An acoustic eigenvalue problem and its application to electrochemistry

Jeffrey K. Landgren

University of Iowa

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AN ACOUSTIC EIGENVALUE PROBLEM AND ITS APPLICATION TO
ELECTROCHEMISTRY

by
Jeffrey K. Landgren

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Applied Mathematical and Computational Sciences
in the Graduate College of
The University of Iowa

August 2016

Thesis Supervisor: Professor Gerhard Strohmer
CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Jeffrey K. Landgren

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Applied Mathematical and Computational Sciences at the August 2016 graduation.

Thesis Committee:

Gerhard Strohmer, Thesis Supervisor

Johna Leddy

Bruce Ayati

Kendall Atkinson

Robert Merlino
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I am six years deep into graduate school at this moment. I think there is a great deal that can be said about the supporting cast who helped get me here.

First, I would like to thank my mathematics adviser, Professor Gerhard Strohmer, for all of his patient support, guidance and advice throughout my time as a graduate student. There is no doubt, I have learned a lot as his student and sincerely appreciate the time and effort he has invested in me. This project was made possible because of him. Not only do I feel like Dr. Strohmer and I have a common interest in math (pde’s and applied sciences), but also in life. Thus as his apprentice I feel as though I have connected with him in so many ways (math and life included).

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To all of my friends, new and old, you have provided me with constant encouragement, much-needed distractions, and a sense of perspective. I hope that I have returned the favor.

My father, John Landgren, received his PhD in mathematics in 1976 from the University of Minnesota (maybe someday I’ll have a child who will receive their PhD in mathematics). Simply put, I would not be receiving mine if it were not for him. We spent countless hours and late nights in his office when I was an undergraduate student. Graduate school was made possible by him. Thank you Dad for all of your dedication and hard work in helping me. I only hope to return the same love and passion to you and possibly new generations of Landgrens.
My mother, Karen Landgren, is one of the most loving warm people in my
supporting cast. She has put up with my math conversations and long distance for
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in Iowa have been amazing with you and I look forward to the many more to come
in Georgia!
ABSTRACT

The fundamental process that lies at the foundation of batteries, capacitors, and solar cells is the electron transfer process. This takes place at an interface or boundary in each device and is governed by its corresponding chemical reaction. Making these devices more efficient can help decrease our negative impact on the environment. Recent experiments in the field of Electrochemistry demonstrate that sound waves act as a catalyst for these electron transfer reactions. A model is developed using an Euler Equation (1), Conservation of Mass Equation (2), boundary motion equation (3), and surface tension equation (4).

\[ u_t + u \cdot \nabla u + \left( \frac{1}{\rho} \right) \nabla P = 0 \quad (1) \]

\[ \rho_t + \nabla \cdot (\rho u) = 0 \quad (2) \]

\[ \Psi_t + u \cdot \nabla \Psi = 0 \quad (3) \]

\[ P(\rho) = P_0 + \beta H \quad (4) \]

Chemically, it is clear that the catalytic phenomenon is derived from the sound waves and how they are affected by the top boundary. When combining these four equations we arrive at a boundary condition involving the top boundary only. We place this condition and the other contributing boundary and initial conditions on the wave equation to understand the interaction that occurs between the waves and the cell. We establish a self-adjoint operator and further use its inverse. Overall, using the Variational form and the Galerkin Method an approximation converges to the solution of the wave equation. With the help of MATLAB these eigenfunctions can be articulated as standing waves.
PUBLIC ABSTRACT

In thinking about how a typical battery works, we notice there is a location at which electrons cross over an interface from inside of the battery to the outside. Changing the rate at which the electrons come out of the battery to a faster rate can provide for better batteries. Recently, in a chemistry lab at The University of Iowa using sound waves has helped enhance this rate.

A mathematical model was constructed to better understand how the sound waves are utilized to enhance this process. The model consists of a list of equations that describe particular characteristics about this enhancement. In solve a system of equations each one is substituted into the next until the final equation is used. In addition, a set of conditions, known as boundary and initial conditions, are needed to approximate the solution to all of the equations. The estimation is done using a programming language called MATLAB and primarily a technique in mathematics called the Finite Element Method. The results show the possible standing waves that exist in the testing cell for a typical battery. These are waves over when waves traveling in opposite directions collide. In the end, the actual waves that occur in the cell are a linear combination of the standing waves from the model.
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CHAPTER 1
ELECTROCHEMICAL BACKGROUND

Before we formulate the core of this project (the mathematical model), let’s begin by examining the physical phenomena we wish to describe. This thesis investigates a topic arising from the subject of electrochemistry. A subset of analytical chemistry, electrochemistry is the study of chemical systems.\(^8\)

1.1 Testing Procedures in Electrochemistry

The fundamental process that lies at the foundation of batteries, capacitors, and solar cells is the electron transfer process coupled with the motion of ions under electrostatics. This process can occur at interfaces in these devices where an electrochemical reaction occurs. It is of note that other processes occur in these electronic devices that are also important, but arguably this chemical transformation driven by electrons across the electrode interface by way of a reaction is the critical part. The flow of electrons is the current. This is where our focus will lie. More specifically, electrochemists create new devices tailored to greater efficiency.

To test devices and electrochemical systems, electrochemists use the three electrode system. As shown in Figure 1.1, the system includes a working electrode, reference electrode, and a counter electrode, where all three electrodes are connected to a single measurement instrument, the potentiostat. On the surface of the working electrode (lower left inset of Figure 1.1) the electrochemical reaction of interest takes place. During an electrochemical reaction, other concomitant events occur such as motion of ions to balance the charge of electron transfer. For cyclic voltammetric experiments described here, the reaction is driven by a voltage applied to the working electrode surface. The voltage ramp is shown in the upper left corner of Figure 1.1. The potential is set between the reference and the working electrode. Over time the potential is ramped and then reversed back to its original potential. In contrast, the reference electrode potential is invariant and fixed. The reference electrode does
not measure the change in potential, but is built with a fixed potential such that when connected to the potentiostat, the potentiostat measures potential with respect to the reference electrode. Current flow occurs between the working and counter electrodes as electrons cross the interfaces between each electrode and electrochemical species in the electrolyte. At one electrode, electrons are added to the chemical species (reduction). To allow balance of change and flow of electrons, electrons are lost from a chemical species (oxidation) to the electrode. flow of electrons between the working and counter electrodes with motion of ions in the electrolyte completes the electrochemical circuit. By measuring the current (flow of electrons) as a function of the applied voltage, the measurement, known as a cyclic voltammogram, is made. The choice of applied potential is determined by the electrochemical reaction of interest. The applied potential provides the energy needed to force the electrochemical
reactants to a higher energy product. As the potential is ramped, the rate of the electrochemical reaction increases and the reactant molecules about the electrode surface are consumed. The difference in reactant concentration at the electrode surface relative to the bulk electrolyte induces diffusion of reactant from the bulk electrode surface toward the electrode. Diffusion is a mass transport process. Mass transport to a flat electrode replenishes the reactants, but after a time becomes rate determining as the supply of reactant from the bulk diminishes. These concepts highlight cyclic voltammetric measurements that are needed to evaluate the electrochemical system described here.

1.2 Investing in Sound Energy

In general, our aim is to increase the electron transfer process. Since the three electrode system provides a means by which to test this process, let's now focus our attention on what to implement that will provide an enhancement.

Figure 1.2: Collision of particles generated by sound waves (from top to bottom).

Let us start by proposing a question. What are sound waves? If we look at Figure 1.2 it is easy to see how sound waves can be thought of as the collision
of particles in the given medium that the waves are traveling through. Another way of thinking about them is the oscillation of pressure. If pressure is changing then energy is changing. Thus, energy is associated with sound waves. This is not an original idea, but a rather important one for this project. The amount of energy can be tested in many ways experimentally. An interesting example of testing comes by way of cavitation. If sound waves are at a high enough intensity they can produce cavitation. Cavitation is the formation and deformation of vapor cavities or bubbles. More precisely, cavitation occurs when a medium is oscillating slower than the frequency of the waves traveling through it. This causes medium to collide with other medium i.e. creating cavities. Commonly found inside these bubbles are temperatures similar to those on the surface of the sun. That is, several thousand Celsius. Similarly, several hundred atmospheres of pressure are found inside of the bubbles. That kind of pressure is found at the bottom of the Mariana Trench. In total, this suggests that sound waves can produce enormous amounts of energy. While this is great news we still have no way of harnessing the energy.

![Figure 1.3: The sonoelectrochemical cell.](image)

In the Leddy Lab, one thought to utilize the energy associated with sound waves, was to run a cyclic voltammogram in a sonicating bath. A sonicating bath
is a devise that uses the energy from sound waves to clean items like silver jewelry. However, there were drawbacks to the sonicating bath idea (for the purpose of our experimentation). In terms of scale, it is a relatively large cell, being typically 12 by 6 by 4 inches in length, width, and height, respectively. Precisely, the drawback is that the sound waves are not concentrated. As a result, the Leddy Lab built a smaller cell seen in Figure 1.3. Here we can see the three electrode system that is used for testing. What is also displayed in the figure is a hollowed out teflon ring. The ring has three carved holes for each electrode. At the base of the ring is an electronic device called a transducer. A transducer is a sound wave generating device. The ring has dimensions much smaller being 4 milimeters in height from the top of the transducer to the top of the ring and having an inner diameter of 1.8 centimeters. Since sound diminishes over time this design, a thin layer cell, is optimal to maintain maximum intensity with limited energy dissipation.

For an overall approach to the setup we have Figure 1.4. The same electrochemical cell can be seen zoomed in. In addition, we see how it is integrated into the system that does the measuring of current and potential. The diagram is best interpreted by starting from the bottom. The oscilloscope displays the intensity of the signal generated from the function generator, the function generator provides the potential for the transducer to generate the sound waves, next we have the cell, then the potentiostat (this generates the potential for the working electrode), and finally the computer which records the data given from the potentiostat. Now it is clear how a cyclic voltammogram is done from a global perspective for this sonoelectrochemical cell.

1.3 Cyclic Voltammetric Results with Sonication (Motivation)

The experiments done above, in Figure 1.5, are the before, during, and after sonication with two minutes of separation. In other words, we consider the results from cyclic voltammograms before turning on the transducer to produce sound waves (sonication), during sonication, and after turning the transducer off. When looking
at each graph in Figure 1.5 there is a top curve and a bottom curve. The top is called the forward wave and the bottom is called the backward wave. The forward wave corresponds to starting at a particular potential (in this case starting at 600 mV seen along the x-axis) and decreasing the potential until a given stopping point (−600 mV). After this is done the potential is inverted or drawn back the other direction to its starting point, the backward wave. As stated before, while the potential is changing the current is being measured (as seen on the y-axis). Another important note is that the x-axis is flipped. Meaning, the negative numbers are on the right-hand side of the axis while the positive numbers are on the left-hand side of the axis. One reason this is done is because voltage can be thought of as negative energy through Gibbs Free Energy equation. This means that decreasing electric potential (from left to right) equates to increasing energy.
Now, let's look at the forward wave (top portion) of the Before sonication graph and we see the current begins increasing rapidly around 0 mV and peaks at $-200$ mV, as indicated by the black arrow. For each reaction taking place, at the surface of the working electrode, one electron is given off. At $-200$ mV the reaction is most favorable. That is, the majority of ions we are interested in have migrated to the electrode to react at that potential. As this is not done in a vacuum other unwanted side reactions may occur in the process as stated before. The nitrogen purging comes in here to create a nitrogen atmosphere of which helps prevent these unwanted side reactions. Following this peak most of the ions are used up and less
and less of them are available as we continue decreasing the potential (increasing energy). Now if we look at the corresponding peak for the During sonication graph and compare it to the Before sonication graph we observe relatively the same amount of current at the cost of a greater potential (less energy). This indicates that the sound waves, During sonication, have an energy effect on this process enabling this similar current output for a greater potential. Following the During sonication cyclic voltammogram there is not testing for two minutes and then we run the final cyclic voltammogram, After sonication. Here the transducer is turned off and the forward and backward wave return to their position as seen in the Before graph. Another positive outcome of the sound waves can be seen in the graphs. If the peak of the forward wave moves closer along the potential axis to the peak of the backward wave this is an indication that the reaction has become more reversible. This means the opposite direction of the reaction of interest has become more favorable. If we are talking about a battery, more reversible is lingua for the charging process of the battery and discharge become more efficient (less side reactions take place and we can expect similar results for cyclic voltammograms in the future). Looking at the During graph we see that it has become more reversible when compared with the Before or After. The slope at which the current increases just before each peak is an indication of the rate at which the electrons transfer across the boundary. Lastly, from this graph it is difficult to tell if the rate has changed when comparing the forward wave of the Before graph to the During graph.

Above lies Figure 1.6. This graph has additional results that, in particular, illuminate the rate of change of the electron transfer process. The graph also provides similar peak current results to that of the previous graph. It is of note that for this graph a more detailed analysis was done on the sonication i.e. there is a before sonication (presono), during sonication (sono1), after sonication (post4min, post10min, post20min), during sonication (final sono), and a background wave (solution with no ions of interest). Now, looking at the before sonication (presono) we see the slope is less than the slope from the during sonication graph (sono1). As we continue and
Figure 1.6: Sonication of $O_2$ saturated with $0.1M \text{H}_2\text{SO}_4$ at 41kHz. $O_2$ envelope (scan rate of 100mV/s).

repeat the process the slope continues to increase or the rate at which the electron transfer process fires increases. This is another indication that sound waves act as a catalyst in these experiments (making them more efficient).

Experimental data suggests via the cyclic voltammograms that sound waves have a catalytic effect on the current. That is to say that for a specific reaction there is a favorable potential at which the maximum output of current occurs and when sound waves are introduced the same amount of current can be achieved under a greater potential (less energy). Another contributing factor to this investigation is that we tend to see this trend if kinetics are slow enough. As stated before, the
reaction also becomes more reversible. On another note, we do not see the same enhancement when placing a lid on top of the electrochemical cell. When doing this the sound seems not to affect the experiments at all. Thus, these experiments show that the height of the cell and shape of the meniscus play a role in the results.

Chemically and physically, what are some possible reasons for these observations? One possibility is additional energy at the electrode surface. As noted above the meniscus plays a key role and so perhaps the sound waves are focused onto the surface of the working electrode (where the reaction of interest occurs). Another possibility is that the oxide layer that covers the working electrode is being removed allowing electron transport to be more efficient. Thirdly, cavitation may be happening that cannot be seen to the naked eye. This can have a profound effect on mass transport issues. As we face these tough questions we may find it useful to model an aspect of these physical phenomena.

As we proceed with the mathematics we must hone our interest first. Thus, two questions arise that can help the chemists better navigate. The first involves the importance of the meniscus. Can we develop a model that describes the role the meniscus plays as it interacts with the sound waves? The second question is a bit more general. How do the sound waves affect the interior of the cell (the liquid on the inside)? From this point on as we proceed, these two questions will be in our thoughts.

### 1.4 Derivation of the Surface Equation $\Psi_0$

As a starting point, we need a way of describing the shape of the meniscus. The surface equation, $\Psi_0$, in generality, can then be defined as:

$$\Psi_0(x, y, z, t) = z - f_0(x, y, t)$$  \hspace{1cm} (1.1)

Here we use subscript zero to mean the original structure or unperturbed structure. We say unperturbed and not initial because later we wish to solve the eigenvalue problem that only involves the spacial variables and not the time variable. This
eigenvalue problem gives a good interpretation of what the waves are doing throughout the cell and on the top boundary. Thus, if we do calculations in this text now that will be put into the description of the model after separation of variables we will simply drop the $t$-variable during those calculations. Now we drop the time variable for the moment as we start a calculation with $\Psi_0$ and $f_0$. For a more precise description of $f_0$ it defines the geometry of the unperturbed surface as the graph of a function and $\Psi_0$ acts as a level set of another function. For a further understanding lets visualize what the surface looks like. Viewing the right-hand side of Figure 1.7 we can easily see that the shape of the top boundary of the liquid or the meniscus is the top section of a sphere. That means the general equation for $\Psi_0$ can be defined using the equation of a sphere and the radius and height of the cell.

Figure 1.7: Sonoelectrochemical cell filled with sound waves versus the liquid from the cell by itself.
Recall from Subsection 1.1.2, the radius of the inner diameter of the cell is 0.09dm (which we denote later as $r_0$) and the cell height is 4mm. Now as stated before we can think of the meniscus being extended to a sphere in three dimensions. In Figure 1.8 one can see the outline of this sphere and how it lines up with the cell. The first step in defining $\Psi_0$ precisely is knowing the radius of the sphere. Already from Figure 1.8 it is easy to see one equation that will help us get to the radius of the sphere ($R$):

$$R = h + 0.02$$

If we can find another equation without introducing any new variables then we can solve for $h$ which will lead us to $R$. Observe the Pythagorean theorem at work:

$$h^2 + 0.09^2 = R^2$$
Now solving for $R$ and substituting into the first equation we see that $\sqrt{h^2 + 0.09^2} = h + 0.02$ and $h$ comes out to 0.1925dm. From this we know $R = 0.2125$dm by using the first equation. Thus, the corresponding equation of the circle in Figure 1.8, $(x - 0)^2 + (z + h)^2 = R^2$, is:

$$(x - 0)^2 + (z + 0.1925)^2 = 0.2125^2$$

Going back to the three dimensional version of what we are addressing. Since this is spherical we can use the equation of a sphere

$$x^2 + y^2 + (z + h)^2 = R^2$$

to articulate the specifics of the three dimensional meniscus

$$z = -h + \sqrt{R^2 - (x^2 + y^2)} \quad (1.2)$$

If we include our recent data equation (1.2) becomes:

$$z = -0.1925 + \sqrt{0.2121^2 - (x^2 + y^2)} \quad (1.3)$$

Now, as we stated before $f_0$ describes the shape of the meniscus as the graph of a function. Thus, what we truly have is:

$$f_0 = -h + \sqrt{R^2 - (x^2 + y^2)} = -0.1925 + \sqrt{0.2121^2 - (x^2 + y^2)} \quad (1.4)$$

The $z$-variable was merely to demonstrate the spherical shape, but was actually $f_0$ the whole time. The $z$-variable from equation (1.1) is used to generate the level set, $\Psi_0$, of another function. Meaning, $\Psi_0$ is actually (without the time variable)

$$\Psi_0(x, y, z) = z - f_0(x, y) \equiv 0 \quad (1.5)$$
Now that we have established questions we want to answer and begun quantifying the phenomena using mathematics we can easily segue into formulating a model.
CHAPTER 2
QUANTIFICATION OF THE MODEL AND BEGINNING STEPS

The first step to modeling is to layout the most important physical characteristics that define the phenomena.

2.1 Choosing a System of Equations (Choosing a Model)

As written in the previous chapter there are two questions of interest that harp on the physical phenomena. The second question was how do the sound waves move throughout the main portion of the liquid? Really what we are asking here is how sound moves in liquid. Also stated before is that sound waves are the oscillation of pressure. In other words an equation that describes change in pressure and how it relates to fluid would be ideal. A standard equation that describes fluid dynamics in this way is the Euler momentum equation seen below.

\[ u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla P = \tilde{g} \]

In this Euler equation, \( u \) represents velocity of a parcel of fluid, \( \rho \) is density, and \( P \) is pressure. In our problem the term \( \tilde{g} \) will be neglected. \( \tilde{g} \) here represents body accelerations and this concept does not play a vital role in affecting the liquid or sound waves under the circumstances. An example of a body acceleration is acceleration due to gravity among other forces. To further elaborate, the body or liquid in this case would have to be accelerating as a result of gravity (technically it is due to earth’s rotation, but it is not accelerating like it would be if it were being dropped off a cliff). Since it is stationary in the cell this is simply not the case. Another important fact about this equation is that it already neglects friction. If friction were included the equation would then be the Navier-Stokes equation and it would include the possibility of vorticity. During the experiments vortices are not created and so we start without its corresponding term. Finally, neglecting the body accelerations
and friction the final equation we work with is:

\[ u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla P = 0 \]

This equation is an important one as it is a translation of conservation of momentum.

Before we depart from the question pertaining to movement of fluid in general. The second equation we chose is the conservation of mass equation:

\[ \rho_t + \nabla \cdot (\rho u) = 0 \]

Conservation of momentum and conservation of mass tend to go hand in hand. Similar to the previous equation, this equation also has velocity of a parcel of fluid \((u)\) and density \((\rho)\). This equation derives directly from thinking about the distribution of density throughout a given volume:

\[ m(x_1, x_2, x_3, t) = \int_V \rho(x_1, x_2, x_3, t) dV \quad (2.1) \]

where \(m\) is mass. Taking the derivative w.r.t time then brings about the conservation of mass equation. In general, there is no loss of mass throughout the experiments and momentum is conserved. What also makes this equation vital to the model is as we will see later combining these first two equations brings about the wave equation. Thus, we do not need the equation that describes waves as it is intrinsic to the problem already. Moreover, we are using equations that describe the general movement of fluid and how that relates to pressure changes or the movement of sound. That is, we have resolved the second question by way of equations!

Now we turn our attention to how the meniscus interacts with the sound waves (the first question). As observed chemically it is critical to evaluate the importance of the meniscus (the top/free boundary) and thus we provide a better description to the short system of equations with the use of the third and fourth equation. The third equation describes the surface of the meniscus and is used because the boundary
simply moves with the flow of the fluid. The third equation originally starts with the
equation that describes the geometric shape of the meniscus (seen in Chapter 1) and
then interprets its movement. Thus, the name for the third equation is the boundary
motion equation. To begin our derivation it must be stated that the equation for \( \Psi \) is
chosen to be identically zero i.e. implicitly defined as follows (this is the most general
form of \( \Psi \)):

\[
\Psi = z - f(x, y, t) \equiv 0
\]  

(2.2)

Notice how it looks very similar to equation (1.1). Two other ways to think about this
definition for \( \Psi \) are that we choose the specific level set \{ \( (x, y, z, t) \mid \Psi (x, y, z, t) = 0 \) \}
(here we think in terms of level sets or contours) or that \( z \) is the graph of a function
(namely a set of points that generates a graph). Moving the pieces of the function
around so that it is identically zero is primarily done for convenience. This way when
linearizing the mean curvature that we do below our computations get a bit easier.
If it not for this construction, we would need some means of describing the surface
as it moves in time and the calculations for this linearization would become a bit
more messy. As stated previously, but now with precision, \( z \) represents the graph of
the function \( f(x, y, t) \) i.e. \{ \( (x, y, f(x, y, t)) \) \}. Also, we will rewrite the function \( \Psi \)
for simplicity in a calculation. Instead we will write \( \Psi(x, y, z, t) = \Psi(T_i(x_i), t) \) where
\( T : \mathbb{R} \rightarrow \mathbb{R}^3 \) is position of parcel \( x_i \). This way when we compute \( \frac{\partial \Psi}{\partial t} \) the chain rule
clearly yields the velocity, \( \frac{\partial T}{\partial t} = u \). That is,

\[
\frac{\partial}{\partial t} \Psi(T_i(x_i), t) = \frac{\partial \Psi}{\partial T} \cdot \frac{\partial T}{\partial t} + \frac{\partial \Psi}{\partial t} \frac{\partial T}{\partial t} + \frac{\partial T}{\partial t} \cdot \frac{\partial \Psi}{\partial T} = \Psi_t + u \cdot \nabla \Psi = 0
\]

This equation equals zero because \( \Psi \) is zero to begin with. As we see in the next
section as well as this one, \( \Psi \) describes the geometry of the surface and this surface
changes over time. As a consequence, we introduce the perturbation function \( \Phi \).
More precisely in terms of a linearization,

\[
\Psi = \Psi_0 + \Phi
\]
where $\Psi_0$ is the equation of the unperturbed surface and $\Phi$ is the second term in the linearization or the perturbation. Collecting previous definitions and new ones:

$$\Psi(x, y, z, t) = z - f(x, y, t) \equiv 0 \text{ and } \Psi = \Psi_0 + \Phi$$

$$\Psi_0(x, y, z, t) = z - f_0(x, y, t)$$

$$\Phi(x, y, z, t) := -\phi(x, y, t)$$

where $f(x, y, t) := f_0(x, y, t) + \phi(x, y, t)$

Now we see that the equation $\Psi = \Psi_0 + \Phi$ is the same as $\Psi(x, y, z, t) = z - f(x, y, t)$.

The final note that describes this equation, $\Psi = \Psi_0 + \Phi$, is the derivation for the actual function $\Psi_0$. As this relates directly to the measured radius of the cell and cell height it can be found at the end of the background chapter (Chapter 1).

The fourth and final equation incorporates surface tension. A simple depiction of surface tension can be seen in Figure 2.1. Surface tension arises primarily
from the inter-molecular forces that exist in the medium of interest and the forces
generated from the surrounding atmospheric pressure. Surface tension is not only
present in our project, but, surprisingly, it represents a major contributing factor to
the movement of the meniscus if we really dig deep. Thus, using an equation that
deals with surface tension is an excellent way to factor in the question about how
the meniscus communicates with the sound waves. Additionally, the fourth equation
involves mean curvature! In short, mean curvature is the average of the principle
curvatures. A physical interpretation of mean curvature is an estimation of changes
in the direction of a surface. It can also be derived from the Weingarten Mapping or
the Shape operator). Here lies the final equation for the model:

\[
P(\rho) = P_0 + \beta H
\]

where \(H\) is mean curvature, and \(\beta\) is the surface tension coefficient. By definition:

\[
H = \nabla \cdot \left( \frac{\nabla (\Psi)}{2 |\nabla (\Psi)|} \right)
\]

In total we have the following four equations for our model:

\[
u_t + u \cdot \nabla u + \left( \frac{1}{\rho} \right) \nabla P = 0 \tag{2.3}
\]

\[
\rho_t + \nabla \cdot (\rho u) = 0 \tag{2.4}
\]

\[
\Psi_t + u \cdot \nabla \Psi = 0 \tag{2.5}
\]

\[
P(\rho) = P_0 + \beta H \tag{2.6}
\]

To help quantify the phenomena described in the previous chapter and in particular,
the questions that arose, these four equations set the foundation for the model. It is
my understanding that they provide a balanced description of the sound waves and
their iteration with the meniscus and the fluid in the cell.
2.2 Approximating Techniques

The system as it stands is a complex one, and thus, simplification is needed where possible. For this project that simplification comes by way of linearization and neglecting small quantities. Additionally, the description of velocity is at the onset of potential flows. That is, \( u = -\nabla U \) (mathematically). The physical intuition of this can be best described by an analogous situation. We know the nature of the relationship between an electric field and its electric potential. The electric field is a vector field. Meaning, at any given point in the region we can ask in what direction are the electrons traveling and with what magnitude. The electric field is then related to its electric potential by the same equation, but with the functions that describe the electric field and electric potential. For our case, the velocity is represented as a vector field and along with that comes its velocity potential or potential flow, \( U \). The potential flow also has units of area per second. Lastly, it is of note that the velocity is smooth and so applying derivatives to it is of no consequence.

2.2.1 Linearization

To begin reducing our system of equations to a single equation we first linearize the terms in each equation. The following is a description of how this approximation is done.

If we look at the equations in total we see that pressure, density, the surface equation, and mean curvature are the primary contributors. Linearizing pressure makes sense because equation (2.6) already includes a few terms in the linearization of pressure (as we will see). Thus, as we substitute one equation into the next it only makes sense to start with a linearization. The surface equation (2.5) gets linearized because the initial equation of the surface is one we can define very easily and will be important in our calculations later. It is important to represent terms in each equation as a linearization because they are constantly changing and we have no or very little other ways to describe this. So in the case of the surface equation, as we saw in Chapter 1, we linearize by stating the initial structure (an equation we
know) plus the perturbation (the portion we don’t know). Density is linearized as well, but since the overall density does not vary much we simply say that it is equal to its initial value. Lastly, we linearize the mean curvature. This is done because otherwise interpreting mean curvature mathematically would be very labor intensive. In general, if we think about other models that involve mean curvature it makes sense to linearize ours. Thinking about the model for a bridge, which includes a description of bending via mean curvature, we note that all bridges do not move or bend very much. If they are permitted to twist and bend then fractures occur. Thus bridges are built with bending possible, but kept to a minimum. This says that the mean curvature will not be a great deal more significant than its first few terms in a Taylor approximation (the linear terms). The same can be said about the meniscus of our problem. Bending may occur, but not a significant amount past the first two terms. Thus we linearize the mean curvature. For a clear derivation and description of mean curvature see reference [12].

The linearization for both the surface, and pressure are straight forward and are seen as:

\[ \Psi \approx \Psi_0 + \Phi \]

\[ P(\rho) \approx P(\rho_0) + P'(\rho_0)(\rho - \rho_0) \]

\( \Phi \) here represents the perturbation in the surface. So we can think of equation \( \Psi \) as its initial surface plus the change that occurs. Of course the change that occurs is a slightly modified version of the standard second term in the taylor series, but this encompasses the concept. The Pressure approximation is as expected, simply the linear terms in the taylor expansion.

Mean Curvature, on the other hand, gets linearized in the following way:

\[ H \approx H_{lin} = H_0 + \frac{\partial}{\partial \varepsilon} \text{div} \left( \frac{\nabla (\Psi_0 + \varepsilon \Phi)}{2|\nabla (\Psi_0 + \varepsilon \Phi)|} \right) \bigg|_{\varepsilon=0} \]

Again, that is to say, the initial value plus the change. We will see shortly that our
approximation at the end of the day just involves the second term in this linearization

\[ \varepsilon DH := \frac{\partial}{\partial \varepsilon} \text{div} \left( \frac{\nabla (\Psi_0 + \varepsilon \Phi)}{2 |\nabla (\Psi_0 + \varepsilon \Phi)|} \right) \bigg|_{\varepsilon=0} \]

and so we focus our attention on this term for now. The simplification of this term proceeds as follows:

\[
\varepsilon DH = \frac{\partial}{\partial \varepsilon} \text{div} \left( \frac{\nabla (\Psi_0 + \varepsilon \Phi)}{2 |\nabla (\Psi_0 + \varepsilon \Phi)|} \right) \bigg|_{\varepsilon=0} = \text{div} \left( \frac{|\nabla (\Psi_0 + \varepsilon \Phi)| \nabla \Phi - \left( \frac{\partial}{\partial \varepsilon} |\nabla (\Psi_0 + \varepsilon \Phi)| \right) \frac{\nabla (\Psi_0 + \varepsilon \Phi)}{2} }{2 |\nabla (\Psi_0 + \varepsilon \Phi)|^2} \right) \bigg|_{\varepsilon=0} = \text{div} \left( \frac{|\nabla \Psi_0| \nabla \Phi - \frac{\nabla \Psi_0 \cdot \nabla \Phi}{|\nabla \Psi_0|^2} \nabla \Psi_0}{2 |\nabla \Psi_0|^2} \right) = \text{div} \left[ \frac{1}{2} \left( \frac{\nabla \Phi}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla \Phi) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \right]
\]

As we proceed with the algebra it is important to consider physical quantities that exist in the algebra. Of note, the numerator of the second term is in fact a projection. Recall that the projection is an indication of how much there is a change in a direction. As is here, this change in a direction is in a sense an indication of the angle of change (which is an important physical quantity for the meniscus). Throughout the calculation we not only want to find \( \Phi \) (our perturbation and in particular where all the parcels move to), but also at what angle or in what direction i.e. the trajectory.

In total, we have the following linearized equations:

\[ \Psi = \Psi_0 + \Phi \quad (2.7) \]

\[ P(\rho) = P(\rho_0) + P'(\rho_0)(\rho - \rho_0) \quad (2.8) \]

\[ H_{lin} = H_0 + \text{div} \left[ \frac{1}{2} \left( \frac{\nabla \Phi}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla \Phi) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \right] \quad (2.9) \]
2.2.2 Accounting for Small Quantities

For this portion of the project we collect all small quantities and if two get multiplied we neglect the term. The chemist performing the experiment has the ability to make $\Phi$ very small simply by turning down the intensity of the sonicator. In fact, the intensity is relatively low during experiments. $\nabla\Phi$, the rate of change along $x$, $y$, and $z$ is small as a result. Physically, it makes sense for the velocity to be small because there is no movement of the fluid to the naked eye and again the sound waves coming out are at a relatively low intensity.

As stated earlier we can neglect terms that have a small quantity times another. Here is a list of those terms that show up in the algebra to come:

$$u \cdot \nabla u$$ (2.10)

$$u \cdot \nabla \Phi$$ (2.11)

2.2.3 Rewriting the System of Equations

Here we wish to utilize our linearization and small quantities. To begin, consider the first equation in our system, (2.3). As listed above, the middle term is a small quantity and so it is left out. If we then plug in the linearized form of pressure we arrive at:

$$u_t + \left(\frac{1}{\rho}\right) \nabla \left(P(\rho_0) + P'(\rho_0)(\rho - \rho_0)\right) = 0$$

For the next step in modifying this equation we have the following description. $P(\rho_0)$ is a fixed quantity and so taking the derivative with regard to position is zero. Similarly, $P'(\rho_0)$ is fixed. Let the change in density be defined as $\tilde{\rho} = \rho - \rho_0$. Now in the physical sense, $\rho \approx \rho_0$, and so our linearization is simply density is replaced with the first term in the taylor series. Also, density is close to its initial value, but not so much that the subtraction of the two can be neglected. Density is a key player here, but does not change much over time. Simply put, $\rho$ gets replaced with $\rho_0$. Next we substitute in potential flows and pull off the gradient. The gradient disappears with-
out a leftover constant function of $t$ because eventually we need to take the derivative w.r.t $x$. Our new equation is as follows:

$$
0 = u_t + \left( \frac{1}{\rho_0} \right) P'(\rho_0) \nabla \tilde{\rho}
= -\nabla U_t + \left( \frac{1}{\rho_0} \right) P'(\rho_0) \nabla \tilde{\rho}
= -U_t + \left( \frac{1}{\rho_0} \right) P'(\rho_0) \tilde{\rho}
$$

Let's now consider our second equation (2.4). Applying a time derivative to $\tilde{\rho} = \rho - \rho_0$ yields $\tilde{\rho}_t = \rho_t$. Since $\rho \approx \rho_0$, and using potential flows we have the modified equation (2.4):

$$
0 = \tilde{\rho}_t + \rho_0 \nabla \cdot u
= \tilde{\rho}_t + \rho_0 \nabla \cdot (-\nabla U)
= \tilde{\rho}_t - \rho_0 \Delta U
$$

Equation (2.5) also involves simplifications. We start with the linearization of $\Psi$. Then simplifying (remember since we are interested in the time independent $\Psi_0$ in the end, then the time derivative vanishes) and neglecting small quantities. At the end we top it off by including potential flows:

$$
0 = \frac{\partial}{\partial t} \left( \Psi_0 + \Phi \right) + u \cdot \nabla \Psi_0 + u \cdot \nabla \Phi
= \Phi_t + u \cdot \nabla \Psi_0
= \Phi_t - \nabla U \cdot \nabla \Psi_0
$$

Finally, let's take a look at the final equation (2.6). We see that it includes the linearized mean curvature and linearized pressure. Before we inject both note that the original equation is

$$
P(\rho) - P_0 = \beta H$$
and the equation with initial values at the start of the experiment is

\[ P(\rho_0) - P_0 = \beta H_0 \]

Now subtracting the two we arrive at the following equation:

\[ P(\rho) - P(\rho_0) = \beta (H - H_0) \]

Now using both linearizations (pressure and mean curvature) we get:

\[ \frac{P'(\rho_0) \tilde{\rho}}{\beta} = \text{div} \left[ \frac{1}{2} \left( \frac{\nabla \Phi}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla \Phi) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \right] \]

In total we have:

\[ -U_t + \left( \frac{1}{\rho_0} \right) P'(\rho_0) \tilde{\rho} = 0 \quad (2.12) \]

\[ \tilde{\rho}_t - \rho_0 \Delta_3 U = 0 \quad (2.13) \]

\[ \Phi_t - \nabla U \cdot \nabla \Psi_0 = 0 \quad (2.14) \]

\[ \frac{P'(\rho_0) \tilde{\rho}}{\beta} = \text{div} \left[ \frac{1}{2} \left( \frac{\nabla \Phi}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla \Phi) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \right] \quad (2.15) \]

### 2.2.4 Substitution

Now we will begin the process of substitution. Equation (2.12) can be solved for \( P'(\rho_0) \tilde{\rho} \) and plugged into equation (2.15) leaving:

\[ \frac{\rho_0 U_t}{\beta} = \text{div} \left[ \frac{1}{2} \left( \frac{\nabla \Phi}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla \Phi) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \right] \]

Taking a time derivative of equation (2.12) we get

\[ U_{tt} = \frac{1}{\rho_0} P'(\rho_0) \tilde{\rho}_t \] If we then solve for \( \tilde{\rho}_t \), substitute it into equation (2.13), and multiply by \( \frac{P'(\rho_0)}{\rho_0} \) we derive our specific wave equation:

\[ U_{tt} - P'(\rho_0) \Delta_3 U = 0 \]
Two things are of note here. First, reassurance comes from this equation because the change in pressure w.r.t. time at the equilibrium density should be very large. Here we get that because the coefficient or $\Delta_3 U$ is the speed of sound squared i.e. a large number. Second, now that we have the wave equation we choose to solve it using separation of variables later to get at the corresponding eigenfunctions that provide a description of the sound waves. Before we move to the next reduction just as a reminder, total we have the following three equations:

\[ U_{tt} - P'(\rho_0)\Delta_3 U = 0 \]  
\[ \Phi_t - \nabla U \cdot \nabla \Psi_0 = 0 \]  
\[ U_t = \frac{\beta}{2\rho_0} \text{div} \left( \frac{\nabla \Phi}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla \Phi) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \]

In equation (2.18) we take another time derivative and substitute in equation (2.17):

\[ U_{tt} = \frac{\beta}{2\rho_0} \text{div} \left( \frac{\nabla (\nabla U \cdot \nabla \Psi_0)}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla (\nabla U \cdot \nabla \Psi_0)) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \]

Using (2.16) now we finally obtain a single equation!

\[ P'(\rho_0)\Delta_3 U = \frac{\beta}{2\rho_0} \text{div} \left( \frac{\nabla (\nabla U \cdot \nabla \Psi_0)}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla (\nabla U \cdot \nabla \Psi_0)) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \]

For simplicity let

\[ \tilde{l} = \frac{2\rho_0 P'(\rho_0)}{\beta} \]

It is of note that the Laplacian present in equation (2.19) is the three dimensional Laplacian. This will be particularly important later. Also, present in (2.19) we have the linear operator:

\[ L(w) = \text{div} \left( \frac{\nabla (w)}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla (w)) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \]
where \( w = \Phi_t = \nabla U \cdot \nabla \Psi_0 \). Note that \( \Phi_t \) is the rate of change of the perturbation in the surface. So also, \( \Phi_t = -\phi_t(x, y, t) \).

Our surface is given implicitly as a set of points satisfying \( \Psi(T_i(\tilde{x}_i), t) = 0 \) where \( T \) represents position given of \( x, y, \) and \( z \). A normal point \((x, y, z)\) on the surface is given by \( \nabla \Psi \). This is true because the gradient at any point is perpendicular to the level set and \( \Psi(x, y, z, t) = 0 \) (the surface) is a level set of \( \Psi \). This is important as our final boundary condition results from the definition of the normal derivative and from the operator of interest being invertible (which we show later). I say here our operator of interest because we will switch from using the operator \( L \) to an operator which we call \( L \). These operators are the same with one minor difference. \( L \) is the original operator and it is a three dimensional operator. When simplifying the details involved in \( L \) we will see that it can be written as a two dimensional operator, \( L \). The details of this can be seen at the beginning of Chapter 3. As we proceed for the rest of the current calculations we will switch to the simplified two dimensional version as that is the operator that is invertible (as needed for our calculations). As a result of the upcoming normal derivative boundary condition we define our other boundary conditions similarly. For the final steps we recall where we left off:

\[
\tilde{l}(\Delta_3 U) = \text{div} \left( \frac{\nabla (\nabla U \cdot \nabla \Psi_0)}{|\nabla \Psi_0|} - \frac{\nabla \Psi_0 \cdot \nabla (\nabla U \cdot \nabla \Psi_0)}{|\nabla \Psi_0|^3} \right) \mathbf{\nabla} \Psi_0 \tag{2.20}
\]

By definition of the normal derivative:

\[
\nabla U \cdot \nabla \Psi_0 = \frac{\partial U}{\partial n} |\nabla \Psi_0|
\]

Now rewriting equation (2.20) by including the newly defined operator and including the normal derivative we get that

\[
\tilde{l}(\Delta_3 U) = L(\nabla U \cdot \nabla \Psi_0)
\]
becomes
\[ \tilde{l}(\Delta_3 U) = L(\frac{\partial U}{\partial n} |\nabla \Psi_0|) \]

That is,
\[ \frac{\partial U}{\partial n} = \frac{1}{|\nabla \Psi_0|} \tilde{l} L^{-1}(\Delta_3 U) \]

After substituting in all equations we arrive at a boundary condition defined in terms of the normal derivative as mentioned above. It is a boundary condition because we include two equations that describe movement of fluid in general and then two equations that describe a specific boundary. In other words, using them all means describing fluid on that boundary. While this represents the boundary condition on the top what are the boundary conditions for the sides and bottom of the cell? Recall the vector product just used in the normal derivative and how it relates to the other boundaries of the cell.

Figure 2.2 displays the normal derivative in the top portion while the bottom half lists an impossible scenario for the fluid to travel (bottom left) along with the three possibilities for each boundary (bottom right). Velocity of a fluid parcel \((u \text{ or } -\nabla U)\) on the side of the cell cannot move in the direction indicated on the bottom far left portion of Figure 2.2. In other words, if we now consider the possible boundary conditions for the lateral part of the cell we observe that the fluid cannot move towards the center of the cell if it begins on the side. If this were a possibility then the cell wall would have to be removed. Similarly, the fluid cannot move in a direction that is slightly off of moving towards the center because a component of that direction would still be towards the center. Thus a parcel of fluid that starts on the lateral cell walls can only move in the direction perpendicular to the normal vector \(\frac{\partial U}{\partial n} = 0\) as indicated with a 1. in the bottom right side of Figure 2.2. In the second portion (2.) we see the only possible movement of the fluid as a result of the change in displacement generated by the transducer is given by the following normal derivative \(\frac{\partial U}{\partial n} = \tilde{f}\). Thirdly, seen in Figure 2.2, we conclude with our boundary condition for the meniscus reproduced below. Also listed below are the other boundary and initial
conditions. The first initial condition arises from the lack of movement or velocity of the fluid at time zero. The second initial condition is a consequence of the first i.e. since the gradient of potential flow is zero (from the first initial condition), then the potential flow is a constant before any time has passed. In total and mathematically, they can be observed as follows:

\[ u(x, 0) = 0 \]  \hspace{1cm} \text{(Initial Condition I)}

\[ U(x, 0) = U_0(x) \]  \hspace{1cm} \text{(Initial Condition II)}

\[ \Psi(T_i(x_i), t) = 0 \]  \hspace{1cm} \text{(Top B.C.I.)}

\[ \frac{\partial U}{\partial n} = 0 \]  \hspace{1cm} \text{(Lateral B.C.)}

\[ \frac{\partial U}{\partial n} = f \]  \hspace{1cm} \text{(Bottom B.C.)}

\[ \frac{\partial U}{\partial n} = \frac{1}{|\nabla \Psi_0|} \tilde{L}^{-1}(\Delta_3 U) \]  \hspace{1cm} \text{(Top B.C.II.)}

While we have finalized our boundary and initial conditions we did this with a key assumption. \( L^{-1} \) cannot exist a priori; in order to use \( L^{-1} \) we must first establish the properties of \( L \).
CHAPTER 3
PROPERTIES OF THE OPERATORS AT WORK

3.1 Properties of $L$

First, we begin by recalling the definitions of our operators.

\[ L\Phi_t := \text{div} \left( \frac{\nabla \Phi_t}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla \Phi_t) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \]

\[ Lw := \text{div} \left( \frac{\nabla w}{\sqrt{1 + |\nabla f_0|^2}} - \frac{(\nabla f_0 \cdot \nabla w) \nabla f_0}{(1 + |\nabla f_0|^2)^{3/2}} \right) \]

If we plug in the information we know about $\Phi_t$ and $\Psi_0$ from the previous chapters and take into account that $\nabla w$ has a zero for its third component, it is easy to see how $L$ is defined as as a two dimensional operator (for $w$ in the domain defined below).

3.1.1 The Solvability Problem

Simply from the definition we can construct quite a bit. Since the divergence gives a scalar and $w : \mathbb{R}^2 \rightarrow \mathbb{R}$, then $L : H^2 \cap H_0^1 \rightarrow L^2$. From here we can establish a solvability condition on $L$ stated as

\[ Lw = \hat{f}, \quad w|_{\partial O} = 0 \]

where $O := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq r_0 \}$ is the restricted domain of the surface described by the graph of the function $\Psi = 0$. Now $w$ is zero on the boundary of this domain for several reasons. The first and primary reason for this is because we assume that the liquid meets right at the edge of the cylinder. Visually in the experimentation this is accurate. Recall what $w$ is here, $w = -\phi_t$. That is, $w$ is the rate of change of the surface. As a result of the position of the liquid, the rate of change of the surface here must be zero. In other words, this is true because the liquid is stationary along the edge of the surface. Also, this manner in which the liquid rests at the corner does not allow us to agree on a well defined contact angle. Lastly, this condition has heavy
implications on the solution and thus we must categorize it properly. Since it is tied to the solution it is a Dirichlet boundary condition. As we proceed, if we use the differential form of the operator $L$ we will observe that if $\hat{f}$ satisfies the solvability condition and $\hat{f} \in H^s(O)$ then $w$ is regulated by being in $H^{2+s}(O)$ space.

3.1.2 Formulating the Working Hilbert Space

Illustrated here we see the domain of definition for $B := -L$:

$$D (B) = \{ w \in H^2 (O) \mid w|_{\partial O} = 0 \}$$

where $H^2$ is the standard Hilbert space. Also we define our bilinear form as the following scalar product:

$$a (w, v) : = \langle Bw, v \rangle : = \int_O -Lwv \, dx$$

(3.1)

for $w, v \in H^1_0 (O)$. Thus, $a : H^1_0 \times H^1_0 \to \mathbb{R}$.

3.1.3 Ellipticity of $L$

For this section, we wish to demonstrate the operator’s ellipticity.

**Definition 3.1.** We say that partial differential operator $L$ is (uniformly) elliptic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^n \tilde{a}^{ij} (x) \xi_i \xi_j \geq \theta \mid \xi \mid^2$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$.

Ellipticity means for each point $x \in U$, the symmetrical $n \times n$ matrix $A (x) = ((\tilde{a}^{ij} (x)))$ is positive definite, with the smallest eigenvalue greater than or equal to $\theta$. There is also an equivalent definition for an elliptic bilinear form. We choose to show the first rather than the latter here. For purposes of ellipticity with our operator $L$ we only care about how $L$ acts on functions (its not necessary to consider what the function
actually is here). Since the function is not of high importance we will continue using $w$ in place of $-\phi_t$ for simplicity. Here starts the importance of the difference between the two operators. For the following proof we start with $\Psi$ from the definition of the first operator, $L$, and show how it changes to $f_0$ for the operator $L$. Along with that comes the change in the number of variables involved and thus we get the operator $L$. We carry out the following calculation to fully demonstrate the transition. It is also of note that this transition has to be done because $L$ is not uniformly elliptic (only $L$ is). For $w \in D(B)$:

$$Lw = \text{div} \left( \frac{\nabla w}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla w) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right)$$

$$= \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \frac{1}{|\nabla \Psi_0|} \left( w_{x_i} - \frac{1}{|\nabla \Psi_0|^2} (\Psi_0)_{x_i} \sum_{j=1}^{3} (\Psi_0)_{x_j} w_{x_j} \right) \right)$$

$$= \sum_{j,i=1}^{3} \frac{\partial}{\partial x_j} \left( \frac{1}{|\nabla \Psi_0|} \left( \delta_{ij} w_{x_j} - \frac{1}{|\nabla \Psi_0|^2} (\Psi_0)_{x_i} (\Psi_0)_{x_j} w_{x_j} \right) \right)$$

$$= \sum_{j,i=1}^{3} \left( \frac{1}{|\nabla \Psi_0|} \left( \delta_{ij} w_{x_j x_i} + \text{l.o.t.} - \frac{1}{|\nabla \Psi_0|^2} \frac{\partial^2 w}{\partial x_i \partial x_j} \left[ \nabla \Psi_0 \otimes \nabla \Psi_0 \right]_{ij} - \text{l.o.t.} \right) \right)$$

$$= \sum_{j,i=1}^{3} \left( \frac{1}{|\nabla \Psi_0|} \left( \delta_{ij} w_{x_j x_i} - \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{[\nabla \Psi_0 \otimes \nabla \Psi_0]_{ij}}{|\nabla \Psi_0|^2} \right) + \text{l.o.t.} \right)$$

$$= \frac{1}{|\nabla \Psi_0|} \sum_{i,j=1}^{2} \left( \delta_{ij} w_{x_j x_i} - \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{[\nabla \Psi_0 \otimes \nabla \Psi_0]_{ij}}{|\nabla \Psi_0|^2} \right) + \text{l.o.t.}$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

and l.o.t. is short for lower order terms (these are order of differentiation terms).
Recall the divergence form:

\[ Lu = - \sum_{i,j=1}^{n} \left( a^{ij}(x) w_{x_i} w_{x_j} \right) + \sum_{i=1}^{n} b^i(x) w_{x_i} + c(x) w \]

and look back at the definition of ellipticity from above. For ellipticity we only consider the \( \tilde{a}^{ij} \) or more clearly the part that includes both \( i \) and \( j \). Hence, we categorize and eventually drop the lower order terms. Also, we replace \( w \) with \( \xi \) to follow the definition of ellipticity and input the data on \( \Psi_0 \). Now picking up from where we left off and including these modifications we can see how \( L \) is elliptic.

**Lemma 3.2.** \( L \) is Uniformly Elliptic.

**Proof.**

\[
\mathcal{L}w = \frac{1}{|\nabla \Psi_0|} \sum_{i,j=1}^{2} \left( \delta_{ij} w_{x_i x_j} - \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{[\nabla \Psi_0 \otimes \nabla \Psi_0]_{ij}}{|\nabla \Psi_0|^2} \right) + \text{l.o.t.}
\]

\[
\Rightarrow \frac{1}{|\nabla \Psi_0|} \sum_{i,j=1}^{2} \left( |\xi|^2 - \xi_i \xi_j \frac{[\nabla \Psi_0 \otimes \nabla \Psi_0]_{ij}}{|\nabla \Psi_0|^2} \right)
\]

\[
= \frac{1}{\sqrt{1 + |\nabla f_0|^2}} \sum_{i,j=1}^{2} \left( |\xi|^2 - \xi_i \xi_j \frac{[\nabla f_0 \otimes \nabla f_0]_{ij}}{1 + |\nabla f_0|^2} \right)
\]

\[
= \frac{1}{\sqrt{1 + |\nabla f_0|^2}} \left( |\xi|^2 - \frac{|\nabla f_0 \cdot \xi|^2}{1 + |\nabla f_0|^2} \right)
\]

\[
\geq \frac{1}{\sqrt{1 + |\nabla f_0|^2}} \left( 1 - \frac{|\nabla f_0|^2}{1 + |\nabla f_0|^2} \right) |\xi|^2
\]

\[
= \frac{1}{\sqrt{1 + |\nabla f_0|^2}} \left( \frac{1}{1 + |\nabla f_0|^2} \right) |\xi|^2
\]

\[
\geq \theta |\xi|^2
\]

\[
\geq 0, \text{ for } \theta = 1
\]
Notice that we started with \( L \), but got to \( L \) and actually showed that \( L \) was elliptic.

So With Ellipticity in the bag we still do not have the existence of \( L^{-1} \).

### 3.1.4 Existence of \( L^{-1} \)

To start off this section before we demonstrate the ellipticity of the bilinear form \( a \) we derive a formulation of the bilinear form to be used many times. The bilinear form \( a \), as defined in equation (3.1) for \( w,v \in H^1_0(O) \), can be written as:

\[
a(w,v) = \langle Bw,v \rangle
\]

\[
= \int_O -Lwv \, dx
\]

\[
= -\int_O \text{div} \left( \frac{\nabla^2 w}{\sqrt{1 + |\nabla^2 f_0|^2}} - \frac{(\nabla^2 f_0 \cdot \nabla^2 w \nabla^2 f_0)}{(1 + |\nabla^2 f_0|^2)^{3/2}} \right) v \, dx
\]

\[
= -\int_{\partial O} \left( \frac{\nabla^2 w}{\sqrt{1 + |\nabla^2 f_0|^2}} - \frac{(\nabla^2 f_0 \cdot \nabla^2 w \nabla^2 f_0)}{(1 + |\nabla^2 f_0|^2)^{3/2}} \right) v \cdot n \, ds
\]

\[
+ \int_O \left( \frac{\nabla^2 w}{\sqrt{1 + |\nabla^2 f_0|^2}} - \frac{(\nabla^2 f_0 \cdot \nabla^2 w \nabla^2 f_0)}{(1 + |\nabla^2 f_0|^2)^{3/2}} \right) \nabla^2 v \, dx
\]

\[
= \int_O \left( \frac{\nabla^2 w}{\sqrt{1 + |\nabla^2 f_0|^2}} - \frac{(\nabla^2 f_0 \cdot \nabla^2 w \nabla^2 f_0)}{(1 + |\nabla^2 f_0|^2)^{3/2}} \right) \nabla^2 v \, dx
\]

\[
= \int_O \left( \frac{(\nabla^2 w \cdot \nabla^2 v)}{\sqrt{1 + |\nabla^2 f_0|^2}} - \frac{(\nabla^2 f_0 \cdot \nabla^2 w \nabla^2 f_0)}{(1 + |\nabla^2 f_0|^2)^{3/2}} \right) \, dx \quad (3.2)
\]

The integral term over the boundary of the surface domain, \( O \), drops out because of the boundary condition from the solvability problem. Now we will demonstrate that the bilinear form is V-elliptic.
Definition 3.3. \( a(\cdot, \cdot) \) is a V-elliptic bilinear form if \( a(w, w) \geq \alpha \|w\|^2 \ \forall w \in H_0^1(O) \), for some constant \( \alpha > 0 \).

Lemma 3.4. The bilinear form, \( a(\cdot, \cdot) \), from equation (3.1) is V-elliptic.

Proof. From the result in equation (3.2) we now can see how the following is true:

\[
a(w, w) = \langle Bw, w \rangle \geq \int_M \frac{1}{\sqrt{1 + |\nabla_2 f_0|^2}} \left( \frac{|\nabla w|^2}{1 + |\nabla_2 f_0|^2} - \frac{|\nabla_2 f_0|^2 |\nabla w|^2}{(1 + |\nabla_2 f_0|^2)^3} \right) dx \geq \frac{\tilde{\alpha}}{\alpha} \|\nabla w\|^2_{L^2(O)} \geq \frac{\alpha}{\alpha} \|w\|^2_{H_0^1(O)}
\]

The first inequality is due to Cauchy-Schwartz and the last is due to the Poincare Inequality.\(^1\) In the end, our bilinear form is V-elliptic.

We next demonstrate that the bilinear form is bounded or coercive.

Definition 3.5. The bilinear form \( a \) is said to be bounded if for constant \( M \), \( a(w, v) \leq M \|w\| \|v\| \ \forall w, v \in V \).

Lemma 3.6. The bilinear form, \( a(\cdot, \cdot) \), from equation (3.1) is bounded.

Proof. We begin with equation (3.2) again:

\[
a(w, v) = \langle Bw, v \rangle
\]
\[
\int_{\Omega} \left( \frac{(\nabla w \cdot \nabla v)}{\sqrt{1 + |\nabla f_0|^2}} - \frac{(\nabla f_0 \cdot \nabla w)(\nabla f_0 \cdot \nabla v)}{(1 + |\nabla f_0|^2)^{3/2}} \right) \, dx
\]

\[
\leq \int_{\Omega} |\nabla w| |\nabla v| \, dx + \int_{\Omega} |(\nabla f_0 \cdot \nabla w)(\nabla f_0 \cdot \nabla v)| \, dx
\]

\[
\leq \int_{\Omega} |\nabla w| |\nabla v| \, dx + \int_{\Omega} |\nabla f_0|^2 |\nabla w| |\nabla v| \, dx
\]

\[
\leq \|\nabla w\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|\nabla f_0\|_{L^2(\Omega)}^2 \|\nabla w\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}
\]

\[
\leq M \|w\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)}
\]

where \( M = \max_{\Omega} \left( 1 + \|f_0\|_{H^1(\Omega)}^2 \right) \). Simply using the Cauchy-Schwarz inequality and the fact that \( f_0 \) is continuous on a compact domain (i.e. bounded) we can conclude that the bilinear form is indeed bounded.

\[\square\]

Recall the well established Lax-Milgram Theorem.\(^1\)

**Theorem 3.7** (Lax-Milgram Theorem). Assume \( V \) is a Hilbert space, \( a(\cdot, \cdot) \) is a bounded, \( V \)-elliptic bilinear form on \( V \), \( l \in V' \) (the dual). Then there is a unique solution of the problem

\[
w \in V, \ a(w, v) = l(v) \quad \forall \ v \in V
\]

**Theorem 3.8** (Main Theorem I). \( L \) is invertible.

**Proof.** By lemma (3.4), lemma (3.6), and then Theorem (3.7), \( L^{-1} \) exists.

\[\square\]

Thus far, we have attacked our model with the assumption that \( L \) is invertible. Now, with the help of the Lax-Milgram Theorem our assumption seizes to exist. It
should be noted that we can show that $L^{-1}$ exists in an easier fashion using the Riesz Representation Theorem.

3.1.5 $L$ and $L^{-1}$ Are Both Symmetric

For the final properties of $L$ and $L^{-1}$ we demonstrate symmetry.

**Lemma 3.9.** $L$ is symmetric.

**Proof.** Jumping to equation (3.2) we see:

\[
\langle Bw, v \rangle = \int_O -Lw v dx
\]

\[
= \int_O \left( \frac{\nabla^2 w \cdot \nabla^2 v}{\sqrt{1 + |\nabla^2 f_0|^2}} - \frac{(\nabla^2 f_0 \cdot \nabla^2 w) (\nabla^2 f_0 \cdot \nabla^2 v)}{(1 + |\nabla^2 f_0|^2)^{3/2}} \right) dx
\]

\[
= \int_O \left( \frac{\nabla^2 v}{\sqrt{1 + |\nabla^2 f_0|^2}} - \frac{(\nabla^2 f_0 \cdot \nabla^2 v) \nabla^2 f_0}{(1 + |\nabla^2 f_0|^2)^{3/2}} \right) \nabla^2 w dx
\]

\[
= -\int_{\partial O} \left( \frac{\nabla^2 v}{\sqrt{1 + |\nabla^2 f_0|^2}} - \frac{(\nabla^2 f_0 \cdot \nabla^2 v) \nabla^2 f_0}{(1 + |\nabla^2 f_0|^2)^{3/2}} \right) w \cdot nds
\]

\[+ \int_O \left( \frac{\nabla^2 v}{\sqrt{1 + |\nabla^2 f_0|^2}} - \frac{(\nabla^2 f_0 \cdot \nabla^2 v) \nabla^2 f_0}{(1 + |\nabla^2 f_0|^2)^{3/2}} \right) \nabla^2 w dx
\]

\[
= -\int_O w \text{ div} \left( \frac{\nabla^2 v}{\sqrt{1 + |\nabla^2 f_0|^2}} - \frac{(\nabla^2 f_0 \cdot \nabla^2 v) \nabla^2 f_0}{(1 + |\nabla^2 f_0|^2)^{3/2}} \right) dx
\]

\[
= \int_O -wLv dx
\]

\[
= \langle w, Bv \rangle
\]

This shows that $L$ is symmetric.

\[\blacksquare\]
Lemma 3.10. With $B : D(B) \to \mathbb{R}$ and $B^{-1} : \mathbb{R} \to D(B)$, let $w, v \in D(B)$ and $x, y \in \mathbb{R}$ such that $x = Bw$ and $v = B^{-1}y$, then $L^{-1}$ is symmetric.

Proof.

$$\langle Bw, v \rangle = \langle w, Bv \rangle$$

becomes

$$\langle x, B^{-1}y \rangle = \langle B^{-1}x, y \rangle$$

with the application of $B^{-1}$ to $x = Bw$ and $B$ to $v = B^{-1}y$, $L^{-1}$ is also symmetric. ■

3.2 Properties of $\Delta$

As there are two operators in this model (See equation (Top B.C.II.) from Chapter 1) we must investigate the other now. Recall the definition of this second operator of interest, the Laplacian, in three dimensions:

$$\Delta_3 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \quad (3.3)$$

3.2.1 The Solvability Problem

Now we begin our investigation by establishing the solvability problem for $\Delta_3$.

To start let $\tilde{f}$ solve $\Delta_3 U$ and $\tilde{g}$ is the function $U$ on the boundary. That is,

$$\Delta_3 U = \tilde{f}, \quad \frac{\partial U}{\partial n} = \tilde{g}$$

with

$$\int_{\Omega} \tilde{f} dx = \int_{\Omega} \Delta_3 U dx = \int_{\partial \Omega} \frac{\partial U}{\partial n} dS = \int_{\partial \Omega} \tilde{g} dS$$

The second equality is from the divergence theorem and by the definition of the normal derivative.
3.2.2 Formulating the Working Hilbert Space

As we proceed in this section we will place the letter D as a subscript to the operator \( L \) or \( L^{-1} \) (like so \( L^{-1}_D \)) to indicate that the operator is involved in the dirichlet boundary condition. This is true since the solvability problem for \( L \) involves the dirichlet boundary condition as mentioned at the beginning of this chapter in section 3.1.1 The Solvability Problem.

We define our space of functions or working Hilbert space to be:

$$\tilde{H} = \left\{ \tilde{f} \in H^1(\Omega) \mid \int_{\Omega} \tilde{f} dx - \int_{\partial\Omega} \tilde{f} L^{-1}_D(1)dx = 0 \right\}$$  \hspace{1cm} (3.4)

where \( \tilde{f}(x,y) = \tilde{f}(x,y,f_0(x,y)) \). We define \( H \) this way because of three key concepts. First,

$$\int_{\partial\Omega_{top}} L^{-1}_D(\Delta_3 U)dx$$

shows up throughout the model (in fact it shows up a great deal) and so as a result we must incorporate it into the definition. To see precisely how the condition in the definition is constructed observe the following:

$$\int_{\Omega} \tilde{f}dx = \int_{\Omega} \Delta_3 Udx$$

$$= \int_{\partial\Omega} \frac{\partial U}{\partial n} dS$$

$$= \int_{\Omega} L^{-1}_D(\Delta_3 U)dx$$

$$= \int_{\Omega} L^{-1}_D(\tilde{f})dx$$

$$= \int_{\Omega} \tilde{f} L^{-1}_D(1)dx$$

since \( L^{-1}_D \) is symmetric. Lastly, this equation directly related to the conservation of
mass. Subtracting the R.H.S from the L.H.S we get the desired condition in (3.4).

The second part of our definition for $H$ involves why $\tilde{f} \in H^1(\Omega)$. This comes from an expression that shows up later, $\int_\Omega \nabla (-\Delta_3 U) \cdot \nabla V dx = \int_\Omega \nabla \left( -\tilde{f} \right) \cdot \nabla V dx$. That is, we require $U$ to be $C^3$ or $\tilde{f} \in H^1(\Omega)$. The suitable domain of definition for $A = -\Delta_3$ is as follows:

$$D(A) = \left\{ U \in H \mid \Delta_3 U \in H, \right\}$$

$$\int_\Omega \Delta_3 UV dx = \int_\Omega L^{-1}_D (\Delta_3 U) V dS - \int_\Omega \nabla U \cdot \nabla V dx, \forall V \in H^1(\Omega)$$

Let us go back to the definition of $H$ for a moment (seen in (3.4)). Since we are interested in the position of the cell where the most kinetic energy accumulates we will equip this space with the following inner product:

$$\langle U, U \rangle = \int_\Omega \nabla U \cdot \nabla U dx$$

This inner product can be easily seen as kinetic energy by checking the units. $U$ is a general notation that represents velocity potential as seen before. Thus, the integrand is $\nabla U \cdot \nabla U$ or velocity squared. These units are then meters squared divided by seconds squared. For simplicity in this inner product we do not show density since we are for the moment labeling it as having magnitude 1. Meaning density is a part of the integrand, but we do not see it. After integrating and including everything we end with units of

$$\langle U, U \rangle = \int_\Omega \nabla U \cdot \nabla U dx$$

$$= \int_\Omega velocity \ast velocity \ast density \ast dV$$

$$= velocity \ast velocity \ast density \ast volume$$
\[ \frac{m^2 \text{ kg}}{s^2 \text{ m}^3} = \frac{m^2}{s^2} \text{ kg} \]  \( \text{(Joules)} \)

As we proceed, mathematically, we must first show that this is in fact an inner product with the addition of our boundary condition i.e. this inner product is well-defined for our system. We do not show that our space is a vector space (as it is easy), but the combination of inner product and vector space gives us an inner product space. The first property is

For any \( \tilde{f} \in H \), \( \langle \tilde{f}, \tilde{f} \rangle \geq 0 \) and \( \langle \tilde{f}, \tilde{f} \rangle = 0 \) iff \( \tilde{f} = 0 \)

It is clear that the first part is satisfied, but what about the if and only if statement? The forward direction of the if and only if statement is the only difficult direction and so we choose to show it. Suppose \( \langle \tilde{f}, \tilde{f} \rangle = 0 \) then

\[ \int_{\Omega} \nabla \tilde{f} \cdot \nabla \tilde{f} dx = \int_{\Omega} |\nabla \tilde{f}|^2 dx = 0 \]

\[ \implies |\nabla \tilde{f}|^2 = 0 \]

\[ \implies \nabla \tilde{f} = 0 \]

\[ \implies \tilde{f} = c \]

To finish this proof we require that \( c = 0 \). We also wish this true with the stipulation that our constraint from (3.4) makes sense or coincides. Thus, we use this constraint
as a part of our proof. Since \( \tilde{f} = c \):

\[
0 = \int_{\Omega} \tilde{f} \, dx - \int_{\partial\Omega} \tilde{f} \, [L^{-1}_D(1)] \, dx
\]

\[
= c \left[ \int_{\Omega} \, dx - \int_{\partial\Omega} \, [L^{-1}_D(1)] \, dx \right]
\]

\[\Rightarrow c = 0\]

To show that \( c = 0 \) we use the Weak Maximum Principle.\(^2\) That is, let \( \zeta = L^{-1}_D(1) \) where \( \zeta|_{\partial\Omega} = 0 \) from the Dirichlet Condition, then by the Weak Maximum Principle the maximum occurs on the boundary. Since \( \zeta \) is zero on the boundary then it is negative in the interior. Thus, the integrals form a positive quantity implying \( c \) must be zero. The other two properties of an inner product are trivial so we will refrain from showing those as well. Now our inner product space induces an equivalent norm:

\[
\|\tilde{f}\|_{H(\Omega)} = \sqrt{\langle \tilde{f}, \tilde{f} \rangle}
\]

What is most important about this section is that our chosen Hilbert space makes sense with our inner product and, in general, is applicable to our model through the solvability problem.

### 3.3 The Self-Adjointness of the Laplacian Under the Top B.C.

We wish to demonstrate self-adjointness now, but first we must define the bilinear form. \( \tilde{B}(U, V) \) from our inner product above is \( \tilde{B}(U, V) : = \langle AU, V \rangle \) where \( U \) and \( V \) lie in the \( D(A) \) and \( A = -\Delta_3 \).

**Theorem 3.11** (Main Theorem II). The Laplacian is self-adjoint with our boundary conditions.
Proof. The following is true by definition and commutativity in $L^1$:

$$\langle AU, V \rangle = \tilde{B}(U, V)$$

$$\tilde{B}(V, U) = \langle AV, U \rangle = \langle U, AV \rangle$$

Now we demonstrate the truth of the second equality in the following string:

$$\langle AU, V \rangle = \tilde{B}(U, V) = \tilde{B}(V, U) = \langle AV, U \rangle = \langle U, AV \rangle$$

i.e. we demonstrate the truth of self-adjointness. We do this by using Green’s Identity, the Divergence Theorem, and the boundary condition for the meniscus. Observe:

$$\langle AU, V \rangle = \int_{\Omega} \nabla(-\Delta_3 U) \cdot \nabla V \, dx$$

$$= \int_{\partial \Omega} (-\Delta_3 U) \nabla V \cdot ndS - \int_{\Omega} (-\Delta_3 U) \nabla \cdot \nabla V \, dx$$

$$= \int_{\partial \Omega} (-\Delta_3 U) \frac{\partial V}{\partial n} dS - \int_{\Omega} (-\Delta_3 U) \, \text{div}(\nabla V) \, dx$$

$$= \int_{\partial \Omega_{\text{top}}} (-\Delta_3 U) \frac{1}{|\nabla \Psi_0|} l L^{-1} (\Delta_3 V) \, dS - \int_{\Omega} (-\Delta_3 U) \, \text{div}(\nabla V) \, dx$$

$$= \int_{\partial \Omega_{\text{top}}} (-\Delta_3 U) \frac{1}{|\nabla \Psi_0|} l L^{-1} (\Delta_3 V) \, |\nabla \Psi_0| \, dx dy - \int_{\Omega} (-\Delta_3 U) \, \text{div}(\nabla V) \, dx$$

$$= l \int_{\partial \Omega_{\text{top}}} (-\Delta_3 U) L^{-1} (\Delta_3 V) \, dx dy - \int_{\Omega} (-\Delta_3 U) \, \text{div}(\nabla V) \, dx$$

$$= l \int_{\partial \Omega_{\text{top}}} L^{-1} (\Delta_3 U) (-\Delta_3 V) \, dx dy - \int_{\Omega} (-\Delta_3 U) \, \text{div}(\nabla V) \, dx$$
\[
\begin{align*}
&= \int_{\partial \Omega_{\text{top}}} \frac{1}{|\nabla \Psi_0|} LL^{-1} (\Delta_3 U) (-\Delta_3 V) |\nabla \Psi_0| \, dxdy - \int_{\Omega} (-\Delta_3 U) \text{div} (\nabla V) \, dx \\
&= \int_{\partial \Omega_{\text{top}}} \frac{1}{|\nabla \Psi_0|} LL^{-1} (\Delta_3 U) (-\Delta_3 V) \, dS - \int_{\Omega} \text{div}(\nabla U) (-\Delta_3 V) \, dx \\
&= \int_{\partial \Omega} \frac{\partial U}{\partial n} (-\Delta_3 V) \, dS - \int_{\Omega} \text{div}(\nabla U) (-\Delta_3 V) \, dx \\
&= \int_{\partial \Omega} (\nabla U \cdot n) (-\Delta_3 V) \, dS - \int_{\Omega} (\nabla \cdot \nabla U) (-\Delta_3 V) \, dx \\
&= \int_{\partial \Omega} \nabla U \cdot \nabla (-\Delta_3 V) \, dx \\
&= \langle U, AV \rangle
\end{align*}
\]

That is, \( \langle AU, V \rangle = \langle U, AV \rangle \) and so \( A \) is self-adjoint.

Throughout the proof we used an underlying surface element \( (dS = |\nabla \Psi_0| \, dxdy) \) and used the fact that \( L^{-1} \) is symmetric.

The surface element switch is a bit involved and so here’s how it comes in starting from generality. Given a curve \( r(t) \) in space from the point \( a \) to \( b \) it can then be written as \( \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} \) where \( x(t) \) and \( y(t) \) make up a set of points governed by time. This can be seen in Figure 3.1:

This way for any time we have a point on the curve designated by the vector \( x(t) \hat{i} + y(t) \hat{j} \). If we differentiate with respect to time and then multiply by the differential \( dt \) we get the form \( dr = dx \hat{i} + dy \hat{j} \). That is, an infinitesimal amount displayed in each respective direction \( x \) and \( y \). This provides for the overall infinitesimal change along the curve \( \vec{r}(t) \). As is custom with integrals we wish to chop up the 3 dimensional surface into parallelograms where each is parameterized by two curves.
For $r_1(t)$ and $r_2(t)$ the surface element is

$$\frac{d\vec{A}}{dt} = \frac{d\vec{r}_1}{dt} \times \frac{d\vec{r}_2}{dt}$$

or

$$d\vec{A} = d\vec{r}_1 \times d\vec{r}_2$$

We actually want the magnitude of $d\vec{A}$ and so $|d\vec{A}| = |d\vec{r}_1 \times d\vec{r}_2|$. The reason we seek the magnitude comes from the definition of the cross product. The length or magnitude of the cross product gives the area of the parallelogram (Figure 3.2). So really what we are computing is the infinitesimal area of the surface that we then add up across the surface (to approximate the surface integral). For purposes of this problem we use the following notation where $f$ is the $f_0$ we are familiar with.

$$\vec{r}(t) = (x(t), y(t), z(t)) = (x, y, f(x, y, t))$$
The magnitude of our surface element is then

$$|d\vec{r} (t)| = \left| \frac{d\vec{r} (t)}{dx} dx \times \frac{d\vec{r} (t)}{dy} dy \right|$$

$$= |(1, 0, f_x) dx \times (0, 1, f_y) dy|$$

$$= |(-f_x, -f_y, 1) dxdy|$$

$$= |(-f_x, -f_y, 1)| dxdy$$

$$= \sqrt{1 + |\nabla f_0|^2} dxdy$$

$$= |\nabla \Psi_0| dxdy$$
and thus

\[ dS = |\nabla \Psi_0| dxdy \]

as it does in the proof of self-adjointness.
CHAPTER 4
SOLVING THE EIGENVALUE PROBLEM

To start off the chapter lets recall our main objective.

4.1 Separation of Variables Generates the Eigenvalue Problem

The wave equation, $U_{tt} - P'(\rho_0)\Delta_3 U = 0$, we found earlier is what we truly wish to investigate as this is the equation that describes the propagation of our waves (on the surface and in the interior of the cell). The previous chapter is merely organizing our top boundary condition i.e. making sure our operators are well-defined. It is well known that we can solve this equation by using separation of variables. However, when we throw in the specifics from this model it becomes very difficult. What we can say for the moment is that beginning with separation of variables yields one ordinary differential equation (an equation relating to time) and one partial differential equation (equation relating to the spacial variables). For a short derivation of this o.d.e. and p.d.e. please see reference 4. Now for the p.d.e., the equation of primary interest, it can be represented as an eigenvalue problem:

$$\Delta_3 \tilde{U} = \lambda \tilde{U}$$

where $\tilde{U} := R(r,z) \Theta(\theta)$ is not a function of time, but only a function of the spacial variables. To get at the eigenvalue problem we investigate everything using cylindrical coordinates (since our domain is essentially a cylinder).

4.1.1 Rayleigh Quotient

Now to investigate the spacial portion of the wave equation we turn our attention to the Rayleigh Quotient. This will give us a glimpse of the spectrum. Here we wish to get a better understanding of the eigenfunction expansion that corresponds to the spacial partial differential equation generated by separation of variables or more specifically the Laplacian. In particular, we use MATLAB to calculate the eigenval-
ues and corresponding eigenvectors that contribute to the eigenfunction expansion. The quotient is calculated with the inverse operator, $K = \Delta_3^{-1}$. We do this because the operator is compact (a necessary condition for an easy spectrum to work with). Observe for $V \subset H$ where $V$ is a subspace of dimension $k - 1$ and $U \neq 0$:

$$
\lambda_k = \inf_{U \perp u \forall u \in V} \left( \sup_{V \subset H} \frac{\langle KU, U \rangle}{\langle U, U \rangle} \right)
$$

$$
= \inf_{U \perp u \forall u \in V} \left( \sup_{V \subset H} \frac{\int_{\Omega} \nabla (KU) \cdot \nabla U dx}{\int_{\Omega} |\nabla U|^2 dx} \right)
$$

$$
= \inf_{U \perp u \forall u \in V} \left( \sup_{V \subset H} \frac{\int_{\partial \Omega} U \frac{1}{\sqrt{\nu_0}} \tilde{L}^{-1} (\Delta_3 (-\Delta_3^{-1}U)) dS + \int_{\Omega} -\Delta_3 (KU) U dx}{\int_{\Omega} |\nabla U|^2 dx} \right)
$$

$$
= \inf_{U \perp u \forall u \in V} \left( \sup_{V \subset H} \frac{\int_{\Omega} U^2 dx - \int_{\Omega} U \tilde{L}^{-1} (U) dx}{\int_{\Omega} |\nabla U|^2 dx} \right)
$$

Now we restrict the quotient with our finite dimensional space: Since $V$ is our subspace of $H$, let $V_h = \{v_h \in V \mid v_h|_{K_i} \in \varrho(K_i), 1 \leq i \leq N\}$ where the basis functions on this space are pyramid functions associated with the nodes. That is, $V_h = \text{span} \{\phi_j | 1 \leq j \leq N\}$. We restrict $U$ to this space and then we can write $U = U_h = \sum_{j=1}^N U_j \phi_j$ where $U_j = U_h(x_j)$ for $1 \leq j \leq N$ and $x_j$ are the nodes of triangles $K_i$ (the $K_i$ form a triangularization of the domain). Now lets make sure we are clear about these finite elements before we proceed. They are piecewise continuous linear functions defined in the following manner $\varphi_j (r, z) := ar + bz + c$. Also, there are three of them per triangle $K_i$. If they are not defined on a particular triangle then they are set to zero there. The coefficients are chosen so that $\varphi_j$ is continuous, $\varphi_j (x_j, y_j) = 1$ on one of the nodes and set to zero, $\varphi_j = 0$, on the other two nodes. Lastly, we will drop the supremum and infimum to save space.
Then

\[ \lambda_k = \inf_{U \perp u \forall u \in V} \left( \sup_{U \subset H} \frac{\int_{\Omega} (U)^2 \, dx - \int_{\Omega} (U) \tilde{L}^{-1} (U) \, dx}{\int_{\Omega} |\nabla U|^2 \, dx} \right) \]

\[ \Rightarrow \frac{\int_{\Omega} \left( \sum_{j=1}^{N} U_j \varphi_j \right)^2 \, dx - \int_{\Omega} \left( \sum_{j=1}^{N} U_j \varphi_j \right) \tilde{L}^{-1} \left( \sum_{i=1}^{N} U_i \varphi_i \right) \, dx}{\int_{\Omega} |\nabla \sum_{j=1}^{N} U_j \varphi_j|^2 \, dx} \]

\[ = \frac{\sum_{i,j=1}^{N} U_j U_i \int_{\Omega} \varphi_j \varphi_i \, dx - \sum_{i,j=1}^{N} U_j U_i \int_{\Omega} \varphi_j \tilde{L}^{-1} (\varphi_i) \, dx}{\sum_{i,j=1}^{N} U_j U_i \int_{\Omega} |\nabla \varphi_j| |\nabla \varphi_i| \, dx} \]

\[ = \frac{\sum_{i,j=1}^{N} U_j U_i \left( \int_{\Omega} \varphi_j \varphi_i \, dx - \int_{\Omega} \varphi_j \tilde{L}^{-1} (\varphi_i) \, dx \right)}{\sum_{i,j=1}^{N} U_j U_i \left( \int_{\Omega} |\nabla \varphi_j| |\nabla \varphi_i| \, dx \right)} \]

\[ = \frac{\sum_{i,j=1}^{N} U_j \left( \int_{\Omega} \varphi_j \varphi_i \, dx - \int_{\Omega} \varphi_j \tilde{L}^{-1} (\varphi_i) \, dx \right) U_i}{\sum_{i,j=1}^{N} U_j \left( \int_{\Omega} |\nabla \varphi_j| |\nabla \varphi_i| \, dx \right) U_i} \]

\[ = \frac{x Ax^t}{x B x^t} \]

At this final step we see that this problem is a quadratic form divided by another.

4.1.2 The Rayleigh Quotient and the Generalized Eigenvalue Problem

The role of this subsection is to discuss the relationship between the rayleigh quotient and the generalized eigenvalue problem. So lets consider \( \frac{x Ax^t}{x B x^t} \). If there is a minimum/maximum then taking the derivative w.r.t. \( x \) and setting it equal to zero will bring it about its corresponding eigenvector. Lets define \( f \) and make use of the directional derivative:

\[ f (x) := \frac{x Ax^t}{x B x^t} \]
then

\[ \nabla_v f(x) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} \]

\[ = \frac{\partial}{\partial t} f(x + tv) \bigg|_{t=0} \]

\[ = \frac{\partial}{\partial t} \left[ \frac{(x + tv) A (x + tv)^T}{(x + tv) B (x + tv)^T} \right] \bigg|_{t=0} \]

\[ = \frac{(xAv^T + vAx^T)(xBx^T) - (xAx^T)(xBv^T + vBx^T)}{(x + tv) B (x + tv)^T)^2} \]

\[ = 0 \]

We set this equal to zero (as we are after the minimum/maximum). Now

\[ \Rightarrow (xAv^T + vAx^T)(xBx^T) = (xAx^T)(xBv^T + vBx^T) \]

\[ \Rightarrow xAv^T + vAx^T = \lambda (xBv^T + vBx^T) \]

\[ \Rightarrow 2vAx^T = \lambda (2vBx^T) \]

\[ \Rightarrow vAx^T = \lambda vBx^T \quad \forall v \in V_h \]

\[ \Rightarrow Ax^T = \lambda Bx^T \]

the third from the last step being true since \( A \) and \( B \) are symmetric. What we see here is that if we wish to obtain the eigenvector that corresponds to the minimum/maximum eigenvalue then we can do that by using the rayleigh quotient and taking the derivative. What we get in the end, however, is the generalized eigenvalue
problem that corresponds to that eigenvalue. In general and of particular note, we can find the eigenvalues and eigenvectors of the generalized eigenvalue problem by solving it straightforwardly. We can do this because our matrix A and B are both symmetric, and because B is positive definite.

4.2 Well-Defined Meaning of $L^{-1}(\varphi)$

Lets turn away from theory and back to calculating. Before I address how we carry out calculating the eigenvalues from the Rayleigh Quotient, (4.1), notice $L^{-1}(\varphi)$ shows up. Now $L$ itself generates an ODE. Thus, we must address the ODE then we can compute the quotient. More precisely we need to compute $Lw$ before we can interpret $L^{-1}\varphi_i$. As we move forward we will write $L\Phi_t$ (even though from before we used $L$) so that we remember the physical features that are attached to that function $\Phi_t$. Along with this we will also use $\nabla \Psi_0$ instead of $\nabla f_0$ for the same reason. For the o.d.e. generated by $L\Phi_t$ we continue with the previous theme. That is, we consider the shape of the domain. Since we have a cylindrical object we convert the o.d.e. to a cylindrical coordinate system.

4.2.1 $L$ in Cylindrical Coordinates

Recall the conversions from cartesian to cylindrical:

$$x = r \cos (\theta), \quad y = r \sin (\theta), \quad z = z$$

$$r = \sqrt{x^2 + y^2}, \quad \tan (\theta) = \frac{y}{x}, \quad z = z$$

Now as we are using separation of variables we know what remains is a function of spacial coordinates. That being said all functions involved have to be written accordingly:

$$\Phi_t(x, y, z) := \cos(k\theta) \Phi(r, z)$$

$$\varphi_i(x, y, z) := \varphi_i(r, z) = ar + bz + c$$
\[ \Psi_0(x, y, z) := \Psi_0(r, z, \theta) \]
\[ = z - f_0(r, \theta) \]
\[ = z - f_0(r) \]
\[ = z + h - \sqrt{R^2 - r^2} \]

Since the cylinder is rotationally symmetric \( \phi_0 \) and \( \Psi_0 \) have no \( \theta \) component. Also, note that \( r^2 = x^2 + y^2 \).

Observe the ODE:

\[
L \Phi_t = \text{div} \left( \frac{\nabla \Phi_t}{|\nabla \Psi_0|} - \frac{(\nabla \Psi_0 \cdot \nabla \Phi_t) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right)
\]
\[= \frac{1}{|\nabla \Psi_0|} \Delta \Phi_t + \nabla \Phi_t \cdot \nabla \left( \frac{1}{|\nabla \Psi_0|} \right) \quad (4.3) \]
\[-\nabla \left( \frac{\nabla \Phi_t \cdot \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \cdot \nabla \Psi_0 - \left( \frac{\nabla \Phi_t \cdot \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \Delta_3 \Psi_0 \quad (4.4) \]

For purposes of simplicity we will use from this point forward \( \Psi \) in place of \( \Psi_0 \) and \( \Phi \) in place of \( \Phi_t \) (until otherwise said so). Now let’s break (4.3) and (4.4) into their individual terms. Looking at the first term in (4.3) we know we need to compute \( |\nabla \Psi_0| \) and \( \Delta \Phi_t \):

\[
|\nabla \Psi| = \left| \left( \frac{-x}{\sqrt{R^2 - x^2 - y^2}}, \frac{-y}{\sqrt{R^2 - x^2 - y^2}}, 1 \right) \right|
\]
\[= \frac{R}{\sqrt{R^2 - r^2}} \quad (4.5) \]
∇\Phi = \left( \frac{\partial}{\partial x} \Phi_r + \frac{\partial}{\partial x} \Phi_\theta, \ \frac{\partial}{\partial y} \Phi_r + \frac{\partial}{\partial y} \Phi_\theta, \ \Phi_z \right)

= \left( \cos(\theta) \Phi_r - \frac{\sin(\theta)}{r} \Phi_\theta, \ \sin(\theta) \Phi_r + \frac{\cos(\theta)}{r} \Phi_\theta, \ \Phi_z \right)

= \left( \cos(\theta) \cos(k\theta) \Phi_r + \frac{k \sin(\theta) \sin(k\theta)}{r} \Phi_\theta, \ \sin(\theta) \cos(k\theta) \Phi_r - \frac{k \cos(\theta) \sin(k\theta)}{r} \Phi_\theta, \ 0 \right)

(4.6)

To calculate \( \Delta \Phi \) we need a bit of work first:

\[ \Phi_{xx} = \frac{\partial^2 \Phi}{\partial r^2} \left( \frac{\partial r}{\partial x} \right)^2 + \frac{\partial \Phi}{\partial r} \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 \Phi}{\partial \theta^2} \left( \frac{\partial \theta}{\partial x} \right)^2 + \frac{\partial \Phi}{\partial \theta} \frac{\partial^2 \theta}{\partial x^2} \]

(4.7)

\[ \Phi_{yy} = \frac{\partial^2 \Phi}{\partial r^2} \left( \frac{\partial r}{\partial y} \right)^2 + \frac{\partial \Phi}{\partial r} \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 \Phi}{\partial \theta^2} \left( \frac{\partial \theta}{\partial y} \right)^2 + \frac{\partial \Phi}{\partial \theta} \frac{\partial^2 \theta}{\partial y^2} \]

(4.8)

\[ \Phi_{zz} = \Phi_{zz} \]

(4.9)

By adding (4.7),(4.8), and (4.9) we obtain \( \Delta \Phi \):

\[ \Delta \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{\partial \Phi}{\partial r} \left( \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} \right) + \frac{\partial^2 \Phi}{\partial \theta^2} \left( \frac{1}{x^2 + y^2} \right) + \frac{\partial \Phi}{\partial \theta} \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) + \Phi_{zz} \]
\[ \frac{\partial^2 \Phi}{\partial r^2} + \frac{\partial \Phi}{\partial r} \left( \frac{1}{\sqrt{x^2 + y^2}} \right) \]

\[ + \frac{\partial^2 \Phi}{\partial \theta^2} \left( \frac{1}{x^2 + y^2} \right) + \frac{\partial \Phi}{\partial \theta} (0) + \Phi_{zz} \]

\[ = \frac{\partial^2 \Phi}{\partial r^2} + \frac{\partial \Phi}{\partial r} \frac{1}{r} + \frac{\partial^2 \Phi}{\partial \theta^2} \frac{1}{r^2} + 0 \]

\[ = \cos(k\theta) \hat{\Phi}_{rr} + \frac{\cos(k\theta)}{r} \hat{\Phi}_r - \frac{k^2 \cos(k\theta)}{r^2} \hat{\Phi} \quad (4.10) \]

Combining (4.5) and (4.10) the first term in \( L \Phi_t \) is

\[ \frac{1}{|\nabla \Psi_0|} \Delta_3 \Phi_t = \frac{\sqrt{R^2 - r^2}}{R} \left( \cos(k\theta) \hat{\Phi}_{rr} + \frac{\cos(k\theta)}{r} \hat{\Phi}_r - \frac{k^2 \cos(k\theta)}{r^2} \hat{\Phi} \right) \quad (4.11) \]

Now for the second term in (4.3) we need \( \nabla \Phi_t \) and \( \nabla \left( \frac{1}{|\nabla \Psi_0|} \right) \). Since we have already calculated most if not all of these we start with:

\[ \nabla \Phi_t \cdot \nabla \left( \frac{1}{|\nabla \Psi_0|} \right) = \nabla \Phi \cdot \nabla \left( \frac{\sqrt{R^2 - r^2}}{R} \right) \]

\[ = \nabla \Phi \cdot \frac{1}{2} \frac{-\nabla r^2}{R \sqrt{R^2 - r^2}} \]

\[ = \nabla \Phi \cdot \left( -\nabla (x^2 + y^2) \right) \frac{1}{2R \sqrt{R^2 - r^2}} \]

\[ = \nabla \Phi \cdot \left( -\frac{2(x, y, 0)}{2R \sqrt{R^2 - r^2}} \right) \]

\[ = -\frac{1}{R \sqrt{R^2 - r^2}} \nabla \Phi \cdot \frac{(x, y, 0)r}{r} \]

\[ = -\frac{r}{R \sqrt{R^2 - r^2}} \nabla \Phi \cdot \frac{(x, y, 0)}{|(x, y, 0)|} \]
\[
\begin{align*}
\frac{-r}{R\sqrt{R^2 - r^2}} \frac{\partial \Phi}{\partial r} \\
= \frac{-r}{R\sqrt{R^2 - r^2}} \cos(k\theta) \hat{\Phi}_r 
\end{align*}
\]

(4.12)

The first term in (4.4) can be calculated in the following fashion. Recall that \(\nabla \Phi\) has a zero in the third component. Looking at:

\[
\nabla \Phi \cdot \nabla \Psi_0 = \nabla \Phi \cdot \left( \frac{-1}{\sqrt{R^2 - r^2}} (x, y, -\sqrt{R^2 - r^2}) \right)
\]

\[
= \nabla \Phi \cdot \left( \frac{-r}{\sqrt{R^2 - r^2}} \frac{(x, y, 0)}{r} \right)
\]

\[
= \frac{-r}{\sqrt{R^2 - r^2}} \frac{\partial \Phi}{\partial r} 
\]

(4.13)

Now looking at:

\[
|\nabla \Psi_0|^3 = |\nabla \Psi|^3
\]

\[
= \frac{R^3}{(R^2 - r^2)^{3/2}}
\]

\[
\rightarrow \frac{1}{|\nabla \Psi|^3} = \frac{(R^2 - r^2)^{3/2}}{R^3} 
\]

(4.14)

Finally combining the two ((4.13) and (4.14)):

\[
-\nabla \left( \frac{\nabla \Phi \cdot \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \cdot \nabla \Psi_0 = \nabla \left( \frac{(R^2 - r^2)^{3/2}}{R^3} \frac{-r}{\sqrt{R^2 - r^2}} \frac{\partial \Phi}{\partial r} \left( \frac{-r}{\sqrt{R^2 - r^2}} \frac{(x, y, 0)}{r} \right) \right)
\]
\[
\n= \nabla \left( \frac{-(R^2 - r^2)r \partial \Phi}{R^3} \right) \cdot \left( \frac{-r}{\sqrt{R^2 - r^2}} \left( \frac{x, y, 0}{r} \right) \right) \\
= \frac{r}{R^3 \sqrt{R^2 - r^2}} \frac{\partial}{\partial r} \left( r(R^2 - r^2) \Phi_r \right) \\
= \frac{r}{R^3 \sqrt{R^2 - r^2}} \left( (R^2 - r^2) \Phi_r + r(-2r) \Phi_r \right) \\
+ r(R^2 - r^2) \Phi_{rr} \\
= \frac{r}{R^3 \sqrt{R^2 - r^2}} \left( (R^2 - r^2) \cos(k\theta) \Phi_r \right) \\
\quad + r(-2r) \cos(k\theta) \Phi_r \\
\quad + r(R^2 - r^2) \cos(k\theta) \Phi_{rr} \right) \quad (4.15)
\]

The final term in (4.4) is done like so:

\[
\left( \frac{\nabla \Phi_t \cdot \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \Delta_3 \Psi_0 = \Delta_3 \Psi \left( \frac{-(R^2 - r^2)r \partial \Phi}{R^3} \right) \\
= \nabla \cdot \left( \frac{-x}{\sqrt{R^2 - r^2}}, \frac{-y}{\sqrt{R^2 - r^2}}, -1 \right) \left( \frac{-(R^2 - r^2)r \partial \Phi}{R^3} \right) \\
= \left( -(R^2 - r^2)^{-1/2} + \frac{-x^2}{(R^2 - r^2)^{3/2}} + -(R^2 - r^2)^{1/2} \right) \\
\quad + \frac{-y^2}{(R^2 - r^2)^{3/2}} \left( \frac{-(R^2 - r^2)r}{R^3} \Phi_r \right)
\]
\[
L \Phi_t = \text{div} \left( \nabla \Phi_t - \frac{(\nabla \Psi_0 \cdot \nabla \Phi_t) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right)
\]

\[
= \frac{1}{|\nabla \Psi_0|} \Delta_3 \Phi_t + \nabla \Phi_t \cdot \nabla \left( \frac{1}{|\nabla \Psi_0|} \right)
\]

\[
- \nabla \left( \frac{\nabla \Phi_t \cdot \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \cdot \nabla \Psi_0 - \left( \frac{\nabla \Phi_t \cdot \nabla \Psi_0}{|\nabla \Psi_0|^3} \right) \Delta_3 \Psi_0
\]

\[
= \frac{\sqrt{R^2 - r^2}}{R} \left( \cos(k\theta) \hat{\Phi}_{rr} + \frac{\cos(k\theta)}{r} \hat{\Phi}_r - \frac{k^2 \cos(k\theta)}{r^2} \hat{\Phi} \right)
\]

\[+ \frac{-r}{R \sqrt{R^2 - r^2}} \cos(k\theta) \hat{\Phi}_r \]

\[+ \frac{r}{R^3 \sqrt{R^2 - r^2}} \left( (R^2 - r^2) \cos(k\theta) \hat{\Phi}_r + r(-2r) \cos(k\theta) \hat{\Phi}_r \right)
\]

As the last step to computing \(L \Phi_t\) we combine all four terms. That is:

\[
L \Phi_t = \text{div} \left( \nabla \Phi_t - \frac{(\nabla \Psi_0 \cdot \nabla \Phi_t) \nabla \Psi_0}{|\nabla \Psi_0|^3} \right)
\]
\[ + r(R^2 - r^2) \cos(k\theta) \hat{\Phi}_{rr} - \left( \frac{(2R^2r - r^3) \cos(k\theta) \hat{\Phi}_r}{R^3 \sqrt{R^2 - r^2}} \right) \]

\[ = \frac{\cos(k\theta)}{R} \left[ -k^2 \sqrt{R^2 - r^2} \hat{\Phi} + \left( \frac{\sqrt{R^2 - r^2}}{r^2} - \frac{r}{\sqrt{R^2 - r^2}} \right) \right. \]

\[ - \frac{r(R^2 - r^2) - 2r^3}{R^2 \sqrt{R^2 - r^2}} - \left( \frac{(2R^2r - r^3)}{R^2 \sqrt{R^2 - r^2}} \right) \hat{\Phi}_r \]

\[ + \left( \frac{\sqrt{R^2 - r^2}}{1} + \frac{-r^2(R^2 - r^2)}{R^2 \sqrt{R^2 - r^2}} \right) \hat{\Phi}_{rr} \]

\[ = \frac{\cos(k\theta)}{R} \left[ -k^2 \sqrt{R^2 - r^2} \hat{\Phi} \right. \]

\[ + \left( \frac{R^2(R^2 - r^2) - r^2R^2 - (r^2(R^2 - r^2) - 2r^4) - r(2R^2r - r^3)}{rR^2 \sqrt{R^2 - r^2}} \right) \hat{\Phi}_r \]

\[ + \frac{R^2(\sqrt{R^2 - r^2})^2 - r^2(R^2 - r^2)}{R^2 \sqrt{R^2 - r^2}} \hat{\Phi}_{rr} \]

\[ = \frac{\cos(k\theta)}{R} \left[ -k^2 \sqrt{R^2 - r^2} \hat{\Phi} \right. \]

\[ + \frac{R^4 - r^2R^2 - r^2R^2 + r^2R^2 - r^4 + 2r^4 + 2R^2r^2 + r^4}{rR^2 \sqrt{R^2 - r^2}} \hat{\Phi}_r \]

\[ + \frac{R^4 - 2r^2R^2 + r^4}{R^2 \sqrt{R^2 - r^2}} \hat{\Phi}_{rr} \]

\[ = \frac{\cos(k\theta)}{R} \left[ -k^2 \sqrt{R^2 - r^2} \hat{\Phi} + \frac{R^4 - 5r^2R^2 + 4r^4}{rR^2 \sqrt{R^2 - r^2}} \hat{\Phi}_r \right. \]

\[ + \frac{R^4 - 2r^2R^2 + r^4}{R^2 \sqrt{R^2 - r^2}} \hat{\Phi}_{rr} \]

\[ = \frac{\cos(k\theta)}{R} \left[ -k^2 \sqrt{R^2 - r^2} \hat{\Phi} + \frac{(R^2 - 4r^2)(R^2 - r^2)}{rR^2 \sqrt{R^2 - r^2}} \hat{\Phi}_r \right. \]
\[ + \frac{(R^2 - r^2)^2}{R^2 \sqrt{R^2 - r^2}} \hat{\Phi}_{rr} \]

\[ := \hat{h}(r) \cos(k\theta) \quad (4.17) \]

In total we have

\[ \hat{h}(r) \cos(k\theta) = \frac{\cos(k\theta)}{R} \left[ \frac{-k^2 \sqrt{R^2 - r^2}}{r^2} \hat{\Phi} + \frac{(R^2 - 4r^2)(R^2 - r^2)}{r R^2 \sqrt{R^2 - r^2}} \hat{\Phi}_r \right. \]

\[ + \left. \frac{(R^2 - r^2)^2}{R^2 \sqrt{R^2 - r^2}} \hat{\Phi}_{rr} \right] \quad (4.18) \]

Now we redefine this portion of the inhomogeneous case (4.18). More specifically, we rewrite the inhomogeneous case by defining a new function that consists of \( \hat{h}(r) \) times another function to continue cleaning up the now R.H.S. We can see this through algebra. Multiplying both the R.H.S. and the L.H.S. by \( \frac{R^3}{\cos(k\theta)(R^2 - r^2)^{3/2}} \) we end up with:

\[ \frac{R^3 \hat{h}(r) \cos(k\theta)}{\cos(k\theta)(R^2 - r^2)^{3/2}} = \hat{\Phi}_{rr} + \frac{R^2 - 4r^2}{r (R^2 - r^2)} \hat{\Phi}_r - \frac{k^2 R^2}{r^2 (R^2 - r^2)} \hat{\Phi} \]

The final clean up is a result of a change of variable. Let \( s = \frac{r}{R} \), then:

\[ \frac{R^3 \hat{h}(sR) \cos(k\theta)}{\cos(k\theta) R^3 (1 - s^2)^{3/2}} = \frac{1}{R^2} \hat{\Phi}_{ss} + \frac{R^2 (1 - 4s^2)}{s R^4 (1 - s^2)} \hat{\Phi}_s - \frac{k^2 R^2}{s^2 R^4 (1 - s^2)} \hat{\Phi} \]

Now multiplying by \( R^2 \) we obtain the new or final inhomogeneous case:

\[ \hat{g}(s) := \frac{R^3 \hat{h}(sR)}{(1 - s^2)^{3/2}} \]

\[ = \hat{\Phi}_{ss} + \frac{1 - 4s^2}{s(1 - s^2)} \hat{\Phi}_s - \frac{k^2}{s^2 (1 - s^2)} \hat{\Phi} \quad (4.19) \]

Now this is the o.d.e. generated by \( L\Phi \), in cylindrical coordinates!
4.3 Approximations to the ODE

For this section we wish to approximate the solution for the o.d.e. This is quite a challenging issue.

4.3.1 Mathematica Estimates

Our aim now is to approximate \( \hat{\Phi} \) from equation (4.19). We manipulated the o.d.e. to its current state because once here it can be easily compared with the bessel’s equations. In particular, we can use knowledge about the solutions to the bessel’s equations to then build intuition about how our solutions should behave. For now lets begin taking a look at the homogeneous solutions calculated by mathematica for \( k = 0 \) to \( k = 5 \):

<table>
<thead>
<tr>
<th>( k )</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 0 )</td>
<td>( c_1 \left( \frac{1}{\sqrt{1-s^2}} \right) + \log(s) - \log(1 + \sqrt{1-s^2}) + c_2 )</td>
</tr>
<tr>
<td>( k = 1 )</td>
<td>( c_1 \left( \frac{s}{\sqrt{1-s^2}} \right) + c_2 \left( \frac{-\sqrt{1-s^2} + s^2 \log(s) - \log(1 + \sqrt{1-s^2})}{2s\sqrt{1-s^2}} \right) )</td>
</tr>
<tr>
<td>( k = 2 )</td>
<td>( c_1 \left( \frac{1}{s^2\sqrt{1-s^2}} \right) + c_2 \left( \frac{-2-s^2}{3s^2} \right) )</td>
</tr>
<tr>
<td>( k = 3 )</td>
<td>( c_1 \left( \frac{1}{s^3} \right) + c_2 \left( \frac{8-4s^2-s^4}{3s^3\sqrt{1-s^2}} \right) )</td>
</tr>
<tr>
<td>( k = 4 )</td>
<td>( c_1 \left( \frac{-6+5s^2}{5s^4\sqrt{1-s^2}} \right) + c_2 \left( \frac{24-8s^2-s^4}{3s^4} \right) )</td>
</tr>
<tr>
<td>( k = 5 )</td>
<td>( c_1 \left( \frac{-8}{5s^5} + \frac{1}{s^3} \right) + c_2 \left( \frac{\sqrt{1-s^2}(64-72s^2+12s^4+s^6)}{3s^5(-1+s^2)} \right) )</td>
</tr>
</tbody>
</table>

Table 4.1: Homogeneous solutions for \( k = 0 \) to \( k = 5 \).

We can see that as \( k \) grows so does the power of \( s \) in the denominator. This implies that as \( k \) grows we will have to be more and more careful near zero (the only value in our domain that may cause a problem since \( s \) can be thought of as \( r \) and \( r \) varies from zero to \( R \)). As we proceed now, we will use these solutions as a guide for
our approximations.

4.3.2 Method of Frobenius

Let’s look into using the method of Frobenius to derive solutions for all k. The method of Frobenius uncovers an infinite series solution to a second order o.d.e. of a particular form (Equation (4.20)). It provides the power series solution provided the coefficients $q(t)$ and $r(t)$ are analytic at the regular singular point (zero). Recall the definition of a regular singular point.

**Definition 4.1.** We say that $t = t_0$ is a regular singular point of an o.d.e with polynomial coefficients provided the equation can be written in the form

$$[(t - t_0)^2 p(t) D^2 + (t - t_0) q(t) D + r(t)] y = 0 \quad (4.20)$$

where $p(t)$, $q(t)$, and $r(t)$ are polynomials and $p(t_0) \neq 0$.

In the case of our o.d.e. the criteria for using the method of Frobenius is satisfied. We start with the inhomogeneous form

$$y'' + \frac{1 - 4s^2}{s(1 - s^2)} y' - \frac{k^2}{s^2(1 - s^2)} y = \hat{g} \quad (4.21)$$

where $y$ is holomorphic on the set $\{ s \in \mathbb{C} \mid 0 < |s| < 1 \}$. Then we move towards the homogeneous form of interest (at the moment). From (4.21) we start by multiplying by $1 - s^2$ and calling the R.H.S. a series:

$$(1 - s^2) y'' + \left( \frac{1}{s} - 4s \right) y' - \frac{k^2}{s^2} y = \sum_{\mu = -\infty}^{\infty} b_{\mu} s^\mu \quad (4.22)$$

Now since we are dealing with singular points the series solution needs negative exponents i.e. we are really after the Laurent series

$$y = \sum_{\mu = -\infty}^{\infty} a_{\mu} s^\mu \quad (4.23)$$
that is assumed to be convergent for $0 < |s| < 1$. We assume convergence because otherwise our solution is meaningless (not helpful in terms of finding the solution) if it diverges. Now plugging in our series solution we get:

\[
\sum_{\mu = -\infty}^{\infty} b_{\mu} s^\mu = (1 - s^2) \sum_{\mu = -\infty}^{\infty} a_{\mu} \mu (\mu - 1) s^{\mu - 2} + \left( \frac{1}{s} - 4s \right) \sum_{\mu = -\infty}^{\infty} a_{\mu} s^{\mu - 1} \]
\[- \frac{k^2}{s^2} \sum_{\mu = -\infty}^{\infty} a_{\mu} s^\mu \]

\[
= \sum_{\mu = -\infty}^{\infty} a_{\mu} \mu (\mu - 1) s^{\mu - 2} - \sum_{\mu = -\infty}^{\infty} a_{\mu} \mu (\mu - 1) s^\mu + \sum_{\mu = -\infty}^{\infty} a_{\mu} \mu s^{\mu - 2} \]
\[- 4 \sum_{\mu = -\infty}^{\infty} a_{\mu} \mu s^\mu - k^2 \sum_{\mu = -\infty}^{\infty} a_{\mu} s^{\mu - 2} \]

\[
= \sum_{\mu = -\infty}^{\infty} a_{\mu} \mu (\mu - 1) s^{\mu - 2} + \sum_{\mu = -\infty}^{\infty} a_{\mu} \mu s^{\mu - 2} - k^2 \sum_{\mu = -\infty}^{\infty} a_{\mu} s^{\mu - 2} \]
\[- \sum_{\mu = -\infty}^{\infty} a_{\mu} \mu (\mu - 1) s^\mu - 4 \sum_{\mu = -\infty}^{\infty} a_{\mu} \mu s^\mu \]

\[
= \sum_{\mu = -\infty}^{\infty} \left[ a_{\mu} \mu (\mu - 1) + a_{\mu} \mu - k^2 a_{\mu} \right] s^{\mu - 2} \]
\[- \sum_{\mu = -\infty}^{\infty} \left[ a_{\mu} \mu (\mu - 1) + 4a_{\mu} \mu \right] s^\mu \]

\[
= \sum_{\mu = -\infty}^{\infty} a_{\mu} \left[ \mu^2 - k^2 \right] s^{\mu - 2} - \sum_{\mu = -\infty}^{\infty} a_{\mu} \mu (\mu + 3) s^\mu \]

\[
= \sum_{\mu = -\infty}^{\infty} a_{\mu + 2} \left[ (\mu + 2)^2 - k^2 \right] s^{\mu} - \sum_{\mu = -\infty}^{\infty} a_{\mu} \mu (\mu + 3) s^\mu \]

\[
= \sum_{\mu = -\infty}^{\infty} \left[ a_{\mu + 2} \left[ (\mu + 2)^2 - k^2 \right] - a_{\mu} \mu (\mu + 3) \right] s^\mu \quad (4.24)
\]
Thus we have a solution exactly if

$$a_{\mu+2} \left[ (\mu + 2)^2 - k^2 \right] = a_{\mu}\mu(\mu + 3) + b_{\mu} \quad (4.25)$$

for $\mu \in \mathbb{Z}$.

Again, let's consider the case when $\hat{g} = 0$. We have that $\sum_{\mu=-\infty}^{\infty} |a_{\mu}s^\mu| < \infty$ for $0 < |s| < 1$. In particular, consider the even lower half, $\sum_{\nu=-\infty}^{0} |a_{2\nu}| |s|^{2\nu} < \infty$, and the odd lower half, $\sum_{\nu=-\infty}^{0} |a_{2\nu+1}| |s|^{2\nu+1}$. Looking at the even first by letting $\mu = 2\nu - 2$, then by Equation (4.25) we have

$$a_{2\nu} \left[ (2\nu)^2 - k^2 \right] = a_{2\nu-2} (2\nu - 2) (2\nu + 1)$$

At this point our aim is to show that there are a finite number of non-zero coefficients in the series. Here we do a proof by contradiction. Assume $a_{2\nu} \neq 0$ for sufficiently large $\nu$ (meaning as $\nu$ becomes very negative). We can see that for large $\nu$ we have the following behavior:

$$\frac{a_{2\nu-2}}{a_{2\nu}} = \frac{(2\nu)^2 - k^2}{(2\nu - 2) (2\nu + 1)} \to 1,$$

Using this fact and the Ratio Test we see

$$\left| \frac{a_{2\nu-2} s^{2\nu-2}}{a_{2\nu} s^{2\nu}} \right| = \frac{a_{2\nu-2}}{a_{2\nu}} |s|^{-2} \to |s|^{-2} > 1.$$  

since $0 < |s| < 1$. That is, the series is divergent. This is a contradiction of the fact that this is a convergent series. Thus our assumption, $a_{2\nu} \neq 0$, is wrong and the sequence of non-zero coefficients actually has to be finite. This also tells us that we are dealing with a pole at zero and not an essential singularity (a critical component to the behavior of the solution). The same works for all odd indices. Thus 0 is actually a pole of the overall solution if any exists, and Equation (4.25) implies that $y$ is odd if $k$ is odd and even if $k$ is even.

Consider Table 4.2 as a guide as we proceed. Here we are still focusing on the
List of Coefficients - Homogeneous Case

<table>
<thead>
<tr>
<th>$k = 2$</th>
<th>$k = 4$</th>
<th>$k = 1$</th>
<th>$k = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3a_1 = -2a_{-1}$</td>
<td>$-15a_1 = -2a_{-1}$</td>
<td>$0a_1 = -2a_{-1}$</td>
<td>$-8a_1 = -2a_{-1}$</td>
</tr>
<tr>
<td>$-4a_0 = -2a_{-2}$</td>
<td>$-16a_0 = -2a_{-2}$</td>
<td>$-1a_0 = -2a_{-2}$</td>
<td>$-9a_0 = -2a_{-2}$</td>
</tr>
<tr>
<td>$-3a_{-1} = 0a_{-3}$</td>
<td>$-15a_{-1} = 0a_{-3}$</td>
<td>$0a_{-1} = 0a_{-3}$</td>
<td>$-8a_{-1} = 0a_{-3}$</td>
</tr>
<tr>
<td>$0a_{-2} = 4a_{-4}$</td>
<td>$-12a_{-2} = 4a_{-4}$</td>
<td>$3a_{-2} = 4a_{-4}$</td>
<td>$-5a_{-2} = 4a_{-4}$</td>
</tr>
<tr>
<td>$5a_{-3} = 10a_{-5}$</td>
<td>$-7a_{-3} = 10a_{-5}$</td>
<td>$8a_{-3} = 10a_{-5}$</td>
<td>$0a_{-3} = 10a_{-5}$</td>
</tr>
<tr>
<td>$12a_{-4} = 18a_{-6}$</td>
<td>$0a_{-4} = 18a_{-6}$</td>
<td>$15a_{-4} = 18a_{-6}$</td>
<td>$7a_{-4} = 18a_{-6}$</td>
</tr>
<tr>
<td>$21a_{-5} = 28a_{-7}$</td>
<td>$9a_{-5} = 28a_{-7}$</td>
<td>$24a_{-5} = 28a_{-7}$</td>
<td>$16a_{-5} = 28a_{-7}$</td>
</tr>
</tbody>
</table>

Table 4.2: List of coefficients for the homogeneous o.d.e.

homogeneous case. For $k = 0$ we have

$$a_{\mu + 2}(\mu + 2)^2 = a_\mu \mu (\mu + 3), \quad (4.26)$$

therefore going from $-\infty$ to $\infty$ the first non-zero element is $a_0$. We can also see that for $k = 0$ we have that $a_{-1} = 0$ which then makes $a_1 = 0$ (this statement is actually true for $k$ even). We also have that $2^2a_2 = 0a_0 = 0$ and thus only $a_0$ is unequal to zero. Therefore, the space of solutions with the postulated properties is spanned by $y(s) = 1$.

For $k = 1$ we have

$$a_{\mu + 2} (\mu^2 + 4\mu + 3) = a_{\mu + 2} (\mu + 3) (\mu + 1)$$

$$= a_\mu \mu (\mu + 3),$$

thus the first possibly non-zero coefficient is $a_{-1}$ with $\mu = -3$. Then we have with
\[ \mu = -1 \]

\[ a_1 ((1)^2 - 1) = -2a_{-1}, \]

thus actually also \( a_{-1} = 0 \). This does allow \( a_1 \) to be non-zero, and then we have non-zero odd-numbered elements all the way to infinity. So far we have only found one-dimensional solution spaces.
CHAPTER 5
THE MODEL RESULTS

The most significant set of results can be summed in a short paragraph. First lets recall the two questions we were after at the start of this model. The first is what happens as the sound waves move throughout the fluid. The second is what happens as the sound waves interact with the meniscus. This significant result is a consequence of having a self-adjoint operator (Main Theorem II). This fact implies we should get standing waves in the experiments. In other words, we partially answered question two because we can say that the waves are in fact reflecting back down as they interact with the meniscus causing standing waves to occur. The connection from self-adjointness to standing waves is as follows. Having a self-adjoint positive definite operator along with more general theory says that our eigenvalues are real and positive. Moreover, we get a solution(eigenfunction expansion) that behaves periodically and is of the form $e^{\sqrt{-\lambda}x}u_\lambda(x)$. This solution set defines the periodic motion attributed to standing waves. Thus the sound waves interact with the meniscus by reflecting back down causing standing waves. Later in this chapter we provide more details regarding this phenomena while partially answering the other question (both of which via graphs). Lastly, we were able to show that for a flat meniscus we have conservation of energy and uniqueness of solutions, a good sign that our model is sound.

5.1 Results and Analysis of the Method of Frobenius

Using MATLAB we developed several codes (as seen in the Appendix) that generate the set of coefficients from the method of Frobenius. Recall these coefficients that depend on $k$ for the homogeneous case i.e. for the method of Frobenius:

$$a_{\mu+2} \left[(\mu + 2)^2 - k^2\right] = a_\mu \mu (\mu + 3)$$  \hspace{1cm} (5.1)
After calculating these coefficients, $a_\mu$, the code then calculates the corresponding Laurent series by separating its negative half and its positive:

$$y(s) = \sum_{\mu=-\infty}^{\infty} a_\mu s^\mu$$

$$= \sum_{\mu=-\infty}^{0} a_\mu s^\mu + \sum_{\mu=1}^{\infty} a_\mu s^\mu$$

In short, the program Frob1 calculates the well-behaved portion of the series (meaning it calculates the positive exponent portion because it is well-behaved near zero) while Frob2 calculates the undefined portion (negative exponent portion). When we say well-behaved what we really mean is bounded. It is of note that in the code we simply used the exact solution (for the homogeneous case) given by Mathematica for $k = 0, 1$, but for the other $k$-values we used the method of Frobenius. We did this to maximize precision especially near zero. Seen in the next few pages are graphs of these estimations for $k = 0, 1, \text{ and } 2$: 
As we can see the left graph in Figure (5.1) is indeed bounded and represents the case when $k = 0$ for the bounded solution. Likewise, we can see how the right graph may become unbounded near zero as it represents a power series of negative exponents that gets evaluated near zero. It is of note that since we have a second order o.d.e. here the positive half and the negative half are the terms in the general solution or more precisely the general solution is a linear combination of the two. Lastly, the bounded solution is simply the constant solution as it should be when $k = 0$. This is simply a result of calculating the coefficients.
Again here, the unbounded solution comes from the Mathematica solution set. Very quickly, we can see the unbounded issues near zero and the bounded solution is almost linear. The unbounded while very negative near zero flattens out rather quickly. This is not dissimilar to the Bessel’s functions. We will see this phenomena continuing as we increase the parameter $k$. 

Figure 5.2: Estimates of solution to the o.d.e. for $k = 1$. 
Our final example here validates the correct behavior. Meaning, as the Bessel functions flatten as $k$ grows so do our solutions.

Now to examine the accuracy of these results we have calculated an approximation to the solution using a slightly different approach for an estimation on the error.
Figure 5.4: Error (in m/s) estimates for \( k = 0, 1, 2, \) and 3.

Here we can see that the error is negligible (on the order of \( 10^{-14}(dm/s) \)). In particular, again following the Bessel functions we see the error after \( k = 2 \) begins to flatten out as the actual function does. This second estimation we do involves a corresponding constant function for the R.H.S. of the inhomogeneous equation. The error is then calculated by subtracting the two. As the error is tiny we suspect we are doing our calculation correctly. Thus, our approximation to the solution is accurate and useful for the results we are after.

5.2 Solution to the Wave Equation (Main Results)

A series of MATLAB codes were written to estimate the solution to the wave equation with the boundary and initial conditions from Chapter 2. The algorithm uses the function in MATLAB known as eig to solve the associated generalized eigenvalue problem from the variational form (estimates the rayleigh quotient). The codes first invoke a set of discrete points for evaluating. The points are the vertices of the triangularization or mesh over half of a slice of the cell as seen in Figure (5.5).
This is done because these estimates can then be rotated about $2\pi$ to generate the estimates in the entire cell as it is rotationally symmetric. Once the mesh is computed the algorithm generates the finite elements that get used to prob the solution. Following this the algorithm estimates the solution to the o.d.e. that is inside of the rayleigh quotient. The solution to this o.d.e as stated before is a linear combination of the unbounded and bounded series. Using Variation of Parameters for nonhomogeneous equations the codes generate the particular and general solution. They then compute the overall solution. This solution then gets used in the estimation for the integrals in the rayleigh quotient and then the eigenvalues and eigenvectors can be approximated using eig.

As a refresher, recall that standing waves occur when two waves traveling in
opposite directions collide. The resulting wave that forms when this happens is then called a standing wave. It is precisely computed by adding the two intersecting waves. If we think for a moment, as two waves propagate towards each other there are times when they will be exactly equal in magnitude, but opposite directions. When this happens the standing wave will have amplitude zero. In any case, below are a sample of the possible standing waves that can occur in the cell. This is the main portion of the project (finding the solution to the wave equation). The solution generates possible standing waves given the conditions:

Figure 5.6: The first four possible standing waves for \( k = 0 \).

Here lies the main results. What we are looking at is the first four possible standing waves that occur when \( k = 0 \). The height and depth of the function or amplitude is the measure of velocity potential. Note, velocity potential, \( U \), has a
direct relationship with velocity $u$. Precisely, it is:

$$u = -\nabla U$$  \hspace{1cm} (5.2)

Thus, the graphs themselves can be pictured as velocity in the following way. Since the relationship, (5.2), holds that means if we fix an $r$-value and only look along that plane, then we can image what the velocity would look like by taking note of the slopes (derivatives) along that plane. This can be done similarly for other directions ($r$-values and $z$-values).

![Figure 5.7: The first four possible standing waves for $k = 1$.](image)

Here is another set of the first four possible standing waves, but with $k = 1$. Not much has changed here other than the first three standing waves are similar to the $k = 0$ case.
Figure 5.8: The first four possible standing waves for $k = 2$.

For the last graph we have again the first four eigenvectors, but this time for $k = 2$. Here its important to consider $k = 2$ as you might recall for the unbounded portion of the solution (the negative exponent terms in the summation) we chose the solution that was generated from Mathematica, but after $k = 0$ and $k = 1$ all approximations are done by the method of Frobenius, our approximation. Not much has changed again though, looking at the results we can see yet another set of possible standing waves that are not far off from the previous two cases.

It is of note that the number of triangles determines the number of standing waves. Really what we mean is if we increase the number of triangles i.e. increase the mesh, then we increase the number of possible standing waves that the program generates. If the number of them increases then the more similar they look. Thus, with the current size we have 109. This is a significant number and already it is easy to see that many are similar.
CHAPTER 6
CONCLUDING ARGUMENTS AND FUTURE WORK

6.1 Conclusions

Overall, based on the results, the model seems to work well. At all possible steps we have tested the validity of that which we calculate. We have run minor tests to demonstrate that each code is working as it should and the calculations are sound because of the corresponding proofs that enable them.

It is clear that the possible standing waves are the result due to the self-adjointness and the graphs. While this answers precisely how the sound waves interact with the meniscus there is a great deal still to be done. The results, unfortunately, do not fully tie the model back to the experiments (at least not as much as they could). We need a bit more to fill in gaps and doing so would help answer our two questions of interest. One way to begin thinking about this is to draw correspondence with a video we have. The video shows the cell during sonication and when the transducer is off. Visually, it seems that during sonication (as the video shows) the liquid oscillates between different standing waves. After enough time has passed while running the transducer the waves finalize to one specific wave (they reach a carrying capacity so to speak). It is our thought that this is simply a dominant standing wave that guides the movement of the liquid more than the others, depending on the intensity of the transducer, how much liquid is in the cell, what liquid is in the cell, and many other factors. If in fact we do observe standing waves in this video then one possibility is to make an attempt at identifying those standing waves based on the intensity and frequency of the sound waves coming in. If this can be done, the next step is to match this standing wave with one that is generated from the model. Then we can verify the accuracy of the model and form predictions. We can then use the models estimates to generate data about the velocity of parcels in the cell.

Additional items that could be of use to the chemists are as follows. One possibility is coding up the distribution of the kinetic energy density in the cell. In
particular, if we can locate the dominant standing wave and identify it in the list provided by the model then perhaps we can calculate the kinetic energy density off of the velocity potential values. Of course this would be done over the mesh first, but then as the cell is rotationally symmetric we can then calculate the kinetic energy density throughout the entire cell, as desired. Another calculation that is of interest is estimating the standing waves for the flat meniscus. In theory, the solution to the flat meniscus case can be easily interpreted physically since the boundary is simple. This again would better enable us to interpret the results.

6.2 Future Work

In the future several items can be calculated and proved that might be of assistance in understanding the phenomena. Among them are showing conservation of energy with the actual meniscus, showing the regularity of the operator $L$, proving existence, and modifying or changing the model.

6.2.1 Conservation of Energy

We were able to show conservation of energy and uniqueness with the flat meniscus, but still have yet to show both with the actual meniscus. Below is a start on the derivation for conservation of energy for the actual meniscus:

$$U_{tt} - P'(\rho_0) \Delta U = 0$$

Now multiplying by $U_t$ and integrating (a standard p.d.e. technique for finding the equation that describes energy) we get:

$$\int_{\Omega} (U_{tt}U_t) \, dx - P'(\rho_0) \int_{\Omega} (\Delta UU_t) \, dx = 0$$

Note, within the first term lies a derivative with respect to time.

$$\int_{\Omega} \frac{\partial}{\partial t} \frac{1}{2} (U_t)^2 \, dx - P'(\rho_0) \int_{\Omega} (\Delta UU_t) \, dx = 0$$
Now by the Leibniz Rule we can push the derivative past the integral. That is,

$$\frac{\partial}{\partial t} \int_{\Omega} \frac{1}{2} (U_t)^2 \, dx - P'(\rho_0) \int_{\Omega} (\Delta U_t) \, dx = 0 \quad (6.1)$$

Now we will take a short detour with the second integral. In general for any two functions $U, V \in C^2$ integration by parts, the Divergence theorem, the definition of the normal derivative, and simplifying provide the following:

$$- \int_{\Omega} (\Delta UV) \, dx = - \int_{\Omega} (\nabla \cdot \nabla U) V \, dx = - \int_{\Omega} [\nabla \cdot (\nabla UV) - \nabla U \cdot \nabla V] \, dx$$

$$= - \int_{\Omega} \text{div} (\nabla UV) \, dx + \int_{\Omega} \nabla U \cdot \nabla V \, dx$$

$$= - \int_{\partial \Omega} (\nabla UV) \cdot ndS + \int_{\Omega} \nabla U \cdot \nabla V \, dx$$

$$= - \int_{\partial \Omega} V (\nabla U) \cdot ndS + \int_{\Omega} \nabla U \cdot \nabla V \, dx,$$

$$= - \int_{\partial \Omega} V \frac{\partial U}{\partial n} dS + \int_{\Omega} \nabla U \cdot \nabla V \, dx. \quad (6.2)$$

Using equation (6.2) and its L.H.S. we can improve equation (6.1) as follows:

$$\frac{\partial}{\partial t} \int_{\Omega} \frac{1}{2} (U_t)^2 \, dx - P'(\rho_0) \int_{\partial \Omega} \frac{\partial U}{\partial n} U_t dS + P'(\rho_0) \int_{\Omega} \nabla U \cdot \nabla U_t \, dx = 0$$

We simplify with a derivative and push it through the integral as before with the Leibniz rule.

$$\frac{\partial}{\partial t} \int_{\Omega} \left( \frac{1}{2} (U_t)^2 + P'(\rho_0) |\nabla U|^2 \right) \, dx - P'(\rho_0) \int_{\partial \Omega} \frac{\partial U}{\partial n} U_t dS = 0$$

Overall we have the following three equations:
\[
\frac{\partial}{\partial t} \int_{\Omega} \left( \frac{1}{2} (U_t)^2 + P'(\rho_0) |\nabla U|^2 \right) \, dx - P'(\rho_0) \int_{\partial \Omega_{\text{bottom}}} \partial_{n} U_t \, dS - P'(\rho_0) \int_{\partial \Omega_{\text{top}}} \partial_{n} U_t \, dS = 0
\]

\[
\Phi_t - \nabla U \cdot \nabla \Psi_0 = 0
\]

\[
U_t = \frac{\beta}{\rho_0} \nabla \cdot \left( \frac{\nabla (\Psi_0 + \Phi)}{2 |\nabla (\Psi_0 + \Phi)|} \right)
\]

As described above by properties of normal derivatives:

\[
\Phi_t = \frac{\partial U}{\partial n} |\nabla \Psi_0|
\]

6.2.2 Alternate Approach to the Model

A last possibility as stated above is to consider changing the model from the beginning. This can be done in a variety of ways. For example, we can include viscosity and the negligible body forces in the Euler equation giving way to the Navier-Stokes equation. This would easily add a great deal of work, but is still possible to approximate. An alternative has been started below. This model already seems less thorough as less equations are involved, but sheds light on the density distribution of the cell. For example, only two equations are involved. This only gives us a look at the interior as described by the given equations. Observe the two equations from before:

\[
\rho [u_t + u \cdot \nabla u] = -\nabla P
\]

\[
\rho_t + \nabla \cdot (\rho u) = 0
\]

Now pulling off a \(d\rho\) on the R.H.S. of the first equation, pulling out \(\rho\) on the L.H.S. in the second equation, and neglect small quantities we see:

\[
\rho u_t = \frac{dP}{d\rho} \frac{d\rho}{dx} = -\frac{dP}{d\rho} \nabla \rho
\]
\( \rho_t + \rho \nabla \cdot (u) = 0 \)

Now take a derivative with respect to time of the second equation and substitute \( u_t \) into this modified second:

\[
\rho_{tt} + \rho \nabla \cdot \left( -\frac{1}{\rho} \frac{dP}{d\rho} \nabla \rho \right) = 0
\]

That is:

\[
\rho_{tt} - \frac{dP}{d\rho} \Delta \rho = 0
\]

Overall, we have the wave equation, but instead of solutions describing velocity they estimate the density:

\[
\rho_{tt} - \frac{dP}{d\rho} \Delta \rho = 0
\]

On the lateral sides of the cell recall we have velocity of a parcel of fluid as being zero. So since our wave equation provides a description of the density we will need boundary conditions that describe density. To initiate this we recall how to introduce our boundary conditions. Dotting the unit normal with the gradient of density and multiplying out the density

\[
-\frac{1}{\rho} \frac{dP}{d\rho} \nabla \rho = 0
\]

becomes

\[
\nabla \rho \cdot n = \frac{\partial \rho}{\partial n} = 0
\]

We could incorporate similar schemes as before with linearization:

\[
P(\rho) = P(\rho_0) + P'(\rho_0)(\rho - \rho_0)
\]

inside

\[
\frac{\partial^2 P}{\partial t^2} = c^2 \nabla^2 P
\]
\[ P(t, x) = P_0 + \text{Re}(e^{i\omega t} \hat{P}(x)) \]

where we get the Helmholtz equation on the inside

\[ \omega^2 \hat{P} = c^2 \nabla^2 \hat{P} \]

with at least temporarily having the top boundary condition set to \(\hat{P} = 0\), and \(\hat{P}\) given on base based on the amplitude of the transducer.

In any case, this is the start of another possible model that can shed some light on another aspect of the phenomena we are after. While this is the start, developing it into a well oiled machine is no small task for the weak hearted.
function Generate_Mesh()

% The function Generate_Mesh() builds a particular shape in the PDE Toolbox. As seen below we are interested in the intersection of a rectangle and a circle. This intersection defines the half-slice of the cell used in the electrochemical experiments. Once the PDE Toolbox opens a few more steps are needed to actually generate the mesh that is used in the finite element process. Notice in the Set formula bar on the Toolbox it says R1+C1. What we want is actually a '*' instead of a '+'. The star indicates intersection. Make this change and then hit the single triangle at the top. Now you have generated the mesh. To see it a bit better click Options, then Axes Limits, and lastly click the auto boxes and hit apply. Then you will have the desired mesh that can be easily seen. The final step is to send this list of edges, triangles, and nodes to the command window. This can be done by selecting Mesh, Export Mesh, and then ok. You will now see the edges, points, and triangles needed to run the program.

% Calling on the MATLAB PDE Toolbox.
pdeinit;
pdetool('appl_cb',1);
pdetool('snapon');

% Generating the rectangle and circle.
pderect([0 0.09 0.1525 0.2125],'R1');
pdecirc(0,0,0.2125,'C1');
function Main(e, p, t, k)
% This is the main function for calculating the standing waves.
% All other functions will be initiated from this program. In
% other words if you wish follow any part of the code begin here.
% The function Main requires an input. This input consists of the
% edges, points (nodes), and triangles generated from the mesh
% and then returns graphs of the standing waves. The function can
% pull in any number of edges, points, triangles, and k value.

% Calculating the three elements for each triangle and
% determining their values (0 or 1 for each node in that
% triangle).
[Z] = Elements(e, p, t);

% Calculating the coefficients that define each element.
[Z] = Coefs(Z, p, t);

% Defining the matrix in the numerator of the Rayleigh
% Quotient and the matrix in its denominator. Then finding
% the eigenvalues and eigenvectors that define the
% eigenfunction expansion.
[V, D] = EigenMatrix(p, t, Z, k);

% Generating the x (radius), y (height), and z (velocity
% potential) values to be plotted.
[a, b, c] = VelPotential(V, Z, p, t, 1);
[d, e, f] = VelPotential(V, Z, p, t, 2);
[g, h, i] = VelPotential(V, Z, p, t, 3);
[j, k, l] = VelPotential(V, Z, p, t, 4);
[m, n, o] = VelPotential(V, Z, p, t, 5);
[P, q, r] = VelPotential(V, Z, p, t, 6);
[ss, tt, uu] = VelPotential(V, Z, p, t, 7);
[vv, ww, xx] = VelPotential(V, Z, p, t, 8);
[yy, zz, aa] = VelPotential(V, Z, p, t, 9);

% Plotting the first 9 standing waves using the MATLAB
% function stem3.
F = figure;
set(F,'name','Behold, K=0 Standing Waves! (Style1)',...
    'numbertitle','off')
set(F, 'Position', [100, 100, 1349, 895]);

subplot(3,3,1)
stem3(a, b, c,'k.')
title('K=0, 1st E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,2)
stem3(d,e,f,'k.')
title('K=0, 2nd E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,3)
stem3(g,h,i,'k.')
title('K=0, 3rd E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,4)
stem3(j,k,l,'k.')
title('K=0, 4th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,5)
stem3(m,n,o,'k.')
title('K=0, 5th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,6)
stem3(P,q,r,'k.')
title('K=0, 6th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,7)
stem3(ss,tt,uu,'k.')
title('K=0, 7th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,8)
stem3(vv,ww,xx,'k.'
title('K=0, 8th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,9)
stem3(yy,zz,aa,'k.'
title('K=0, 9th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

% Plotting the same graphs, but using the MATLAB function
% trimesh
G = figure;
set(G,'name','Behold, K=0 Standing Waves! (Style2)',...
    'numbertitle','off')
set(G, 'Position', [100, 100, 1349, 895]);

s = delaunay(a,b);
T = delaunay(d,e);
u = delaunay(g,h);
v = delaunay(j,k);
w = delaunay(m,n);
x = delaunay(P,q);
bb = delaunay(ss,tt);
cc = delaunay(vv,ww);
dd = delaunay(yy,zz);

subplot(3,3,1)
trimesh(s,a,b,c)
title('K=2, 1st E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,2)
trimesh(T,d,e,f)
title('K=2, 2nd E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,3)
trimesh(u,g,h,i)
title('K=2, 3rd E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,4)
trimesh(v,j,k,l)
title('K=2, 4th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,5)
trimesh(w,m,n,o)
title('K=0, 5th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,6)
trimesh(x,P,q,r)
title('K=0, 6th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,7)
trimesh(bb,ss,tt,uu)
title('K=0, 7th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,8)
trimesh(cc,vv,ww,xx)
title('K=0, 8th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

subplot(3,3,9)
trimesh(dd,yy,zz,aa)
title('K=0, 9th E.Vector.')
xlabel('Radius of Cell(dm)')
ylabel('Height of Cell(dm)')
zlabel('Velocity Potential')

% One plot of the 8th standing wave
H = figure;
trimesh(cc,vv,ww,xx)
title('K=0, 8th E.Vector.')
xlabel('x-axis (dm)')
ylabel('z-axis (dm)')
zlabel('Velocity Potential (dm^2/s)')
function [Z] = Elements(eds,pts,trs)
% This function pulls in the edges, points, and triangles
% generated by the mesh. It then uses that information to
% construct the elements in the finite element method. Each
% element, denoted phi, is given three values. It is given the
% value 1, 0, and 0 on the three vertices of its corresponding
% triangle. There are three phi’s for each triangle. Information
% about each triangle can be found in each column of the matrix
% denoted Z. As an example, recorded in column 1 of matrix Z is
% information about the three phi’s. First the function locates
% the first triangle listed in trs, looks up its nodes, and then
% calculates when phi_1 is 1, 0, and 0 and stores those values
% in row 1, 2, and 3 of matrix Z. It then leaves rows 4, 5, and
% 6 blank for later use. Following this it enters phi_2s’ values
% into rows 7, 8, and 9 and phi_3 into rows 13, 14, and 15. The
% program does this for every triangle i or every column i and
% then returns the matrix Z.

% Defining my initials.
N_eds = length(eds);  % number of edges
N_pts = length(pts);  % number of points
N_trs = length(trs);  % number of triangles

lambda_1 = @(x,y)(1-x-y);  % reference triangle lambda 1
lambda_2 = @(x,y)(x);    % reference triangle lambda 2
lambda_3 = @(x,y)(y);    % reference triangle lambda 3

coeff = 0;    % 1 over the det(B_inverse)
B_k_inv = zeros(2,2);  % Inverse of B
x_hat_1 = zeros(2,1);  % initializing the x_hat_1
x_hat_2 = zeros(2,1);  % initializing the x_hat_2
x_hat_3 = zeros(2,1);  % initializing the x_hat_3

P = zeros(2,1);    % node 1 in a triangle
Q = zeros(2,1);    % node 2 in a triangle
R = zeros(2,1);    % node 3 in a triangle

phi = zeros(32,N_pts);  % rows = phi’s, col’s = nodes

Z = zeros(18,N_trs);  % holds all 3 phi’s for each tri

% Initializing phi so that the rest of the code knows when
% we have already used space and when we haven’t.
for j = 5:4:32
\[ \phi(j,:) = 1; \]

end

% Running over all columns of the triangle matrix \( \text{trs} \).
for \( i = 1:N_{\text{trs}} \)

\[ a_{11} = \text{pts}(1,\text{trs}(1,i)); \quad \% \text{x-value for node 1 of tri } i \]
\[ a_{21} = \text{pts}(2,\text{trs}(1,i)); \quad \% \text{y-value for node 1 of tri } i \]
\[ a_{12} = \text{pts}(1,\text{trs}(2,i)); \quad \% \text{x-value for node 2 of tri } i \]
\[ a_{22} = \text{pts}(2,\text{trs}(2,i)); \quad \% \text{y-value for node 2 of tri } i \]
\[ a_{13} = \text{pts}(1,\text{trs}(3,i)); \quad \% \text{x-value for node 3 of tri } i \]
\[ a_{23} = \text{pts}(2,\text{trs}(3,i)); \quad \% \text{y-value for node 3 of tri } i \]

% Calculating the inverse of the transformation matrix.
\[ \text{coeff} = \frac{1}{((a_{12}-a_{11})*(a_{23}-a_{21}) - (a_{22}-a_{21})*(a_{13}-a_{11}))}; \]
\[ B_{k_{\text{inv}}} = \text{coeff} * \begin{bmatrix} a_{23}-a_{21} & a_{11}-a_{13} \\ a_{21}-a_{22} & a_{12}-a_{11} \end{bmatrix}; \]
\[ x_{\text{hat}_1} = B_{k_{\text{inv}}}(\begin{bmatrix} a_{11}; a_{21} \end{bmatrix} - \begin{bmatrix} a_{11}; a_{21} \end{bmatrix}); \]
\[ x_{\text{hat}_2} = B_{k_{\text{inv}}}(\begin{bmatrix} a_{12}; a_{22} \end{bmatrix} - \begin{bmatrix} a_{11}; a_{21} \end{bmatrix}); \]
\[ x_{\text{hat}_3} = B_{k_{\text{inv}}}(\begin{bmatrix} a_{13}; a_{23} \end{bmatrix} - \begin{bmatrix} a_{11}; a_{21} \end{bmatrix}); \]

\[ j = 1; \]
% Calculating the \( \phi_1, \phi_2, \) and \( \phi_3 \) at node 1
while \( i \)
    if \( \phi(j,\text{trs}(1,i)) = 0 \)
        % Calc lambda 1,2, & 3 for node 1 of tri i.
        \[ \phi(j+1,\text{trs}(1,i)) = \lambda_1(x_{\text{hat}_1}(1),x_{\text{hat}_1}(2)); \]
        \[ \phi(j+2,\text{trs}(1,i)) = \lambda_2(x_{\text{hat}_1}(1),x_{\text{hat}_1}(2)); \]
        \[ \phi(j+3,\text{trs}(1,i)) = \lambda_3(x_{\text{hat}_1}(1),x_{\text{hat}_1}(2)); \]

        % Looking at node 1 and storing the value of each \( \phi \) into the \( \text{Z} \) matrix.
        \[ \text{Z}(1,i) = \phi(j+1,\text{trs}(1,i)); \]
        \[ \text{Z}(7,i) = \phi(j+2,\text{trs}(1,i)); \]
        \[ \text{Z}(13,i) = \phi(j+3,\text{trs}(1,i)); \]

    end
    \[ \phi(j+4,\text{trs}(1,i)) = 0; \quad \% \text{Replacing the 1 with a 0} \]
    \[ \phi(j,\text{trs}(1,i)) = i; \quad \% \text{Replacing 0 with the tri } i \]
end
\[ j = j + 4; \]

\[ j = 1; \]

\% Calculating the \( \phi_1 \), \( \phi_2 \), and \( \phi_3 \) at node 2
\while i
  \if \phi(j, \text{trs}(2, i)) == 0
    \% Calc lambda 1, 2, & 3 for node 2 of triangle i
    \phi(j + 1, \text{trs}(2, i)) = \lambda_1(x_{\text{hat}}_2(1), x_{\text{hat}}_2(2));
    \phi(j + 2, \text{trs}(2, i)) = \lambda_2(x_{\text{hat}}_2(1), x_{\text{hat}}_2(2));
    \phi(j + 3, \text{trs}(2, i)) = \lambda_3(x_{\text{hat}}_2(1), x_{\text{hat}}_2(2));

    \% Looking at node 2 and storing the value of each \( \phi \) into the \( Z \) matrix.
    Z(2, i) = \phi(j + 1, \text{trs}(2, i));
    Z(8, i) = \phi(j + 2, \text{trs}(2, i));
    Z(14, i) = \phi(j + 3, \text{trs}(2, i));

    \phi(j + 4, \text{trs}(2, i)) = 0; \% Replacing the 1 with a 0
    \phi(j, \text{trs}(2, i)) = i; \% Replacing 0 with the tri i
    \break
  \endif
\endwhile

\j = j + 4;

\j = 1;

\% Calculating the \( \phi_1 \), \( \phi_2 \), and \( \phi_3 \) at node 3
\while i
  \if \phi(j, \text{trs}(3, i)) == 0
    \% Calc lambda 1, 2, & 3 for node 3 of triangle i
    \phi(j + 1, \text{trs}(3, i)) = \lambda_1(x_{\text{hat}}_3(1), x_{\text{hat}}_3(2));
    \phi(j + 2, \text{trs}(3, i)) = \lambda_2(x_{\text{hat}}_3(1), x_{\text{hat}}_3(2));
    \phi(j + 3, \text{trs}(3, i)) = \lambda_3(x_{\text{hat}}_3(1), x_{\text{hat}}_3(2));

    \% Looking at node 3 and storing the value of each \( \phi \) into the \( Z \) matrix.
    Z(3, i) = \phi(j + 1, \text{trs}(3, i));
    Z(9, i) = \phi(j + 2, \text{trs}(3, i));
    Z(15, i) = \phi(j + 3, \text{trs}(3, i));

    \phi(j + 4, \text{trs}(3, i)) = 0; \% Replacing the 1 with a 0
phi(j, trs(3, i)) = i; \% Replacing 0 with the tri i
break
end
j = j + 4;
end
end
end
function [Z] = Coeffs(Z,pts,trs)
% The function Coeffs pulls in Z and the points and triangles
% from the mesh. To see what Z contains up to this point go to
% the function Elements. Here we find the coefficients of the
% functions phi where phi = ax+by+c or more precisely since we
% are using cylindrical coordinates phi = ar+bz+c. Consider
% phi_1 of the first triangle. Already are its three values
% stored in the matrix Z in column 1 row 1, 2, and 3. Now a, b,
% and c get stored in rows 4, 5, and 6. Phi_2 and phi_3 are
% dealt with in a similar manner.

a = 0;  % Initializing a
b = 0;  % Initializing b
c = 0;  % Initializing c

N_trs = length(trs);  % number of triangles

% In the function Elements all phi’s were given the values
% 1, 0, and 0. Since a calculation was done to find this the
% actual values are simply close to 0 and 1, but not
% actually 0 and 1 like they should be. This double for-loop
% rewrites Z so that the values are exactly 0 and 1.
for i=1:N_trs
    for j=1:18
        if Z(j,i) < 0.5
            Z(j,i) = 0;
        else
            Z(j,i) = 1;
        end
    end
end

% Calculating a, b, and c for each phi.
for i = 1:N_trs

    a_11 = pts(1,trs(1,i));  % x-value node 1 of tri (i)
    a_21 = pts(2,trs(1,i));  % y-value node 1 of tri (i)
    a_12 = pts(1,trs(2,i));  % x-value node 2 of tri (i)
    a_22 = pts(2,trs(2,i));  % y-value node 2 of tri (i)
    a_13 = pts(1,trs(3,i));  % x-value node 3 of tri (i)
    a_23 = pts(2,trs(3,i));  % y-value node 3 of tri (i)

    % Finding a, b, and c for phi_1 of triangle i.
    if Z(1,i) == 1

% end
% end
A = [a_{11} a_{21} 1; a_{12} a_{22} 1; a_{13} a_{23} 1];
B = [1; 0; 0];
X = linsolve(A,B);
a = X(1,1);
b = X(2,1);
c = X(3,1);
Z(4,i) = a;
Z(5,i) = b;
Z(6,i) = c;
end

% Finding a, b, and c for phi_2 of triangle i.
if Z(8,i) == 1
    A = [a_{11} a_{21} 1; a_{12} a_{22} 1; a_{13} a_{23} 1];
    B = [0; 1; 0];
    X = linsolve(A,B);
    a = X(1,1);
    b = X(2,1);
    c = X(3,1);
    Z(10,i) = a;
    Z(11,i) = b;
    Z(12,i) = c;
end

% Finding a, b, and c for phi_3 of triangle i.
if Z(15,i) == 1
    A = [a_{11} a_{21} 1; a_{12} a_{22} 1; a_{13} a_{23} 1];
    B = [0; 0; 1];
    X = linsolve(A,B);
    a = X(1,1);
    b = X(2,1);
    c = X(3,1);
    Z(16,i) = a;
    Z(17,i) = b;
    Z(18,i) = c;
end
end
function [V,D] = EigenMatrix(p,t,Z,k)
% The function EigenMatrix pulls in the points and triangles
% from the mesh, the Z matrix (includes all information about
% the finite elements on that mesh), and the k-value we are
% interested in. After this it initiates the calculating of the
% three integrals involved in the Rayleigh Quotient. Following
% this it uses the MATLAB function eig to compute the eigenvalues
% and eigenvectors of the generalized eigenvalue problem used to
% build the quotient.

N_trs = length(t); % Number of triangles
N_pts = length(p); % Number of points (repetition allowed)

A = zeros(N_pts,N_pts); % Initializing numerator matrix
B = zeros(N_pts,N_pts); % Initializing denominator matrix

% Iterating over every triangle because each integral can
% be done on only one triangle at a time.
for j = 1:N_trs
    x = t(1,j); % node 1
    y = t(2,j); % node 2
    z = t(3,j); % Node 3

    P1 = [p(1,x),p(2,x)]; % x-value and y-value at node 1
    P2 = [p(1,y),p(2,y)]; % x-value and y-value at node 1
    P3 = [p(1,z),p(2,z)]; % x-value and y-value at node 1

    J = j; % Matlab being annoying

    % All combinations of phi's per triangle. If you look
    % at the integrals each is done with two phi's at a
    % time. Thus there are 9 possibilities with 3 phi's and
    % 3 nodes. Matrix A is the integral in the numerator of
    % the Rayleigh Quotient and B is denoted for the matrix
    % of the denominator.

    % These are the 3 combinations with node 1. That is,
    % phi_1 & phi_1, phi_1 & phi_2, and phi_1 & phi_3
    % on node 1.
    [a,b] = IntegralCalc(Z(4,j),Z(5,j),Z(6,j),...% int(Z_4, Z_5, Z_6, P1, P2, P3, Z, J, k);% A(x,x) = A(x,x) + a;
    B(x,x) = B(x,x) + b;

    A(x,x) = A(x,x) + a;
    B(x,x) = B(x,x) + b;
\[ [a, b] = \text{IntegralCalc}(Z(4,j), Z(5,j), Z(6,j), \ldots, Z(10,j), Z(11,j), Z(12,j), P1, P2, P3, Z, J, k); \]
\[ A(x, y) = A(x, y) + a; \]
\[ B(x, y) = B(x, y) + b; \]

% These are the 3 combinations with node 2. That is, phi_2 & phi_1, phi_2 & phi_2, phi_2 & phi_3 on node 2.
\[ [a, b] = \text{IntegralCalc}(Z(10,j), Z(11,j), Z(12,j), \ldots, Z(4,j), Z(5,j), Z(6,j), P1, P2, P3, Z, J, k); \]
\[ A(y, x) = A(y, x) + a; \]
\[ B(y, x) = B(y, x) + b; \]

% These are the 3 combinations with Node3. That is, phi_3 & phi_1, phi_3 & phi_2, phi_3 & phi_3 on node 3.
\[ [a, b] = \text{IntegralCalc}(Z(16,j), Z(17,j), Z(18,j), \ldots, Z(10,j), Z(11,j), Z(12,j), P1, P2, P3, Z, J, k); \]
\[ A(z, x) = A(z, x) + a; \]
\[ B(z, x) = B(z, x) + b; \]
Z(16,j), Z(17,j), Z(18,j), P1, P2, P3, Z, J, k);
A(z,z) = A(z,z) + a;
B(z,z) = B(z,z) + b;
end

% Solving the generalized eigenvalue problem found in the
% Rayleigh Quotient using the MATLAB function eig.
[V,D] = eig(A,B);
function [a,b] = IntegralCalc(a,b,c,d,e,f,P1,P2,P3,Z,j,k)
% The function IntegralCalc pulls in the coefficients of two
% finite elements, the points associated with the triangle
% where they exist, the Z matrix which carries information
% about their values on those nodes, which triangle we are
% looking at (j), and the k-value of interest. It uses this
% information to then calculate the three integrals in the
% Rayleigh Quotient. They are denoted by NumL for the integral
% in the numerator on the left, NumR for the integral in the
% numerator on the right, and finally Den for the integral in
% the denominator. This program is sectioned into three parts
% for each of them. It then sends back the numerator value and
% the denominator value of the quotient.

NumL = 0; % Initializing Left Numerator Integral
NumR = 0; % Initializing Right Numerator Integral
Den = 0; % Initializing Denominator Integral
AreaT = 0; % Initializing Area of the tri of interest
Rsqrd = (0.2125)^2; % Radius squared

% Left Integral in the Numerator of the Rayleigh Quotient.
% We start with the calculation of G = (a*X + b*Y + c)
% and H = (d*X + e*Y + f). G and H are compositions of X
% and Y where X and Y are the first and second components
% (respectively) of the transformation F. That is,
% F = [a_11;a_21] + B_k*[r;z]) := [X;Y]. After we use this
% fact in our calculations we multiply the functions G
% and H, convert to cylindrical (this involves
% r*dr*dtheta*dz and since there is no theta because we
% have rotational symmetry we simply get 2*pi*r*dr*dz),
% integrate over the reference triangle, and multiply be
% the determinant of the corresponding matrix B_k.

a_11 = P1(1); % x-value for node 1 of triangle (i)
a_21 = P1(2); % y-value for node 1 of triangle (i)
a_12 = P2(1); % x-value for node 2 of triangle (i)
a_22 = P2(2); % y-value for node 2 of triangle (i)
a_13 = P3(1); % x-value for node 3 of triangle (i)
a_23 = P3(2); % y-value for node 3 of triangle (i)

B_k = [a_12-a_11, a_13-a_11; a_22-a_21, a_23-a_21];
Det = det(B_k);
% Recall: \( g = \forall(r,z)(a*r + b*z + c) \) before the composition.
% Recall: \( h = \forall(r,z)(d*r + e*z + f) \) before the composition.
\[
G = \forall(r,z)(a*(a_{11} + (a_{12}-a_{11})*r + (a_{13}-a_{11})*z) + b* ... \\
(a_{21} + (a_{22}-a_{21})*r + (a_{23}-a_{21})*z) + c);
\]
\[
H = \forall(r,z)(d*(a_{11} + (a_{12}-a_{11})*r + (a_{13}-a_{11})*z) + e* ... \\
(a_{21} + (a_{22}-a_{21})*r + (a_{23}-a_{21})*z) + f);
\]
\[
F1 = \forall(r,z)(2\pi*r.*G(r,z).*H(r,z));
\]
\[
y = \forall(r)(-r+1);
\]
\[
NumL = Det*integral2(F1,0,1,0,y);
\]
% Integral in the Denominator.
\[
F2 = \forall(r,z)(2\pi*r.(a*d + b*e));
\]
\[
y = \forall(r)(-r+1);
\]
\[
Den = Det*integral2(F2,0,1,0,y);
\]
% Right Integral in the Numerator.
% Since this integral is over the top boundary (meniscus)
% it only involves the phi's that show up on that boundary.
% We have excluded the phi’s that show up on the top
% boundary only for one point, but the others that are
% there for more. Also, notice this means we only care
% about two of the nodes at a time. To carry out this
% integral we invoke that which computes the o.d.e.,
% VPFRob.
if ( abs(P1(1)^2+P1(2)^2-Rsqrd)<1.00e-15 && ... \\
abs(P2(1)^2+P2(2)^2-Rsqrd)<1.00e-15 )
if (P1(1)==0 || P2(1)==0)
elseif (a==Z(16,j) || d==Z(16,j))
else
if (a==Z(4,j))
    NumR = VPFRob(a,b,c,d,e,f,P1,P2,k);
else
    NumR = VPFRob(d,e,f,a,b,c,P1,P2,k);
end
end
elseif ( abs(P1(1)^2+P1(2)^2-Rsqrd)<1.00e-15 && ... \\
abs(P3(1)^2+P3(2)^2-Rsqrd)<1.00e-15 )
if (P1(1)==0 || P3(1)==0)
elseif (a==Z(10,j) || d==Z(10,j))

else
else
    if (a==Z(4,j))
        NumR = VPFrob(a,b,c,d,e,f,P1,P3,k);
    else
        NumR = VPFrob(d,e,f,a,b,c,P1,P3,k);
    end
end
elseif ( abs(P2(1)^2+P2(2)^2-Rsqrd)<1.00e-15 && ...
        abs(P3(1)^2+P3(2)^2-Rsqrd)<1.00e-15 )
    if (P2(1)==0 || P3(1)==0)
    elseif (a==Z(4,j) || d==Z(4,j))
    else
        if (a==Z(10,j))
            NumR = VPFrob(a,b,c,d,e,f,P2,P3,k);
        else
            NumR = VPFrob(d,e,f,a,b,c,P2,P3,k);
        end
    end
end

% Overall Fraction (Collecting the numerator and % denominator).
a = NumL + NumR;
b = Den;
function [Int] = VPFrob(a,b,c,d,e,f,pt1,pt2,k)
% This function calculates the right integral in the numerator 
% of the Rayleigh Quotient. Specifically, this is the integral 
% that involves the o.d.e. generated by the operator L. It 
% pulls in the coefficients of the two phi's of interest, the 
% two points along the top boundary where they exist, and the 
% k-value of interest. Here we use Simpsons Rule to find 
% approximations to the integral. This is done because we only 
% have a set of discrete points to do this with. Fortunately 
% we can have as many as we wish for accuracy. Meaning the 
% unbounded and bounded functions generate overall solution 
% to the o.d.e., but we only have those functions in the form 
% of a power series. This says that we can only see how they 
% behave at a set of discrete points.

Rsqrd = (0.2125)^2; % Radius squared

% When bringing the two points in we want to make sure the 
% calculations are done the same way everytime. Meaning if 
% the points come in with the right point on the left and 
% the left point on the right then we want to flip them so 
% that we dont get an unnecesssary negative.
if (pt1(1)<pt2(1))
    r1 = pt1(1);
    r2 = pt2(1);
else
    r1 = pt2(1);
    r2 = pt1(1);
end

x = a;
y = b;
z = c;

a = d;
b = e;
c = f;

d = x;
e = y;
f = z;
end

r = linspace(r1,r2,100); % 100pts btwn two x-values
C1 = zeros(1,length(r)-1);  % Coeff. of bounded term
C2 = zeros(1,length(r)-1);  % Coeff. of unbounded term
rab = zeros(1,length(r)-1);  % values in btwn each
    % interval
y1ab = zeros(1,length(r)-1);  % bdd val’s btwn each
    % interval
y2ab = zeros(1,length(r)-1);  % unbdd val’s btwn each
    % interval
dy1ab = zeros(1,length(r)-1);  % bounded derivative
    % values in between each
    % interval
dy2ab = zeros(1,length(r)-1);  % unbounded derivative
    % values in between
    % each interval

% Upper right corner boundary condition with H + p = 0
% there.
BC = [0.09, 0.1925];

% Finding all of the midpoint values. Wab is the wronskian,
% Zab is from the equation z = z_0 + sqrt(R^2 - r^2) (the
% equation of the sphere), and RHSab is the RHS of the
% o.d.e. All of which are being calculated at the midpoint
% (hence the ab notation).
for i=1:99
    rab(i)=(r(i+1)-r(i))/2;
    [y1ab(i),dy1ab(i)]=Frob1(k,rab(i));
    [y2ab(i),dy2ab(i)]=Frob2(k,rab(i));
    Wab(i) = y2ab(i)*dy1ab(i)-dy2ab(i)*y1ab(i);
    Zab(i) = -0.1925+sqrt(Rsqrd-rab(i)^2);
    RHSab(i) = (a*rab(i)+b*Zab(i)+c)*(Rsqrd/(
        ((1-rab(i)^2)^(3/2))));
end

% Calculating the values at the endpoints of the intervals.
for i=1:100
    [y1(i),dy1(i)]=Frob1(k,r(i));
    [y2(i),dy2(i)]=Frob2(k,r(i));
    W(i) = y2(i)*dy1(i)-dy2(i)*y1(i);
    Z(i) = -0.1925+sqrt(Rsqrd-r(i)^2);
    RHS(i) = (a*r(i)+b*Z(i)+c)*(Rsqrd/((1-r(i)^2)^(3/2)));
end
% Using Simpsons Rule to find C2.
C2(1)=0;
for i=1:99
    C2(i+1) = ((r(i+1)-r(i))/6)*( RHS(i)*y1(i)/W(i)...
              + 4*RHSab(i)*y1ab(i)/Wab(i)...
              + RHS(i+1)*y1(i+1)/W(i+1) )
    C2(i+1) = C2(i+1)+C2(i);
end

% Using Simpsons Rule to find C1. Note that C1(99) should
% be the smallest magnitude and C1(1) should be the largest.
C1(100)=0;
for i=1:99
    j=-i;
    C1(100+j) = ((r(100+j)-r(101+j))/6)*(...
                 RHS(101+j)*y2(101+j)/W(101+j)...
                 + 4*RHSab(100+j)*y2ab(100+j)/Wab(100+j)...
                 + RHS(100+j)*y2(100+j)/W(100+j) )
    C1(100+j) = C1(100+j)+C1(101+j);
end

% Particular soln (the technique for calculating this is
% called the variation of parameters).
for i=1:99
    p(i) = -C1(i+1)*y1(i) + C2(i+1)*y2(i);
end
Const = p(99)/y1(99);

% General solution, H, and overall solution, L_inverse(phi)
% labeled LI.
for i=1:99
    H(i) = Const*y1(i);
    LI(i) = -p(i) + H(i);
end

% Recall the constant L = 2*rho_0*P'(pho_0)/beta
% speed of sound = 3402.9 dm/s
rho0 = 1;
dPdt_rho0 = sqrt(3402.9);
beta = 1; %7.199 surface tension mN/dm
L = 2*rho0*dPdt_rho0/beta;

% Actual third integral with everything.
I(1) = 0;
for i=1:99
    I(i+1) = 2*pi*L*( ((r(i+1)-r(i))/6)*(...
        (d*r(i)+e*Z(i)+f)*LI(i)*r(i)...
        + 4*(d*rab(i)+e*Zab(i)+f)*LI(i)*rab(i)...
        + (d*r(i+1)+e*Z(i+1)+f)*LI(i)*r(i+1)  ) );
    I(i+1) = I(i+1) + I(i);
end

% Actual value of the integral is stored in.
Int = I(100);
function [Y1,dY1] = Frob1( k,x )
% This function is the bounded function or well behaved series
% that defines the upper-half of the solution to the o.d.e. that
% is generated by L(Phi). By upper-half we mean the iterates in
% the infinite sum that are positive. It is simply calculated
% by the sequence of coefficients that define it seen in the
% chapters that discuss the o.d.e.

% Initiating values.
k=k+1;
a=linspace(0,0,10);
a(k)=1;
a(k+1)=0;

% Calculating the coefficients.
for n=k:18
    a(n+2)=a(n)*(n-1)*(n+2)/((n+1)^2-(k-1)^2);
end

% Calculating the solution at a set of x-values.
Y1=0;
for n=1:20
    Y1=Y1+a(n)*x^(n-1);
end

% Calculating the derivative at a set of x-values.
dY1=0;
for n=1:20
    dY1=dY1+a(n)*(n-1)*x^(n-2);
end
end
function [y2,dy2] = Frob2( k, x )
% This function calculates the unbounded portion of the
% infinite sum that solves the o.d.e. generated by L
% acting on Phi. Here we use the actual solution to the
% o.d.e. for k = 0 and 1. These solutions were provided
% by to us from mathematica.

% Initiating values.
y2=0;
dy2=0;

% Mathematica solution if k = 0.
if (k == 0)
  Y2 = @(s)( 1/((1-s^2)^(1/2)) + log(s) -... 
             log(1+sqrt(1-s^2)) );
  dY2 = @(s)( s/((1-s^2)^(3/2)) + 1/s +... 
              s/(sqrt(1-s^2)+1-s^2) );
  y2 = Y2(x);
  dy2 = dY2(x);

% Mathematica solution if k = 1.
elseif (k == 1)
  Y2 = @(s)( -(1-s^2)^(1/2) + (s^2)*log(s) -... 
             (s^2)*log(1+sqrt(1-s^2))/(2*s*((1-s^2)^(1/2))) );
  dY2 = @(s)(0.5*s*sqrt(1-s^2)*(s^2)*log(s) -... 
              2*s*log(1+sqrt(1-s^2)) )... 
    +0.5*sqrt(1-s^2)*(sqrt(1-s^2)+(1-s^2)+2*s*log(s)-... 
    sqrt(1-s^2)+(1-s^2))/(2*sqrt(1-s^2))... 
    +0.5*sqrt(1-s^2)*(-sqrt(1-s^2)+(1-s^2)+2*s*log(s)-... 
    sqrt(1-s^2)+(1-s^2))/(2*sqrt(1-s^2))... 
    +0.5*sqrt(1-s^2)*(-sqrt(1-s^2)+... 
    (s^2)*log(s)-(s^2)*log(1+sqrt(1-s^2)) ) ;
  y2 = Y2(x);
  dy2 = dY2(x);

% Here we use the sequence of terms generated by the
% method of
% Frobenius.
else
  a(1) = 1;  % a_0
  a(2) = 0;  % a_-1
  a(4) = 1;  % a_-3
  for i=1:18
\( j=-(i+1) \);
if (mod(k,2)==0) \% k is even and > than 1
  if \( j=-3 \)
    \( a(4)=0; \)
  else
    if (mod(i+2,2)==0) \% i is even
      \% meaning nu is odd
      \% (if i=4, nu=j=-5)
      \( a(i+2) = a(i)*((j+2)^2 \ldots -k^2))/(j*(j+3)); \)
    else \% i is odd meaning nu is even
      \% (if i=1, nu=j=-2)
      \( a(i+2) = a(i)*((j+2)^2 \ldots -(k^2))/(j*(j+3)); \)
    end
  end
else
  if \( j=-3 \)
    else
      if (mod(i+2,2)==0 \&\& i=-2) \% i is even
        \% even meaning nu is odd
        \% (if i=4, nu=j=-5)
        \( a(i+2) = a(i)*((j+2)^2 \ldots (k^2))/(j*(j+3)); \)
      else \% i is odd meaning nu is even
        \% (if i=1, nu=j=-2)
        \( a(i)=0; \)
        \( a(i+2) = a(i)*((j+2)^2 \ldots (k^2))/(j*(j+3)); \)
      end
    end
  end
end

\% Calculating the series at a set of x-values.
for \( n=1:20 \)
  \( m = -n; \)
  \( y2 = y2+a(n)*x^{-}(m+1); \)
end

\% Calculating the derivative of the series at
\% a set of x-values.
for \( n=1:20 \)
  \( m = -n; \)
\[
dy2 = dy2 + a(n) \times (n-1) \times x^m;
\]
end
dend
dend
function [L,M,N] = VelPotential(V,Z,p,t,C)
% This function denoted VelPotential is what it sounds like. It
% takes the eigenvalues and eigenvectors calculated from the
% MATLAB eig function on the Rayleigh Quotient and then finds
% the corresponding velocity potential.

% Initializing my terms.
N_trs = length(t);
N_pts = length(p);
ZL = 3*length(Z);
Z_new = zeros(3,3*N_trs);

% Since we want Phi_t = L_inverse(Phi) then we need Phi.
% Meaning we need the coefficients of the Phi’s from the
% Z matrix.
i=1;
for j=1:3:ZL
    Z_new(1,j) = Z(4,i);
    Z_new(2,j) = Z(5,i);
    Z_new(3,j) = Z(6,i);
    Z_new(1,j+1) = Z(10,i);
    Z_new(2,j+1) = Z(11,i);
    Z_new(3,j+1) = Z(12,i);
    Z_new(1,j+2) = Z(16,i);
    Z_new(2,j+2) = Z(17,i);
    Z_new(3,j+2) = Z(18,i);
    i = i+1;
end

% Initializing.
A = zeros(N_pts,1);
B = zeros(3,3);
i=1;
k=0;
m=1;
a = zeros(1,500);
b = zeros(1,500);
c = zeros(1,500);

% Calculating the velocity potential.
for j = 1:N_trs
    x = t(1,j);   % Node1
    y = t(2,j);   % Node2
    z = t(3,j);   % Node3
    
    r1 = p(1,x);  
    z1 = p(2,x);  
    r2 = p(1,y);  
    z2 = p(2,y);  
    r3 = p(1,z);  
    z3 = p(2,z);  

    % Calculating U at the first node
    a(x) = r1;
    b(x) = z1;
    c(x) = c(x) + V(x,C);

    % Calculating U at the second node
    a(y) = r2;
    b(y) = z2;
    c(y) = c(y) + V(y,C);

    % Calculating U at the third node
    a(z) = r3;
    b(z) = z3;
    c(z) = c(z) + V(z,C);
end

% Counting the number of times we hit the same x and y value
% in all of a and b, respectively. If it is used more then
% once we give the second a-value the number 10.
for n=1:length(a)
    d = a(n);
    e = b(n);
    J = n+1;
    if (J==length(a)+1)
    else
        for m = J:length(a)
            if (abs(a(m) - d)<1e-14 && abs(b(m)-e)<1e-14)
                a(m) = 10;
            end
        end
    end
end
end
end

% If the same x-value and y-value were used then delete
% those extra values. So basically rewriting a into f
% without the duplicates.
counter=0;
for i=1:length(a)
    if a(i)==10
        counter = counter+1;
    else
        f(i-counter)=a(i);
        g(i-counter)=b(i);
        h(i-counter)=c(i);
    end
end

% Getting ride of the point (0,0,0).
counter=0;
for i=1:length(f)
    if (f(i)==0 && g(i)==0 && h(i)==0)
        counter = counter+1;
    else
        L(i-counter)=f(i);
        M(i-counter)=g(i);
        N(i-counter)=h(i);
    end
end
REFERENCES


