Summer 1928

A study of the hyper-quadrics in Euclidean space of four dimensions

Clarence Selmer Carlson
University of Iowa

No known copyright restrictions.

This thesis is available at Iowa Research Online: https://ir.uiowa.edu/etd/5382

Recommended Citation
Carlson, Clarence Selmer. 'A study of the hyper-quadrics in Euclidean space of four dimensions.' MS (Master of Science) thesis, State University of Iowa, 1928.
https://doi.org/10.17077/etd.y5z5n8s8.

Follow this and additional works at: https://ir.uiowa.edu/etd
Part of the Geometry and Topology Commons
A STUDY OF THE HYPER-QUADRICS

IN

EUCLIDEAN SPACE OF FOUR DIMENSIONS

by

Clarence Selmer Carlson

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science, in the Graduate College of the State University of Iowa.

July 1928
I wish to express my appreciation to Dr. Roscoe Woods for his liberal gifts of time and energy to help, guide, and encourage me in the preparation of this thesis.
Table of Contents.

Section 1. Introduction pp 1-3
Section 2. Some Properties of Euclidean
Four-Space. pp 3-11
Section 3. Systems of Hyperplanes pp 11-14
Section 4. Systems of Hyperspheres pp 15-19
Section 5. Classification of Hyperquadrics pp 20-32
Section 6. Generation of Hyper-quadrics in $S_4$ pp 33-38
Section 7. Tangent Hyperplanes and Polar Hyperplanes pp 39-45
Section 8. Some Theorems, Proofs and Numerical Examples pp 45-56
Section 1

Introduction

The idea of a space of four-dimensions is quite easily developed, in which many of the ideas and methods of the usual three-dimensional geometry can be readily extended to the new space. Naturally it is necessary to make a few fundamental definitions and assumptions.

A point (also referred to in this thesis as an $S_0$) in a four-dimensional space (also referred to as an $S_4$) is defined as any set of values of the four ratios $x_1 : x_2 : x_3 : x_4 : x_5$ of the five variables. In non-homogeneous form the point is a set of values of the four variables $(x,y,z,w)$.

A straight line in $S_4$ is defined as a one-dimensional extent $(S_1)$ determined by the equations

$$px_i = y_i + lz_i \quad (i=1,2,3,4,5) \ldots \ldots \ldots (1)$$

where $y_i$ and $x_i$ are two fixed points and $l$ is an independent parameter.

A plane is defined as a two-dimensional extent determined by the equations

L. Woods, Frederick S. Higher Geometry Pg. 362
\[ px_i = y_i + l z_i + mw_i \quad (i=1,2,3,4,5,\ldots)(2) \]

where \( y_i, z_i, \) and \( w_i \) are three fixed points not on the same straight line and \( l, m, \) and \( w_i \) are independent parameters.

A hyperplane (also referred to as a three-fold or three-space) is defined as a three-dimensional extent \((S_3)\) determined by the equations

\[ px_i = y_i + l z_i + mw_i + nu_i \quad (i=1,2,3,4,5,\ldots)(3) \]

where \( y_i, z_i, w_i, \) and \( u_i \) are fixed points not in the same plane and \( l, m, n, \) are independent parameters.

On the basis of these definitions it is evident that a straight line is completely and uniquely determined by two of its points, a plane by any three of its points and a hyperplane by any four of its points which are not coplanar. Also, if two points lie in a plane, the line determined by them lies in the plane, if three points lie in a hyperplane, the plane determined by them lies in the hyperplane.

Eliminating \( p, l, m, n, \) from the equations of the hyperplane we find that any hyperplane may be represented by a linear equation in the co-ordinates \( x_i \quad (i=1,2,3,4,5,) \) and conversely, any linear equation in \( x_i \) as \( \sum a_i x_i = 0 \quad (i=1,2,3,4,5,) \) represents a hyperplane.

In like manner, any plane may be represented by
two linear equations in the co-ordinates $x_i$ ($i=1,2,3,4,5,$) and conversely; any straight line may be represented by three linear equations and conversely; and finally, four linear equations of the type mentioned in general represent a point, the common point of intersection.
Section 2

Some Properties of Euclidean Four-Space.

Article 1. Transition. To arrive at some of the properties of Euclidean four-space let us write
\[ x_i = x_i / x_5 \quad (i=1,2,3,4). \] If \( x_5 \neq 0 \) the co-ordinates are finite, and are said to represent a point in finite space.

In this thesis it has been found to be very advantageous for the sake of brevity and efficiency to make frequent shifts from homogeneous to non-homogeneous co-ordinates. Where ambiguity would not result the transition has been made by making \( x_5 = 1 \) without mention of the fact and vice versa.

Article 2. The Line. From the equations (1) defining a line we have the symmetric form
\[ \frac{x_1 - y_1}{z_1 - y_1} = \frac{x_2 - y_2}{z_2 - y_2} = \frac{x_3 - y_3}{z_3 - y_3} = \frac{x_4 - y_4}{z_4 - y_4} \quad \ldots \ldots \text{(4)} \]
where \( P(y_1) \) and \( Q(z_1) \) are two points on the line (\( i=1,2,3,4, \)). This equation may also be written
\[ \frac{x_1 - y_1}{A_1} = \frac{x_2 - y_2}{A_2} = \frac{x_3 - y_3}{A_3} = \frac{x_4 - y_4}{A_4} \quad \ldots \ldots \text{(5)} \]
where the numbers \( A_i \) (\( i=1,2,3,4, \)) are determined from the two points used to determine the line and are defined as the directions of the line.
Two lines with directions \( A_i \) and \( B_i \) respectively are said to make an angle \( \theta \) with each other defined by the equation

\[
\cos \theta = \frac{\sum A_i B_j}{\sqrt{\sum A_i^2} \sqrt{\sum B_j^2}} 
\]

\( i=1,2,3,4 \)

Now if we write \( l_{ij} = \frac{A_i}{\sqrt{\sum A_i^2}}, \) \( i,j=1,2,3,4 \)

and likewise \( m_{ij} = \frac{B_i}{\sqrt{\sum B_i^2}}, \) \( i,j=1,2,3,4 \)

the equation of the straight line takes the form

\[
\frac{x_1 - y_1}{l_{11}} = \frac{x_2 - y_2}{l_{21}} = \frac{x_3 - y_3}{l_{31}} = \frac{x_4 - y_4}{l_{41}} \ldots \ldots (7)
\]

Clearly, with this form of the straight line equation, the angle between the lines is defined by the equation

\[
\cos \theta = \sum l_{ij} m_{ij} \quad \{i=1,2,3,4\} \ldots \ldots (8)
\]

From the values of \( l_{ij} \) and \( m_{ij} \) above it is clear that

\[
\sum l_{ij}^2 = 1 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (9)
\]

where \( l_{ij} \) is the cosine of the angle which the line makes with the axis \( x_i \). Likewise the sine of the angle which the line makes with this axis becomes

\[
\sin^2 \theta = \sum (l_{ij} m_{ji} - l_{ij} m_{ij})^2 \ldots \ldots (10)
\]

Article 3. The Hyperplane. It was seen that the

2. Woods, Frederick S. Higher Geometry pg. 369
3. Schoute, Dr. P. E. Mehrdimensionale Geometrie pg. 133
3. Ibid page 128
hyperplane could be represented by a single linear equation in five homogeneous variables. The normal form of the hyperplane is shown to be

\[ \sum_{j} a_{j} x_{j} = p = 0 \] (11)

where \( p \) is the normal. If (11) is equated to \( d \), \( d \) will represent the distance from a point \( x_{i} \) to the \( S_{3} \) given by (11).

**Article 4. Distance Relations.** The distance between two points \( P \) and \( Q \) is defined for a rectangular system of co-ordinates as

\[ (PQ)^{2} = \sum_{i} (x_{i} - y_{i})^{2} \] (12)

where \( x_{i} \) and \( y_{i} \) are the co-ordinates of the points \( P \) and \( Q \) respectively.

To find the distance from a point to a line, let the line be given in the form (7) and the point \( P(b_{1}, b_{2}, b_{3}, b_{4}) \) not lying on the line, where the \( l_{1} = \cos \alpha_{1}, \alpha_{1} \) being the angle with the axis of \( x_{1} \).

Now if \( P_{1}(a_{1}, a_{2}, a_{3}, a_{4}) \) be a point on the line, and if we let \( P \) be the foot of the perpendicular from \( P_{2} \) to the line, and let \( \theta \) be the angle between the given line and the line \( P_{1}P_{2} \), while at the same time \( d \) is

4. Schoute, Dr. P.H. Mehrdimensionale Geometrie p. 168
5. Ibid p. 127
is the length of the segment $P_1P_2$, then

$$(P_2P) = (P_1P_2)\sin \theta$$

$$(P_2P)^2 = (P_1P_2)^2 \sin^2 \theta = d^2 - d^2 \cos^2 \theta$$

The direction cosines of the line $P_1P_2$ are $m_i = (b_i - a_i)/d$, $(i=1,2,3,4,...)$ and thus $\cos \theta = \frac{(b_i - a_i)}{d}$

from (10), so that we have

$$(P_2P)^2 = \sum_{j} (b_j - a_j)^2 - \left(\sum_{j} \frac{b_j - a_j}{d}\right)^2 \ldots (13)$$

Schoute\textsuperscript{6} has a beautiful generalization for n-dimensional geometry, which should be mentioned here. He says that if in $S_n$, a space of $S_{n-1}$ and a point are given, the locus of the line through $P$ and perpendicular to $S_{n-1}$ is an $S_d$.

In order to find the system of equations representing the $S_d$, let the co-ordinates of the point be $(a_1, a_2, ... , a_n)$ and let the equations of the $S_{n-d}$ be given by the independent equations

$$x_1 \cos a_1 \theta_1 + x_2 \cos a_2 \theta_2 + ... + x_n \cos a_n \theta_n - p = 0$$

$(k=1,2,3,4,...d)$ when referred to a rectangular system of co-ordinates.

By the familiar properties of the normal form (Hesse) the equations of the directed perpendicular from 0 to these d spaces are

$$x_1/\cos a_1 \theta_1 = x_2/\cos a_2 \theta_2 = ... = x_n/\cos a_n \theta_n \quad (k=1,2,...,d)$$

6. Schoute, Dr. M. P. Mehrdimensionale Geometrie pg 168
Then, the determinant
\[
\begin{vmatrix}
 x_1 & x_2 & \ldots & x_n & 1 \\
 a_1 & a_2 & \ldots & a_n & 1 \\
 \rho_1\cos\alpha_1 & \rho_1\cos\alpha_2 & \ldots & \rho_1\cos\alpha_n & 1 \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 \rho_d\cos\alpha_1 & \rho_d\cos\alpha_2 & \ldots & \rho_d\cos\alpha_n & 1
\end{vmatrix} = 0
\]
when \( \rho_k \) is infinite, becomes the equation of the \( S_d \) through \( P \) and the infinite point of the \( d \) perpendicular, for in that case it simplifies to the form
\[
\begin{vmatrix}
 x_1-a_1 & x_2-a_2 & \ldots & x_n-a_n \\
 \cos\alpha_1 & \cos\alpha_2 & \ldots & \cos\alpha_n \\
 \vdots & \vdots & \ddots & \vdots \\
 \cos\alpha_1 & \cos\alpha_2 & \ldots & \cos\alpha_n
\end{vmatrix} = 0
\]
where each determinant of the highest rank which can be taken from this form constitutes one equation of the desired system.

Article 5. Parallelism. Any two of the configurations, line, plane or hyperplane are said to be parallel if their complete intersection is infinitely distant. A few theorems on parallelism may be useful in the main problem of this thesis. These theorems have been taken from Woods Higher Geometry page 371.

1. If a line is parallel to a plane the two lie in the same hyperplane and determine that hyperplane. Through any point in space there passes a pencil of lines parallel to a fixed plane.

2. Two planes are said to be simply parallel if they intersect in a single infinite point and completely
parallel if they intersect in a line at infinity, thus two completely parallel planes lie in the same hyperplane \((S_3)\). In the case of completely parallel planes, any line of one is parallel to a pencil of lines of the other, while if the planes are simply parallel, there is a unique direction in each plane such that lines with that direction in either plane are parallel to lines with the same direction in the other, but lines with any other direction in one plane are parallel to no lines of the other.

3. If a plane and a hyperplane are parallel, any line in the plane is parallel to each line of a bundle in the hyperplane, and any plane in the hyperplane is at least simply parallel to the given plane.

4. If two hyperplanes are parallel, any plane of one is completely parallel to some plane and hence to a pencil of planes of the other, and any plane of one is simply parallel to any plane of the other to which it is not completely parallel.

The condition that two lines be parallel is that the direction cosines be alike or their direction numbers be proportional. Two hyperplanes are parallel if their coefficients in their equations be proportional.

If we have two planes given by the equations
the necessary and sufficient condition that they intersect in a finite point, i.e., be simply parallel is that

$$\begin{align*}
P_1 &= \sum_{i=1}^{n} A_i x_i = 0 \\
P_2 &= \sum_{i=1}^{n} B_i x_i = 0 \\
P_3 &= \sum_{i=1}^{n} C_i x_i = 0 \\
P_4 &= \sum_{i=1}^{n} D_i x_i = 0
\end{align*}$$

(14) (15)

shall be of rank four, while if the determinants of the matrix

$$\begin{vmatrix} A_1 & \cdots & A_5 \\ B_1 & \cdots & B_5 \\ C_1 & \cdots & C_5 \\ D_1 & \cdots & D_5 \end{vmatrix} = 0$$

and the matrix

$$\begin{vmatrix} A_1 & \cdots & A_5 \\ B_1 & \cdots & B_5 \\ C_1 & \cdots & C_5 \\ D_1 & \cdots & D_5 \end{vmatrix}$$

shall be of rank four, while if the determinants of the matrix

$$\begin{vmatrix} A_1 & \cdots & A_5 \\ B_1 & \cdots & B_5 \\ C_1 & \cdots & C_5 \\ D_1 & \cdots & D_5 \end{vmatrix}$$

have a constant ratio to the corresponding determinants of

$$\begin{vmatrix} C_1 & \cdots & C_5 \\ D_1 & \cdots & D_5 \end{vmatrix}$$

the planes (14) and (15) are completely parallel.

Article 5. Perpendicularity. We have already discussed the angle between two lines. Other perpendicular properties may be given in theorem form:

1. A line, perpendicular to three lines of a hyper-plane, which are non-coplanar and no two of which are parallel, is perpendicular to the hyper-plane, while a line perpendicular to every plane in a hyper-plane is perpendicular to the hyper-plane.

2. Two planes such that every line of one is perpendicular to every line of the other is said to be completely perpendicular, while two planes such that each
contains a line perpendicular to the other are called semi-perpendicular planes

3. Two semi-perpendicular planes may be simply parallel. The direction of the parallel lines of the two planes is then orthogonal to the directions of the perpendicular lines.

4. If a plane is perpendicular to a hyperplane, it is completely perpendicular to each plane of a pencil of parallel planes of the hyperplane and semi-perpendicular to every other plane of the hyperplane.

5. Finally, in four-dimensional Euclidean space the geometry in any hyperplane, for which $A^2 + B^2 + C^2 + D^2 \neq 0$ is that of the usual Euclidean three-dimensional geometry.
Section 3.

Systems of Hyperplanes.

Article 7. System of Hyperplanes Through a Plane. If

\[ P_1 = \sum A_i x_i = 0 \]
\[ P_2 = \sum B_i x_i = 0 \]

are equations of two intersecting \(S_3\)'s, the equation

\[ k_1 P_1 + k_2 P_2 = 0 \]

is for all real values of \(k_1\) and \(k_2\), the equation of a hyperplane passing through the plane \((S_2)\) determined by (16). The totality of hyperplanes passing through a plane is called a pencil of hyperplanes, and any two of these hyperplanes may be used to define the plane.

The two equations \(P_1 = 0, P_2 = 0\), will represent the same hyperplane when the coefficients are proportional i.e., every second order determinant of the matrix shall vanish

\[ \begin{vmatrix} A_1 & \cdots & A_5 \\ B_1 & \cdots & B_5 \end{vmatrix} \]

In this case the multipliers \(k_1\) and \(k_2\) can be found such that \(k_1 P_1 + k_2 P_2 = 0\) is satisfied identically and conversely.

Article 8. Bundles of Hyperplanes. Three hyperplanes not in the same pencil intersect is a straight line. All hyperplanes through the same line form a
a bundle, and any three of them not in the same pencil determine the line. Let

\[
\begin{align*}
P_1 &= \sum A_i x_i = 0 \\
P_2 &= \sum B_i x_i = 0 \\
P_3 &= \sum C_i x_i = 0
\end{align*}
\] .......... (17)

be the equations of three hyperplanes belonging to a bundle. Then \(k_1 P_1 + k_2 P_2 + k_3 P_3 = 0\) .......... (18) is the equation of the bundle.

Article 9. A Web of Hyperplanes. Four linearly independent hyperplanes intersect in a point. All hyperplanes through the same point form a three-dimensional extent. On the other hand, we may say that any four hyperplanes not in the same bundle determine the point. Given

\[
\begin{align*}
P_1 &= \sum A_i x_i = 0 \\
P_2 &= \sum B_i x_i = 0 \\
P_3 &= \sum C_i x_i = 0 \\
P_4 &= \sum D_i x_i = 0
\end{align*}
\] .......... (19)

be the equation of four hyperplanes. If \(\begin{vmatrix} A_1 & B_2 & C_3 & D_4 \end{vmatrix} \neq 0\) this system has a point in common, where \(\begin{vmatrix} A_1 & B_2 & C_3 & D_4 \end{vmatrix}\) is written for the determinant of the coefficients of \(x_1, x_2, x_3, x_4\). This point is called the vertex. From Art.5. we recall that the planes in (19) will be simply parallel when \(\begin{vmatrix} A_1 & B_2 & C_3 & D_4 \end{vmatrix} = 0\) and when not all of the other fourth order determinants of the matrix

7. We shall say that systems with one, two, or three essential parameters shall be called pencils, bundles, and webs respectively.
8. This abridged notation will be adopted in this thesis.
\[ |A_1 B_2 C_3 D_4 E_5| \] (in the abridged notation) are not zero. If one of the fourth order determinants of the matrix is different from zero and \[ |A_1 B_2 C_3 D_4| \neq 0 \] the system \( \sum k_i P_i = 0 \) is called a web of hyperplanes.

**Article X  A System of Five Hyperplanes.** The condition that five hyperplanes

\[
\begin{align*}
P_4 &= \sum A_i x_i = 0 \\
P_5 &= \sum B_i x_i = 0 \\
P_3 &= \sum C_i x_i = 0
\end{align*}
\]

all pass through a point if five numbers \( x_1 x_2 x_3 x_4 x_5 \), not all zero, exist which satisfy the five equations simultaneously. This condition requires that

\[ |A_1 B_2 C_3 D_4 E_5| = 0. \]

The hyperplanes are said to be independent, should this condition not be satisfied. When the given hyperplanes are independent, five numbers \( k_1 k_2 k_3 k_4 k_5 \) can always be found such that every hyperplane can be expressed by the equation

\[ \sum k_i P_i = 0 \] .............................. (21)

To show this let \( \sum a_i x_i = 0 \) be the equation of any hyperplane. This last equation and (21) will represent the same hyperplane if their coefficients are proportional. This condition is fulfilled when
\[ a_1 = k_1A_1 + k_2B_1 + k_3C_1 + k_4D_1 + k_5E_1 \]
\[ a_2 = k_1A_2 + k_2B_2 + k_3C_2 + k_4D_2 + k_5E_2 \]
\[ a_3 = k_1A_3 + k_2B_3 + k_3C_3 + k_4D_3 + k_5E_3 \]
\[ a_4 = k_1A_4 + k_2B_4 + k_3C_4 + k_4D_4 + k_5E_4 \]
\[ a_5 = k_1A_5 + k_2B_5 + k_3C_5 + k_4D_5 + k_5E_5 \]

Since the determinant of the co-efficients is not zero, and the five \( k \)'s can be determined so as to satisfy these equations.

Article 11. The matrix of a System of Hyperplanes.

The material of the previous articles of this section may be summarized thus: The necessary and sufficient condition that a system of hyperplanes

(a) have no point in common is that the matrix formed of their co-efficients is of rank five;

(b) belong to a web of hyperplanes if the matrix is of rank four, i.e., we have all the hyperplanes on a point, or in our abbreviated notation a system of \( S_3 \)'s on \( S_0 \);

(c) constitute a bundle of hyperplanes if the matrix is of rank three, all the \( S_3 \)'s on \( S_1 \);

(d) form a pencil of hyperplanes if the matrix is of rank two and we have all \( S_3 \)'s on \( S_2 \);

(e) are all coincident hyperplanes in the matrix is of rank one.
Section 4.

Systems of Hyperspheres.

Article 12. The Hypersphere. The equation in the four variables $x_1, x_2, x_3, x_4$,

$$\sum_{i=1}^{4} (x_i - a_i)^2 = r^2 \quad \quad \quad \quad (23)$$

is the equation of the hypersphere of radius $r$ and center $(a_1, a_2, a_3, a_4)$. It is the locus of all points equidistant from the fixed point, its center. See eq. (12)

Any equation of the form

$$a_{11} \sum_{i=1}^{4} x_i^2 + 2 \sum_{i=1}^{4} a_{15} x_i + a_{55} = 0 \quad \quad \quad \quad (24)$$

may be written in the form

$$\sum_{i=1}^{4} (x_i + a_{15}/a_{11})^2 = \sum_{i=1}^{4} a_{i5}^2/a_{11}^2 - a_{55}/a_{11} \quad \quad \quad \quad (25)$$

where $a_{11} \neq 0$. By comparing with (23) this is seen to be the equation of a hypersphere provided the right hand member is greater than zero, with center

$(-a_{15}/a_{11}, -a_{25}/a_{11}, -a_{35}/a_{11}, -a_{45}/a_{11})$ and radius

$$\left(\frac{a_{15}^2}{a_{11}^2} - a_{55}/a_{11}\right)^{1/2}/a_{11}.$$

Article 13. The Absolute. If we write the equation of the hypersphere

$$a_{11} \sum_{i=1}^{4} x_i^2 + 2 \sum_{i=1}^{4} a_{15} x_i x_5 + a_{55} x_5 = 0 \quad \quad \quad \quad (26)$$

($a_{11} \neq 0$), the intersection of this hypersphere with the infinite $S_3$ is represented by the equations

$$\sum_{i=1}^{4} x_i^2 = 0, x_5 = 0.$$

Since these equations are independent of the coefficients $a_{11}, a_{15}, (i=1, 2, 3, 4)$, which appear in the equation of the hypersphere (26), we conclude
that all hyperspheres intersect the infinite hyper-plane in the same trace.

This trace is called the absolute and contains no real points. The equation of any second degree surface which contains the absolute may be written

$$a_11 \sum_1^4 x_i^2 + x_5 \sum_1^4 a_{i5} x_i = 0 \quad \text{(27)}$$

If $a_{11} \neq 0$ this is the equation of a hypersphere.

If $a_{11} = 0$ the locus of the equation is two $S_3$'s, of which one, at least, is $x_5 = 0$. In the latter case we shall call the surface a hypersphere, since it contains, or passes through the absolute, and is of the second degree. It may also be called a composite hypersphere.

Conversely, every surface of the second degree that contains that absolute is a hypersphere.

Any hyperplane $\sum_1^5 u_i x_i = 0$ other than $x_5 = 0$ intersects the absolute in a circle which in turn contains the circular points mentioned in ordinary plane analytical geometry.

Article 14. Systems of Hyperspheres. Let $A = a_0 \sum_1^4 x_i^2 + 2 \sum_1^4 a_{i5} x_i x_5 = 0$ be the equation of a hypersphere with similar equations for other hyperspheres.

then

$$k_1 A + k_2 B = 0 \quad \text{............... (28)}$$

is the equation of all hyperspheres containing the intersection of $A = 0, B = 0$. Now if $k_1 a_0 + k_2 b_0 = 0$ the hypersphere (28) is composite and consists of the infinite $S_3$ and the hyperplane

$$\sum_1^4 (b_0 a_{i5} - b_{i5} a_0) x_i + (b_0 a_{55} - a_0 b_{55}) = 0 \quad \text{(29)}$$
which intersects all the hyperspheres of the system (28) in a fixed quadric common to A = 0, and B = 0. The hyperplane (29) is to be called the radical hyperplane.

But further, the radical hyperplane is the locus of the centers of the hyperspheres intersecting A = 0 and B = 0 orthogonally. If we define the angle between two hyperspheres at a point P of their trace of intersection, as the angle between their tangent $S_3$'s at the point P. Then by the same method as used in three-space geometry, using the relation for the cosine of the angle between two $S_3$'s, we have the relation for two hyperspheres to be orthogonal

$$2 \sum_{i} a_{15}b_{15} - a_0b_{55} - b_0a_{55} = 0 \quad \ldots \ldots \ldots (30)$$

where $A = 0$, and $B = 0$, are the hyperspheres.

Now if $C = 0$ be a hypersphere orthogonal to $A = 0$, we must have $2 \sum_{i} a_{15}c_{15} - a_0c_{55} - c_0a_{55} = 0$, and if it is orthogonal to $B = 0$ then $2 \sum_{i} b_{15}c_{15} - b_0c_{55} - c_0b_{55} = 0$. If we eliminate $c_{55}$ between these equations we get

$$\sum_{i} 2(b_0a_{15} - b_{15}a_0)c_{15} - (b_0a_{55} - b_{55}a_0)c_{55} = 0$$

which is the very condition that the center ($-c_{15}/c_{55}$) ($i=1,2,3,4,$) of the orthogonal hypersphere lie on the radical hyperplane (29). Conversely, if $c_0, c_{15}$, ($i=1,2,3,4,$) are given numbers which satisfy (28) a value of $c_{55}$ can be found such that the corresponding hypersphere $C = 0$ is orthogonal to every hypersphere of the system

$$k_1A + k_2B = 0$$
Likewise, if $D = 0$ is a hyper-sphere whose center does not lie on the line of centers of $A = 0$ and $B = 0$ then every hypersphere of the system

$$k_1A + k_2B + k_3D = 0 \quad \cdots \cdots \cdots \cdots (31)$$

passes through the intersection of the hyperspheres $A = 0, B = 0, D = 0$.

Every hypersphere of the system (31) for which $k_1a_0 + k_2b_0 + k_3d_0 = 0$ is composite and consists of the infinite $S_3$ and the $S_3$ on the pencil given by the radical hyperplanes

$$\sum_{i} 2(b_0a_{1i} - b_{1i}a_0)x_i + (b_0a_{55} - b_{55}a_0) \quad \cdots \cdots \cdots \cdots (32)$$

$$\sum_{i} 2(d_0a_{1i} - d_{1i}a_0)x_i + (d_0a_{55} - d_{55}a_0)$$

The $S_3 (32)$ is called the radical plane of the system.

In the same manner as we showed that the radical hyperplane was the locus of the centers of all hyperspheres orthogonal to the system (28), so the radical plane is the locus of the centers of all hyperspheres orthogonal to the system (31).

Again, if $E = 0$ is another hypersphere whose center is not on the plane determined by the hyperspheres $A = 0, B = 0, D = 0$, the relation

$$k_1A + k_2B + k_3D + k_4E = 0 \quad \cdots \cdots \cdots \cdots (33)$$

is composite if $k_1a_0 + k_2b_0 + k_3d_0 + k_4e_0 = 0$. The centers of all spheres orthogonal to this system lie on the radical axis defined by
After the same manner the radical center becomes the center of the hypersphere cutting all the hyperspheres of the linear system

\[ k_1A + k_2B + k_3D + k_4E + k_5F = 0 \quad \ldots \quad (35) \]

orthogonally.

Article 15. A System of Six Hyperspheres. Given six hyperspheres in \( S_4 \), given by the equations

\[ \sum_{i=1}^{6} (a_{i5} - a_{15})x_i + (b_{i5} - b_{15}) = 0 \]

\[ \sum_{i=1}^{6} (d_{i5} - d_{15})x_i + (d_{i5} - d_{15}) = 0 \ldots \quad (34) \]

\[ \sum_{i=1}^{6} (e_{i5} - e_{15})x_i + (e_{i5} - e_{15}) = 0 \]

How if we require that a hypersphere be a linear combination of these so that

\[ q = k_1q_1 + k_2q_2 + k_3q_3 + k_4q_4 + k_5q_5 + k_6q_6 \quad \ldots \quad (37) \]

we must find \( k_1, k_2, k_3, k_4, k_5, k_6 \), such that

\[ \sum_{i=1}^{6} k_i q_j = q_j \quad (j=1, 2, 3, 4, 5, 6, \ldots) \quad (38) \]

where the \( q_j \) without the superscript is a coefficient of \( Q \) without the superscript.

The condition that these equations have a solution in the \( k \)'s other than zeros is that

\[ \begin{vmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \\ q_1^2 & q_2^2 & q_3^2 & q_4^2 & q_5^2 & q_6^2 \\ q_1^3 & q_2^3 & q_3^3 & q_4^3 & q_5^3 & q_6^3 \\ q_1^4 & q_2^4 & q_3^4 & q_4^4 & q_5^4 & q_6^4 \end{vmatrix} = 0 \]

Hence, any hypersphere can be written as the linear combination of six independent hyperspheres in \( S_4 \).
Section 5.

The Classification of Hyperquadrics in $S_4$.

Article 16. Definition of a Hyperquadric. The locus representing an equation of the second degree in $x_1, x_2, x_3, x_4$, (non-homogeneous co-ordinates) is called a hyperquadric in $S_4$.

The most general equation of the second degree in five homogeneous variables can be written

$$\sum a_{ij}x_ix_j = 0$$

It is the purpose of this section to reduce this equation to a simpler form and classify the surfaces according to canonical forms.

Article 17. The Intersection of a Hyperquadric and a Line. Given the general equation of a hyperquadric in $S_4$

$$F(x) = \sum a_{ij}x_ix_j = 0 \quad (a_{ij} = a_{ji}) \ldots \ldots \ldots (39)$$

and let us solve simultaneously with the line

$$x_i = y_i + l_ir \quad (i=1,2,3,4,5) \ldots \ldots \ldots (40)$$

where $P_0(y_i)$ is a point on the line, and assuming that the coefficients of the second degree terms are not all zero and all the coefficients are real we have
The roots of (41) are the distances from the point \( P_0(y_1) \) on the line (40) to the points in which the line intersects the hyperquadric.

If \( Q \neq 0 \), equation (41) is a quadratic in \( r \). If \( Q = 0 \), but \( R \) and \( S \) are not both zero, (41) is still to be considered a quadratic, with one or more infinite roots.

If \( Q = R = S = 0 \), (41) is satisfied for all values of \( r \) and the corresponding line (40) lies entirely on the surface. From which we can conclude that every line which does not lie on a given hyperquadric has (distinct or coincident) points in common with the surface while if a given line has more than two points in common with a given hyperquadric, it lies entirely on that surface.

**Article 18. Diametral Hyperplanes and Center.**

Let \( P_1 \) and \( P_2 \) be the points of intersection of the line (40) with the hyperquadric. Then \( P_1 P_2 \) will be called a chord of the hyperquadric. If \( r_1 \) and \( r_2 \) be the roots of (41) so that \( P_0 P_1 = r_1 \) and \( P_0 P_2 = r_2 \) and if we require that \( P_0 \) be the middle point of \( P_1 P_2 \), then \( r_1 + r_2 = 0 \) and hence

\[
\sum_i (\partial F/\partial y_i) l_i = 0. \tag{43}
\]
If the parameters \( l_i \) are constant but \( y_i \) allowed to vary the line (40) describes a system of parallel lines. Equation (43) is the equation of the locus of the middle points of this system of chords. Now (43) is linear in \( y_i \), hence it is the equation of a hyperplane. We shall call it a hyper-diametral-plane, or a diametral hyperplane.

If we consider (43) as a system we have the system of all \( S_3 \)'s on the intersection of four hyperplanes, for all values of \( l_i \) (43) passes through the intersection of the hyperplanes

\[
\sum_{j=1,2,3,4} a_{ij} x_i = 0 \quad (j=1,2,3,4,)
\] ............. (44)

Let us use the following notation for brevity,

\[
A_{55} = \begin{vmatrix} a_{11} & a_{22} & a_{33} & a_{44} \\ \end{vmatrix}, \quad A_{54} = \begin{vmatrix} a_{11} & a_{22} & a_{33} & a_{45} \\ \end{vmatrix},
\]

\[
A_{53} = \begin{vmatrix} a_{11} & a_{22} & a_{34} & a_{45} \\ \end{vmatrix}, \quad A_{52} = \begin{vmatrix} a_{11} & a_{23} & a_{34} & a_{45} \\ \end{vmatrix},
\]

\[
A_{51} = \begin{vmatrix} a_{22} & a_{23} & a_{34} & a_{45} \\ \end{vmatrix}.
\]

If \( A_{55} \neq 0 \) the hyperplanes intersect in a single finite point, \((y_i) = \{-1\} A_{5i} \div A_{55} \ (i=1,2,3,4,)\) (non-homogeneous co-ordinates), and \((y_i)\) is the middle point of every chord through it. If this point does not lie on the surface it is called the center of the hyperquadric. In either case, the system of hyperplanes (44) is a three parameter system with vertex at \((y_i)\).

**Article 18. Discussion of A\(_{55}\).** If \( A_{55} = 0 \), but \( A_{5i} \) are not all zero, the hyperplanes (44) intersect in an infinitely distant point. See Art 5. And
the system is a parallel web. The hyperquadric is now said to be non-central.

Recalling the summary in Article 11, if \( A_{55} \) is of rank three, the system (44) intersect in a line. If this line is finite and does not lie on the hyperquadric it is called a line of centers, and the surface is of the cylinder type. If the line lies on the surface it is called a line of vertices.

If the system is of rank two, the diametral planes \((a_3's)\) lie on a plane \((a_2)\), and we have a plane of centers if the plane does not lie on the surface, and a plane of vertices if the plane lies on the surface.

If the system is of rank one, the diametral hyperplanes coincide and we have a hyperplane of centers or vertices.

Article 20. Discriminating Quartic. The condition that the diametral hyperplanes (43) be perpendicular to the chord it bisects is that

\[
\sum l_1 l_i = \sum a_i l_i^4 + \sum (a_3 l_4) l_i = \sum (a_4 l_1) l_i = 0
\]

If we denote the common ratio by \( k \) we may write in place of (45).

\[
\begin{align*}
(a_{11} - k) l_1 + a_{12} l_2 + a_{13} l_3 + a_{14} l_4 &= 0 \\
(a_{21} l_1 + (a_{22} - k) l_2 + a_{23} l_3 + a_{24} l_4 &= 0 \\
(a_{31} l_1 + a_{32} l_2 + (a_{33} - k) l_3 + a_{34} l_4 &= 0 \\
(a_{41} l_1 + a_{42} l_2 + a_{43} l_3 + (a_{44} - k) l_4 &= 0
\end{align*}
\]
In order that these equations have a solution in \(l_1, l_2, l_3, l_4\), other than 0, 0, 0, 0, we must have
\[
\begin{vmatrix}
(a_{11}-k) & (a_{12}-k) & (a_{13}-k) & (a_{14}-k) \\
(a_{21}-k) & (a_{22}-k) & (a_{23}-k) & (a_{24}-k) \\
(a_{31}-k) & (a_{32}-k) & (a_{33}-k) & (a_{34}-k) \\
(a_{41}-k) & (a_{42}-k) & (a_{43}-k) & (a_{44}-k)
\end{vmatrix} = 0 \quad \ldots \ldots \ldots (47)
\]
which, developed and arranged in powers of \(k\) gives us the quartic
\[
k^4 - J_0 k^3 + J_1 k^2 - J_2 k + A_{55} = 0 \quad \ldots \ldots \ldots (48)
\]
where
\[
J_0 = \sum_{i} a_{ii}
\]
\[
J_1 = a_{11}a_{22} - a_{12}^2 + a_{11}a_{33} - a_{13}^2 + a_{11}a_{44} - a_{14}^2 + a_{22}a_{33} - a_{23}^2 + a_{22}a_{44} - a_{24}^2 + a_{33}a_{44} - a_{34}^2
\]
\[
J_2 = |a_{11}a_{33}a_{44}| + |a_{11}a_{33}a_{44}| + |a_{11}a_{22}a_{44}|
\]
\[
A_{55} = |a_{11}a_{22}a_{33}a_{44}|
\]
Let us also write \(\Delta = |a_{11}a_{22}a_{33}a_{44}a_{55}|\).

In the determinant \(A_{55}\) let us multiply the elements of the first, second, and third rows by \(t_0, t_1, t_2\), respectively, and add the first three terms so formed in each column and equate this sum to the last element of the column. Solving for \(a_{44}\) in terms of the other relations we find
\[
a_{44} = a_{11} + 2t_0 a_{12} + 2t_1 a_{13} + t_0^2 a_{22} + 2t_0 t_1 a_{32} + a_{33} t_1^2
\]

Then if in \(J_2\), the coefficient of the linear term in the discriminating quartic (48) we substitute the relations obtained above we find that
\[
J_2 = A_{44}' \left( t_0^2 + t_1^2 + 2t_0^2 \right) \ldots \ldots \text{where } A_{44}' \text{ is the cofactor of } a_{44} \text{ within } A_{55}. \text{ If } J_2 \text{ vanishes two of the roots of the quartic (48) are zero and since } t_0, t_1, t_2,
are real numbers evidently \( A_{44} \) must vanish. If \( A_{44} \) vanishes we can write the row and column \( a_{31} \) \( a_{32} \) \( a_{33} \) as a row and column of zeros. Likewise if \( A_{55} \) vanishes (the condition that one root of the quartic vanish), the row and column \( a_{41} \) \( a_{42} \) \( a_{43} \) \( a_{44} \) can be written as a row and column of zeros. Further, if \( A_{55} \) and \( A_{44} \) vanish simultaneously and hence two roots of the quartic be zero it is necessary that \( \Delta = 0 \).

It is interesting to note that equation (48) has only real roots. Also if we square \( J_0 \) and subtract \( J_1 \) we see that \( a_{ij} = 0 \) \((i,j, = 1,2,3,4,)\) which is contrary to hypothesis that the coefficients of the second degree terms could not be zero. Looking further \( J_0, J_1, J_2, \) are invariant under rotation and translation.

Article 21. The Hyperquadric and its Center.

Let the center of the hyperquadric (if it has a center) be \( (y_1) \) \((i=1,2,3,4,5,)\) and let us translate the origin to this center by using \( x_i = x'_i + y_1 \) \((i=1,2,3,4,5,)\). The equation of \( F(x'_i) \) then becomes

\[
\sum a_{ij} x'_i x'_j + \sum \sum a_{ij} y_1 x'_i + E = 0 \text{ where } E = F(y_1).
\]

But \( y_1 \) is the center so from (44) \( \sum \sum a_{ij} y_1 = 0 \) \((j=1,2,3,4,)\) and the equation of the hyperquadric takes the form

\[
\sum a_{ij} x'_i x'_j + E = 0 \ldots \ldots \ldots \ldots \ldots (50)
\]

9. Kowaleski, G. Einführung in Die Determinenten Theorie pg 126
10. Ibid page 245, Bocher, Maxime, Higher Algebra, pg. 154
11. Schoute Dr. H. F. Mehrdimensionale Geometrie, pg. 131
Now \( E = F(y_i) = \sum_{i=1}^{5} \sum_{j=1}^{5} a_{ij} y_i y_j \) whence from (49)

\[ E = \frac{\sum a_{ij} y_i}{\sum a_{ij}} \]  

(51)

By eliminating \((y_i)\) from (49) and (51) we obtain

\[ |a_{11} a_{22} a_{33} a_{44} a_{55} - E| = 0 \]  

which can be written

\( A_{55} E = \Delta \), where the \( \Delta \) is the determinant

\[ |a_{11} a_{22} a_{33} a_{44} a_{55}| \]  
or if \( A_{55} \neq 0 \) \( E = \Delta / A_{55} \). 

(52)

The determinant \( \Delta \) is called the discriminant of the surface. The surface is singular when \( \Delta = 0 \), and nonsingular when \( \Delta \neq 0 \)

If \( A_{55} \neq 0 \) and \( \Delta = 0 \), then \( E = 0 \) and our equation is homogeneous of the second degree in four variables and represents a hypercone with vertex at the origin. The vertex of the hypercone and the hyperquadric are then the same.

If \( \Delta = 0 \) and \( E \neq 0 \), then \( A_{55} = 0 \). Since \((y_i)\) \((i=1,2,3,4,5)\) was assumed to be a finite point, then \( A_{54} = A_{53} = A_{52} = A_{51} = 0 \) (see Art 16), the system (44) is of rank three and the surface has a hyperplane of centers, or it may be of rank two and the surface have a plane of centers.

If \( \Delta = E = A_{55} = 0 \) the surface is composite and the equation is factorable into the equations of two hyperplanes.

**Article 21.** The Discriminating Quartic again.

Let the equation of the quadric be so reduced that in
the homogeneous form it is \( \sum_{i} a_i x_i^2 + 2 \sum_{j=k}^{5} a_j x_j = 0 \). The discriminating quartic now becomes

\[
k^2 - J_0 k^3 + J_1 k^2 - J_2 k + A_5 = 0
\]

where \( J_0 = \sum a_i^2 \)

\[
J_1 = a_{11} a_{33} + a_{11} a_{33}^* + a_{11} a_{33}^* + a_{11} a_{33}^* + a_{11} a_{33}^* \\
J_2 = a_{11} a_{33} + a_{11} a_{33} + a_{11} a_{33} + a_{11} a_{33} + a_{11} a_{33}
\]

From this fact we can say further that if we translate so that the center is taken as the new origin the equation becomes

\[
\sum_{i} a_i x_i + \hat{a}_5 = 0
\]

and the roots of the discriminating quartic are the coefficients of the squared terms and \( \hat{a}_5 \) becomes \( \Delta/A_5 \) if \( A_5 \neq 0 \).

Article 22. Reduction to Type Forms.

From the previous article we see that the hyperquadric can be written in the form

\[
k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2 + k_4 x_4^2 + \Delta/A_5 = 0
\]

where the \( k_i \) are the roots of the discriminating quartic and \( A_5 = k_1 k_2 k_3 k_4 \)

The problem of reduction to type forms is divided into several cases. In our study we shall designate each type by a number and rather than characterize each by a name we shall characterize each by its projecting cylinder in one of the coordinate \( S_3 \)'s. A tabulation of these types will be made at the end of the section.
In the discussion of the various cases we shall use $k_1$ to designate the absolute values of the roots of the discriminating quartic for the sake of brevity and ease of discussion.

Case 1. If all the roots $k_1, k_2, k_3, k_4$, are different from zero and distinct, with $\Delta > 0$ the equation reduces to

Type 1. \[ \sum_{i=1}^{4} k_1 x_i^2 + \Delta/ k_1 k_2 k_3 k_4 = 0 \]

Type 2. \[ \sum_{i=1}^{4} k_1 x_i^2 - k_4 x_4^2 - \Delta/ k_1 k_2 k_3 k_4 = 0 \]

Type 3. \[ k_1 x_1^2 + k_2 x_2^2 - k_3 x_3^2 - k_4 x_4^2 + \Delta/ k_1 k_2 k_3 k_4 = 0 \]

Type 4. \[ k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 - k_4 x_4^2 - \Delta/ k_1 k_2 k_3 k_4 = 0 \]

Type 5. \[ -\sum_{i=1}^{4} k_1 x_i^2 + \Delta/ k_1 k_2 k_3 k_4 = 0 \]

If $\Delta$ were negative we would have obtained the same types but slightly shifted in their position with respect to the co-ordinate system, i.e., type 1, going over into type 5, type 2 going over into type 4, and type 3 going over into itself.

Case 2. If in case 1, two of the roots, say $k_1$ and $k_2$, were alike we would have five additional types.

(\(\Delta\) still positive and \(\neq 0\))

Type 6. \[ k_1 (x_1^2 + x_2^2) + k_3 x_3^2 + k_4 x_4^2 + \Delta/ k_1 k_2 k_3 k_4 = 0 \]

Type 7. \[ k_1 (x_1^2 + x_2^2) + k_3 x_3^2 - k_4 x_4^2 - \Delta/ k_1 k_2 k_3 k_4 = 0 \]

Type 8. \[ k_1 (x_1^2 + x_2^2) - k_3 x_3^2 - k_4 x_4^2 + \Delta/ k_1 k_2 k_3 k_4 = 0 \]

Type 9. \[ -k_1 (x_1^2 + x_2^2) - k_3 x_3^2 - k_4 x_4^2 + \Delta/ k_1 k_2 k_3 k_4 = 0 \]

Type 10. \[ -k_1 (x_1^2 + x_2^2) + k_3 x_3^2 - k_4 x_4^2 - \Delta/ k_1 k_2 k_3 k_4 = 0 \]
Case 3. If in case 1, three of the roots say \( k_1, k_2, k_3 \), were alike we would obtain four additional cases \((\Delta \text{ still positive and } \neq 0)\)

Type 11. \( k_1(x_1^2 + x_2^2 + x_3^2) + k_4x_4^2 + \Delta/k_1^2k_4 = 0 \)

Type 12. \( k_1(x_1^2 + x_2^2 + x_3^2) - k_4x_4^2 - \Delta/k_1^2k_4 = 0 \)

Type 13. \( -k_1(x_1^2 + x_2^2 + x_3^2) - k_4x_4^2 + \Delta/k_1^2k_4 = 0 \)

Type 14. \( -k_1(x_1^2 + x_2^2 + x_3^2) + k_4x_4^2 - \Delta/k_1^2k_4 = 0 \)

Case 4. If in case 1, all the roots are alike we would obtain two types \((\Delta > 0, \Delta \neq 0)\)

Type 15. \( k_1 \sum_{i=1}^{4} x_i^2 + \Delta/k_1^4 = 0 \)

Type 16. \( -k_1 \sum_{i=1}^{4} x_i^2 + \Delta/k_1^4 = 0 \)

Case 5. If in the four cases above \( \Delta \) were zero, each of the types would become a cone (hypercone) with vertex at the origin.

If \( \Delta \) is negative we obtain in general, the same surfaces but differently situated with respect to the axis. In the tabulation at the end of the section this possible change will be noted.

Case 6. If three of the roots are all distinct and \( \neq 0 \) while one is zero say \( k_4 \), we obtain the types

Type 17. \( k_1x_1^2 + k_2x_2^2 + k_3x_3^2 + 2a_{145}x_4 = 0 \)

Type 18. \( k_1x_1^2 + k_2x_2^2 - k_3x_3^2 + 2a_{145}x_4 = 0 \)

Type 19. \( k_1x_1^2 - k_2x_2^2 - k_3x_3^2 + 2a_{145}x_4 = 0 \)

Type 20. \( -k_1x_1^2 - k_2x_2^2 - k_3x_3^2 + 2a_{145}x_4 = 0 \)

where \( a_{145} = (-\Delta/k_1k_2k_3)^{1/2} \) allowing for the change of sign in the \( k_i \)'s. If any of the roots of the quartic are equal the above types will show surfaces of revolution in the projecting cylinders as we discussed then in cases 1-4.
Case 7. If in case 6, \( k_4 = 0 \) and \( \Delta = 0 \), we could determine the type forms if we could evaluate the indeterminate form \( \Delta / k_1 k_2 k_3 k_4 = 0/0 \). The method of evaluating the indeterminate form is developed in section 8, article 27. The type forms resulting, when \( C \) is the constant obtained from the evaluation of \( \Delta / k_1 k_2 k_3 k_4 = 0/0 \) are

- **Type 21.** \( k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2 + C = 0 \)
- **Type 22.** \( k_1 x_1^2 + k_2 x_2^2 - k_3 x_3^2 + C = 0 \)
- **Type 23.** \( k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 + C = 0 \)
- **Type 24.** \( -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 + C = 0 \)

Again, if any of the \( k_i \)'s are equal, the projections or sections will be surfaces of revolution in \( S_3 \).

Case 8. When three roots of the discriminating quartic are zero, two of the diametral hyperplanes are undetermined since the third order determinant \( |a_{11} \ a_{22} \ a_{33}| \) must vanish in this case.

A short study of the discriminant \( \Delta \) shows that if we choose \( k_4 = k_3 = 0 \), that by making six successive two-dimensional rotations (if all the coefficients of the second degree terms in the products of the variables taken two at a time, are not zero) we can simplify the equation so that \( \Delta \) now becomes,
after two translations to remove two of the linear terms

\[ \Delta = \begin{vmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a'_{35} \\ 0 & 0 & 0 & 0 & a'_{45} \\ 0 & 0 & a_{35} & a'_{45} & 0 \end{vmatrix} \]

where \( \Delta \) was originally

\[ \begin{vmatrix} a_{11} & \cdots & \cdots & \cdots & a_{15} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{51} & \cdots & \cdots & \cdots & a_{55} \end{vmatrix} \]

Our equation accordingly has been reduced to the form

\[ k_1 x_1^2 + k_2 x_5^2 + 2a'_{35} x_3 x_5 + 2a'_{45} x_4 x_5 = 0 \]

where \( k_1 \) and \( k_2 \) are the roots of the discriminating quartic.

If both \( a_{35} \) and \( a_{45} \) are zero we have a two dimensional cylinder, if either \( a_{35} \) or \( a_{45} \) are zero but not both zero, we have a three dimensional quadric cylinder in \( S_4 \).

Using then \( k_1 \) and \( k_2 \) as the absolute values of the roots of the quartic we can distinguish the types

Type 25. \( k_1 x_1^2 + k_2 x_5^2 + 2a'_{35} x_3 + 2a'_{45} x_4 = 0 \)

Type 26. \( k_1 x_1^2 - k_2 x_5^2 + 2a'_{35} x_3 + 2a'_{45} x_4 = 0 \)

Type 27. \( -k_1 x_1^2 - k_2 x_5^2 + 2a'_{35} x_3 + 2a'_{45} x_4 = 0 \).
Case 9. If three roots of the discriminating quartic are zero, the second degree terms for a perfect square and the equation of the surface takes the form

\[ \sum_{i=1}^{4} (a_{1i}x_i + \beta)^2 + 2 \sum_{i=1}^{4} (a_{15} - a_{1\beta})x_i + a_{55} - \beta^2 = 0 \]

...........(53)

If the hyperplanes

\[ \sum_{i=1}^{4} a_{1i}x_i + \beta = 0 \]
\[ 2 \sum_{i=1}^{4} (a_{15} - a_{1\beta})x_i + a_{55} + \beta^2 = 0 \]

...........(54)

are not parallel, we may choose \( \beta \) so that they are perpendicular. Taking these hyperplanes, when \( \beta \) is so chosen as the \( x_1 = 0 \) and \( x_\alpha = 0 \) hyperplanes, the equation reduces to

Type 34. \[ \sum_{i=1}^{4} a_{1i}^2x_i^2 + 2 \left[ \sum_{i=1}^{4} (a_{15} - a_{1\beta})^2 \right]^{1/2}x_\alpha = 0 \ldots ...(55) \]

If the hyperplanes are parallel \( \beta \) may be so chosen that \( (a_{15} - a_{1\beta}) = 0 \) \( (i=1,2,3,4) \) and the equation becomes

Type 35. \[ \sum_{i=1}^{4} a_{1i}^2x_i^2 + a_{55} - \beta^2 = 0 \ldots \ldots \ldots \ldots (56) \]

where \[ \sum_{i=1}^{4} a_{1i}x_i + \beta = 0 \] is the new \( x_1 = 0 \).
Characterization of Type Forms.

We shall characterize the type forms by their section in the various co-ordinate $S_3$'s forming the co-ordinate System. In our tabulation we shall use the following symbols.

- $E =$ Real Ellipsoid
- $E_i =$ Imaginary Ellipsoid
- $H_1 =$ Hyperboloid of one sheet
- $H_2 =$ Hyperboloid of two sheets
- $S_i =$ Imaginary Sphere
- $S =$ Sphere
- $EP =$ Elliptic Paraboloid
- $HP =$ Hyperbolic Paraboloid
- $C_r =$ Real Quadric Cone
- $C_i =$ Imaginary Quadric Cone
- $EC =$ Elliptic Cylinder
- $HC =$ Hyperbolic Cylinder
- $PC =$ Parabolic Cylinder
- $E_i C =$ Imaginary Elliptic Cylinder
- $2p =$ Two planes.

If an $R$ is placed after any of these symbols it indicates that the surface is a surface of revolution in that $S_3$. 
### 32.(ii)

**Characterization of Type Forms**

<table>
<thead>
<tr>
<th>Type</th>
<th>$x_1 = k$</th>
<th>$x_2 = k$</th>
<th>$x_3 = k$</th>
<th>$x_4 = k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$\Delta &gt; 0$</td>
<td>$E_1$</td>
<td>$E_1$</td>
<td>$E_1$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &lt; 0$</td>
<td>$E$</td>
<td>$E$</td>
<td>$E$</td>
</tr>
<tr>
<td>2.</td>
<td>$\Delta &gt; 0$</td>
<td>$H_1$</td>
<td>$H_1$</td>
<td>$H_1$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &lt; 0$</td>
<td>$H_2$</td>
<td>$H_2$</td>
<td>$H_2$</td>
</tr>
<tr>
<td>3.</td>
<td>$\Delta &gt; 0$</td>
<td>$H_1$</td>
<td>$H_1$</td>
<td>$H_2$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &lt; 0$</td>
<td>$H_1$</td>
<td>$H_1$</td>
<td>$H_2$</td>
</tr>
<tr>
<td>4.</td>
<td>$\Delta &gt; 0$</td>
<td>$E_1$</td>
<td>$E_2$</td>
<td>$H_1$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &lt; 0$</td>
<td>$E$</td>
<td>$H_1$</td>
<td>$E_1$</td>
</tr>
<tr>
<td>5.</td>
<td>$\Delta &gt; 0$</td>
<td>$E$</td>
<td>$E_2$</td>
<td>$E_1$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &lt; 0$</td>
<td>$E_1$</td>
<td>$E_1$</td>
<td>$E_1$</td>
</tr>
<tr>
<td>6.</td>
<td>$\Delta &gt; 0$</td>
<td>$E$</td>
<td>$E_2$</td>
<td>$E_1R$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &lt; 0$</td>
<td>$E$</td>
<td>$E$</td>
<td>$ER$</td>
</tr>
<tr>
<td>7.</td>
<td>$\Delta &gt; 0$</td>
<td>$H_1$</td>
<td>$H_1$</td>
<td>$H_1R$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &lt; 0$</td>
<td>$H_2$</td>
<td>$H_2$</td>
<td>$H_2R$</td>
</tr>
<tr>
<td>8.</td>
<td>$\Delta &gt; 0$</td>
<td>$H_1$</td>
<td>$H_1$</td>
<td>$H_2R$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &lt; 0$</td>
<td>$H_2$</td>
<td>$H_2$</td>
<td>$H_1R$</td>
</tr>
<tr>
<td>9.</td>
<td>$\Delta &gt; 0$</td>
<td>$E$</td>
<td>$E$</td>
<td>$ER$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &lt; 0$</td>
<td>$E_1$</td>
<td>$E_1$</td>
<td>$E_1R$</td>
</tr>
<tr>
<td>10.</td>
<td>$\Delta &lt; 0$</td>
<td>$H_1$</td>
<td>$H_1$</td>
<td>$ER$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &lt; 0$</td>
<td>$E_1R$</td>
<td>$E_1R$</td>
<td>$E_1R$</td>
</tr>
<tr>
<td>11.</td>
<td>$\Delta &lt; 0$</td>
<td>$ER$</td>
<td>$ER$</td>
<td>$ER$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &gt; 0$</td>
<td>$H_1R$</td>
<td>$H_1R$</td>
<td>$H_1R$</td>
</tr>
<tr>
<td>12.</td>
<td>$\Delta &lt; 0$</td>
<td>$H_2R$</td>
<td>$H_2R$</td>
<td>$H_2R$</td>
</tr>
<tr>
<td>13.</td>
<td>$\Delta &gt; 0$</td>
<td>$ER$</td>
<td>$ER$</td>
<td>$ER$</td>
</tr>
<tr>
<td></td>
<td>$\Delta &lt; 0$</td>
<td>$E_1R$</td>
<td>$E_1R$</td>
<td>$E_1R$</td>
</tr>
<tr>
<td>Type</td>
<td>$x_1 = k$</td>
<td>$x_2 = k$</td>
<td>$x_3 \leq k$</td>
<td>$x_4 = k$</td>
</tr>
<tr>
<td>------</td>
<td>-----------</td>
<td>-----------</td>
<td>---------------</td>
<td>-----------</td>
</tr>
<tr>
<td>14.</td>
<td>$H_2R$</td>
<td>$H_2R$</td>
<td>$H_2R$</td>
<td>$S_1$</td>
</tr>
<tr>
<td>$&gt;0$</td>
<td>$&lt;0$</td>
<td>$H_1R$</td>
<td>$H_1R$</td>
<td>$S$</td>
</tr>
<tr>
<td>15.</td>
<td>$S_i$</td>
<td>$S_i$</td>
<td>$S_i$</td>
<td>$S_i$</td>
</tr>
<tr>
<td>$&gt;0$</td>
<td>$&lt;0$</td>
<td>$S$</td>
<td>$S$</td>
<td>$S$</td>
</tr>
<tr>
<td>16.</td>
<td>$S_i$</td>
<td>$S_i$</td>
<td>$S_i$</td>
<td>$S_i$</td>
</tr>
<tr>
<td>$&gt;0$</td>
<td>$&lt;0$</td>
<td>$S$</td>
<td>$S$</td>
<td>$S$</td>
</tr>
<tr>
<td>18.</td>
<td>HP</td>
<td>HP</td>
<td>HP</td>
<td>$C_r$</td>
</tr>
<tr>
<td>19.</td>
<td>EP</td>
<td>HP</td>
<td>HP</td>
<td>$C_r$</td>
</tr>
<tr>
<td>21.</td>
<td>$C&gt;0$</td>
<td>EC</td>
<td>EC</td>
<td>$E_i$</td>
</tr>
<tr>
<td>$C&lt;0$</td>
<td>$E_1C$</td>
<td>$E_1C$</td>
<td>$E_1C$</td>
<td>$E$</td>
</tr>
<tr>
<td>22.</td>
<td>$H_1C$</td>
<td>$H_1C$</td>
<td>$E_1C$</td>
<td>$H_2$</td>
</tr>
<tr>
<td>$C&gt;0$</td>
<td>$C&lt;0$</td>
<td>$H_1C$</td>
<td>$H_1C$</td>
<td>$H_2$</td>
</tr>
<tr>
<td>23.</td>
<td>$E_1C$</td>
<td>$E_1C$</td>
<td>$E_1C$</td>
<td>$E_1$</td>
</tr>
<tr>
<td>$C&gt;0$</td>
<td>$C&lt;0$</td>
<td>$E_1C$</td>
<td>$E_1C$</td>
<td>$E_1$</td>
</tr>
<tr>
<td>24.</td>
<td>$E_1C$</td>
<td>$E_1C$</td>
<td>$E_1C$</td>
<td>$E_1$</td>
</tr>
<tr>
<td>$C&gt;0$</td>
<td>$C&lt;0$</td>
<td>$E_1C$</td>
<td>$E_1C$</td>
<td>$E_1$</td>
</tr>
<tr>
<td>25.</td>
<td>PC</td>
<td>PC</td>
<td>PC</td>
<td>PC</td>
</tr>
<tr>
<td>26.</td>
<td>PC</td>
<td>PC</td>
<td>HP</td>
<td>HP</td>
</tr>
<tr>
<td>27.</td>
<td>PC</td>
<td>PC</td>
<td>EP</td>
<td>EP</td>
</tr>
<tr>
<td>28.</td>
<td>PC</td>
<td>PC</td>
<td>$2p$</td>
<td>-</td>
</tr>
<tr>
<td>29.</td>
<td>PC</td>
<td>PC</td>
<td>$2p$</td>
<td>-</td>
</tr>
</tbody>
</table>

Other types are trivial.
Section 6.

Generation of Hyperquadrics in $S_4$.

Article 23. Methods. When studying the hyperquadric in $S_4$ it was quite natural that we should inquire about methods of generating the surface. The method that first suggested itself was that of extending some of the loci problems of two and three dimensional space to the analogous problem in four-space. Both special and general surfaces resulted.

(1). The first method was that of finding the locus of a point moving equidistant from a point $P$ as $(0,0,0,a)$ and from an $S_3$ as $x_4 = -a$. This locus is

$$x_1^2 + x_2^2 + x_3^2 - 4ax_4 = 0$$

(2). The locus of a point moving so that its distance from a point say $(-ae,0,0,0)$ and its distance from a given $S_3$, say $x_1 = -a/e$ are in a constant ratio $e$, we obtain

$$\frac{x_1^2}{a^2} + \frac{x_2^2 + x_3^2 + x_4^2}{a^2(1 - e^2)} = 1$$

(3). (a) The locus of a point moving such that the sum of the squares of its distances from the points $(ia,0,0,0)$ be a constants resulted in a hypersphere.
(b) If we had asked for the difference instead of the sum in (a) we would have obtained
\[ x_1 = C/-4a \]

(c) If \( d_1 \) and \( d_0 \) designate the respective distances in (a) and we require \( kd_1 + ld_0 = C \) we obtain a hypersphere where \( k \) and \( l \) are real numbers.

(d) If in (a) we had taken the eight points symmetric about the origin we would also have obtained a hypersphere.

(4) (a) The locus of the point moving such that the sum of its distances from two \((\pm a, 0, 0, 0)\) is constantly \( 2k \) gives us the form
\[ \frac{x_1^2}{k^2} + \left( x_2^2 + x_3^2 + x_4^2 \right) / (k^2 - a^2) = 1 \]

(b) If the difference is desired in (a) instead of the sum we obtain the same form of the equation.

(5) A. The locus of the point moving so that its distances from a point and a line have a constant ratio gives
\[ (1 - e^2)(x_1^2 + x_2^2 + x_3^2) + x_4^2 - 2ax_1 + a^2 = 0 \]
where the line is \( x_1 = 0, x_2 = 0, x_3 = 0 \) and the point is \((a, 0, 0, 0)\).

This equation can be discussed for the cases where \( e \) is =, >, or < 1.

B. If the line is the general line
\[ \frac{(x_1-a_1)/l_1}{(x_2-a_2)/l_2} = \frac{(x_3-a_3)/l_3}{(x_4-a_4)/l_4} \]
and the fixed point be \((b_i)\) \((i=1,2,3,4,\ldots)\) the equation becomes

\[\sum_{i=1}^{4} (x_i - a_i)^2 - \left[\sum_{i=1}^{4} (x_i - a_i)ight]^2 = \sum_{i=1}^{4} (x_i - b_i)^2\]

which is the equation of a general hyperquadric.

(6). The locus of the point that moves equidistant from the two lines

\[
\begin{align*}
\frac{x_1-a_1}{l_1} &= \frac{x_2-a_2}{l_2} = \frac{x_3-a_3}{l_3} = \frac{x_4-a_4}{l_4} \\
\frac{x_1-b_1}{m_1} &= \frac{x_2-b_2}{m_2} = \frac{x_3-b_3}{m_3} = \frac{x_4-b_4}{m_4}
\end{align*}
\]

is given by the equation

\[
\sum_{i=1}^{4} (x_i-a_i)^2 - (x_i-b_i)^2 - \left[\sum_{i=1}^{4} (x_i-a_i)ight] \cdot \left[\sum_{i=1}^{4} (x_i-b_i)ight] = 0
\]

which is the equation of a general hyperquadric.

(7). The locus of a point moving so that the distance \(d_1\) from a point say \(P=(a,o,o,o,..)\) and the distance \(d_2\) from an \(S_3\) say \(x_1 = 0\) have a constant ratio \(e\), so that

\[
d_1 = ed_2,
\]

is represented by the equation

\[
x_1^2(1-e^2) + x_2^2 + x_3^2 + x_4^2 - 2ax_1 + a^2 = 0
\]

(8) The locus of a point moving such that the sum of the squares of its distances from two \(S_3\)'s be constant is a hyperquadric surface. If we take the hyperplanes as

\[
\sum_{i=1}^{4} x_i \cos \alpha_i - p_1 = 0
\]

\[
\sum_{i=1}^{4} x_i \cos \theta_i - p_2 = 0
\]

the equation of the locus becomes

\[
\sum_{i=1}^{4} x_i x_j (\cos \alpha_i \cdot \cos \alpha_j + \cos \theta_i \cdot \cos \theta_j) + \sum_{i=1}^{4} 2x_i (p_1 \cos \alpha_i + p_2 \cos \theta_i) + p_1^2 + p_2^2 = 0
\]
(9). If we require that a line move such that four fixed points on it remain, one in each of the four $S_3's$ forming the rectangular co-ordinate system, and if the points are $A_1, A_2, A_3, A_4$, with distances $a_1, a_2, a_3, a_4$, respectively from $P(x_1, x_2, x_3, x_4)$, the locus of $P$ becomes

$$\sum x_i^2/a_i^2 = 1$$

(10). This is the method of two related projective pencils of hyperplanes.

Let $A_x = \sum a_i x_i = 0 \ (i=1,2,3,4,5)$ and similarly $B_x = 0, D_x = 0, E_x = 0$ be the equations of four hyperplanes. Let us write the two pencils of hyperplanes

$$A_x + mB_x = 0$$
$$D_x + nE_x = 0$$

and require that they be related thus

$$m = (rn + s)/(tn + v) \ \text{where} \ rv-st \neq 0 \ \text{then}$$

$$A_x + (rn + s)/(tn + v) B_x = 0 \ \text{and this gives}$$

$$n = -(vA_x + sB_x)/(tA_x + rB_x).$$

From the second pencil we obtain the quadratic in $x_i$

$$D_x (tA_x + rB_x) - (vA_x + sB_x)E_x = 0$$

Now if we denote this surface by $Q$ and differentiate with respect to the $x_i$ in turn we obtain the four equations of the form

$$\delta Q/\delta x_i = A_x(td_i + ve_i) + B_x(rd_i - se_i) + D_x(a_i t + rb_i)$$

$$- E_x(a_i v + sb_i) = 0 \ \ (i=1,2,3,4,).$$

These diametral hyperplanes intersect in a point
which is common to the four given hyperplanes, and since the point satisfies the given hyperplanes, it also satisfies the hyperquadric. Hence the hyperquadric is a cone (hypercone). It is also quite evident that there are $S_9$'s on this hypercone.

(11). This method is closely related to that in (10). Let there be two planes $P_1$ and $P_2$ and a line $P_3$ given by the equations

$$P_1 = \begin{cases} \sum a_i x_i = 0 \\ \sum b_i x_i = 0 \\ \sum c_i x_i = 0 \end{cases} \quad P_2 = \begin{cases} \sum c_i x_i = 0 \\ \sum d_i x_i = 0 \end{cases}$$

$$P_3 = \begin{cases} \sum f_i x_i = 0 \\ \sum g_i x_i = 0 \end{cases}$$

Let $(y)$ be a point in $P_3$, then $\sum e_i y_i = 0$, $\sum f_i y_i = 0$, $\sum g_i y_i = 0$ and form the equation $\sum e_i x_i + k \sum b_i x_i = 0$. \[ \cdots \cdots \cdots (58) \]

and let $k = -\frac{\sum e_i y_i}{\sum b_i y_i}$ by requiring the pencil (58) to pass through the point $(y)$. The we have

$$\sum a_i x_i \sum b_i y_i - \sum a_i x_i \sum b_i y_i = 0 \quad \cdots \cdots \cdots \cdots (59)$$

$$\sum c_i x_i \sum d_i x_i - \sum c_i x_i \sum d_i x_i = 0 \quad \cdots \cdots \cdots \cdots (60)$$

where (59) in the $S_9$ determined by $(y)$ and $P_1$ and (60) is the $S_9$ determined by $(y)$ and $P_2$.

If now we write $X_1 = \sum a_i x_i = 0$, $X_2 = \sum b_i x_i = 0$, $X_3 = \sum c_i x_i = 0$, $X_4 = \sum d_i x_i = 0$, and on eliminating $(y)$ between these equations
\[
\begin{vmatrix}
X_1b_1 - X_2a_1 & b_2X_1 - a_2X_2 & b_3X_1 - a_3X_2 & b_4X_1 - a_4X_2 & b_5X_1 - a_5X_2 \\
X_3d_1 - X_4c_1 & d_2X_3 - a_2X_4 & d_3X_3 - a_3X_4 & d_4X_3 - a_4X_4 & d_5X_3 - a_5X_4 \\
e_1 & e_2 & e_3 & e_4 & e_5 \\
f_1 & f_2 & f_3 & f_4 & f_5 \\
g_1 & g_2 & g_3 & g_4 & g_5
\end{vmatrix}
= 0
\]

which yields a second degree equation in \( x_1 \) and its locus is a hyperquadric surface.

We can show that this surface is a hypercone by the same argument used in (10)
Section 7.

Tangent Hyperplanes and Polar Hyperplanes.

Article 24. Tangent Lines and Tangent Hyperplanes.

If, when the equation of the hyperquadric \( F(x_1, x_2, x_3, x_4) = 0 \) (see Art 16) is solved with the line \( x_1 = y_1 + l_1r \) through the point \( P_0(y_1) \) we require that the point \( P_0 \) lie on the surface, it is necessary that both roots of the equation (41) vanish. Since \( F(y_1, y_2, y_3, y_4) = 0 \) because \( P_0 \) is to lie on the surface, we need only to require that (eq 42)

\[
R = \frac{1}{2} \sum_{i=1}^{5} \frac{\partial F}{\partial y_i} l_i = 0 \quad \text{..........................(61)}
\]

If from the equation of the line we substitute the values of \( l_i \) in (61) we obtain

\[
\sum_{i=1}^{5} (x_i - y_1) \frac{\partial F}{\partial y_i} = 0 \quad \text{..........................(62)}
\]

which must be satisfied by the co-ordinates of every point of every line tangent to the surface at \( P_0 \). Conversely, if \( (x_i) \) is any point distinct from \( P_0 \) whose co-ordinates satisfy \( S \) in (42), the line determined by \( (x_i) \) and \( P_0 \) is tangent to the surface at \( P_0 \).

Since (62) is of the first degree in \( (x_i) \) its equation represents a hyperplane and is called the tangent hyperplane at \( P_0 \).

The equation (62) of the tangent hyperplane may be simplified by multiplying out the equivalents of the
partial derivatives and adding $a_{15}y_1 + a_{25}y_2 + a_{35}y_3 + a_{45}y_4 + a_{55}y_5$ to each side of the equation and transposing the constant term to the right hand side. Then we obtain

$$
\sum_{i=1}^{5} a_{i1}x_1y_i + \sum_{i=1}^{5} a_{i1}j(x_iy_j + x_jy_i) + \sum_{i=1}^{5} a_{15}x_5y_5(x_i + y_i) = 0
$$

...........(63)

Article 25. Polar Hyperplanes. Two points $P$ and $R$ are said to be conjugate with regard to a hyperquadric surface when they are divided harmonically by the points where the line connecting them meets the surface. If through $P$ and $R$ a line is drawn intersecting the surface in two points $Q$ and $S$, the line given by the equations, $x_i = z_i + ky_i$ ($i=1,2,3,4,5$), where $(y_i)$ and $(z_i)$ are the co-ordinates of $P$ and $R$ respectively. Solving this line with the hyperquadric (39) and requiring the the cross-ratio $(PQ,RS)$ be -1, i.e., the condition that the roots in $k$, $k_1 = k_2$, we obtain

$$\sum_{i=1}^{5} a_{i1}x_1y_i = 0 \quad .................(65)$$

This equation is linear in $x_1$ and is called the polar hyperplane. It is the locus of $R$. 
Article 26. Relations of Hyperplanes and Planes to a Hyperquadric Surface.

If we have a tangent hyperplane to the hyperquadric, each given by the respective equations

\[ \sum u_i x_i = 0 \]  \hspace{1cm} \text{(66)}

\[ \sum a_{ij} x_i x_j = 0 \]  \hspace{1cm} \text{(67)}

If \((y_1)\) be the point of contact of (66) then

\[ \sum a_{ij} x_i y_j = 0 \]  \hspace{1cm} \text{(68)}

the equation of the polar hyperplane of \((y_1)\) coincides with the tangent hyperplane, which algebraically, states that the coefficients of the two linear equations (66) and (68) are proportional and

\[ \sum a_{ij} y_j - ku_1 = 0 \]

\[ \sum a_{ij} y_j - ku_2 = 0 \]

\[ \sum a_{ij} y_j - ku_3 = 0 \]  \hspace{1cm} \text{(69)}

\[ \sum a_{ij} y_j - ku_4 = 0 \]

\[ \sum a_{ij} y_j - ku_5 = 0 \]

and if \((y_1)\) lies on the hyperplane (66) as we stated in the hypothesis then

\[ \sum u_i y_i = 0 \]  \hspace{1cm} \text{(70)}

If the hyperquadric is of the cone type, (66) will not be a true tangent hyperplane. If \((y_1)\) is the vertex and since it lies on (66) equation is fulfilled. Since equation (68) is identically zero, equations (69) will be fulfilled if we let \(k = 0\). Thus if (66) is a
tangent plane to (67) there exists six constants $y_1$, $y_2$, $y_3$, $y_4$, $y_5$, $k$ of which the first five are not all zero and which satisfy the equations (69) and (70) i.e., the condition is fulfilled

\[
\begin{vmatrix}
  a_{11} & \cdots & a_{15} & u_1 \\
  \vdots & \ddots & \vdots & \vdots \\
  \vdots & & \ddots & \vdots \\
  a_{51} & \cdots & a_{55} & u_5 \\
  u_1 & u_2 & u_3 & u_4 & u_5 & 0
\end{vmatrix} = 0 \quad \text{(71)}
\]

The converse of (71) can readily be shown so that we can state the proposition: Equation (71) is the necessary and sufficient condition that the hyperplane (66) be tangent to the hyperquadric (67).

Now if the three-fold (66) and that given by the equation

\[
\sum_{i} y_{i} x_{i} = 0 \quad \text{(72)}
\]

intersect in a plane (5$\gamma$) and if this plane be tangent to the hyperquadric (67) then there is a point ($y_{i}$) ($i=1,2,3,4,5$) the point of contact, such that its polar (68) contains the plane. Then it must be possible to write the equation of this polar plane

\[
\sum_{i} (mu_{i} + nv_{i}) x_{i} = 0 \quad \text{(73)}
\]

Now we may choose $m$ and $n$ so that the coefficients of (73) may be proportional to the coefficients of (68),
and further, the coefficients of (73) can be equal to those of "68) so that

\[ \sum a_{ij}y_j - mu_i - nv_i = 0 \]

\[ \sum a_{2j}y_j - mu_2 - nv_2 = 0 \]

\[ \sum a_{3j}y_j - mu_3 - nv_3 = 0 \]

\[ \sum a_{4j}y_j - mu_4 - nv_4 = 0 \]

\[ \sum a_{5j}y_j - mu_5 - nv_5 = 0 \]

\( j=1,2,3,4,5, \)

But \( (y_1) (i=1,2,3,4,5,) \) lie both on \((66( and (72)\)

so that

\[ \sum u_i y_i = 0 \]

\[ \sum v_i y_i = 0 \]

Since we have found seven constants, \( (y_1) (i=1, 2,3,4,5,) \) and \( m, n, \) not all zero, then surely

\[
\begin{vmatrix}
  a_{11} & \cdots & a_{15} & u_1 & v_1 \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  a_{51} & \cdots & a_{55} & u_5 & v_5 \\
  u_1 & u_2 & u_3 & u_4 & u_5 & 0 & 0 \\
  u_1 & u_2 & u_3 & u_4 & u_5 & 0 & 0
\end{vmatrix} = 0
\]

This equation was deduced on the supposition that the plane of intersection of \((66) \) and \((72) \) is a true tangent to \((67) . \) But in the same manner as we argued for equation \((71) \) for the case of a pseudo tangent we can prove \((76) \) holds if the plane is a
pseudo tangent or a ruling of (67). The converse is also quite evident. Our proposition then becomes:

The necessary and sufficient condition that the plane of intersection of the hyperplane (66) and (72) be a tangent plane or ruling of (67) is that (76) be fulfilled.
Section 8.

Some Theorems, Proofs, and Numerical Examples.

Article 27. An Indeterminate Form Evaluated.

In case 7 Article 22 Section 5 a certain indeterminate form was mentioned: \( \frac{\Delta}{k_1 k_2 k_3 k_4} \) where \( \Delta = 0 \) and \( k_4 = 0 \). To evaluate this form we take recourse to a method of differential calculus.

It is quite evident that if we insert a \( \delta \) in any one of the elements \( a_{11}, a_{22}, a_{33}, a_{44} \), of \( \Delta \) or \( k_1, k_2, k_3, k_4 \), of the transformed \( \Delta \), that neither the \( \Delta \) nor \( A_{55} \) will vanish but yet approach zero as \( \delta \) approaches zero. Since we can choose any of the \( k_1 \)'s to be the root that is zero let us choose \( k_4 \) as that root and accordingly insert \( \delta \) in the position of this element. The we have

\[
\Delta = \begin{vmatrix}
 a_{11} & \cdots & \cdots & \cdots & a_{15} \\
 \cdots & a_{22} & \cdots & \cdots & \cdots \\
 \cdots & \cdots & a_{33} & \cdots & \cdots \\
 \cdots & \cdots & \cdots & a_{44} + \delta & \cdots \\
 a_{51} & \cdots & \cdots & \cdots & a_{55}
\end{vmatrix}
\]
To establish the validity of this limit we must show that the denominator cannot vanish. From the conditions that one root of the discriminating quartic be zero we have $J_2 \neq 0$ (See eq 48 pg 24). But since $A_{55} = 0$ we can write

\[ \Sigma_l a_{1l} = a_{14} \quad (i=1, 2, 3) \]
\[ \Sigma_l a_{2l} = a_{24} \]
\[ \Sigma_l a_{3l} = a_{34} \]
\[ \Sigma_l a_{4l} = a_{44} \]

From these relations we find that

\[ J_2 = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\cdot 1) |a_{11} a_{22} a_{33}| \neq 0 \]

whence, our limit must exist when $\sigma$ is placed in the position of $a_{44}$. Likewise it can be shown that if $\sigma$ is placed in any of the other positions our result is still true.

By the same token we can prove that this method will
not work for the case of two roots zero for then \( J_0 = 0 \)
and we have shown that \( |a_{11} a_{22} a_{33}| \) vanishes for this
case (see art 20) and the limit does not exist.

Article 28. Miquel Theorem. In space of four
dimensions, all three dimensional quadrics which pass
through eleven general points are expressible linearly,
when their equations are in point co-ordinates, by four
linearly independent quadrics passing through these points.
All such quadrics therefore pass through five other points.
It is assumed of course that these four quadrics have no
common curve or surface and intersect in sixteen points.

Article 29. Isotropic Lines. Darboux calls the centers
of the two spheres of zero radius which pass through a
circle in \( S_3 \) foci. In like manner, in \( S_4 \), the foci of
a hypersphere may be defined as the centers of two
hyperspheres of zero radius which pass through the sphere.
If two spheres lying in the same \( S_3 \) touch, then the line
joining a focus of one to one of the foci of the other
is an isotropic line, i.e., a line of zero length.

Article 30. On an Hyperquadric In \( S_5 \). A hexahedron is
defined as that configuration formed by six points in a
space of five dimensions where each of the five points
determine a \( S_4 \) and the six points and six \( S_4 \)'s are called
respectively the vertices and faces of the hexahedron.

13. Bateman H. British Association for Advancement of
Science, 80th meet, 1910 pg 532-33
H.W. Richmond proves an interesting theorem: If in $S_5$ a quadric passes through all the vertices of a hexahedron and touches all of its faces, it must touch the sixth face also. This theorem, algebraically, states that if in a symmetrical determinant of order six, the elements in the leading diagonal all vanish and the first minors of five of these elements vanish, the minor of the remaining element must vanish.

Article 31. On the Condition that Five Straight Lines in $S_4$ Should Lie on a Quadric.

A Quadric in $S_5$ has on it an infinite number of $S_9$'s which form two distinct families, an $S_9$ from each family passing through every line that lies on the quadric. Two $S_9$'s from the same family always have a point in common, while two $S_9$'s from different families have no common point at all, generally, but have a line in common.

If we require five lines $a,b,c,d,e$, to lie on a quadric $Q$ in $S_5$ we arrive at the theorem: In order that five lines in $S_4$ should lie on a quadric, it is necessary and sufficient that they should be sections of five $S_9$'s in $S_5$ of which each two have one point in common.

Article 32. Quadric Point Cone. In space of three dimensions we have the two definitions of a cone, the envelope of its tangent lines and the dual, the quadric

15 Ibid vol 10 pg 210-13
cone, consisting of its generating lines passing through its vertex. In particular, a solid lying in a space of four dimensions we may have an ordinary quadric surface. We may treat it as two sets of generating lines, where any line of either set meets every line of the second set. The dual will be two sets of planes all passing through some point V, so that any plane of either set meets every plane of the other set in a line, passing through V, with two planes of the same set meeting in V only. Through every general line lying in a quadric surface in four dimensions there passes two planes, each containing two lines of the quadric surface, two points on this line are those where two of the four generators meet. Dually, an arbitrary plane through V contains two lines passing through V, each of which is the intersection of a plane of one set with a plane of the other set; and there are two solids containing this plane each of which contains two of the four planes so arising. The aggregate of points lying in these sets of planes is called a quadric point cone with vertex V. Just as two intersecting generating lines determine a point of the quadric surface and a tangent plane, so also, two generating planes which meet in a line lie in a solid, the tangent solid of the point cone, the line of contact of this tangent solid with the
point cone being the line of intersection of the planes. There are a double infinity of tangent solids, the line of contact of each passing through the vertex. It is clear that the generating planes of the point cone meet in an arbitrary solid in the generating lines of a quadric surface lying in that solid. Conversely, the planes joining a point, not lying in the solid in which the quadric surface lies, to the generating lines of a quadric surface, generate a point cone.

Article 33. Quadric Line Cone. The dual of the tangent lines of a conic in S₄ consists of the single infinity of planes passing through a line 1, two of these planes lying in an arbitrary solid which contains the line 1. The aggregate of the points of these planes is calles a quadric line cone.

(1) Classify the hyperquadric given by the equation
\[2x_2x_3 + 2x_1x_3 + 2x_1x_2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4 + 1 = 0\]
Here \(\Delta = -3\) and the discriminating quartic is
\[k^4 - 6k^2 + 8k - 3 = 0\]
with roots 1, 1, 1, -3. and the equation becomes
\[3x_1^2 - x_2^2 - x_3^2 - x_4^2 - 1 = 0\]

(2) Classify the surface
\[x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_1x_2 + 2x_1x_4 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4 + 4x_1 + 4x_2 + 4x_3 + 4x_4 + 3 = 0\]
Here \(\Delta = 0\) and the quartic is \(k^4 - 4k^3 = 0\) with roots \(k_1 = k_2 = k_3 = 0\) and \(k_4 = 4\) whence
The surface is composite since
\[(x_1 + x_2 + x_3 + x_4 + 1)(x_1^2 + x_2^2 + x_3^2 + x_4^2 + 3) = 0\]

(3) Classify the surface
\[x_1^2 + x_2^2 + 2x_3x_4 + 4x_3 + 4x_4 + 3 = 0\]
Here \(\Delta = 5\) and the quartic becomes
\[k^4 - 2k^3 + 2k - 1 = 0\]
with roots \(k = 1, 1, 1, -1\), and the surface becomes
\[x_1^2 + x_2^2 + x_3^2 - x_4^2 - 5 = 0\]

(4) Classify the surface
\[x_1^2 + 2x_2x_3 + 4x_4 + 3 = 0\]
Here \(\Delta = 4\) and the quartic is \(k^4 - k^3 - k^2 + k = 0\)
with roots \(k = 0, 1, 1, -1\), then \(a_4 = 4\) and the equation becomes \(x_1^2 + x_2^2 - x_3^2 + 8x_4 = 0\)
(5) Classify the surface

\[ 2yz + 4x + 2y + 4z + 2t + 1 = 0 \]

Here \( \Delta = 0 \) and the quartic becomes \( k^4 - k^2 = 0 \) with roots \( k = 0, 0, 1, -1 \).

Rotating by the formulas

\[
\begin{align*}
x &= x' \\
y &= y'\cos 45 - z'\cos 45 \\
t &= t' \\
z &= y'\sin 45 + z'\cos 45
\end{align*}
\]

we obtain

\[ y'^2 - z'^2 + 3\sqrt{2}y' + \sqrt{2}z' + 4x' + 2t' + 1 = 0 \]

and on completing squares and translating

\[ y'^2 - z'^2 + 4x'' + 2t'' = 0 \]

(6) Classify the surface

\[ x^2 + y^2 + 2yz + 4x + 6y + 5 = 0 \]

This equation has \( \Delta = 0 \) with \( k^4 - 2k^3 + k = 0 \) as the discriminant quartic whose roots are \( 0, 1, -1 \pm \sqrt{5}/2 \).

By the method of article 27

\[
C = \frac{A}{k_1 k_2 k_3 k_4} = \left( \frac{\partial A}{\partial \sigma} \right)_{\sigma = 0} = 10
\]

The equation becomes

\[ x^2 + (1 + \sqrt{5})/2 y^2 + (1 - \sqrt{5})/2 z^2 + 10 = 0 \]
(7) Given in $S_4$ a plane and a line parallel to this plane represented by the equations in homogeneous coordinates.

$$
x_1 = x_3 - 4x_4 + 3x_5 \quad \ldots \ldots S_2
$$

$$
x_2 = 2x_3 - 7x_4 + 4x_5
$$

$$
x_1 = x_4 + 2x_5 \quad \ldots \ldots S_1
$$

$$
x_2 = 3x_4 + 4x_5
$$

$$
x_3 = 5x_4 + 6x_5
$$

Let us determine the equations of the plane, which is the locus of the line containing the distance, and let us find the length of the distance from the line to the plane.

It is immediately seen that the given plane contains the infinite point of the line given by the equation

$$L_1 \equiv \frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{5} = \frac{x_4}{1} = p \quad \text{where } p \text{ is infinite.}
$$

The $S_3$ given by the equation

$$x_1 + kx_2 = (1 + 2k)x_3 - (4 + 7k)x_4 + (3 + 4k)x_5$$

passes through the point $(2,4,6,0,1)$ of the given line for $k = -7/12$ since the line $(S_1)$ may be written in the form

$$\left(\frac{x_1}{1} - 2x_5\right)/1 = \left(\frac{x_2 - 4x_5}{3} = \left(\frac{x_3 - 6x_5}{5} = \frac{x_4}{1}.\right.ight.$$

Also $12x_1 - 7x_2 + 2x_3 - x_4 - 8 = 0$ is the equation of the $S_3$ relating the given elements, i.e., it is the $S_3$ obtained by substituting $k = -7/12$ in $L_1$.

18. Exercises 7, 8, 9, are taken from Schoute, H.P. Mehr-dimensionale Geometrie page 171.
Therefore, 
\[
x_1/12 = x_2/7 = x_3/2 = x_4/-1 \]
is the equation of the perpendicular from 0 to this \( S_3 \). Now each line containing the desired distance while it lies in this space is perpendicular to this normal. Moreover it is perpendicular to the given plane and consequently also to the \( S_3 \) given by 
\[
\begin{vmatrix}
x_1 & x_2 & x_3 & x_4 \\
12 & -7 & 2 & -1 \\
1 & 2 & 1 & 0 \\
-4 & -7 & 0 & 1 \\
\end{vmatrix} = 0 \quad \text{...............} \quad S_3'
\]
which is obtained by choosing the points 
\( x_1 = -4, \ x_2 = -7, \ x_3 = 0, \ x_4 = 1 \) and \( x_5 = 0 \)
\( x_1 = 1, \ x_2 = 2 \ x_3 = 1, \ x_4 = 0 \) and \( x_5 = 0 \)
\( S_3 \) also passes through this perpendicular parallel to the given plane. \( S_3' \) reduces to 
\[
3x_1 + x_2 - 5x_3 + 19x_4 = 0
\]
whence 
\[
x_1/3 = x_2/1 = x_3/-5 = x_4/19 \quad \text{gives the direction of the line containing the distance from 0 to } S_3'. \quad \text{This perpendicular is parallel to the given line.}
\]
The plane 
\[
\begin{cases}
25x_1 = 4x_3 + 5x_4 + 26x_5 \\
25x_2 = 14x_3 + 5x_4 + 16x_5
\end{cases}
\]
through the given line and the infinite point of this normal becomes the locus of the line containing the distance.
And the length of the distance will be found as the distance of the point $(2, 4, 6, 0)$ from the $S_3$

$$3(x_1 - 3) + (x_3 - 4) - 5x_3 + 19x_4 = 0 \quad \cdots \cdots (i)$$

through the point $(3, 4, 0, 0)$ of the given plane perpendicular to the line containing the distance. When the equation (i) is in the normal form we have

$$L = 33/ (3^2 + 1 + 5^2 + 19^2)^{1/2} = 1/2 \sqrt{11}$$

upon substituting $(2, 4, 6, 0)$ for the variables.

(8) Determine the projection of the given line upon the given plane in (7)

From (7) the locus of the line containing the distance is given by the plane:

(i) $25x_1 = 4x_3 + 5x_4 + 26$ \quad \cdots \cdots S_3$

(ii) $25x_2 = 14x_3 + 5x_4 + 16$

The plane given in the problem is

(iii) $x_1 = x_3 - 4x_4 + 3$ \quad \cdots \cdots S_2$

(iv) $x_2 = 2x_3 - 7x_4 + 4$ \quad \cdots \cdots S_2$

From (i) and (ii) $x_1 = x_4 + 2/3$

From (ii) and (iv) $x_2 = 3x_4 - 2/3$ \quad \cdots \cdots S_1$

From (i) and (iii) $x_3 = 5x_4 - 7/3$

Now $S_1'$ is parallel to $S_1$ of exercise (7) and we find that $S_1'$ is the section of the given plane with the locus of the line containing the distance.

(9) Determine the distance between the parallel lines $S_1$ of exercise (7) and $S_1'$ of exercise (8).

Let us write
and then write
\[ \tilde{s} = (x_1-2)/1 = (x_2-4)/3 = (x_3-6)/5 = x_4 \]
\[ S_1' = (x_1-2/3)/1 = (x_2+2/3)/3 = (x_3+7/3)/5 = x_4 \]
and then write \( x_1 + 3x_2 + 5x_3 + x_4 = k \) as the system of \( S_3 \)'s perpendicular to the two parallel lines and passing through \((2,4,6,0)\) which gives us
\[ x_1 + 3x_2 + 5x_3 + x_4 = 4k \] \[ \text{(i)} \]
Now (i) and \( S_1' \) determine the point
\[ (27/12,49/12,67/12,19/12) \] and hence
\[ l^2 = (2-27/12)^2 + (4-49/12)^2 + (6 - 67/12)^2 + (19/12)^2 \]
\[ l = 1/12 \quad 396 = 1/2 \sqrt{11} \]
Article 34, exercise 10.

As a three-dimensional exercise illustrating the generalized theorem of Schoute, mentioned in article 4 let us find the distance from the point \((2,2,2,2)\) to the plane \( x-2y-z+3 = 0 \).

The equation of the normal to the plane through \(0\) is \( x/l = y/-2 = z/-1 \) while the equation of the line along which the distance is measured is
\[ \begin{vmatrix} x-2 & y-2 & z-2 \\ 1 & -2 & -1 \end{vmatrix} = 0 \]
whence
\[ (x-2)/1 = (y-2)/-2 = (z-2)/-1 \] is the equation of the line along which the distance is measured. This line intersects the plane in the point \((13/6,5/3,11/6)\). The distance between the two points is \( \sqrt{12}/12 \).
Bibliography

I. Books referred to in this thesis.


Kowalewski, Dr. Gerhard, "Einführung in die Determinanten Theorie," Leipzig, von Veit, 1909

Schoute, Dr. P. H. "Mehrdimensionale Geometrie" 1. teil, Die linearen Raume, Leipzig, Gosch-sche, 1902

Woods, Frederick, Higher Geometry, New York, Ginn, 1922.

II. Periodical articles referred to in this thesis.

Batemen, H. "Foci of a circle in space and some geometrical theorems connected therewith" British Association for the Advancement of Science, 20th meet, Sheffield, 1910, Pp. 532-33

Richmond, H. W. "Condition that five straight lines situated in S_A lie on a quadric", Proceedings of Cambridge Philosophical Society, Volume 10, pp. 210-13

III. Other books consulted.

IV. Other periodicals consulted.
Coolidge, J. L. Annals of Mathematics, "On the


Richmond, H. W. "Figure formed from Six Points in Space of Four Dimensions", Mathematische Annalen, volume 53, pg 161, Leipzig, Tuebner, 1900


Sommer, J. "Focaleigenschaften Quadratischer Mannigfaltigkeiten im Vier Dimensional Raum" Mathematische Annalen, volume 53, pg 113, Leipzig, Tuebner, 1900.