)^{\wedge}(y)dy + O\big(K^{-M}\big) \end{aligned}$$

by a Fourier transform formula in [1] (volume 1, p. 59). We can also write

$$\begin{aligned}
&\Big(h\Big(\frac{r-K}{L}\Big)|r| + h\Big(-\frac{r+K}{L}\Big)|r|\Big)^{\wedge}(y) \\
&= \int_{\mathbb{R}} h\Big(\frac{r-K}{L}\Big)|r|e(ry)dr + \int_{\mathbb{R}} h\Big(-\frac{r+K}{L}\Big)|r|e(ry)dr \\
&= e(yK)L\Big(h(u)|uL+K|\Big)^{\wedge}(yL) + e(-yK)L\Big(h(u)|uL+K|\Big)^{\wedge}(-yL),
\end{aligned}$$

where we have made the substitutions $\pm r = uL + K$. We may drop the absolute value signs since $L \leq K^{1-\varepsilon}$ and $h(u)$ isolates u to $O(1)$.

Setting $y = t/L$ we have

$$\begin{aligned}
V_{K,L}(x) &= \frac{1}{i}\int_{\mathbb{R}} \cos\left(x\cosh\left(\frac{\pi t}{L}\right)\right) e\Big(\frac{tK}{L}\Big)(h(u)(uL+K))^{\wedge}(t)dt \qquad (3.8)\\
&= \frac{1}{2i}\int_{\mathbb{R}} \left[e\left(\frac{x}{2\pi}\cosh\left(\frac{\pi t}{L}\right)\right) + e\left(\frac{-x}{2\pi}\cosh\left(\frac{\pi t}{L}\right)\right)\right] e\Big(\frac{tK}{L}\Big)(h(u)(uL+K))^{\wedge}(t)dt \\
&= \frac{1}{2i}\left(W_{K,L}(x) + W_{K,L}(-x)\right),
\end{aligned}$$

where

$$W_{K,L}(x) = \int_{\mathbb{R}} e\left(\frac{tK}{L} + \frac{x}{2\pi}\cosh\left(\frac{\pi t}{L}\right)\right)(h(u)(uL+K))^{\wedge}(t)dt. \qquad (3.9)$$

Taking one term in (3.8) and substituting (2.8) and (2.9) for (3.9), we have reduced to approximating a typical term

$$\begin{aligned}
T_\nu^\eta &= \sum_{n,m} \phi\Big(\frac{n}{X}\Big)\bar{\phi}\Big(\frac{m}{X}\Big) e\Big(\alpha(n^\beta - m^\beta)\Big) \sum_{\substack{1\leq c\leq \frac{X}{LK^{1-\varepsilon}} \\ N|c}} \frac{S(m,n,c)}{c} \qquad (3.10)\\
&\times \frac{L^{2\nu+1}}{\eta^{\nu+1/2}\Big(K^2 + \frac{4\pi^2 nm}{c^2}\Big)^{\nu/2+1/4}} K H_{2\nu}\Big(-\frac{\eta L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mn}}\Big)\Big) \\
&\times e\left(\frac{\eta}{\pi c}\sqrt{c^2K^2 + 4\pi^2 nm} - \frac{\eta K}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mn}}\Big)\right).
\end{aligned}$$

Here we have plugged in $4\pi\sqrt{mn}/c$ for x in (2.9). Opening up the Kloosterman sum in (3.10) and ignoring constant coefficients, we can rearrange the order of summation as

$$L^{2\nu+1}K\sum_{m}\bar{\phi}\Big(\frac{m}{X}\Big)e\Big(-\alpha m^{\beta}\Big)\sum_{\substack{1\le c\le\frac{X}{LK^{1-\varepsilon}}\\ N|c}}c^{\nu-\frac{1}{2}}\sum_{\substack{z \bmod c\\ (z,c)=1}}e\Big(\frac{m\bar{z}}{c}\Big) \tag{3.11}$$

$$\times\sum_{n}\phi\Big(\frac{n}{X}\Big)e(\alpha n^{\beta})e\Big(\frac{nz}{c}\Big)\Big(c^2K^2+4\pi^2nm\Big)^{-\frac{\nu}{2}-\frac{1}{4}}H_{2\nu}\Big(-\frac{\eta L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mn}}\Big)\Big)$$

$$\times e\left(\frac{\eta}{\pi c}\sqrt{c^2K^2+4\pi^2nm}-\frac{\eta K}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mn}}\Big)\right).$$

Now we apply Poisson's summation formula (Theorem 2.1) to convert the sum over n in (3.11) to

$$\sum_{n}\int_{\mathbb{R}}\phi\Big(\frac{t}{X}\Big)e\Big(\alpha t^{\beta}+\frac{tz}{c}\Big)\Big(c^2K^2+4\pi^2mt\Big)^{-\nu/2-1/4}H_{2\nu}\Big(-\frac{\eta L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mt}}\Big)\Big)$$

$$\times e\left(\frac{\eta}{\pi c}\sqrt{c^2K^2+4\pi^2mt}-\frac{\eta K}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mt}}\Big)\right)e(-nt)dt,$$

which, after changing the variable t to tX, becomes

$$X^{\frac{1}{2}-\nu}\sum_{n}\int_{1}^{2}\phi(t)\Big(\Big(\frac{cK}{X}\Big)^2+4\pi^2t\frac{m}{X}\Big)^{-\frac{\nu}{2}-\frac{1}{4}}H_{2\nu}\Big(-\frac{\eta L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mtX}}\Big)\Big) \tag{3.12}$$

$$\times e\left(\frac{\eta}{\pi c}\sqrt{c^2K^2+4\pi^2mtX}-\frac{\eta K}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mXt}}\Big)\right)$$

$$\times e\Big(\alpha X^{\beta}t^{\beta}+\frac{zXt}{c}-nXt\Big)dt.$$

We introduce the notation $\widetilde{H}(t)$, $\theta(t)$ and $\psi(t)$ in an obvious way in (3.12) so that we may rewrite T_{ν}^{η} more succinctly as

$$T_{\nu}^{\eta} = L^{2\nu+1}KX^{\frac{1}{2}-\nu}\sum_{m}\bar{\phi}\Big(\frac{m}{X}\Big)e\Big(-\alpha m^{\beta}\Big) \tag{3.13}$$

$$\times\sum_{\substack{1\le c\le\frac{X}{LK^{1-\varepsilon}}\\ N|c}}c^{\nu-\frac{1}{2}}\sum_{\substack{0\le z<c\\ (z,c)=1}}e\Big(\frac{m\bar{z}}{c}\Big)\sum_{n}\int_{1}^{2}\widetilde{H}(t)e(\theta(t))e(\psi(t))dt.$$

We are now faced with the task of bounding the integral in (3.13). Using Theorem 2.7 we will show that in fact this integral is negligible for all but a finite number of values of n. In order to do this, we need to be able to control the size of the derivatives of $\widetilde{H}(t)$, $\theta(t)$ and $\psi(t)$.

First consider

$$\widetilde{H}(t) = \phi(t)\Big(\Big(\frac{cK}{X}\Big)^2 + 4\pi^2 t\frac{m}{X}\Big)^{-\frac{\nu}{2}-\frac{1}{4}} H_{2\nu}(\gamma_0), \tag{3.14}$$

where

$$\gamma_0 = -\frac{\eta L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mtX}}\Big).$$

All derivatives of $\phi(t)$ are $O(1)$. As

$$\frac{cK}{X} \le \frac{K^{\varepsilon}}{L} \to 0, \quad 1 \le t \le 2, \quad \text{and} \quad 1 \le \frac{m}{X} \le 2,$$

it follows that the derivatives of the second factor will be $O(1)$ as well. By (2.17) we see that, up to constant coefficients, $H_{2\nu}(\gamma_0) = H^{(2\nu)}(\gamma_0)$ plus a sum of products of the form $H^{(\ell)}\sigma^{(n_1)}\cdots\sigma^{(n_k)}(\sigma^{(2)})^{-k}$, all evaluated at γ_0, with $n_i \ge 3$ and $k \ge 1$. The derivative of γ_0 is

$$\gamma_0' = \eta\frac{LKc}{X}\frac{1}{4\pi^2 t\sqrt{\frac{m}{X}t + \left(\frac{cK}{2\pi X}\right)^2}} \asymp \frac{LKc}{X},$$

and we see that its higher derivatives are of the same size. For the derivatives of the terms in the product, note from (2.13) that

$$\begin{aligned}\sigma^{(r)}(\gamma_0) &= \begin{cases} \eta\frac{2X}{c}\left(\frac{\pi}{L}\right)^r\sqrt{\left(\frac{Kc}{2\pi X}\right)^2 + \frac{m}{X}t} & r \text{ is even;} \\ \frac{-K}{\pi}\left(\frac{\pi}{L}\right)^r & r \text{ is odd,}\end{cases} \\ &\ll \frac{X}{cL^r}\end{aligned}$$

Each derivative of $\sigma^{(n_i)}$ produces L^{-1}, as does each derivative of $(\sigma^{(2)})^{-k}$, and by the chain rule we get the additional factor LKc/X. Likewise, each derivative of $H^{(\ell)}(\gamma_0)$ picks up an LKc/X. Hence,

$$H_{2\nu}^{(s)}(\gamma_0) \ll (\max\{1, LKc/X\})^s \ll K^{s\varepsilon}.$$

Thus the derivatives of $\widetilde{H}(t)$ are bounded by $\widetilde{H}^{(s)}(t) \ll K^{s\varepsilon}$.

For the derivative of $\theta(t)$, we use Lemma 2.8. Taking $A = Kc$, $B = \frac{\eta}{\pi c}$ and $p(t) = 2\pi\sqrt{mtX}$ we obtain

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{\eta}{c}\sqrt{\frac{mX}{t}}\sqrt{\left(\frac{Kc}{2\pi\sqrt{mXt}}\right)^2 + 1} \\ &= \frac{\eta X}{c}\sqrt{\frac{m}{Xt}}\sqrt{\left(\frac{Kc}{2\pi\sqrt{mXt}}\right)^2 + 1}. \end{aligned} \tag{3.15}$$

Since $0 < Kc/\sqrt{mX} < 1$ and $X \leq m \leq 2X$, we see that

$$\frac{X}{\sqrt{2}c}\left(1 + O\left(\frac{K^\varepsilon}{L^2}\right)\right) \leq \left|\frac{d\theta}{dt}\right| \leq \frac{\sqrt{2}X}{c}\left(1 + O\left(\frac{K^\varepsilon}{L^2}\right)\right). \tag{3.16}$$

Moreover,

$$\theta^{(r)}(t) \asymp \frac{X}{c} \quad \text{for} \quad r \geq 1. \tag{3.17}$$

Lastly, we have

$$\psi'(t) = \alpha\beta X^\beta t^{\beta-1} + \left(\frac{z}{c} - n\right)X,$$

so $\psi^{(r)}(t) \asymp |\alpha| X^\beta$ for $r \geq 2$ and $0 < \beta < 1$.

3.2 Weighted Derivative Tests

We use Theorem 2.7 to show that the integral in (3.13) is negligible in most cases. In our application, we take $g(t) = \widetilde{H}(t)$ as in (3.14). Since $\phi(t)$ and all of its derivatives vanish at 1 and 2, the boundary terms in the expansion will in fact be zero.

Let $D = z - cn$ so that

$$\psi'(t) = \alpha\beta X^{\beta} t^{\beta-1} + D\frac{X}{c}. \tag{3.18}$$

Since $0 < \beta < 1$ we may assume $X^{\beta+\varepsilon} \ll LK$. As $1 \le c \le X/LK^{1-\varepsilon}$, we know that $X/c \ge LK^{1-\varepsilon}$ and is therefore a power larger than X^{β}. Thus $|\theta'(t) + \psi'(t)| \gg X/c$ for $1 \le t \le 2$ if $|D| \notin [1/\sqrt{2}, \sqrt{2}]$, that is, if $|D| \ne 1$.

If $n \ge 2$, then $D \le (c-1) - 2c \le -c - 1 \le -2$. Likewise, if $n \le -2$ then $D \ge z + 2c \ge 2c \ge 2$. So, there are only three values for n where $|D| = 1$ is possible.

- $n = -1 : |D| = 1 \iff |z + c| = 1 \iff c = 1$ and $z = 0$
- $n = 0 : |D| = 1 \iff |z| = 1 \iff c \ge 2$ and $z = 1$
- $n = 1 : |D| = 1 \iff |z - c| = 1 \iff z = c - 1$

In each of these cases we could have significant cancellation in our phase function, so we can't use Theorem 2.7 to argue that the integral is negligible. However, in these cases there is only one non-negligible choice for z given c. Furthermore, we still have $|\theta^{(r)}(t) + \psi^{(r)}(t)| \ll X/c$ for $r \ge 2$.

Thus we may take $T = X/c$, $M = 1$, $U = 1$ and $N = K^{-\varepsilon}$. By Theorem 2.7, the integral in (3.13) is bounded by $O((|n|X)^{-n_1} K^{\varepsilon})$ for $|n| \ge 2$, and by $O((c/X)^{n_1} K^{\varepsilon})$ for $|n| \le 1$ for the non-exceptional cases, where n_1 is an arbitrarily large positive integer.

For the non-exceptional cases with $|n| \le 1$, the contribution to (3.13) is

$$L^{2\nu+1} K X^{\frac{1}{2}-\nu} \sum_m \bar{\phi}\Big(\frac{m}{X}\Big) e\Big(-\alpha m^\beta\Big) \sum_{\substack{1 \le c \le \frac{X}{LK^{1-\varepsilon}} \\ N|c}} c^{\nu-\frac{1}{2}} \sum_{\substack{0 \le z < c \\ (z,c)=1}} e\Big(\frac{m\bar{z}}{c}\Big) \sum_{|n| \le 1} \Big(\frac{c}{X}\Big)^{n_1} K^\varepsilon$$

$$\ll L^{2\nu+1} K^{1+\varepsilon} X^{\frac{1}{2}-\nu-n_1} \sum_m \Big|\bar{\phi}\Big(\frac{m}{X}\Big)\Big| \sum_{\substack{1 \le c \le \frac{X}{LK^{1-\varepsilon}} \\ N|c}} c^{\nu+\frac{1}{2}+n_1}$$

$$\ll L^{2\nu+1} K^{1+\varepsilon} X^{\frac{3}{2}-\nu-n_1} \left(\frac{X}{LK^{1-\varepsilon}}\right)^{\nu+\frac{3}{2}+n_1}$$

$$\ll \left(\frac{L}{K^{1-\varepsilon}}\right)^{\nu} \frac{K^\varepsilon X^3}{(LK^{1-\varepsilon})^{\frac{1}{2}+n_1}}.$$

Likewise, when $|n| \ge 2$ the contribution is

$$L^{2\nu+1} K X^{\frac{1}{2}-\nu} \sum_m \bar{\phi}\Big(\frac{m}{X}\Big) e\Big(-\alpha m^\beta\Big) \sum_{\substack{1 \le c \le \frac{X}{LK^{1-\varepsilon}} \\ N|c}} c^{\nu-\frac{1}{2}} \sum_{\substack{0 \le z < c \\ (z,c)=1}} e\Big(\frac{m\bar{z}}{c}\Big) \sum_{|n| \ge 2} \frac{K^\varepsilon}{(|n|X)^{n_1}}$$

$$\ll L^{2\nu+1} K^{1+\varepsilon} X^{\frac{1}{2}-\nu-n_1} \sum_m \Big|\bar{\phi}\Big(\frac{m}{X}\Big)\Big| \sum_{\substack{1 \le c \le \frac{X}{LK^{1-\varepsilon}} \\ N|c}} c^{\nu+\frac{1}{2}} \sum_{|n| \ge 2} \frac{1}{|n|^{n_1}}$$

$$\ll L^{2\nu+1} K^{1+\varepsilon} X^{\frac{3}{2}-\nu-n_1} \left(\frac{X}{LK^{1-\varepsilon}}\right)^{\nu+\frac{3}{2}}$$

$$\ll \left(\frac{L}{K^{1-\varepsilon}}\right)^{\nu} \frac{K^\varepsilon X^{3-n_1}}{(LK^{1-\varepsilon})^{\frac{1}{2}}}$$

Thus we may choose n_1 large enough so that the contribution to T_ν^η is negligible.

Now we consider the three exceptional cases. Recall that there is only one non-negligible choice for z given c, so the sum over z reduces to one term, which is of norm 1. Since $\frac{d^2}{dt^2}(\theta + \psi) \asymp X/c$, by the second derivative test (Lemma 2.5) it follows that the integral in (3.13) is bounded by $(X/c)^{-1/2}$. Therefore

$$\begin{aligned} T_\nu^\eta &\ll L^{2\nu+1} K X^{\frac{1}{2}-\nu} \sum_m \left|\bar{\phi}\left(\frac{m}{X}\right)\right| \sum_{\substack{1\le c\le \frac{X}{LK^{1-\varepsilon}} \\ N|c}} c^{\nu-\frac{1}{2}} \left(\frac{X}{c}\right)^{-1/2} \\ &\ll L^{2\nu+1} K X^{1-\nu} \sum_{1\le c\le \frac{X}{LK^{1-\varepsilon}}} c^{\nu} \ll \left(\frac{L}{K^{1-\varepsilon}}\right)^{\nu} K^{\varepsilon} X^2. \end{aligned} \tag{3.19}$$

Consequently, the off-diagonal terms for (3.1) are bounded by $O(K^\varepsilon X^2)$. Together with the diagonal term (3.7), we have proven that (3.1) is bounded by

$$\sum_{K-L^{1+\varepsilon}\le k_j\le K+L^{1+\varepsilon}} |S_X(f_j;\alpha,\beta)|^2 \ll L^{1+\varepsilon} K X + K^\varepsilon X^2$$

for $0<\beta<1$ subject to $LK \gg X^{\beta+\varepsilon}$. This completes the proof of Theorem 1.1.

3.3 The Holomorphic Case

Suppose now that f is a holomorphic cusp form of weight k. We will work through the necessary changes in order for Theorem 1.1 to still hold. As f is no longer a Maass form, the Kuznetsov trace formula is no longer applicable. Instead we form our smooth average with respect to an orthonormal basis $\mathcal{F}_k$ of $S_k(\Gamma_0(N))$ so that we can apply the Petersson formula. We will follow the proof in [42] starting in §2.

Let h be a positive Schwartz function and set

$$\begin{aligned} \sum_{K,L} &:= \sum_k h\left(\frac{k-1-K}{L}\right) \sum_{f\in\mathcal{F}_k} \frac{2\pi^2}{(k-1)L(\mathrm{sym}^2 f,1)} |S_X(f;\alpha,\beta)|^2 \\ &= \sum_k h\left(\frac{k-1-K}{L}\right) \sum_{m,n} e\left(\alpha(n^\beta - m^\beta)\right) \phi\left(\frac{n}{X}\right) \overline{\phi}\left(\frac{m}{X}\right) \\ &\quad \times \sum_{f\in\mathcal{F}_k} \frac{2\pi^2}{(k-1)L(\mathrm{sym}^2 f,1)} \lambda_f(n)\overline{\lambda_f}(m). \end{aligned} \tag{3.20}$$

The factor $L(\mathrm{sym}^2 f, 1)$ is the symmetric square L-function at 1, whose value is given in [17, p. 251] as

$$L(\mathrm{sym}^2 f, 1) = \frac{\pi}{2}\frac{(4\pi)^k}{\Gamma(k)}\|f\|^2.$$

Since our basis $\mathcal{F}_k$ is orthonormal, it follows that this normalization is exactly what is required to apply the Petersson trace formula. We choose this presentation as it makes explicit the effect of k. (By [42, Eq. (39)] we know that $k^{-\varepsilon} \ll L(\mathrm{sym}^2 f, 1) \ll k^{\varepsilon}$)

The outermost sum in (3.20) is over positive even integers k, and the rapid decay of h will further ensure that $K+1-L^{1+\varepsilon} \le k \le K+1+L^{1+\varepsilon}$, so there will be approximately $L^{1+\varepsilon}$ terms.

Applying Theorem 2.3 to the inner most sum gives

$$\begin{aligned}\sum_{K,L} &= \sum_k h\left(\frac{k-1-K}{L}\right)\sum_{m,n} e\left(\alpha(n^\beta - m^\beta)\right)\phi\left(\frac{n}{X}\right)\overline{\phi}\left(\frac{m}{X}\right) \\ &\times\left(\delta_{m,n} + 2\pi i^{-k}\sum_{\substack{c>0\\ N|c}} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{K(m,n;c)}{c}\right).\end{aligned} \tag{3.21}$$

For the diagonal terms we get

$$\sum_k h\left(\frac{k-1-K}{L}\right)\sum_n \left|\phi\left(\frac{n}{X}\right)\right|,$$

which is $O(L^{1+\varepsilon}X)$.

The off-diagonal terms are

$$\begin{aligned}&\pi\sum_{m,n} e\left(\alpha(n^\beta - m^\beta)\right)\phi\left(\frac{n}{X}\right)\overline{\phi}\left(\frac{m}{X}\right)\sum_{\substack{c>0\\ N|c}}\frac{K(m,n;c)}{c} \\ &\qquad\times 2\sum_k h\left(\frac{k-1-K}{L}\right) i^{-k} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).\end{aligned} \tag{3.22}$$

Since k is even, the generalized Kloosterman sum $K(m,n;c)$ reduces to the classical one $S(m,n;c)$, and $i^{-k} = i^k$. We will apply Poisson summation to the k-sum. First we write

$$V(x) = 2 \sum_{l \equiv 1(2)} e\left(\frac{l+1}{4}\right) h_0(l) J_l(x)$$

where

$$h_0(l) = h\left(\frac{l-K}{L}\right),$$

so that $V(x)$ is the k-sum, with $l = k-1$. Observe that

$$\begin{aligned} 2 \sum_{l \equiv 1(2)} e\left(\frac{l+1}{4}\right) h_0(l) J_l(x) &= \sum_{l \in \mathbb{Z}} e\left(\frac{l+1}{4}\right) h_0(l) J_l(x) \\ &\quad + \sum_{l \in \mathbb{Z}} e\left(\frac{l+1}{4}\right) e\left(\frac{l-1}{2}\right) h_0(l) J_l(x). \end{aligned}$$

By [17, p. 86], the right-hand side is equal to

$$\begin{aligned} & i \int_{\mathbb{R}} \hat{h}_0(t) \left[e^{ix \sin 2\pi(1/4-t)} - e^{ix \sin 2\pi(3/4-t)}\right] dt \\ &= i \int_{\mathbb{R}} \hat{h}_0(t) \left[e^{ix \cos 2\pi t} - e^{-ix \cos 2\pi t}\right] dt \end{aligned}$$

Now,

$$\hat{h}_0(t) = \int_{\mathbb{R}} h\left(\frac{w-K}{L}\right) e(wt) dw = Le(Kt)\hat{h}(Lt),$$

after making the change of variables $w \mapsto Lw + K$. Replacing t with t/L yields

$$V(x) = W(x) - W(-x), \tag{3.23}$$

where

$$W(x) = i \int_{\mathbb{R}} \hat{h}(t) e\left(\frac{Kt}{L}\right) e^{ix \cos 2\pi t/L} dt. \tag{3.24}$$

In [42], Sarnak takes the Taylor expansion of $\cos 2\pi t/L$, keeping only the first two terms, as the rest of them contribute an allowable error term. In doing so, the remaining phase function has a single stationary point and one can then argue that $W(x)$ is negligible for $x < LK^{1-\varepsilon}$. Thus we may still assume $c \leq X/(LK^{1-\varepsilon})$. However, we wish to obtain a better asymptotic expansion analagous to Theorem 2.9, wherefore we apply Theorem 2.10 to (3.24).

For $W(\eta x)$, the derivative of the phase function is

$$\sigma'(t) = \frac{K}{L} - \frac{\eta x}{L} \sin\left(\frac{2\pi t}{L}\right), \tag{3.25}$$

and so the sole stationary point is

$$\gamma := \eta \frac{L}{2\pi} \arcsin\left(\frac{K}{x}\right). \tag{3.26}$$

(Note that $K/x = Kc/(4\pi\sqrt{mn}) \leq K^{\varepsilon}/(4\pi L) < 1$)

Since K/x is close to zero, we use the approximation $\sin^{-1}(z) = z + O(z^3)$. Thus

$$|\gamma| = \frac{L}{2\pi}\sin^{-1}\left(\frac{K}{x}\right) = \frac{L}{2\pi}\left(\frac{K}{x} + O\left(\left(\frac{K}{x}\right)^3\right)\right) \leq \frac{K^{\varepsilon}}{2\pi} + O\left(\frac{K^{3\varepsilon}}{L^2}\right) < K^{\varepsilon},$$

and we see that γ will be contained in our region of integration.

Now, we have

$$\begin{aligned} \sigma(\gamma) &= \frac{x}{2\pi}\sqrt{1 - \left(\frac{K}{x}\right)^2} + \eta\frac{K}{2\pi}\sin^{-1}\left(\frac{K}{x}\right) \\ &= \frac{1}{2\pi}\sqrt{x^2 - K^2} + \eta\frac{K}{2\pi}\sin^{-1}\left(\frac{K}{x}\right) \end{aligned}$$

and for $r \geq 2$

$$\sigma^{(r)}(t) = \begin{cases} (-1)^{\frac{r}{2}} \frac{x}{2\pi}\left(\frac{2\pi}{L}\right)^r \cos\left(\frac{2\pi t}{L}\right) & r \text{ is even} \\ (-1)^{\frac{r+1}{2}} \frac{x}{2\pi}\left(\frac{2\pi}{L}\right)^r \sin\left(\frac{2\pi t}{L}\right) & r \text{ is odd.} \end{cases} \tag{3.27}$$

As in the Maass form case, we take our parameters to be $T = X/c$, $M = L$, and $U = N = 1$. For $|t| \leq K^\varepsilon$, it follows from (3.27) that

$$
\begin{aligned}
|\sigma^{(r)}(t)| &\leq \frac{x}{2\pi}\left(\frac{2\pi}{L}\right)^r \leq 8\pi(2\pi)^r\frac{X/c}{L^r} = C_r\frac{T}{M^r},\\
|\sigma^{(2)}(t)| &\geq \frac{x}{2\pi}\left(\frac{2\pi}{L}\right)^2\cos\left(\frac{2\pi K^\varepsilon}{L}\right) \geq 8\pi^2\frac{X/c}{L^2}\cos\left(\frac{\pi}{3}\right) \geq \frac{X/c}{32\pi^3L^2} = \frac{T}{C_2M^2}.
\end{aligned}
$$

For the weight function $\hat{h}(t)$, which is a Schwartz function, we have

$$|\hat{h}^{(s)}(t)| \leq C_s.$$

We may now invoke Theorem 2.10 to obtain the following asymptotic expansion of $W(x)$ by cutting off negligible tails:

$$
\begin{aligned}
\int_{-K^\varepsilon}^{K^\varepsilon} e(\sigma(t))g(t)dt &= \frac{e\Big(\sigma(\gamma)+\frac{1}{8}\Big)}{\Big|-\frac{2\pi}{L^2}\sqrt{x^2-K^2}\Big|^{\frac{1}{2}}}\left[g(\gamma)+\sum_{\nu=1}^{n}\varpi_{2\nu}\frac{(-1)^\nu(2\nu-1)!!}{(4\pi i\lambda_2)^\nu}\right] \qquad (3.28)\\
&+\text{Boundary terms} + O\Big(\frac{L^{2n+5}}{(X/c)^{n+2}}\sum_{j=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{1}{(K^\varepsilon)^{n+2+j}}\sum_{t=j}^{n+1-j}\frac{1}{L^t}\Big)\\
&+O\Big(\frac{L^{2n+4}}{(X/c)^{n+2}}\Big(L+1\Big)\Big(\frac{1}{(K^\varepsilon)^{n+2}}\Big)\Big)\\
&+O\Big(\frac{L^{2n+4}}{(X/c)^{n+2}}\sum_{j=1}^{n+1}\Big(\frac{1}{(K^\varepsilon)^{n+2+j}}\Big)\sum_{t=0}^{n+1-j}\frac{1}{L^t}\Big)+O\Big(\frac{L^{2n+2}}{(X/c)^{n+1}}\Big).
\end{aligned}
$$

The boundary terms in (3.28) will be negligible since $g(t) = \hat{h}(t)$ is a Schwartz function, and the largest error term in (3.28) will be the last one. Evaluating $e(1/8)$, simplifying the main term, and using that $X/c \geq LK^{1-\varepsilon}$ in the error terms reduces (3.28) to

$$
\begin{aligned}
&\frac{L(1+i)e(\sigma(\gamma))}{2\sqrt{\pi}(x^2-K^2)^{1/4}}\left[g(\gamma)+\sum_{\nu=1}^{n}\varpi_{2\nu}\frac{(-1)^\nu(2\nu-1)!!}{(4\pi i\lambda_2)^\nu}\right]+O\Big(\frac{L^{2n+2}}{(LK^{1-\varepsilon})^{n+1}}\Big)\\
&=\sum_{\nu=0}^{n}\widetilde{W}_\nu(x)+O\Big(\Big(\frac{L}{K^{1-\varepsilon}}\Big)^{n+1}\Big). \qquad (3.29)
\end{aligned}
$$

Since σ and g are both infinitely differentiable, we may take n to be as large as we need in (3.29) to ensure that the error term is negligible. We rewrite $\varpi_{2\nu}$ in (2.10) as $H_{2\nu}(\gamma)$, where

$$\begin{aligned} H_{2\nu}(\gamma) \;&=\; \frac{\hat{h}^{(2\nu)}(\gamma)}{(2\nu)!} \\ &+ \sum_{\ell=0}^{2\nu-1} \frac{\hat{h}^{(\ell)}(\gamma)}{\ell!} \sum_{k=1}^{2\nu-\ell} \frac{2^k C_{2\nu,\ell,k}}{\Big(-\frac{2\pi}{L^2}\sqrt{x^2-K^2}\Big)^k} \sum_{\substack{3\le n_1,\ldots,n_k\le 2n+3 \\ n_1+\cdots+n_k=2\nu-\ell+2k}} \lambda_{n_1}\cdots\lambda_{n_k} \end{aligned} \tag{3.30}$$

with λ_{n_i} being given in (2.11) and $H(t)$ in (2.14).

We now have our analogous expansion, which we insert into (3.22). Ignoring constant coefficients and plugging in $x = 4\pi\sqrt{mn}/c$, a typical term is

$$\begin{aligned} T_\nu^\eta \;&=\; \sum_{m,n} e\left(\alpha(n^\beta - m^\beta)\right)\phi\left(\frac{n}{X}\right)\overline{\phi}\left(\frac{m}{X}\right) \sum_{\substack{1\le c\le X/LK^{1-\varepsilon} \\ N|c}} \frac{S(m,n;c)}{c} \\ &\times \frac{L^{2\nu+1}c^{\nu+1/2}}{(16\pi^2 mn - c^2K^2)^{\nu/2+1/4}} H_{2\nu}\left(\eta\frac{L}{2\pi}\arcsin\left(\frac{Kc}{4\pi\sqrt{mn}}\right)\right) \\ &\times e\left(\frac{1}{2\pi c}\sqrt{16\pi^2mn - c^2K^2} + \eta\frac{K}{\pi}\sin^{-1}\left(\frac{Kc}{4\pi\sqrt{mn}}\right)\right). \end{aligned} \tag{3.31}$$

Opening up the Kloosterman sum in (3.31) we can rearrange the order of summation as

$$\begin{aligned} &L^{2\nu+1}\sum_m \bar{\phi}\Big(\frac{m}{X}\Big)e\Big(-\alpha m^\beta\Big) \sum_{\substack{1\le c\le \frac{X}{LK^{1-\varepsilon}} \\ N|c}} c^{\nu-\frac{1}{2}} \sum_{\substack{z \bmod c \\ (z,c)=1}} e\Big(\frac{m\bar{z}}{c}\Big) \\ &\times \sum_n \phi\Big(\frac{n}{X}\Big)e(\alpha n^\beta)e\Big(\frac{nz}{c}\Big)\Big(16\pi^2nm - c^2K^2\Big)^{-\frac{\nu}{2}-\frac{1}{4}} H_{2\nu}\Big(\frac{\eta L}{2\pi}\arcsin\Big(\frac{Kc}{4\pi\sqrt{mn}}\Big)\Big) \\ &\times e\Bigg(\frac{1}{2\pi c}\sqrt{16\pi^2nm - c^2K^2} + \frac{\eta K}{\pi}\arcsin\Big(\frac{Kc}{4\pi\sqrt{mn}}\Big)\Bigg). \end{aligned} \tag{3.32}$$

Now we apply Poisson's summation formula (Theorem 2.1) to convert the sum over

n in (3.32) to

$$\sum_n \int_{\mathbb{R}} \phi\Big(\frac{t}{X}\Big) e\Big(\alpha t^\beta + \frac{tz}{c}\Big)\Big(16\pi^2 mt - c^2K^2\Big)^{-\nu/2-1/4} H_{2\nu}\Big(\frac{\eta L}{2\pi}\arcsin\Big(\frac{Kc}{4\pi\sqrt{mt}}\Big)\Big)$$
$$\times e\left(\frac{1}{2\pi c}\sqrt{16\pi^2 mt - c^2K^2} + \frac{\eta K}{\pi}\arcsin\Big(\frac{Kc}{4\pi\sqrt{mt}}\Big)\right) e(-nt)dt,$$

which, after changing the variable t to tX, becomes

$$X^{\frac{1}{2}-\nu}\sum_n \int_1^2 \phi(t)\Big(16\pi^2 t\frac{m}{X} - \Big(\frac{cK}{X}\Big)^2\Big)^{-\frac{\nu}{2}-\frac{1}{4}} H_{2\nu}\Big(\frac{\eta L}{2\pi}\arcsin\Big(\frac{Kc}{4\pi\sqrt{mtX}}\Big)\Big) \quad (3.33)$$
$$\times e\left(\frac{1}{2\pi c}\sqrt{16\pi^2 mtX - c^2K^2} + \frac{\eta K}{\pi}\arcsin\Big(\frac{Kc}{4\pi\sqrt{mtX}}\Big)\right)$$
$$\times e\Big(\alpha X^\beta t^\beta + \frac{zXt}{c} - nXt\Big)dt.$$

We introduce the notation $\widetilde{H}(t)$, $\theta(t)$ and $\psi(t)$ in an obvious way in (3.33) so that we may rewrite T_ν^η more succinctly as

$$T_\nu^\eta = L^{2\nu+1}X^{\frac{1}{2}-\nu}\sum_m \bar{\phi}\Big(\frac{m}{X}\Big)e\Big(-\alpha m^\beta\Big) \quad (3.34)$$
$$\times \sum_{\substack{1\le c\le \frac{X}{LK^{1-\varepsilon}} \\ N|c}} c^{\nu-\frac{1}{2}} \sum_{\substack{0\le z<c \\ (z,c)=1}} e\Big(\frac{m\bar{z}}{c}\Big)\sum_n \int_1^2 \widetilde{H}(t)e(\theta(t))e(\psi(t))dt.$$

Once we have control of the derivatives of these functions, we may again wield Theorem 2.7 to demonstrate that this integral is negligible for all but a finite number of values of n. Setting

$$\gamma_0 = \frac{\eta L}{2\pi}\arcsin\left(\frac{Kc}{4\pi\sqrt{mtX}}\right), \quad (3.35)$$

we may proceed in a similar manner as the Maass form case. We have

$$\sigma^{(r)}(\gamma_0) = \begin{cases} (-1)^{\frac{r}{2}}\frac{1}{2\pi}\Big(\frac{2\pi}{L}\Big)^r \frac{X}{c}\sqrt{16\pi^2 t\frac{m}{X} - \big(\frac{Kc}{X}\big)^2} & r \text{ is even} \\ (-1)^{\frac{r+1}{2}}\frac{\eta K}{2\pi}\Big(\frac{2\pi}{L}\Big)^r & r \text{ is odd,} \end{cases}$$

so $\sigma^{(r)} \ll X/(cL^r)$. Thus, $\widetilde{H}^{(s)}(\gamma_0) \ll K^{s\varepsilon}$.

It can be verified that

$$\frac{d\theta}{dt} = \frac{4\pi m}{c\sqrt{16\pi^2 t\frac{m}{X} - \left(\frac{Kc}{X}\right)^2}} \left(1 - 2\eta\left(\frac{Kc}{4\pi\sqrt{mtX}}\right)^2\right),$$

and thus $\theta'(t)$ is bounded in the range

$$\frac{X}{c}\left(\frac{1}{2} + O\left(\frac{K^{2\varepsilon}}{L^2}\right)\right) \leq \frac{d\theta}{dt} \leq \frac{X}{c}\left(2 + O\left(\frac{K^{2\varepsilon}}{L^2}\right)\right).$$

With D defined as in (3.18), a similar line of reasoning shows that the number of values for n such that $1 \leq |D| \leq 2$ is finite. By (3.34), our estimation will produce the same bound as (3.19), but without the factor of K, whence our final bound for the off-diagonal terms is

$$\left(\frac{L}{K^{1-\varepsilon}}\right)^{\nu} \frac{X^2}{K^{1-\varepsilon}}. \tag{3.36}$$

We have shown that

$$\sum_k h\left(\frac{k-1-K}{L}\right) \sum_{f\in\mathcal{F}_k} \frac{2\pi^2}{(k-1)L(\mathrm{sym}^2 f, 1)} |S_X(f;\alpha,\beta)|^2 \ll L^{1+\varepsilon}X + \frac{X^2}{K^{1-\varepsilon}},$$

and the existence of k in the denominator implies

$$\sum_{K-L\leq k\leq K+L} |S_X(f_k;\alpha,\beta)|^2 \ll L^{1+\varepsilon}KX + K^{\varepsilon}X^2, \tag{3.37}$$

exactly as in Theorem 1.1.

CHAPTER 4
RESONANCE FOR $GL(2) \times GL(2)$

In this chapter we consider the case of resonance sums correlated to the Dirichlet coefficients of a Rankin-Selberg L-function. Let f be a Maass form as before, and let g be a holomorphic cusp form for $SL_2(\mathbb{Z})$ of weight l. The resonance sum is

$$S_X(f \times g; \alpha, \beta) = \sum_n \lambda_f(n)\lambda_g(n)e(\alpha n^\beta)\phi\Big(\frac{n}{X}\Big). \tag{4.1}$$

We follow the strategy in Section 3.1 up until the application of Poisson's summation formula. At that point we instead utilize Voronoi's summation formula due to the presence of the Fourier coefficients of g.

Suppose $K^{2\varepsilon} \leq L = K^\mu \leq K^{1-2\varepsilon}$. We form our smooth average

$$\begin{aligned}\sum_{f_j} \Big[h\Big(\frac{k_j - K}{L}\Big) + h\Big(-\frac{k_j + K}{L}\Big)\Big] &|S_X(f_j \times g; \alpha, \beta)|^2 \qquad (4.2)\\ = \sum_{n,m} e\Big(\alpha(n^\beta - m^\beta)\Big)&\lambda_g(n)\overline{\lambda_g}(m)\phi\Big(\frac{n}{X}\Big)\bar{\phi}\Big(\frac{m}{X}\Big)\\ \times \sum_{f_j}\Big[h\Big(\frac{k_j - K}{L}\Big) &+ h\Big(-\frac{k_j + K}{L}\Big)\Big]\lambda_{f_j}(n)\bar{\lambda}_{f_j}(m).\end{aligned}$$

We may then apply Kuznetsov's trace formula to the inner sum, which is exactly the same as before. The resulting terms have therefore already been dealt with. In (3.5) we now have the additional factor of $|\lambda_g(n)|^2$, which does not effect the positivity. In (3.6), the presence of $|\lambda_g(n)|^2$ increases our bound for the diagonal terms by a factor of X^ε, and they are therefore $O(L^{1+\varepsilon}KX^{1+\varepsilon})$.

The computations for the off-diagonal terms hold up through (3.9), but now instead of (3.10) we have

$$T_\nu^{\eta_1} = \sum_{n,m} \phi\Big(\frac{n}{X}\Big)\bar{\phi}\Big(\frac{m}{X}\Big) e\Big(\alpha(n^\beta - m^\beta)\Big)\lambda_g(n)\overline{\lambda_g}(m) \sum_{\substack{1\le c\le \frac{X}{LK^{1-\varepsilon}} \\ N|c}} \frac{S(m,n,c)}{c}$$
$$\times \frac{L^{2\nu+1}}{\eta_1^{\nu+1/2}\Big(K^2+\frac{4\pi^2 nm}{c^2}\Big)^{\nu/2+1/4}} K H_{2\nu}\Big(-\eta_1 \frac{L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mn}}\Big)\Big)$$
$$\times e\Bigg(\frac{\eta_1}{\pi c}\sqrt{K^2c^2+4\pi^2 nm} - \eta_1\frac{K}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mn}}\Big)\Bigg). \tag{4.3}$$

4.1 Voronoi Summation

We open up the Kloosterman sum and group together our sum over m, so $T_\nu^{\eta_1}$ is

$$\eta_1^{\nu+3/2} L^{2\nu+1} K \sum_{\substack{1\le c\le \frac{X}{LK^{1-\varepsilon}} \\ N|c}} c^{\nu-1/2} \sum_n \phi\Big(\frac{n}{X}\Big) e\Big(\alpha n^\beta\Big)\lambda_g(n) \sum_{\substack{z \bmod c \\ (z,c)=1}} e\left(\frac{n\bar{z}}{c}\right) \tag{4.4}$$
$$\times \sum_m \overline{\lambda_g}(m) e\left(\frac{mz}{c}\right)\bar{\phi}\Big(\frac{m}{X}\Big) e\left(-\alpha m^\beta\right)$$
$$\times \Big(K^2c^2+4\pi^2 nm\Big)^{-\nu/2-1/4} H_{2\nu}\Big(-\eta_1\frac{L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mn}}\Big)\Big)$$
$$\times e\Bigg(\frac{\eta_1}{\pi c}\sqrt{K^2c^2+4\pi^2 nm} - \eta_1\frac{K}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{mn}}\Big)\Bigg).$$

We apply a Voronoi-type summation formula, Theorem 2.2, to the inner sum over m to transform it into

$$\frac{2\pi i^l}{c}\sum_{r\ge 1}\bar{\lambda}_g(r) e\left(-\frac{r\bar{z}}{c}\right)\int_0^\infty J_{l-1}\left(\frac{4\pi\sqrt{rw}}{c}\right)\bar{\phi}\Big(\frac{w}{X}\Big)e\left(-\alpha w^\beta\right)$$
$$\times\Big(K^2c^2+4\pi^2 nw\Big)^{-\nu/2-1/4} H_{2\nu}\Big(-\eta_1\frac{L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{nw}}\Big)\Big)$$
$$\times e\Bigg(\frac{\eta_1}{\pi c}\sqrt{K^2c^2+4\pi^2 nw} - \eta_1\frac{K}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi\sqrt{wn}}\Big)\Bigg)dw.$$

We change w to w^2X

$$\frac{4\pi i^l}{c}X\sum_{r\geq 1}\bar{\lambda}_g(r)e\left(-\frac{r\bar{z}}{c}\right)\int_1^{\sqrt{2}} J_{l-1}\left(\frac{4\pi w\sqrt{rX}}{c}\right)w\bar{\phi}(w^2)e\left(-\alpha w^{2\beta}X^{\beta}\right)$$
$$\times\Big(K^2c^2+4\pi^2w^2nX\Big)^{-\nu/2-1/4}H_{2\nu}\Big(-\eta_1\frac{L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi w\sqrt{nX}}\Big)\Big)$$
$$\times e\left(\frac{\eta_1}{\pi c}\sqrt{K^2c^2+4\pi^2w^2nX}-\eta_1\frac{K}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi w\sqrt{nX}}\Big)\right)dw,$$

and then pull out $(2\pi wX)^2$ from the first factor in the second line

$$\frac{2^{3/2-\nu}i^l}{c}(\pi X)^{1/2-\nu}\sum_{r\geq 1}\bar{\lambda}_g(r)e\left(-\frac{r\bar{z}}{c}\right)\int_1^{\sqrt{2}} J_{l-1}\left(\frac{4\pi w\sqrt{rX}}{c}\right)w^{1/2-\nu}\bar{\phi}(w^2) \quad (4.5)$$
$$\times\left(\left(\frac{Kc}{2\pi wX}\right)^2+\frac{n}{X}\right)^{-\nu/2-1/4}H_{2\nu}\Big(-\eta_1\frac{L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi w\sqrt{nX}}\Big)\Big)$$
$$\times e\left(-\alpha w^{2\beta}X^{\beta}+\frac{\eta_1}{\pi c}\sqrt{K^2c^2+4\pi^2w^2nX}-\eta_1\frac{K}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi w\sqrt{nX}}\Big)\right)dw.$$

We use the following asymptotic expansion for the J-Bessel function:

$$\begin{aligned}J_{l-1}(x) &= \frac{1}{\sqrt{2\pi x}}e^{i(x-(2l-1)\pi/4)}\sum_{0\leq j<2N}\frac{i^j\Gamma(l+j-\frac{1}{2})}{j!\Gamma(l-j-\frac{1}{2})}(2x)^{-j}\\ &+ \frac{1}{\sqrt{2\pi x}}e^{-i(x-(2l-1)\pi/4)}\sum_{0\leq j<2N}\frac{i^j\Gamma(l+j-\frac{1}{2})}{j!\Gamma(l-j-\frac{1}{2})}(-2x)^{-j}\\ &+ O(x^{-2N-1/2})\end{aligned}$$

Ignoring constant coefficients, a typical term on the right hand side is of the form

$$\frac{1}{x^{j+1/2}}e\left(\eta_2\left(\frac{x}{2\pi}-\frac{2l-1}{8}\right)\right) \tag{4.6}$$

where $\eta_2=\pm 1$ as determined by the sign in the exponent of e. With $x=4\pi w\sqrt{rX}/c$

and ignoring constant coefficients we may therefore replace $J_{l-1}(x)$ with

$$\frac{c^{j+1/2}}{w^{j+1/2}(rX)^{j/2+1/4}}e\left(\eta_2\left(\frac{2w\sqrt{rX}}{c}-\frac{2l-1}{8}\right)\right)+O\left(\left(\frac{rX}{c^2}\right)^{-N-\frac{1}{4}}\right).$$

By choosing N sufficiently large, our error term will be negligible provided $X/c^2 > K^{\varepsilon}$. We may therefore assume $\sqrt{X} < LK^{1-\varepsilon}$. We need not worry that this assumption is too restrictive, as it is weaker than the restraints we will eventually impose on μ in (4.42) and (4.62).

Plugging in a term from the asymptotic expansion turns (4.5) into

$$\begin{aligned}
&\frac{2^{3/2-\nu}i^l}{c}(\pi X)^{1/2-\nu}\sum_{r\geq 1}\bar{\lambda}_g(r)e\left(-\frac{r\bar{z}}{c}\right)\\
&\times\int_1^{\sqrt{2}}\left[\frac{c^{j+1/2}}{w^{j+1/2}(rX)^{j/2+1/4}}e\left(\eta_2\left(\frac{2w\sqrt{rX}}{c}-\frac{2l-1}{8}\right)\right)+O\left(K^{-M}\right)\right]\\
&\times w^{1/2-\nu}\bar{\phi}(w^2)\left(\left(\frac{Kc}{2\pi wX}\right)^2+\frac{n}{X}\right)^{-\nu/2-1/4}H_{2\nu}\Big(-\eta_1\frac{L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi w\sqrt{nX}}\Big)\Big)\\
&\times e\left(-\alpha w^{2\beta}X^{\beta}+\frac{\eta_1}{\pi c}\sqrt{K^2c^2+4\pi^2w^2nX}-\eta_1\frac{K}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi w\sqrt{nX}}\Big)\right)dw.
\end{aligned}$$

We may ignore the constant factor $2^{3/2-\nu}i^l\pi^{1/2-\nu}e(-\eta_2(2l-1)/8)$ and the negligible error term, thus simplifying this expression to

$$\begin{aligned}
&c^{j-1/2}X^{1/4-j/2-\nu}\sum_{r\geq 1}\frac{\bar{\lambda}_g(r)}{r^{j/2+1/4}}e\left(-\frac{r\bar{z}}{c}\right)\\
&\qquad\times\int_1^{\sqrt{2}}\frac{\bar{\phi}(w^2)}{w^{j+\nu}}\frac{H_{2\nu}\Big(-\eta_1\frac{L}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi w\sqrt{nX}}\Big)\Big)}{\left(\left(\frac{Kc}{2\pi wX}\right)^2+\frac{n}{X}\right)^{\nu/2+1/4}}e(\Phi(w))dw,
\end{aligned}\tag{4.7}$$

where

$$\Phi(w)=-\alpha w^{2\beta}X^{\beta}+\eta_2\frac{2w\sqrt{rX}}{c}+\frac{\eta_1}{\pi c}\sqrt{K^2c^2+4\pi^2w^2nX}-\eta_1\frac{K}{\pi}\sinh^{-1}\Big(\frac{Kc}{2\pi w\sqrt{nX}}\Big).$$

Substituting (4.7) into (4.4), we see that the goal of bounding the off-diagonal terms has been reduced to bounding

$$T^{\eta}_{\nu,j} = L^{2\nu+1}KX^{\frac{1}{4}-\frac{j}{2}-\nu} \sum_{\substack{1\le c\le \frac{X}{LK^{1-\varepsilon}} \\ N|c}} c^{j+\nu-1} \sum_{n,r\ge 1} \frac{\lambda_g(n)\bar{\lambda}_g(r)}{r^{j/2+1/4}} \phi\Big(\frac{n}{X}\Big) e\Big(\alpha n^{\beta}\Big) \quad (4.8)$$
$$\times \sum_{\substack{z \mod c \\ (z,c)=1}} e\left(\frac{\bar{z}(n-r)}{c}\right) B^{j,\nu}_{\eta,X,c}(n,r),$$

where $\eta = (\eta_1, \eta_2)$ and

$$B^{j,\nu}_{\eta,X,c}(n,r) = \int_1^{\sqrt{2}} e(\Phi(w)) \frac{H_{2\nu}\Big(-\eta_1 \frac{L}{\pi} \sinh^{-1}\Big(\frac{Kc}{2\pi w\sqrt{nX}}\Big)\Big)}{\Big(\big(\frac{Kc}{2\pi w X}\big)^2 + \frac{n}{X}\Big)^{\nu/2+1/4}} \frac{\bar{\phi}(w^2)}{w^{j+\nu}} dw. \quad (4.9)$$

To proceed, we first take advantage of the phase function in (4.9) to control the size of r and $|r-n|$. As usual, Theorem 2.7 will be the weapon of choice. Using Lemma 2.8 with $B = \eta_1/\pi c$, $A = Kc$ and $p(w) = 2\pi w\sqrt{nX}$, we have

$$\begin{aligned} \Phi'(w) &= -2\alpha\beta w^{2\beta-1}X^{\beta} + \eta_2 \frac{2\sqrt{Xr}}{c} + \frac{\eta_1}{c} 2\sqrt{nX} \sqrt{\left(\frac{Kc}{2\pi w\sqrt{nX}}\right)^2 + 1} \\ &= -2\alpha\beta w^{2\beta-1}X^{\beta} + \frac{2X}{c}\left(\eta_2\sqrt{\frac{r}{X}} + \eta_1\sqrt{\left(\frac{Kc}{2\pi w X}\right)^2 + \frac{n}{X}}\right). \end{aligned}$$

Henceforth we assume that α is fixed (independent of X or β) and, when $\alpha \neq 0$, we set

$$|\alpha|X^{\beta} = \frac{K^{1+2\varepsilon}}{L}. \quad (4.10)$$

Note that this implies

$$\frac{K^2c}{X} \le \frac{K^{1+\varepsilon}}{L} = \frac{|\alpha|X^{\beta}}{K^{\varepsilon}} \quad \text{and} \quad |\alpha|X^{\beta} = \frac{K^{1+2\varepsilon}}{L} \le \frac{K^{1+2\varepsilon}}{L}\frac{X}{cLK^{1-\varepsilon}} \le \frac{X}{c}\frac{K^{3\varepsilon}}{L^2}. \quad (4.11)$$

That is, $|\alpha|X^{\beta}$ is greater than K^2c/X and less than X/c, each by a factor of at least K^{ε}. We are now ready to prove the following lemma:

Lemma 4.1. *Assume (4.10).*

1. *If* $\eta_2 = \eta_1$*, then*

$$B^{j,\nu}_{\eta,X,c}(n,r) \ll K^{-M} \tag{4.12}$$

for any $M > 0$.

2. *If* $r \notin [\frac{3}{4}X, \frac{9}{4}X]$*, then (4.12) holds for any* $M > 0$.

3. *Suppose* $\alpha = 0$*. If* $|r-n| \gg K^{2+\varepsilon}c^2/X$ *then (4.12) holds.*

4. *Suppose that* $\alpha \neq 0$ *and* $\beta < 1$*. If* $|r-n| \geq 7cK^{1+2\varepsilon}/L$*, then (4.12) holds.*

Proof. All estimates are proven by applying Theorem 2.7 above to (4.9). Note that the derivatives of $\frac{\overline{\phi}(w^2)}{w^{j+\nu}}$ and $\left(\left(\frac{cK}{2\pi wX}\right)^2 + \frac{n}{X}\right)^{-(\nu/2+1/4)}$ will be bounded by a constant, and the Nth derivative of $H_{2\nu}\left(\frac{-L}{\pi}\sinh^{-1}\left(\frac{\eta_1 Kc}{2\pi w\sqrt{nX}}\right)\right)$ will be bounded by $(LKc/X)^N \leq K^{N\epsilon}$. Also, $\Phi^{(j)} \ll K^2c/X$ for all $j \geq 2$ when $\alpha = 0$, and $\Phi^{(j)} \ll |\alpha|X^\beta$ otherwise. Thus we set

$$T = \max\{|\alpha|X^\beta, \frac{K^{2+\varepsilon}c}{X}\}, \qquad M = 1, \qquad U = 1, \qquad \text{and } N = \frac{1}{K^\varepsilon}. \tag{4.13}$$

Then our error term will be $O(K^{-M})$ so long as $\min|\Phi'| \gg T$.

(1) If $\eta_1 = \eta_2$ we have

$$\begin{aligned} |\Phi'| \quad &\geq \frac{2X}{c}\left|\eta_2\sqrt{\frac{r}{X}} + \eta_1\sqrt{\left(\frac{Kc}{2\pi wX}\right)^2 + \frac{n}{X}}\right| - |\alpha|2\beta w^{2\beta-1}X^\beta \\ &\geq \frac{2X}{c}\sqrt{\frac{n}{X} + \left(\frac{Kc}{2\pi wX}\right)^2} - 4|\alpha|X^\beta \\ &\gg \frac{X}{c}, \end{aligned}$$

in light of (4.11) and the fact that the square root in line two is bounded between 1 and 3. By (4.11) then, $\Phi' \gg TK^\varepsilon$. Hereafter we assume $\eta_2 = -\eta_1$.

(2) We still have $\Phi' \gg X/c$ as above so long as

$$\left|\sqrt{\frac{r}{X}} - \sqrt{\frac{n}{X} + \left(\frac{Kc}{2\pi w X}\right)^2}\right| \gg 1, \tag{4.14}$$

which is true for $r < \frac{3}{4}X$ or $r > \frac{9}{4}X$.

(3) Suppose $\alpha = 0$, so that $T = K^{2+\varepsilon}c/X$. We may assume $r \in [\frac{3}{4}X, \frac{9}{4}X]$. Multiplying by the conjugate we rewrite the left-hand side of (4.14) as

$$\left|\frac{\frac{r-n}{X} - \left(\frac{Kc}{2\pi w X}\right)^2}{\sqrt{\frac{r}{X}} + \sqrt{\frac{n}{X} + \left(\frac{Kc}{2\pi w X}\right)^2}}\right|. \tag{4.15}$$

The denominator is bounded between 1 and 3, and the numerator is $\asymp |r-n|/X$ if $|r-n|/X \gg K^{2+\varepsilon}c^2/X^2$. With this assumption we have

$$|\Phi'| \gg \frac{2X}{c}\frac{|r-n|}{X} = \frac{2|r-n|}{c} \gg \frac{K^{2+\varepsilon}c}{X}. \tag{4.16}$$

(4) Suppose $\alpha \neq 0$ so $T = |\alpha|X^{\beta}$. The argument holds as in the previous case, but now we need an additional assumption to ensure that the $r-n$ term will dominate. Hence, we assume

$$\frac{2X}{c}\frac{|r-n|}{3X} > 2\beta w^{2\beta-1}|\alpha|X^{\beta}.$$

Since $\beta < 1$, this holds if $|r-n| \geq 7cK^{1+2\varepsilon}/L$. □

Following section 5 of [28], we let $h = r-n$, and apply the well-known formula for the Ramanujan sum (e.g., [16, Eq. (2.26)]):

$$\sum_{\substack{z \mod c \\ (z,c)=1}} e\left(\frac{zN}{c}\right) = \sum_{\delta|(N,c)} \mu\left(\frac{c}{\delta}\right)\delta.$$

We may therefore replace (4.8) with

$$T^{\eta}_{\nu,j} \ll L^{2\nu+1} K X^{1/4-j/2-\nu} \sum_{\delta \leq \frac{X}{LK^{1-\epsilon}}} \delta \sum_{\substack{c \leq \frac{X}{LK^{1-\epsilon}} \\ \delta | c}} c^{j+\nu-1} \sum_{\substack{1-2X<h \\ \delta | h}} |P(c,h,X)| \tag{4.17}$$

where

$$P(c,h,X) = \sum_{n \geq \max(1,1-h)} \frac{\lambda_g(n)\overline{\lambda_g}(n+h)}{(n+h)^{j/2+1/4}} e(\alpha n^{\beta}) \phi\left(\frac{n}{X}\right) B^{j,\nu}_{\eta,X,c}(n,n+h).$$

Due to the above lemma, we may assume $|h| \leq 7cK^{1+2\varepsilon}/L$. We now focus on bounding $P(c,h,X)$.

4.2 Computation of $P(c,h,X)$

Following [31] we write

$$P(c,h,X) = \sum_{n \geq \max(1,1-h)} \lambda_g(n)\overline{\lambda_g}(n+h) \left(\frac{\sqrt{n(n+h)}}{2n+h}\right)^{l-1} G(2n+h),$$

where

$$G(2n+h) = \left(\frac{2n+h}{\sqrt{n(n+h)}}\right)^{l-1} \frac{\phi(\frac{n}{X}) e(\alpha n^{\beta})}{(n+h)^{j/2+1/4}} B^{j,\nu}_{\eta,X,c}(n,n+h).$$

That is

$$\begin{aligned} G(x) \quad &= \quad \left(\frac{2x}{\sqrt{x^2-h^2}}\right)^{l-1} \left(\frac{2}{x+h}\right)^{\frac{j}{2}+\frac{1}{4}} \phi\left(\frac{x-h}{2X}\right) \\ &\quad \times e\left(\alpha\left(\frac{x-h}{2}\right)^{\beta}\right) B^{j,\nu}_{\eta,X,c}\left(\frac{x-h}{2}, \frac{x+h}{2}\right). \end{aligned}$$

The Mellin transform of $G(x)$ is

$$\begin{aligned} \widetilde{G}(s) = \int_{\max(1,1-h)}^{\infty} G(x) x^{s-1} dx &= (2X)^s \int_0^{\infty} G(2Xz+h)\left(z+\frac{h}{2X}\right)^{s-1} dz \\ &= (2X)^s X^{-(j/2+1/4)} \int_0^{\infty} G_0(z)\left(z+\frac{h}{2X}\right)^{s-1} dz, \end{aligned} \tag{4.18}$$

where $z = (x-h)/(2X)$ and

$$G_0(z) = \left(\frac{2z + \frac{h}{X}}{\sqrt{z(z+\frac{h}{X})}} \right)^{l-1} \left(z + \frac{h}{X} \right)^{-\frac{j}{2}-\frac{1}{4}} e\left(\alpha (zX)^{\beta}\right) \phi(z) B^{(j,\nu)}_{\eta,X,c}\left(zX, zX+h\right). \tag{4.19}$$

Consider the integral

$$\frac{1}{2\pi i} \int_{(\sigma)} D_g(s,1,1,h) \widetilde{G}(s) ds, \tag{4.20}$$

for $\sigma > 1/2$, where $D_g(s,1,1,h)$ is a convolution sum defined by

$$D_g(s,\nu_1,\nu_2,h) = \sum_{\substack{m,n>0 \\ \nu_1 m - \nu_2 n = h}} \lambda_g(n) \overline{\lambda_g}(m) \left(\frac{\sqrt{\nu_1 \nu_2 mn}}{\nu_1 m + \nu_2 n} \right)^{l-1} (\nu_1 m + \nu_2 n)^{-s},$$

for $\nu_1, \nu_2 > 0$. (This series is absolutely convergent by Theorm 4.2 below) With this definition (4.20) becomes

$$\frac{1}{2\pi i} \int_{(\sigma)} \sum_{n \geq \max(1,1-h)} \lambda_g(n) \overline{\lambda_g}(n+h) \left(\frac{\sqrt{n(n+h)}}{2n+h} \right)^{l-1} (2n+h)^{-s} \widetilde{G}(s) ds$$

$$= \sum_{n \geq \max(1,1-h)} \lambda_g(n) \overline{\lambda_g}(n+h) \left(\frac{\sqrt{n(n+h)}}{2n+h} \right)^{l-1} \left[\frac{1}{2\pi i} \int_{(\sigma)} (2n+h)^{-s} \widetilde{G}(s) ds \right].$$

The term inside the brackets is the inverse Mellin transform of $\widetilde{G}(s)$ at $2n+h$ and is therefore equal to $G(2n+h)$, so (4.20) is in fact equal to $P(c,h,X)$.

4.2.1 Shifted Convolution Sums

We take a moment to provide a heavily abridged survey of these shifted convolution sums and the relevant results that we need. The interested reader should consult [27] for an in-depth discussion of the history as well as the various strategies undertaken to deal with them.

The investigation of these sums began with [44], and they have since found applications in the study of L-functions. In [42], Sarnak's subconvexity estimate depends crucially on the ability to obtain an upper bound, and the appendix of his paper is dedicated to that goal. He was able to prove the following.

Theorem 4.2. *Let g be a holomorphic cusp form for $\Gamma_0(N)$ of even weight l. Then $D_g(s,\nu_1,\nu_2,h)$ extends to a holomorphic function for $\sigma \geq 1/2+\theta+\varepsilon$, for any $\varepsilon > 0$. Moreover, in this region it satisfies*

$$D_g(s,\nu_1,\nu_2,h) \ll_{N,g,\varepsilon} (\nu_1\nu_2)^{\frac{1}{2}+\varepsilon}|h|^{\frac{1}{2}+\theta+\varepsilon-\sigma}(1+|t|)^3. \tag{4.21}$$

(We have omitted the part of Sarnak's theorem dealing with the case where g is a Maass form.)

In [28], Lau-Liu-Ye improved upon this result in two important regards. Firstly, they allowed for the existence of poles, and were thus able to meromorphically continue the sum to a larger region. Secondly, they significantly refined Sarnak's bound in the t-aspect.

Theorem 4.3. *With the notation as above, the function $D_g(s,\nu_1,\nu_2,h)$ admits an analytic continuation to a meromorphic function on $\sigma > 1/2$, with at most a finite number of poles $s_j \in (1/2, 1/2+\theta]$ due to possible exceptional eigenvalues $\lambda_j = s_j(1-s_j)$ of the Laplacian Δ. Moreover, for $\sigma \geq 1/2+\varepsilon$ and $|t| \geq 1$, we have*

$$D_g(s,\nu_1,\nu_2,h) \ll_{N,\nu_1,\nu_2,g,\varepsilon} |h|^{\frac{1}{2}+\theta+\varepsilon-\sigma}|t|^{1+\varepsilon}. \tag{4.22}$$

We lose the explicit dependence on ν_1 and ν_2, but this will be irrelevant in our application as we take $\nu_1 = 1 = \nu_2$. We also point out that this theorem only holds for $|t| \geq 1$. This will be a slight nuisance in our computations, but nothing more, as we have recourse to Theorem 4.2 for the situation of $|t| \leq 1$.

4.2.2 The Function $\widetilde{G}(s)$

We now deal with $\widetilde{G}(s)$. Plugging (4.19) into (4.18) gives us

$$\begin{aligned}\widetilde{G}(s) &= (2X)^s X^{-(j/2+1/4)} \int_1^2 \left(\frac{2z+h/X}{\sqrt{z(z+h/X)}}\right)^{l-1} \left(z+\frac{h}{X}\right)^{-j/2-1/4} \\ &\quad \times e\left(\alpha(zX)^{\beta}\right)\phi(z) B^{(j,\nu)}_{\eta,X,c}\left(zX, zX+h\right)\right) \left(z+\frac{h}{2X}\right)^{s-1} dz.\end{aligned}$$

Inserting (4.9) we have

$$\begin{aligned}\widetilde{G}(s) &= (2X)^s X^{-\frac{j}{2}-\frac{1}{4}} \int_1^2 \left(\frac{2z+h/X}{\sqrt{z(z+h/X)}}\right)^{l-1} \left(z+\frac{h}{X}\right)^{-\frac{j}{2}-\frac{1}{4}} e\left(\alpha(zX)^{\beta}\right)\phi(z) \\ &\quad \times \int_1^2 e(\Phi_0(w,z)) \frac{H_{2\nu}\left(\frac{-L}{\pi}\sinh^{-1}\left(\frac{\eta_1 Kc}{2\pi w X\sqrt{z}}\right)\right)}{\left(\left(\frac{cK}{2\pi wX}\right)^2+z\right)^{\nu/2+1/4}} \frac{\overline{\phi}(w^2)}{w^{j+\nu}} dw \left(z+\frac{h}{2X}\right)^{s-1} dz \\ &= (2X)^s X^{-\frac{j}{2}-\frac{1}{4}} \int_1^2\int_1^2 \left(\frac{2z+h/X}{\sqrt{z(z+h/X)}}\right)^{l-1} \left(z+\frac{h}{X}\right)^{-\frac{j}{2}-\frac{1}{4}} \left(z+\frac{h}{2X}\right)^{s-1} \\ &\quad \times e(\Phi_0(w,z)+\alpha(zX)^{\beta}) \left(\frac{H_{2\nu}\left(\frac{-L}{\pi}\sinh^{-1}\left(\frac{\eta_1 Kc}{2\pi w X\sqrt{z}}\right)\right)}{\left(\left(\frac{cK}{2\pi wX}\right)^2+z\right)^{\nu/2+1/4}}\right) \frac{\overline{\phi}(w^2)\phi(z)}{w^{j+\nu}} dw dz,\end{aligned}$$

where now

$$\begin{aligned}\Phi_0(w,z) &= -\alpha w^{2\beta} X^{\beta} + \eta_2 \frac{2wX\sqrt{z+h/X}}{c} + \frac{\eta_1}{\pi c}\sqrt{K^2c^2+4\pi^2w^2zX^2} \\ &\quad -\eta_1 \frac{K}{\pi}\sinh^{-1}\left(\frac{Kc}{2\pi wX\sqrt{z}}\right).\end{aligned}$$

Change w to $w/\sqrt{z}$ and write $(z+h/(2X))^{it}$ as $(2X)^{-it}e\left(\frac{t}{2\pi}\log(2Xz+h)\right)$ to make this

$$\begin{aligned}\widetilde{G}(s) &= (2X)^{\sigma}X^{-(\frac{j}{2}+\frac{1}{4})}\int_1^2\int_1^2\left(\frac{2z+h/X}{\sqrt{z(z+h/X)}}\right)^{l-1}\left(z+\frac{h}{X}\right)^{-\frac{j}{2}-\frac{1}{4}}\left(z+\frac{h}{2X}\right)^{\sigma-1} \\ &\quad\times e(\psi(w,z,t))\left(\frac{H_{2\nu}\left(\frac{-L}{\pi}\sinh^{-1}\left(\frac{\eta_1 Kc}{2\pi wX}\right)\right)}{\left(\left(\frac{cK}{2\pi wX}\right)^2+1\right)^{\nu/2+1/4}}\right)\frac{\overline{\phi}\left(\frac{w^2}{z}\right)\phi(z)}{w^{j+\nu}}z^{\frac{j}{2}-\frac{3}{4}}dwdz \\ &= 2X^{\sigma-(\frac{j}{2}+\frac{1}{4})}\int_1^2\int_1^2\frac{(2z+h/X)^{l+\sigma-2}}{(z+h/X)^{1/2(l+j-1/2)}}e(\psi(w,z,t)) \\ &\quad\times\left(\frac{H_{2\nu}\left(\frac{-L}{\pi}\sinh^{-1}\left(\frac{\eta_1 Kc}{2\pi wX}\right)\right)}{\left(\left(\frac{cK}{2\pi wX}\right)^2+1\right)^{\nu/2+1/4}}\right)\frac{\overline{\phi}\left(\frac{w^2}{z}\right)\phi(z)}{w^{j+\nu}}z^{\frac{j-l}{2}-\frac{1}{4}}dwdz \end{aligned}\tag{4.23}$$

where $s=\sigma+it$, $1\le z,w\le 2$ and

$$\begin{aligned}\psi(w,z,t) &= -\alpha\left(\frac{w^2}{z}\right)^{\beta}X^{\beta}+\eta_2\frac{2wX\sqrt{1+h/(zX)}}{c}+\frac{\eta_1}{\pi c}\sqrt{K^2c^2+4\pi^2w^2X^2} \\ &\quad -\frac{K}{\pi}\sinh^{-1}\left(\frac{\eta_1 Kc}{2\pi wX}\right)+\alpha(zX)^{\beta}+\frac{t}{2\pi}\log(2Xz+h) \\ &= \alpha X^{\beta}\left(z^{\beta}-\left(\frac{w^2}{z}\right)^{\beta}\right)+\eta_2\frac{2wX}{c}\sqrt{1+\frac{h}{Xz}}+\frac{t}{2\pi}\log(2Xz+h) \\ &\quad +\frac{\eta_1}{\pi c}\sqrt{K^2c^2+4\pi^2w^2X^2}-\eta_1\frac{K}{\pi}\sinh^{-1}\left(\frac{Kc}{2\pi wX}\right).\end{aligned}$$

To determine when this integral is negligible we'll need lower bounds on the partial derivatives of ψ:

$$\frac{\partial\psi}{\partial z}=\alpha\beta X^{\beta}\left(z^{\beta-1}+\frac{w^{2\beta}}{z^{\beta+1}}\right)-\eta_2\frac{h}{c}\frac{w}{z^2\sqrt{1+\frac{h}{Xz}}}+\frac{t}{2\pi}\frac{1}{z+\frac{h}{2X}},\tag{4.24}$$

$$\begin{aligned}\frac{\partial\psi}{\partial w} &= -2\alpha\beta X^{\beta}\frac{w^{2\beta-1}}{z^{\beta}}+\frac{2X}{c}\left(\eta_2\sqrt{1+\frac{h}{Xz}}+\eta_1\sqrt{1+\left(\frac{Kc}{2\pi wX}\right)^2}\right) \\ &= -2\alpha\beta X^{\beta}\frac{w^{2\beta-1}}{z^{\beta}}+\eta_2\frac{\frac{h}{cz}-\frac{K^2c}{4\pi^2w^2X}}{\sqrt{1+\frac{h}{Xz}}+\sqrt{1+\left(\frac{Kc}{2\pi wX}\right)^2}}.\end{aligned} \tag{4.25}$$

In this last equation we used Lemma 4.1(1) so that $\eta_2 = -\eta_1$.

Under various conditions on h, c and t, we can argue that $\widetilde{G}(s)$ is negligible, as in Lemma 5.1 of [28]. We first treat the case of $\alpha = 0$. This serves the dual purpose of improving upon the subconvexity bound in [28], as well as outlining the framework for the case of $\alpha \neq 0$.

Lemma 4.4. *Suppose $\alpha = 0$ and $s = \sigma + it$ with $1/2 \leq \sigma \leq 2$.*

1. *We have*

$$\widetilde{G}(s) \ll X^{\sigma-j/2-1/4}K^{\varepsilon}\left(\frac{K^2c}{X}\right)^{-1/2}. \tag{4.26}$$

2. *If $|h| \leq cK^{\varepsilon}$ and $c \geq X/K^{2-2\varepsilon}$ then*

$$\widetilde{G}(s) \ll_{M,\varepsilon} K^{-M} \tag{4.27}$$

   for any $M > 0$.

3. *If $|t| \leq 1$ and $|h| \geq cK^{\varepsilon}$, then (4.27) holds.*

4. *If $|t| \geq K^{\varepsilon}|h|/c$ and $c \leq |h|$, then*

$$\widetilde{G}(s) \ll_{M,\varepsilon} |t|^{-M}. \tag{4.28}$$

5. *If $|h| \leq cK^{\varepsilon}$ and $|t| \geq K^{2\varepsilon}$, then (4.28) holds.*

6. *If $|h| \geq cK^{\varepsilon}$ and $|t| \geq K^{2+2\varepsilon}c/X$, then (4.28) holds.*

Proof. For parts *(2) - (6)* we apply Theorem 2.7 above to (4.23). Note that the derivatives of $\frac{\overline{\phi}(w^2/z)\phi(z)}{w^{j+\nu}} z^{\frac{j-l}{2}-\frac{1}{4}}$ and $\left(\left(\frac{cK}{2\pi w X}\right)^2+1\right)^{-(\nu/2+1/4)}$ will be bounded by a constant, as well as $\frac{(2z+h/X)^{l+\sigma-2}}{(z+h/X)^{1/2(l+j-1/2)}}$, due to the restriction on $|h|$. In particular, $|h| < K^{3\varepsilon}X/L^2$. The Nth derivative of $H_{2\nu}\left(\frac{-L}{\pi}\sinh^{-1}\left(\frac{\eta_1 Kc}{2\pi w\sqrt{nX}}\right)\right)$ will be bounded by $(LKc/X)^N \leq K^{N\epsilon}$.

Now, for *(2)* we will use (4.25). Since $\psi_w^{(j)} \ll \frac{K^2c}{X}$ for all $j \geq 2$, we set

$$T = \frac{K^2c}{X}, \qquad M = 1, \qquad U = 1, \qquad \text{and } N = \frac{1}{K^{\varepsilon}}. \tag{4.29}$$

As in the proof of Lemma 3.3, it suffices to prove $\min \psi' \gg T$.

For *(3)-(6)* we use (4.24) and the fact that $\psi_z^{(j)} \ll \max\{|h|/c, |t|\} =: T$.

(1) The estimate (4.26) follows from the second derivative test (Lemma 2.5). We have

$$\frac{\partial^2\psi}{\partial w^2} = -\eta_1 \frac{\frac{K^2c}{2\pi^2 w^3 X}}{\sqrt{1+\left(\frac{Kc}{2\pi w X}\right)^2}} \asymp \frac{K^2c}{X}, \tag{4.30}$$

and by the comments above, the derivative of the weight function is bounded by K^{ε}. By Proposition 2.6, the total variation is also K^{ε}.

(2) With these conditions on h and c it follows that

$$\frac{K^2c}{4\pi^2w^2X} > \frac{K^2c}{8\pi^2w^2X} \geq \frac{K^{2\varepsilon}}{32\pi^2} > K^{\varepsilon} \geq \frac{|h|}{cz},$$

and so

$$\frac{\partial\psi}{\partial w} \asymp \left|\frac{h}{cz} - \frac{K^2c}{4\pi^2w^2X}\right| \gg \frac{K^2c}{X} = T.$$

(3) Since $|h|/c \geq K^{\varepsilon} \geq K^{\varepsilon}|t|$, it follows that

$$\frac{\partial\psi}{\partial z} \gg \frac{|h|}{c} - |t| \gg \frac{|h|}{c} \gg \frac{\partial^j\psi}{\partial z^j}.$$

Setting $T = |h|/c$ then yields the result.

(4) The argument is the same as in (3), but with the roles of $|t|$ and $|h|/c$ reversed.

(5) We have $|t| \geq K^{2\varepsilon} \geq K^{\varepsilon}|h|/c$, and so the argument holds as in the previous two

cases, with $T = |t|$.

(6) By part (4) of Lemma 4.1, we know $|h|/c \leq K^{2+\varepsilon}c/X$, whence $|t| - |h|/c \gg |t|$, and we set $T = |t|$.

□

4.3 The Main Estimation for $\alpha = 0$

Armed with Lemma 4.4 and Theorems 4.2 and 4.3, we estimate the integral (4.20) in several steps.

Step 1. $|h| \leq cK^{\varepsilon}$ *and* $|t| \geq K^{2\varepsilon}$. We take $\sigma = 1/2 + \theta + \varepsilon$, so that $D_g(\sigma + it, 1, 1, h) \ll |t|^3$ by the theorem in [42]. In conjunction with Lemma 4.4(5), this yields

$$\frac{1}{2\pi i} \int\limits_{|t| \geq K^{2\varepsilon}} D_g(\sigma + it, 1, 1, h)\widetilde{G}(\sigma + it)dt \ll_{M,\varepsilon} \int\limits_{|t| \geq K^{2\varepsilon}} |t|^3 |t|^{-M} dt \ll_{M,\varepsilon} K^{2\varepsilon(4-M)}, \tag{4.31}$$

which is negligible.

Step 2. $|h| \leq cK^{\varepsilon}$, $|t| < K^{2\varepsilon}$, *and* $c \leq X/K^{2-2\varepsilon}$. With $\sigma = 1/2 + \theta + \varepsilon$, we have $D_g(s, 1, 1, h) \ll_{\varepsilon} K^{6\varepsilon}$. We apply the estimate for $\widetilde{G}(s)$ in Lemma 4.4(1) to get

$$P(c, h, X) \ll K^{9\varepsilon - 1} X^{3/4 + \theta - j/2 + \varepsilon} c^{-1/2}. \tag{4.32}$$

Step 3. $|h| \leq cK^{\varepsilon}$, $|t| < K^{2\varepsilon}$, *and* $c > X/K^{2-2\varepsilon}$. In view of $D_g(s, 1, 1, h) \ll_{\varepsilon} K^{6\varepsilon}$ for $\sigma = 1/2 + \theta + \varepsilon$ and Lemma 4.4(2), we have

$$\frac{1}{2\pi i} \int\limits_{|t| < K^{2\varepsilon}} D_g(\sigma + it, 1, 1, h)\widetilde{G}(\sigma + it)dt \ll_{M,\varepsilon} K^{-M} \int\limits_{|t| < K^{2\varepsilon}} K^{6\varepsilon} dt \ll_{M,\varepsilon} K^{8\varepsilon - M}, \tag{4.33}$$

which is negligible.

Step 4. $cK^{\varepsilon} < |h| \leq K^{2+\varepsilon}c^2/X$. Specifying $\sigma = 1/2 + \theta + \varepsilon$ in (4.20), we can write (4.20) as

$$P(c,h,X) = \frac{1}{2\pi}\left\{ \int\limits_{|t|\le 1} + \int\limits_{1<|t|<K^{1+3\varepsilon}/L} + \int\limits_{K^{1+3\varepsilon}/L\le|t|} \right\} D_g(\sigma+it,1,1,h)\widetilde{G}(\sigma+it)dt.$$

By Lemma 4.4(3) the first integral is

$$\int\limits_{|t|\le 1} K^{6\varepsilon}K^{-M}dt \ll_{M,\varepsilon} K^{6\varepsilon-M}. \tag{4.34}$$

The last integral can be estimated by Lemma 4.4(7) as

$$\ll \int\limits_{K^{1+3\varepsilon}/L\le|t|} |t|^3|t|^{-M}dt \ll_{M,\varepsilon} \left(\frac{K^{1+\varepsilon}}{L}\right)^{4-M}, \tag{4.35}$$

which is negligible because $L \le K^{1-2\varepsilon}$. Therefore, for h in this range, (4.20) becomes

$$P(c,h,X) = \frac{1}{2\pi}\int\limits_{1<|t|<K^{1+3\varepsilon}/L} D_g(\frac{1}{2}+\theta+\varepsilon+it,1,1,h)\widetilde{G}(\frac{1}{2}+\theta+\varepsilon+it)dt \tag{4.36}$$
$$+ \text{ negligible terms.}$$

Now, we write Theorem 2 of [28] in the form

$$D_g(s,1,1,h) \ll |h|^{1/2+\theta-\sigma+\varepsilon}|t|^{1+\varepsilon}, \tag{4.37}$$

which holds for $\sigma > 1/2$ and $|t| \ge 1$. Since $D_g(s,1,1,h)$ has no poles in this region, we move the line segments to $\sigma = 1/2+\varepsilon$, with the horizontal parts controlled by Lemma 4.4(3) and (6). Applying the bound Lemma 4.4(1), we obtain

$$\begin{aligned} P(c,h,X) &\ll K^{-1}c^{-1/2}X^{3/4-j/2+\varepsilon}|h|^{\theta+\varepsilon}\int\limits_{1<|t|<K^{1+3\varepsilon}/L}|t|^{1+\varepsilon}dt \\ &\ll K^{-1}c^{-1/2}X^{3/4-j/2+\varepsilon}|h|^{\theta+\varepsilon}\left(\frac{K^{1+3\varepsilon}}{L}\right)^{2+\varepsilon}. \end{aligned} \tag{4.38}$$

Now we turn back to (4.17). Due to Lemma 4.1(3) we impose a smaller range for h:

$$T^{\eta}_{\nu,j} \ll L^{2\nu+1}KX^{1/4-j/2-\nu} \sum_{\delta\leq\frac{X}{LK^{1-\epsilon}}} \delta \sum_{\substack{c\leq\frac{X}{LK^{1-\epsilon}}\\ \delta|c}} c^{j+\nu-1} \sum_{\substack{|h|\leq K^{2+\varepsilon}c^2/X\\ \delta|h}} |P(c,h,X)|. \quad (4.39)$$

Using (4.31) through (4.38), we can compute the inner most sums over c and h as

$$\begin{aligned}
&\sum_{\substack{c\leq\frac{X}{LK^{1-\epsilon}}\\ \delta|c}} c^{j+\nu-1} \sum_{\substack{|h|\leq K^{2+\varepsilon}c^2/X\\ \delta|h}} |P(c,h,X)| \\
&\qquad \ll \sum_{\substack{c\leq\frac{X}{K^{2-2\epsilon}}\\ \delta|c}} c^{j+\nu-1} \sum_{\substack{|h|\leq cK^{\varepsilon}\\ \delta|h}} K^{9\varepsilon-1}X^{\frac{3}{4}+\theta-\frac{j}{2}+\varepsilon}c^{-\frac{1}{2}} \\
&\qquad\quad + \sum_{\substack{c\leq\frac{X}{LK^{1-\epsilon}}\\ \delta|c}} c^{j+\nu-1} \sum_{\substack{cK^{\varepsilon}\leq|h|\leq K^{2+\varepsilon}c^2/X\\ \delta|h}} K^{-1}c^{-\frac{1}{2}}X^{\frac{3}{4}-\frac{j}{2}+\varepsilon}|h|^{\theta+\varepsilon}\left(\frac{K^{1+2\varepsilon}}{L}\right)^{2+\varepsilon} \\
&\qquad \ll \frac{1}{\delta}K^{10\varepsilon-1}X^{\frac{3}{4}+\theta-\frac{j}{2}+\varepsilon} \sum_{\substack{c\leq\frac{X}{K^{2-2\epsilon}}\\ \delta|c}} c^{j+\nu-\frac{1}{2}} \\
&\qquad\quad + \frac{1}{\delta}\frac{K^{1+5\varepsilon}}{L^{2+\varepsilon}}X^{\frac{3}{4}-\frac{j}{2}+\varepsilon} \sum_{\substack{c\leq\frac{X}{LK^{1-\epsilon}}\\ \delta|c}} c^{j+\nu-\frac{3}{2}}\left(\frac{K^{2+\varepsilon}c^2}{X}\right)^{1+\theta+\varepsilon} \\
&\qquad \ll \frac{1}{\delta^2}\frac{X^{5/4+\theta+j/2+\nu+\varepsilon}}{K^{2+2(j+\nu)-\varepsilon(2j+2\nu+11)}} \\
&\qquad\quad + \frac{1}{\delta^2}X^{5/4+j/2+\nu+\theta+3\varepsilon}\frac{K^{3/2-j-\nu+\varepsilon(10+j+\nu)}}{L^{7/2+j+\nu+2\theta+\varepsilon}}. \qquad (4.40)
\end{aligned}$$

Inserting (4.40) into (4.39), we get

$$\begin{aligned}
T^{\eta}_{\nu,j} \ll\ & L^{2\nu+1}KX^{1/4-j/2-\nu}\sum_{\delta\leq\frac{X}{LK^{1-\epsilon}}}\delta \qquad (4.41)\\
&\times\left(\frac{1}{\delta^2}\frac{X^{5/4+\theta+j/2+\nu+\varepsilon}}{K^{2+2(j+\nu)-\varepsilon(2j+2\nu+11)}} + \frac{1}{\delta^2}X^{5/4+j/2+\nu+\theta+3\varepsilon}\frac{K^{3/2-j-\nu+\varepsilon(10+j+\nu)}}{L^{7/2+j+\nu+2\theta+\varepsilon}}\right).
\end{aligned}$$

The sum over δ will contribute $O(X^\varepsilon)$, and so the sum simplifies to

$$T^{\eta}_{\nu,j} \ll \left(\frac{L}{K^{1-\varepsilon}}\right)^{2\nu} \frac{X^{1/2+\theta}}{K^{2+2j-\varepsilon(2j+11)}} LKX^{1+\varepsilon} + \left(\frac{L}{K^{1-\varepsilon}}\right)^{\nu} X^{\frac{1}{2}+\theta} \frac{K^{3/2-j+\varepsilon(10+j)}}{L^{7/2+j+2\theta+\varepsilon}} LKX^{1+3\varepsilon}.$$

It is clear that the largest this can be is when $j = 0 = \nu$. The first term will be $O(LKX^{1+3\varepsilon})$ provided $X^{1/2+\theta} \leq K^2$, i.e., $K \geq X^{1/4+\theta/2}$. Likewise, the second term will also be $O(LKX^{1+\varepsilon})$ so long as $X^{1/2+\theta}K^{3/2-\mu(7/2+2\theta)} \leq 1$, where μ is defined by $L = K^\mu$. This is true when $\mu(7+4\theta) - 3 > 0$ and

$$K \geq X^{\frac{1/2+\theta}{\mu(7/2+2\theta)-3/2}} = X^{\frac{1+2\theta}{\mu(7+4\theta)-3}}. \tag{4.42}$$

We note that

$$\frac{1+2\theta}{\mu(7+4\theta)-3} > \frac{1+2\theta}{4} \quad \text{if} \quad \frac{3}{7+4\theta} < \mu < \frac{7}{7+4\theta},$$

and

$$\frac{1+2\theta}{\mu(7+4\theta)-3} \leq \frac{1+2\theta}{4} \quad \text{if} \quad \frac{7}{7+4\theta} \leq \mu < 1.$$

This proves Theorem 1.2.

To prove Theorem 1.3 fix K in (1.13) to find the smallest possible μ. That is,

$$\frac{1+2\theta}{\mu(7+4\theta)-3} \leq \frac{\log K}{\log X}.$$

This implies

$$\mu \geq \frac{3}{7+4\theta} + \frac{1+2\theta}{7+4\theta}\frac{\log X}{\log K}, \tag{4.43}$$

and hence $\mu > 3/(7+4\theta)$ holds. To have $\mu < 7/(7+4\theta)$, we must have

$$\frac{1+2\theta}{7+4\theta}\frac{\log X}{\log K} < \frac{4}{7+4\theta}, \quad \text{i.e.,} \quad K > X^{\frac{1+2\theta}{4}}. \tag{4.44}$$

Taking the smallest μ available in (4.43) we get

$$K^{1+\mu} = K^{\frac{10+4\theta}{7+4\theta}+\frac{1+2\theta}{7+4\theta}\frac{\log X}{\log K}} = K^{\frac{10+4\theta}{7+4\theta}} X^{\frac{1+2\theta}{7+4\theta}}.$$

Consequently the right side of (1.12) becomes

$$K^{\frac{10+4\theta}{7+4\theta}} X^{\frac{8+6\theta}{7+4\theta}+\varepsilon}$$

Taking only one term on the left-hand side of (1.12) we conclude from (4.44) that

$$S_X(f \times g; 0, 0) \ll k^{\frac{5+2\theta}{7+4\theta}} X^{\frac{4+3\theta}{7+4\theta}+\varepsilon} \tag{4.45}$$

for $k > X^{(1+2\theta)/4}$. The bound in (4.45) will be less than X (and therefore non-trivial) if $K < X^{(3+\theta)/(5+2\theta)}$. This proves Theorem 1.3.

4.4 The Main Estimation for $\alpha \neq 0$

First, we obtain the analogue to Lemma 4.4. We will see that $|\alpha|X^\beta$ will be our dominating term, so we will have fewer cases to consider.

Lemma 4.5. *Suppose $\alpha \neq 0$ and $s = \sigma + it$ with $1/2 \leq \sigma \leq 2$.*

1. *We have*

$$\widetilde{G}(s) \ll X^{\sigma - j/2 - 1/4} K^{\varepsilon} (|\alpha| X^\beta)^{-1/2}. \tag{4.46}$$

2. *If $|h| \leq K^2 c^2 / X$ then*

$$\widetilde{G}(s) \ll_{M,\varepsilon} K^{-M} \tag{4.47}$$

for any $M > 0$.

3. If $|t| \gg K^{1+3\varepsilon}/L$*, then*

$$\widetilde{G}(s) \ll_{M,\varepsilon} |t|^{-M}. \tag{4.48}$$

Proof. The proof is very similar to the previous lemma.

(1) We use Lemma 2.5 on the w-integral. The second derivative of ψ is

$$-2\beta(2\beta-1)\alpha X^{\beta}\frac{w^{2\beta-2}}{z^{\beta}} - \eta_1 \frac{\frac{K^2c}{2\pi^2 w^3 X}}{\sqrt{1+\left(\frac{Kc}{2\pi w X}\right)^2}} \asymp |\alpha| X^{\beta}. \tag{4.49}$$

As in Lemma 4.4(1), the derivative of the weight function, and thus the total variation, is K^{ε}.

(2) By our assumption on h,

$$\left|\frac{h}{cz} - \frac{K^2c}{4\pi^2 w^2 X}\right| \le 2\frac{K^2c}{X}. \tag{4.50}$$

As $|\alpha|X^{\beta} \ge K^{2+\varepsilon}c/X$, we have $\psi'_w \gg |\alpha|X^{\beta} \gg \psi_w^{(j)}$ for $j \ge 2$.

(3) We now consider ψ'_z. By Lemma 4.1(4) we know $|h| \le 7cK^{1+2\varepsilon}/L$. So as long as $|t|$ is larger than $K^{1+3\varepsilon}/L$ the t term will dominate the other two. Set $T = |t|$ in Lemma 2.7. The higher derivatives are readily seen to be $\asymp |t|$ so that

$$\frac{\partial \psi}{\partial z} \gg |t| =: T \gg \frac{\partial^{(r)}\psi}{\partial z^{(r)}}.$$

□

We now estimate the integral (4.20) in several steps.

Step 1. $|h| \le K^2c^2/X$ *and* $|t| \ge K^{1+3\varepsilon}/L$. We take $\sigma = 1/2 + \theta + \varepsilon$, so that $D_g(\sigma + it, 1, 1, h) \ll |t|^3$ by the theorem in [42]. In conjunction with Lemma 4.5(3), this yields

$$P(c,h,X) = \frac{1}{2\pi i} \int\limits_{|t|\geq K^{1+3\varepsilon}/L} D_g(\sigma+it,1,1,h)\widetilde{G}(\sigma+it)dt$$
$$\ll_{M,\varepsilon} \int\limits_{|t|\geq K^{1+3\varepsilon}/L} |t|^3|t|^{-M}dt \ll_{M,\varepsilon} \left(\frac{K^{1+3\varepsilon}}{L}\right)^{4-M}, \tag{4.51}$$

which is negligible.

Step 2. $|h| \leq K^2c^2/X$, $|t| < K^{1+3\varepsilon}/L$. With $\sigma = 1/2+\theta+\varepsilon$, we have $D_g(s,1,1,h) \ll_\varepsilon |t|^3$. We apply the estimate for $\widetilde{G}(s)$ in Lemma 4.5(2) to get

$$P(c,h,X) \ll \int_{|t|\leq K^{1+3\varepsilon}/L} |t|^3K^{-M}dt$$
$$\ll K^{-M}\left(\frac{K^{1+3\varepsilon}}{L}\right)^4, \tag{4.52}$$

which is again negligible.

Step 3. $K^2c^2/X \leq |h| \leq 7cK^{1+2\varepsilon}/L$. Specifying $\sigma = 1/2+\theta+\varepsilon$ in (4.20), we can write (4.20) as

$$P(c,h,X) = \frac{1}{2\pi}\left\{\int\limits_{|t|\leq 1} + \int\limits_{1<|t|<K^{1+3\varepsilon}/L} + \int\limits_{K^{1+3\varepsilon}/L\leq|t|}\right\}D_g(\sigma+it,1,1,h)\widetilde{G}(\sigma+it)dt.$$

By Lemma 4.5(1) the first integral is

$$\int\limits_{|t|\leq 1} X^{\frac{1}{4}+\theta-\frac{j}{2}+\varepsilon}K^\varepsilon(|\alpha|X^\beta)^{-1/2}|t|^3dt \ll_{M,\varepsilon} X^{\frac{1}{4}+\theta-\frac{j}{2}+\varepsilon}K^\varepsilon(|\alpha|X^\beta)^{-1/2}. \tag{4.53}$$

The last integral can be estimated by Lemma 4.5(3) as

$$\ll \int\limits_{K^{1+3\varepsilon}/L\leq|t|} |t|^3|t|^{-M}dt \ll_{M,\varepsilon} \left(\frac{K^{1+3\varepsilon}}{L}\right)^{4-M}, \tag{4.54}$$

which is negligible. Therefore, for h in this range, (4.20) becomes

$$P(c,h,X) = \frac{1}{2\pi} \int\limits_{1<|t|<K^{1+3\varepsilon}/L} D_g(\frac{1}{2}+\theta+\varepsilon+it,1,1,h)\widetilde{G}(\frac{1}{2}+\theta+\varepsilon+it)dt.$$

Now, we write Theorem 4.3 in the form

$$D_g(s,1,1,h) \ll |h|^{1/2+\theta-\sigma+\varepsilon}|t|^{1+\varepsilon},$$

which holds for $\sigma > 1/2$ and $|t| \geq 1$. Since $D_g(s,1,1,h)$ has no poles in this region, we move the line segments to $\sigma = 1/2+\varepsilon$, with the top horizontal line segment controlled by Lemma 4.5(3). For the bottom line segment, $|t| = 1$, and so it is bounded by

$$X^{\frac{1}{4}+\theta-\frac{j}{2}+\varepsilon}K^{\varepsilon}(|\alpha|X^{\beta})^{-1/2}. \tag{4.55}$$

Applying the bound in Lemma 4.5(1) to the line integral over $\sigma = 1/2+\varepsilon$, its contribution to $P(c,h,X)$ is

$$\begin{aligned} &\ll X^{\frac{1}{4}-\frac{j}{2}+\varepsilon}K^{\varepsilon}(|\alpha|X^{\beta})^{-\frac{1}{2}}|h|^{\theta+\varepsilon} \int\limits_{1<|t|<K^{1+3\varepsilon}/L} |t|^{1+\varepsilon}dt \\ &\ll X^{\frac{1}{4}-\frac{j}{2}+\varepsilon}K^{\varepsilon}(|\alpha|X^{\beta})^{-\frac{1}{2}}|h|^{\theta+\varepsilon}\left(\frac{K^{1+3\varepsilon}}{L}\right)^{2+\varepsilon} \end{aligned} \tag{4.56}$$

Thus by (4.53), (4.55) and (4.56), we conclude that

$$\begin{aligned} P(c,h,X) \ll\ & X^{\frac{1}{4}+\theta-\frac{j}{2}+\varepsilon}K^{\varepsilon}(|\alpha|X^{\beta})^{-\frac{1}{2}} \\ & +X^{\frac{1}{4}-\frac{j}{2}+\varepsilon}K^{\varepsilon}(|\alpha|X^{\beta})^{-\frac{1}{2}}|h|^{\theta+\varepsilon}\left(\frac{K^{1+3\varepsilon}}{L}\right)^{2+\varepsilon} \end{aligned} \tag{4.57}$$

for $K^2c^2/X < |h| \leq 7cK^{1+2\varepsilon}/L$, and negligible for other h.

Now we turn back to (4.17). Due to Lemma 4.1(4) we impose a smaller range for h:

$$T^{\eta}_{\nu,j} \ll L^{2\nu+1} K X^{\frac{1}{4}-\frac{j}{2}-\nu} \sum_{\delta \le \frac{X}{LK^{1-\epsilon}}} \delta \sum_{\substack{c \le \frac{X}{LK^{1-\epsilon}} \\ \delta | c}} c^{j+\nu-1} \sum_{\substack{|h| \le 7c\frac{K^{1+2\varepsilon}}{L} \\ \delta | h}} |P(c,h,X)|. \tag{4.58}$$

Using (4.51) through (4.57), we can compute the inner most sums over c and h as

$$\begin{aligned}
&\sum_{\substack{c \le \frac{X}{LK^{1-\epsilon}} \\ \delta | c}} c^{j+\nu-1} \sum_{\substack{|h| \le 7c\frac{K^{1+2\varepsilon}}{L} \\ \delta | h}} |P(c,h,X)| \\
&\quad \ll \sum_{\substack{c \le \frac{X}{LK^{1-\epsilon}} \\ \delta | c}} c^{j+\nu-1} \sum_{\substack{|h| \le \frac{K^2c^2}{X} \\ \delta | h}} K^{-M} \\
&\quad + \sum_{\substack{c \le \frac{X}{LK^{1-\epsilon}} \\ \delta | c}} c^{j+\nu-1} \sum_{\substack{\frac{K^2c^2}{X} \le |h| \le 7c\frac{K^{1+2\varepsilon}}{L} \\ \delta | h}} \Bigg(X^{\frac{1}{4}+\theta-\frac{j}{2}+\varepsilon} K^{\varepsilon} (|\alpha| X^{\beta})^{-\frac{1}{2}} \\
&\qquad\qquad + X^{\frac{1}{4}-\frac{j}{2}+\varepsilon} K^{\varepsilon} (|\alpha| X^{\beta})^{-\frac{1}{2}} |h|^{\theta+\varepsilon} \left(\frac{K^{1+3\varepsilon}}{L}\right)^{2+\varepsilon} \Bigg) \\
&\quad \ll \frac{1}{\delta} \frac{K^{2-M}}{X} \sum_{\substack{c \le \frac{X}{LK^{1-\epsilon}} \\ \delta | c}} c^{j+\nu+1} \\
&\quad + \frac{1}{\delta} \sum_{\substack{c \le \frac{X}{LK^{1-\epsilon}} \\ \delta | c}} c^{j+\nu-1} \left(\frac{cK^{1+2\varepsilon}}{L}\right) X^{\frac{1}{4}+\theta-\frac{j}{2}+\varepsilon} K^{\varepsilon} (|\alpha| X^{\beta})^{-\frac{1}{2}} \\
&\quad + \frac{1}{\delta} \sum_{\substack{c \le \frac{X}{LK^{1-\epsilon}} \\ \delta | c}} c^{j+\nu-1} \left(\frac{cK^{1+2\varepsilon}}{L}\right)^{1+\theta+\varepsilon} X^{\frac{1}{4}-\frac{j}{2}+\varepsilon} K^{\varepsilon} (|\alpha| X^{\beta})^{-\frac{1}{2}} \left(\frac{K^{1+3\varepsilon}}{L}\right)^{2+\varepsilon} \\
&\quad \ll \frac{1}{\delta^2} \frac{K^{2-M}}{X} \left(\frac{X}{LK^{1-\varepsilon}}\right)^{j+\nu+2} \\
&\quad + \frac{1}{\delta^2} \left(\frac{X}{LK^{1-\varepsilon}}\right)^{j+\nu+1} \left(\frac{K^{1+2\varepsilon}}{L}\right) X^{\frac{1}{4}+\theta-\frac{j}{2}+\varepsilon} K^{\varepsilon} (|\alpha| X^{\beta})^{-\frac{1}{2}} \\
&\quad + \frac{1}{\delta^2} \left(\frac{X}{LK^{1-\varepsilon}}\right)^{j+\nu+1+\theta+\varepsilon} \left(\frac{K^{1+2\varepsilon}}{L}\right)^{1+\theta+\varepsilon} X^{\frac{1}{4}-\frac{j}{2}+\varepsilon} K^{\varepsilon} (|\alpha| X^{\beta})^{-\frac{1}{2}} \left(\frac{K^{1+3\varepsilon}}{L}\right)^{2+\varepsilon}
\end{aligned}$$

The first term will be negligible. For the remaining two we make the substitution

(4.10) so this becomes

$$\ll \frac{1}{\delta^2}\frac{X^{5/4+\theta+j/2+\nu+\varepsilon}}{K^{1/2+j+\nu-\varepsilon}L^{3/2+j+\nu}} + \frac{1}{\delta^2}\frac{X^{5/4+\theta+j/2+\nu+\varepsilon}K^{3/2-j-\nu-\varepsilon}}{L^{7/2+2\theta+j+\nu}} \tag{4.59}$$

Inserting (4.59) into (4.58), we get

$$\begin{aligned} T^{\eta}_{\nu,j} &\ll L^{2\nu+1}KX^{\frac{1}{4}-\frac{j}{2}-\nu}\sum_{\delta\leq\frac{X}{LK^{1-\epsilon}}}\delta^{-1}\frac{X^{5/4+\theta+j/2+\nu+\varepsilon}}{(LK^{1-\varepsilon})^{j+\nu}}\left(\frac{1}{K^{1/2-\varepsilon}L^{3/2}}+\frac{K^{3/2-\varepsilon}}{L^{7/2+2\theta}}\right) \\ &\ll \left(\frac{L}{K^{1-\varepsilon}}\right)^{\nu}\left(\frac{1}{LK^{1-\varepsilon}}\right)^{j}LKX^{1+\varepsilon}\left[X^{\frac{1}{2}+\theta}\left(\frac{1}{K^{1/2-\varepsilon}L^{3/2}}+\frac{K^{3/2-\varepsilon}}{L^{7/2+2\theta}}\right)\right]. \end{aligned} \tag{4.60}$$

The largest such term is when $j = 0 = \nu$. We have established then that (4.60) is $O(LKX^{1+\varepsilon})$ so long as we assume

$$X^{\frac{1}{2}+\theta+\varepsilon} \ll K^{\frac{7}{2}\mu-\frac{3}{2}} \quad \text{and} \quad X^{\frac{1}{2}+\theta+\varepsilon} \ll K^{\frac{3}{2}\mu+\frac{1}{2}}. \tag{4.61}$$

(Recall that $L = K^{\mu}$) In particular, we need $\frac{7}{2}\mu - \frac{3}{2} > 0$, or $\mu > \frac{3}{7}$. Note that the first relation in (4.61) implies the second (as $\mu < 1$), and so we require

$$K \geq X^{\frac{1+2\theta}{7\mu-3}+\varepsilon}. \tag{4.62}$$

Since $\alpha \neq 0$ is fixed, (4.10) implies

$$K \asymp X^{\frac{\beta}{1-\mu}-\varepsilon}. \tag{4.63}$$

Comparing (4.62) and (4.63) we see that for $\frac{3}{7} < \mu < 1$ we have

$$\frac{\beta}{1-\mu} > \frac{1+2\theta}{7\mu-3} \iff \mu > \frac{1+2\theta+3\beta}{1+2\theta+7\beta}. \tag{4.64}$$

This is consistent with the requirement $\mu > \frac{3}{7}$ as $\beta > 0$. The conditions (4.63) and

(4.64) dictate that the smallest allowable K is

$$K = X^{\frac{1+2\theta+7\beta}{4}}. \tag{4.65}$$

This completes the proof of Theorems 1.5 and 1.6.

REFERENCES

[1] H. Bateman, *Tables of Integral Transforms*, McGraw-Hill, New York, 1954.

[2] V. Blomer, Rankin-Selberg L-functions on the critical line, *Manuscripta Math.*, **117** (2005), 111-133.

[3] D. Bump, *Automorphic Forms and Representations*, Cambridge University Press, Cambridge, 1998.

[4] K. Czarnecki, Resonance sums for Rankin–Selberg products of Maass cusp forms, *Journal of Number Theory*, **163** (2016), 359 - 374.

[5] W. Duke, J. Friedlander, H. Iwaniec, Bounds for automorphic L-functions, *Invent. Math.* **112** (1993), no. 1, 1-8.

[6] W. Duke, J. Friedlander, H. Iwaniec, Bounds for automorphic L-functions. II. *Invent. Math.* **115** (1994), no. 2, 219-239.

[7] W. Duke, J. Friedlander, H. Iwaniec, Bounds for automorphic L-functions. III. *Invent. Math.* **143** (2001), no. 2, 221-248.

[8] W. Duke, J. Friedlander, H. Iwaniec, Class group L-functions. *Duke Math. J.* **79** (1995), no. 1, 1-56.

[9] W. Duke, J. Friedlander, H. Iwaniec, The subconvexity problem for Artin L-functions. *Invent. Math.* **149** (2002), no. 3, 489-577.

[10] A. M. Ernvall-Hytönen. On certain exponential sums related to $GL(3)$ cusp forms, *C. R. Math. Acad. Sci. Paris*, **348** (1-2):5–8, 2010.

[11] A. M. Ernvall-Hytönen, J. Jääsaari, and E. V. Vesalainen. Resonances and Ω-results for exponential sums related to Maass forms for $SL(n, Z)$. *J. Number Theory*, **153**: 135–157, 2015.

[12] D. Goldfeld, *Automorphic Forms and L-functions for the Group* $Gl(n, r)$. S.I.: Cambridge Univ, 2015. Print.

[13] A. Good, The square mean of Dirichlet series associated with cusp forms, *Mathematika*, **29** (1982), 225 -320.

[14] J. Hafner, Some remarks on odd Maass wave forms (and a correction to: "Zeros of L-functions attached to Maass forms" [*Math. Z.* **190** (1985), no. 1, 113128; MR0793354 (86m:11034)] by Hafner, C. Epstein and P. Sarnak). *Math. Z.* **196** (1987), no. 1, 129 -132.

[15] M. N. Huxley, *Area, lattice points, and exponential sums.* London Mathematical Society Monographs. New Series, 13. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[16] H. Iwaniec, *Introduction to the spectral theory of automorphic forms*, Biblioteca de la Revista Matemàtica Iberoamericana, Madrid, 1995.

[17] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, **17**. American Mathematical Society, Providence, RI, 1997.

[18] H. Iwaniec, E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, 53.

[19] H. Iwaniec, W.Z. Luo, P. Sarnak, Low lying zeros of families of L-functions. *Inst. Hautes Études Sci. Publ. Math.* **No. 91** (2000), 55 -131 (2001).

[20] M. Jutila and Y. Motohashi, Uniform bounds for Rankin-Selberg L-functions, in: Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory, edited by Friedberg, Bump, Goldfeld, and Hoffstein, *Proc. Symp. Pure Math*, **75** (2006), 243-256.

[21] J. Kaczorowski and A. Perelli, On the structure of the Selberg class VI: non-linear twists, *Acta Arith.* **116** (2005), 315-341.

[22] H. Kim, Functoriality for the exterior square of GL_4 and symmetric fourth of GL_2, *J. Amer. Math. Soc.* **16** (2003), 139-183.

[23] H. Kim and P. Sarnak, Refined estimates towards the Ramanujan and Selberg conjectures, appendix to [22].

[24] E. Kowalski, P. Michel, J. VanderKam, Rankin-Selberg L-functions in the level aspect, *Duke Math. J.* **114** (2002), no. 1, 123-191.

[25] N.V. Kuznetsov, Petersson's conjecture for cusp forms of weight zero and Linnik's conjecture. Sums of Kloosterman sums, *Math. USSR Sbornik* **29** (1981), 299-342.

[26] Y. K. Lau, J. Liu, Y. Ye, A new bound $k^{2/3+\varepsilon}$ for Rankin-Selberg L-functions for Hecke congruence subgroups, *Int. Math. Res. Pap.* 2006 (2006) 1-78, Article ID 35090.

[27] Y. K. Lau, J. Liu, Y. Ye, Shifted convolution sums of Fourier coefficients of cusp forms. In *Number Theory (Series on Number Theory and its Applications* **2**), World Sci. Publ., Hackensack, NJ, 2007.

[28] Y.-K. Lau, J. Liu, and Y. Ye, Subconvexity bounds for Rankin-Selberg L-functions for congruence subgroups, *Journal of Number Theory* **121** (2006), 204 - 223.

[29] X. Li, Bounds for $GL(2) \times GL(3)$ L-functions and $GL(3)$ L-functions. *Annals of Mathematics*, **173** (2011), No 1. pp 301-336.

[30] J. Liu, Y. Ye, Petersson and Kuznetsov trace formulas. *Lie groups and automorphic forms*, 147 -168, AMS/IP Stud. Adv. Math., **37**, Amer. Math. Soc., Providence, RI, 2006.

[31] J. Liu, Y. Ye, Subconvexity for Rankin-Selberg L-functions of Maass forms, *Geom. Funct. Anal.* **12** (2002) 1296-1323.

[32] M. McKee, H. Sun, Y. Ye, Weighted stationary phase of higher orders, *preprint*, arXiv:1601.07887.

[33] T. Meurman, On the order of the Maass L-function on the critical line, *Number Theory, Vol. I, Budapest Colloq. Math. Soc.*, **51**, (1990).

[34] P. Michel, Analytic number theory and families of automorphic L-functions, in: *Automorphic Forms and Applications*, IAS/Park City Math. Ser., vol. 12, edited by P. Sarnak and F. Shahidi, Amer. Math. Soc. and Inst. Adv. Study, 2007, 179-295

[35] S.D. Miller and W. Schmid, The highly oscillatory behavior of automorphic distributions for $SL(2)$, *Lett. Math. Phys.* **69** (2004), 265-286.

[36] H. Petersson, Über die Entwicklungskoeffizienten der automorphen Formen, Acta Math. **58** (1932), 169 - 215.

[37] X. Ren and Y. Ye, Asymptotic Voronoi's summation formulas and their duality for $SL_3(\mathbb{Z})$, *"Arithmetic in Shangri-La" -Proceedings of the 6th China-Japan Seminar on Number Theory held in Shanghai Jiaotong University, August 15-17, 2011,* edited by S. Kanemitsu, Hongze Li, and Jianya Liu, Series on Number Theory and its Applications, volume **9**, World Scientific, Singapore, 2012.

[38] X. Ren, Y. Ye, Resonance and rapid decay of exponential sums of Fourier coefficients of a Maass form for $GL_m(\mathbb{Z})$. (English summary) Sci. China Math. **58** (2015), no. 10, 2105–2124.

[39] X. Ren and Y. Ye, Resonance between automorphic forms and exponential functions, *Sci. China Math.* **53** (2010), no. 9, 2463 -2472.

[40] X. Ren, Y. Ye, Resonance of automorphic forms for GL(3). (English summary) Trans. Amer. Math. Soc. **367** (2015), no. 3, 2137–2157.

[41] X. Ren, Y. Ye, Sums of Fourier coefficients of a Maass form for $SL_3(\mathbb{X})$ twisted by exponential functions. (English summary) Forum Math. **26** (2014), no. 1, 221–238.

[42] P. Sarnak, Estimates for Rankin-Selberg L-functions and quantum unique ergodicity, *J. Functional Analysis* **184** (2001), 419-453.

[43] P. Savala, Computing the Laplace eigenvalue and level for Maass cusp forms, *preprint.*

[44] A. Selberg, On the estimation of Fourier coefficients of modular forms *(Proceedings of Symposia in Applied Mathematics* **VIII**), American Mathematical Society (Providence, RI, 1965), 1-15.

[45] Q. Sun, On cusp form coefficients in nonlinear exponential sums, *Quart. J. Math.* **61**(3) (2010), 363-372.

[46] Q. Sun, Y. Wu, Exponential sums involving Maass forms, *Front. Math. China* **9** (2014), no. 6, 1349 - 1366.