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On the Green rings of pointed, coserial Hopf algebras

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ON THE GREEN RINGS OF POINTED, COSERIAL HOPF ALGEBRAS

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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ABSTRACT

The Green ring is a powerful mathematical tool used to codify the interactions between representations of groups and algebras. This ring is spanned by isomorphism classes of finite-dimensional indecomposable representations that are added together via direct sums and multiplied via tensor products.

In this thesis, we explore the Green rings of a class of Hopf algebras that form an extension of the Taft algebras. These Hopf algebras are pointed and coserial, meaning their simple comodules are 1-dimensional, and their comodules possess unique composition series respectively. The comodules of these Hopf algebras thus have a particularly well-behaved structure.

We present results giving structure to the comodule Green ring of the Hopf algebra H_s and in particular fully classify the Green rings of H_s where $s \leq 6$. More generally, we classify the indecomposable comodules of H_s and their composition series and prove how the composition series may be used to classify the tensor product of indecomposable comodules.

Additionally, for these Hopf algebras we classify the Grothendieck rings, the subrings of the corresponding Green rings consisting only of isomorphism classes of projective indecomposable comodules. We describe a simpler presentation of these Grothendieck rings and the multiplication in the ring.

PUBLIC ABSTRACT

Representation theory is a powerful branch of mathematics in which abstract mathematical objects are considered in a simpler manner by "representing" them using linear transformations of vector spaces. In particular, representations of Hopf algebras, which have the structure of both algebras and the dual structure of coalgebras can be used to model many different natural phenomena such as the Standard Model of particle physics. Hopf algebra representations are of particular interest because they can be added and multiplied together, creating a ring structure called the Green ring.

This thesis asks what ring structures Green rings may take. We explore the Green ring structures of Hopf algebras that are pointed and coserial, meaning their representation interact in a particularly nice manner. We classify these rings as quotients of integer polynomial rings. In doing so, we learn valuable information about how these Hopf algebra representations behave like more familiar integer polynomials.

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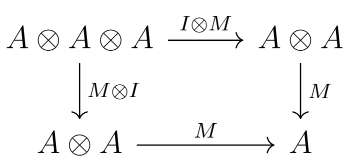
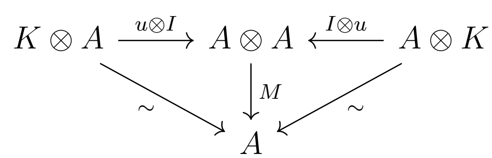
1 Hopf algebras and their representations

1.1 Algebras and coalgebras

Let K be an algebraically closed field with $\text{char}(K) = 0$. All tensor products are taken to be over K.

Definition 1.1. *A K-algebra is a triple (A, M, u) where A is a K-vector space and $M : A \otimes A \to A$ and $u : K \to A$ are K-vector space morphisms called the multiplication and unit maps such that the following diagrams commute:*

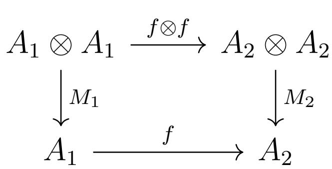


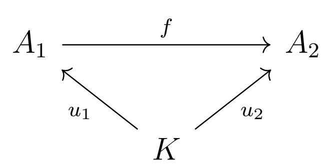


Here the unnamed arrows are the canonical isomorphisms.

Example 1.2. *Consider K as a vector space over itself. K forms a K-algebra with M the field multiplication and u the identity map. We call this algebra the canonical structure of K as a K-algebra.*

Definition 1.3. *Given K-algebras (A_1, M_1, u_1) and (A_2, M_2, u_2), an algebra morphism $f : A_1 \to A_2$ is a K-linear map that makes the following diagrams commute:*





We now dualize the notion of an algebra to produce coalgebras.

Definition 1.4. *A K-coalgebra is a triple (C, Δ, ϵ) where C is a K-vector space and $\Delta : C \to C \otimes C$ and $\epsilon : C \to K$ are K-vector space morphisms called the comultiplication and counit maps such that the following diagrams commute:*

$$
\begin{array}{ccc}
C & \xrightarrow{\quad \Delta \quad} & C \otimes C \\
\big\downarrow{\scriptstyle \Delta} & & \big\downarrow{\scriptstyle I\otimes\Delta} \\
C \otimes C & \xrightarrow{\Delta\otimes I} & C \otimes C \otimes C
\end{array}
\qquad\qquad
\begin{array}{ccccc}
K \otimes C & \xleftarrow{\quad \sim \quad} & C & \xrightarrow{\quad \sim \quad} & C \otimes K \\
 & {\scriptstyle \epsilon\otimes I}\nwarrow & \big\downarrow{\scriptstyle \Delta} & \nearrow{\scriptstyle I\otimes\epsilon} & \\
 & & C \otimes C & &
\end{array}
$$

Again, the unnamed arrows are the canonical isomorphisms.

Example 1.5. *Consider K as a K-vector space. By defining the maps $\Delta(r) = r \otimes 1$ and $\epsilon(r) = 1$ for all $r \in K$, we endow K with the canonical structure of a K-coalgebra.*

Example 1.6. *Let V be a vector space with basis B. Then V forms a coalgebra via the maps $\Delta(b) = b \otimes b$ and $\epsilon(b) = 1$ for all $b \in B$. Thus, we see that any vector space can be formed into a coalgebra.*

Example 1.7. *Let W be a vector space with basis $\{g_i, x_i : i \in \mathbb{N}\}$. We define the maps $\Delta : W \to W \otimes W$ and $\epsilon : W \to K$ by*

$$
\begin{aligned}
\Delta(g_i) &= g_i \otimes g_i \\
\Delta(x_i) &= g_i \otimes x_i + x_i \otimes g_{i+1} \\
\epsilon(g_i) &= 1 \\
\epsilon(x_i) &= 0
\end{aligned}
$$

Then (W, Δ, ϵ) forms a K-coalgebra.

Definition 1.8. *Given K-coalgebras $(C_1, \Delta_1, \epsilon_1)$ and $(C_2, \Delta_2, \epsilon_2)$, a coalgebra morphism $g : C_1 \to C_2$ is a K-linear map that makes the following diagrams commute:*

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\quad g \quad} & C_2 \\
\big\downarrow{\scriptstyle \Delta_1} & & \big\downarrow{\scriptstyle \Delta_2} \\
C_1 \otimes C_1 & \xrightarrow{g\otimes g} & C_2 \otimes C_2
\end{array}
\qquad\qquad
\begin{array}{ccc}
C_1 & \xrightarrow{\quad g \quad} & C_2 \\
 {\scriptstyle \epsilon_1}\searrow & & \swarrow{\scriptstyle \epsilon_2} \\
 & K &
\end{array}
$$

We now introduce new notation to simplify that of the comultiplication map. Given a coalgebra (C, Δ, ϵ) and $c \in C$, with the usual conventions we would write $\Delta(c)$ in the form

$$\Delta(c) = \sum_{i=1}^{n} c_{i_1} \otimes c_{i_2}.$$

However, this double indexing can be quite tedious in longer computations. Using the *Sweedler notation*, we suppress the index "i", leading to the notation

$$\Delta(c) = \sum c_1 \otimes c_2.$$

This provides a simplified way to write long compositions using the comultiplication.

We now introduce the standard representations of algebras and coalgebras. These representations give us a way to describe the actions of algebras and coalgebras on other algebraic objects.

Definition 1.9. *Let (A, M, u) be a K-algebra. Then a left A-module is a pair (B, μ) where B is a K-vector space and $\mu : A \otimes B \to B$ is a morphism of K-vector spaces such that the following diagrams commute:*

$$\begin{array}{ccc} A \otimes A \otimes B & \xrightarrow{I \otimes \mu} & A \otimes B \\ \Big\downarrow{\scriptstyle M \otimes I} & & \Big\downarrow{\scriptstyle \mu} \\ A \otimes B & \xrightarrow{\mu} & B \end{array} \qquad\qquad \begin{array}{ccc} K \otimes B & \xrightarrow{u \otimes I} & A \otimes B \\ & \searrow{\scriptstyle \sim} & \Big\downarrow{\scriptstyle \mu} \\ & & B \end{array}$$

Note 1.10. *One can similarly define a right A-module B with the difference that the map μ now has the form $\mu : B \otimes A \to B$.*

Example 1.11. *Let (A, M, u) be a K-algebra. Then A itself forms an A-module with the map $\mu : A \otimes A \to A$ defined to be the multiplication map M.*

Example 1.12. *Let (A, M, u) be a K-algebra. Then $A \otimes X$ forms an A-module with the map $\mu : A \otimes (A \otimes X) \to A \otimes X$ defined by*

$$\mu(a_1 \otimes a_2 \otimes x) = M(a_1 \otimes a_2) \otimes x.$$

We can dualize the notion of modules to give us comodules over a coalgebra.

Definition 1.13. *Let (C, Δ, ϵ) be a K-coalgebra. A right C-comodule is a pair (M, ρ) where M is a K-vector space and $\rho : M \to M \otimes C$ is a morphism of K-vector spaces such that the following diagrams commute:*

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes C \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle I\otimes\Delta} \\ M \otimes C & \xrightarrow{\rho\otimes I} & M \otimes C \otimes C \end{array} \qquad\qquad \begin{array}{ccc} M & \xrightarrow{\sim} & M \otimes K \\ \downarrow{\scriptstyle \rho} & \nearrow{\scriptstyle I\otimes\epsilon} & \\ M \otimes C & & \end{array}$$

Note 1.14. *Again, we can similarly define a left C-comodule M with the difference that Δ now has the form $\Delta : M \to C \otimes M$.*

Given a right C-comodule M and $m \in M$, we would normally write

$$\rho(m) = \sum_{i=1}^{k} m_{0_i} \otimes m_{1_i}$$

for some $m_{0_i} \in M$ and $m_{1_i} \in C$. However, we may again apply *Sweedler notation* to suppress the index i and instead write this expression in the form

$$\rho(m) = \sum m_0 \otimes m_1.$$

Example 1.15. *Any coalgebra C forms a comodule (left or right) over itself with the comodule map $\rho : C \to C \otimes C$ defined to be the coproduct Δ.*

Example 1.16. *Let C be a coalgebra and X any K-vector space. Then $X \otimes C$ forms* *a right C-comodule via the map $\rho : X \otimes C \to (X \otimes C) \otimes C$ given by*

$$\rho(x \otimes c) = \sum x \otimes c_1 \otimes c_2.$$

Definition 1.17. *Let A be an algebra and (B_1, μ_1) and (B_2, μ_2) be A-modules. An A-module morphism $f : B_1 \to B_2$ is a K-linear map such that the following diagram commutes:*

$$\begin{array}{ccc} A \otimes B_1 & \xrightarrow{I \otimes f} & A \otimes B_2 \\ \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} \\ B_1 & \xrightarrow{\quad f \quad} & B_2 \end{array}$$

Definition 1.18. *Let C be a coalgebra and (M_1, ρ_1) and (M_2, ρ_2) be C-comodules. A C-comodule morphism $g : M_1 \to M_2$ is a K-linear map such that the following diagram commutes:*

$$\begin{array}{ccc} M_1 & \xrightarrow{\quad g \quad} & M_2 \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\ M_1 \otimes C & \xrightarrow{g \otimes I} & M_2 \otimes C \end{array}$$

Definition 1.19. *Let C be a K-coalgebra. Given a (right) C-comodule (M, ρ) and N a subgroup of M, we call (N, ρ) a subcomodule of M if $\rho(N) \subseteq N \otimes C$.*

Let (C, Δ, ϵ) be a K-coalgebra. We will now classify some classes of C-comodules that are of particular importance to us.

Definition 1.20. *A C-comodule M is called simple if the only proper subcomodule of M is 0.*

Definition 1.21. *A C-comodule M is called indecomposable if there is no way to write $M = M_1 \oplus M_2$ where M_1 and M_2 are proper subcomodules of M.*

Note 1.22. *Clearly if M is simple, then M must be indecomposable. However, M may be indecomposable without being simple.*

Definition 1.23. *Let Q be a C-comodule. Then Q is injective if any exact sequence of comodules of the form*

$$0 \longrightarrow Q \longrightarrow M \longrightarrow N \longrightarrow 0$$

splits, meaning M is isomorphic to the direct sum of Q and N.

Definition 1.24. *Let P be a C-comodule. Then P is projective if any exact sequence of comodules of the form*

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$$

splits.

Definition 1.25. *Given a C-comodule M, we call (P, ρ) a projective cover for M if P is a projective C-comodule and $\rho : P \to M$ is a morphism of comodules satisfying the condition that if T is a subcomodule of P such that $P = T + \ker(\rho)$, then $P = T$.*

More intuitively, a projective cover of a comodule M is the best approximation of M by a projective comodule. We note that in the coalgebra structure we study later, the projective and injective comodules considered coincide; in particular, the indecomposable projective comodules are the projective covers for their unique simple subcomodules.

In the following definitions, let (C, Δ, ϵ) be a K-coalgebra and (M, ρ) be a (right) C-comodule. We will describe some properties of M in terms of its subcomodules.

Definition 1.26. *Given a chain of subcomodules of M*

$$M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k,$$

we define the length of this chain to be k.

Definition 1.27. *We define the length of M to be the maximal length of any chain of subcomodules. If no such maximal length exists, we say that M has infinite length.*

Definition 1.28. *We define a composition series of M to be a sequence of subcomodules of M*

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

such that each M_i is a maximal subcomodule of M_{i+1} or equivalently that M_{i+1}/M_i forms a simple comodule. In this case, the comodules M_{i+1}/M_i are called composition factors of M.

We note that in general, a comodule does not necessarily have a composition series, or if it does so, the composition series need not be unique. However, the comodules we will later study all have a unique composition series that will be key to our study.

Definition 1.29. *A coalgebra C is said to be right/left serial if each indecomposable injective right/left comodule is uniserial, meaning it has a unique composition series up to isomorphism. If C is both right and left serial, then we say C is serial.*

Up to this point, the definitions and properties we have given for comodules and modules have been very similar. We will now examine a key property that distinguishes comodules from modules as seen in [5].

Theorem 1.30. *(Fundamental Theorem of Comodules) Let C be a K-coalgebra and M be a right C-comodule. Then any element $m \in M$ must lie in some finite dimensional subcomodule of M.*

Proof. Let $\{c_i\}_{i \in I}$ be a basis for C. Given $m \in M$, we write $\rho(m) = \sum m_i \otimes c_i$ where all but finitely many m_i are zero. Let N be the k-subspace of M spanned by the m_i. Note that N is finite-dimensional. We claim that N in fact forms a subcomodule of M. To see this, for each c_i, we write $\Delta(c_i) = \sum a_{ijk} c_j \otimes c_k$. Due to the commutativity of the first comodule diagram for ρ, we know that for each m_i and c_i,

$$\sum \rho(m_i) \otimes c_i = \sum m_i \otimes \Delta(c_i) = \sum m_i \otimes a_{ijk} c_j \otimes c_k.$$

Hence we must have for any k that $\rho(m_k) = \sum m_i \otimes a_{ijk} c_j \in N \otimes C$. We can thus conclude that N in fact forms a finite-dimensional subcomodule of M. Finally, since we can write $m = (I \otimes \epsilon)\rho(m)$, we know $m \in N$, completing our proof. □

We note that this property does not generally hold for modules over an algebra: for a module N, there may exist $n \in N$ such that n does not lie in any finite dimensional submodule of N.

We now describe some ways to form filtrations of a comodule by building towers of subcomodules that ultimately lead to the whole comodule.

Definition 1.31. *Let M be a right C-comodule. The socle of M, which we denote $s(M)$, is the sum of all of the simple subcomodules of M.*

Definition 1.32. *Given M a C-comodule, we define the Loewy series of M as follows. Let $M_0 = s(M)$. For any $n \geq 0$, we define M_{n+1} to be the subcomodule of M satisfying $s(M/M_n) = M_{n+1}/M_n$.*

By Theorem 1.30, we know that M can be written as the union of all of its subcomodules of finite dimension. Hence, given the terms M_n from its Loewy series, we can write $M = \bigcup_{n \geq 0} M_n$, giving us a filtration of M.

Definition 1.33. *The smallest index k such that $M_k = M$ in a Loewy filtration of M (if it exists) is called the Loewy length of M.*

Definition 1.34. *Given a comodule M, we define the Jacobson radical of M, which we call $J(M)$, to be the intersection of all maximal subcomodules of M (where $N \subset M$ is maximal if M/N is simple).*

Definition 1.35. *Given a coalgebra C over K and subspaces U and V of C, the wedge product of U and V is defined by*

$$U \wedge V = \Delta^{-1}(U \otimes C + C \otimes V).$$

Definition 1.36. *For U a subspace of the coalgebra C, we recursively define the wedge product*

$$\wedge^0 U = 0$$

$$\wedge^1 U = U$$

$$\wedge^n U = (\wedge^{n-1} U) \wedge U$$

Definition 1.37. *For a coalgebra C, the coradical C_0 of C is the sum of all simple subcoalgebras of C. The coradical filtration of C has terms given by $C_n = \wedge^{n+1} C_0$ for $n \geq 0$.*

Note that given a comodule M with a Loewy filtration, we may use the following result from [10] to find a Loewy filtration for subcomodules of M:

Proposition 1.38. *If the comodule M has Loewy filtration $L_0(M) \subset L_1(M) \subset \cdots \subset L_n(M) = M$, then any subcomodule N of M will have Loewy filtration given by*

$$L_k(N) = N \cap L_k(M).$$

Note 1.39. *The coradical filtration of a coalgebra C also gives the Loewy filtration of C as both a left and right C-comodule as seen in [5].*

1.2 Bialgebras and Hopf algebras

We will now discuss mathematical objects that have the structure of both algebras and coalgebras.

Definition 1.40. *A K-bialgebra is a K-vector space H endowed with the structure of both an algebra (H, M, u) and a coalgebra (H, Δ, ϵ) such that Δ and ϵ are H-algebra morphisms.*

We note that Δ being an algebra morphism means that the following two diagrams commute:

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\quad M \quad} & H \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & \\ H \otimes H \otimes H \otimes H & & \downarrow{\scriptstyle \Delta} \\ \downarrow{\scriptstyle I\otimes T\otimes I} & & \\ H \otimes H \otimes H \otimes H & \xrightarrow{M\otimes M} & H \otimes H \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\quad u \quad} & H \\ {\scriptstyle \phi^{-1}}\downarrow & & \downarrow{\scriptstyle \Delta} \\ k \otimes k & \xrightarrow{u\otimes u} & H \otimes H \end{array}$$

where $T : H \otimes H \to H \otimes H$ is the twisting map given by $T(c_1 \otimes c_2) = c_2 \otimes c_1$. We also know that ϵ being an algebra morphism means that the following two diagrams commute:

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\quad M \quad} & H \\ \downarrow{\scriptstyle \epsilon\otimes\epsilon} & & \\ k \otimes k & & \downarrow{\scriptstyle \epsilon} \\ \downarrow{\scriptstyle \phi} & & \\ k & \xrightarrow{\quad I \quad} & k \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\quad u \quad} & H \\ & {\scriptstyle I}\searrow \quad \swarrow{\scriptstyle \epsilon} & \\ & k & \end{array}$$

We note that a bialgebra can be defined in an equivalent manner:

Proposition 1.41. *A K-vector space H endowed with the structure of an algebra (H, M, u) and coalgebra (H, Δ, ϵ) forms a K-bialgebra if and only if M and u are H-coalgebra morphisms.*

Proof. M is a morphism of coalgebras if and only if the first and third of these diagrams commute. Also, u is a morphism of coalgebras if and only if the second and fourth of these diagrams commute. Thus, H forms a K-bialgebra if and only if M and u form H-coalgebra morphisms, meaning all four commutative diagrams are satisfied in the definition.

□

Example 1.42. *With the canonical algebra and coalgebra structures, K forms a K-bialgebra.*

Given an algebra (A, M, u) and coalgebra (C, Δ, ϵ), we define an algebra structure on $\mathrm{Hom}(C, A)$ by defining multiplication in the following way for $f, g \in Hom(C, A)$:

$$(f * g)(c) = \sum f(c_1)g(c_2)$$

where

$$\Delta(c) = \sum c_1 \otimes c_2.$$

Note that the above multiplication is associative and that under this product, the identity element of $\mathrm{Hom}(C, A)$ is $u\epsilon$.

We note that this convolution product (in the case where A is the field K) defines the algebra structure of C^*, the dual of C. If M is a C-comodule, then M forms a C^*-module with module action defined by

$$c^* \cdot m = \sum c^*(c_2)c_1.$$

In fact, we know from [5] that if M is a C-comodule, then N is a subcomodule of M if and only if N forms a C^*-submodule of M. Thus, all of our statements about comodules in fact form statements about modules over C^*.

Definition 1.43. *Given a K-bialgebra H, we call a linear map $S : H \to H$ an* *antipode of H if $S * I = I * S = u\epsilon$, meaning that S is the inverse of the identity map with respect to this convolution product. Using Sweedler's notation, this condition can be written that for $h \in H$,*

$$\sum S(h_1)h_2 = \sum h_1 S(h_2) = \epsilon(h)1$$

Definition 1.44. *A bialgebra H that possesses an antipode is called a Hopf algebra.*

A Hopf algebra is a powerful mathematical tool that has been used in many different branches of mathematics and physics. For instance, Hopf algebras have been studied under the guise of quantum groups in topology. Hopf algebras have even been used to model the Standard Model of particle physics.

Example 1.45. *Let G be a group. We define $K[G]$ to be the K-vector space with basis G and elements formal sums of the form $\sum_{g \in G} \alpha_g g$ where $(\alpha_g)_{g \in G}$ is a family of elements of G with only finitely many nonzero elements. We call $K[G]$ the group algebra where multiplication is taken by linearly extending the map*

$$(\alpha g) \cdot (\beta h) = (\alpha\beta)(gh)$$

for $\alpha, \beta \in K$, $g, h \in G$.

We can also form $K[G]$ into a coalgebra by linearly extending the maps $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ for all $g \in G$. In this way, $K[G]$ forms a bialgebra. The map $S : K[G] \to K[G]$ taken by linearly extending $S(g) = g^{-1}$ then is an antipode, giving $K[G]$ the structure of a Hopf algebra.

Example 1.46. *Let $n \geq 2$ and q be a primitive n^{th} root of unity. We define the algebra structure*

$$H_n(q) = \langle g, h | g^n = 1, h^n = 0, hg = qgh \rangle.$$

We introduce a coalgebra structure and antipode to $H_n(q)$ via the following maps:

$$\Delta(g) = g \otimes g \qquad \epsilon(g) = 1 \qquad S(g) = g^{-1}$$

$$\Delta(h) = 1 \otimes h + h \otimes g \qquad \epsilon(h) = 0 \qquad S(h) = -g^{-1}h$$

These maps give $H_n(q)$ the structure of a Hopf algebra called the Taft algebra. Note that this Hopf algebra has dimension n^2 with basis $\{g^i h^j | 0 \leq i, j \leq n-1\}$.

In the particular case where $n = 2$ and $q = -1$, we get Sweedler's 4-dimensional Hopf algebra

$$H = \langle g, h | g^2 = 1, h^2 = 0, hg = -gh \rangle.$$

Let H be a bialgebra. The dual of H, H^*, can then be considered to be an algebra that is dual in a sense to the coalgebra structure of H. The multiplication on H^* is given by the convolution product described before, namely that given $f^*, g^* \in H^*$, we have that

$$(f^* g^*)(c) = \sum f^*(c_0) g^*(c_1).$$

Let $q \in K$ be invertible. We will now define a q-analog to the standard binomial coefficients known as Gauss polynomials. This q-analog is critical to the study of our Hopf algebras of interest as well as the Taft algebras.

Let $n > 0$ be an integer. We define

$$(n)_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}.$$

We define the q-factorial of n by

$$(n)!_q = (1)_q (2)_q \cdots (n)_q = \frac{(q-1)(q^2-1)\cdots(q^n-1)}{(q-1)^n}$$

and define $(0)!_q = 1$. Now, for $0 \leq k \leq n$, we define the *Gauss polynomials* by

$$\binom{n}{k}_q = \frac{(n)!_q}{(n-k)!_q \, (k)!_q}.$$

We note that for $q = 1$, we have that $(n)_q = n$ and thus that $(n)!_q = n!$ and $\binom{n}{k}_q = \binom{n}{k}$. We state without proof the following proposition from [13] which gives a way to define $\binom{n}{k}$ recursively:

Lemma 1.47. *For $0 \leq k \leq n$, we have that*

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q.$$

We will use this result about Gauss polynomials later where q will be defined to be a root of unity.

1.3 Green rings of Hopf algebras

We will now define Green rings, which describe the interactions between the representations of a Hopf algebra. Let H be a Hopf algebra. We define

$$S = \{[N] : N \text{ a finite dimensional, indecomposable comodule of H}\}$$

where $[N]$ denotes the comodule isomorphism class of N.

Now, let $R(H)$ be the free abelian group with basis S; note that the addition is given by

$$[N_1] + [N_2] = [N_1 \oplus N_2].$$

Definition 1.48. *If H is a Hopf algebra and (N_1, ρ_{N_1}) and (N_2, ρ_{N_2}) are H-comodules, then we define the tensor product $(N_1 \otimes N_2, \rho_{N_1 \otimes N_2})$ as the comodule formed via the map $\rho_{N_1 \otimes N_2} : N_1 \otimes N_2 \to N_1 \otimes N_2 \otimes H$ defined by*

$$\rho_{N_1 \otimes N_2}(m \otimes n) = \sum m_0 \otimes n_0 \otimes M(m_1 \otimes n_1)$$

where $\rho_{N_1}(m) = \sum m_0 \otimes m_1$ and $\rho_{N_2}(n) = \sum n_0 \otimes n_1$.

Now, using this structure, we introduce multiplication to $R(H)$ via

$$[N_1] \cdot [N_2] = [N_1 \otimes N_2].$$

Definition 1.49. *The ring structure induced on $R(H)$ by the above addition and multiplication is called the Green ring or representation ring of H.*

Note 1.50. *We note that a similar Green ring structure can be formed using modules and the algebra structure rather than comodules and the coalgebra structure of a Hopf algebra. These will generally give rise to nonisomorphic rings. We choose to look at the comodule Green rings rather than the module Green rings due to the fact that given a Hopf algebra H, an H-comodule M, and $m \in M$, we know that m must be contained in some finite-dimensional subcomodule of M by the Fundamental Theorem of Comodules. In other words, there is no element in M not contained in some finite-dimensional subcomodule. This property is not true for modules.*

This tells us that the finite-dimensional comodules of H somehow give us more information about the category of comodules as a whole than the finite-dimensional modules of H tell us about the module category of H. As the Green ring consists of isomorphism classes of sums of indecomposable modules or comodules, thus the Green ring formed over comodules may in a sense give us more information about the underlying Hopf algebra than the module Green ring.

We will now describe the module Green rings of some well-known Hopf algebras.

Example 1.51. *Consider the group algebra $K[G]$ for a group G.*

Any indecomposable comodule of $K[G]$ has the form $K\{g\}$ for some $g \in G$ where $K\{g\}$ is the one-dimensional subspace spanned by g. Given two indecomposable comodules $K\{g_1\}$ and $K\{g_2\}$, we note that

$$
\begin{aligned}
K\{g_1\} \otimes K\{g_2\} &\cong K\{g_1 \otimes g_2\} \\
&\cong K\{g_1 g_2\}
\end{aligned}
$$

In this way, the multiplication of indecomposable comodules is analogous to the product of the corresponding elements of the group. Thus, the comodular Green ring $R(K[G])$ is isomorphic to $K[G]$, the group ring.

Example 1.52. *Consider the Taft algebra $H_n(q)$ as defined in Example 1.46.*

Let $\mathbb{Z}[y, z]$ be the polynomial algebra over two variables. We recursively define a generalized Fibonacci polynomial $f_n(y, z)$ over our algebra for $n \geq 1$ by

$$
f_1(y, z) = 1, f_2(y, z) = z, \textit{and } f_n(y, z) = z f_{n-1}(y, z) - y f_{n-2}(y, z)
$$

Then the modular Green ring $R(H_n(q))$ has been shown in [4] to be isomorphic to the ring $\mathbb{Z}[y, x]/I$ where I is the ideal generated by all polynomials of the form $y^n - 1$ and $(z - y - 1)f_n(y, z)$.

In addition, we note that the modular Green rings of many other famous Hopf algebras have been classified. For instance, the Green rings of the generalized Taft algebras have been studied in [15], and the Green ring of the Drinfeld double $D(H_4)$ has been studied in [3].

2 Representations of the Hopf algebra H_s

2.1 The Hopf algebra H_s

We will start by giving some characterizations of Hopf algebras based on the structures of their representations, namely their comodules.

Definition 2.1. *A Hopf algebra H is called pointed if every simple comodule of H is one-dimensional.*

Definition 2.2. *A Hopf algebra H is called coserial if every injective indecomposable comodule of H is uniserial, meaning it possesses a unique composition series.*

Definition 2.3. *An element $x \in H$ is (a, b)-skew primitive if $\Delta(x) = b \otimes x + x \otimes a$.*

Definition 2.4. *An element $x \in H$ is primitive if $\Delta(x) = 1 \otimes x + x \otimes 1$, meaning x is $(1, 1)$-skew primitive.*

The properties of being pointed and coserial mean that the comodules of a Hopf algebra are endowed with a particular well-behaved structure. The classification of such Hopf algebras follows from this result from [11]:

Theorem 2.5. *Let H be a pointed coserial Hopf algebra. Then one of the following holds:*

- *H is isomorphic to a group algebra $K[G]$.*
- *$H \cong K[x]*K[G]/\langle xh = \chi(h)hx \text{ for all } h \in G, x^{s+1} = \alpha(g^{s+1}-1)\rangle$ with $g \in Z(G)$ fixed, x a $(g, 1)$-skew primitive, $S(x) = -xg^{-1}$, χ an invertible character of G with $\chi(g) = q$ a primitive $s+1^{th}$ root of unity, and $\alpha \in K$ can be nonzero only if $\chi^{s+1} = 1$.*
- *$H \cong K[x] * K[G]/\langle xh = \chi(h)hx \text{ for all } h \in G, x^{s+1} = \alpha(g^{s+1} - 1), g^n = 1\rangle$ with $g \in Z(G)$ fixed, x a $(g, 1)$-skew primitive, $S(x) = -xg^{-1}$, χ an invertible*

character of G with $\chi(g) = q$ a primitive $s+1^{th}$ root of unity and $\alpha \in K$ can be nonzero only if $\chi^{s+1} = 1$.

- *H is isomorphic to the Ore extension (of $K[G]$) $H \cong K[x]*K[G]/\langle xh = \chi(h)hx\rangle$ with χ an invertible character of G and x primitive.*
- *H is isomorphic to the generalized Ore extension (of $K[G]$) $H \cong K[x]*K[G]/\langle xh = \chi(h)hx + \lambda(h)(h - gh)$ for all $h \in G\rangle$ with $g \in Z(G)$ fixed, x a $(g,1)$-skew primitive, $S(x) = -xg^{-1}$, χ an invertible character of G with $\chi(g) = 1$ or $\chi(g)$ not a root of unity, and λ an (ϵ, χ)-skew primitive element of $K[G]^0$.*

We will consider Hopf algebras of the second type defined above. More particularly, we consider the Hopf algebra of the form

$$H \cong K[x] * K[G]/\langle xh = \chi(h)hx, x^{s+1} = 0\rangle$$

where $G = \mathbb{Z}$, g is a generator of $\mathbb{Z}$ (which we do not call 1 to avoid confusion with the unit of the field K), χ is a character of G such that $\chi(g) = 1^{\frac{1}{s+1}}$, $\alpha = 0$, and x is defined to be a $(g,1)$-skew primitive, meaning $\Delta(x) = 1 \otimes x + x \otimes g$. Simplifying the notation then gives us Hopf algebras of the form

$$H_s = K[x] * K[\mathbb{Z}]/\langle xg = \epsilon gx, x^s = 0\rangle$$

where $\epsilon = \chi(g) = 1^{\frac{1}{s}}$. The Hopf algebra structure comes from defining

$\Delta(g) = g \otimes g$ $\qquad$ $\Delta(x) = 1 \otimes x + x \otimes g$

$\epsilon(g) = 1$ $\qquad$ $\epsilon(x) = 0$

This Hopf algebra can be written as the path coalgebra of the "line quiver" ${}_\infty A_\infty$:

$$\cdots \longrightarrow (i-1) \longrightarrow i \longrightarrow (i+1) \longrightarrow \cdots$$

where g^i represents a point in the quiver and $g^i x$ corresponds to the arrow connecting the points g^i and g^{i+1}.

Remark 2.6. *We note that the structure of H_s is remarkably similar to that of the Taft algebras $H_n(q)$ as defined in Example 1.46. In fact, the Taft algebras can be written as quotients of these classes of algebras, giving additional motivation to their study.*

Finally, the Hopf algebras H_s are of particular interest because of their categories of comodules as described by the following theorem of [16].

Theorem 2.7. *The Hopf algebra H_2 is the unique Hopf algebra such that H_2-Comod, the category of comodules over H_2, and Com(K), the category of K-chain complexes, are naturally isomorphic as monoidal categories.*

This result was extended in [2] to H_s for $s \geq 2$ in the following way:

Theorem 2.8. *The Hopf algebra H_s is the unique Hopf algebra such that H_s-Comod, the category of comodules over H_s, and s-Com(K), the category of K s-chain complexes, are naturally isomorphic as monoidal categories. This category generalizes K-complexes in that the composition of s consecutive maps in these complexes yields zero.*

Thus, the Green rings of H_s are of particular interest to us because the comodules of H_s correspond to K-complexes in this manner.

We will now classify the indecomposable comodules of H_s. Note that we can write H_s in the form

$$H_s = \bigoplus_{n \in \mathbb{Z}} \mathrm{Span}\{g^n, g^n x, \cdots, g^n x^{s-1}\}.$$

Let $T_n = \text{Span}\{g^n, g^n x, \cdots, g^n x^{s-1}\}$. It is easy to show that T_n is closed under comultiplication and so forms a comodule of H_s under the mapping

$$\rho_{T_n} = \Delta|_{T_n}.$$

Note that each T_n is indecomposable as a comodule of H_s. Then since H_s is injective over itself, and each T_n is a summand of H_s, thus the comodules T_i must be the injective indecomposable comodules of H_s.

We claim that the indecomposable subcomodules of the T_i will give us all other indecomposable subcomodules of H_s. To show this, we will first show that H_s is coserial. We start by noting that the coradical filtration of H_s is given by

$$\begin{aligned} H_{s,0} &= \text{Span}\{g^n\}_{n\in\mathbb{Z}} \\ H_{s,1} &= \text{Span}\{g^n, g^n x\}_{n\in\mathbb{Z}} \\ &\vdots \\ H_{s,s-1} &= \text{Span}\{g^n, g^n x, \cdots, g^n x^{s-1}\}_{n\in\mathbb{Z}}. \end{aligned}$$

Now, for $n \in \mathbb{Z}$ and $0 \leq i < s$, we define

$$T_{n,i} = \text{Span}\{g^n, g^n x, \cdots, g^n x^i\}.$$

Note that for any n and i, we have that

$$T_{n,i} = T_n \cap H_{s,i}.$$

Then since the $H_{s,i}$ form a Loewy filtration of H_s, we know by Proposition 1.38 that the $T_{n,i}$ form a Loewy filtration of T_n for each n. We also know that for each n and

i, we have that

$$\dim(T_{n,i}/T_{n,i-1}) = \text{length}(T_{n,i}/T_{n,i-1}) = 1,$$

implying the successive quotients $T_{n,i}/T_{n,i-1}$ are simple. We know then from the following proposition that the $T_{n,i}$ must be uniserial:

Proposition 2.9. *Let U be a finite dimensional comodule with Loewy series $L_k(U)$ such that $L_i(U)/L_{i-1}(U)$ is simple for all i. Then U is uniserial.*

Proof. Suppose that V is a nonzero subcomodule of U. We will show that V must equal $L_i(U)$ for some i. Since V is nonzero, we know that $s(V) \neq 0$ where $s(V)$ denotes the socle of V. By Proposition 1.38, we know that

$$\begin{aligned} s(V) &= s(U) \cap V \\ &= L_0(U) \cap V. \end{aligned}$$

Then we must have that $L_0(U) = s(V)$, implying $L_0(U) \subset V$.

Now, consider the quotient subcomodule $V/L_0(U)$ of $U/L_0(U)$. Since $L_i(U)/L_{i-1}(U)$ is simple for all i, we know that $U/L_0(U)$ must be uniserial, and so its only nonzero subcomodules are of the form $L_i/(U)/L_0(U)$. But then we know that for some k, $V/L_0(U) = L_k(U)/L_0(U)$, and hence $V = L_k(U)$. Thus, the only nonzero subcomodules of U are the terms of its Loewy series, implying U must be uniserial as desired. □

Now, we know that all of the $T_{n,i}$ must be uniserial. This implies that H_s must be right serial as a coalgebra as all of its right comodules are uniserial. By a similar argument, we know that H_s is left serial as well. Thus H_s forms a serial coalgebra, meaning the indecomposable comodules of H_s can be classified by the following theorem of [11]:

Theorem 2.10. *If C is a serial coalgebra, then all indecomposable C-comodules are uniserial. Therefore, the indecomposable C-comodules are the injective indecomposable comodules and the terms of their Loewy series.*

Hence, we know that the indecomposable comodules of H_s are all isomorphic to comodules of the form

$$M_i^j = K\{g^i, g^i x, g^i x^2, \cdots, g^i x^{j-1}\}$$

where $i, j \in \mathbb{Z}$ and $0 < j \leq s$ and whose comodule map

$$\rho : M_i^j \to M_i^j \otimes H_s$$

is defined by

$$\rho(g^i x^j) = \Delta(g^i x^j).$$

We note that the indecomposable comodules of dimension 1 are precisely the simple comodules, meaning H_s is pointed, and the indecomposable comodules of maximal dimension s are precisely the injective indecomposable comodules. We note these comodules are also the projective indecomposables, forming projective covers of their corresponding simple subcomodules.

Since all indecomposable comodules are isomorphic to comodules of the form M_i^j, we will restrict ourselves to the study of these comodules. In addition, we will write M_i^j to denote the isomorphism class $[M_i^j]$ for calculations in the Green (and later Grothendieck) rings.

We will now begin our classificiation of the tensor product of indecomposable comodules as sums of indecomposable comodules. We will start by classifying the tensor product of indecomposable comodules where one of the comodules is simple.

Lemma 2.11. *For any $i, j, k \in \mathbb{Z}$, $M_i^1 \otimes M_k^j \cong M_{i+k}^j$.*

Proof. Given M_i^1 and M_k^j, we note that

$$\begin{aligned} M_i^1 \otimes M_k^j &= K\{g^i\} \otimes K\{g^k, g^k x, \cdots, g^k x^{j-1}\} \\ &\cong K\{g^i \otimes g^k, g^i \otimes g^k x, \cdots, g^i \otimes g^k x^{j-1}\} \\ &\cong K\{M(g^i \otimes g^k), M(g^i \otimes g^k x), \cdots, M(g^i \otimes g^k x^{j-1})\} \\ &= K\{g^{i+k}, g^{i+k} x, \cdots, g^{i+k} x^{j-1}\} \\ &= M_{i+k}^j. \end{aligned}$$

□

In other words, multiplication by a simple comodule acts as a translation, producing a new comodule of the same dimension as the original.

Corollary 2.12. *M_0^1 is a multiplicative identity in the Green ring $R(H_s)$. Thus, $R(H_s)$ is a unital ring.*

We will now show that in order to classify the tensor product of indecomposable comodules, we may restrict the problem to a smaller class of indecomposables, namely ones of subindex 0.

Theorem 2.13. *Suppose that for $i, j \in \mathbb{Z}$,*

$$M_0^i \otimes M_0^j \cong M_{r_1}^{k_1} \oplus M_{r_2}^{k_2} \oplus \cdots \oplus M_{r_n}^{k_n}.$$

Then for any $a, b \in \mathbb{Z}$,

$$M_a^i \otimes M_b^j \cong M_{r_1+a+b}^{k_1} \oplus M_{r_2+a+b}^{k_2} \oplus \cdots \oplus M_{r_n+a+b}^{k_n}.$$

Proof. This follows from the preceding lemma and the commutativity of the tensor product:

$$
\begin{aligned}
M_a^i \otimes M_b^j &\cong (M_a^1 \otimes M_0^i) \otimes (M_b^1 \otimes M_0^j) \\
&\cong (M_a^1 \otimes M_b^1) \otimes (M_0^i \otimes M_0^j) \\
&\cong M_{a+b}^1 \otimes (M_{r_1}^{k_1} \oplus M_{r_2}^{k_2} \oplus \cdots \oplus M_{r_n}^{k_n}) \\
&\cong (M_{a+b}^1 \otimes M_{r_1}^{k_1}) \oplus (M_{a+b}^1 \otimes M_{r_2}^{k_2}) \oplus \cdots \oplus (M_{a+b}^1 \otimes M_{r_n}^{k_n}) \\
&\cong M_{r_1+a+b}^{k_1} \oplus M_{r_2+a+b}^{k_2} \oplus \cdots \oplus M_{r_n+a+b}^{k_n},
\end{aligned}
$$

giving us our desired result.

□

In other words, by determining the tensor product of indecomposable comodules of the form M_0^i, we can classify the tensor product of all of the indecomposable comodules of H_s. Thus, many of our remaining results will work towards classifying the tensor product of such comodules.

2.2 The comultiplication and comodule maps of H_s

In order to more fully understand the comodule structures of H_s, we will find explicit formulas for the comultiplication map Δ in H_s as well as the comodule map $\rho : M \to M \otimes H_s$ for M an indecomposable comodule of H_s. To start, we will use the linearity of Δ to generalize our formulas.

Lemma 2.14. *For any* $n > 0$, $\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i}_\epsilon x^{n-i} \otimes g^{n-i}x^i$.

Proof. Proof proceeds by induction. We note that given $\Delta(x) = 1 \otimes x + x \otimes g = \sum_{i=0}^{1} \binom{1}{i}_\epsilon x^{1-i} \otimes g^{1-i}x^i$, we know this result to be true by definition for $n = 1$. Now, suppose that for some fixed $n > 0$, we have that

$$\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i}_\epsilon x^{n-i} \otimes g^{n-i}x^i.$$

We will prove $\Delta(x^{n+1})$ has the desired formula with explanations for each numbered line following the proof:

$$
\begin{aligned}
\Delta(x^{n+1}) &= \Delta(x)\Delta(x^n) \\
&= (1\otimes x + x\otimes g)\left(\sum_{i=0}^{n}\binom{n}{i}_{\epsilon} x^{n-i}\otimes g^{n-i}x^i\right) \\
&= \sum_{i=0}^{n}\binom{n}{i}_{\epsilon}((1\otimes x)(x^{n-i}\otimes g^{n-i}x^i) + (x\otimes g)(x^{n-i}\otimes g^{n-i}x^i)) \\
&= \sum_{i=0}^{n}\binom{n}{i}_{\epsilon}(x^{n-i}\otimes xg^{n-i}x^i + x^{n-i+1}\otimes g^{n-i+1}x^i) \\
&= \sum_{i=0}^{n}\binom{n}{i}_{\epsilon}((\epsilon^{n-i})(x^{n-i}\otimes g^{n-i}x^{i+1}) + (x^{n+1-i}\otimes g^{n+1-i}x^i)) \qquad (1) \\
&= \binom{n}{0}_{\epsilon}(x^{n+1}\otimes g^{n+1}) + \binom{n}{n}_{\epsilon}(\epsilon^0)(1\otimes x^{n+1}) \\
&\quad + \sum_{i=1}^{n}\binom{n}{i-1}_{\epsilon}(\epsilon^{n-(i-1)})(x^{n-(i-1)}\otimes g^{n-(i-1)}x^{(i-1)+1}) + \binom{n}{i}_{\epsilon}(x^{n+1-i}\otimes g^{n+1-i}x^i) \\
&\qquad (2) \\
&= \binom{n+1}{0}_{\epsilon}(x^{n+1}\otimes g^{n+1}) + \binom{n+1}{n+1}_{\epsilon}(1\otimes x^{n+1}) \\
&\quad + \sum_{i=1}^{n}\left(\epsilon^{n+1-i}\binom{n}{i-1}_{\epsilon} + \binom{n}{i}_{\epsilon}\right) x^{n+1-i}\otimes g^{n+1-i}x^i \qquad (3) \\
&= \binom{n+1}{0}_{\epsilon}(x^{n+1}\otimes g^{n+1}) + \binom{n+1}{n+1}_{\epsilon}(\epsilon^0)(1\otimes x^{n+1}) \\
&\quad + \sum_{i=1}^{n}\binom{n+1}{i}_{\epsilon} x^{n+1-i}\otimes g^{n+1-i}x^i \\
&= \sum_{i=0}^{n+1}\binom{n+1}{i}_{\epsilon} x^{(n+1)-i}\otimes g^{(n+1)-i}x^i
\end{aligned}
$$

- In (1), we split up our sum into three components: the second summand in the term where $i=0$, the first summand in the term where $i=n$, and a sum whose individual summands consist of the first summand from the $i-1$ term added to the second summand from the i term.

- In (2), we use that for any n, $\binom{n}{0}_\epsilon = 1 = \binom{n+1}{0}_\epsilon$ and $\binom{n}{n}_\epsilon = 1 = \binom{n+1}{n+1}_\epsilon$.
- In (3), we apply Lemma 1.47, giving us $\epsilon^{n+1-i}\binom{n}{i-1}_\epsilon + \binom{n}{i}_\epsilon = \binom{n+1}{i}_\epsilon$.

By inducting on n then, we can conclude that $\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i}_\epsilon x^{n-i} \otimes g^{n-i}x^i$ for any $n > 0$ as desired.

□

Lemma 2.15. *For any positive integer i, $\Delta(g^i) = g^i \otimes g^i$.*

Proof. This formula follows quickly by induction. We note this result is true by definition for $i = 1$. Suppose that $\Delta(g^k) = g^k \otimes g^k$. Then we know that

$$\begin{aligned}\Delta(g^{k+1}) &= \Delta(g)\Delta(g^k)\\ &= (g \otimes g)(g^k \otimes g^k)\\ &= g^{k+1} \otimes g^{k+1}.\end{aligned}$$

Thus, by inducting on i, we arrive at our desired result.

□

Theorem 2.16. *For any positive integer i, n with $n < s - 1$,*

$$\Delta(g^i x^n) = \sum_{k=0}^{n} \binom{n}{k}_\epsilon g^i x^{n-k} \otimes g^{i+n-k} x^k.$$

Proof. Given $g^i x^n$, we note that

$$\begin{aligned}\Delta(g^i x^n) &= \Delta(g^i)\Delta(x^n)\\ &= (g^i \otimes g^i)\left(\sum_{k=0}^{n}\binom{n}{k}_{\epsilon} x^{n-k}\otimes g^{n-k}x^k\right)\\ &= \sum_{k=0}^{n}\binom{n}{k}_{\epsilon}(g^i\otimes g^i)(x^{n-k}\otimes g^{n-k}x^k)\\ &= \sum_{k=0}^{n}\binom{n}{k}_{\epsilon} g^i x^{n-k}\otimes g^{i+n-k}x^k,\end{aligned}$$

giving us our desired result.

□

We will now use our generalized formula for the comultiplication map Δ to find a formula for the map defining the comodule structure of the tensor product of indecomposable comodules M_0^i and M_0^j.

Lemma 2.17. *The map* $\rho : M_0^a \otimes M_0^b \to M_0^a \otimes M_0^b \otimes H_s$ *maps* $x^r \otimes x^t$ *to*

$$\sum_{i=0}^{r}\sum_{j=0}^{t}\binom{r}{i}_{\epsilon}\binom{t}{j}_{\epsilon}(\epsilon^{i(t-j)})(x^{r-i}\otimes x^{t-j})\otimes(g^{r+t-i-j}x^{i+j}).$$

Proof. We start by recalling the comodule structure of $M_0^a \otimes M_0^b$: given $p \in M_0^a$ and $q \in M_0^b$, we have

$$\rho(p\otimes q) = \sum p_0 \otimes q_0 \otimes M(p_1 \otimes q_1)$$

for $\rho(p) = \sum p_0 \otimes p_1$ and $\rho(q) = \sum q_0 \otimes q_1$. We know by the previous lemmata that

$$\rho(x^r) = \Delta(x^r) = \sum_{i=0}^{r}\binom{r}{i}_{\epsilon} x^{r-i}\otimes g^{r-i}x^i$$

and

$$\rho(x^t) = \Delta(x^t) = \sum_{j=0}^{t} \binom{t}{j}_\epsilon x^{t-j} \otimes g^{t-j}x^j.$$

Using these facts, we know that

$$\begin{aligned}
\rho(x^r \otimes x^t) &= \sum_{i=0}^{r}\sum_{j=0}^{t} \left(\binom{r}{i}_\epsilon x^{r-i}\right) \otimes \left(\binom{t}{j}_\epsilon x^{t-j}\right) \otimes M(g^{r-i}x^i \otimes g^{t-j}x^j) \\
&= \sum_{i=0}^{r}\sum_{j=0}^{t} \binom{r}{i}_\epsilon \binom{t}{j}_\epsilon (x^{r-i} \otimes x^{t-j}) \otimes (g^{r-i}x^i g^{t-j}x^j) \qquad (4) \\
&= \sum_{i=0}^{r}\sum_{j=0}^{t} \binom{r}{i}_\epsilon \binom{t}{j}_\epsilon (x^{r-i} \otimes x^{t-j}) \otimes (\epsilon^{i(t-j)})(g^{r-i}g^{t-j}x^i x^j) \\
&= \sum_{i=0}^{r}\sum_{j=0}^{t} \binom{r}{i}_\epsilon \binom{t}{j}_\epsilon (\epsilon^{i(t-j)})(x^{r-i} \otimes x^{t-j}) \otimes (g^{r+t-i-j}x^{i+j}).
\end{aligned}$$

- In (4), we note that to commute x^i past g^{t-j}, we must commute an x past g^{t-j} i times. This produces $i(t-j)$ copies of ϵ.

We note that for $s \le r+t$, we will produce powers of x, namely x^k in H_s where $k \ge s$; these whole terms will be zero in our formula due to the structure of H_s.

□

2.3 The composition series of M_0^i

Since H_s is coserial, we know that the composition series of its indecomposable comodules is unique up to isomorphism. In fact, we see that M_i^j has composition series given by

$$0 \subset M_i^1 \subset M_i^2 \subset \cdots \subset M_i^{j-1} \subset M_i^j,$$

noting that quotients of subsequent terms must be 1-dimensional and hence simple. We will determine which simple comodules make up its composition terms by classifying comodules of the form M_i^{k+1}/M_i^k up to isomorphism.

Theorem 2.18. *Given the comodule M_i^j with $j > 1$, $M_i^j/M_i^{j-1} \cong M_{i+j-1}^1$ as comodules.*

Proof. We note that by definition, $M_i^j = K\{g^i, g^i x, \cdots, g^i x^{j-1}\}$, while $M_i^{j-1} = K\{g^i, g^i x, \cdots, g^i x^{j-2}\}$. Thus, the only element in the spanning set of M_i^j that is not an element of M_i^{j-1} is $g^i x^{j-1}$. We know that this quotient must be a simple comodule; the question is which one. For $a \in M_i^j$, let $\overline{a}$ denote the natural projection of a in the quotient space M_i^j/M_i^{j-1}. Then we note that

$$\begin{aligned}
\Delta(\overline{g^i x^{j-1}}) &= \sum_{k=0}^{j-1} \binom{j-1}{k}_\epsilon \overline{g^i x^{(j-1)-k}} \otimes g^{(j-1)-k+i} x^k \\
&= \binom{j-1}{0}_\epsilon \overline{g^i x^{j-1}} \otimes g^{(j-1)+i} + \sum_{k=1}^{j-1} \binom{j-1}{k}_\epsilon \overline{g^i x^{(j-1)-k}} \otimes g^{(j-1)-k+i} x^k \quad (5) \\
&= \overline{g^i x^{j-1}} \otimes g^{i+j-1}.
\end{aligned}$$

- In line (5), we know that $\overline{g^i x^{(j-1)-k}} = 0$ for all $k \in \{1, 2, \cdots, j-1\}$ as for all of these values of k, $g^i x^{(j-1)-k}$ is an element of M_i^{j-1}.

Let $z = g^i x^{j-1}$. Then we know that

$$\Delta(\overline{z}) = \overline{z} \otimes g^{i+j-1}.$$

We can conclude that if ϕ is the isomorphism mapping M_i^j/M_i^{j-1} to a simple comodule, then $\Delta(\phi(\overline{z})) = \overline{z} \otimes g^{i+j-1}$. Then by our formulas for the coproduct Δ, we must have $\Delta(\phi(\overline{z})) = g^{i+j-1} \otimes g^{i+j-1}$, as we know $\Delta(g^k) = g^k \otimes g^k$ for any power k. This in turn implies that $\overline{z} = g^{i+j-1}$. We can thus conclude that the simple comodule to which M_i^j/M_i^{j-1} is isomorphic must be $K\{g^{i+j-1}\} = M_{i+j-1}^1$ as desired.

∎

Corollary 2.19. *The composition factors of any indecomposable comodule M_i^j are of the form*

$$M_i^1, M_{i+1}^1, \cdots, M_{i+j-1}^1.$$

Proof. This follows immediately from the fact that M_i^j has composition series given by

$$0 \subset M_i^1 \subset M_i^2 \subset \cdots \subset M_i^{j-1} \subset M_i^j$$

and by Theorem 2.18.

□

We now have a way of uniquely classifying each indecomposable comodule by its composition series: M_i^j is precisely the indecomposable comodule which has j composition factors (and is thus of length j) and whose composition factors are the j simple comodules with consecutive indices starting at $S_i = M_i^1$. From now on, we refer to S_i as the *bottom* composition factor of M_i^j and $S_{i+j-1} = M_{i+j-1}^1$ to be the *top* composition factor of M_i^j.

We can use the composition series of our comodules to show that the indecomposable comodules of H_s can be uniquely written in the form M_a^i.

Theorem 2.20. *The indecomposable comodules M_a^i and M_b^j are isomorphic if and only if $a = b$ and $i = j$.*

Proof. If $a = b$ and $i = j$, then $M_a^i \cong M_b^j$ by the identity map. We will show that if $a \neq b$ or $i \neq j$, then $M_a^i \ncong M_b^j$. We first note that if $i \neq j$, then these comodules have composition series of differing lengths; by uniqueness of the composition series then, $M_a^i \ncong M_b^j$.

Now, suppose that $a \neq b$. We know that the bottom composition term of M_a^i is the simple comodule S_a and the bottom composition term of M_b^j is S_b. By uniqueness of the composition series then, we need only show that $S_a \ncong S_b$ in order to prove $M_a^i \ncong M_b^j$. We will do so by directly showing no such isomorphism can exist. Recall that by definition, we know $S_a = K\{g^a\}$ and $S_b = K\{g^b\}$. Let $\phi : S_a \to S_b$ be a homomorphism of comodules. Then we know that

$$\phi(g^a) = rg^b$$

for some $r \in K$. In order for ϕ to be a comodule homomorphism, we need that $(\phi \otimes I) \circ \rho(g^a) = \Delta \circ \phi(g^a)$. However, notice that

$$\begin{aligned}(\phi \otimes I) \circ \rho(g^a) &= (\phi \otimes I)(g^a \otimes g^a) \\ &= rg^b \otimes g^a\end{aligned}$$

while

$$\begin{aligned}\Delta \circ \phi(g^a) &= \Delta(rg^b) \\ &= r\Delta(g^b) \\ &= rg^b \otimes g^b.\end{aligned}$$

Then if $\phi \circ \rho(g^a) = \Delta \circ \phi(g^a)$, we must have that

$$\begin{aligned}0 &= rg^b \otimes g^a - rg^b \otimes g^b \\ &= rg^b \otimes (g^a - g^b).\end{aligned}$$

Since $a \neq b$, we know that $g^a \neq g^b$. Thus, in order for this tensor product to be zero, we must have that $r = 0$. However, this implies that the only comodule homomorphism from S_a to S_b is the zero map, implying that $S_a \ncong S_b$ and thus that $M_a^i \ncong M_b^j$ as desired.

□

2.4 The composition terms of $M_0^i \otimes M_0^j$

Now that we have determined the composition terms for any indecomposable comodule, we will prove some results about the composition terms of the indecomposable summands of the tensor product of such comodules. More specifically, we will classify all such composition terms, then determine which terms form the top composition factors of individual summands and which terms form the bottom composition factors of summands.

We know that $M_0^i \otimes M_0^j$ must have ij simple comodules as composition terms since $M_0^i \otimes M_0^j$ is of dimension ij. We will first determine which comodules appear as these composition terms.

Theorem 2.21. *S_k appears r times as a composition factor of $M_0^i \otimes M_0^j$ where r is the number of times k can be written in the form $k = a+b$ where $a \in \{0, 1, \cdots, i-1\}$ and $b \in \{0, 1, \cdots, j-1\}$.*

Proof. Note that the composition factors of M_0^i are precisely the simple comodules of the form S_a where $a \in \{0, 1, \cdots, i-1\}$. The composition factors of M_0^j are precisely the simple comodules of the form S_b where $b \in \{0, 1, \cdots, j-1\}$. Thus, the composition factors of $M_0^i \otimes M_0^j$ are precisely the simple comodules of the form $S_a \otimes S_b \cong S_{a+b}$ where a ranges from 0 to $i-1$ and b ranges from 0 to $j-1$. Thus, we know that S_k appears as a composition factor once for each pairing of a and b such that $k = a + b$ as desired.

□

We note that this result does not by itself fully determine the indecomposable summands of $M_0^i \otimes M_0^j$, as these composition terms can belong to different indecomposable summands in different ways.

Example 2.22. *Note that $M = M_1^3 \oplus M_2^1$ and $N = M_1^2 \oplus M_2^2$ have the same composition terms, namely one copy of S_1 and S_3 and two copies of S_2. However, $M \ncong N$.*

Theorem 2.21 does give us a way to begin looking at the possible structures of such summands. We will next prove which of these composition factors appear as the bottom composition term of an indecomposable summand and which ones appear as the top composition term:

Theorem 2.23. *Let S_k denote the simple comodule $M_k^1 = K\{g^k\}$. Then S_k appears as the bottom composition factor of a summand of $M_0^i \otimes M_0^j$ if and only if $0 \leq k \leq n-1$ where $n = \min\{i, j\}$. For such k, S_k appears as a bottom composition factor in only one summand.*

Proof. We start by noting that S_k appears as the bottom composition factor of a summand of $M_0^i \otimes M_0^j$ if and only if $\mathrm{Hom}(S_k, M_0^i \otimes M_0^j)$ is nonzero. We note that the dual of M_0^i, $(M_0^i)^*$ satisfies $(M_0^i)^* = M_{-i+1}^i$, the comodule of the same length whose top composition term S_0 equals the bottom composition term of M_0^i. We use the result of [7] that for comodules A, B, C, we must have that

$$\mathrm{Hom}(A, B \otimes C) \cong \mathrm{Hom}(B^* \otimes A, C).$$

This result gives us that

$$\begin{aligned}\mathrm{Hom}(S_k, M_0^i \otimes M_0^j) &\cong \mathrm{Hom}((M_0^i)^* \otimes S_k, M_0^j) \\ &\cong \mathrm{Hom}(M_{-i+1}^i \otimes S_k, M_0^j) \\ &\cong \mathrm{Hom}(M_{-i+k+1}^i, M_0^j).\end{aligned}$$

We note that M_{-i+k+1}^i has composition factors

$$S_{-i+k+1}, S_{-i+k+2}, \cdots, S_{-i+k+1+(i-1)} = S_k,$$

while M_0^j has composition factors

$$S_0, S_1, \cdots, S_{j-1}.$$

We know that $\mathrm{Hom}(M_{-i+k+1}^i, M_0^j)$ is nonzero if a string of composition factors from the *top* of M_{-i+k+1}^i matches up with a string of composition factors from the *bottom* of M_0^j. Thus, it must be the case that the top composition factor of M_{-i+k+1}^i, namely S_k, appears as a composition factor in M_0^j. In other words, we need that $0 \leq k \leq j-1$.

However, as M_{-i+k+1}^i has i composition factors, in order for a string of its top composition factors to appear as a string of bottom composition factors in M_0^j, we must also have that S_k appears in the bottom i composition factors of M_0^j. In other words, we need that $k \in \{0, 1, \cdots, i-1\}$. Thus, we additionally need $0 \leq k \leq i-1$.

Thus, in the case where $0 \leq k \leq n-1$ where $n = \min\{i, j\}$, we know that a string of composition factors at the top of M_{-i+k+1}^i appears as a string of composition factors at the bottom of M_0^j. This does not occur if $k < 0$ or if $k \geq n$. We can thus conclude that S_k appears as the bottom composition factor of a summand of $M_0^i \otimes M_0^j$ if and only if $0 \leq k \leq n-1$.

Now, to see that each unique simple comodule can only appear once as the bottom composition term of a summand, we note that $\mathrm{Hom}(S_k, M_0^i \otimes M_0^j) \cong \mathrm{Hom}(M^i_{-i+k+1}, M_0^j)$ is one-dimensional at most as there is only one way to match a string of composition factors from the top of M^i_{-i+k+1} with the bottom of M_0^j as desired. □

Now, we will similarly determine the simple comodules appearing as the top composition terms of summands of $M_0^i \otimes M_0^j$:

Theorem 2.24. *Let S_k again denote the simple comodule $M_k^1 = K\{g^k\}$. Then S_k appears as the top composition factor of a summand of $M_0^i \otimes M_0^j$ if and only if $m-1 \leq k \leq i+j-2$ where $m = \max\{i,j\}$. For such k, S_k appears as a top composition factor in only one summand.*

Proof. We start by noting that S_k appears as the top composition factor of a summand of $M_0^i \otimes M_0^j$ if and only if $\mathrm{Hom}(M_0^i \otimes M_0^j, S_k)$ is nonzero. We then use the following result from [7] that for comodules A, B, C, we have that

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(B, A^* \otimes C).$$

Notice then that

$$\begin{aligned} \mathrm{Hom}(M_0^i \otimes M_0^j, S_k) &\cong \mathrm{Hom}(M_0^j, (M_0^i)^* \otimes S_k) \\ &\cong \mathrm{Hom}(M_0^j, M^i_{-i+1} \otimes S_k) \\ &\cong \mathrm{Hom}(M_0^j, M^i_{-i+k+1}). \end{aligned}$$

As in the previous lemma, we know that M_0^j has composition factors

$$S_0, S_1, \cdots, S_{j-1},$$

while M^i_{-i+k+1} has composition factors

$$S_{-i+k+1}, S_{-i+k+2}, \cdots, S_{-i+k+1+(i-1)} = S_k.$$

We know that $\mathrm{Hom}(M^j_0, M^i_{-i+k+1})$ is nonzero if a string of composition factors from the *top* of M^j_0 matches up with a string of composition factors from the *bottom* of M^i_{-i+k+1}. Thus, it must be the case that the top composition factor of M^j_0, namely S_{j-1} appears as a composition factor in M^i_{-i+k+1}. In other words, we need that $-i+k+1 \leq j-1 \leq k$, leading to the necessary inequalities $k \geq j-1$ and $k \leq i+j-2$.

However, as M^j_0 has j composition factors, in order for a string of its top composition factors to appear as a string of bottom composition factors in M^i_{-i+k+1}, we must also have that S_{j-1} appears in the bottom j composition factors of M^i_{-i+k+1}. In other words, we need that $(j-1) \in \{-i+k+1, -i+k+2, \cdots, -i+k+1+(j-1)\}$. Thus, we need that $j-1 \geq -i+k+1$ (meaning $k \leq i+j-2$ as above) and that $j-1 \leq -i+k+1+(j-1)$, or that $k \geq i-1$.

Thus, in the case where $m-1 \leq k \leq i+j-2$ where $m = \max\{i,j\}$, we know that a string of composition factors at the top of M^j_0 appears as a string of composition factors at the bottom of M^i_{-i+k+1}. This does not occur if $k < m-1$ or if $k > i+j-2$. We can thus conclude that S_k appears as the top composition factor of a summand of $M^i_0 \otimes M^j_0$ if and only if $m-1 \leq k \leq i+j-2$.

Now, to see that each unique simple comodule can only appear once as the top composition term of a summand, we note that $\mathrm{Hom}(M^i_0 \otimes M^j_0, S_k) \cong \mathrm{Hom}(M^j_0, M^i_{-i+k+1})$ is one-dimensional at most as there is only one way to match a string of composition factors from the top of M^j_0 with the bottom of M^i_{-i+k+1} as desired.

□

Corollary 2.25. *$M^i_0 \otimes M^j_0$ has precisely min(i,j) indecomposable summands.*

Proof. This follows immediately from the face that $M^i_0 \otimes M^j_0$ has precisely $\min(i,j)$

simple comodules that appear as the bottom/top of an indecomposable summand, and each such simple comodule is the bottom/top composition term precisely once.

□

Given indecomposable comodules M_0^i and M_0^j, we now know which simple comodules appear as the top and bottom composition factors in the indecomposable summands of their tensor product. However, as in Example 2.22, this is not sufficient to fully classify their tensor product, as we do not yet know which bottom composition terms match up with which top composition terms to determine these summands.

2.5 Injective summands of $M_0^i \otimes M_0^j$

Our next objective is to determine the summands of maximal dimension in the case where $M_0^i \otimes M_0^j$ has at most one injective indecomposable summand. This will further narrow down the possibilities for the tensor product.

Lemma 2.26. *An injective comodule M_k^s may only appear once at most as a summand of $M_0^i \otimes M_0^j$.*

Proof. We note that given any M_k^s, M_k^s has bottom composition term S_k. Since S_k can only appear once as the bottom composition term of $M_0^i \otimes M_0^j$ by Theorem 2.23, this result follows.

□

Before classifying the injective summands of $M_0^i \otimes M_0^j$, we will determine the indecomposable summand of maximal dimension in this product in the case $M_0^i \otimes M_0^j$ has at most one injective indecomposable summand.

Lemma 2.27. *Let M_0^i, M_0^j be comodules such that $i + j \leq s + 1$. Then $M_0^i \otimes M_0^j$ contains M_0^{i+j-1} as an indecomposable summand.*

Proof. We know that M_0^i contains the element x^{i-1} and M_0^j contains the element x^{j-1}. Applying our formula for ρ in the tensor product gives us that

$$\rho(x^{i-1} \otimes x^{j-1}) = \sum_{k=0}^{i-1} \sum_{r=0}^{j-1} \binom{i-1}{k}_\epsilon \binom{j-1}{k}_\epsilon (\epsilon^{k(j-1-r)})(x^{i-1-k} \otimes x^{j-1-r}) \otimes (g^{i+j-2-k-r} x^{k+r}).$$

We note that since $i + j \leq s + 1$, thus $k + r < s$ for all values of k and r and hence no terms vanish in the above sum. Now, note in the above formula that when $k = i - 1$ and $r = j - 1$, we have a summand of $(1 \otimes 1) \otimes x^{i+j-2}$. This implies that the comodule generated by $x^{i-1} \otimes x^{j-1}$ in the tensor product has Loewy length at least $i + j - 2$ and hence length at least $i + j - 1$. In other words, $M_0^i \otimes M_0^j$ has an indecomposable summand of length at least $i + j - 1$.

However, we know that out of all the summands of $M_0^i \otimes M_0^j$, S_0 is the bottom composition term of smallest index, and S_{i+j-2} is the top composition term of maximal index. If these simple comodules were respectively the bottom and top of the same indecomposable summand, this summand would be of length $i + j - 1$; in fact, this is the only way to have a summand of this length or more. Thus, we know that S_0 must be the bottom composition term of the summand for which S_{i+j-2} is the top composition term. This corresponding indecomposable summand of $M_0^i \otimes M_0^j$ then must be M_0^{i+j-1}, concluding our proof.

□

Lemma 2.28. *Given* $0 < i < s$, $M_0^i \otimes M_0^{s+1-i}$ *has no injective indecomposable summands other than* M_0^s.

Proof. By Lemma 2.23, we know that the bottom composition terms of the indecomposable summands of $M_0^i \otimes M_0^{s+1-i}$ are precisely the simple comodules

$$S_0, S_1, \cdots, S_{\min\{i, s+1-i\}}.$$

By Lemma 2.24, the top composition terms of the these summands are precisely

$$S_{\max\{i,s+1-i\}}, S_{\max\{i,s+1-i\}+1}, \cdots, S_{i+(s+1-i)-2} = S_{s-1}.$$

The only possible combination of a top and a bottom composition terms from these lists that result in a indecomposable summand of length s (hence injective) is a bottom term of S_0 and a top term of S_{s-1}. This produces an injective summand of M_0^s; no other possible summands of length s exist from combinations of these bottom and top composition factors.

□

Now, we consider the injective indecomposable summands of $M_0^i \otimes M_0^j$. We will classify these for arbitrary i and j.

Theorem 2.29. *Let M_0^i and M_0^j be comodules such that $i+j < s+1$. Then $M_0^i \otimes M_0^j$ has no injective indecomposable summands.*

Proof. By Lemma 2.23 again, we know the bottom composition term of minimal index of the indecomposable summands of $M_0^i \otimes M_0^j$ is S_0, while by Lemma 2.24, the composition term of maximal index is S_{i+j-2}. Since $i+j < s+1$, we know that $i+j-2 \leq s-2$. Then the indecomposable summand of $M_0^i \otimes M_0^j$ of maximal possible length would be the summand formed by taking S_0 as its bottom and S_{s-2} as its top. However, this summand would be of length $s-1$. Thus, no indecomposable summands of $M_0^i \otimes M_0^j$ can be of length s, meaning $M_0^i \otimes M_0^j$ has no injective indecomposable summands. □

Theorem 2.30. *Let M_0^i and M_0^j be comodules such that $i+j \geq s+1$. Then $M_0^i \otimes M_0^j$ has M_k^s as an injective summand if and only if $0 \leq k \leq i+j-s-1$. M_k^s may only appear once as a summand.*

Proof.

$\Longleftarrow$

Let i be fixed; proof then proceeds by induction on j. First, consider $j = s + 1 - i$. Then $i + j = s + 1$. By Lemma 2.27, we know that $M_0^i \otimes M_0^j$ contains $M_0^{i+j-1} = M_0^s$ as a direct summand.

Now, suppose that for some i, k such that $i + k \geq s + 1$, we know that $M_0^i \otimes M_0^k$ has $M_0^s, M_1^s, \cdots, M_{i+k-s-1}^s$ as injective summands. We will show that $M_0^i \otimes M_0^{k+1}$ has all of these injective comodules as summands as well as M_{i+k-s}^s.

First, we note that M_0^k is a subcomodule of M_0^{k+1}. Hence $M_0^i \otimes M_0^k$ forms a subcomodule of $M_0^i \otimes M_0^{k+1}$. Then since M_r^s is an injective summand of $M_0^i \otimes M_0^k$ for all $0 \leq r \leq i + k - s - 1$, we know they must be summands of $M_0^i \otimes M_0^{k+1}$ as well.

We now need only show that $M_0^i \otimes M_0^{k+1}$ additionally has M_{i+k-s} as an injective summand. Recall that the composition terms of M_0^{k+1} are $S_0, S_1, S_2, \cdots, S_k$ while the composition terms of M_1^k are $S_1, S_2, \cdots, S_k$. Hence M_1^k forms a quotient of M_0^{k+1} by modding out by the bottom composition term S_0. Thus $M_0^i \otimes M_1^k$ must also form a quotient of $M_0^i \otimes M_0^{k+1}$. But then we note that

$$\begin{aligned} M_0^i \otimes M_1^k &\cong M_0^i \otimes (S_1 \otimes M_0^k) \\ &\cong S_1 \otimes (M_0^i \otimes M_0^k). \end{aligned}$$

Thus, since $M_0^i \otimes M_0^k$ has $M_{i+k-s-1}^s$ as a direct summand by hypothesis, we know that $M_0^i \otimes M_1^k$ must have $S_1 \otimes M_{i+k-s-1}^s \cong M_{i+k-s}^s$ as a direct summand. As this summand is injective and $M_0^i \otimes M_1^k$ is a quotient of $M_0^i \otimes M_0^{k+1}$, we know that $M_0^i \otimes M_0^{k+1}$ must also have M_{i+k-s}^s as a direct summand as well. By Lemma 2.26, these summands may only appear once. Thus, by inducting, we know that for any i, j such that $i + j \geq s + 1$, $M_0^i \otimes M_0^j$ must have precisely one copy of M_k^s as an injective summand for each $0 \leq k \leq i+j-s-1$.

$\Longrightarrow$

We will now show that if $k < 0$ or $k > i+j-s-1$, then M_k^s does not appear as a summand of $M_0^i \otimes M_0^j$. Suppose that $k < 0$. Then we know that S_k is not the bottom of any indecomposable summand of $M_0^i \otimes M_0^j$ by Lemma 2.23. This in turn implies that M_k^s is not a summand of $M_0^i \otimes M_0^j$.

Now, suppose that $k > i+j-s-1$. Suppose that $M_0^i \otimes M_0^j$ has a summand of M_k^s. Then the composition series of M_k^s consists of the simple comodules $S_k, S_{k+1}, \cdots, S_{k+s-1}$. Note that since $k > i+j-s-1$, we know that

$$\begin{aligned} k+s-1 &> i+j-s-1+s-1 \\ &= i+j-2. \end{aligned}$$

However, this contradicts Lemma 2.24, as we know that the top composition terms of $M_0^i \otimes M_0^j$ are of the form S_r where $r \leq i+j-2$; thus all composition terms of $M_0^i \otimes M_0^j$ must satisfy this condition. Thus, we know that $M_0^i \otimes M_0^j$ has no summand of the form M_k^s where $k > i+j-s-1$ as desired.

□

Now that we have classified the injective summands in the tensor product of any indecomposable comodules, we will fully determine the tensor product when at least one of the comodules is itself injective.

Theorem 2.31. *Let $1 \leq i \leq s$. Then $M_0^i \otimes M_0^s \cong M_0^s \oplus M_1^s \oplus \cdots \oplus M_{i-1}^s$.*

Proof. Proof proceeds by induction. We know already that $M_0^1 \otimes M_0^s \cong M_0^s$. Suppose that for all $j < i$, we have that $M_0^j \otimes M_0^s \cong M_0^s \oplus M_1^s \oplus \cdots \oplus M_{j-1}^s$. We note that M_0^i has the composition series

$$0 \subset M_0^1 \subset M_0^2 \subset \cdots \subset M_0^{i-1} \subset M_0^i$$

where $M_0^j/M_0^{j-1} \cong M_j^1$ for each $j \leq i$. Now, consider the short exact sequence

$$0 \longrightarrow M_0^{i-1} \longrightarrow M_0^i \longrightarrow M_0^i/M_0^{i-1} \longrightarrow 0.$$

By tensoring with M_0^s, this gives rise to the short exact sequence

$$0 \longrightarrow M_0^{i-1} \otimes M_0^s \longrightarrow M_0^i \otimes M_0^s \longrightarrow M_0^i/M_0^{i-1} \otimes M_0^s \longrightarrow 0.$$

Since M_0^i/M_0^{i-1} is simple, we know that the product $M_0^i/M_0^{i-1} \otimes M_0^s$ is projective. Thus, the above sequence splits, giving us the isomorphism:

$$\begin{aligned} M_0^i \otimes M_0^s &\cong (M_0^{i-1} \otimes M_0^s) \oplus (M_0^i/M_0^{i-1} \otimes M_0^s) \\ &\cong (M_0^{i-1} \otimes M_0^s) \oplus (M_i^1 \otimes M_0^s) \\ &\cong (M_0^s \oplus M_1^s \oplus \cdots \oplus M_{i-2}^s) \oplus (M_{i-1}^s) \\ &\cong M_0^s \oplus M_1^s \oplus \cdots \oplus M_{i-1}^s. \end{aligned}$$

□

We note the previous theorem also follows directly as a corollary of Theorem 2.30, as by this theorem the sum of the injective indecomposable summands of $M_0^i \otimes M_0^s$ must have dimension $s((i+s-s-1)+1) = is$, while we know that $M_0^i \otimes M_0^s$ is also of dimension is, meaning $M_0^i \otimes M_0^s$ must be the sum of its injective indecomposable summands.

Corollary 2.32. *For any $i, j, k \in \mathbb{Z}$, we have* $M_i^k \otimes M_j^s \cong \bigoplus_{r=0}^{k-1} M_{i+j+r}^s$.

Corollary 2.33. *Given any injective indecomposable comodules M_i^s and M_j^s, we have* $M_i^s \otimes M_j^s \cong \bigoplus_{k=0}^{s-1} M_{i+j+k}^s$.

We will later use the above formula for the tensor product of injective comodules to classify an important subring of the Green ring called the Grothendieck ring.

2.6 The Jacobson radical's action on H_s

We will now develop one last tool that will allow us to further decompose the tensor product of indecomposable comodules. Recall that the Jacobson radical of a comodule M is defined to be the intersection of all maximal subcomodules of M. We can use the Jacobson radical to determine a comodule decomposition into indecomposable summands by noting that the indecomposable summands of $J(M)$ are isomorphic to the indecomposable summands of M but with the top composition terms removed.

Example 2.34. *Consider the comodule*

$$M = M_0^3 \oplus M_1^2 \oplus M_2^4 \oplus M_3^1.$$

We know that M then has dimension 10*. We note that*

$$J(M) \cong M_0^2 \oplus M_1^1 \oplus M_2^3$$

and hence has dimension 6*. If we want, we can take the Jacobson radical a second time and find that*

$$J^2(M) \cong M_0^1 \oplus M_2^2$$

and so is of dimension 3*.*

Now, consider the comodule

$$N = M_0^2 \oplus M_1^2 \oplus M_2^2 \oplus M_3^4.$$

We know that $\dim(M) = \dim(N) = 10$. *We also see that*

$$J(N) \cong M_0^1 \oplus M_1^1 \oplus M_2^1 \oplus M_3^3$$

and so has dimension 6. *However, by applying the Jacobson radical a second time, we see that*

$$J^2(N) \cong M_3^2$$

and so only has dimension 2, *meaning* $\dim(J^2(N)) \neq \dim(J^2(M))$.

Now, we will try to reverse this process. If we can determine the dimension of $J^i(M)$ for different values of i, we can use this to give us information about the structure of M itself, allowing us to distinguish between different possibilities for the decomposition of M into indecomposable comodules. Consider the case now where $M = M_0^i \otimes M_0^j$ for some i, j. The following result will allow us to compute the Jacobson radical of the tensor product $M_0^i \otimes M_0^j$.

Proposition 2.35. *Let* H *be a Hopf algebra and* M *a finite dimensional comodule over* H. *Then* $J(M) = J(H^*) \cdot M$.

Proof. First, let $N = M/(J(H^*) \cdot M)$. We know that $J(H^*) \cdot N = 0$. By [5], we know that $J(H^*) = H_0^\perp$ where H_0 is the coradical of H and

$$H_0^\perp = \{f \in H^* : f(H_0) = 0\}.$$

Again by [5], this implies that $\rho(N) \subseteq N \otimes H_0$, implying that N forms an H_0-comodule. However, H_0 is cosemisimple, meaning N must be semisimple. But this implies then that $0 = J(N) = J(M/(J(H^*) \cdot M))$. We can thus conclude that $J(H^*) \cdot M \supseteq J(M)$.

Now, suppose that B is a maximal subcomodule of M. Then M/B is simple so that $\rho_{M/B}(M/B) \subset M/B \otimes H_0$, implying that $J(H^*) \cdot M/B = 0$ (using that

$J(H^*)^{\perp} = (H_0)^{\perp\ \perp} = H_0$ by [5]), hence that $J(H^*) \cdot M \subseteq B$. But then $J(H^*) \cdot M$ must be contained in every maximal subcomodule of M, so we must have $J(H^*) \cdot M \subseteq J(M)$. We can thus conclude that $J(H^*) \cdot M = J(M)$ as desired.

□

Now, in our case, the above proposition tells us that $J(M_0^i \otimes M_0^j) = J(H_s^*) \cdot (M_0^i \otimes M_0^j)$ where H_s^*, the dual of H_s, is spanned by the set of functions $(g^i x^j)^*$ defined by

$$(g^i x^j)^*(g^k x^m) = \begin{cases} 1 & : i = k, j = m \\ 0 & : i \neq k \text{ or } j \neq m. \end{cases}$$

For a comodule M with $m \in M$, we write

$$(g^i x^j)^* \cdot m = \sum (g^i x^j)^*(m_1)(m_0)$$

for $\rho(m) = \sum m_0 \otimes m_1$ where $m_0 \in M$ and $m_1 \in H_s$.

Example 2.36. *Consider the element $x \otimes x$ in $M_0^2 \otimes M_0^3$ over the Hopf algebra H_5. Then since*

$$\rho(x \otimes x) = (1 \otimes 1) \otimes x^2 + ((x \otimes 1) + \epsilon(1 \otimes x)) \otimes gx + (x \otimes x) \otimes g^2,$$

we know that

$$\begin{aligned}(gx)^* \cdot (x \otimes x) &= (gx)^*(x^2)(1 \otimes 1) + (gx)^*(gx)((x \otimes 1) + \epsilon(1 \otimes x)) + (gx)^*(g^2)(x \otimes x) \\ &= 0(1 \otimes 1) + 1((x \otimes 1) + \epsilon(1 \otimes x)) + 0(x \otimes x) \\ &= x \otimes 1 + \epsilon(1 \otimes x).\end{aligned}$$

Now, we note that $J(H_s^*) = \text{span}((g^i x^j)^*)$ satisfying that $j \geq 1$ and in general that $J^k(H_s^*) = \text{span}((g^i x^j)^*)$ satisfying $j \geq k$. This will allow us to calculate the dimension of $J^k(M)$ for our comodules $M = M_0^i \otimes M_0^j$.

Example 2.37. *Let $M = M_0^3 \otimes M_0^3$ be a comodule over H_5. We wish to write M as the sum of indecomposable comodules. By Theorem 2.30, we know that M_0^5 must be an injective summand of M. By Theorems 2.23 and 2.24, we know that S_1 and S_2 are the bottom composition terms of the remaining indecomposable summands of M, and S_3 and S_2 are the top composition terms of the remaining summands. This gives us two possibilities for our decomposition, namely*

$$M_0^3 \otimes M_0^3 \cong M_0^5 \oplus M_1^3 \oplus M_2^1$$

and

$$M_0^3 \otimes M_0^3 \cong M_0^5 \oplus M_1^2 \oplus M_2^2.$$

We will use the Jacobson radical as described previously to eliminate one of these possibilities. First, we note that

$$J^2(M_0^5 \oplus M_1^3 \oplus M_2^1) \cong M_0^3 \oplus M_1^1$$

is of dimension 4 while

$$J^2(M_0^5 \oplus M_1^2 \oplus M_2^2) \cong M_0^3$$

is of dimension 3.

We know that since $\{1, x, x^2\}$ forms a K-basis of M_0^3, thus the set

$$\{1 \otimes 1, 1 \otimes x, 1 \otimes x^2, x \otimes 1, x \otimes x, x \otimes x^2, x^2 \otimes 1, x^2 \otimes x, x^2 \otimes x^2\}$$

forms a basis of $M_0^3 \otimes M_0^3$. By Theorem 2.17, we know that

$$
\begin{aligned}
\rho(1\otimes 1) &= (1\otimes 1)\otimes 1\\
\rho(1\otimes x) &= (1\otimes 1)\otimes x + (1\otimes x)\otimes g\\
\rho(1\otimes x^2) &= (1\otimes 1)\otimes x^2 + (1+\epsilon)(1\otimes x)\otimes gx + (1\otimes x^2)\otimes g^2\\
\rho(x\otimes 1) &= (1\otimes 1)\otimes x + (x\otimes 1)\otimes g\\
\rho(x\otimes x) &= (1\otimes 1)\otimes x^2 + ((x\otimes 1)+\epsilon(1\otimes x))\otimes gx + (x\otimes x)\otimes g^2\\
\rho(x\otimes x^2) &= (1\otimes 1)\otimes x^3 + ((x\otimes 1)+(\epsilon+\epsilon^2)(1\otimes x))\otimes gx^2\\
&\quad + ((1+\epsilon)(x\otimes x)+\epsilon^2(1\otimes x^2))\otimes g^2x + (x\otimes x^2)\otimes g^3\\
\rho(x^2\otimes 1) &= (1\otimes 1)\otimes x^2 + (1+\epsilon)(x\otimes 1)\otimes gx + (x^2\otimes 1)\otimes g^2\\
\rho(x^2\otimes x) &= (1\otimes 1)\otimes x^3 + ((1+\epsilon)(x\otimes 1)+\epsilon^2(1\otimes x))\otimes gx^2\\
&\quad + ((x^2\otimes 1)+(\epsilon+\epsilon^2)(x\otimes x))\otimes g^2x + (x^2\otimes x)\otimes g^3\\
\rho(x^2\otimes x^2) &= (1\otimes 1)\otimes x^4 + ((1+\epsilon)(x\otimes 1)+(\epsilon^2+\epsilon^3)(1\otimes x))\otimes gx^3\\
&\quad + ((x^2\otimes 1)+(\epsilon+2\epsilon^2+\epsilon^3)(x\otimes x)+\epsilon^4(1\otimes x^2))\otimes g^2x^2\\
&\quad + ((1+\epsilon)(x^2\otimes x)+(\epsilon^2+\epsilon^3)(x\otimes x^2))\otimes g^3x + (x^2\otimes x^2)\otimes g^4
\end{aligned}
$$

We know that $J^2(M_0^3\otimes M_0^3)$ is spanned by the set

$$S = \{(g^ix^j)^*\cdot(x^k\otimes x^m) : j\geq 2,\ k,m\leq 2\}.$$

We can conclude that S is spanned by the multiset

$$
\begin{aligned}
S' = \{&(1\otimes 1),(1\otimes 1),(1\otimes 1),((x\otimes 1)+(\epsilon+\epsilon^2)(1\otimes x)),(1\otimes 1),(1\otimes 1),\\
&((1+\epsilon)(x\otimes 1)+\epsilon^2(1\otimes x)),(1\otimes 1),((1+\epsilon)(x\otimes 1)+(\epsilon^2+\epsilon^3)(1\otimes x)),\\
&z = ((x^2\otimes 1)+(\epsilon+2\epsilon^2+\epsilon^3)(x\otimes x)+\epsilon^4(1\otimes x^2))\}.
\end{aligned}
$$

We note that $Span(S')$ then has basis

$$S'' = \{(1 \otimes 1), (1 \otimes x), (x \otimes 1), z\}.$$

We thus know that $J^2(M_0^3 \otimes M_0^3) = J^2(H_5^*) \cdot (M_0^3 \otimes M_0^3)$ has dimension 4. Given that only one of our possible decompositions of $M_0^3 \otimes M_0^3$ into indecomposable summands satisfies this condition, we can thus conclude that

$$M_0^3 \otimes M_0^3 \cong M_0^5 \oplus M_1^3 \oplus M_2^1.$$

In fact, we note that the above proof gives us the same decomposition of $M_0^3 \otimes M_0^3$ in H_6, as the set S still has the corresponding basis S'' despite ϵ now being a sixth root of unity rather than a fifth root of unity.

3 The Grothendieck ring of H_s

3.1 The ring structure of K_0

Consider the Hopf algebra H_s as defined in the previous chapter.

$$H_s = K[x] * K[\mathbb{Z}]/\langle xg - \epsilon gx, x^s \rangle,$$

where g is a generator of $\mathbb{Z}$, and ϵ is a primitive s^{th} root of unity for $s \geq 2$. Recall that the indecomposable comodules of H_s are (up to isomorphism) of the form

$$M_i^j = K\{g^i, g^i x, g^i x^2, \cdots, g^i x^{j-1}\}$$

In this chapter, we will classify an interesting subring of the Green ring that consists of the sums of projective indecomposable comodules.

Definition 3.1. *Given a Hopf algebra H with Green ring $R(H)$, the Grothendieck ring $K_0(H)$ is the subring of $R(H)$ whose K-basis is given by the collection of all projective indecomposable (co)modules.*

Recall that the projective indecomposable comodules of H_s are precisely the comodules of the form

$$M_i^s = \{g^i, g^i x, g^i x^2, \cdots, g^i x^{s-1}\},$$

in other words the ones of maximal dimension s. We showed in Corollary 2.33 that the tensor product of projective comodules has the form

$$M_i^s \otimes M_j^s \cong \bigoplus_{k=0}^{s-1} M_{i+j+k}^s.$$

We note that the proof of the above result used calculations that took place inside the larger Green ring, namely products and sums involving non-projective comodules.

However, this tensor product in the Green ring descends to the Grothendieck ring as well, though comodules of smaller dimension do not exist there.

We denote $Q_i = M_i^s$, the projective comodule with bottom composition term S_i. Under this notation, our above tensor product formula can be rewritten

$$Q_i \otimes Q_j \cong \bigoplus_{k=0}^{s-1} Q_{i+j+k}.$$

This decomposition of the product of projective comodules gives us the classification of our Grothendieck ring $K_0[H_s]$.

Theorem 3.2. *The Hopf algebra H_s has Grothendieck ring*

$$K_0(H_s) \cong \mathbb{Z}[(Q_i)_{i\in\mathbb{Z}}]^+ / \left\langle Q_iQ_j = \sum_{k=0}^{s-1} Q_{i+j+k} \right\rangle.$$

In the above notation, $\mathbb{Z}[(Q_i)_{i\in\mathbb{Z}}]^+$ indicates that we are considering the ring of non-constant polynomials in the Q_i. We note this is the case as the Grothendieck ring is non-unital: the unit in the Green ring, M_0^1, is non-projective and thus does not appear in the Grothendieck ring. We write K_0 to denote $K_0(H_s)$ when the value of s is understood.

3.2 A simpler presentation of K_0

We will now show that the Grothendieck ring $K_0(H_s)$ can be written using a simpler presentation, namely one in which only s generators are used. More specfically, we claim that certain sets of s consecutive Q_i's will generate the whole ring.

Theorem 3.3. *The Grothendieck ring $K_0(H_s)$ can be generated by the set $P_i = \{Q_{i-(s-1)}, Q_{i-(s-2)}, \cdots, Q_{i-1}, Q_i\}$ if and only if $1 - s < i < s - 1$.*

Proof. $\Longleftarrow$

First, we will show that P_i generates $K_0 = K_0(H_s)$ for $1 - s < i \leq 0$. Proof proceeds by induction.

First, suppose $i = 0$, meaning we have the set $P_0 = \{Q_{-(s-1)}, Q_{-(s-2)}, \cdots, Q_{-1}, Q_0\}$. For all $j \in \mathbb{Z}$, we note that

$$\begin{aligned} Q_0 Q_{j-(s-1)} &= \sum_{k=0}^{s-1} Q_{j-(s-1)+k} \\ &= Q_j + \sum_{k=0}^{s-2} Q_{j-(s-1)+k}. \end{aligned}$$

Thus, for $j > 0$, we have

$$Q_j = Q_0 Q_{j-(s-1)} - \sum_{k=0}^{s-2} Q_{j-(s-1)+k}.$$

Thus, inductively we see that $Q_j \in \mathbb{Z}[P_0]$ for all $j > 0$.

Similarly, for all $j \in \mathbb{Z}$, we note that

$$\begin{aligned} Q_{-(s-1)} Q_{j+(s-1)} &= \sum_{k=0}^{s-1} Q_{j+k} \\ &= \sum_{k=-1}^{s-2} Q_{j+1+k} \\ &= Q_j + \sum_{k=0}^{s-2} Q_{j+1+k}. \end{aligned}$$

Thus, for $j < -(s-1)$, we have

$$Q_j = Q_{-(s-1)} Q_{j+(s-1)} - \sum_{k=0}^{s-2} Q_{j+1+k}.$$

Thus, we inductively see that $Q_j \in \mathbb{Z}[P_0]$ for all $j < -(s-1)$ as well. We conclude that we can use the set P_0 to generate Q_j for all $j \in \mathbb{Z}$, implying P_0 generates K_0.

Now, let n be fixed such that $1 - s < n < 0$. Suppose that for all m such that $n < m \leq 0$, P_m generates K_0. We will prove that $P_n = \{Q_{n-(s-1)}, Q_{n-(s-2)}, \cdots, Q_n\}$ generates K_0 as well. We need only show that $Q_{n+1} \in \mathbb{Z}[P_n]$, as then we would have $\mathbb{Z}[P_n] \supseteq \mathbb{Z}[P_{n+1}] = K_0$. Note that since $n > 1 - s$, we have that $n \geq 2 - s$. Also,

$$\begin{aligned} n - (s-1) &< 0 - (s-1) \\ &= 1 - s \\ &< 2 - s. \end{aligned}$$

Hence, we know that

$$2 - s \in \{n - (s-1), n - (s-2), \cdots, n\}$$

and thus that $Q_{2-s} \in P_n$.

Now notice that

$$\begin{aligned} Q_{2-s} Q_n &= \sum_{k=0}^{s-1} Q_{2-s+n+k} \\ &= Q_{n+1} + \sum_{k=0}^{s-2} Q_{(2-s)+n+k}. \end{aligned}$$

Hence we know that

$$Q_{n+1} = Q_{2-s} Q_n - \sum_{k=0}^{s-2} Q_{2-s+n+k}.$$

Finally, we note that for all $k \in \{0, 1, \cdots, s-2\}$,

$$\begin{aligned} 2 - s + n + k &\geq 2 - s + n \\ &= n - (s-2) \end{aligned}$$

and

$$\begin{aligned} 2 - s + n + k &\leq 2 - s + n + (s-2) \\ &= n; \end{aligned}$$

hence $Q_{(2-s)+n+k} \in P_n$ for all $k \in \{0, 1, \cdots, s-2\}$. We can thus conclude that $Q_{n+1} \in \mathbb{Z}[P_n]$ and thus that $\mathbb{Z}[P_n] \supseteq \mathbb{Z}[P_{n+1}] = K_0$. By induction, we thus know that P_i generates K_0 for all $1 - s < i \leq 0$.

Next, we will show that P_i generates K_0 for $0 \leq i < s-1$. Proof again proceeds by induction, noting that we have already proven our result in the case $i = 0$.

Let n be fixed such that $0 < n < s-1$. Suppose that for all m such that $0 \leq m < n$, P_m generates K_0. We will prove that $P_n = \{Q_{n-(s-1)}, Q_{n-(s-2)}, \cdots, Q_n\}$ generates K_0. Similar to before, we need only show that $Q_{n-s} \in \mathbb{Z}[P_n]$, as then we would have that $\mathbb{Z}[P_n] \supseteq \mathbb{Z}[P_{n-1}] = K_0$. Note that since $n < s-1$, we have that

$$n - (s-1) < (s-1) - (s-1) = 0,$$

hence $n - (s-1) \leq -1$. In addition, $n > 0$ implies that $n > -1$. Thus, we know that $Q_{-1} \in P_n$.

Now notice that

$$\begin{aligned} Q_{-1}Q_{n-(s-1)} &= \sum_{k=0}^{s-1} Q_{n-s+k} \\ &= Q_{n-s} + \sum_{k=1}^{s-1} Q_{n-s+k}. \end{aligned}$$

Thus, we can write

$$Q_{n-s} = Q_{-1}Q_{n-(s-1)} - \sum_{k=1}^{s-1} Q_{n-s+k}.$$

Finally, we note that for all $k \in \{1, \cdots, s-1\}$,

$$\begin{aligned} n - s + k &\geq n - s + 1 \\ &= n - (s-1) \end{aligned}$$

and

$$\begin{aligned} n - s + k &\leq n - s + (s-1) \\ &= n - 1 \\ &< n, \end{aligned}$$

implying $Q_{n-s+k} \in \mathbb{Z}[P_n]$. We can thus conclude that $Q_{n-s} \in \mathbb{Z}[P_n]$ and hence that $\mathbb{Z}[P_n] \supseteq \mathbb{Z}[P_{n-1}] = K_0$. We thus know by induction that P_i generates K_0 for all $0 \leq i < s-1$ and hence for all $1-s < i < s-1$ as desired.

$\Longrightarrow$

Now, suppose that $i \leq 1-s$. Consider the set $P_i = \{Q_{i-(s-1)}, Q_{i-(s-2)}, \cdots, Q_i\}$. We will show P_i does not generate K_0. Let $a, b \leq i$. We know that

$$Q_a Q_b = \sum_{k=0}^{s-1} Q_{a+b+k}.$$

Now, we notice that for $k \in \{0, 1, \cdots, s-1\}$, we have that

$$\begin{aligned} a + b + k &\leq i + i + (s-1) \\ &\leq 2(1-s) + (s-1) \\ &= 1 - s. \end{aligned}$$

Hence we know that $Q_a Q_b$ can be written as a sum of elements of $\mathbb{Z}[Q_i : i \leq 1-s]$. Since $a, b \leq i$ were arbitrary, this shows that $\mathbb{Z}[Q_i : i \leq 1-s]$ is closed under multiplication and hence forms a proper subring of K_0. Thus $\mathbb{Z}[P_i] \subseteq \mathbb{Z}[Q_i : i \leq 1-s] \subsetneq K_0$. Hence $\mathbb{Z}[P_i] \neq K_0$.

Finally, suppose that $i \geq s-1$. Consider the set $P_i = \{Q_{i-(s-1)}, Q_{i-(s-2)}, \cdots, Q_i\}$. As before, we will show that P_i does not generate K_0. Let $a, b \geq i-(s-1)$. We know as above that

$$Q_a Q_b = \sum_{k=0}^{s-1} Q_{a+b+k}.$$

Now, notice that for $k \in \{0, 1, \cdots, s-1\}$, we have that (using our assumption that $i-(s-1) \geq 0$)

$$\begin{aligned} a + b + k &\geq a + b \\ &\geq 2(i-(s-1)) \\ &\geq i-(s-1) \end{aligned}$$

Hence we know that Q_aQ_b can be written as a sum of elements of $\mathbb{Z}[Q_i : i \geq (s-1)-(s-1) = 0]$. Since $a, b \geq s-1$ were arbitrary, this shows that $\mathbb{Z}[Q_i : i \geq 0]$ is closed under multiplication and hence forms a proper subring of K_0. Thus $\mathbb{Z}[P_i] \subseteq \mathbb{Z}[Q_i : i \geq 0] \subsetneq K_0$. Hence $\mathbb{Z}[P_i] \neq K_0$, completing our proof.

□

We now know that the elements of K_0 can be generated by particular sets of consecutive projective comodules. The problem of determining which sets of non-consecutive Q_i generate K_0 is much more challenging, and we do not address it here. We will now show that for one of these sets, namely $P_0 = \{Q_{-s+1}, Q_{-s+2}, \cdots, Q_{-1}, Q_0\}$, that the relations satisfied only by the elements of P_0 in K_0 are enough to determine all of the relations of K_0.

Lemma 3.4. *Let R be the polynomial ring $\mathbb{Z}[X_{-s+1}, X_{-s+2}, \cdots, X_{-1}, X_0]$ in s variables. Consider the ring $R/I = \mathbb{Z}[X_{-s+1}, X_{-s+2}, \cdots, X_{-1}, X_0]/I$ where I is the ideal generated by all relations of the form $X_iX_j - X_aX_b$ with $i + j = a + b$ for $i, j, a, b \in \{-s+1, -s+2, \cdots, -1, 0\}$ as well as the relation*

$$X_{-s+1}X_0 - \sum_{k=0}^{s-1} X_{-s+1+k}.$$

For $i > 0$, we recursively define

$$X_i = X_0X_{i-(s-1)} - \sum_{k=0}^{s-2} X_{i-(s-1)+k},$$

and for $j < -s+1$, we recursively define

$$X_j = X_{-s+1}X_{j+(s-1)} - \sum_{k=0}^{s-2} X_{j+1+k}.$$

Then for all $i, j, a, b \in \mathbb{Z}$ satisfying $i + j = a + b$, we have $X_iX_j = X_aX_b$.

Proof. We will proceed via three inductive arguments. First, we will prove this result for all $i, j, a, b > -s+1$. We will then prove this result for all $i, j, a, b < 0$. Finally, we will prove this result in the most general case where i, j, a, b can range over the integers.

(I)

First, let $n > 0$ be fixed. Suppose that for all m, j satisfying $-s+1 \leq j < n-1, -s+1 < m < n$, we know that $X_m X_j = X_{m-1} X_{j+1}$. We know this result to be true already for $n = 1$ by the relations given in the ring R. Now, given our fixed n, we pick j such that $-s+1 \leq j < n$. We will show that $X_n X_j = X_{n-1} X_{j+1}$.

Without loss of generality, suppose that $j < n-1$; otherwise we would have that $j = n-1$ and our desired result is given by $X_n X_{n-1} = X_{n-1} X_n$ automatically. Our proof proceeds by the following argument, with explanations for numbered lines detailed below.

$$\begin{aligned}
X_n X_j &= X_j \left(X_0 X_{n-(s-1)} - \sum_{k=0}^{s-2} X_{n-(s-1)+k} \right) \\
&= X_j X_0 X_{n-(s-1)} - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} \\
&= (X_j X_{n-(s-1)}) X_0 - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} && (6) \\
&= X_{j+1} X_{n-s} X_0 - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} \\
&= X_{j+1} (X_0 X_{n-s}) - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} && (7) \\
&= X_{j+1} \left(\sum_{k=0}^{s-1} X_{n-s+k} \right) - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} \\
&= \sum_{k=0}^{s-1} X_{n-s+k} X_{j+1} - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k}
\end{aligned}$$

$$
\begin{aligned}
&= X_{n-1}X_{j+1} + \sum_{k=0}^{s-2} X_{n-s+k}X_{j+1} - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} \\
&= X_{n-1}X_{j+1} + \sum_{k=0}^{s-2} \left(X_{n-s+k}X_{j+1} - X_j X_{n-(s-1)+k}\right) \qquad (8) \\
&= X_{n-1}X_{j+1} + \sum_{k=0}^{s-2} \left(X_{n-(s-1)+k}X_j - X_j X_{n-(s-1)+k}\right) \\
&= X_{n-1}X_{j+1}
\end{aligned}
$$

- In order to apply the induction hypothesis in (6), we note that by assumption, $-s+1 \leq j < n-1$, $n-(s-1) < n$, and $n-(s-1) = n-s+1 > -s+1$.

- In (7), we note that (either by our given relation in the ring if $n = 1$ or by definition if $n > 1$)

$$X_{n-1} = X_0 X_{n-s} - \sum_{k=0}^{s-2} X_{n-s+k},$$

implying

$$
\begin{aligned}
X_0 X_{n-s} &= X_{n-1} + \sum_{k=0}^{s-2} X_{n-s+k} \\
&= \sum_{k=0}^{s-1} X_{n-s+k}.
\end{aligned}
$$

- Finally, in order to apply the induction hypothesis in (8), we note that for all $k \in \{0, 1, \cdots, s-2\}$,

$$
\begin{aligned}
n-s+k &\leq n-s+(s-2) \\
&= n-2 \\
&< n-1
\end{aligned}
$$

and

$$n - s + k \geq n - s$$
$$\geq 1 - s.$$

In addition, $j + 1 > j \geq -s + 1$, and $j + 1 < n$ (as $j < n - 1$ by assumption).

Knowing that $X_n X_j = X_{n-1} X_{j+1}$, by repeated application of our hypothesis that $X_m X_j = X_{m-1} X_{j+1}$ for $-s + 1 < m < n$, we get that $X_n X_j = X_a X_b$ for any $a, b \in \{-s+1, -s+2, \cdots, -1, 0, \cdots, n\}$ satisfying $n + j = a + b$ and $-s + 1 \leq j < n - 1$:

$$X_n X_j = X_{n-1} X_{j+1} = X_{n-2} X_{j+2} = \cdots = X_{n-(n-j-1)} X_{j+(n-j-1)} = X_j X_n,$$

noting that in the above string of equalities, the first equality follows from our inductive argument, the final equality follows from commutativity of the X_i, and all the other equalities follow from our inductive hypothesis. Applying induction then tells us that this result must hold true too for any $n > 0$.

(II)

Next, let $n < -s + 1$ be fixed. Suppose that for all m, j satisfying $n + 1 \leq m < 0$, $n + 1 < j \leq 0$, we know that $X_m X_j = X_{m+1} X_{j-1}$. We know this result to be true already for $n = -s$ by the relations given in the ring R. Now, given our fixed n, we pick j such that $n < j \leq 0$. We will show that $X_n X_j = X_{n+1} X_{j-1}$.

Without loss of generality, suppose that $j > n + 1$; otherwise we would have $j = n + 1$ and our desired result is given by $X_n X_{n+1} = X_{n+1} X_n$ automatically. Our proof proceeds by the following argument, with explanations for numbered lines detailed below.

$$
\begin{aligned}
X_n X_j &= X_j \left(X_{-s+1} X_{n+s-1} - \sum_{k=0}^{s-2} X_{n+1+k} \right) \\
&= X_j X_{-s+1} X_{n+s-1} - \sum_{k=0}^{s-2} X_j X_{n+1+k} \\
&= (X_j X_{n+s-1}) X_{-s+1} - \sum_{k=0}^{s-2} X_j X_{n+1+k} && (9) \\
&= (X_{j-1} X_{n+s}) X_{-s+1} - \sum_{k=0}^{s-2} X_j X_{n+1+k} \\
&= X_{j-1} (X_{-s+1} X_{n+s}) - \sum_{k=0}^{s-2} X_j X_{n+1+k} && (10) \\
&= X_{j-1} \left(\sum_{k=-1}^{s-2} X_{n+k+2} \right) - \sum_{k=0}^{s-2} X_j X_{n+1+k} \\
&= \sum_{k=-1}^{s-2} X_{n+k+2} X_{j-1} - \sum_{k=0}^{s-2} X_j X_{n+1+k} \\
&= X_{n+1} X_{j-1} + \sum_{k=0}^{s-2} X_{n+k+2} X_{j-1} - \sum_{k=0}^{s-2} X_j X_{n+1+k} \\
&= X_{n+1} X_{j-1} + \sum_{k=0}^{s-2} (X_{n+k+2} X_{j-1} - X_j X_{n+1+k}) && (11) \\
&= X_{n+1} X_{j-1} + \sum_{k=0}^{s-2} (X_{n+k+1} X_j - X_j X_{n+1+k}) \\
&= X_{n+1} X_{j-1}
\end{aligned}
$$

- In order to apply our induction hypothesis in (9), we note that $0 \geq j > n+1$ by hypothesis, $n+s-1 > n$ since $s \geq 2$, and

  $$n + s - 1 \leq -s + s - 1 = -1 < 0,$$

  again by assumption.

- In (10), we use that (by a relation from the ring if $n = -s$ or by definition if otherwise)

$$X_{n+1} = X_{-s+1}X_{n+s} - \sum_{k=0}^{s-2} X_{n+k+2},$$

implying that

$$\begin{aligned} X_{-s+1}X_{n+s} &= X_{n+1} + \sum_{k=0}^{s-2} X_{n+k+2} \\ &= \sum_{k=-1}^{s-2} X_{n+k+2}. \end{aligned}$$

- Finally, in order to apply our induction hypothesis in (11), we note that for all $k \in \{0, 1, \cdots, s-2\}$,

$$\begin{aligned} n + k + 2 &\leq n + 2 + (s - 2) \\ &= n + s \\ &\leq -s + s = 0, \end{aligned}$$

  $n + k + 2 \geq n + 2 > n + 1$, $j - 1 < j \leq 0$, and $j - 1 > n$ (as by assumption $j > n + 1$).

Knowing that $X_nX_j = X_{n+1}X_{j-1}$, by repeated application of our induction hypothesis that $X_mX_j = X_{m+1}X_{j-1}$ for $n+1 \leq m < 0$, we thus know that $X_nX_j = X_aX_b$ for any $a, b \in \{n, n+1, \cdots, -1, 0\}$ and $0 \geq j > n$:

$$X_nX_j = X_{n+1}X_{j-1} = X_{n+2}X_{j-2} = \cdots = X_{n+(j-n-1)}X_{j-(j-n-1)} = X_jX_n,$$

noting that in the above string of equalities, the first equality follows from our inductive argument, the final equality follows from commutativity of the X_i, and all the other equalities follow from our inductive hypothesis. Applying induction then tells us that this result must hold true too for any $n < -s + 1$.

(III)

Now, we have our desired result that $X_n X_j = X_a X_b$ if $n + j = a + b$ in the cases where either n, j, a, b are all less than 0 or all greater than $-s + 1$. We will now generalize these cases to include all possibilities for $n, j, a, b \in \mathbb{Z}$. Suppose that for fixed $n > 0$, we know that for all m, j satisfying $m < n$, $j < n - 1$, we have that $X_m X_j = X_{m-1} X_{j+1}$. We know this to be true for $n = 1$ by case (II) in our proof. We will show that $X_n X_j = X_{n-1} X_{j+1}$ We will complete our proof then by inducting on n.

Without loss of generality, we assume that $j < -s + 1$; otherwise, we know this result to be true by the first case of the proof. Our proof proceeds by the following argument, with explanations for numbered lines detailed below.

$$
\begin{aligned}
X_n X_j &= X_j \left(X_0 X_{n-(s-1)} - \sum_{k=0}^{s-2} X_{n-(s-1)+k} \right) \\
&= X_j X_0 X_{n-(s-1)} - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} \\
&= (X_j X_{n-(s-1)}) X_0 - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} && (12) \\
&= X_{j-s+2} X_{n-1} X_0 - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} \\
&= X_{n-1} (X_{j-s+2} X_0) - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} && (13) \\
&= X_{n-1} (X_{-s+1} X_{j+1}) - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} && (14)
\end{aligned}
$$

$$
\begin{aligned}
&= X_{n-1}\left(\sum_{k=0}^{s-1} X_{j-s+2+k}\right) - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} \\
&= \sum_{k=0}^{s-1} X_{n-1} X_{j-s+2+k} - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} \\
&= X_{n-1} X_{j-s+2+(s-1)} + \sum_{k=0}^{s-2} X_{n-1} X_{j-s+2+k} - \sum_{k=0}^{s-2} X_j X_{n-(s-1)+k} \\
&= X_{n-1} X_{j+1} + \sum_{k=0}^{s-2} \left(X_{n-1} X_{j-s+2+k} - X_j X_{n-(s-1)+k}\right) \qquad (15) \\
&= X_{n-1} X_{j+1} + \sum_{k=0}^{s-2} \left(X_{n-1} X_{j-s+2+k} - X_{n-1} X_{j-s+2+k}\right) \\
&= X_{n-1} X_{j+1}
\end{aligned}
$$

- In order to apply our induction hypothesis in (12), we first note that $j < n$, and so $j - k < n$ for all $k \geq 0$. Additionally, $n - (s-1) + k < n - 1$ for all $k \leq s - 3$; hence, we can apply our induction hypothesis $s - 2$ consecutive times (once for each $k \in \{0, 1, \cdots, s-3\}$) to $X_j X_{n-(s-1)}$ in order to get $X_{j-s+2} X_{n-1}$.

- In order to apply our induction hypothesis (as well as our result in case (II)) in (13), we note that for all $k \in \{0, 1, \cdots, -(j+1)\}$,

$$
\begin{aligned}
j - s + 2 + k &\leq j - s + 2 - (j+1) \\
&= -s + 1 \\
&\leq -2 + 1 < 0 \leq n - 1.
\end{aligned}
$$

  Also, for all $k \in \{0, 1, \cdots, -(j+1)\}$, we know that $0 - k \leq 0 < n$. Thus, we can apply our induction hypothesis $-j$ times (once for each $k \in \{0, 1, \cdots, -(j+1)\}$) to $X_{j-s+2} X_0$ to get $X_{-s+1} X_{j+1}$.

- In (14), we use that

$$X_{j-s+2} := X_{-s+1}X_{j+1} - \sum_{k=0}^{s-2} X_{j-s+3+k},$$

implying that

$$\begin{aligned} X_{-s+1}X_{j+1} &= X_{j-s+2} + \sum_{k=0}^{s-2} X_{j-s+3+k} \\ &= \sum_{k=-1}^{s-2} X_{j-s+3+k} \\ &= \sum_{k=0}^{s-1} X_{j-s+2+k}. \end{aligned}$$

- Finally, in order to apply our induction hypothesis in (15), we note that for each $r \in \{0, 1, \cdots, n-j-2\}$,

$$\begin{aligned} j + r &\leq j + (n - j - 2) \\ &= n - 2 \\ &< n - 1, \end{aligned}$$

and

$$\begin{aligned} n - (s-1) + k - r &\leq n - (s-1) + k \\ &\leq n - (s-1) + (s-2) \\ &= n - 1 < n. \end{aligned}$$

Thus, we can safely apply our induction hypothesis $n-j-1$ times (once for each $r \in \{0, 1, \cdots, n-j-2\}$) to $X_j X_{n-(s-1)+k}$ to get $X_{n-1} X_{j-s+2+k}$.

Knowing that $X_n X_j = X_{n-1} X_{j+1}$, by repeated application of our induction hypothesis that $X_m X_j = X_{m-1} X_{j+1}$ for $m < n$ and $j < n-1$, we thus know that $X_n X_j = X_a X_b$ for any $a, b \leq n$:

$$X_n X_j = X_{n-1} X_{j+1} = X_{n-2} X_{j+2} = \cdots = X_{j+1} X_{n-1} = X_j X_n,$$

noting that in the above string of equalities, the first equality follows from our inductive argument, the final equality follows from commutativity of the X_i, and all the other equalities follow from our inductive hypothesis. Applying induction then tells us that this result must hold true too for any $n > 0$. By combining all of the cases we have shown, we know that $X_i X_j = X_a X_b$ for all $i, j, a, b \in \mathbb{Z}$ as desired.

□

Lemma 3.5. *For any $i, j \in \mathbb{Z}$, in the ring R/I from Lemma 3.4, we have $X_i X_j = \sum_{k=0}^{s-1} X_{i+j+k}$.*

Proof. First, suppose that $i + j < -s + 1$. Then we know that

$$X_{i+j} = X_{-s+1} X_{i+j+s-1} - \sum_{k=0}^{s-2} X_{i+j+1+k}$$

and hence that

$$\begin{aligned} X_i X_j &= X_{-s+1} X_{i+j+s-1} \\ &= X_{i+j} + \sum_{k=0}^{s-2} X_{i+j+1+k} \\ &= \sum_{k=-1}^{s-2} X_{i+j+k+1} \end{aligned}$$

$$= \sum_{k=0}^{s-1} X_{i+j+k}.$$

Now, suppose that $i + j \geq -s + 1$. Then we know that

$$X_{i+j+(s-1)} = X_0 X_{i+j} - \sum_{k=0}^{s-2} X_{i+j+k}.$$

This implies that

$$\begin{aligned} X_i X_j &= X_0 X_{i+j} \\ &= X_{i+j+(s-1)} + \sum_{k=0}^{s-2} X_{i+j+k} \\ &= \sum_{k=0}^{s-1} X_{i+j+k}. \end{aligned}$$

□

We are now ready to show that the Grothendieck ring K_0 can be presented in the form R/I.

Theorem 3.6. *Let $R = \mathbb{Z}[X_{-s+1}, X_{-s+2}, \cdots, X_0]$. Then $R/I = \mathbb{Z}[X_{-s+1}, X_{-s+2}, \cdots, X_0]/I$ and $\mathbb{Z} \oplus K_0$ are isomorphic as rings, where $\mathbb{Z} \oplus K_0$ is the unital extension of K_0 and I is the ideal of Lemma 3.4 generated by all elements of the form $X_i X_j - X_a X_b$ where $i + j = a + b$ and by $X_{-s+1} X_0 - \sum_{k=0}^{s-1} X_{-s+1+k}$.*

Proof. First, we define the map $\phi : R \to \mathbb{Z} \oplus K_0$ by $\phi(X_i) = Q_i$. By Theorem 3.3, we know that ϕ is surjective, as K_0 is generated by the set $\{Q_{-s+1}, Q_{-s+2}, \cdots, Q_0\}$, hence $K_0 \oplus \mathbb{Z}$ is generated by the set $\{Q_{-s+1}, Q_{-s+2}, \cdots, Q_0, 1\} = \{\phi(X_{-s+1}), \phi(X_{-s+2}), \cdots, \phi(X_0), \phi(1)\}$. We also know by Lemma 3.4 that $\ker \phi$ contains the ideal I: for $i, j, a, b \in \mathbb{Z}$ satisfying $i + j = a + b$, we know that

$$\begin{aligned}\phi(X_iX_j - X_aX_b) &= \phi(X_i)\phi(X_j) - \phi(X_a)\phi(X_b)\\ &= Q_iQ_j - Q_aQ_b\\ &= 0.\end{aligned}$$

In addition, we know that

$$\begin{aligned}\phi\left(X_{-s+1}X_0 - \sum_{k=0}^{s-1} X_{-s+1+k}\right) &= \phi(X_{-s+1})\phi(X_0) - \sum_{k=0}^{s-1}\phi(X_{-s+1+k})\\ &= Q_{-s+1}Q_0 - \sum_{k=0}^{s-1} Q_{-s+1+k}\\ &= 0.\end{aligned}$$

Thus, ϕ induces a homomorphism

$$\overline{\phi} : R/I = \mathbb{Z}[X_{-s+1}, X_{-s+2}, \cdots, X_0]/I \to \mathbb{Z} \oplus K_0.$$

Now, for each $i \in \mathbb{Z}$ such that $i > 0$, we (recursively) define

$$X_i = X_0X_{i-(s-1)} - \sum_{k=0}^{s-2} X_{i-(s-1)+k}.$$

For each $j \in \mathbb{Z}$ such that $j < -s+1$, we define

$$X_j = X_{-s+1}X_{j+s-1} - \sum_{k=0}^{s-2} X_{i+1+k}.$$

This gives a definition for X_m for each $m \in \mathbb{Z}$ in terms of our set of generators. Now, notice that by the relations in the Grothendieck ring K_0, we must have that

$\overline{\phi}(X_m) = Q_m$ for each $m \in \mathbb{Z}$.

We note that by Lemma 3.4, in R/I we have that $X_iX_j = X_mX_n$ for all $i, j, m, n \in \mathbb{Z}$ such that $i + j = m + n$, and more importantly, we know by Lemma 3.5 that

$$X_iX_j = \sum_{k=0}^{s-1} X_{i+j+k}.$$

In other words, in the ring R/I, we know that any polynomial of the X_i is a monomial, ie: can be written as a sum of the form $\sum X_i$. This tells us that as a $\mathbb{Z}$-module, R/I is generated by the set $(X_n)_{n\in\mathbb{Z}}$.

Note too that since $(Q_i)_{i\in\mathbb{Z}}$ forms a linearly independent set in K_0 and $Q_i = \overline{\phi(X_i)}$, it follows that the set $(X_i)_{i\in\mathbb{Z}}$ must be linearly independent over $\mathbb{Z}$ in R/I. Thus, the set $(X_i)_{i\in\mathbb{Z}} \cup \{1\}$ forms a $\mathbb{Z}$-basis of R/I. As $(Q_i)_{i\in\mathbb{Z}} \cup \{1\}$ forms a $\mathbb{Z}$-basis of $\mathbb{Z} \oplus K_0$, we can thus conclude that $\overline{\phi}$ maps a basis of R/I to a basis of $\mathbb{Z} \oplus K_0$ and thus is an isomorphism as desired.

□

Thus, we note that the ring R/I gives us a simpler presentation for the unital extension of K_0.

3.3 The generalized multiplicative property of K_0

We know that the product of any two projective comodules Q_i and Q_j is determined fully by the sum $i + j$. We will now prove that this property holds true for products of arbitrarily many comodules as long as the products have the same number of indecomposable factors.

Theorem 3.7. *Let $A = \{a_1, \cdots a_k\}$ and $B = \{b_1, \cdots b_k\}$ be multisets of integers. If $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i$, then $\prod_{i=1}^{k} Q_{a_i} = \prod_{i=1}^{k} Q_{b_i}$, hence $\bigotimes_{i=1}^{k} M^s_{a_i} = \bigotimes_{i=1}^{k} M^s_{b_i}$.*

Proof. Proof proceeds by induction. We have proven this result in the case where

$n = 2$. Let $n \geq 2$ be fixed. Suppose now that for all $m \leq n$, we know this result to be true for m-fold products. Now, consider the sets $A = \{a_1, \cdots, a_{n+1}\}$ and $B = \{b_1, \cdots, b_{n+1}\}$. Suppose that $\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n+1} b_i$. Then we must have $b_{n+1} = \sum_{i=1}^{n+1} a_i - \sum_{j=1}^{n} b_j$. We see then that

$$\begin{aligned}
\prod_{i=1}^{n+1} Q_{a_i} &= \prod_{i=1}^{n-1} Q_{a_i} \times (Q_{a_n} \times Q_{a_{n+1}}) \\
&= \prod_{i=1}^{n-1} Q_{a_i} \times \left(\sum_{k=0}^{s-1} Q_{a_n + a_{n+1} + k} \right) \\
&= \sum_{k=0}^{s-1} \left(\prod_{i=1}^{n-1} Q_{a_i} \times Q_{a_n + a_{n+1} + k} \right)
\end{aligned}$$

For each $k \in \{0, 1, \cdots, s-1\}$, we note that each product $S_k = \prod_{i=1}^{n-1} Q_{a_i} \times Q_{a_n + a_{n+1} + k}$ in the above summand has index sum $\left(\sum_{i=1}^{n+1} a_i \right) + k$. Now, we consider the product of the elements of B:

$$\begin{aligned}
\prod_{i=1}^{n+1} Q_{b_i} &= \prod_{i=1}^{n-1} Q_{b_i} \times (Q_{b_n} \times Q_{b_{n+1}}) \\
&= \prod_{i=1}^{n-1} Q_{b_i} \times \left(\sum_{k=0}^{s-1} Q_{b_n + b_{n+1} + k} \right) \\
&= \sum_{k=0}^{s-1} \left(\prod_{i=1}^{n-1} Q_{b_i} \times Q_{b_n + b_{n+1} + k} \right)
\end{aligned}$$

For each $k \in \{0, 1, \cdots, s-1\}$, we note that each product $T_k = \prod_{i=1}^{n-1} Q_{b_i} \times Q_{b_n + b_{n+1} + k}$

in the above summand has index sum

$$\left(\sum_{i=1}^{n} b_i + b_{n+1}\right) + k = \sum_{i=1}^{n} b_i + \left(\sum_{i=1}^{n+1} a_i - \sum_{i=1}^{n} b_i\right) + k$$
$$= \left(\sum_{i=1}^{n+1} a_i\right) + k.$$

Thus, S_k and T_k have the same index sum. In addition, as both S_k and T_k are both n-fold products of Q_i's, by hypothesis we know that $S_k = T_k$. Thus, we can conclude that

$$\prod_{i=1}^{n+1} Q_{a_i} = \sum_{k=0}^{s-1} S_k = \sum_{k=0}^{s-1} T_k = \prod_{i=1}^{n+1} Q_{b_i}.$$

By induction then, we can conclude that for any $k \geq 2$, any two k-fold products satisfy $\prod_{i=1}^{k} Q_{a_i} = \prod_{i=1}^{k} Q_{b_i}$ as long as $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i$ as desired.

□

We note that it is not sufficient for two multisets of indices to have the same sum; they must also have the same number of elements. As an example, we see that while $Q_3 \times Q_3 = \sum_{k=0}^{s-1} Q_{6+k}$, we know

$$Q_2 \times Q_2 \times Q_2 = Q_2 \times \left(\sum_{k=0}^{s-1} Q_{4+k}\right)$$
$$= \sum_{k=0}^{s-1} (Q_2 \times Q_{4+k})$$
$$= \sum_{k=0}^{s-1} \sum_{j=0}^{s-1} Q_{(6+k)+j}$$

Since $s \geq 2$, we know then that these expressions are not equal, implying $Q_3 \times Q_3 \neq Q_2 \times Q_2 \times Q_2$.

4 The Green rings of H_s for $s \leq 6$

Using the techniques we have previously defined, we are now prepared to fully classify the Green rings of the Hopf algebras H_2, H_3, H_4, H_5, and H_6.

4.1 The Green ring of H_2

We begin by considering the Green ring of H_2. H_2 has two indecomposable comodules whose bottom composition series terms are S_0, namely $S_0 = M_0^1$ and $Q_0 = M_0^2$. We will now classify the products of these indecomposables:

- $M_0^1 \otimes M_0^1 \cong M_0^1$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^2 \cong M_0^2$ as M_0^1 is the unit.
- $M_0^2 \otimes M_0^2 \cong M_0^2 \oplus M_1^2$ as the product of injective comodules.

This gives us the multiplication table

$\bigotimes$	$\boldsymbol{M_0^1}$	$\boldsymbol{M_0^2}$
$\boldsymbol{M_0^1}$	M_0^1	M_0^2
$\boldsymbol{M_0^2}$	M_0^2	$M_0^2 \oplus M_1^2$

Table 1: Multiplication table of H_2

and gives rise to the Green ring:

$$R(H_2) \cong \mathbb{Z}[X, X^{-1}, T_2]/\langle T_2^2 = T_2 + T_2 X \rangle$$

under the isomorphism mapping

$$M_1^1 \mapsto X$$
$$M_{-1}^1 \mapsto X^{-1}$$
$$M_0^2 \mapsto T_2.$$

For H_2, we have also calculated explicit isomorphisms for the tensor products leading to the multiplication table seen above. These isomorphisms $\Phi_{i,j}$ map basis elements of $M_0^i \otimes M_0^j$ to basis elements of our sums of indecomposable comodules and satisfy $\Phi \circ \rho = \Delta \circ \Phi$.

For $M_0^1 \otimes M_0^1 \cong M_0^1$, we have the isomorphism

$$(1 \otimes 1) \mapsto 1.$$

For $M_0^1 \otimes M_0^2 \cong M_0^2$, we have the isomorphisms

$$(1 \otimes 1) \mapsto 1$$
$$(1 \otimes x) \mapsto x.$$

For $M_0^2 \otimes M_0^1 \cong M_0^2$, we have the isomorphisms

$$(1 \otimes 1) \mapsto 1$$
$$(x \otimes 1) \mapsto x.$$

Finally, for $M_0^2 \otimes M_0^2 \cong M_0^2 \oplus M_1^2$, we have the isomorphisms

$$
\begin{aligned}
(1 \otimes 1) &\mapsto 1 \\
(1 \otimes x) &\mapsto \frac{x+g}{2} \\
(x \otimes 1) &\mapsto \frac{x-g}{2} \\
(x \otimes x) &\mapsto -gx.
\end{aligned}
$$

4.2 The Green ring of H_3

Next, we will consider the Green ring of H_3. H_3 has two indecomposable comodules whose bottom composition series terms are S_0, namely $S_0 = M_0^1$, M_0^2 and $Q_0 = M_0^3$. We will now classify the products of these indecomposables:

- $M_0^1 \otimes M_0^1 \cong M_0^1$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^2 \cong M_0^2$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^3 \cong M_0^3$ as M_0^1 is the unit.
- $M_0^2 \otimes M_0^2 \cong M_0^3 \oplus M_1^1$. We know that M_0^3 must be an injective summand by Theorem 2.30. We are left with S_1 as the only remaining top and bottom composition term of the remaining summands of our tensor product. Thus, our only remaining summand must be $M_1^1 = S_1$.
- $M_0^2 \otimes M_0^3 \cong M_0^3 \oplus M_1^3$ as M_0^3 is injective.
- $M_0^3 \otimes M_0^3 \cong M_0^3 \oplus M_1^3 \oplus M_2^3$ as M_0^3 is injective.

This gives us the multiplication table

$\bigotimes$	$\boldsymbol{M_0^1}$	$\boldsymbol{M_0^2}$	$\boldsymbol{M_0^3}$
$\boldsymbol{M_0^1}$	M_0^1	M_0^2	M_0^3
$\boldsymbol{M_0^2}$	M_0^2	$M_0^3 \oplus M_1^1$	$M_0^3 \oplus M_1^3$
$\boldsymbol{M_0^3}$	M_0^3	$M_0^3 \oplus M_1^3$	$M_0^3 \oplus M_1^3$ $\oplus M_2^3$

Table 2: Multiplication table of H_3

and gives rise to the Green ring

$$R(H_3) \cong \mathbb{Z}[X, X^{-1}, T_2, T_3]/I$$

where I consists of the relations

$$T_2^2 = T_3 + X$$
$$T_2T_3 = T_3 + T_3X$$
$$T_3^2 = T_3 + T_3X + T_3X^2$$

under the isomorphism mapping

$$M_1^1 \mapsto X$$
$$M_{-1}^1 \mapsto X^{-1}$$
$$M_0^2 \mapsto T_2$$
$$M_0^3 \mapsto T_3.$$

4.3 The Green ring of H_4

We will next consider the Green ring of H_4. H_4 has four indecomposable comodules whose bottom composition series terms are S_0, namely $S_0 = M_0^1$, M_0^2 , M_0^3, and $Q_0 = M_0^4$. We will now classify the products of these indecomposables:

- $M_0^1 \otimes M_0^1 \cong M_0^1$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^2 \cong M_0^2$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^3 \cong M_0^3$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^4 \cong M_0^4$ as M_0^1 is the unit.
- $M_0^2 \otimes M_0^2 \cong M_0^3 \oplus M_1^1$. We first note that $M_0^2 \otimes M_0^2$ must have M_0^3 as a summand by Theorem 2.27. The remaining composition term and hence summand must then be $M_1^1 = S_1$.
- $M_0^2 \otimes M_0^3 \cong M_0^4 \oplus M_1^2$. By Theorem 2.30, we know M_0^4 must be an injective summand of this product. We are left over with S_1 as a bottom summand and S_2 as a top summand; by process of elimination, they must belong to the same summand, M_1^2.
- $M_0^2 \otimes M_0^4 \cong M_0^4 \oplus M_1^4$ as M_0^4 is injective.
- $M_0^3 \otimes M_0^3 \cong M_0^4 \oplus M_1^4 \oplus M_2^1$. By Theorem 2.30, we know M_0^4 and M_1^4 must both be injective summands of this product. We are left over with just S_2 as the only remaining bottom and top summand; thus, our last summand is $M_2^1 = S_2$.
- $M_0^3 \otimes M_0^4 \cong M_0^4 \oplus M_1^4 \oplus M_2^4$ as M_0^4 is injective.
- $M_0^4 \otimes M_0^4 \cong M_0^4 \oplus M_1^4 \oplus M_2^4 \oplus M_3^4$ as M_0^4 is injective.

This gives us the multiplication table

$\bigotimes$	$\boldsymbol{M_0^1}$	$\boldsymbol{M_0^2}$	$\boldsymbol{M_0^3}$	$\boldsymbol{M_0^4}$
$\boldsymbol{M_0^1}$	M_0^1	M_0^2	M_0^3	M_0^4
$\boldsymbol{M_0^2}$	M_0^2	$M_0^3 \oplus M_1^1$	$M_0^4 \oplus M_1^2$	$M_0^4 \oplus M_1^4$
$\boldsymbol{M_0^3}$	M_0^3	$M_0^4 \oplus M_1^2$	$M_0^4 \oplus M_1^4 \oplus M_2^1$	$M_0^4 \oplus M_1^4 \oplus M_2^4$
$\boldsymbol{M_0^4}$	M_0^4	$M_0^4 \oplus M_1^4$	$M_0^4 \oplus M_1^4 \oplus M_2^4$	$M_0^4 \oplus M_1^4 \oplus M_2^4 \oplus M_3^4$

Table 3: Multiplication table of H_4

and gives rise to the Green ring

$$R(H_4) \cong \mathbb{Z}[X, X^{-1}, T_2, T_3, T_4]/I$$

where I consists of the relations

$$\begin{aligned}
T_2^2 &= T_3 + X \\
T_2T_3 &= T_4 + T_2X \\
T_2T_4 &= T_4 + T_4X \\
T_3^2 &= T_4 + T_4X + X^2 \\
T_3T_4 &= T_4 + T_4X + T_4X^2 \\
T_4^2 &= T_4 + T_4X + T_4X^2 + T_4X^3
\end{aligned}$$

under the isomorphism mapping

$$\begin{aligned}
M_1^1 &\mapsto X \\
M_{-1}^1 &\mapsto X^{-1}
\end{aligned}$$

$$M_0^2 \mapsto T_2$$

$$M_0^3 \mapsto T_3$$

$$M_0^4 \mapsto T_4.$$

4.4 The Green ring of H_5

Next, we will consider the Green ring of H_5. H_5 has five indecomposable comodules whose bottom composition series terms are S_0, namely $S_0 = M_0^1$, M_0^2 , M_0^3, M_0^4, and $Q_0 = M_0^5$. We will now classify the products of these indecomposables:

- $M_0^1 \otimes M_0^1 \cong M_0^1$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^2 \cong M_0^2$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^3 \cong M_0^3$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^4 \cong M_0^4$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^5 \cong M_0^5$ as M_0^1 is the unit.
- $M_0^2 \otimes M_0^2 \cong M_0^3 \oplus M_1^1$ by the same reasoning as in H_4.
- $M_0^2 \otimes M_0^3 \cong M_0^4 \oplus M_1^2$. We know from Theorem 2.27 that M_0^4 must be a summand of maximal dimension. The only remaining top composition term is S_2 and the only remaining bottom composition term is S_1; they therefore must belong to the same summand, namely M_1^2.
- $M_0^2 \otimes M_0^4 \cong M_0^5 \oplus M_1^3$. We know from Theorem 2.30 that M_0^5 must be an injective summand of this product. This leaves us with a remaining bottom composition term of S_1 and top of S_3; they therefore must belong to the same summand, namely M_1^3.
- $M_0^2 \otimes M_0^5 \cong M_0^5 \oplus M_1^5$ by Theorem 2.32 since M_0^5 is injective.

- $M_0^3 \otimes M_0^3 \cong M_0^5 \oplus M_1^3 \oplus M_2^1$ by Example 2.37.

- $M_0^3 \otimes M_0^4 \cong M_0^5 \oplus M_1^5 \oplus M_2^2$. We know from Theorem 2.30 that M_0^5 and M_1^5 must be injective summands. This leaves us S_2 as the only remaining bottom composition and S_3 as the only remaining top; thus, they belong to the same summand M_2^2.

- $M_0^3 \otimes M_0^5 \cong M_0^5 \oplus M_1^5 \oplus M_2^5$ by Theorem 2.32 since M_0^5 is injective.

- $M_0^4 \otimes M_0^4 \cong M_0^5 \oplus M_1^5 \oplus M_2^5 \oplus M_3^1$. We know from Theorem 2.30 that M_0^5, M_1^5, and M_2^5 must be injective summands. This leaves $S_3 = M_3^1$ as the only remaining composition term and hence summand.

- $M_0^4 \otimes M_0^5 \cong M_0^5 \oplus M_1^5 \oplus M_2^5 \oplus M_3^5$ by Theorem 2.32 since M_0^5 is injective.

- $M_0^5 \otimes M_0^5 \cong M_0^5 \oplus M_1^5 \oplus M_2^5 \oplus M_3^5 \oplus M_4^5$ by Theorem 2.32 since M_0^5 is injective.

This gives us the multiplication table

$\bigotimes$	$\boldsymbol{M_0^1}$	$\boldsymbol{M_0^2}$	$\boldsymbol{M_0^3}$	$\boldsymbol{M_0^4}$	$\boldsymbol{M_0^5}$
$\boldsymbol{M_0^1}$	M_0^1	M_0^2	M_0^3	M_0^4	M_0^5
$\boldsymbol{M_0^2}$	M_0^2	$M_0^3 \oplus M_1^1$	$M_0^4 \oplus M_1^2$	$M_0^5 \oplus M_1^3$	$M_0^5 \oplus M_1^5$
$\boldsymbol{M_0^3}$	M_0^3	$M_0^4 \oplus M_1^2$	$M_0^5 \oplus M_1^3 \oplus M_2^1$	$M_0^5 \oplus M_1^5 \oplus M_2^2$	$M_0^5 \oplus M_1^5 \oplus M_2^5$
$\boldsymbol{M_0^4}$	M_0^4	$M_0^5 \oplus M_1^3$	$M_0^5 \oplus M_1^5 \oplus M_2^2$	$M_0^5 \oplus M_1^5 \oplus M_2^5 \oplus M_3^1$	$M_0^5 \oplus M_1^5 \oplus M_2^5 \oplus M_3^5$
$\boldsymbol{M_0^5}$	M_0^5	$M_0^5 \oplus M_1^5$	$M_0^5 \oplus M_1^5 \oplus M_2^5$	$M_0^5 \oplus M_1^5 \oplus M_2^5 \oplus M_3^5$	$M_0^5 \oplus M_1^5 \oplus M_2^5 \oplus M_3^5 \oplus M_4^5$

Table 4: Multiplication table of H_5

and gives rise to the Green ring

$$R(H_5) \cong \mathbb{Z}[X, X^{-1}, T_2, T_3, T_4, T_5]/I$$

where I consists of the relations

$$
\begin{aligned}
T_2^2 &= T_3 + X \\
T_2T_3 &= T_4 + T_2X \\
T_2T_4 &= T_5 + T_3X \\
T_2T_5 &= T_5 + T_5X \\
T_3^2 &= T_5 + T_3X + X^2 \\
T_3T_4 &= T_5 + T_5X + T_2X^2 \\
T_3T_5 &= T_5 + T_5X + T_5X^2 \\
T_4^2 &= T_5 + T_5X + T_5X^2 + X^3 \\
T_4T_5 &= T_5 + T_5X + T_5X^2 + T_5X^3 \\
T_5^2 &= T_5 + T_5X + T_5X^2 + T_5X^3 + T_5X^4
\end{aligned}
$$

under the isomorphism mapping

$$
\begin{aligned}
M_1^1 &\mapsto X \\
M_{-1}^1 &\mapsto X^{-1} \\
M_0^2 &\mapsto T_2 \\
M_0^3 &\mapsto T_3 \\
M_0^4 &\mapsto T_4 \\
M_0^5 &\mapsto T_5.
\end{aligned}
$$

4.5 The Green ring of H_6

Finally, we will consider the Green ring of H_6. H_5 has six indecomposable comodules whose bottom composition series terms are S_0, namely $S_0 = M_0^1$, M_0^2 , M_0^3, M_0^4, M_0^5, and $Q_0 = M_0^6$. We will now classify the products of these indecomposables:

- $M_0^1 \otimes M_0^1 \cong M_0^1$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^2 \cong M_0^2$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^3 \cong M_0^3$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^4 \cong M_0^4$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^5 \cong M_0^5$ as M_0^1 is the unit.
- $M_0^1 \otimes M_0^6 \cong M_0^6$ as M_0^1 is the unit.
- $M_0^2 \otimes M_0^2 \cong M_0^3 \oplus M_1^1$ by the same reasoning as in H_4 and H_5.
- $M_0^2 \otimes M_0^3 \cong M_0^4 \oplus M_1^2$ by the same reasoning as in H_5.
- $M_0^2 \otimes M_0^4 \cong M_0^5 \oplus M_1^3$. We know from Theorem 2.27 that M_0^5 must be a summand of maximal dimension. The only remaining top composition term is S_3 and the only remaining bottom composition term is S_1; they therefore must belong to the same summand, namely M_1^3.
- $M_0^2 \otimes M_0^5 \cong M_0^6 \oplus M_1^4$. We know from Theorem 2.30 that M_0^6 must be an injective summand of this product. The only remaining top composition term is S_4 and the only remaining bottom composition term is S_1; they therefore must belong to the same summand, namely M_1^4.
- $M_0^2 \otimes M_0^6 \cong M_0^6 \oplus M_1^6$ by Theorem 2.32 since M_0^6 is injective.
- $M_0^3 \otimes M_0^3 \cong M_0^5 \oplus M_1^3 \oplus M_2^1$ by Example 2.37.

- $M_0^3 \otimes M_0^4 \cong M_0^6 \oplus M_1^4 \oplus M_2^2$. We know from Theorem 2.30 that M_0^6 must be an injective summand of this product. The remaining top composition terms of this product are S_4 and S_3, while the remaining bottom composition terms are S_1 and S_2. Thus, we know that $M_0^3 \otimes M_0^4$ decomposes as either

$$M_0^6 \oplus M_1^4 \oplus M_2^2$$

or

$$M_0^6 \oplus M_1^3 \oplus M_2^3.$$

We know that the first decomposition M satisfies $J^3(M)$ has dimension 4 while the second decomposition N satisfies $J^3(N)$ has dimension 3. By the same process as in Example 2.37, we know that $J^3(M_0^3 \otimes M_0^4)$ has basis

$$S = \{(1\otimes 1), (x\otimes 1), (1\otimes x), ((x^2\otimes 1)+(\epsilon+2\epsilon^2+2\epsilon^3+\epsilon^4)(x\otimes x)+(\epsilon^4+\epsilon^5+1)(1\otimes x^2))\}$$

and thus has dimension 4, proving our decomposition.

- $M_0^3 \otimes M_0^5 \cong M_0^6 \oplus M_1^6 \oplus M_2^3$. We know from Theorem 2.30 that M_0^6 and M_1^6 must be injective summands. This leaves us S_2 as the only remaining bottom composition and S_4 as the only remaining top; thus, they belong to the same summand M_2^3.

- $M_0^3 \otimes M_0^6 \cong M_0^6 \oplus M_1^6 \oplus M_2^6$ by Theorem 2.32 since M_0^6 is injective.

- $M_0^4 \otimes M_0^4 \cong M_0^6 \oplus M_1^6 \oplus M_2^3 \oplus M_3^1$. We know from Theorem 2.30 that M_0^6 and M_1^6 must be injective summands of this product. The remaining top composition terms of this product are S_4 and S_3, while the remaining bottom composition terms are S_2 and S_3. Thus, we know that $M_0^4 \otimes M_0^4$ decomposes as either

$$M_0^6 \oplus M_1^6 \oplus M_2^3 \oplus M_3^1$$

or

$$M_0^6 \oplus M_1^6 \oplus M_2^2 \oplus M_3^2.$$

We know that the first decomposition M satisfies $J^2(M)$ has dimension 9 while the second decomposition N satisfies $J^2(N)$ has dimension 8. By the same process as in Example 2.37, we know that $J^2(M_0^4 \otimes M_0^4)$ has basis

$$\begin{aligned}
S = \{&(1 \otimes 1), (x \otimes 1), (1 \otimes x), (x^2 \otimes 1), (x \otimes x), (1 \otimes x^2),\\
&((1+\epsilon+\epsilon^2)(x^2 \otimes x) + (\epsilon^2 + 2\epsilon^3 + 2\epsilon^4 + \epsilon^5)(x \otimes x^2) + (1 \otimes x^3)),\\
&((x^3 \otimes 1) + (\epsilon + 2\epsilon^2 + 2\epsilon^3 + \epsilon^4)(x^2 \otimes x) + (\epsilon^4 + \epsilon^5 + 1)(x \otimes x^2)),\\
&((1+\epsilon+\epsilon^2)(x^3 \otimes x) + (\epsilon^2 + 2\epsilon^3 + 3\epsilon^4 + 2\epsilon^5 + 1)(x^2 \otimes x^2) + (1+\epsilon+\epsilon^2)(x \otimes x^3)\}
\end{aligned}$$

and thus has dimension 9, proving our decomposition.

- $M_0^4 \otimes M_0^5 \cong M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^2$. We know from Theorem 2.30 that M_0^6, M_1^6, and M_2^6 must be injective summands. This leaves S_3 as the only remaining bottom composition term and S_4 as the only remaining top; thus they belong to the same summand M_3^2.

- $M_0^4 \otimes M_0^6 \cong M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^6$ by Theorem 2.32 since M_0^6 is injective.

- $M_0^5 \otimes M_0^5 \cong M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^6 \oplus M_4^1$. We know from Theorem 2.30 that M_0^6, M_1^6, M_2^6, and M_3^6 must be injective summands. This leaves $S_4 = M_1^4$ as the last remaining composition term and hence summand.

- $M_0^5 \otimes M_0^6 \cong M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^6 \oplus M_4^6$ by Theorem 2.32 since M_0^6 is injective.

- $M_0^6 \otimes M_0^6 \cong M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^6 \oplus M_4^6 \oplus M_5^6$ by Theorem 2.32 since M_0^6 is injective.

This gives us the multiplication table

$\bigotimes$	$\boldsymbol{M_0^1}$	$\boldsymbol{M_0^2}$	$\boldsymbol{M_0^3}$	$\boldsymbol{M_0^4}$	$\boldsymbol{M_0^5}$	$\boldsymbol{M_0^6}$
$\boldsymbol{M_0^1}$	M_0^1	M_0^2	M_0^3	M_0^4	M_0^5	M_0^6
$\boldsymbol{M_0^2}$	M_0^2	$M_0^3 \oplus M_1^1$	$M_0^4 \oplus M_1^2$	$M_0^5 \oplus M_1^3$	$M_0^6 \oplus M_1^4$	$M_0^6 \oplus M_1^6$
$\boldsymbol{M_0^3}$	M_0^3	$M_0^4 \oplus M_1^2$	$M_0^5 \oplus M_1^3 \oplus M_2^1$	$M_0^6 \oplus M_1^4 \oplus M_2^2$	$M_0^6 \oplus M_1^6 \oplus M_2^3$	$M_0^6 \oplus M_1^6 \oplus M_2^6$
$\boldsymbol{M_0^4}$	M_0^4	$M_0^5 \oplus M_1^3$	$M_0^6 \oplus M_1^4 \oplus M_2^2$	$M_0^6 \oplus M_1^6 \oplus M_2^3 \oplus M_3^1$	$M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^2$	$M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^6$
$\boldsymbol{M_0^5}$	M_0^5	$M_0^6 \oplus M_1^4$	$M_0^6 \oplus M_1^6 \oplus M_2^3$	$M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^2$	$M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^6 \oplus M_4^1$	$M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^6 \oplus M_4^6$
$\boldsymbol{M_0^6}$	M_0^6	$M_0^6 \oplus M_1^6$	$M_0^6 \oplus M_1^6 \oplus M_2^6$	$M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^6$	$M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^6 \oplus M_4^6$	$M_0^6 \oplus M_1^6 \oplus M_2^6 \oplus M_3^6 \oplus M_4^6 \oplus M_5^6$

Table 5: Multiplication table of H_6

and gives rise to the Green ring

$$R(H_6) \cong \mathbb{Z}[X, X^{-1}, T_2, T_3, T_4, T_5, T_6]/I$$

where I consists of the relations

$$\begin{aligned}
T_2^2 &= T_3 + X \\
T_2T_3 &= T_4 + T_2X \\
T_2T_4 &= T_5 + T_3X \\
T_2T_5 &= T_6 + T_4X \\
T_2T_6 &= T_6 + T_6X \\
T_3^2 &= T_5 + T_3X + X^2 \\
T_3T_4 &= T_6 + T_4X + T_2X^2
\end{aligned}$$

$$
\begin{aligned}
T_3T_5 &= T_6 + T_6X + T_3X^2 \\
T_3T_6 &= T_6 + T_6X + T_6X^2 \\
T_4^2 &= T_6 + T_6X + T_3X^2 + X^3 \\
T_4T_5 &= T_6 + T_6X + T_6X^2 + T_2X^3 \\
T_4T_6 &= T_6 + T_6X + T_6X^2 + T_6X^3 \\
T_5^2 &= T_6 + T_6X + T_6X^2 + T_6X^3 + X^4 \\
T_5T_6 &= T_6 + T_6X + T_6X^2 + T_6X^3 + T_6X^4 \\
T_6^2 &= T_6 + T_6X + T_6X^2 + T_6X^3 + T_6X^4 + T_6X^5
\end{aligned}
$$

under the isomorphism mapping

$$
\begin{aligned}
M_1^1 &\mapsto X \\
M_{-1}^1 &\mapsto X^{-1} \\
M_0^2 &\mapsto T_2 \\
M_0^3 &\mapsto T_3 \\
M_0^4 &\mapsto T_4 \\
M_0^5 &\mapsto T_5 \\
M_0^6 &\mapsto T_6.
\end{aligned}
$$

We note that while the methods we have developed have proved sufficient to classify the Green rings for H_s with $s \leq 6$, they do not yet fully classify tensor products and hence the Green ring for larger values of s.

5 Future work

5.1 The Green ring formula of H_s

We have shown that while we can fully classify the Green ring structures of H_2, H_3, H_4, H_5, and H_6, our methods are so far insufficient to classify the tensor product of arbitrary indecomposable comodules of H_s for $s \geq 7$. Nonetheless, we have conjectured a formula for this product that fits with the known products in H_s for $s \leq 6$.

Conjecture 5.1. *Over the Hopf algebra H_s, the tensor product of indecomposable comodules can be decomposed by the following formula where $n = \min(i, j)$:*

$$M_0^i \otimes M_0^j \cong \begin{cases} \bigoplus_{r=0}^{n-1} M_r^{i+j-(2r+1)} & : i+j \leq s+1 \\ \left(\bigoplus_{r=0}^{i+j-s-1} M_r^s \right) & : i+j > s+1 \\ \oplus \left(\bigoplus_{r=i+j-s}^{n-1} M_r^{i+j-(2r+1)} \right) & \end{cases}$$

In this formula, for the tensor product of comodules with no injective summands, the bottom composition term of minimal index is matched with the top composition term of maximal index in the summand. The remaining bottom composition term of minimal index is then matched with the remaining top composition term of maximal index to form the second summand, and so on. For the tensor product of comodules resulting in injective summands, the injective summands as determined by Theorem 2.30 are first considered, then the remaining bottom composition terms are matched with the remaining top composition terms as described in the previous case.

We claim that in terms of the lengths of the summands, the above decomposition of the tensor product $M_0^i \otimes M_0^j$ is unique.

Theorem 5.2. *Out of all possible remaining decompositions of $M_0^i \otimes M_0^j$ into indecomposable summands, the decomposition in Conjecture 5.1 is unique in the lengths of its summands.*

Proof. First, if $M_0^i \otimes M_0^j$ has any injective indecomposable summands, these summands then already been determined by Theorem 2.30. We shall consider the remaining indecomposable summands.

If $M_0^i \otimes M_0^j$ has no injective summands, then by Lemma 2.27, the longest summand in our decomposition is M_0^{i+j-1} and is of length $i+j-1$. By Theorem 2.23, we know that the bottom composition term of minimal index is S_0. By Theorem 2.24, we know that the top composition term of maximal index is S_{i+j-2}. Thus the only way for a summand to be of length $i+j-1$ is if this summand had bottom composition term S_0 and top S_{i+j-2}, as any other combination of top and bottom terms would produce a summand of smaller dimension. Now, the remaining bottom composition term of minimal index is S_1, while the remaining top composition term of maximal index is S_{i+j-3}. Thus, in order to produce a summand of length $i+j-3$ as found in the formula of Conjecture 5.1, these two terms must correspond to the top and bottom the same summand. This process repeats, noting that each consecutive summand has length two less than the preceding summand; as we remove top and bottom composition terms from our remaining list, we must always take the top term of maximal index and bottom term of minimal index to maintain these summand lengths.

Now, suppose that $M_0^i \otimes M_0^j$ has at least one injective summand. After considering these, the longest remaining summand in the formula in Conjecture 5.1 is $M_{i+j-s}^{2s-i-j-1}$, which is of length $2s-i-j-1$. By Theorem 2.23 and Theorem 2.24, we know that the remaining bottom composition term of minimal index is $S_{(i+j-s-1)+1} = S_{i+j-s}$, while the top composition term of maximal index is $S_{(s-1)-1} = S_{s-2}$. Thus, using the remaining composition terms, in order to have a summand of length $2s-i-j-1$, we need that S_{i+j-s} and S_{s-2} are the bottom and top of the same summand. Then as in the previous case, to produce the next summand, which need be length $2s-i-j-3$, we need that the remaining minimal index bottom composition term $S_{i+j-s+1}$ and the remaining maximal index top composition term S_{s-3} must belong to the same summand and so on, noting that as in the case of no injective summands, as the length of the summands in Conjecture 5.1 decreases each time by two, the new maximal index top and minimal index bottom must always belong to the same indecomposable summand.

□

Under this formula for the tensor product of indecomposable comodules, the resulting Green rings have the following form:

Conjecture 5.3. *The comodule Green ring for the Hopf algebra H_s satisfies the formula*

$$R(H_s) \cong \mathbb{Z}[T_2, T_3, \cdots, T_s][X, X^{-1}]/I$$

where I is generated by all relations of the following form where $n = \min(i, j)$:

$$T_iT_j = \begin{cases} \sum_{r=0}^{n-1} T_{i+j-(2r+1)}X^r & : i+j \leq s+1 \\ \\ \sum_{r=0}^{i+j-s-1} T_kX^r + \sum_{r=i+j-s}^{n-1} T_{i+j-(2r+1)}X^r & : i+j > s+1 \end{cases}$$

We note that H_2, H_3, H_4, H_5, and H_6 all satisfy these formulas for both the tensor product decomposition of indecomposable comodules and the Green ring structures. We hope to later prove these formulas for the tensor product and general Green ring structures of H_s for arbitrary s.

5.2 The Taft algebras

Recall that the Taft algebras are defined as follows:

Let $n \geq 2$ and q be a primitive n^{th} root of unity.

$$H_n(q) = \langle g, h | g^n = 1, h^n = 0, hg = qgh \rangle$$

The coalgebra structure and antipode to $H_n(q)$ are defined by

$$\Delta(g) = g \otimes g \qquad \epsilon(g) = 1 \qquad S(g) = g^{-1}$$

$$\Delta(h) = 1 \otimes h + h \otimes g \qquad \epsilon(h) = 0 \qquad S(h) = -g^{-1}h.$$

Of particular interest to our work is that the Taft algebra $H_n(q)$ can be written as a quotient of the Hopf algebra

$$H_n = K[x] * K[\mathbb{Z}]/\langle xg = \epsilon gx, x^n = 0 \rangle$$

by taking the quotient given by the relation $g^n = 1$.

The Green rings and Grothendieck rings of the Taft algebras have been previously classified in [4]; however, different constructions were used in their classification that are not applicable to the classification of these rings for H_s. We will show how our results may descend to the Taft algebras.

We define ϕ to be the surjective morphism projecting from H_s to $H_s(q)$. Then ϕ defines a functor $F : \mathcal{M}^{H_s} \to \mathcal{M}^{H_s(g)}$, the categories of right H_s- and right $H_s(q)$-comodules defined on objects in the following way:

Let M be a right H_s-comodule. Then $F(M)$ is defined to be the right $H_s(q)$-comodule with the same group structure and whose comodule structure is given by the map

$$\rho_{H_s(q)} = (1_M \otimes \phi)\rho_{H_s}.$$

Thus, if $x \in M$ satisfies

$$\rho_M(x) = \sum x_0 \otimes x_1,$$

then the corresponding $x \in F(M)$ satisfies

$$\rho_{F(M)}(x) = \sum x_0 \otimes \phi(x_1).$$

Proposition 5.4. *Let H_1, H_2 be Hopf algebras with functor $F : \mathcal{M}^{H_1} \to \mathcal{M}^{H_2}$ between comodule categories. Let M, M' be right H_1-comodules. Then*

$$F(M \otimes M') \cong F(M) \otimes F(M')$$

as right H_2-comodules.

Proof. Let $x \otimes y \in M \otimes M'$. Write $\rho_M(x) = \sum x_0 \otimes x_1$ and $\rho_{M'}(y) = \sum y_0 \otimes y_1$. Thus in the tensor product $M \otimes M'$, we have

$$\rho_{M \otimes M'}(x \otimes y) = \sum x_0 \otimes y_0 \otimes x_1 y_1.$$

Then we know that

$$\begin{aligned}
\rho_{F(M \otimes M')}(x \otimes y) &= \sum x_0 \otimes y_0 \otimes \phi(x_1 y_1) \\
&= \sum x_0 \otimes y_0 \otimes \phi(x_1)\phi(y_1) \\
&= \left(\sum x_0 \otimes \phi(x_1)\right) \left(\sum y_0 \otimes \phi(y_1)\right) \\
&= (\rho_{F(M)}(x))(\rho_{F(M')}(y)),
\end{aligned}$$

proving our claim.

□

Thus, if we write the $H_s(q)$-comodules N_0^i as

$$N_0^i = F(M_0^i),$$

we have that

$$\begin{aligned} N_0^i \otimes N_0^j &= F(M_0^i) \otimes F(M_0^j) \\ &= F(M_0^i \otimes M_0^j). \end{aligned}$$

So by determining the tensor product formula of $M_0^i \otimes M_0^j$ of H_s, we will be able to find the tensor product of the corresponding comodules $N_0^i \otimes N_0^j$ of $H_s(q)$. We hope to then rederive the Green ring formulas found in [4] using this tensor product decomposition.

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