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Cofree objects in the categories of comonoids in certain abelian monoidal categories

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COFREE OBJECTS IN THE CATEGORIES OF COMONOIDS IN CERTAIN ABELIAN MONOIDAL CATEGORIES

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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ABSTRACT

We investigate cofree coalgebras, and limits and colimits of coalgebras in some abelian monoidal categories of interest, such as bimodules over a ring, and modules and comodules over a bialgebra or Hopf algebra. We find concrete generators for the categories of coalgebras in these monoidal categories, and explicitly construct cofree coalgebras, products and limits of coalgebras in each case. This answers an open question in [4] on the existence of a cofree coring, and constructs the cofree (co)module coalgebra on a B-(co)module, for a bialgebra B.

PUBLIC ABSTRACT

Significantly, free and cofree objects play a crucial role in reshaping the adjunctions of faithful functors in terms of comma categories controlled by universal properties. In this thesis, we use the dual of the special adjoint functor theorem as a reasonable machinery to study three important concepts that imply the existence of cofree objects in each concrete category of interest. This is not only helpful in investigating the existence of such objects, but it is also crucial in finding an explicit construction for them. We start our investigation by showing that if $\mathcal{C}$ is a cocomplete monoidal category, then so is the category $CoMon(\mathcal{C})$ of comonoids in $\mathcal{C}$. We also show that the concept of co-wellpoweredness is hereditary in the sense that if $\mathcal{C}$ is a co-wellpowered (monoidal) category, then the category $CoMon(\mathcal{C})$ is co-wellpowered. The most critical point in our work is to deal with the generating sets. The reason why they are demanding is that if the monoidal category $\mathcal{C}$ has a generating set, then the category $CoMon(\mathcal{C})$ needs not have a generating set. Thus, one might needs to explicitly find generating sets for the corresponding categories of comonoids in each category of interest. We find generators for these important categories and construct the cofree objects in terms of the corresponding generators and colimits. Furthermore, we find a more explicit description to the cogenerators of comodule algebras and the generators of comodule coalgebras over Hopf algebras. Finally, we show that each category of interest is complete, and we explicitly construct their limits.

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CHAPTER 1
INTRODUCTION, PRELIMINARY DEFINITIONS AND NOTATIONS

1.1 Introduction and Background

Universal properties are significantly considered as one of the most important concepts in mathematics. In deed, they can be thought of as the skeleton of all mathematics concepts. They show how the objects and the morphisms being unified and described compatibly relate the whole category that they live in. Many influential concepts, such as kernels, cokernels, products, coproducts, limits, colimits, etc, are essentially involved with universal properties. Perhaps the most important notion concerned with them is the concept of adjoint functors.

Kan's discovery was a substantial try to give the category theory a systematic authority to control all branches of mathematics by using a brilliant mechanism that bridges two categories in a compatible way that allows all objects and morphisms to move back and forth from one category to another in a compatible way.

Crucially, Kan has emphasized the importance of his discovery when he says "An important property of adjoint functors is that each determines the other up to a unique natural equivalence" [36, p. 294]. Mac Lane, furthermore, has pointed out the significance of adjoints by saying " We [Eilenberg & Mac Lane] did not then regard it [category theory] as a field for further research effort, but just as a language and an orientation -a limitation which we followed for a dozen years or so, till the advent of adjoint functors" (Mac Lane [40]). He also highlights this facts when he says "Adjoint functors arise everywhere" [40, p. vii].

The reason is all fundamentals notions of mathematics can systematically be reshaped in terms of adjunctions. Many dominant notions, such as reflections functors, Coxeter functors, restriction and idempotent functors, which play a crucial role in representation theory, are simply are examples of adjoint functors. Briefly, adjoint functors take a pivotal role with pertinent associations to manifolds, knot theory, quantum theory, number theory, monoidal categories, Hopf algebras, etc., and they "provide the general idea and the principle from which various starting points can be defined and deduced" (Marquis [44]).

Significantly, free and cofree objects play a crucial role in recasting the adjunctions of forgetful functors in terms of comma categories. For fundamentals concepts and examples of adjoint functors, we refer the reader to [38], [40], [8], [55], [48], or [45]. For the basic notions of comma categories, we refer to [38] and [40].

Let $\mathfrak{X}$ be a category. A *concrete category* over $\mathfrak{X}$ is a pair $(\mathfrak{A}, \mathfrak{U})$, where $\mathfrak{A}$ is a category and $\mathfrak{U} : \mathfrak{A} \to \mathfrak{X}$ is a faithful functor [3, p. 61]. Let $(\mathfrak{A}, \mathfrak{U})$ be a concrete category over $\mathfrak{X}$. Following [3, p. 140-143], a *free object* over $\mathfrak{X}$-object X is an $\mathfrak{A}$-object A such that there exists a *universal arrow* (A, u) over X; that is, $u : X \to \mathfrak{U}A$ such that for every arrow $f : X \to \mathfrak{U}B$, there exists a unique morphism $f' : A \to B$ in $\mathfrak{A}$ such that $\mathfrak{U}f'u = f$. We also say that (A, u) is the free object over X. A concrete category $(\mathfrak{A}, \mathfrak{U})$ over $\mathfrak{X}$ is said to *have free objects* provided that for each $\mathfrak{X}$-object X, there exists a universal arrow over X. For example, the category $Vect_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$ has free objects. So do the category **Top** of topological spaces and the category of **Grp** of groups. However, some interesting categories do not have free objects [3, p. 142]).

Dually, *co-universal arrows*, *cofree objects* and categories that *have cofree objects* can be defined. For the basic concepts of concrete categories, free objects, and cofree objects, we refer the reader to [37, p. 138-155].

It turns out that a concrete $(\mathfrak{A}, \mathfrak{U})$ over $\mathfrak{X}$ has (co)free objects if and only if the functor that builds up (co)free object is a (right) left adjoint to the faithful functor $\mathfrak{U} : \mathfrak{A} \to \mathfrak{X}$.

Although cofree objects are the dual of free objects, the behavior of cofree objects is more complicated than the one of free objects. Furthermore, studying such behavior cannot be obtained by studying free objects because "the categories considered are not selfdual generally" [37, p. 149]. All categories considered in this thesis are locally small. Let $(\mathcal{C}, \otimes, I)$ a monoidal category and V be an object of $\mathcal{C}$. We refer to [27], [40] for basics on monoidal categories. A monoid in the monoidal category $(\mathcal{C}, \otimes, I)$ is a triple (A, m, u), where $m : A \otimes A \to A$ and $u : I \to A$ are the multiplication and unit respectively, satisfying the associativity and unital conditions which are the categorical analogue of the usual monoid/algebra axioms; morphisms of monoids are also defined analogously. A comonoid in $\mathcal{C}$ is simply a monoid in the dual category $\mathcal{C}^0$. Let $Mon(\mathcal{C})$ and $CoMon(\mathcal{C})$ be the categories of of monoids and comonoids in $\mathcal{C}$ respectively. We note that $CoMon(\mathcal{C}) = Mon(\mathcal{C}^0)$. Both monoids and comonoids have a dominant influence on developing many branches of mathematics, such as algebra, category theory, representation theory, topology, quantum theory, algebraic geometry, etc.

Explicitly, the problem formulates as follows: given a monoidal category $\mathcal{C}$, does the forgetful functor $Mon(\mathcal{C}^0) \to \mathcal{C}^0$ have a left adjoint, or equivalently, does the forgetful $CoMon(\mathcal{C}) \to \mathcal{C}$ have a right adjoint, the "cofree" comonoid?

This is frequently an interesting question involved with the same kinds of categories in which one considers monoids and to which Mac Lane's construction of the free monoid applies.

Nevertheless, obtain cofree comonoids in $\mathcal{C}$ using this construction, one needs that $\otimes$ commutes with denumerable products in $\mathcal{C}$. However, this is almost never the case for any monoidal category of interest; this fails, for example, even in the case of vector spaces [2].

Therefore, one needs to consider a different strategy to construct cofree comonoids. Naturally, Freyd's adjoint functor theorem is an efficient method to do such kind of construction. We will regard this question for several valuable categories, namely, the category ${}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$ of bimodules over a ring A, and the categories of modules ${}_{B}\mathcal{M}$ or comodules $\mathcal{M}^B$ over a bialgebra (or Hopf algebra) B.

As a usual terminology, when $\mathcal{C}$ is abelian, monoids and comonoids are simply called algebras and coalgebras respectively. We denote the categories of algebras and coalgebras in $\mathcal{C}$ by $Alg(\mathcal{C}) = Mon(\mathcal{C})$ and $CoAlg(\mathcal{C}) = CoMon(\mathcal{C})$ respectively.

We recall that an $\mathbb{A}$-coring is simply a coalgebra in the category ${}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$. We denote by $Crg_{\mathbb{A}}$ the category of $\mathbb{A}$-corings.

Corings were introduced by Sweedler [60], and revived by the paper [16]. Their importance has significantly been emphasized since M. Takeuchi observed that any entwining phenomena (the compatibility condition between an algebra and a coalgebra) can be reshaped in terms of a coring [16, p. vii]. It turns out that corings suggest a distinctive approach to understand and characterize many notions with pertinent connection to algebra, representation theory, noncommutative ring theory, comodule theory, category theory, Hopf algebras, geometry, quantum group theory, and topology. Many important categories can be identified as comodules over suitable corings: modules, categories of actions and co-actions of bialgebras, modules over algebras in certain monoidal categories, chain complexes, (differentially) graded modules, quasi-coherent sheaves, generalized quiver representations, etc.; we refer the reader to [16] and references therein.

The existence of cofree coalgebras over vector spaces has been introduced for a long time [12], [24]; it was obtained for coalgebras over arbitrary commutative rings in [10], and some constructions for the non-coassociative case are given in [28]. The cofree coring over $\mathbb{A}^n$ is constructed in [32], and the problem is answered for von Neumann regular rings in [50] (see also [49]). The main difficulty in generalizing to arbitrary bimodules ([28], [50]) is to deal with the pathological behavior of the tensor product over an arbitrary ring $\mathbb{A}$. The question of whether a cofree coring always exists is proposed as an open problem in [4]. A closely related question turns out to be whether the category of comonoids in a monoidal category is complete, and in particular, if limits of $\mathbb{A}$-corings exist [4](see also [5, 48, 51, 52]).

A reasonably expected machinery for studying such questions is, of course, to use the general categorical framework of the dual of The Special Adjoint Functor Theorem [SAFT]; (see also [50]). We investigate cofree coalgebras and completeness for the category of coalgebras $CoAlg(\mathcal{C})$ for the case when $\mathcal{C} = {}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$, or $\mathcal{C} = {}_{B}\mathcal{M}$ or $\mathcal{C} = \mathcal{M}^{B}$, which are of particular interest among monoidal categories. We will explicitly see in Chapter 3 that if $\mathcal{C}$ is co-wellpowered and cocomplete, then $CoAlg(\mathcal{C})$ is cocomplete, co-wellpowered and the forgetful is cocontinuous. Although existence of the adjoints can significantly be considered as an interesting phenomena, we emphasize the importance of finding concrete constructions of these universal objects, which can be written out in terms of generators of the respective category of coalgebras.

The contents of the thesis can be summarized as follows.

Chapter 1 basically supplies some categorical concepts that play a crucial role in making the rest of the chapters approachable for the reader and easy to follow.

In chapter 2, we provide some important concepts involved with coring and comodule theory. Chapter 1 and chapter 2 serve as complementary parts of an introduction of our research.

Chapter 3 is concentrated on fundamental subjects: existence and construction of cofree corings. Most importantly, it stresses the significance of the technique used in showing the existence and the construction of cofree corings as a substantial procedure that could be adapted to be applicable for other monoidal categories of interest.

Chapter 4 is congenial development for the consequences obtained in chapter three. Although it inherits the same appliance used in the previous chapter, its ingredients are remarkably distinct, and some of them need to be stablished differently.

It is worth mentioning that constructing a generating set for a category of module or comodule over a bialgebra or Hopf algebra is reasonably considered as a critical substance not only to show the existence of cofree objects in such kind of categories, but also to establish a construction of such objects.

In chapter 5, an important construction inspired by the proof of the dual of The Special Adjoint Functor Theorem is introduced. We use this construction not only to construct limits for monoidal categories that subject to some certain conditions. In particular, we use this construction to show that each monoidal category of the categories we are involved in is complete. This construction, furthermore, contributes in producing a magnificently explicit description for cofree objects in such monoidal categories.

Chapter 6 highlights some future directions and very recent work involved with the main work of my thesis. Explicitly, we shed light on three important themes: cofree objects in the centralizer and the center categories, establishing generators, limits, and colimits of Coalgebras in the categories of Yetter-Drinfeld Modules over a quasi-bialgebra, and universal investigation for endofunctors categories. Ending our work with pointing out to these themes is compatible with their significance as very influential notions.

Throughout this thesis, $\mathbb{A}$ is a ring with a unit, and $\mathbb{K}$ is a field.

1.2 Preliminary Definitions and Notation

Definition 1.1. [55, p. 8] A *category* $\mathfrak{C}$ consists of three ingredients: a class $Obj(\mathfrak{C})$ of objects, a set of morphisms $Hom_{\mathfrak{C}}(A, B)$ for every ordered pair (A, B) of objects, and composition $Hom_{\mathfrak{C}}(A, B) \times Hom_{\mathfrak{C}}(B, C) \to Hom_{\mathfrak{C}}(A, C)$, denoted by $(f, g) \mapsto gf$, for every ordered triple A, B, C of objects. (We often write $f : A \to B$ or $A \xrightarrow{f} B$ instead of $f \in Hom_{\mathfrak{C}}(A, B)$.) These ingredients are subject to the following axioms:

1. the Hom sets are pairwise disjoint; that is, each $f \in Hom_{\mathfrak{C}}(A, B)$ has a unique domain A and a unique target B;

2. for each object A, there is an identity morphism $id_A \in Hom_{\mathfrak{C}}(A, A)$ such that $fid_A = f$ and $id_B f = f$ for all$f : A \to B$;

3. composition is associative: given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then $h(gf) = (hg)f$.

We denote to the class of all morphisms of $\mathfrak{C}$ by $Mor(\mathfrak{C})$.

Example 1.2.

1. There is a category $\bullet$ with one object and only the identity map [38, p. 13].

2. The category **Set** is the category whose object class is the class of all sets; $Hom_{\mathbf{Set}}(A, B)$ is the set of all functions from A to B, id_A is the identity function on A, and a composition is the usual composition of functions [3, p. 22].

3. There is a category **Grp** of groups, whose objects are groups and whose morphisms are group homomorphisms. [38, p. 11].

4. The category **Ab** is the category whose Objects are abelian groups, morphisms are homomorphisms, and composition is the usual composition [55, p. 12].

5. There is a category $Vec_{\mathbb{K}}$ of vector spaces over k and linear maps between them [38, p. 12].

6. There is a category **Top** of topological spaces and continuous maps [38, p. 12].

7. The category R-*Mod* of all left R-modules (where R is a ring) has as its objects all left R-modules, as its morphisms all R-homomorphisms, and as its composition the usual composition of functions [55, p. 15].

8. Let $(P, \leq)$ be a preordered set. P defines a category $\mathcal{P}$ with the elements of P as objects. For $a, b \in P = Ob(\mathcal{P})$, we define

$$Hom_{\mathcal{P}}(a,b) = \{ \begin{matrix} \{(a,b)\} \\ \\ \phi \end{matrix} \}$$

   The possibility of defining a (unique) composition law is just the transitivity axiom of the partial order; the existence of identities is just the reflexivity axiom. Since $Hom_{\mathcal{P}}(a, b)$ has at most one element, the composition is associative [14, p. 7].

⊞

Definition 1.3. [3, p. 48]

1. A category $\mathcal{A}$ is said to be a *subcategory* of a category $\mathcal{B}$ provided that the following conditions are satisfied:

(a) $Ob(\mathcal{A}) \subseteq Ob(\mathcal{B})$,

(b) for each $A, A' \in Ob(\mathcal{A}), Hom_{\mathcal{A}}(A, A') \subseteq Hom_{\mathcal{B}}(A, A')$,

(c) for each $\mathcal{A}$-object A, the $\mathcal{B}$-identity on A is the $\mathcal{A}$-identity on A,

(d) the composition law in $\mathcal{A}$ is the restriction of the composition law in $\mathcal{B}$ to the morphisms of $\mathcal{A}$.

2. $\mathcal{A}$ is called a *full subcategory* of $\mathcal{B}$ if, in addition to the above, for each $A, A' \in Ob(\mathcal{A}), Hom_{\mathcal{A}}(A, A') = Hom_{\mathcal{B}}(A, A')$.

Example 1.4. [3, p. 48-49]

1. The class of all Hausdorff spaces specifies the full subcategory **Haus** of **Top**; likewise, all Tychonoff spaces (i.e., completely regular T_1 spaces) yields a full subcategory **Tych** of **Haus**; and **HComp**,the category of compact Hausdorff spaces (and continuous functions), is a full subcategory of **Tych**.

2. The category **Rel** is the category whose objects all pairs (X, ρ), where X is a set and ρ is a (binary) relation on X. Morphisms $f : (X, \rho) \to (Y, \sigma)$ are relation-preserving maps; i.e., maps $f : X \to Y$ such that $x\rho x$ implies $f(x)\sigma f(x)$.

   The class of all preordered sets (i.e., all sets supplied with a reflexive and transitive relation) determines a full subcategory **Prost** of **Rel**.

   The class of all partially ordered sets (i.e., all sets supplied with a reflexive, transitive and antisymmetric relation) determines a full subcategory **Pos** of **Prost**.

⊞

Definition 1.5. [45, p. 49] Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories. A *(covariant) functor* $\mathfrak{F} : \mathfrak{A} \to \mathfrak{B}$ is an assignment of an object $\mathfrak{F}(A) \in Ob(\mathfrak{B})$ to each object $A \in Ob(\mathfrak{A})$ and a morphism $\mathfrak{F}(f) : \mathfrak{F}(A) \to \mathfrak{F}(B) \in Mor(\mathfrak{B})$ to each morphism $f : A \to B \in Mor(\mathfrak{A})$, subject to the following conditions:

1. Preservation of composition. If gf is defined in $\mathfrak{A}$,then $\mathfrak{F}(gf) = \mathfrak{F}(g)\mathfrak{F}(f)$

2. Preservation of identities. For each $A \in Ob(\mathfrak{A})$, we have $\mathfrak{F}(id_A) = id_{\mathfrak{F}(A)}$.

Example 1.6.

1. Given two categories $\mathscr{A}$, $\mathscr{B}$ and a fixed object $B \in Ob(\mathscr{B})$, we define the *constant functor* $\Omega_B : \mathscr{A} \to \mathscr{B}$ to B by $\Omega_B(A \xrightarrow{f} B) = B \xrightarrow{id_B} B$ for every morphism f of $\mathscr{A}$ [14, p. 9].

2. For any category $\mathscr{A}$, there is the identity functor $\mathcal{I}_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ defined by $\mathscr{I}_{\mathscr{A}}(A \xrightarrow{f} B) = A \xrightarrow{f} B$ [3, p. 30].

⊞

We are going to give more examples of interesting functors whence we define the notion of adjoint functors.

Definition 1.7. [38, p. 48] Let $\mathscr{A}$ be a category. An object $I \in \mathscr{A}$ is *initial* if for every $A \in \mathscr{A}$, there is exactly one map $I \to A$. An object $T \in \mathscr{A}$ is terminal if for every $A \in \mathscr{A}$, there is exactly one map $A \to T$.

Example 1.8. [38, p. 48] The empty set is initial in **Set**, the trivial group is initial in **Grp** while the one-element set is terminal in **Set**, the trivial group is terminal (as well as initial) in **Grp**. ⊞

Definition 1.9. [55, p. 23] Let $\mathfrak{S}, \mathfrak{T} : \mathfrak{A} \to \mathfrak{B}$ be (covariant) functors. A *natural transformation* $\tau : \mathfrak{S} \to \mathfrak{T}$ is a one-parameter family of morphisms in $\mathfrak{B}$,

$$\tau = (\tau_A : \mathfrak{S}A \to \mathfrak{T}A)_{A \in Ob(\mathfrak{A})}$$

making the following diagram commute for all $f : A \to A'$ in $\mathfrak{A}$:

$$\begin{array}{ccc} \mathfrak{S}A & \xrightarrow{\tau_A} & \mathfrak{T}A \\ {\scriptstyle \mathfrak{S}f}\downarrow & & \downarrow{\scriptstyle \mathfrak{T}f} \\ \mathfrak{S}A' & \xrightarrow[\tau_{A'}]{} & \mathfrak{T}A' \end{array}$$

A*natural isomorphism* is a natural transformation τ for which each τ_A is an isomorphism.

Definition 1.10. [34, p. 23] A *monoidal* (or a *tensor*)category is a tuple $(\mathcal{M},\otimes,I,a,l,r)$, where

- $\mathcal{M}$ is a category
- $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a bifunctor called (*tensor product*)
- I is an object in $\mathcal{M}$ called (*unit*) of $\mathcal{M}$
- a is a functorial isomorphism called (*associativity constraint*):

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

- l is a functorial isomorphism called (*left unit constraint*): $l_X : I \otimes X \to X$

- r is a functorial isomorphism called (*right unit constraint*): $r_X : X \otimes I \to X$

The functorial morphism a satisfies the Pentagon Axiom, that is, for every $A, B, C, D \in Ob(\mathcal{M})$, the following diagram is commutative:

$$\begin{array}{ccccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{a_{A,B,C\otimes D}} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{a_{A,B\otimes C,D}} & A \otimes ((B \otimes C) \otimes D) \\ \downarrow{\scriptstyle a_{A\otimes B,C,D}} & & & & \downarrow{\scriptstyle 1_A \otimes a_{B,C,D}} \\ (A \otimes B) \otimes (C \otimes D) & & \xrightarrow[a_{A,B,C\otimes D}]{} & & A \otimes (B \otimes (C \otimes D)) \end{array}$$

The morphisms l and r satisfy the *Triangle Axiom*, that is, for every $A, B \in Ob(\mathcal{M})$, the following diagram is commutative:

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{a_{A,I,B}} & A \otimes (I \otimes B) \\ & {\scriptstyle r_A \otimes B} \searrow \quad \swarrow {\scriptstyle 1_A \otimes l_B} & \\ & A \otimes B & \end{array}$$

Notation 1.11. The notation $(\mathcal{M},\otimes,I)$ is often used in place of $(\mathcal{M},\otimes,I,a,l,r)$; especially, when a,l,r are obvious.

Definition 1.12. [27, p. 22] A *monoidal subcategory* of a monoidal category $(\mathcal{M},\otimes,I,a,l,r)$, is a tuple $(\mathfrak{N},\otimes,I,a,l,r)$, where $\mathfrak{N}$ is a subcategory of $\mathcal{M}$ such that $\mathfrak{N}$ a subcategory closed under the tensor product of objects and morphisms and containing I and l_I.

Example 1.13.

1. [27, p. 26] The category **Sets** of sets is a monoidal category, where the tensor product is the Cartesian product and the unit object is a one element set; the structure morphisms a,l,r are obvious. The same holds for the subcategory of

finite sets. This example can be widely generalized: one can take the category of sets with some structure, such as groups, topological spaces, etc.

2. [27, p. 26] The category $Vec_{\mathbb{K}}$ of all $\mathbb{K}$-vector spaces is a monoidal category, where $\otimes = \otimes_{\mathbb{K}}$, $I = \mathbb{K}$, and the morphisms a, l, r are the obvious ones. The same is true about the category of finite dimensional vector spaces over $\mathbb{K}$. More generally, if R is a commutative unital ring, then replacing k by R we can define monoidal categories $R - Mod$ of R-modules.

3. [15, p. 299] Let $\mathcal{C}$ be a small category. Then the category $Fun(\mathcal{C})$ endofunctors (i.e. functors from the category $\mathcal{C}$ to itself) and natural transformations is monoidal when choosing the composition of two functors as their tensor product.

4. [27, p. 29] The category ${}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$ of $\mathbb{A}$-bimodules over $\mathbb{A}$ is a monoidal category, with tensor product $\otimes = \otimes_{\mathbb{A}}$. The unit object in this category is the ring $\mathbb{A}$ itself (regarded as an $\mathbb{A}$-bimodule). If $\mathbb{A}$ is commutative, this category has a full monoidal subcategory ${}_{\mathbb{A}}\mathscr{M}_{\mathbb{A}}$, consisting of $\mathbb{A}$-modules, regarded as $\mathbb{A}$-bimodules in which the left and right actions of $\mathbb{A}$ coincide.

$\boxplus$

Definition 1.14. [27, p. 40] Let $(\mathcal{C},\otimes,I,a,l,r)$ be a monoidal category.

1. An object X in $\mathcal{C}$ is said to be a *left dual* of X if there exist morphisms $ev_X : X^* \otimes X \to I$ and $coev_X : I \to X \otimes X^*$ called the *evaluation* and *coevaluation*, such that the compositions

$$X \xrightarrow{coev_X \otimes id_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{id_X \otimes ev_X} X$$

$$X^* \xrightarrow{id_{X^*} \otimes coev_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{ev_X \otimes id_{X^*}} X^*$$

are the identity morphisms.

2. An object ${}^*X$ in $\mathcal{C}$ is said to be a *right dual* of X if there exist morphisms $ev'_X : X \otimes {}^*X \to I$ and $coev'_X : I \to {}^*X \otimes X$ such that the compositions

$$X \xrightarrow{id_X \otimes coev'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{ev'_X \otimes id_X} X$$

$${}^*X \xrightarrow{coev'_X \otimes id_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{id_{{}^*X} \otimes ev'_X} {}^*X$$

are the identity morphisms.

3. An object in $\mathcal{C}$ is called *rigid* if it has left and right duals. A monoidal category $\mathcal{C}$ is called *rigid* if every object of $\mathcal{C}$ is rigid.

Example 1.15. [27, p. 42-43] The category $fdVec_{\mathbb{K}}$ of finite dimensional $\mathbb{K}$-vector spaces is rigid: the right and left dual to a finite dimensional vector space V are its dual space V^*, with the evaluation map $ev_V : V^* \otimes V \to \mathbb{K}$ being the contraction, and the coevaluation map $coev_V : \mathbb{K} \to V \otimes V^*$ being the usual embedding. On the other hand, the category $Vec_{\mathbb{K}}$ of all $\mathbb{K}$-vector spaces is not rigid, since for infinite dimensional spaces there is no coevaluation maps. ⊞

Definition 1.16. [59, p. 97] Let $(\mathcal{M},\otimes,I,a,l,r)$ be a monoidal category. A *monoid* is a triple (M, m, u), where M is an object in $\mathcal{M}$, and

$$m : M \otimes M \to M \text{ (multiplication) } u : I \to M \text{ (unit)}$$

are morphisms in $\mathcal{M}$ subject to the associativity and unity axioms:

$$\begin{array}{ccc} M\otimes M\otimes M & \xrightarrow{id_M\otimes m} & M\otimes M \\ {\scriptstyle m\otimes id_M}\downarrow & & \downarrow{\scriptstyle m} \\ M\otimes M & \xrightarrow{m} & M \end{array} \qquad \begin{array}{ccccc} I\otimes M & \xrightarrow{l_M} & M & \xleftarrow{r_M} & M\otimes I \\ & {\scriptstyle u\otimes id_M}\searrow & \uparrow{\scriptstyle m} & \swarrow{\scriptstyle id_M\otimes u} & \\ & & M\otimes M & & \end{array}$$

A *comonoid* is simply a monoid in the dual monoidal category $(\mathcal{M}^o,\otimes,I,a^{-1},l^{-1},r^{-1})$ If $\mathcal{M}$ is abelian, then a monoid is often called an *algebra* in $\mathcal{M}$, and a comonoid is often called a *coalgebra* in $\mathcal{M}$.

Example 1.17. [6, p. 12] Consider Example 1.13. A monoid in the monoidal category $(Vec_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$ is just a (usual) $\mathbb{K}$-algebra. ⊞

Notation 1.18. We denote by $Mon(\mathcal{C})$ the category of monoids in $\mathcal{C}$. Similarly, we denote $CoMon(\mathcal{C})$ the category of comonoids in $\mathcal{C}$.

Definition 1.19. [34, p. 24-25] Let $(\mathcal{M},\otimes,I,a,l,r)$, $(\mathcal{M}',\otimes',I',a',l',r')$ be two monoidal categories. A *monoidal functor* from $(\mathcal{M},\otimes,I,a,l,r)$ to $(\mathcal{M}',\otimes',I',a',l',r')$ is a triple $(\mathfrak{F},\Phi,\Psi)$, where $\mathfrak{F}:\mathcal{M}\to\mathcal{M}'$ is a functor, $\Phi: I'\to\mathfrak{F}(I)$ is an isomorphism, and $\Psi_{X,Y}:\mathfrak{F}(X)\otimes'\mathfrak{F}(Y)\to\mathfrak{F}(X\otimes Y)$ is isomorphism, natural in $X,Y,\forall X,Y\in Ob(\mathcal{M})$, subject to the following commutative diagram:

$$\begin{array}{ccccc} (\mathfrak{F}(X)\otimes'\mathfrak{F}(Y))\otimes'\mathfrak{F}(Z) & \xrightarrow{\Psi_{X,Y}\otimes' 1_Z} & \mathfrak{F}(X\otimes Y)\otimes'\mathfrak{F}(Z) & \xrightarrow{\Psi_{X\otimes Y,Z}} & \mathfrak{F}((X\otimes Y)\otimes Z) \\ {\scriptstyle a'_{\mathfrak{F}(X),\mathfrak{F}(Y),\mathfrak{F}(Z)}}\downarrow & & & & \downarrow{\scriptstyle \mathfrak{F}(a_{X,Y,Z})} \\ \mathfrak{F}(X)\otimes'(\mathfrak{F}(Y)\otimes'\mathfrak{F}(Z)) & \xrightarrow{1_X\otimes'\Psi_{Y,Z}} & \mathfrak{F}(X)\otimes'\mathfrak{F}(Y\otimes Z) & \xrightarrow{\Psi_{X,Y\otimes Z}} & \mathfrak{F}(X\otimes(Y\otimes Z)) \end{array}$$

$$\begin{array}{ccc} \mathfrak{F}(X)\otimes' I' & \xrightarrow{r'_{\mathfrak{F}(X)}} & \mathfrak{F}(X) \\ {\scriptstyle 1_{\mathfrak{F}(X)}\otimes'\Phi}\downarrow & & \uparrow{\scriptstyle \mathfrak{F}(r_X)} \\ \mathfrak{F}(X)\otimes'\mathfrak{F}(I) & \xrightarrow[\Psi_{X,I}]{} & \mathfrak{F}(X\otimes I) \end{array} \qquad \begin{array}{ccc} I'\otimes'\mathfrak{F}(X) & \xrightarrow{l'_{\mathfrak{F}(X)}} & \mathfrak{F}(X) \\ {\scriptstyle \Phi\otimes' 1_{\mathfrak{F}(X)}}\downarrow & & \uparrow{\scriptstyle \mathfrak{F}(l_X)} \\ \mathfrak{F}(I)\otimes'\mathfrak{F}(X) & \xrightarrow[\Psi_{I,X}]{} & \mathfrak{F}(I\otimes X) \end{array}$$

Example 1.20. [27, p. 32]

1. An important class of examples of monoidal functors is forgetful functors (e.g. functors of forgetting the structure, from the categories of groups, topological spaces, etc., to the category of sets). Such functors have an obvious monoidal structure.

2. Let A be a $\mathbb{K}$-algebra with unit, and $\mathcal{C} = {}_A\mathcal{M}$ be the category of left A-modules. Then we have a functor

$$\mathcal{F} : {}_A\mathcal{M}_A \to End(\mathcal{C}) \text{ given by } \mathcal{F}(M) = M \otimes_A -$$

This functor is naturally monoidal. A similar functor $\mathcal{G} : {}_{\mathbb{B}}\mathcal{M}_{\mathbb{B}} \to End(\mathcal{D})$ can be defined if $\mathbb{B}$ is a finite dimensional $\mathbb{K}$ -algebra, and $\mathcal{D}$ is the category of finite dimensional left $\mathbb{B}$-modules.

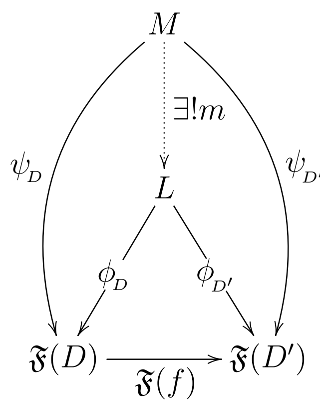
$\boxplus$

Definition 1.21. [14, p. 56] Given a functor $\mathfrak{F} : \mathfrak{D} \to \mathfrak{C}$, a *cone* on $\mathfrak{F}$ consists of

1. an object $C \in Ob(\mathfrak{C})$;

2. for every object $D \in Ob(\mathfrak{D})$, a morphism $\phi_D : C \to \mathfrak{F}(D) \in Mor(\mathfrak{C})$, in such a way that for every morphism $f : D \to D' \in Mor(\mathfrak{D})$, $\mathfrak{F}(f)\phi_D = \phi_{D'}$.

Definition 1.22. [14, p. 56] Given a functor $\mathfrak{F} : \mathfrak{D} \to \mathfrak{C}$, a *limit* of $\mathfrak{F}$, denoted by $\varprojlim \mathfrak{F}$, is a cone $(L, (\phi_D))$ on $\mathfrak{F}$ such that, for every cone $(M, (\psi_D)_{D \in Ob(\mathfrak{D})})$ on $\mathfrak{F}$, there exists a unique morphism $m : M \to L \in Mor(\mathfrak{C})$ such that for every object $D \in Mor(\mathfrak{D})$, $\phi_{_D} m = \psi_{_D}$.

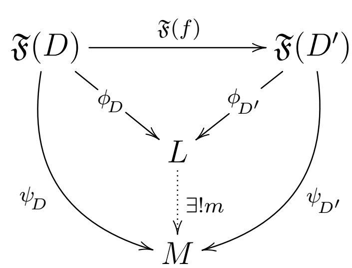


Definition 1.23. [14, p. 57] Given a functor $\mathfrak{F} : \mathfrak{D} \to \mathfrak{C}$, a *cocone* on $\mathfrak{F}$ consists of

1. an object $C \in Ob(\mathfrak{C})$;

2. for every object $D \in Ob(\mathfrak{D})$, a morphism $\phi_{_D} : \mathfrak{F}(D) \to C \in Mor(\mathfrak{C})$, in such a way that for every morphism $f : D \to D' \in Mor(\mathfrak{D})$, $\phi_{_{D'}} \mathfrak{F}(f) = \phi_{_D}$.

Definition 1.24. [14, p. 57] Given a functor $\mathfrak{F} : \mathfrak{D} \to \mathfrak{C}$, a *colimit* of $\mathfrak{F}$, denoted by $\varinjlim \mathfrak{F}$, is a cocone $(L, (\phi_{_D}))$ on $\mathfrak{F}$ such that, for every cocone $(M, (\psi_{_D})_{_{D \in Ob(\mathfrak{D})}}))$ on $\mathfrak{F}$, there exists a unique morphism $m : L \to M \in Mor(\mathfrak{C})$ such that for every object $D \in Mor(\mathfrak{D})$, $m\phi_{_D} = \psi_{_D}$.



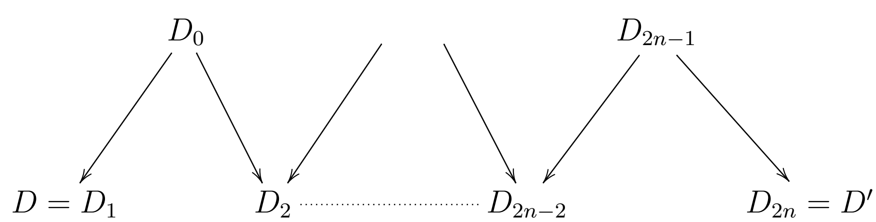
Remark 1.25. Let $\mathfrak{F} : \mathfrak{D} \to \mathfrak{C}$ be a functor. Then a limit (colimit) of $\mathfrak{F}$ (if it exists) is called **small** if $\mathfrak{D}$ is small category. ⊞

Example 1.26. [14, p. 57-58]

1. Let $\mathcal{A}$ be the category $\bullet \rightrightarrows \bullet$

   that is, a category with two objects X and Y and four morphisms $id_X, id_Y, f : X \to Y$, and $g : X \to Y$. Let $\mathcal{C}$ be a Category and $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ be a covariant functor. Then $\varprojlim \mathcal{F} = equalizer(\mathcal{F}(f), \mathcal{F}(g))$. Dually, $\varinjlim \mathcal{F} = coequalizer(\mathcal{F}(f), \mathcal{F}(g))$.

2. A category $\mathfrak{D}$ is connected when it is non-empty and, given two objects $D, D' \in \mathfrak{D}$, there exists a finite "zigzag" in $\mathfrak{D}$ as in the following diagram.

   Consider an object C of a category $\mathfrak{C}$ and the corresponding constant functor $\mathfrak{F}_C : \mathfrak{C} \to \mathfrak{D}$ on C. If $(\phi_D : \mathfrak{F}_C(D) \to M)_{D \in \mathfrak{D}}$ is a cocone on $\mathfrak{F}_C$ and D, D' are connected by the above zigzag, one immediately gets $\phi_D = \phi_{D_0} = \phi_{D_1} = \phi_{D_2} = \phi_{D_{2n-2}} = \phi_{D_{2n-1}} = \phi_{D_{2n}} = \phi_{D'}$; Thus the cocone is a constant one. Therefore $(C, (id_C)_{D \in \mathfrak{D}})$ is the colimit of $\mathfrak{F}_C$.

$\boxplus$

Definition 1.27. [14, p. 59] A category $\mathfrak{C}$ is *complete* when every functor

$$\mathfrak{F} : \mathfrak{D} \to \mathfrak{C},$$

with $\mathfrak{D}$ a small category has a limit. By duality, we get the notion of a *cocomplete* category; that is, a category $\mathfrak{C}$ is cocomplete when every functor

$$\mathfrak{F} : \mathfrak{D} \to \mathfrak{C},$$

with $\mathfrak{D}$ a small category has a colimit.

Theorem 1.28. *[14, p. 60] A category $\mathfrak{A}$ is complete precisely when each family of objects has a product and each pair of parallel arrows has an equalizer.*

Example 1.29.

1. Each of the categories **Set**, $Vec_{\mathbb{K}}$, **Top**, **Pos**, and **Grp** is complete and cocomplete [3, p. 213].

2. Every non-trivial group, considered as a category, has pullbacks and pushouts, but not equalizers or coequalizers, nor products or coproducts of pairs, nor terminal or initial objects [3, p. 213].

3. A poset is (co)complete as a category precisely when it is complete as a poset. Thus, for posets, completeness and cocompleteness coincide [14, p. 63].

4. The category of finite sets is neither complete nor cocomplete [3, p. 213].

$\boxplus$

Definition 1.30. [3, p. 122] Let $\mathfrak{M}$ be a class of all monomorphisms of a category $\mathfrak{A}$. A *subobject* of an object $B \in Ob(\mathfrak{A})$ is a pair (A, m), where $A \xrightarrow{m} B$ belongs to $\mathfrak{M}$.

Definition 1.31. [3, p. 122] Let (A, m) and (A', m') be subobjects of $B \in Ob(\mathfrak{A})$. Then (A, m) and (A', m') are called *isomorphic* provided that there exists an isomorphism $h : A \to A'$ with $m = m'h$.

Definition 1.32. [3, p. 123] Let $\mathfrak{M}$ be a class of all monomorphisms of a category $\mathfrak{A}$. Then $\mathfrak{A}$ is called *wellpowered* provided that no $\mathfrak{A}$-object has a proper class of pairwise non-isomorphic subobjects. In other words, for every object the subobjects form a set [57, p. 92].

Definition 1.33. [3, p. 124] Let $\mathfrak{E}$ be a class of all epimorphisms of a category $\mathfrak{A}$. A *quotient* of an object A is a pair (e, B), where $A \xrightarrow{e} B$ belongs to $\mathfrak{E}$.

Definition 1.34. [3, p. 124] Let $\mathfrak{A}$ be a category, and let (e, B) and (e', B') be quotients of $A \in Ob(\mathfrak{A})$. Then (e, B) and (e', B') are called *isomorphic* provided that there exists an isomorphism $h : B \to B'$ with $e' = he$.

Definition 1.35. [3, p. 125] Let $\mathfrak{E}$ be a class of all epimorphisms of a category $\mathfrak{A}$. Then $\mathfrak{A}$ is called *co-wellpowered* provided that no $\mathfrak{A}$-object has a proper class of pairwise non-isomorphic quotients. In other words, for every object the quotients form a set [57, p. 92,95].

Remark 1.36. Some people (like Pareigis in [48] and Mitchell in [45]) use the terminology "locally small category" and "locally cosmall category" (or "colocally small category") instead of " wellpowered category" and " co-wellpowered category" respectively. This terminology, however, seems confusing since most mathematicians use "locally small" for the category with small hom sets. ⊞

Example 1.37.

1. **Set**, **Top**, **Ab**, $R{-}Mod$ are (co)wellpowered, and so are the categories of groups **Grp** and of rings **Rng** [57, p. 95].

2. The construct of Urysohn spaces is not co-wellpowered. The proof is nontrivial [3, p. 126].

3. Let $\mathfrak{A}$ be the full subcategory of **Top** that consists of those topological spaces in which every compact subspace is Hausdorff. Then $\mathfrak{A}$ not co-wellpowered [3, p. 126].

$\boxplus$

Definition 1.38. [40, p. 104] A set $\mathcal{G}$ of objects of the category $\mathscr{C}$ is said to *generate* $\mathscr{C}$ when to any parallel pair $f, g : X \to Y$ of arrows of $\mathscr{C}$, $f \neq g$ implies that there is an $G \in \mathcal{G}$ and an arrow $h : G \to X$ with $fh \neq gh$ (the term "generates" is well established but poorly chosen; "*separates*" would have been better).

Definition 1.39. [40, p. 104] A set $\mathcal{Q}$ of objects of the category $\mathscr{C}$ is *cogenerating set* $\mathscr{C}$ when to any parallel pair $f, g : X \to Y$ of arrows of $\mathscr{C}$, $f \neq g$ implies that there is an $Q \in \mathcal{Q}$ and an arrow $k : Y \to Q$ with $kf \neq kg$.

Remark 1.40. Some authors, [3, p. 104] for example, prefer to use "separating set" (resp. coseparating set) in stead of "generating set" (resp. "cogenerating set"). $\boxplus$

Example 1.41.

1. Any one-point set generates Set, $\mathbb{Z}$ generates **Ab** and **Grp**, and R generates $R - Mod$. The set of finite cyclic groups is a generator for the category of all finite abelian groups [40, p. 127].

2. Any two-point set is a cogenerator in **Set**[40, p. 127].

3. In the category **Ab**, $\mathbb{Q}/\ \mathbb{Z}$ is an injective cogenerator [40, p. 131].

4. In the category **Top** of topological spaces and continuous mappings, the singleton is a generator [14, p. 161].

5. The cogenerators in **Top** are precisely the non$-T0-$spaces [3, p. 105].

6. In $Vec_{\mathbb{K}}$, the cogenerators are precisely the nonzero vector spaces [3, p. 105].

7. None of the categories **Rng**, **Grp**, or **Haus** has a cogenerating set [3, p. 105].

⊞

Definition 1.42. [38, p. 58-59] Given categories and functors

$$\begin{array}{ccc} & & \mathscr{B} \\ & & \downarrow{\scriptstyle \mathcal{P}} \\ \mathscr{A} & \xrightarrow[\mathcal{Q}]{} & \mathscr{C} \end{array}$$

the comma category $\mathcal{P} \downarrow \mathcal{Q}$ is the category defined as follows:

- objects are triples (A, h, B) with $A \in Ob(\mathscr{A})$, $B \in Ob(\mathscr{B})$, and $h : \mathcal{P}A \to \mathcal{Q}B$ in $\mathscr{C}$;

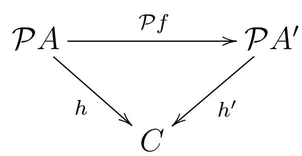
- maps $(A, h, B) \to (A', h', B')$ are pairs $(f : A \to A', g : B \to B')$ of maps such that the square

$$\begin{array}{ccc} \mathcal{P}A & \xrightarrow{\mathcal{P}f} & \mathcal{P}A' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ \mathcal{Q}B & \xrightarrow[\mathcal{Q}g]{} & \mathcal{Q}B' \end{array}$$

  commutes.

Remark 1.43. An important special case is where $\mathcal{Q} = \Omega_C$, the constant functor to C, for some $C \in Ob(\mathscr{C})$. The comma category $\mathcal{P} \downarrow \Omega_C$ is reduced to the category defined as follows:

- objects are pairs (A, h) with $A \in Ob(\mathscr{A})$, and $h : \mathcal{P}A \to C$ in $\mathscr{C}$;
- maps $(A, h) \to (A', h')$ are morphisms $f : A \to A'$ in $\mathscr{A}$ such that the following diagram is commutative:



⊞

Notation 1.44. We would rather use the notation $\mathcal{P} \downarrow C$ than $\mathcal{P} \downarrow \Omega_C$ since it is very commonly used.

Definition 1.45. [8, p. 110-111] A functor $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ is said to *create limits of type* $\mathcal{J}$ if for every diagram (functor) $\mathcal{D} : \mathcal{J} \to \mathfrak{A}$ and limit $(L, p_j : L \to \mathcal{F}B_j)_{j \in \mathcal{J}}$ in $\mathfrak{B}$ there is a unique cone $(\bar{L}, \bar{p_j} : \bar{L} \to B_j)_{j \in \mathcal{J}}$ in $\mathfrak{B}$ with $\mathcal{F}\bar{L} = L$ and $\mathcal{F}\bar{p_j} = p_j$, which, furthermore, is a limit for $\mathcal{D}$.

The notion of *creating colimits* is defined analogously.

Definition 1.46. [29, p. 142] A functor is *continuous* if it preserves all small limits. When its source is complete, a functor is continuous iff it preserves equalizers and arbitrary products. Dually, a functor is *cocontinuous* if it preserves all small colimits. If its source is cocomplete then it is cocontinuous iff it preserves coequalizers and arbitrary coproducts.

There are many important examples of continuous and cocontinuous functors. However, we restrict our examples to the following examples since forgetful functors play a significant role in our research. For further examples, we refer the reader to [3], [40], [45], [48],[29], or [57].

Example 1.47.

1. The forgetful functor $\mathcal{U} : \textbf{Top} \rightarrow \textbf{Set}$ mapping a topological space to its underlying set is continuous and cocontinuous [14, p. 65].

2. The forgetful functor $\mathcal{U} : \textbf{Ab} \rightarrow \textbf{Set}$ mapping an abelian group to its underlying set is continuous [14, p. 65].

⊞

Definition 1.48. [55, p. 257] Let $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \rightarrow \mathcal{C}$ be (covariant) functors. The ordered pair $(\mathcal{F}, \mathcal{G})$ is an *adjoint* pair if, for each $C \in Ob(\mathcal{C})$ and for each $D \in Ob(\mathcal{D})$, there are bijections

$$\tau_{C,D} : Hom_{\mathcal{D}}(\mathcal{F}C, D) \rightarrow Hom_{\mathcal{C}}(C, \mathcal{G}D)$$

that are natural transformations in C and in D.

In more detail, naturality says that the following two diagrams commute for all $f : C' \rightarrow C$ in $\mathcal{C}$ and $g : D \rightarrow D'$ in $\mathcal{D}$:

$$\begin{array}{ccc} Hom_{\mathcal{D}}(\mathcal{F}C, D) & \xrightarrow{(\mathcal{F}f)^*} & Hom_{\mathcal{D}}(\mathcal{F}C', D) \\ \downarrow{\scriptstyle \tau_{C,D}} & & \downarrow{\scriptstyle \tau_{C',D}} \\ Hom_{\mathcal{C}}(C, \mathcal{G}D) & \xrightarrow[f^*]{} & Hom_{\mathcal{C}}(C', \mathcal{G}D) \end{array}$$

$$\begin{array}{ccc} Hom_{\mathcal{D}}(\mathcal{F}C, D) & \xrightarrow{g_*} & Hom_{\mathcal{D}}(\mathcal{F}C, D') \\ \tau_{C,D} \downarrow & & \downarrow \tau_{C,D'} \\ Hom_{\mathcal{C}}(C, \mathcal{G}D) & \xrightarrow[(\mathcal{G}g)_*]{} & Hom_{\mathcal{C}}(C, \mathcal{G}D') \end{array}$$

If $(\mathcal{F}, \mathcal{G})$ is an adjoint pair, then we say that $\mathcal{F}$ is a *left adjoint* of $\mathcal{G}$ and $\mathcal{G}$ is a *right adjoint* of $\mathcal{F}$. This setup is often denoted by $\mathcal{F} \dashv \mathcal{G}$.

Remark 1.49. [39, p. 25-26]

1. Consider $\mathcal{F} \dashv \mathcal{G}$ as above. The pair of adjoint functors

$$\mathcal{C} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{D}$$

together with a natural isomorphism τ is called *adunction*.

2. Applying τ to $id_{\mathcal{F}(C)} \in Hom_{\mathcal{D}}(\mathcal{F}C, \mathcal{F}C)$ yields a map $\eta_C : C \to \mathcal{G}\mathcal{F}C$. The resulting natural transformation $\eta : id_{\mathcal{C}} \to \mathcal{G}\mathcal{F}$ is called the *unit* of the adjunction. Dually, there is a *counit* $\varepsilon : \mathcal{F}\mathcal{G} \to id_{\mathcal{D}}$. In fact, an adjunction can equivalently be defined as a quadruple $(\mathcal{F}, \mathcal{G}, \eta, \varepsilon)$ where

$$\mathcal{C} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{D}$$

$$\eta : id_{\mathcal{C}} \to \mathcal{G}\mathcal{F},$$

$$\varepsilon : \mathcal{F}\mathcal{G} \to id_{\mathcal{D}}$$

and η and ε satisfy

$$\varepsilon_{\mathcal{F}C} \mathcal{F}\eta_{C} = id_{\mathcal{F}C}, \forall C \in \mathcal{C},$$

$$\mathcal{G}\varepsilon_{D} \eta_{\mathcal{G}D} = id_{\mathcal{G}D}, \forall D \in \mathcal{D}$$

⊞

Example 1.50.

1. [48, p. 56] The following is one of the best known examples which, in fact, led to the development of the theory of adjoint functors. Let R and S be unitary, associative rings. Let A be an (R,S)-bimodule. The set $Hom_R(A,C)$ with a left R-module C is a left S-module by $(sf)(a) = f(as)$. $Hom_R(A,-) : R-Mod \to S-Mod$ is even a functor. To this functor there is a left adjoint functor $A \otimes_S - : S-Mod \to R-Mod$ called the tensor product. Thus there is an isomorphism

$$Hom_R(A \otimes_S B, C) \cong Hom_S(B, Hom_R(A,C))$$

which is natural in B and C. Actually this isomorphism is also natural in A.

2. Let $\mathcal{U}$: **Grp** $\to$ **Set** be the forgetful functor, which assigns to each group F its underlying set, and let $\mathcal{F}$: **Set** $\to$ **Grp** be the functor assigning to each set X the free group generated by the elements of X. Then $\mathcal{F} \dashv \mathcal{U}$ [40, p. 87].

3. Let $\mathcal{U} : \mathbf{Alg}_{\mathbb{K}} \to Vect_{\mathbb{K}}$ be the forgetful functor form the category of $\mathbb{K}$-algebras to the category of $\mathbb{K}$-vector spaces. Then $\mathcal{U}$ has a left adjoint $T : Vect_{\mathbb{K}} \to \mathbf{Alg}_{\mathbb{K}}$ that associates to any $\mathbb{K}$-vector space V the tensor algebra $T(V)$, the free algebra on the vector space V over $\mathbb{K}$, where

$$T(V) = \mathbb{K} \bigoplus V \bigoplus (V \otimes V) \bigoplus (V \otimes V \otimes V) \bigoplus ......$$

Then $T \dashv \mathcal{U}$ [40, p. 87].

4. More generally, if $\mathfrak{A} = {}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$, $\mathcal{U} : \mathbf{Alg}_{\mathfrak{A}} \to \mathfrak{A}$ is the forgetful functor, where $\mathbf{Alg}_{\mathfrak{A}}$ is the category of algebras in the category $\mathfrak{A}$, then $\mathcal{U}$ has a left adjoint $T : \mathfrak{A} \to \mathbf{Alg}_{\mathfrak{A}}$ that associates to any $(\mathbb{A}, \mathbb{A})$-bimodule V the tensor algebra $T(V)$, the free algebra on V over $\mathbb{A}$, where

$$T(V) = \mathbb{A} \bigoplus V \bigoplus (V \otimes V) \bigoplus (V \otimes V \otimes V) \bigoplus ......$$

Then $T \dashv \mathcal{U}$.

$\boxplus$

For further examples, we refer to [40], [55],[14], [3], or [38].

Theorem 1.51. *[38, p. 61] Take categories and functors $\mathcal{F} \dashv \mathcal{G}$ as above. The pair of adjoint functors*

$$\mathscr{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathscr{B}$$

There is a one-to-one correspondence between:

1. *adjunctions between $\mathcal{F}$ and $\mathcal{G}$ (with $\mathcal{F}$ on the left and $\mathcal{G}$ on the right);*

2. *natural transformations $\eta : id_{\mathscr{A}} \to \mathcal{G}\mathcal{F}$ such that $\eta_A : A \to \mathcal{G}\mathcal{F}A$ is initial in $A \downarrow \mathcal{G}$ for every $A \in \mathscr{A}$.*

The following Corollary characterizes the concept of adjunction using the notion of comma categories.

Corollary 1.52. [38, p. 63] Let $\mathcal{G} : \mathscr{B} \to \mathscr{A}$ be a functor. Then $\mathcal{G}$ has a left adjoint if and only if for each $A \in \mathscr{A}$, the category $A \downarrow \mathcal{G}$ has an initial object.

Definition 1.53. [53, p. 1445-1446] Let $\mathfrak{A}$ and $\mathfrak{B}$ be arbitrary categories. A functor $\mathcal{F} : \mathfrak{A} \to \mathfrak{B}$ is said to be *separable* if for all objects $A, A' \in \mathfrak{A}$ there is a map

$$\phi : Hom_{\mathfrak{B}}(\mathcal{F}A, \mathcal{F}A') \to Hom_{\mathfrak{A}}(A, A')$$

satisfying the following axioms:

1. For all $f \in Hom_{\mathfrak{A}}(A, A')$, $\phi(\mathcal{F}f) = f$

2. For every commutative diagram in $\mathfrak{B}$ of type:

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{h} & \mathcal{F}Y \\ {\scriptstyle \mathcal{F}f}\downarrow & & \downarrow{\scriptstyle \mathcal{F}g} \\ \mathcal{F}Z & \xrightarrow[k]{} & \mathcal{F}W \end{array}$$

   the following diagram in $\mathfrak{A}$ is also commutative:

$$\begin{array}{ccc} X & \xrightarrow{\phi(h)} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[\phi(k)]{} & W \end{array}$$

Proposition 1.54. *(Etingof [27, p. 42]) Let $\mathcal{C}$ be a monoidal category and let U, V, W be objects in $\mathcal{C}$. If V has a left dual V^* then there are natural adjunction isomorphisms*

$$Hom_{\mathcal{C}}(U \otimes V, W) \xrightarrow{\sim} Hom_{\mathcal{C}}(U, W \otimes V^*) \tag{1.1}$$

$$Hom_{\mathcal{C}}(V^* \otimes U, W) \xrightarrow{\sim} Hom_{\mathcal{C}}(U, V \otimes W) \tag{1.2}$$

The following theorem is well-known as **The Dual of The Special Adjoint Functor Theorem** [SAFT].

Theorem 1.55 (SAFT)**.** *[29, p. 148] If* $\mathfrak{A}$ *is cocomplete, co-wellpowered and with a generating set, then every cocontinuous functor from* $\mathfrak{A}$ *to a locally small category has a right adjoint.*

CHAPTER 2
COALGEBRAS, COMODULES, HOPF ALGEBRAS, AND CORINGS AND SOME OF THEIR VALUED ASSOCIATIVE CATEGORIES

This chapter provides important preliminary ingredients for the next chapters. Although corings and comodules have been introduced recently, they play a central role in not only developing algebra, but also in supporting topology, representation theory, homology, and algebraic geometry theory.

Furthermore, they introduce an influential device by which many categories of interest can be characterized in more flexible and smooth manner. Their behavior submits a systematic identification for categories with complex structures by using adjunctions to make bridges with categories that can be easy to handle.

Moreover, the notions of corings and comodules are highly considered as a fundamental stone for very recent fields of studies, such as Hopf Galois comodule theory and Hopf Galois extension theory.

To characterize the concept of corings, we need to consider purity conditions on both sides due to the way in which the category ${}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$ behave. In particular, Cohns criterion plays a significant role not only in characterizing corings, but also in constituting a generating set for the category $Crg_{\mathbb{A}}$ of $\mathbb{A}$-corings.

On the other hand, many valued categories can be viewed as categories of modules or comodules over Hopf or bialgebra, and hence the definitions of algebras, coalgebras, bialgebras, and Hopf algebras are needed.

2.1 Algebras, Coalgebras, Comodules, Hopf algebras, and Corings

Definition 2.1. [24, p. 1] A $\mathbb{K}$-*algebra* is a triple (A, m, u), where A is an $\mathbb{K}$-vector space, $m : A \otimes_R A \to A$, and $u : R \to A$ are $\mathbb{K}$-linear maps such that the following diagrams are commutative:

$$\begin{array}{ccc} A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A & \xrightarrow{id_A \otimes m} & A \otimes_{\mathbb{K}} A \\ {\scriptstyle m \otimes id_A} \downarrow & & \downarrow {\scriptstyle m} \\ A \otimes_{\mathbb{K}} A & \xrightarrow[m]{} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{id_A \otimes u} & A \otimes_{\mathbb{K}} A \\ {\scriptstyle u \otimes id_A} \downarrow & \searrow^{id_A} & \downarrow {\scriptstyle m} \\ A \otimes_{\mathbb{K}} A & \xrightarrow[m]{} & A \end{array}$$

Example 2.2. [59, p. 28] Let $\mathbb{A}$ be a ring with 1 and M an $\mathbb{A}$-bimodule. Write

$$M^{\otimes n} = M \otimes_{\mathbb{A}} M \otimes_{\mathbb{A}} ... \otimes_{\mathbb{A}} M \ (n \text{ terms}) .$$

The tensor algebra on M is defined by

$$T(M) = \oplus_{n=0}^{\infty} M^{\otimes n}$$

with multiplication $\mu : T(M) \otimes_{\mathbb{A}} T(M) \to T(M)$ induced by the canonical isomorphisms

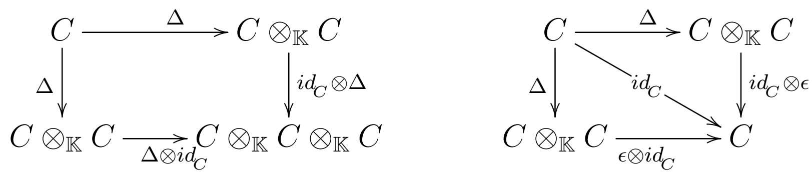
$$M^{\otimes p} \otimes_{\mathbb{A}} M^{\otimes q} \xrightarrow{\cong} M^{\otimes (p+q)}$$

and unit $\eta : \mathbb{A} \to T(M)$ equal to the injection

$$\iota_0 : \mathbb{A} = M^{\otimes 0} \to \oplus_{n=0}^{\infty} M^{\otimes n} = T(M).$$

$\boxplus$

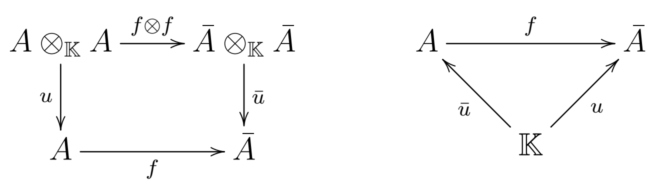
Definition 2.3. [24, p. 2-3] A $\mathbb{K}$-*coalgebra* is a triple (C, Δ, ϵ), where C is an $\mathbb{K}$-vector space, $\Delta : C \to C \otimes_{\mathbb{K}} C$, and $\epsilon : C \to \mathbb{K}$ are $\mathbb{K}$-linear maps, called (coassociative) coproduct and counit respectively, such that the following diagrams are commutative:



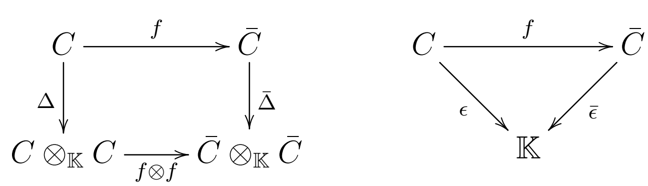
Remark 2.4. [59, p. xv] The dual vector space of a coalgebra is an algebra, however, the usual dual of an algebra need not naturally be a coalgebra. ⊞

Definition 2.5. [24, p. 23] Let (C, Δ, ϵ) be a $\mathbb{K}$-coalgebra. A $\mathbb{K}$-subspace D of C is called a *subcoalgebra* if $\Delta(D) \subseteq D \otimes D$.

Definition 2.6. [24, p. 9] Let (A, m, u) and $(\bar{A}, \bar{m}, \bar{u})$ be $\mathbb{K}$-algebras. A $\mathbb{K}$-linear map $f : C \to \bar{C}$ is said to be a $\mathbb{K}$-*algebra morphism* if the following diagrams are commutative



Definition 2.7. [24, p. 9] Let (C, Δ, ϵ) and $(\bar{C}, \bar{\Delta}, \bar{\epsilon})$ be $\mathbb{K}$-coalgebras. A $\mathbb{K}$-linear map $f : C \to \bar{C}$ is said to be a $\mathbb{K}$-*coalgebra morphism* if the following diagrams are commutative



Notation 2.8. Sweedlers Σ-notation [16, p. 2]

For an elementwise description of the maps, we use the Σ-notation, writing for $c \in C$

$\Delta(c) = \sum_{i=1}^{k} c_i \otimes c'_i = \sum c_1 \otimes c_2$

The first version is more precise; the second version, introduced by Sweedler, turnes out to be very handy in explicit calculations. With this notation, the coassociativity of Δ is expressed by

$\sum \Delta(c_1) \otimes c_2 = \sum c_{11} \otimes c_{12} \otimes c_2 = \sum c_1 \otimes c_{21} \otimes c_{22} = \sum c_1 \otimes \Delta(c_2)$

The conditions for the counit are described by

$\sum \epsilon(c_1)c_2 = c = \sum c_1 \epsilon(c_2)$

Moreover, lots of people use the notation above without writing the symbol $\sum$.

Example 2.9.

1. **Group-like Coalgebra on a Set** [60, p. 6]

   Let $\mathbb{K}$ be a field, and let S be a nonempty set; $\mathbb{K}S$ is the $\mathbb{K}$-vector space with basis S. Then $\mathbb{K}S$ is a coalgebra with comultiplication Δ and counit ϵ defined by $\Delta(s) = s \otimes s, \epsilon(s) = 1$ for any $s \in S$.

   This shows that any vector space can be endowed with a $\mathbb{K}$-coalgebra structure [24, p. 3].

2. **Matrix Coalgebra** [24, p. 3-4]

   Let $n \geq 1$ be a positive integer, and $M^c(n, \mathbb{K})$ a $\mathbb{K}$ -vector space of dimension n^2. We denote by $(e_{ij})_{1 \leq i,j \leq n}$a basis of $M^c(n, \mathbb{K})$. We denote on $M^c(n, \mathbb{K})$ a comultiplication Δ by

$$\Delta(e_{ij}) = \sum_{1 \leq p \leq n} e_{ip} \otimes e_{pj},$$

   for any i, j, and a counit by

$$\epsilon(e_{ij}) = \delta_{ij}$$

In this way, $M^c(n, \mathbb{K})$ becomes a coalgebra, which is called the matrix coalgebra.

$\boxplus$

Definition 2.10. [24, p. 50] Let V be a $\mathbb{K}$-vector space. A cofree coalgebra over V is a pair $(C(V)$,$p)$, where $C(V)$ is $\mathbb{K}$-coalgebra, and $p : C(V) \to V$ is a $\mathbb{K}$-linear map such that for any $\mathbb{K}$-coalgebra C, and any $\mathbb{K}$-linear map $f : C \to V$, there exists a unique morphism $f : C \to C(V)$ of coalgebras with $pf' = f$.

Definition 2.11. [24, p. 33] Let (A, m, u) be $\mathbb{K}$-algebra. Then the following set

$$A^0 = \{f \in A^* \ |ker(f) \text{ contains an ideal of finite codimension}\}$$

is a $\mathbb{K}$-subspace in A^*, and is called the *finite dual* of A.

Remark 2.12.

1. The definition of cofree coalgebras simply means that the functor $Vec_{\mathbb{K}} \to CoAlg_{\mathbb{K}}$, $V \mapsto C(V)$ is a right adjoint to the forgetful functor $CoAlg_{\mathbb{K}} \to Vec_{\mathbb{K}}$.

2. The tensor $\mathbb{K}$-coalgebra over V is not the cofree $\mathbb{K}$-coalgebra over V. For convenience, the reader could see the discussion and the example in [31, p. 110].

3. For an algebra A, $A^0 = \{f \in A^* | \exists\, n,\, g_i, h_i \in A^*,\, i = 1, \ldots, n \text{ such that } f(ab) = \sum_{i=1}^{n} g_i(a)h_i(b),\ \forall\, a, b \in A\}$ (See (Dascalescu [24])).

$\boxplus$

Definition 2.13. [54, p. 166] A bialgebra over the field $\mathbb{K}$, or a $\mathbb{K}$-bialgebra, is a tuple $(A, m, u, \Delta, \epsilon)$, where (A, m, u) is an $\mathbb{K}$-algebra and (A, Δ, ϵ) is a $\mathbb{K}$-coalgebra such that Δ and ϵ are $\mathbb{K} - algebra$ morphisms or, equivalently, m and u are $\mathbb{K}$-coalgebra morphisms.

Definition 2.14. [30, p. 74] A $\mathbb{K}$-bialgebra B is called an $\mathbb{K}$-*Hopf algebra* if it has an *antipode map*; that is there is a $\mathbb{K}$-linear map $S : B \to B$, such that
$m(S \otimes id_B)\Delta = u\epsilon = m(id_B \otimes S)\Delta$.

Example 2.15. [54, p. 212] Let G be a group and $H = \mathbb{K}[G]$ be the group algebra of G over $\mathbb{K}$. Then H is a Hopf algebra with antipode S given by $S(g) = g^{-1}$ for all $g \in G$. ⊞

Definition 2.16. [16, p. 20] Let R be a commutative ring with 1, and let C be R-colagebra. A *right C-comodule* is a pair (M, ρ^M), where $M \in Ob(\mathcal{M}_R)$, and

$$\rho^M : M \to M \otimes_R C$$

is an R-linear map, called *C-coaction*, subject to the following commutative diagrams (called coassociative and counital properties)

$$\begin{array}{ccc} M & \xrightarrow{\rho^M} & M \otimes_R C \\ \downarrow{\scriptstyle \rho^M} & & \downarrow{\scriptstyle id_M \otimes \Delta} \\ M \otimes_R C & \xrightarrow[\rho^M \otimes id_C]{} & M \otimes_R C \otimes_R C \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho^M} & M \otimes_R C \\ & {\scriptstyle id_M} \searrow \quad \swarrow {\scriptstyle id_M \otimes \epsilon} & \\ & M & \end{array}$$

We write $\rho^M(m) = \sum m_0 \otimes m_1 \in M \otimes_R C, \forall m \in M$.

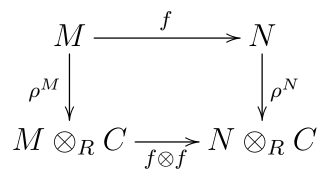
Symmetrically, a *left C-comodule* $(M, {}^M\rho)$ can be defined, where

$${}^M\rho : M \to C \otimes_R M.$$

We also write ${}^M\rho(m) = \sum m_{-1} \otimes m_0 \in C \otimes_R M, \forall m \in M$.

Example 2.17. [47, p. 12] Let $C = \mathbb{K}G$. Then M is a right C-comodule if and only if M is a G-graded module; that is $M = \oplus_{g \in G} M_g$. ⊞

Definition 2.18. [16, p. 21] Let (M, ρ^M), (N, ρ^N) be right C-comodules. An R-linear map $f : C \to \bar{C}$ is called a *C-comodule morphism* or a *morphism of right C-comodules* if and only if the diagram



is commutative.

Symmetrically, a *C-comodule morphism* can be defined.

Notation 2.19. [16, p. 22]

1. Instead of comodule morphism we also say C-morphism or (C-)colinear map.

2. The class of right comodules over C together with the colinear maps form an additive category. This category is denoted by $\mathcal{M}^C$.

   Symmetrically, the class of left comodules over C together with the colinear maps form an additive category. This category is denoted by ${}^C\mathcal{M}$.

Definition 2.20. [24, p. 74] Let C be a $\mathbb{K}$-coalgebra, and C^* the dual algebra. Let M be a left C^*-module, and $\psi_M : C^* \otimes M \to M$ the map giving the module structure of M. Define

$$\rho_M : M \to Hom(C^*, M), \rho_M(m)(c^*) = c^*.m$$

Let $j : C \to C^{**}, j(c)(c^*) = c^*(c)$ be the canonical embedding, and

$$f_M : M \otimes C^{**} \to Hom(C^*, M), f_M(m \otimes c^{**})(c) = c^{**}(c^*).m,$$

which is an injective morphism. It follows that the map

$$\mu_M : M \otimes C \to Hom(C^*, M), \mu_M = f_M(id_M \otimes j)$$

is injective. It is clear from the definition that $\mu_M(m \otimes c)(c^*) = c^*(c).m$ for $c \in C, c^* \in C^*, m \in M$.

The left C^*-module M is called *rational* if

$$\rho_M(M) \subseteq \mu_M(M \otimes C)$$

Theorem 2.21. *[24, p. 75] Let $Rat({}_{C^*}\mathcal{M})$ denote the full subcategory of ${}_{C^*}\mathcal{M}$ having as objects all rational C^*-modules. Then the categories $\mathcal{M}^C$ and $Rat({}_{C^*}\mathcal{M})$ are isomorphic.*

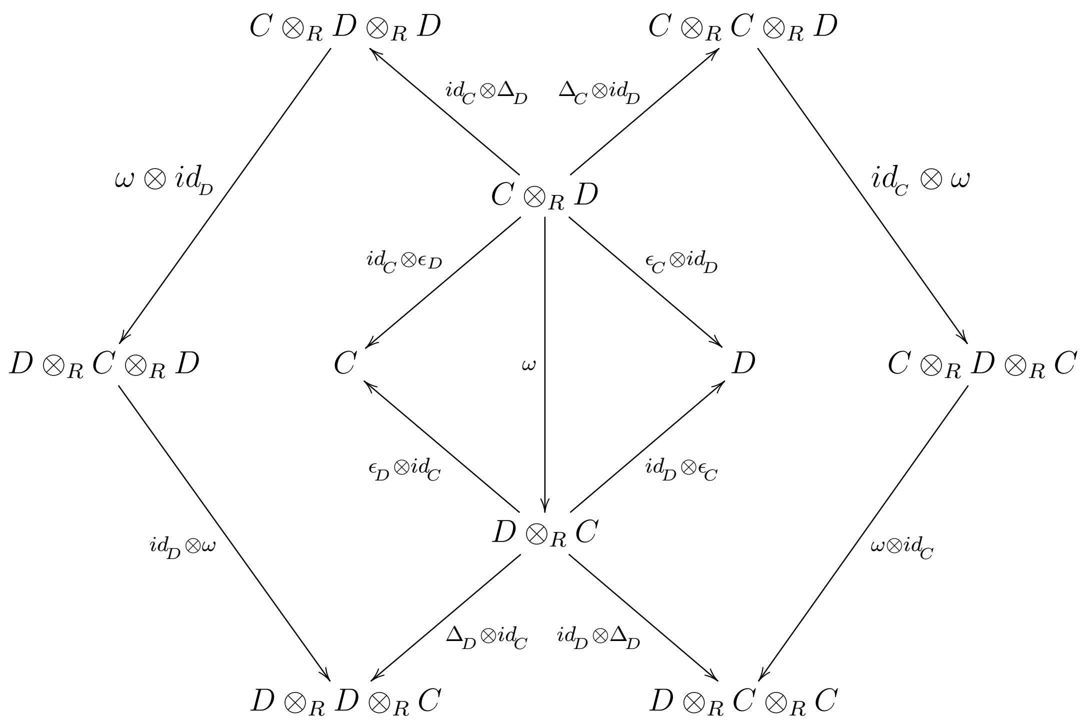
Definition 2.22. [16, p. 14] For R-coalgebras $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$, let $\omega : C \otimes_R D \to D \otimes_R C$ be an R-linear map.

Explicitly on elements we write $\omega(c \otimes d) = \sum d^\omega \otimes c^\omega$. Denote by $C \bowtie_\omega D$ the R-module $(C \otimes_R D)$ endowed with the maps

$$\bar{\Delta} = (id_C \otimes \omega \otimes id_C)(\Delta_C \otimes \Delta_D) : C \otimes_R D \to (C \otimes_R D) \otimes_R (C \otimes_R D),$$

$$\bar{\epsilon} = (\epsilon_C \otimes \epsilon_D) : C \otimes_R D \to R$$

Then $C \bowtie_\omega D$ is an R-coalgebra if and only if the following bow-tie diagram is commutative



If this holds, the coalgebra $C \rtimes_\omega D$ is called a *smash (crossed) coproduct* of C and D.

Remark 2.23. The definition above is a general definition for the crossed coproduct of R-coalgebras $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$. Molnar in [46] has showed that the crossed coproduct of any $\mathbb{K}$-coalgebra C and a $\mathbb{K}$-Hopf algebra H always exists. Furthermore, it is given by the following definition.

$\boxplus$

Definition 2.24. [21, p. 4330-4331] Let C be a $\mathbb{K}$-coalgebra and H a $\mathbb{K}$-Hopf algebra. We say that H *coacts weakly* on C if there is a $\mathbb{K}$-linear map

$$\rho : C \to H \otimes_{\mathbb{K}} C;\ \rho(c) = c_{-1} \otimes c_0$$

satisfying the following conditions, for all $c \in C$:

$$c_{-1} \otimes c_{0,1} \otimes c_{0,2} = c_{1,-1}c_{2,-1} \otimes c_{1,0} \otimes c_{2,0}$$

$$\epsilon_C(c_0)c_{-1} = \epsilon_C(c)1_H$$

$$\epsilon_H(c_{-1})c_0 = c$$

Assume that H coacts weakly on C, and let

$$\alpha : C \to H \otimes_{\mathbb{K}} H;\ \alpha(c) = \alpha_1(c) \otimes \alpha_2(c)$$

be a $\mathbb{K}$-linear map. Define $C \rtimes_\alpha H$ to be the coalgebra whose underlying vector space is $C \otimes_{\mathbb{K}} H$, with comultiplication and counit given by

$$\Delta_\alpha(c \rtimes h) = (c_1 \rtimes c_{2,-1}\alpha_1(c_3)h_1) \otimes (c_{2,0} \rtimes \alpha_2(c_3)h_2)$$

$$\epsilon_\alpha(c \rtimes h) = \epsilon_C(c)\epsilon_H(h)$$

$C \rtimes_\alpha H$ is called the *crossed coproduct coalgebra* of C and H.

Definition 2.25. [23, p. 380] A submodule N of a left $\mathbb{A}$-module M is called a *pure* submodule provided that for any right $\mathbb{A}$-module P, the induced map $P \otimes_{\mathbb{A}} N \to P \otimes_{\mathbb{A}} M$ is mono. Pure right $\mathbb{A}$-modules are defined analogously.

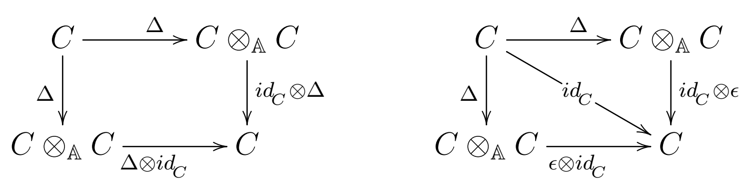
The following theorem is well-known as *Cohn's Criterion for pure left* $\mathbb{A}$-*modules*, which we recall here for convenience [23].

Theorem 2.26. ***Cohn's Criterion for Pure Modules*** *[23, p. 384] Let M be a left $\mathbb{A}$-module, and $\bar{M} \subseteq M$ be a submodule. Then $\bar{M}$ is a pure submodule of M if and only if every finite system of linear equations*

$$\sum_{i=1}^{k_j} \lambda_{ij} x_i = \bar{m}_j, \quad j = 1, 2, \ldots, n \tag{2.1}$$

with $\bar{m}_j \in \bar{M}$, $\lambda_{ij} \in \mathbb{A}$ has a solution in $\bar{M}$ whenever it has one in M.

Definition 2.27. [16, p. 170] An $\mathbb{A}$-*coring* is an $\mathbb{A}$-bimodule C with $\mathbb{A}$-bilinear maps: $\Delta : C \to C \otimes_{\mathbb{A}} C$ and $\epsilon : C \to \mathbb{A}$, called (coassociative) coproduct and counit, respectively, such that the following diagrams commute



Remark 2.28.

1. [11, p. 88] For $\mathbb{A}$ a not necessarily commutative ring, an $\mathbb{A}$-coring is C is defined to be a coalgebra in the monoidal category of $\mathbb{A}$-bimodules, $({}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}, \otimes_{\mathbb{A}}, \mathbb{A})$.

2. [13, p. 1720] An $\mathbb{A}$-coring can be defined as an $\mathbb{A}$-bimodule C such that the tensor endofunctor $- \otimes_{\mathbb{A}} C : \mathcal{M}_{\mathbb{A}} \to \mathcal{M}_{\mathbb{A}}$ is a comonad or a cotriple. On the other hand, the tensor functor $- \otimes_{\mathbb{A}} C$ has a right adjoint, the Hom-functor $Hom_{\mathbb{A}}(C, -)$. By purely categorical arguments (see Eilenberg and Moore [26, Proposition 3.1]), the functor $- \otimes_{\mathbb{A}} C$ is a comonad if and only if its right adjoint $Hom_{\mathbb{A}}(C, -)$ is a monad. Thus, C is an $\mathbb{A}$-coring if and only if $Hom_{\mathbb{A}}(C, -)$ is a monad on $\mathcal{M}_{\mathbb{A}}$.

⊞

Definition 2.29. [16, p. 177] Let (C, Δ, ϵ) and $(\bar{C}, \bar{\Delta}, \bar{\epsilon})$ be $\mathbb{A}$-corings. An $\mathbb{A}$-bilinear map $f : C \to \bar{C}$ is said to be an $\mathbb{A}$-coring morphism provided the following diagrams commute

$$\begin{array}{ccc} C & \xrightarrow{f} & \bar{C} \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \bar{\Delta}} \\ C \otimes_{\mathbb{A}} C & \xrightarrow[f \otimes f]{} & \bar{C} \otimes_{\mathbb{A}} \bar{C} \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{f} & \bar{C} \\ & {\scriptstyle \epsilon}\searrow \quad \swarrow{\scriptstyle \bar{\epsilon}} & \\ & \mathbb{A} & \end{array}$$

This gives rise to a category $Crg_{\mathbb{A}}$ of $\mathbb{A}$-corings.

Example 2.30.

1. [16, p. 251-252] Let R be a commutative ring with 1, and let $B \to A$ be an extension of R-algebras. Then $C = A \otimes_B A$ is an A-coring with coproduct

$$\Delta : C \to C \otimes_{\mathbb{A}} C \simeq A \otimes_B A \otimes_B A,\ a \otimes a' \mapsto a \otimes 1_A \otimes a'$$

and counit

$$\epsilon : C \to A,\ a \otimes a' \mapsto aa'$$

C is called the *canonical* or *Sweedler A-coring* associated to a ring (algebra) extension $B \to A$.

2. [16, p. 171] For any set G, $C = \mathbb{A}^{(G)}$, the free $\mathbb{A}$-bimodule generated by G, is an $\mathbb{A}$-coring by the comultiplication and counit

$$\Delta : C \to C \otimes_{\mathbb{A}} C,\ g \mapsto g \otimes g,$$

$$\epsilon : C \to A,\ g \mapsto 1_A, \text{ for all } g \in G.$$

⊞

Remark 2.31. [10, p. 601] Let M be a right $\mathbb{A}$-module, N be a left $\mathbb{A}$-module, and M' and N' be submodules of M and N respectively. We denote by $N' \cdot M'$ the image of the canonical map $M' \otimes_{\mathbb{A}} N' \to M \otimes_{\mathbb{A}} N$. ⊞

Definition 2.32.

1. Let C be an $\mathbb{A}$-coring. An $\mathbb{A}$-sub-bimodule $M \subseteq C$ is called *invariant* under Δ, or simply invariant, provided $\Delta(M) \subseteq M \cdot M$ [10, p. 602].

2. Let C be an $\mathbb{A}$-coring. An ($\mathbb{A}$-sub-bimodule $D \subseteq C$ is said to be subcoring, provided D has a coring structure $(D, \Delta_{_D}, \epsilon_{_D})$ such that the inclusion map is a coring morphism. We note that if D is an $\mathbb{A}$-sub-bimodule of C that is pure both as a left and as a right $\mathbb{A}$-module, then $D \cdot D$ can be identified with $D \otimes_{\mathbb{A}} D$, and in this case D is a subcoring of C provided that $\Delta_{_D}(D) \subseteq D \otimes_A D \subseteq C \otimes_{\mathbb{A}} C$, and $\varepsilon \mid_{_D} \colon D \to \mathbb{A}$ is a counit for D [16, p. 178].

Remark 2.33. In contrast with the definition of a subcoalgebra, the definition of a subcoring is something one needs to be careful about, due to the non-exactness of the tensor product. ⊞

The following theorem analogously follows from a result of [10] when adapted for (A, A)-bimodules.

Proposition 2.34. *Let M be an $(\mathbb{A}, \mathbb{A})$-bimodule, and let N be a $(\mathbb{A}, \mathbb{A})$-sub-bimodule of M. Then there is an $(\mathbb{A}, \mathbb{A})$-sub-bimodule N^* of M such that $N \subseteq N^* \subseteq M$, N^* is* *a pure both as a left and as a right A-submodule of M, and $|N^*| \leq max\ \{|N|, |\mathbb{A}|, \aleph_0\}$, where $|X|$ denotes the cardinality of X.*

Proof. □

The following proposition can be regarded as a version of the fundamental finiteness theorem of coalgebras; unfortunately, in the absence of flatness, the cardinality of the base-ring has to be factored in. Its proof is similar to the results of [23], and we briefly recall it.

Proposition 2.35. *[A Fundamental Theorem for Corings] Let (C, Δ, ε) be an $\mathbb{A}$-coring and M an $\mathbb{A}$-sub-bimodule of C.*

(i) Then there is an $\mathbb{A}$-sub-bimodule $M^\sim$ such that $M \subseteq M^\sim \subseteq C$, $M^\sim$ is invariant, and $|M^\sim| \leq max\ \{|M|, |\mathbb{A}|, \aleph_0\}$.

(ii) There is a subcoring D of C such that $M \subseteq D \subseteq C$, D is a pure left and right A-submodule of C, and $|D| \leq \max\{|M|, |\mathbb{A}|, \aleph_0\}$

Proof. For $m \in M$ choose representations $\Delta(m) = \sum_{(m)} m_{(1)} \otimes m_{(2)}$ let $\bar{M}$ be the A-sub-bimodule generated by M and all the $m_{(1)}$'s and $m_{(2)}$'s. Then $M \subseteq \bar{M}$, $\Delta(M) \subseteq \bar{M} \cdot \bar{M}$, and $|\bar{M}| \leq \max\ \{|M|, |\mathbb{A}|, \aleph_0\}$. Iterating this, we get a sequence $M \subseteq \bar{M} \subseteq \bar{\bar{M}} \subseteq \cdots \subseteq M^{\overline{(n)}} \subseteq \ldots$ and $M^\sim = \bigcup_n M^{\overline{(n)}}$ clearly has the claimed properties.

For (ii), combine (i) and Proposition 2.34: set $M_0 = M^*$, and $\forall n \geq 1$, define

$$M_n = \begin{cases} (M_{n-1})^* & \text{if } n \text{ is even;} \\ (M_{n-1})^\sim & \text{if } n \text{ is odd.} \end{cases}$$

Then $D = \bigcup_n M_n$ is left and right pure by Cohen's result, $\Delta(D) \subseteq D \cdot D$ and by purity it is a subcoring satisfying the other desired properties. □

2.2 Interesting Monoidal Categories for Our Investigation: ${}_B\mathcal{M}$, $\mathcal{M}_B$, $\mathcal{M}^B$ and ${}^B\mathcal{M}$

In order to investigate existence of (co)free objects in certain abelian monoidal categories, we might need to show how those categories could be viewed as monoidal categories. Explicitly, we need to see how the categories ${}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$, ${}_B\mathcal{M}$, $\mathcal{M}_B$, $\mathcal{M}^B$ and ${}^B\mathcal{M}$ can be identified as monoidal categories. We have showed how the category ${}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$ can be viewed as a monoidal category, and it remains to show how the categories ${}_B\mathcal{M}$, $\mathcal{M}_B$, $\mathcal{M}^B$ and ${}^B\mathcal{M}$ can be identified as monoidal categories.

Let B be a $\mathbb{K}$-bialgebra. Following [7, p. 7-8], we have the following.

1. **The Monoidal Category of Left B-modules: ${}_B\mathcal{M}$.**

   Explicitly we have $({}_B\mathcal{M}, \otimes_{\mathbb{K}}, \mathbb{K})$ is a monoidal category. Given a left B-module $(M, {}_M\mu)$, ${}_M\mu : B \otimes_{\mathbb{K}} M \to M$, we write ${}_M\mu(b \otimes m) = bm, \forall b \in B, m \in M$. For every $M, N \in Ob({}_B\mathcal{M})$, one has $M \otimes_{\mathbb{K}} N \in Ob({}_B\mathcal{M})$ as follows

   $$b(m \otimes n) = \sum b_1 m \otimes b_2 n, \forall b \in B, m \in M, n \in N, \textit{ (diagonal left action).}$$

   The unit $\mathbb{K}$ is a left B-module as follows

   $$bk = \epsilon(b)k, \forall b \in B, k \in \mathbb{K}, \textit{ (trivial left action).}$$

   With these structures the constraints become morphisms in ${}_B\mathcal{M}$.

   An algebra in ${}_B\mathcal{M}$ is called a *left B-module algebra*. A coalgebra in ${}_B\mathcal{M}$ is called a *left B-module coalgebra*.

   We denote the category of algebras in ${}_B\mathcal{M}$ by $Alg({}_B\mathcal{M})$ and the category of coalgebras in ${}_B\mathcal{M}$ by $CoAlg({}_B\mathcal{M})$.

2. **The Monoidal Category of Right B-modules: $\mathcal{M}_B$.**

   Explicitly, we have $(\mathcal{M}_B, \otimes_{\mathbb{K}}, \mathbb{K})$ is a monoidal category. Given a right B-module (M, μ_M), $\mu_M : M \otimes_{\mathbb{K}} B \to M$, we write $\mu_M(m \otimes b) = mb, \forall m \in M, b \in B$. For every $M, N \in Ob(\mathcal{M}_B)$, one has $M \otimes_{\mathbb{K}} N \in Ob(\mathcal{M}_B)$ as follows

   $$(m \otimes n)b = \sum mb_1 \otimes nb_2, \forall m \in M, n \in N, b \in B, \textit{ (diagonal right action).}$$

   The unit $\mathbb{K}$ is a right B-module as follows

   $$kb = k\epsilon(b), \forall k \in \mathbb{K}, b \in B, \textit{ (trivial right action).}$$

   With these structures the constraints become morphisms in $\mathcal{M}_B$.

   An algebra in $\mathcal{M}_B$ is called a *right B-module algebra*. A coalgebra in $\mathcal{M}_B$ is called a *right B-module coalgebra*.

   We denote the category of algebras in $\mathcal{M}_B$ by $Alg(\mathcal{M}_B)$ and the category of coalgebras in $\mathcal{M}_B$ by $CoAlg(\mathcal{M}_B)$.

3. **The Monoidal Category of Right B-comodules: $\mathcal{M}^B$.**

   Explicitly, we have $(\mathcal{M}^B, \otimes_{\mathbb{K}}, \mathbb{K})$ is a monoidal category. Given a right B-comodule (M, ρ^M), $\rho^M : M \to M \otimes_{\mathbb{K}} B$, we write $\rho^M(m) = \sum m_0 \otimes m_1 \in M \otimes_{\mathbb{K}} B, \forall m \in M$. For every $M, N \in Ob(\mathcal{M}^B)$, one has $M \otimes_{\mathbb{K}} N \in Ob(\mathcal{M}^B)$ as follows

   $$\rho^{M \otimes_{\mathbb{K}} N}(m \otimes n) = \sum m_0 n_0 \otimes m_1 n_1, \forall m \in M, n \in N, \textit{ (diagonal right coaction).}$$

   The unit $\mathbb{K}$ is a right B-comodule as follows

$$\rho^{\mathbb{K}}(k) = k \otimes 1_B, \forall k \in \mathbb{K}, \textit{(trivial right coaction).}$$

With these structures the constraints become morphisms in $\mathcal{M}^B$.

An algebra in $\mathcal{M}^B$ is called a *right B-comodule algebra.* A coalgebra in $\mathcal{M}^B$ is called a *right B-comodule coalgebra.*

We denote the category of algebras in $\mathcal{M}^B$ by $Alg(\mathcal{M}^B)$ and the category of coalgebras in $\mathcal{M}^B$ by $CoAlg(\mathcal{M}^B)$.

4. **The Monoidal Category of Left B-comodules: ${}^B\mathcal{M}$.**

Explicitly we have $(\mathcal{M}^B, \otimes_{\mathbb{K}}, \mathbb{K})$ is a monoidal category. Given a left B-comodule $(M, {}^M\rho)$, ${}^M\rho : M \to B \otimes_{\mathbb{K}} M$, we write ${}^M\rho(m) = \sum m_0 \otimes m_1 \in M \otimes_{\mathbb{K}} B, \forall m \in M$. For every $M, N \in Ob(\mathcal{M}^B)$, one has $M \otimes_{\mathbb{K}} N \in Ob(\mathcal{M}^B)$ as follows

$${}^{M\otimes_{\mathbb{K}}N}\rho(m \otimes n) = \sum m_{-1}n_{-1} \otimes m_0 n_0, \forall m \in M, n \in N, \textit{(diagonal left coaction).}$$

The unit $\mathbb{K}$ is a left B-comodule as follows

$${}^{\mathbb{K}}\rho(k) = 1_B \otimes k, \forall k \in \mathbb{K}, \textit{(trivial left coaction).}$$

With these structures the constraints become morphisms in ${}^B\mathcal{M}$.

An algebra in ${}^B\mathcal{M}$ is called a *left B-comodule algebra.* A coalgebra in ${}^B\mathcal{M}$ is called a *left B-comodule coalgebra.*

We denote the category of algebras in $\mathcal{M}^B$ by $Alg({}^B\mathcal{M})$ and the category of coalgebras in ${}^B\mathcal{M}$ by $CoAlg({}^B\mathcal{M})$.

Remark 2.36. Let H be a bialgebra over the field $\mathbb{K}$. The definition of module coalgebras and comodule coalgebras can equivalently be defined as following.

1. A $\mathbb{K}$-coalgebra that is also a left H-module is called a *left H-module coalgebra* if the counit and the comultiplication are left H-linear. This is equivalent to

$$\Delta_C(h.c) = h_{(1)}.c_{(1)} \otimes h_{(2)}.c_{(2)} \text{ and } \epsilon_C(h.c) = \epsilon_H(h)\epsilon_C(c)$$

for all $h \in H$ and $c \in C$ [20, p. 8] .

2. A $\mathbb{K}$-coalgebra C that is also a right H-comodule is called a *right H-comodule coalgebra* if the comultiplication and the counit are right $H - colinear$, or

$$c_{(0)1} \otimes c_{(0)2} \otimes c_{(1)} = c_{(1)0} \otimes c_{(2)0} \otimes c_{(1)1}c_{(2)1} \text{ and } \epsilon_C(c_{(0)})c_{(1)} = \epsilon_C(c)1_H$$

for all $c \in C$ [20, p. 21] .

$\boxplus$

CHAPTER 3
INVESTIGATING THE EXISTENCE AND THE CONSTRUCTION OF COFREE CORINGS

This chapter is devoted for exposing two main themes: the existence and construction of cofree corings. The dual of The Adjoint Functor Theorem suggests an elegantly efficient strategy to investigate cofree objects in concrete categories. To apply this strategy for the forgetful functor $\mathcal{U} : Crg_{\mathbb{A}} \to {}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$, we need to make sure that $\mathcal{U}$ is a cocotinuous functor, and the category of corings $Crg_{\mathbb{A}}$ is cocomplete co-wellpowered with a generating set. Although Wisbauer in [62] points out that " the direct limit of $\mathbb{A}$-corings can be obtained by the corresponding construction for $(\mathbb{A}, \mathbb{A})$-bimodules (Wisbauer in [62]), he did not explicitly show that the category $Crg_{\mathbb{A}}$ is cocomplete. Therefore, it seems reasonable to do that explicitly; especially, we need to understand the behavior of the tensor product $\otimes_{\mathbb{A}}$ which plays a critical role not only with the existence process of cofree corings, but also with their construction. Observably, showing $\mathcal{U}$ a cocotinuous functor and proving $Crg_{\mathbb{A}}$ a cocomplete category are inherent, and this could be attributed to the mechanism by which colimits in $Crg_{\mathbb{A}}$ can be constructed. Following the procedure that Michael Barr has done in [10], we establish a generating set for the category $Crg_{\mathbb{A}}$. After showing that $Crg_{\mathbb{A}}$ is co-wellpowered, the dual of The Adjoint Functor Theorem asserts that it has cofree objects.

The last theme is inspired by the proof of the dual of The Adjoint Functor Theorem. To do the construction, we need to use tools, such as the notion of comma categories and consequences obtained in the first part. Eventually, we get an accurate and clear construction for each cofree object in the category $Crg_{\mathbb{A}}$.

3.1 Verifying The [SAFT]

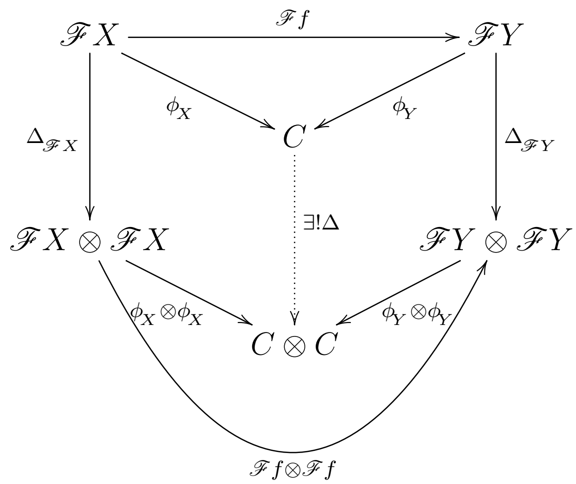
Proposition 3.1. *Let $(\mathcal{C}, \otimes, \mathcal{I})$ be a monoidal category. If $\mathcal{C}$ is cocomplete, then $CoMon(\mathcal{C})$ is cocomplete and the forgetful functor $\mathscr{U} : CoMon(\mathcal{C}) \to \mathcal{C}$ is cocontinuous.*

Proof. Let $\mathscr{D}$ be a small category, and let $\mathscr{F} : \mathscr{D} \to CoMon(\mathcal{C})$ be a functor. Since $\mathcal{C}$ is a cocomplete category, $\mathcal{U}\mathscr{F}$ has a colimit $(C, (\phi_X)_{X \in Ob(\mathscr{D})})$. We note that for any morphism $f : X \to Y$ in $\mathscr{D}$, we have

$$
\begin{aligned}
(\phi_Y \otimes \phi_Y)\Delta_{\mathscr{F}Y}\mathscr{F}f \quad &= (\phi_Y \otimes \phi_Y)(\Delta_{\mathscr{F}Y}\mathscr{F}f) \\
&= (\phi_Y \otimes \phi_Y)(\mathscr{F}f \otimes \mathscr{F}f)\Delta_{\mathscr{F}X} \\
&\text{(since } \mathscr{F}f \text{ is a morphism in } \mathcal{C}) \\
&= (\phi_Y \mathscr{F}f \otimes \phi_Y \mathscr{F}f)\Delta_{\mathscr{F}X} \\
&= (\phi_X \otimes \phi_X)\Delta_{\mathscr{F}X} \\
&\text{(since } (C, (\phi_X)_{X \in Ob(\mathscr{D})}) \text{ is a cocone on } \mathcal{U}\mathscr{F})
\end{aligned}
$$

Hence, $((C \otimes C), ((\phi_X \otimes \phi_X)\Delta_{\mathscr{F}X})_{X \in Ob(\mathscr{D})})$ is a cocone on $\mathcal{U}\mathscr{F}$.

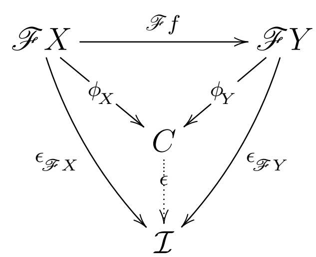
Consider the following diagram



Since $(C, (\phi_X)_{X \in Ob(\mathscr{D})})$ is the colimit of $\mathcal{U}\mathscr{F}$, there exists a unique morphism

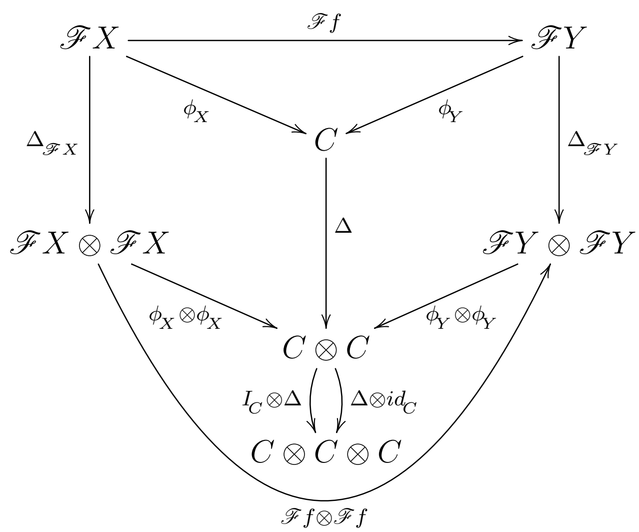
$$\Delta : C \to C \otimes C \text{ in } \mathcal{C} \text{ with } \Delta\ \phi_X = (\phi_X \otimes \phi_X)\Delta_{\mathscr{F}X}.$$

Consider the following diagram



Since $\mathscr{F}f$ is a morphism in $CoMon(\mathcal{C})$, $\epsilon_{\mathscr{F}Y}\ \mathscr{F}f = \epsilon_{\mathscr{F}X}$, hence $(\mathcal{I}, (\epsilon_{\mathscr{F}X})_{X \in Ob(\mathscr{D})})$ is a cocone on $\mathcal{U}\mathscr{F}$. Since $(C, (\phi_X)_{X \in Ob(\mathscr{D})})$ is the colimit of $\mathcal{U}\mathscr{F}$, there exists a unique morphism $\epsilon : C \to \mathcal{I}$ in $\mathcal{C}$ with $\epsilon\ \phi_X = \epsilon_{\mathscr{F}X}$.

To show that Δ is a coproduct map of C, consider the following diagram



Since $(\Delta\otimes id_C)(\phi_Y\otimes\phi_Y)\Delta_{\mathscr{F}Y}\mathscr{F}f = (\Delta\otimes id_C)(\phi_X\otimes\phi_X)\Delta_{\mathscr{F}X}$, $((C\otimes C\otimes C),((\Delta\otimes id_C)(\phi_X\otimes\phi_X)\Delta_{\mathscr{F}X})_{X\in Ob(\mathscr{D})})$ is a cocone on $\mathcal{U}\mathscr{F}$. Since $(C,(\phi_X)_{X\in Ob(\mathscr{D})})$ is a colimit of $\mathcal{U}\mathscr{F}$, there exists a unique morphism $\theta : C \to C\otimes C\otimes C$ in $\mathcal{C}$ with $\theta\ \phi_X = (\Delta\otimes id_C)(\phi_X\otimes\phi_X)\Delta_{\mathscr{F}X}$. Since $(\Delta\otimes id_C)\Delta\phi_X = (\Delta\otimes id_C)(\phi_X\otimes\phi_X)\Delta_{\mathscr{F}X}$, we have $\theta = (\Delta\otimes id_C)\Delta$.

We note that

$$
\begin{aligned}
(\Delta\otimes id_C)\Delta\phi_X &= (\Delta\otimes id_C)(\phi_X\otimes\phi_X)\Delta_{\mathscr{F}X}\\
&= (\Delta\phi_X\otimes id_C\phi_X)\Delta_{\mathscr{F}X}\\
&= (\Delta\phi_X\otimes\phi_X)\Delta_{\mathscr{F}X}\\
&= (((\phi_X\otimes\phi_X)\Delta_{\mathscr{F}X})\otimes\phi_X)\Delta_{\mathscr{F}X}\\
&= ((\phi_X\otimes\phi_X)\otimes\phi_X)(\Delta_{\mathscr{F}X}\otimes id_{\mathscr{F}X})\Delta_{\mathscr{F}X}\\
&= ((\phi_X\otimes\phi_X)\otimes\phi_X)(id_{\mathscr{F}X}\otimes\Delta_{\mathscr{F}X})\Delta_{\mathscr{F}X}\\
&\text{(since } \mathscr{F}X\in Ob(\mathcal{C}))
\end{aligned}
$$

$$= (\phi_X \otimes (\phi_X \otimes \phi_X))(id_{\mathscr{F}X} \otimes \Delta_{\mathscr{F}X})\Delta_{\mathscr{F}X}$$

$$= (\phi_X I_{\mathscr{F}X} \otimes (\phi_X \otimes \phi_X)\Delta_{\mathscr{F}X})\Delta_{\mathscr{F}X}$$

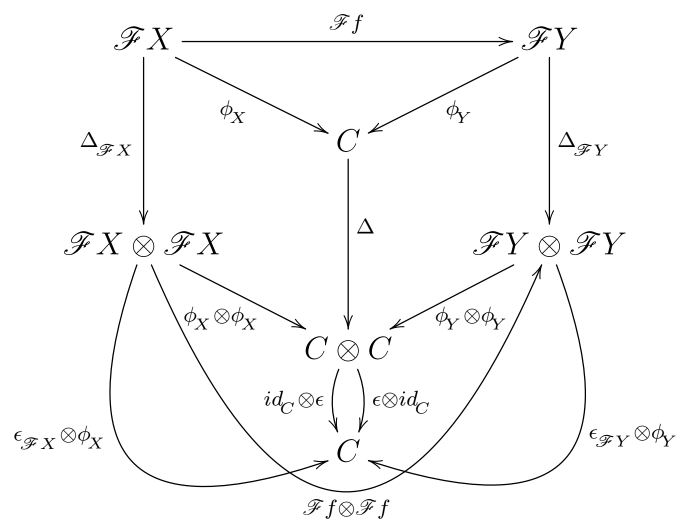
$$= (\phi_X \otimes (\phi_X \otimes \phi_X)\Delta_{\mathscr{F}X})\Delta_{\mathscr{F}X}$$

$$= (\phi_X \otimes \Delta\phi_X)\Delta_{\mathscr{F}X}$$

$$= (id_C \otimes \Delta)(\phi_X \otimes \phi_X)\Delta_{\mathscr{F}X}$$

$$= (id_C \otimes \Delta)\Delta\phi_X$$

It follows $(\Delta \otimes id_C)\Delta = \theta = (id_C \otimes \Delta)\Delta$. Therefore, $(\Delta \otimes id_C)\Delta = (id_C \otimes \Delta)\Delta$. To show that ϵ is a counit of C, consider the following diagram



We note that

$$(\epsilon \otimes id_C)(\phi_X \otimes \phi_X)\Delta_{\mathscr{F}X} = (\epsilon\phi_X \otimes id_C\phi_X)\Delta_{\mathscr{F}X}$$

$$= (\epsilon_{\mathscr{F}X} \otimes \phi_X)\Delta_{\mathscr{F}X}$$

Similarly, we have

$$(id_C \otimes \epsilon)(\phi_X \otimes \phi_X)\Delta_{\mathscr{F}X} = (id_C\phi_X \otimes \epsilon\phi_X)\Delta_{\mathscr{F}X}$$

$$= (\phi_X \otimes \epsilon_{\mathscr{F}X})\Delta_{\mathscr{F}X}$$

On the other hand, we have

$$(\epsilon_{\mathscr{F}Y} \otimes \phi_Y)\Delta_{\mathscr{F}Y}\mathscr{F}f = (\epsilon \otimes id_C)(\phi_Y \otimes \phi_Y)\Delta_{\mathscr{F}Y}\mathscr{F}f$$

$$= (\epsilon \otimes id_C)(\phi_X \otimes \phi_X)\Delta_{\mathscr{F}X}$$

$$= (\epsilon_{\mathscr{F}X} \otimes \phi_X)\Delta_{\mathscr{F}X}$$

Therefore, $(C, ((\epsilon_{\mathscr{F}X} \otimes \phi_X)\Delta_{\mathscr{F}X})_{X \in Ob(\mathcal{D})})$ is a cocone on $\mathcal{U}\mathscr{F}$. There exists a unique morphism $\sigma : C \to C$ in $\mathcal{C}$ with $\sigma\ \phi_X = (\epsilon_{\mathscr{F}X} \otimes \phi_X)\Delta_{\mathscr{F}X}$.

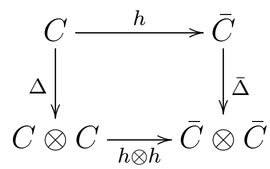
Since $(\epsilon \otimes id_C)\Delta\phi_X = (\epsilon_{\mathscr{F}X} \otimes \phi_X)\Delta_{\mathscr{F}X}$, the uniqueness of σ gives us $\sigma = (\epsilon \otimes id_C)\Delta$. Moreover, $\sigma = id_C$ because $(\epsilon_{\mathscr{F}X} \otimes \phi_X)\Delta_{\mathscr{F}X} = \phi_X = id_C\phi_X$. Therefore, we have $(\epsilon \otimes id_C)\Delta = id_C$.

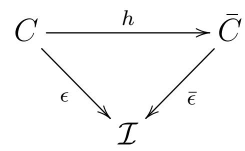
Similarly, we have $(id_C \otimes \epsilon)\Delta = id_C$, hence $(\epsilon \otimes id_C)\Delta = id_C = (id_C \otimes \epsilon)\Delta$. Thus, (C, Δ, ϵ) is a comonoid in $\mathcal{C}$.

To show that $((C, \Delta, \epsilon), (\phi_X)_{X \in Ob(\mathscr{D})})$ is the colimit of $\mathscr{F}$, it remains to show the uniqueness of $((C, \Delta, \epsilon), (\phi_X)_{X \in Ob(\mathscr{D})})$.

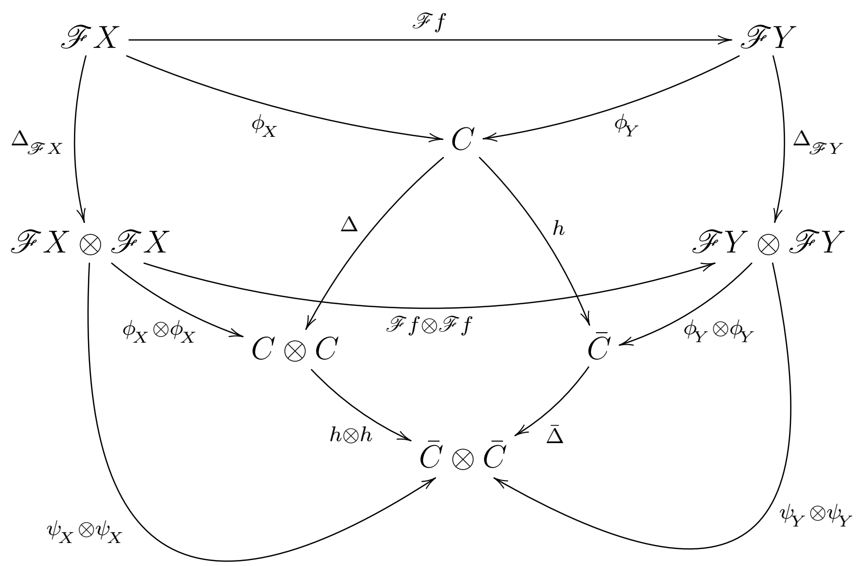
Let $((\bar{C}, \bar{\Delta}, \bar{\epsilon}), (\psi_X)_{X \in Ob(\mathscr{D})})$ be a cocone on $\mathscr{F}$. Since $((C, \Delta, \epsilon), (\phi_X)_{X \in Ob(\mathscr{D})})$ is a colimit of $\mathcal{U}\mathscr{F}$, there exists a unique morphism $h : C \to \bar{C}$ in $\mathcal{C}$ with $h\phi_X = \psi_X$ for every $X \in Ob(\mathscr{D})$. The proof is complete whence we show that h is a morphism in $CoMon(\mathcal{C})$.

Explicitly, we need to show that the following diagrams commute





Consider the following diagram



First, notice that $\bar{\Delta} h \phi_Y \mathscr{F} f = \bar{\Delta} h \phi_X$. This makes $((\bar{C} \otimes \bar{C}), (\bar{\Delta} h \phi_X)_{X \in Ob(\mathscr{D})})$ a cocone on $\mathcal{U}\mathscr{F}$.

Furthermore, for every $X \in Ob(\mathscr{D})$, we have

$$
\begin{aligned}
\bar{\Delta} h \phi_X &= \bar{\Delta} \psi_X \text{ (since } h\phi_X = \psi_X, \forall X \in Ob(\mathscr{D})) \\
&= (\psi_X \otimes \psi_X) \bar{\Delta}_{\mathscr{F}X} \text{ (since } \psi_X \text{ is a morphism in } CoMon(\mathcal{C}), \forall X \in Ob(\mathscr{D})) \\
&= (h\phi_X \otimes h\phi_X) \Delta_{\mathscr{F}X} \text{ (since } h\phi_X = \psi_X, \forall X \in Ob(\mathscr{D})) \\
&= (h \otimes h)((\phi_X \otimes \phi_X) \Delta_{\mathscr{F}X}) \\
&= (h \otimes h) \Delta \phi_X \text{ (since } \phi_X \text{ is a morphism in } CoMon(\mathcal{C}), \forall X \in Ob(\mathscr{D}))
\end{aligned}
$$

From the universal property of the colimit of $\mathcal{U}\mathscr{F}$, we have $\bar{\Delta} h = (h \otimes h)\Delta)$.

On the other hand, we have $\bar{\epsilon}h\phi_Y\mathscr{F}f = \bar{\epsilon}h\phi_X$. Thus, $(\mathcal{I}, (\bar{\epsilon}h\phi_X)_{X\in Ob(\mathscr{D})})$ is a cocone on $\mathcal{U}\mathscr{F}$.

Moreover, for every $X \in Ob(\mathscr{D})$, we have

$$\begin{aligned} \bar{\epsilon}h\phi_X &= \bar{\epsilon}\psi_X \text{ (since } h\phi_X = \psi_X, \forall X \in Ob(\mathscr{D})) \\ &= \epsilon_{\mathscr{F}X} \text{ (since } \psi_X \text{ is a morphism in} CoMon(\mathcal{C}), \forall X \in Ob(\mathscr{D})) \\ &= \epsilon\phi_X \text{ (since } \phi_X \text{ is a morphism in } CoMon(\mathcal{C}), \forall X \in Ob(\mathscr{D})) \end{aligned}$$

From the definition of the colimit of $\mathcal{U}\mathscr{F}$, we have $\bar{\epsilon}h = \epsilon$. Hence, h is a morphism in $CoMon(\mathcal{C})$. Therefore, $((C, \Delta, \epsilon), (\phi_X)_{X\in Ob(\mathscr{D})})$ is the colimit of $\mathscr{F}$. It follows that the category $\mathcal{C}$ is cocomplete, and the functor $\mathcal{U} : \mathcal{C} \to CoMon(\mathcal{C})$ is cocontinuous.

□

Corollary 3.2. The category $Crg_{\mathbb{A}}$ is cocomplete, and the functor $\mathcal{U} : Crg_{\mathbb{A}} \to {}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$ is cocontinuous.

Proof. The category of $\mathbb{A}$-bimodules can be identifies as a monoidal category $({}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}, \otimes_{\mathbb{A}}, \mathbb{A})$, and the category $Crg_{\mathbb{A}} = CoMon({}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}})$. Therefore, $Crg_{\mathbb{A}}$ is cocomplete, and the functor $\mathcal{U}$ is cocontinuous by Proposition 3.1. □

As a quick consequence of Corollary 3.2, we have the following.

Lemma 3.3. *The epimorphisms in $Crg_{\mathbb{A}}$ are precisely the surjective morphisms of corings.*

Proof. Since the forgetful functor $\mathcal{U}$ preserves colimits, it also preserves epimorphisms, and so an epimorphism in $Crg_{\mathbb{A}}$ is surjective. □

The following corollary is obviously an immediate consequence of the proof of Proposition 3.1, but it is worth mentioning.

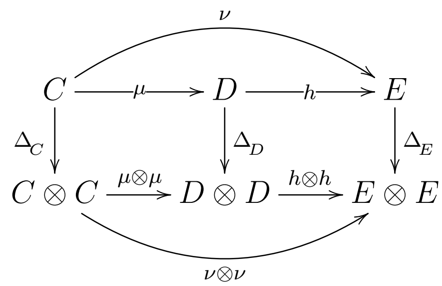
Corollary 3.4. Let $(\mathcal{C}, \otimes, \mathcal{I})$ be a a monoidal category. If $\mathcal{C}$ is a cocomplete category, then the forgetful functor $\mathscr{U} : CoMon(\mathcal{C}) \to \mathcal{C}$ creates all small colimits. In particular, the forgetful functor $\mathcal{U} : Crg_{\mathbb{A}} \to {}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$ creates all small colimits.

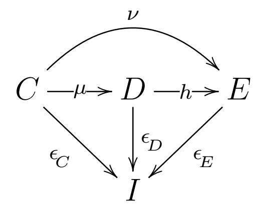
Proposition 3.5. *The $\mathbb{A}$-corings whose cardinality are $\leq$ max $\{|\mathbb{A}|, \aleph_0\}$ generate $Crg_{\mathbb{A}}$.*

Proof. Let $f, g : C_1 \to C_2$ be morphisms in $Crg_{\mathbb{A}}$ with $f \neq g$, and pick $c \in C_1$ with $f(c) \neq g(c)$. Applying Proposition 2.35 for the $\mathbb{A}$-sub-bimodule $\mathbb{A}c\mathbb{A}$ generated by c, there is a $\mathbb{A}$-Subcoring $D \subseteq C$ such that $\mathbb{A}c\mathbb{A} \subseteq D \subseteq C_1$ and $|D| \leq \max\{|\mathbb{A}c\mathbb{A}|, |\mathbb{A}|, \aleph_0\} = \max\{|\mathbb{A}|, \aleph_0\}$. Also, $f|_D \neq g|_D$, i.e. $f \circ (D \hookrightarrow C_1) \neq g \circ (D \hookrightarrow C_1)$, which ends the proof. □

Proposition 3.6. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, $CoMon(\mathcal{C})$ be the category of comonoids of $\mathcal{C}$ and $\mathcal{U} : CoMon(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor. If $\mathcal{C}$ is co-wellpowered, then so is $CoMon(\mathcal{C})$.*

Proof. Let $Epi(CoMon(\mathcal{C}))$ be the class of epimorphisms in the category $CoMon(\mathcal{C}$. It is enough to show that if $\mu : C \to D$ and $\nu : C \to E$ are in $Epi(CoMon(\mathcal{C}))$ and equivalent as epimorphisms in $\mathcal{C}$, then they are equivalent in $CoMon(\mathcal{C})$ (as epimorphisms). Let $h : D \to E$ be an isomorphism in $\mathcal{C}$ for which $h\mu = \nu$ and we show that h is in fact an isomorphism in $CoMon(\mathcal{C})$.





The first and outside diagrams on the left picture are commutative since μ and ν are morphisms in $CoMon(\mathcal{C})$, so $\Delta_E h\mu = \Delta_E \nu = (\nu \otimes \nu)\Delta_C = (h \otimes h)(\mu \otimes \mu)\Delta_C = (h \otimes h)\Delta_D \mu$. Since μ is an epi in $\mathcal{C}$, $\Delta_E h = (h \otimes h)\Delta_D$. Similarly, we have $\epsilon_E h\mu = \epsilon_E \nu = \epsilon_C = \epsilon_D \mu$, and it follows that $\epsilon_E h = \epsilon_D$. Hence, h is a morphism in $CoMon(\mathcal{C})$, and so is its $\mathcal{C}$ inverse. $\square$

Corollary 3.7. The category $Crg_{\mathbb{A}}$ is co-wellpowered.

Proof. The category of $\mathbb{A}$-bimodules can be identifies as a monoidal category $({}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}, \otimes_{\mathbb{A}}, \mathbb{A})$, and the category $Crg_{\mathbb{A}} = CoMon({}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}})$. Therefore, $Crg_{\mathbb{A}}$ is co-wellpowered.

$\square$

Theorem 3.8. *Let $(\mathcal{C}, \otimes, \mathcal{I})$ be a monoidal category. If $\mathcal{C}$ is cocomplete and co-wellpowered, and the category $CoMon(\mathcal{C})$ has a generating set, then the forgetful functor $\mathscr{U} : CoMon(\mathcal{C}) \to \mathcal{C}$ has a right adjoint. In other words, in this case, the concrete category $(CoMon(\mathcal{C}), \mathcal{U})$ over $\mathcal{C}$ has cofree objects.*

Proof. It follows from Proposition 3.1 and Proposition 3.6 that the category $CoMon(\mathcal{C})$ is cocomplete, co-wellpowered and the forgetful functor $\mathcal{U}$ is cocontinuous. Using Proposition 1.55, the functor $\mathscr{U} : CoMon(\mathcal{C}) \to \mathcal{C}$ has a right adjoint. $\square$

Corollary 3.9. The forgetful functor $\mathcal{U} : Crg_{\mathbb{A}} \to {}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$ has a right adjoint. In other words, the concrete category $(Crg_{\mathbb{A}}, \mathcal{U})$ over ${}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$ has cofree objects.

Proof. The category of $\mathbb{A}$-bimodules can be identifies as a monoidal category $({}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}, \otimes_{\mathbb{A}}, \mathbb{A})$, and the category $Crg_{\mathbb{A}} = CoMon({}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}})$. Hence, the assertion follows from Proposition 3.1, Proposition 3.6, Proposition3.5 and Theorem 3.8.

□

Remark 3.10. Let $\mathscr{C}$ be the right adjoint (which exists by corollary 3.9) to the forgetful functor $\mathcal{U} : Crg_{\mathbb{A}} \to {}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$. From Definition 1.48, for each $C \in Ob(Crg_{\mathbb{A}})$ and for each $V \in Ob({}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}})$, there are bijections

$$\tau_{C,V} : Hom_{{}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}}(\mathcal{U}C, V) \to Hom_{Crg_{\mathbb{A}}}(C, \mathscr{C}(V)) \tag{3.1}$$

that are natural transformations in C and in V.

Following [38], the correspondence (3.1) between maps is denoted by a horizontal bar, in both directions:

$$(\mathcal{U}C \xrightarrow{f} V) \mapsto (C \xrightarrow{\bar{f}} \mathscr{C}(V)), \tag{3.2}$$

$$(C \xrightarrow{g} \mathscr{C}(V)) \mapsto (\mathcal{U}C \xrightarrow{\bar{g}} V). \tag{3.3}$$

So $\bar{\bar{f}} = f$ and $\bar{\bar{g}} = g$. We call $\bar{f}$ the *transpose* of f , and similarly for g. The naturality axiom has two parts:

$$\overline{(\mathcal{U}C \xrightarrow{f} V \xrightarrow{\alpha} V')} = (C \xrightarrow{\bar{f}} \mathscr{C}(V) \xrightarrow{\mathscr{C}(\alpha)} \mathscr{C}(V')), \tag{3.4}$$

$$\overline{(C' \xrightarrow{\beta} C \xrightarrow{g} \mathscr{C}(V))} = (\mathcal{U}C' \xrightarrow{\mathcal{U}\beta} \mathcal{U}C \xrightarrow{\bar{g}} V). \tag{3.5}$$

for every morphisms $V \xrightarrow{\alpha} V'$ in ${}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$ and $C' \xrightarrow{\beta} C$ in $Crg_{\mathbb{A}}$. The unit η and counit ε of the adjunction, furthermore, are given by

$$(C \xrightarrow{\eta_C} C(\mathcal{U})) = \overline{(\mathcal{U}C \xrightarrow{id_{\mathcal{U}C}} \mathcal{U}C)}, \tag{3.6}$$

$$(\mathcal{U}\mathscr{C}(V) \xrightarrow{\varepsilon_V} V) = \overline{(\mathscr{C}(V) \xrightarrow{id_{\mathscr{C}(V)}} \mathscr{C}(V))}. \tag{3.7}$$

Therefore, for any cofree $\mathbb{A}$-coring $(C\mathscr{C}(V), p)$ over an $\mathbb{A}$-bimodule V, we have:

$$\begin{aligned} p &= \bar{\bar{p}} \\ &= \overline{\overline{(\mathcal{U}\mathscr{C}(V) \xrightarrow{p} V)}} \\ &= \overline{(\mathscr{C}(V) \xrightarrow{I_{\mathscr{C}(V)}} \mathscr{C}(V))} \\ &= (\mathcal{U}\mathscr{C}(V) \xrightarrow{\varepsilon_V} V) \\ &= \varepsilon_V \end{aligned}$$

⊞

Let $(\mathcal{C}, \otimes, \mathcal{I})$ be a monoidal category with $\mathcal{C}$ cocomplete and co-wellpowered. Assume, furthermore, that the category $CoMon(\mathcal{C})$ has a generating set. Then by Theorem 3.8, the forgetful functor $\mathscr{U} : CoMon(\mathcal{C}) \to \mathcal{C}$ has a right adjoint $\mathscr{C}$. Let $C \in Ob(\mathcal{C})$ and $(\mathscr{C}(C), p)$ the cofree object over C. One can ask when is p an epimorphism? Here is a couple of answers:

Corollary 3.11. Let $\bar{C} \in Ob(CoMon(\mathcal{C}))$ and $(\mathscr{C}(\mathscr{U}\bar{C}), p)$ the cofree object over $\mathscr{U}\bar{C}$. Then the morphism p has a right inverse (hence, p is epimorphism in $(\mathcal{C})$. In other words, p splits epimorphism.

Proof. Consider the following diagram

$$\begin{array}{ccc} \mathscr{U}\bar{C} & \overset{\exists! \mathscr{U}q}{\dashrightarrow} & \mathscr{U}\mathscr{C}(\mathscr{U}\bar{C}) \\ & {\scriptstyle I_C}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathscr{U}\bar{C} & \end{array}$$

It follows from the definition of cofree objects that there exists a unique morphism $q : \bar{C} \to \mathscr{C}(\mathscr{U}\bar{C})$ in $CoMon(\mathcal{C})$ such that the diagram above is commutative. Hence, p has a right inverse. □

Corollary 3.12. Let $\bar{C} \in Ob(CoMon(\mathcal{C}))$ and $(\mathscr{C}(\mathscr{U}\bar{C}), p)$ the cofree object over $\mathscr{U}\bar{C}$. If $\mathscr{C}$ is faithful, then p is an epimorphism. In particular, If $\mathscr{C}$ is separable functor, then p is an epimorphism.

Proof. Let $\bar{C}'$ be an object in $\mathcal{C}$ and $\bar{C} \underset{\alpha'}{\overset{\alpha}{\rightrightarrows}} \bar{C}'$ arrows in $\mathcal{C}$ with $\alpha p = \alpha' p$. Using the same formulation as in Remark 3.10, we have:

$$\overline{(\mathscr{U}\mathscr{C}(\bar{C}) \overset{p}{\to} \bar{C} \underset{\alpha'}{\overset{\alpha}{\rightrightarrows}} \bar{C}')} = (\mathscr{C}(\bar{C}) \xrightarrow{id_{\mathscr{C}(\bar{C})}} \mathscr{C}(\bar{C}) \underset{\mathscr{C}(\alpha')}{\overset{\mathscr{C}(\alpha)}{\rightrightarrows}} \mathscr{C}(\bar{C}'))$$

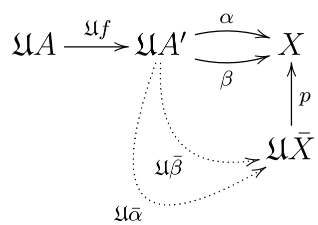
Thus, we have $\mathscr{C}(\alpha) = \mathscr{C}(\alpha')$. Hence, we get $\alpha = \alpha'$ because $\mathscr{C}$ is faithful. The last statement follows from Definition 1.53.

□

Although the Dual of the following proposition is proved in [3], the proposition is still worth being proven because it is helpful to determine the relation between the epimorphisms in a monoidal category $(\mathcal{C}, \otimes, \mathcal{I})$ and the epimorphisms in $CoMon(\mathcal{C})$.

Proposition 3.13. *If a concrete category* $(\mathfrak{A}, \mathfrak{U})$ *over* $\mathfrak{X}$ *has cofree objects and* $f : A \to A'$ *is a morphism in* $\mathfrak{A}$*, then* f *is epimorphism in* $\mathfrak{A}$ *if and only if* $\mathfrak{U}f$ *is epimorphism in* $\mathfrak{X}$*.*

Proof. Let $f : A \to A'$ be an epimorphism morphism in $\mathfrak{A}$ and $\alpha, \beta : \mathfrak{U}A' \to X$ a pair of morphisms in $\mathfrak{X}$ with $\alpha\mathfrak{U}f = \beta\mathfrak{U}f$. Since the concrete category $(\mathfrak{A}, \mathfrak{U})$ over $\mathfrak{X}$ has cofree objects, the object X has a cofree object say $(\bar{X}, p)$. Using the co-universal property of the cofree object $(\bar{X}, p)$, we have unique morphisms $\bar{\alpha}, \bar{\beta} : A' \to \bar{X}$ making the following diagram commutative



From $p\ \mathfrak{U}\bar{\alpha} = \alpha$ and $p\ \mathfrak{U}\bar{\beta} = \beta$, we get $p\ \mathfrak{U}\bar{\alpha}\ \mathfrak{U}f = \alpha\ \mathfrak{U}f$ and $p\ \mathfrak{U}\bar{\beta}\ \mathfrak{U}f = \beta\ \mathfrak{U}f$. Since $\alpha\ \mathfrak{U}f = \beta\ \mathfrak{U}f$, we have $p\ \mathfrak{U}(\bar{\alpha}f) = p\ \mathfrak{U}(\bar{\beta}f)$. By the uniqueness in the definition of co-universal arrows, we have $\bar{\alpha}f = \bar{\beta}f$, and since $f : A \to A'$ is an epimorphism morphism in $\mathfrak{A}$, $\bar{\alpha} = \bar{\beta}$. Thus, $\mathfrak{U}f$ is epimorphism in $\mathfrak{X}$.

The proof of the other direction is very clear. Actually, we do not need the fact that the concrete category $(\mathfrak{A}, \mathfrak{U})$ over $\mathfrak{X}$ has cofree objects to prove it. □

From Theorem 3.8 and Proposition 3.13, we have the following Corollary.

Corollary 3.14. Let $(\mathcal{C}, \otimes, \mathcal{I})$ be a monoidal category $\mathscr{U} : CoMon(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor, and let $f : A \to A'$ be a morphism in $CoMon(\mathcal{C})$. If $\mathcal{C}$ is cocomplete and co-wellpowered, and the category $CoMon(\mathcal{C})$ has a generating set, then f is epimorphism in $CoMon(\mathcal{C})$ if and only if $\mathfrak{U}f$ is epimorphism in $\mathcal{C}$.

Proof. □

Let $(\mathcal{C}, \otimes, \mathcal{I})$ be a monoidal category with $\mathcal{C}$ cocomplete and co-wellpowered. Assume, furthermore, that the category $CoMon(\mathcal{C})$ has a generating set. Then by Theorem 3.8, the forgetful functor $\mathscr{U} : CoMon(\mathcal{C}) \to \mathcal{C}$ has a right adjoint $\mathscr{C}$. From Corollary 1.52 and Corollary 1.51, we deduce:

Corollary 3.15.

1. We have the following intimately related statements:

   (a) There is a one-to-one correspondence between:

      i. adjunctions between $\mathscr{U}$ and $\mathscr{C}$

      ii. natural transformations $\eta : id_{CoMon(\mathcal{C})} \to \mathscr{C}\mathscr{U}$ such that $\eta_{\bar{C}} : \bar{C} \to \mathscr{C}\mathscr{U}\bar{C}$ is initial in $\bar{C} \downarrow \mathscr{C}$ for every $\bar{C} \in CoMon(\mathcal{C})$.

   (b) The functor $\mathscr{C}$ has a left adjoint if and only if for each $C \in CoMon(\mathcal{C})$, the category $C \downarrow \mathscr{C}$ has an initial object.

2. In particular, if the forgetful functor $\mathcal{U} : Crg_{\mathbb{A}} \to {}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$ has a right adjoint $\mathscr{C}$, then we have the following:

   (a) There is a one-to-one correspondence between:

      i. adjunctions between $\mathcal{U}$ and $\mathscr{C}$

      ii. natural transformations $\eta : id_{Crg_{\mathbb{A}}} \to \mathscr{C}\mathcal{U}$ such that $\eta_C : C \to \mathscr{C}\mathcal{U}C$ is initial in $C \downarrow \mathscr{C}$ for every $C \in Crg_{\mathbb{A}}$.

   (b) The category $C \downarrow \mathscr{C}$ has an initial object.

Proof. □

Before we start a new section, we need the following proposition which its dual is available in [38] as an exercise.

Proposition 3.16. *Let $\mathscr{A}$ and $\mathscr{B}$ be categories. Let $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ be a cocontinuous functor. Then if $\mathscr{A}$ is cocomplete then so is each comma category $\mathcal{F} \downarrow B$, $B \in Ob(\mathscr{B})$.*

Proof. Let $\mathcal{J} : \mathcal{D} \to \mathcal{F} \downarrow B$ be a functor with $\mathcal{D}$ a small category. From Remark 1.43, for any $X \in Ob(\mathcal{D})$, $f : X \to Y \in Mor(\mathcal{D})$, we have

$$\mathcal{J}X = (A_X, h_X) \in Ob(\mathcal{F} \downarrow B),\ \mathcal{J}f = \alpha_f,$$

where $\alpha_f : A_X \to A_Y \in Mor(\mathscr{A})$ making the following diagram:

$$\begin{array}{ccc} \mathcal{F}A_X & \xrightarrow{\mathcal{F}\alpha_f} & \mathcal{F}A_Y \\ & {\scriptstyle h_X}\searrow \quad \swarrow{\scriptstyle h_Y} & \\ & B & \end{array}$$

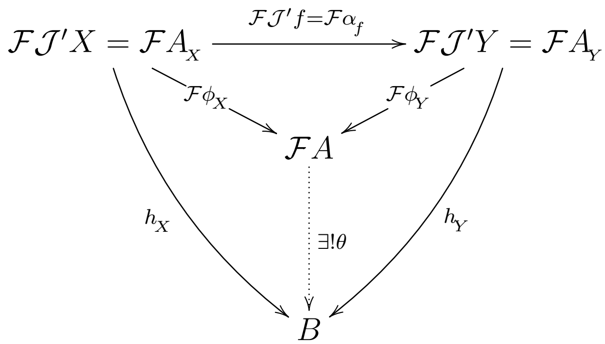
commutes.

Let $\mathcal{J}' : \mathcal{D} \to \mathscr{A}$ be the small functor defined as follows:

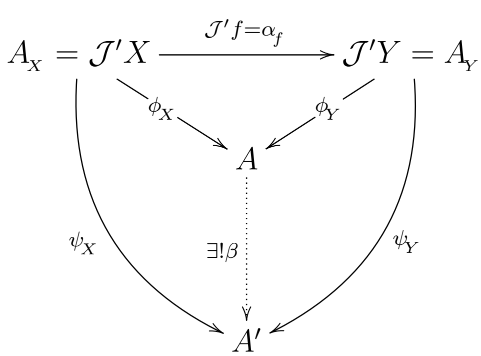
$$\mathcal{J}'X = A_X,\ \mathcal{J}'f = \alpha_f$$

for every object $X \in Ob(\mathcal{D})$, and every morphism $f : X \to Y$ of $\mathcal{D}$.

Since $\mathscr{A}$ is cocomplete, the functor $\mathcal{J}'$ has a colimit. Let $(A, (\phi_X : \mathcal{J}'X \to A)_{X \in Ob(\mathcal{D})})$ be a colimit of $\mathcal{J}'$. Since $\mathcal{F}$ is cocontinuous, $(\mathcal{F}A, (\mathcal{F}\phi_X : \mathcal{J}'X \to \mathcal{F}A)_{X \in Ob(\mathcal{D})})$ is a colimit of $\mathcal{F}\mathcal{J}'$. Consider the following diagram:

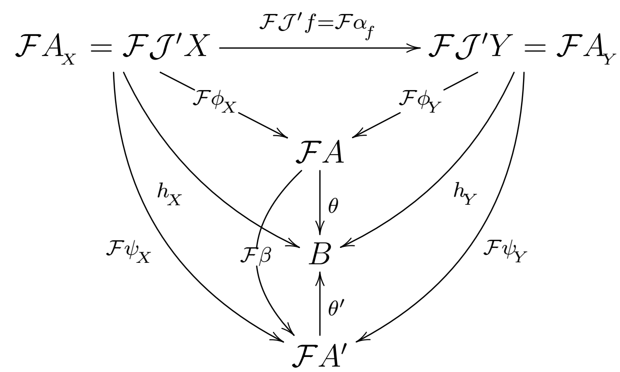


Since α_f is a morphism in the category $\mathcal{F} \downarrow B$, $(B, (h_X : A_X \to B)_{X \in Ob(\mathcal{D})})$ is a cocone on the functor $\mathcal{F}\mathcal{J}'$. Thus, there exists a unique morphism $\theta : \mathcal{F}A \to B$ with $\theta\mathcal{F}\phi_X = h_X$, since $(\mathcal{F}A, (\mathcal{F}\phi_X : \mathcal{J}'X \to \mathcal{F}A)_{X \in Ob(\mathcal{D})})$ is a colimit of $\mathcal{F}\mathcal{J}'$. Hence, $(A, \theta) \in Ob(\mathcal{F} \downarrow B)$. We shall show that $((A, \theta), (\phi_X : A_X \to A)_{X \in Ob(\mathcal{D})})$ is a colimit of $\mathcal{J}$. We note that for any $X \in \mathcal{D}$, ϕ_X can be viewed as a morphism in the category $\mathcal{F} \downarrow B$, since ϕ_X is a morphism in the category $\mathscr{A}$ and $\theta\mathcal{F}\phi_X = h_X$ for every $X \in \mathcal{D}$. Furthermore, α_f is a morphism in the category $\mathcal{F} \downarrow B$, hence $((A, \theta), (\phi_X : A_X \to A)_{X \in Ob(\mathcal{D})})$ is a cocone on the functor $\mathcal{J}$. To show that $((A, \theta), (\phi_X : A_X \to A)_{X \in Ob(\mathcal{D})})$ is a colimit of $\mathcal{J}$, it remains to prove that for any cocone $((A', \theta'), (\psi_X : A_X \to A')_{X \in Ob(\mathcal{D})})$ on $\mathcal{J}$, there exists unique morphism $\beta : A \to A'$ in $\mathcal{F} \downarrow B$ with $\beta\phi_X = \psi_X$. Let $((A', \theta'), (\psi_X : A_X \to A')_{X \in Ob(\mathcal{D})})$ be a cocone on $\mathcal{J}$ and consider the following diagram:



Since $((A', \theta'), (\psi_X : A_X \to A')_{X \in Ob(\mathcal{D})})$ is a cocone on $\mathcal{J}$, it is obviously a cocone on

$\mathcal{J}'$. But then there exists unique morphism $\beta : A \to A'$ in $\mathscr{A}$ with $\beta\phi_X = \psi_X$, since $(A, (\phi_X : \mathcal{J}'X \to A)_{X \in Ob(\mathcal{D})})$ is a colimit of $\mathcal{J}'$. In order to complete our proof, we need to show that $\beta : A \to A'$ is actually a morphism in $\mathcal{F} \downarrow B$; that is, $\theta'\mathcal{F}\beta = \theta$. Applying the functor $\mathcal{F}$ to the last diagram gives us the following diagram:



Since $\mathcal{F}$ is a functor, we have $\mathcal{F}\beta\mathcal{F}\phi_X = \mathcal{F}\psi_X$, hence $\theta'\mathcal{F}\beta\mathcal{F}\phi_X = \theta'\mathcal{F}\psi_X$, hence $(\mathcal{F}A', (\mathcal{F}\psi_X : \mathcal{F}A_X \to \mathcal{F}A')_{X \in Ob(\mathcal{D})})$ is a cocone on $\mathcal{F}\mathcal{J}$. Using the fact that that $\mathcal{F}$ is a cocontinuous functor, we have $(\mathcal{F}A, (\mathcal{F}\phi_X : \mathcal{F}A_X \to \mathcal{F}A)_{X \in Ob(\mathcal{D})})$ is a colimit of the functor $\mathcal{F}\mathcal{J}$. By the definition of the colimit $(\mathcal{F}A, (\mathcal{F}\phi_X : \mathcal{F}A_X \to \mathcal{F}A)_{X \in Ob(\mathcal{D})})$, to show that $\theta'\mathcal{F}\beta = \theta$, it suffices to show that $\theta'\mathcal{F}\beta\mathcal{F}\phi_X = h_X$ for every $X \in Ob(\mathcal{D})$. We note that

$$\theta'\mathcal{F}\beta\mathcal{F}\phi_X = \theta'\mathcal{F}\psi_X = h_X \text{ for every } X \in Ob(\mathcal{D})$$

The last equality follows from the fact that ψ_X is a morphism in the comma category $\mathcal{F} \downarrow B$ for every $X \in Ob(\mathcal{D})$. Therefore, $\beta : A \to A'$ is a morphism in $\mathcal{F} \downarrow B$, and it follows that $((A, \theta), (\phi_X : A_X \to A)_{X \in Ob(\mathcal{D})})$ is a colimit of $\mathcal{J}$. Hence each comma category $\mathcal{F} \downarrow B$, $B \in Ob(\mathscr{B})$ is cocomplete.

□

The proof of Proposition 3.16 clearly implies the following forward consequence, which is a slightly more general version to Proposition 3.16:

Corollary 3.17. Let $\mathscr{A}$ and $\mathscr{B}$ be categories. Let $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ be a cocontinuous functor. Then the projection functor $\mathscr{P}_{B} : \mathcal{F} \downarrow B \to \mathscr{A}$, creates colimits, for each $B \in Ob(\mathscr{B})$.

Proof. □

As an immediately obvious consequence to Proposition 3.16 and Proposition 3.1, we have the following corollary:

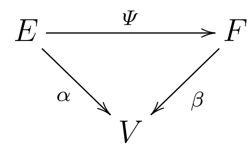
Corollary 3.18. Let $(\mathcal{C}, \otimes, \mathcal{I})$ be a monoidal category and $\mathscr{U} : CoMon(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor. If $\mathcal{C}$ is cocomplete, then so is each comma category $\mathcal{U} \downarrow C$, $C \in Ob(\mathcal{C})$. In particular, if $(\mathcal{C}, \otimes, \mathcal{I}) = ({}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}, \otimes_{\mathbb{A}}, \mathbb{A})$, then each comma category $\mathcal{U} \downarrow V$, $V \in Ob({}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}})$ is cocomplete.

3.2 Construction of Cofree Corings

In what follows, we present the construction of cofree corings, which is inspired by the proof of the SAFT; for convenience, we translate this construction in the current setting. Let Λ be a set of representatives of isomorphism classes of the $\mathbb{A}$-corings whose cardinality is less or equal to $\max\{|\mathbb{A}|, \aleph_0\}$.

Let V be an $\mathbb{A}$-bimodule, and let $\mathcal{E}$ be the category defined as follows. An object of $\mathcal{E}$ is a pair (E, α), where $E \in \Lambda$ and $\alpha : E \to V$ is a morphism in ${}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$. Technically, a morphism between the two objects is a pair $(\Psi, *)$ where $*$ means the above diagram is commutative. Let $(E, \alpha), (F, \beta) \in Ob(\mathcal{E})$. A morphism from (E, α)

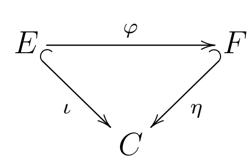
to (F, β) is a morphism $\Psi : E \to F$ of corings which furthermore makes the following diagram commutative



Let $\mathcal{F}' : \mathcal{E} \to Crg_{\mathbb{A}}$ be the functor defined by $\mathcal{F}'((E, \alpha)) = E, \forall (E, \alpha) \in Ob(\mathcal{E})$, and $\mathcal{F}'(\Psi) = \Psi, \forall \Psi \in Mor(\mathcal{E})$ (more precisely, $\mathcal{F}'((\Psi, *)) = \Psi, \forall (\Psi, *) \in Mor(\mathcal{E})$). Since the category $Crg_{\mathbb{A}}$ is cocomplete, and $\mathcal{E}$ is a small category, the colimit of the functor $\mathcal{F}'$ exists. The cofree coring $\mathscr{C}(V)$ on V will be the colimit $\varinjlim \mathcal{F}'$; more precisely, let $(\mathscr{C}(V), \Delta_{\mathscr{C}(V)}, \varepsilon_{\mathscr{C}(V)}), (\sigma_{(E,\alpha)})_{(E,\alpha) \in Ob(\mathcal{E})})$ be the colimit of the functor $\mathcal{F}'$. Also, let $p : \mathscr{C}(V) \to V$ be the canonical map $\varinjlim(\alpha) : \varinjlim \mathcal{F}' \to V$, so defined such that $p\sigma_{(E,\alpha)} = \alpha$ (here, $\sigma_{(E,\alpha)}$ are the canonical maps of the colimit).

We now justify why this is the case. To make things precise, this means that for every coring C and morphism of $\mathbb{A}$-bimodules $f : C \to V$, there is a morphism of $\mathbb{A}$-corings $\bar{f} : C \to \mathscr{C}(V)$ such that $p\bar{f} = f$. Let C be a coring, $f : C \to V$ a morphism of $\mathbb{A}$-bimodules. Let $\mathcal{D}$ be the category defined as follows. An object of $\mathcal{D}$ is a pair (E, ι), where $E \in \Lambda$ and $\iota : E \hookrightarrow C$ is a morphism in $Crg_{\mathbb{A}}$, such that ι is injective and pure both as a morphism of left and a morphism of right $\mathbb{A}$-modules (we note that the following argument works with $\mathcal{D}$ considered without these restrictions).

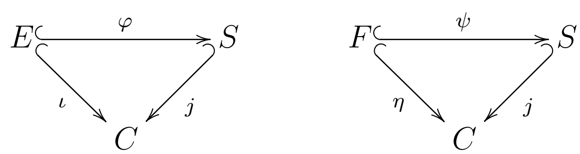
Let $(E, \iota), (F, \eta) \in Ob(\mathcal{D})$. A morphism from (E, ι) to (F, η) is a pair $(\varphi, (*))$, where φ is a morphism in $Crg_{\mathbb{A}}$ and $(*)$ means the following diagram commutes



Let $\mathcal{F} : \mathcal{D} \to Crg_{\mathbb{A}}$ be the functor defined as follows:

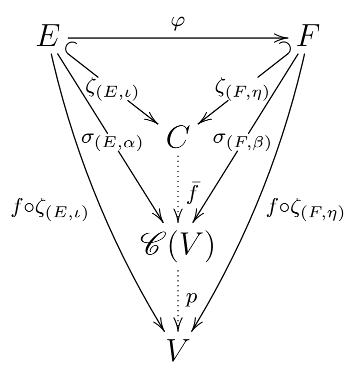
$\mathcal{F}((E, \iota)) = E, \forall (E, \iota) \in Ob(\mathcal{D})$, and $\mathcal{F}((\varphi, (*))) = \varphi, \forall (\varphi, (*)) \in Mor(\mathcal{D})$.

We note that if (E, ι), $(F, \eta) \in Ob(\mathcal{D})$, by Proposition 2.35, the $\mathbb{A}$-sub-bimodule $\iota(E) + \eta(F)$ of C is contained in a subcoring of C, pure as left and right $\mathbb{A}$-submodule and isomorphic to a member of Λ. Hence, there exists $(S, j) \in Ob(\mathcal{D})$ and $\mathbb{A}$-coring morphisms $\varphi : E \to S$, $\psi : F \to S$ (obtained from the inclusions $\iota(E) \subseteq j(S)$, $\eta(F) \subseteq j(S)$) such that the following diagrams commute:



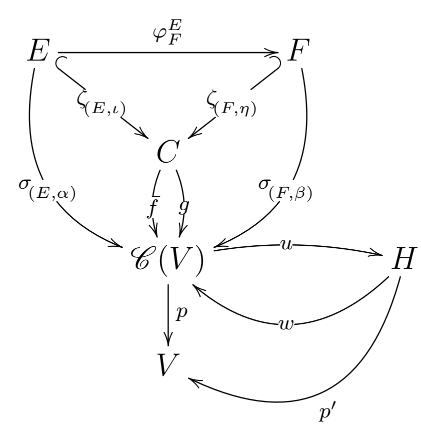
Furthermore, every element of C is contained in $j(S)$ for some $(S, j) \in Ob(\mathcal{D})$. Hence, C is the colimit of the directed system of $\mathbb{A}$-sub-bimodules $j(S), (S, j) \in Ob(\mathcal{D})$, and this allows us to regard $((C, \Delta, \varepsilon), (\zeta_{(E,\iota)})_{(E,\iota) \in Ob(\mathcal{D})})$ as the colimit of the functor $\mathcal{F}$, where $\zeta_{(E,\iota)}$ are the canonical maps of the colimit. Note that, in fact, $\zeta_{(E,\iota)} = \iota$ as morphisms, but the indexing is needed as each object (E, ι) has a canonical map associated with it.

Consider the following diagram:



We note that $(\mathscr{C}(V), \Delta_{\mathscr{C}(V)}, \epsilon_{\mathscr{C}(V)}), (\sigma_{(E, f\circ\zeta_{(E,\iota)})})_{(E,\alpha)\in Ob(\mathcal{E})})$ is a cocone on $\mathcal{F}$, and since the object $((C, \Delta, \epsilon), (\zeta_{(E,\iota)})_{(E,\iota)\in Ob(\mathcal{D})})$ is the colimit of $\mathcal{F}$, there exists a unique morphism $\bar{f} : C \to \mathscr{C}(V)$ in $Crg_{\mathbb{A}}$ such that $\bar{f} \circ \zeta_{(E,\iota)} = \sigma_{(E, f\circ\zeta_{(E,\iota)}))} = \sigma_{(E, f\circ\iota)}$, $\forall (E, \iota) \in Ob(\mathcal{D})$. Note that $p \circ \sigma_{(E,\alpha)} = \alpha$, so for $\alpha = f \circ \iota$ we get $p \circ \sigma_{(E,f\circ\iota)} = f \circ \iota = f \circ \zeta_{(E,\iota)}$. Hence, $p \circ \bar{f} \circ \zeta_{(E,\iota)} = p \circ \sigma_{(E,\iota)} = f \circ \zeta_{(E,\iota)}$ for all $(E, \iota) \in Ob(\mathcal{D})$, and so $p \circ \bar{f} = f$. It only remains to show that $\bar{f}$ is unique such that $p \circ \bar{f} = f$. To show this, let $g : C \to \mathscr{C}(V)$ in $Crg_{\mathbb{A}}$ be another morphism of $\mathbb{A}$-bimodules such that $p \circ g = f$. Since $Crg_{\mathbb{A}}$ is a cocomplete category, by Corollary 3.18, the comma category $\mathcal{U} \downarrow V$ is cocomplete. In particular, each pair of parallel arrows in the comma category $\mathcal{U} \downarrow V$ has a coequalizer. We note that $\bar{f}$ and g can be identified as morphisms $\bar{f}, g : (C, p\bar{f} = pg) \to (\mathscr{C}(V), p)$ in $\mathcal{U} \downarrow V$.

Let $u : (\mathscr{C}(V), p) \to (H, p')$ be the coequalizer of $\bar{f}$ and g in the comma category $\mathcal{U} \downarrow V$. We consider the following commutative diagram:

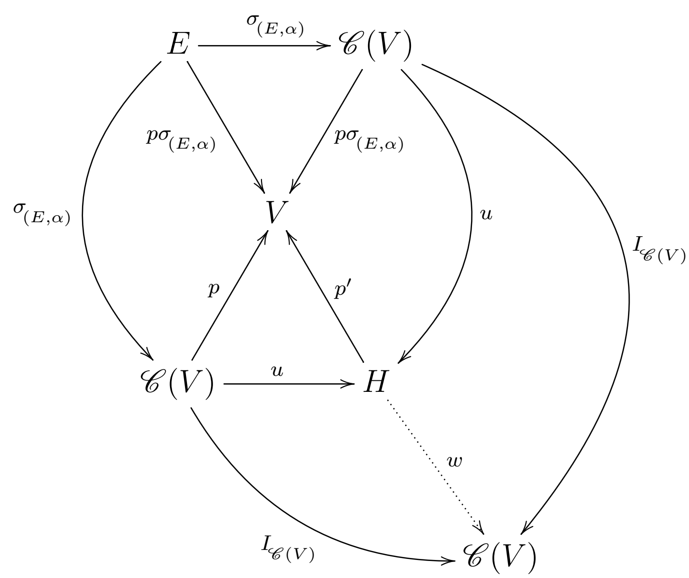


We note that $u : (\mathscr{C}(V), p) \to (H, p')$ can be viewed as a coequalizer of all

$\bar{f} \circ \zeta_{(E,\iota)}; g \circ \zeta_{(E,\iota)}$ in the comma category $\mathcal{U} \downarrow V$.

Furthermore, (H, p') can be viewed as a pushout(colimit) of all $\bar{f} \circ \zeta_{(E,\iota)}; g \circ \zeta_{(E,\iota)}$ in $\mathcal{U} \downarrow V$.

Consider the following commutative diagram:



By the universal property of the pushout (H, p'), we find $w : H \to \mathscr{C}(V)$ in $\mathcal{U} \downarrow V$ such that $w \circ u = id_{\mathscr{C}(V)}$. Hence, u is monomorphism, and we get $\bar{f} \circ \zeta_{(E,\iota)} = g \circ \zeta_{(E,\iota)}$ for all (E, ι), so $\bar{f} = g$, since C is the colimit of the family (E, ι).

Another way to see the uniqueness of f is the following. By the construction of $\mathscr{C}(V)$ we have that $p \circ \sigma_{(E,\alpha)} = \alpha$, and so for $\alpha = g \circ \zeta_{(E,\iota)}$, we get $p \circ \sigma_{(E, p \circ g \circ \zeta_{(E,\iota)})} = g \circ \zeta_{(E,\iota)}$, and since $p \circ g = p \circ \bar{f} = f$, we get $p \circ \sigma_{(E, p \circ \bar{f} \circ \zeta_{(E,\iota)})} = g \circ \zeta_{(E,\iota)}$, and so $g \circ \zeta_{(E,\iota)} = p \circ \sigma_{(E, f \circ \zeta_{(E,\iota)})} = p \circ \sigma_{(E, f \circ \iota)} = f \circ \iota = f \circ \zeta_{(E,\iota)}$. As $g \circ \zeta_{(E,\iota)} = f \circ \zeta_{(E,\iota)}$ for all (E, ι), we get $f = g$.

CHAPTER 4
COFREE COALGEBRAS IN OTHER MONOIDAL CATEGORIES OF INTEREST AND CONSTITUTING EXPLICIT GENERATING SETS FOR THEM

The strategic procedure introduced in the previous chapter can mostly be adapted for other categories of interest. In particular, the existence and construction of cofree objects in categories of modules or comodules over Hopf or bialgebra could similarly be demonstrated.

In order to extend our consequences, however, we need to deal with generating sets for those categories as a critical point. It turns out there is a reasonable need to construct a generating set for each category of our interest. Establishing systems of generators for such kind of important categories could be not only seen as a satisfactory condition to apply the dual of The Special Adjoint Functor Theorem, but it could also be considered as a flexible tool exploited to produce an explicit description for cofree objects in those categories.

4.1 Cofree Coalgebras in Other Monoidal Categories

We begin by noting that the construction of the cofree coalgebra on a vector space over a field $\mathbb{K}$ is recovered and "explained" by the above considerations. If V is a vector space, denote $p : C(V) \to V$ the cofree coalgebra on V. When V is finite dimensional, the usual construction makes use of the self-duality of the category of finite dimensional vector spaces. More precisely, let $T(V^*)$ be the tensor algebra of V^*, and $i : V^* \to T(V^*)$ the canonical map; then $p = i^* : T(V^*)^0 \to (V^*)^* \cong V$

is the cofree coalgebra on V [24, Chapter 1]. Here, $T(V^*)^0$ is the finite dual (or restricted dual) of the algebra $T(V^*)$. We note that this also follows directly from the above considerations. Indeed, for every algebra A, A^0 can be identified with a colimit $A^0 = \varinjlim_I (A/I)^*$ over all cofinite dimensional ideals of A, and the colimit is taken in the category of coalgebras, with $(A/I)^*$ regarded canonically as a coalgebra with dual structure of the finite dimensional algebra A/I.

This gives rise to deal with generating sets as a critical point. Thus, one sees that in order to have a good workable description of the cofree coalgebra, one needs to have a "good" set of generators. We note the following statement is known but perhaps not in this form; it shows that coalgebras in fact have a nice system of generators consisting of comatrix coalgebras. Recall that if n is a positive integer, the coalgebra $M_n^c(\mathbb{K}) = (M_n(\mathbb{K}))^*$ is called the comatrix coalgebra over $\mathbb{K}$.

Proposition 4.1. *The set $(M_n^c(\mathbb{K}))_n$ generates the category $Coalg_{\mathbb{K}}$ of $\mathbb{K}$-coalgebras over a field $\mathbb{K}$.*

Proof. This is a re-statement of the fact that any finite dimensional coalgebra D has a "comatrix basis", i.e. it is generated as a vector space by a set of elements $(c_{ij})_{1\leq i,j\leq n}$ such that $\Delta(c_{ij}) = \sum_k c_{ik} \otimes c_{kj}$, $\varepsilon(c_{ij}) = \delta_{ij}$ (in general, the c_{ij} are not linearly independent).

More precisely, the finite dimensional coalgebras generate $Coalg_{\mathbb{K}}$; if $D \in Coalg_{\mathbb{K}}$ is finite dimensional, then D^* is finite dimensional and so embeds in some $M_n(\mathbb{K})$ (using the left regular representation of D^* for example); hence, there is a surjective coalgebra map $M_n^c(\mathbb{K}) \to D$. □

Over a general ring, the category $Crg_{\mathbb{A}}$ of $\mathbb{A}$-corings is not necessarily generated by corings that are finitely generated as modules or bimodules, which should be a prerequisite if one would expect an analogue of the above corollary to hold. We note that there are some results in this direction. Recall that an $\mathbb{A}$-module M is locally projective if for every epimorphism $p : X \to Y$ of $\mathbb{A}$-modules and every morphism $f : M \to Y$, there are "local extensions" of f i.e. for every finitely generated submodule F of M, there is $g : F \to X$ such that $p \circ g = f|_F$. Given an $\mathbb{A}$-coring C, let C^* and ${}^*C$ denote the right and left duals of C. Then C is a $({}^*C, C^*)$-bimodule. Similar to the coalgebra case, by [16, 19.12], if an $\mathbb{A}$-coring C is locally projective as left or right $\mathbb{A}$-module, then every finite subset of C is contained in a $({}^*C, C^*)$-sub-bimodule of C which is finitely generated as $\mathbb{A}$-sub-bimodule of C. Nevertheless, a $({}^*C, C^*)$-sub-bimodule of C does not necessarily become a subcoring, unless some flatness conditions are further satisfied. By [16, 19.10], if C is left and right locally projective over $\mathbb{A}$, then a $({}^*C, C^*)$-sub-bimodule D of C is a sub-coring if provided $D \subset C$ is a left and right pure submodule of C. Hence, even if C is locally projective both as left and right $\mathbb{A}$-module, one needs extra purity conditions on the $({}^*C, C^*)$-sub-bimodule generated by some finite subset F of C to infer that F is contained in a subcoring D of C which is finitely generated as $\mathbb{A}$-bimodule, and so hope for a nice set of generators for C is strictly tied into this. We note that the "pure closure" constructed in Section 1 to obtain generators of $Crg_{\mathbb{A}}$ is still needed and may make the subcoring generated by F be non-finitely generated. Putting the all above together, we obtain the following:

Theorem 4.2. *Let $\mathbb{A}$ be a von-Neumann regular ring, and C be an $\mathbb{A}$-coring which is left and right locally projective. Then every finite subset of C is contained in a subcoring of C which is finitely generated as left and right $\mathbb{A}$-module, and C is generated by $\mathbb{A}$-corings which are finitely generated as $\mathbb{A}$-bimodules.*

Proof. As above, every finite set F is contained in a $({}^*C, C^*)$-sub-bimodule D of C, which is finitely generated as $\mathbb{A}$-sub-bimodule of C. Since $\mathbb{A}$ is VNR, every module is flat, so D is pure in C, and hence D is a subcoring. □

We also note that the set of generators of $Crg_{\mathbb{A}}$ can be chosen better in certain circumstances.

Theorem 4.3. *Let $\mathbb{A}$ be a countably generated $\mathbb{K}$-algebra over a field $\mathbb{K}$. If C is an $\mathbb{A}$-coring, then for every at most countable dimensional $\mathbb{A}$-sub-bimodule M of C, there is a subcoring D of C such that D is a pure left and right $\mathbb{A}$-submodule of C, $M \subseteq D$ and* $\dim(D) \leq \aleph_0$. *In particular, the $\mathbb{A}$-corings of at most countable dimension generate $Crg_{\mathbb{A}}$.*

Proof. The proof goes along the same lines as Proposition 2.35 and Proposition 2.26. Under the current hypotheses, first note that in Proposition 2.35, if $\dim(N) \leq \aleph_0$, we may choose the submodule N^* such that $\dim(N^*) \leq \aleph_0$. Indeed, one needs to recursively add all solutions to the system of equations 2.1; but at each step, it is enough to consider all such systems with m_{ij}-coefficients in a fixed basis of $N^{(k)}$, and with coefficients $\lambda_{ij} \in A$ coming from from a fixed basis of $\mathbb{A}$. If inductively $\mathbb{A}$ and $N^{(k)}$ are assumed to have an at most countable basis, each time, this produces a new submodule $N^{(k+1)}$ which is at most countable, and so $\dim(N^*) \leq \aleph_0$.

Similarly, in Proposition 2.35 (i), given M with $\dim(M) \leq \aleph_0$, for a basis $(m_i)_{i\geq 1}$ choose representations as finite tensors $\Delta(m_i) = \sum_j m_{ij} \otimes n_{ij}$ and let $\bar{M}$ be the $\mathbb{A}$-sub-bimodule generated by all the m_i, m_{ij}, n_{ij}; it will be at most countable dimensional since $\dim(A) \leq \aleph_0$. Iterating this and taking the countable union will again yield countable dimensional $\mathbb{A}$-submodule.

Finally, the countable iteration in the proof of Proposition 2.35(ii) stays within the class of countable dimensional vector spaces. □

One can easily note that the same proof applies to any $\mathbb{K}$-algebra $\mathbb{A}$ whose dimension is some cardinality γ; in this case, one gets the following version of the fundamental finiteness of corings:

Corollary 4.4. [2, p. 7] If $\mathbb{A}$ is a $\mathbb{K}$ algebra with $\dim(A) = \gamma$ and C is an $\mathbb{A}$-coring, then every subset F of C with $|F| \leq \gamma$ is contained in a subcoring D of C which is a pure left and right $\mathbb{A}$-submodule of C and such that $\dim(D) \leq \max\{\gamma, \aleph_0\}$.

The above applies in particular for algebras which are finitely generated, and shows that in that case the category $Crg_{\mathbb{A}}$ is generated by corings of at most countable dimension; in turn, the construction of the cofree coring on an $\mathbb{A}$-bimodule V will yield $C(V)$ as the colimit $\varinjlim_{\{D \to V\}} D$ over the category of linear maps $D \to V$ from at most countable dimensional corings D to V.

4.2 Cofree Coalgebras in Monoidal Categories of Modules and Comodules Over Bialgebras and Hopf Algebras

As an application of the above ideas, we also look at the case of coalgebras in the category $\mathcal{M}^H$ of right H-comodules and ${}_H\mathcal{M}$ of left H-modules, where H is a bialgebra (or a Hopf algebra). We refer the reader to [24] and [54] for basic Hopf algebra definitions and notations. Let $Coalg(\mathcal{M}^H)$ and $Coalg({}_H\mathcal{M})$ be the category of coalgebras in $\mathcal{M}^H$ and ${}_H\mathcal{M}$, respectively. Recall that from Remark 2.2 $\mathcal{M}^H$ and ${}_H\mathcal{M}$ are monoidal categories.

To fix terminology, we note that coalgebras in $Coalg({}_H\mathcal{M})$ are also called left module coalgebras; C is a left H-module coalgebra if it is a left H-module and a coalgebra such that $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{K}$ are morphisms of H-modules. Let $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ be the Sweedler sigma notation for the comultiplication of Δ, and $\Delta_H : H \to H \otimes H$, $\Delta_H(h) = h_1 \otimes h_2$ with action of H on C denoted by $(h, c) \longmapsto h \cdot c$. The axioms for a left H-module coalgebra translate to

$$\begin{aligned} \sum (h \cdot c)_{(1)} \otimes (h \cdot c)_{(2)} &= \sum h_1 c_{(1)} \otimes h_2 c_{(2)} \\ \varepsilon(h \cdot c) &= \varepsilon_H(h)\varepsilon(c) \end{aligned}$$

Similarly, coalgebras in $Coalg(\mathcal{M}^H)$ are precisely the objects known as right H-comodule coalgebras. In sigma notation, if $\rho_C : C \to C\otimes H$ is the right H-comodule map of C, it means Δ and ε are morphisms of comodules. If $\rho_C(c) = \sum c_0 \otimes c_1$, this translates into sigma equations as

$$\sum c_{(1)0} \otimes c_{(2)0} \otimes c_{(1)1}c_{(2)1} = c_{0(1)} \otimes c_{0(2)} \otimes c_1 \tag{4.1}$$

$$\varepsilon(c_0)c_1 = \varepsilon(c)1 \tag{4.2}$$

First, one can note without difficulty that the coproduct and colimit of coalgebras in both $\mathcal{M}^H$ and ${}_H\mathcal{M}$ exists and is constructed by the coproduct and colimit, respectively, of underlying objects in $\mathcal{M}^H$ and ${}_H\mathcal{M}$; the construction is done similar to the case of corings, and in fact works categorically (see [16, 2.11 and Chapter 17]). Hence, the categories $Coalg(\mathcal{M}^H)$ and $Coalg({}_H\mathcal{M})$ are cocomplete and forgetful functors $F^H : Coalg(\mathcal{M}^H) \longrightarrow \mathcal{M}^H$ and $F_H : Coalg({}_H\mathcal{M}) \longrightarrow {}_H\mathcal{M}$ preserve colimits.

Second, as in the case of corings over rings, we note that the categories $Coalg(\mathcal{M}^H)$ and $Coalg({}_H\mathcal{M})$ are co-wellpowered. Since forgetful functors F^H and F_H preserve colimits, they preserve epimorphisms and this again implies that surjective morphisms are exactly the epimorphisms in $Coalg(\mathcal{M}^H)$ and $Coalg({}_H\mathcal{M})$. Then the proof of Proposition 3.6 applies mutatis mutandis. To summarize, we have

Proposition 4.5. *[2, p. 7] The categories of coalgebras $Coalg(\mathcal{M}^H)$ and $Coalg({}_H\mathcal{M})$ are cocomplete, co-wellpowered, and the forgetful functors $F^H : Coalg(\mathcal{M}^H) \longrightarrow \mathcal{M}^H$ and $F_H : Coalg({}_H\mathcal{M}) \longrightarrow {}_H\mathcal{M}$ are cocontinuous.*

To show the two categories above have generators, we will have somewhat different situations.

Module coalgebras

Proposition 4.6. *[2, p. 7] Let C be a left H-module coalgebra, and F a finite subset of C. Then F is contained in a submodule subcoalgebra D of C (i.e. D is a H-submodule and subcoalgebra of C) which is finitely generated as a left H-module .*

Proof. Let $E \subseteq C$ be the subcoalgebra generated by F; it is finite dimensional by the fundamental theorem of coalgebras. Let $D = HE$ be the left H-submodule of C generated by E. Write $\Delta_H(h) = \sum h_1 \otimes h_2$ and $\Delta_C(c) = \sum c_{(1)} \otimes c_{(2)}$. If $c \in E, h \in H$, then $\sum (h \cdot c)_{(1)} \otimes (h \cdot c)_{(2)} = \sum h_1 c_{(1)} \otimes h_2 c_{(2)} \in H \cdot E \otimes H \cdot E$, so $H \cdot E$ is a H-submodule subcoalgebra of C. □

Corollary 4.7. [2, p. 8] The left H-module coalgebras $f.g.CoAlg({}_H\mathcal{M})$ which are finitely generated as left H-modules form a system of generators for $CoAlg({}_H\mathcal{M})$.

Comodule coalgebras

We can prove a stronger statement for this case. From Remark 2.12, the finite dual of A is a coalgebra A^0 which consists of all $f \in A^*$ for which there are finite families g_i, h_i, $i = 1, \ldots, n$ (n depends on f) such that $f(ab) = \sum_{i=1}^{n} g_i(a)h_i(b)$ for all $a, b \in A$, equivalently, A^0 contains all $f \in A^*$ whose kernel contains an ideal of A of finite codimension. This is also called the coalgebra of representative functions, and is spanned by coefficients of all finite dimensional representations of A. From Remark 2.4, the usual dual of an algebra need not be a coalgebra. Furthermore, in general $A^0 \neq A^*$ unless A is finite dimensional. However, every $f \in A^*$ is "*locally representative*" in the following sense:

Lemma 4.8. *[2, p. 8] Let A be a $\mathbb{K}$-algebra and $f \in A^*$; then for every finite dimensional $V \subseteq A$, there are families g_i, h_i, $i = 1, \ldots, n$ for which $f(ab) = \sum_{i=1}^{n} g_i(a)h_i(b)$ for all $a, b \in V$.*

Proof. This is standard linear algebra and follows easily from the fact that $(V \otimes V)^* \cong V^* \otimes V^*$ when V is finite dimensional, which means that every $\varphi : V \otimes V \to \mathbb{K}$ is of the form $\varphi(a \otimes b) = \sum_i g_i(a)h_i(b)$; then if $i : V \hookrightarrow A$ is the inclusion and $m : A \otimes A \to A$ the multiplication, the induced dual map $A^* \xrightarrow{m^*} (A \otimes A)^* \xrightarrow{i \otimes i} (V \otimes V)^* \cong V^* \otimes V^*$ shows that $m^*(f)$ restricted to $V \otimes V$ is of the desired form as an element coming from $V^* \otimes V^*$. □

The following finiteness theorem may be known, but we were unable to locate a reference. Its proof is straightforward, and we sketch it for completeness. We recall that if C is a coalgebra and $(M, \rho : M \to M \otimes C)$ is a finite dimensional right C-comodule, then there is a finite dimensional subcoalgebra D of C such that $\rho(M) \subseteq M \otimes D$. The smallest such coalgebra $cf_C(M)$ is unique and is called the coalgebra of coefficients of M (see [24, Chapter 1]).

Theorem 4.9 (Finiteness theorem for comodule coalgebras)**.** *[2, p. 8] Let H be a bialgebra and C be a right H-comodule coalgebra. Then every finite subset F of C is contained in a finite dimensional subcomodule subcoalgebra D of C.*

Proof. This can be proved by applying the finiteness theorem of comodules over coalgebras, using the crossed coproduct coalgebra $H \ltimes C$ (see [46]) and regarding an H-subcomodule subcoalgebra D of C as a subcomodule of C over the coalgebra

$(H \ltimes C) \otimes C^{cop}$. However, a direct proof can be done as well. Let E be a finite dimensional subcoalgebra of C with $F \subseteq E$. Then $D = H^* \cdot E$ is the right H-subcomodule of C generated by E and $\dim(D) < \infty$. Since C is a right H-comodule coalgebra, by Remark 2.36, we have $c_{(0)1} \otimes c_{(0)2} \otimes c_{(1)} = c_{(1)0} \otimes c_{(2)0} \otimes c_{(1)1}c_{(2)1}$ for all $c \in C$. In particular, we have $d_{(0)1} \otimes d_{(0)2} \otimes d_{(1)} = d_{(1)0} \otimes d_{(2)0} \otimes d_{(1)1}c_{(2)1}$ for all $d \in D$. Taking the composition

$$C \otimes C \otimes H \xrightarrow{id_C \otimes id_C \otimes \alpha} C \otimes C \otimes \mathbb{K} \xrightarrow[\sim]{\theta} C \otimes C$$

for both sides, we obtain

$$\alpha(d_1)d_{0(1)} \otimes d_{0(2)} = \alpha(d_{(1)1}d_{(2)1})d_{(1)0} \otimes d_{(2)0} \ , \tag{4.3}$$

where $\theta : C \otimes C \otimes \mathbb{K} \to C \otimes C, c \otimes c' \otimes a \mapsto ac \otimes c'$ is the canonical isomorphism.

It is enough to show that D is a subcoalgebra of C. By the proof of Theorem 2.21, we note that D has a left H^*-module structure given by $\alpha.d = \alpha(d_1)d_0$ for $d \in E$. Let $V = cf_H(D) \subseteq H$ (H-coefficients of D), so $\dim(V) < \infty$ as $\dim(D) < \infty$. For $\alpha \in H^*$, let β_i, γ_i, $i = 1, \ldots, n \in H^*$ be such that $\alpha(ab) = \sum_{i=1}^{n} \beta_i(a)\gamma_i(b)$ for $a, b \in V$. Denote $\rho_C(c) = c_0 \otimes c_1$ the H-comodule structure of C and $\Delta_H(h) = h_1 \otimes h_2$, $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$ as before with omitted summation symbol. Then for $d \in E$

$$\begin{aligned}\Delta_C(\alpha \cdot d) &= (\alpha \cdot d)_{(1)} \otimes (\alpha \cdot d)_{(2)} \\ &= (\alpha(d_1)d_0)_{(1)} \otimes (\alpha(d_1)d_0)_{(2)} \\ &= \alpha(d_1)d_{0(1)} \otimes d_{0(2)} \text{ (since } \Delta_C \text{ is bilinear)}\end{aligned}$$

$$= \alpha(d_{(1)1}d_{(2)1})d_{(1)0} \otimes d_{(2)0} \text{ (by 4.3)}$$

$$= \sum_i \beta_i(d_{(1)1})\gamma_i(d_{(2)1})d_{(1)0} \otimes d_{(2)0} \text{ (since the } d_{(1)1}, d_{(1)2}\text{'s are in } V = cf_H(D))$$

$$= \sum_i \beta_i \cdot d_{(1)} \otimes \gamma_i \cdot d_{(2)} \in H^* \cdot D \otimes H^* \cdot D$$

□

This immediately yields

Corollary 4.10. [2, p. 9] The category $Coalg(\mathcal{M}^H)$ (=the category of right H-comodule coalgebras) is generated by objects which are finite dimensional. The cofree coalgebra over a right H-comodule V (that is, the cofree comodule coalgebra over H) is the colimit of the functor

$\{f | f : D \to V, D$ is a finite dimensional comodule coalgebra, $f,\ V \in \mathcal{M}^H\} \to \mathcal{M}^H$.

In the particular case of Hopf algebras, we can see that the category $Coalg(\mathcal{M}^H)$ has a special easier to describe set of generators. Let H be a Hopf algebra with antipode S, and let V be a finite dimensional right H-comodule. Before starting a couple of important theorems about systems of cogenerators for comodule algebras and generators for comodule coalgebras, we need the following remarks.

Remark 4.11.

1. Following [61], the evaluation map ev_V and the coevaluation map $coev_V$ which are associated with V^* can be defined as follows:

$$ev_V : V^* \otimes V \to \mathbb{K}, v^* \otimes v \mapsto v^*(v) \tag{4.4}$$

$$coev_V : \mathbb{K} \to V \otimes V^*, 1_{\mathbb{K}} \mapsto \sum_i v_i \otimes v_i^* \tag{4.5}$$

where $(v_i)_i$ is a finite basis for V, and $(v_i^*)_i$ is its dual basis for V^*, and $ev_V : V^* \otimes V \to \mathbb{K}$ defined by $ev_V(v^* \otimes v) = v^*(v)$ are called co-evaluation and evaluation respectively. They are morphisms of right H-comodules; we refer to the monograph [42, Chapter 9] and [9, Chapter 2] for more examples and characterization of right and left duals in monoidal categories.

2. Since V is a left rational H^*-module, where H^* is the dual (convolution) algebra of the coalgebra H, V^* has a structure of a right H^*-module which is rational, so it becomes a left H-comodule. Using the antimorphism of coalgebras $S : H \to H$, V^* has a right H-comodule structure (V^*, ρ_0) given by $\rho_0(\alpha) = \alpha_{(0)} \otimes S(\alpha_{(-1)})$. Since $\mathcal{M}^H$ is a monoidal category by Remark 2.2, we have $V \otimes V^*, V^* \otimes V \in \mathcal{M}^H$. We refer the reader to [54] for explicit comodule structures for V^* and $V^* \otimes V$.

3. It is known that the right H-comodule $V \otimes V^*$ becomes an algebra in the category of right H-comodules by taking the composition

$$(V \otimes V^*) \otimes (V \otimes V^*) \underset{\cong}{\xrightarrow{\beta}} V \otimes (V^* \otimes V) \otimes V^* \xrightarrow{(id_V \otimes ev_V \otimes id_{V^*})} V \otimes \mathbb{K} \otimes V^* \underset{\cong}{\xrightarrow{\alpha_{V^*}}} (V \otimes V^*) \tag{4.6}$$

as multiplication map, and

$$coev_V : \mathbb{K} \to V \otimes V^* \tag{4.7}$$

as unit, where

$$\beta = a_{V,V^*,V \otimes V^*} \circ a^{-1}_{V \otimes V^*, V, V^*} : (V \otimes V^*) \otimes (V \otimes V^*) \xrightarrow{\cong} V \otimes (V^* \otimes V) \otimes V^* \tag{4.8}$$

and

$$\alpha_{V^*} : \mathbb{K} \otimes V^* \xrightarrow{\cong} V^*, 1_{\mathbb{K}} \otimes v_i^* \mapsto 1_{\mathbb{K}}.v_i^* \tag{4.9}$$

Similarly, the right H-comodule $V^* \otimes V$ becomes a coalgebra in $\mathcal{M}^H$ with comultiplication given by the composition

$$V^* \otimes V \xrightarrow[\cong]{(id_V^* \otimes \alpha'_V)} V^* \otimes \mathbb{K} \otimes V \xrightarrow{(id_V^* \otimes coev_V \otimes id_V)} V^* \otimes (V \otimes V^*) \otimes V \xrightarrow[\cong]{\beta'} (V^* \otimes V) \otimes (V^* \otimes V) \tag{4.10}$$

and counit

$$ev_V : V^* \otimes V \to \mathbb{K} \tag{4.11}$$

where

$$\alpha'_V = \alpha_V^{-1} : V \xrightarrow{\cong} V \otimes \mathbb{K}, v_i \mapsto 1_{\mathbb{K}} \otimes v_i \tag{4.12}$$

and

$$\beta' = a_{V^* \otimes V, V^*, V} \circ a^{-1}_{V^*, V, V^* \otimes V} : V^* \otimes (V \otimes V^*) \otimes V \xrightarrow{\cong} (V^* \otimes V) \otimes (V^* \otimes V) \tag{4.13}$$

We note that $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is simply the associativity constraint in the monoidal category $(Vec_{\mathbb{K}}, \otimes, \mathbb{K})$, where $\otimes$ here is the tensor product over the field $\mathbb{K}$.

⊞

Following Remark 4.11, we have the following theorems:

Theorem 4.12. *[2, p. 9] The finite dimensional algebras of the form $V \otimes V^*$ for finite dimensional H-comodules V, form a system of cogenerators in the category $fdAlg(\mathcal{M}^H)$ of finite dimensional algebras in $\mathcal{M}^H$ (and also in $Alg(\mathcal{M}^H)$).*

Proof. From Remark 2.2, the category $\mathcal{M}^H$ can be identified as a monoidal category. If U, V, W are finite-dimensional H-comodules, then from Example 1.15, we have U, V, W are rigid objects in $\mathcal{M}^H$. By Proposition 1.54, equation 1.1, we have

$$Hom_{\mathcal{M}^H}(U \otimes V, W) \xrightarrow{\sim} Hom_{\mathcal{M}^H}(U, W \otimes V^*) \tag{4.14}$$

In particular, if $U = V = W$, we have

$$Hom_{\mathcal{M}^H}(V \otimes V, V) \xrightarrow{\sim} Hom_{\mathcal{M}^H}(V, V \otimes V^*) \tag{4.15}$$

If A is finite dimensional algebra in $\mathcal{M}^H$, then from the canonical isomorphism (4.15), we have

$$Hom_{\mathcal{M}^H}(A \otimes A, A) \xrightarrow{\sim} Hom_{\mathcal{M}^H}(A, A \otimes A^*) \tag{4.16}$$

If $(v_i)_{i=1,\ldots,n}$ is a basis for A and $(v_i^*)_{i=1,\ldots,n}$ is its dual basis for A^*, then this isomorphism takes $m : A \otimes A \to A$ to $\varphi(m) = (a \longmapsto \sum_i av_i \otimes v_i^*)$. Of course, as a $\mathbb{K}$-algebra, $A \otimes A^*$ is simply isomorphic to $M_n(\mathbb{K})$; hence, if we forget the H-comodule structures, if $m : A \otimes A \to A$ is the multiplication of A, then $\varphi(m) : A \to A \otimes A^*$ is a morphism of algebras corresponding to the left regular representation of A. By the canonical isomorphism (4.16), $\varphi(m)$ is a morphism in $\mathcal{M}^H$ if m is so, and therefore, for a H-comodule algebra A, $\varphi(m) : A \to A \otimes A^*$ is a morphism of H-comodule algebras. Also, $\varphi(m)$ is obviously injective, so it is a monomorphism in $Alg(\mathcal{M}^H)$.

Now let $f, g : X \to Y$ be any pair of distinct morphisms in $Alg(\mathcal{M}^H)$. Since the category $Alg(\mathcal{M}^H)$ is generated by objects which are finite dimensional, there exists a morphism $\zeta : Y \to Z$ with Z a finite dimensional comodule algebra such that $\zeta f \neq \zeta g$. Let m_Z be the corresponding multiplication for the algebra Z. Since

$\varphi(m_Z) : Z \to Z \otimes Z^*$ is a monomorphism in $Alg(\mathcal{M}^H)$ and $\zeta f \neq \zeta g$, we have $\varphi(m_Z)\zeta f \neq \varphi(m_Z)\zeta g$. □

Theorem 4.13. *[2, p. 9] The finite dimensional coalgebras of the form $V^* \otimes V$ for finite dimensional H-comodules V form a system of generators of $CoAlg(\mathcal{M}^H)$ (= the category of H-comodule coalgebras).*

Proof. By Proposition 1.54, equation 1.2, we have

$$Hom_{\mathcal{M}^H}(V^* \otimes U, W) \xrightarrow{\sim} Hom_{\mathcal{M}^H}(U, V \otimes W) \tag{4.17}$$

In particular, if $U = V = W$, we have

$$Hom_{\mathcal{M}^H}(V^* \otimes V, V) \xrightarrow{\sim} Hom_{\mathcal{M}^H}(V, V \otimes V) \tag{4.18}$$

If (C, Δ, ϵ) is finite dimensional coalgebra in $\mathcal{M}^H$, then from the canonical isomorphism (4.18), we have

$$Hom_{\mathcal{M}^H}(C^* \otimes C, C) \xrightarrow{\sim} Hom_{\mathcal{M}^H}(C, C \otimes C) \tag{4.19}$$

If $(w_i)_{i=1,\ldots,n}$ is a basis for C and $(w_i^*)_{i=1,\ldots,n}$ is its dual basis for C^*, then this isomorphism takes $\Delta : C \to C \otimes C$ to $\varphi(m) = (a \longmapsto \sum_i av_i \otimes v_i^*)$. Thus, if we forget the H-comodule structures, if $\Delta : C \to C \otimes C$ is the comultiplication of C, then $\psi(\Delta) : C^* \otimes C \to C$ is a morphism of coalgebras given by the composition

$$C^* \otimes C \xrightarrow{(id_{C^*} \otimes \Delta)} C^* \otimes C \otimes C \xrightarrow{(ev_C \otimes id_{C^*})} \mathbb{K} \otimes C \underset{\cong}{\xrightarrow{\alpha_C}} C \tag{4.20}$$

Explicitly, if we write $\Delta_C(w_i) = w_{i(1)} \otimes w_{i(2)}$, then $\psi(\Delta)$ is given by

$$\psi(\Delta) : C^* \otimes C \to C, w_i^* \otimes w_i \mapsto w_i^*(w_{i(1)})w_{i(2)} \tag{4.21}$$

By the canonical isomorphism (4.19), $\psi(\Delta)$ is a morphism in $\mathcal{M}^H$ if Δ is so, and therefore, for a H-comodule coalgebra C, $\psi(\Delta) : C^* \otimes C \to C$ is a morphism of H-comodule coalgebras. Furthermore, $\psi(\Delta)$ is obviously surjective, so it is a epimorphism in $Coalg(\mathcal{M}^H)$.

Now let $f, g : X \to Y$ be any pair of distinct morphisms in $CoAlg(\mathcal{M}^H)$. Since the category $CoAlg(\mathcal{M}^H)$ is generated by objects which are finite dimensional, there exists a morphism $\xi : G \to X$ with G a finite dimensional comodule coalgebra such that $f\xi \neq g\xi$. Let Δ_G be the corresponding comultiplication for the coalgebra G. Since $\psi(\Delta_G) : G^* \otimes G \to G$ is a epimorphism in $CoAlg(\mathcal{M}^H)$ and $f\xi \neq g\xi$, we have $f\xi\psi(\Delta_G) \neq g\xi\psi(\Delta_G)$. □

It might be notable that in the case of $\mathcal{M}^H$ for a Hopf algebra H, besides using the above systems of generators for the description of the cofree coalgebra on an H-comodule V, one can also do a construction similar to that of vector spaces. Namely, for an algebra A one can define a right finite dual A^0 of A. If the antipode of H is bijective, one can also define a left finite dual ${}^0A$ using left duals. If V is a finite dimensional right H-comodule, then the tensor algebra $T({}^*V)$ in the category $\mathcal{M}^H$ has a right finite dual $T({}^*V)^0$ which is a coalgebra in $\mathcal{M}^H$, and it is the cofree coalgebra on V. This can be extended to arbitrary V's as in the case of coalgebras over fields. We leave the details of this construction to the reader.

Using 3.8, 4.7 and 4.13, we have the following substantial consequences.

Corollary 4.14. Let H be a bialgebra over a field $\mathbb{K}$. Consider the forgetful functors

$$_HF : CoAlg(_H\mathcal{M}) \longrightarrow {}_H\mathcal{M},\ F_H : CoAlg(\mathcal{M}_H) \longrightarrow {}_H\mathcal{M},$$

$$^HF : CoAlg(^H\mathcal{M}) \longrightarrow {}^H\mathcal{M},\ F^H : CoAlg(\mathcal{M}^H) \longrightarrow \mathcal{M}^H$$

Then each of the functors $_HF$, F_H, ^HF, and F^H has a right adjoint.

We also ask whether a cofree coalgebra can be constructed in general monoidal categories, and even in the case when the category is not abelian. The conditions below seem natural to ask, perhaps along other additional ones, and the answer to the following question, seems to only depend on the ability of constructing generators for the category of coalgebras in $\mathcal{C}$.

CHAPTER 5
ESTABLISHING A CONSTRUCTION FOR EXPLICIT DESCRIPTION OF COFREE OBJECTS AND LIMITS IN $CRG_{\mathbb{A}}$ AND SOME ABELIAN MONOIDAL CATEGORIES OF INTEREST

In this chapter, we use the proof of the dual of The Special Adjoint Functor Theorem as an appropriate technique to show that, under certain conditions, monoidal categories get forced to be complete, and it gives an explicit construction for limits in such kind of categories in terms of cofree objects. In particular, this mechanism efficiently shows that the category of corings is complete and gives an accurate establishment for their limits.

Furthermore, it simultaneously allows us to exhibit a more precise picture for cofree objects in each monoidal category of interest in terms of colimits. This could be enormously important to have a semantic interpretation for the behavior of monidal categories of interest under the constraints of their comlits and limits. It also determines how their colimits and limits play an extremely crucial role in shaping their cofree objects.

5.1 Construction Inspired by [SAFT]

Following [2], we recall the construction used in the Special Adjoint Functor Theorem (or SAFT) due to P. Freyd, and use it to provide formulas that will be applied in several situations. A functor U that has a right adjoint commutes with limits, and the SAFT says that under additional natural hypothesis, this is enough to find such an adjoint. Recall that a category $\mathcal{C}$ is called co-wellpowered if for every object C in $\mathcal{C}$, the equivalence classes of epimorphisms with source C form set. The (dual) SAFT asserts that *if $\mathcal{C}$ is a cocomplete (has small colimits), co-wellpowered category and with a generating set, then every cocontinuous functor U (i.e. U commutes with colimits) from $\mathcal{C}$ to a locally small category $\mathcal{A}$ has a right adjoint.*

We also briefly recall the construction of the adjoint, since it will be of importance in our examples. It can be formulated in the following way. Let $\mathcal{G}$ be a set of generators of $\mathcal{C}$, and for each M in $\mathcal{A}$, let $\overline{\mathcal{E}}$ be the category whose objects are pairs (C, f) with C in $\mathcal{C}$, and $f : U(C) \to M$ is a morphism in $\mathcal{A}$. Morphisms in this category are $(\varphi, *) : (C, f) \to (C', f')$ such that $\varphi : C \to C'$ is a morphism in $\mathcal{C}$, and $*$ means simply that the condition $f'U(\phi) = f$ is satisfied. Consider also $\mathcal{E}$ the full subcategory of $\overline{\mathcal{E}}$ of objects of the type (G, f) for $G \in \mathcal{G}$. Let $F : \mathcal{E} \to \mathcal{C}$ be defined by $F(G, f) = G$ and $F(\varphi, *) = \varphi$ on morphisms. Then F is a functor, and since the class of objects of $\mathcal{E}$ is a set (since $\mathcal{G}$ is a set and $\mathcal{A}$ is a locally small category), it has a colimit. Let $(R(M), f_M) = \varinjlim_{(G,f)\in\mathcal{E}} F(G) = \varinjlim F$. Then $(R(M), f_M)$ is a final object in $\mathcal{E}$, which in turn helps proving that it is a final object in $\overline{\mathcal{E}}$. We need to recall this argument, as it will be used later. Given C in $\mathcal{C}$ together with a morphism $f : U(C) \to M$ in $\mathcal{A}$, use the generators $\mathcal{G}$ to write $C = \varinjlim_{p:G\to C, G\in\mathcal{G}} G$ and $(C, f) = \varinjlim_{p:G\to C} (G, fU(p))$. Since $(R(M), f_M) = \varinjlim_{h:U(G)\to M} (G, h)$, and there is a canonical map between the two inductive systems induced by the correspondence $(G, p) \longmapsto (G, fU(p))$,we get a canonical morphism $\psi_C : C \to R(M)$ in $\mathcal{C}$ which turns out unique with $f_M\psi_C = f$.

We will need the following short hand formulas to explicitly compute several functors.

$$R(M) = \varinjlim_{h:U(G)\to M|G\in\mathcal{G}} G \tag{5.1}$$

$$(R(M), f_M) = \varinjlim_{h:U(G)\to M|G\in\mathcal{G}} (G, h : U(G) \to M) \tag{5.2}$$

We will also need the following observation, which is similar to the above discussion. Let $\mathcal{A}, \mathcal{C}$ be categories, $U : \mathcal{C} \to \mathcal{A}$ be a faithful functor. For convenience, we will interpret U as a "forgetful" functor and slightly abuse notation sometimes to write $U(C) = C$; similarly, if $f : U(C) \to U(C')$ is a morphism in $\mathcal{A}$, we say f is a morphism in $\mathcal{C}$ if $f = U(g)$ for some (unique) $g \in \mathrm{Hom}_{\mathcal{C}}(C, C')$. Fix an object N in $\mathcal{A}$ and objects $(C_i)_{i\in I}$ in $\mathcal{C}$ and a family of morphisms $f_i : N \to U(C_i)$ in $\mathcal{A}$; the set I can be empty. Assume $\mathcal{C}$ and $\mathcal{A}$ each have an initial object denoted 0 and $U(0) = 0$. Let $\mathcal{H}$ be the comma category whose objects are $(E, p), E \in \mathcal{C}, p \in \mathrm{Hom}_{\mathcal{A}}(U(E), N)$ and such that $f_i p$ is a morphism in $\mathcal{C}$ for all i; the morphisms $(E, p) \to (E', p')$ of this category are given by $h \in \mathrm{Hom}_{\mathcal{C}}(E, E')$ with $p'h = p$ (i.e. $p'U(h) = p$). We show that this category $\mathcal{H}$ has a final object under the same conditions as in SAFT; when $I = \emptyset$, this is exactly the construction in SAFT.

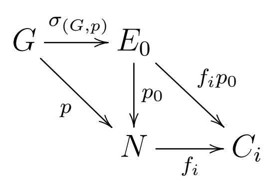
Proposition 5.1. *[2, p. 3] If $\mathcal{C}$ is cocomplete, co-wellpowered and has a set $\mathcal{G}$ of generators, and $\mathcal{A}, \mathcal{C}$ have the same initial object 0. Then the category $\mathcal{H}$ has a final object, which we denote by $(\mathcal{H}_0(N, (f_i)_i), p_0)$.*

Proof. Let $\mathcal{H}_0$ be the full subcategory of $\mathcal{H}$ whose objects (G, p) are such that $G \in \mathcal{G}$.

Note that $\mathcal{H}_0$ has at least one object $0 \in \mathcal{C}$, since 0 is also the initial object in $\mathcal{A}$, so the compositions $0 \to N \overset{f_i}{\to} C_i$ are morphisms in $\mathcal{C}$. Let

$$E_0 = \varinjlim_{(G,p)\in\mathcal{H}_0} G \tag{5.3}$$

be a colimit in $\mathcal{C}$, which is also a colimit in $\mathcal{A}$ (i.e. $U(E_0) = \varinjlim U(G)$), and let $p_0 : E_0 \to N$ be the canonical morphism in $\mathcal{A}$ obtained from this colimit. Let $\sigma_{(G,p)} : G \to E_0$ be the canonical morphisms (in $\mathcal{C}$) of the colimit, so $p_0\sigma_{(G,p)} = p$ (more precisely, $p_0U(\sigma_{(G,p)}) = p$).



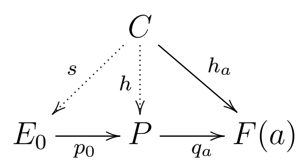
We show that p_0 is a morphism in $\mathcal{H}$. Fix i, and consider the morphisms $f_ip : G \to C_i$ which are morphisms in $\mathcal{C}$ (since (G,p) is in $\mathcal{H}$; that is, $f_ip = U(h_i)$); there is a unique morphism in $\theta : E_0 \to C_i$ in $\mathcal{C}$ such that $\theta\sigma_{(G,p)} = f_ip$ for all (G,p). But f_ip_0 (a morphism in $\mathcal{A}$) already satisfies this equation, and since the colimit in 5.3 is also a colimit in $\mathcal{A}$ by hypothesis, we obtain that $\theta = f_ip_0$ (obviously, here we implicitly use that U is faithful). Hence, f_ip_0 is a morphism in $\mathcal{C}$. This shows that p_0 is in $Mor(\mathcal{H})$. The fact that (E_0, p_0) is the final object in the category $\mathcal{H}$ follows as in the proof of SAFT, along the lines noted in the comments preceding this proposition. □

5.2 Explicit Description of Cofree Objects and Limits in The Categories of Interest

Remark 5.2. [2, p. 4] We will use the above notation $(\mathcal{H}_0(N; (f_i)_i), p_0)$ and Proposition 5.1 to construct limits. ⊞

Proposition 5.3. *[2, p. 4] Let $(\mathcal{C}, U)$ be a concrete category over $\mathcal{A}$ such that $\mathcal{A}$ and $\mathcal{C}$ are cocomplete categories, and $U : \mathcal{C} \to \mathcal{A}$ is cocontinuous. Assume $\mathcal{A}$ is also complete, $\mathcal{C}$ is co-wellpowered, has a set of generators $\mathcal{G}$ and $\mathcal{C}$ and $\mathcal{A}$ have initial objects 0 with $U(0) = 0$. Then $\mathcal{C}$ is complete*

Proof. Let $\mathscr{D}$ be a small category, and let $F : \mathscr{D} \to \mathcal{C}$ be a functor. Let $P = \varprojlim UF$ be the limit (in $\mathcal{A}$) of UF, with canonical morphisms $q_a : P \to UF(a)$ in $\mathcal{A}$, $a \in \mathcal{A}$, i.e. we may write $q_a : P \to F(a)$ are morphisms in $\mathcal{A}$. Let $(E_0, p_0) = (\mathcal{H}_0(P, (q_a)_a), p_0)$, $p_0 : E_0 \to P$ in $\mathcal{A}$ (more precisely $p_0 : U(E_0) \to P$). We note that $(E_0, (\pi_a)_a)$ with $\pi_a = q_a p_0$ is the limit of F. Of course, π_a are also morphisms in $\mathcal{C}$ by construction. To see E_0 is this limit, consider $h_a : C \to F(a)$ a family of morphisms in $\mathcal{C}$ which is a cone on F (so if $x : a \to b$ then $h_b F(x) = h_a$).



Then $U(h_a) \in Mor(\mathcal{A})$ and $U(h_a)$ is a cone on UF, and by applying the limit property we get a unique morphism h in $\mathcal{A}$ with $q_a h = h_a$. Since $q_a h = h_a$ are in fact morphism in $\mathcal{C}$ (images of U), there is a unique s in $\mathcal{C}$ with $p_0 s = h$, so $\pi_a s = q_a h = h_a$. The uniqueness of s with this property follows again easily by backward chase using the universal properties of E_0 and P. □

Remark 5.4. [2, p. 4] We note that a similar process is used in [4] for describing products of coalgebras over a field, but with an extra-intermediate step: the canonical map $p_0 : E_0 \to P$ is obtained using the right adjoint R of U and the canonical map

$c_P : R(P) \to P$; of course, our p_0 factors through $R(P)$, but this intermediate step is not necessary in general. ⊞

Corollary 5.5. The category $Crg_{\mathbb{A}}$ is complete. Moreover, limits in $Crg_{\mathbb{A}}$ could be explicitly given.

Proof. Since the category $Crg_{\mathbb{A}}$ can be identified as the concrete category $(Crg_{\mathbb{A}}, \mathcal{U})$ over ${}_{\mathbb{A}}\mathcal{M}_{\mathbb{A}}$. To construct limits in $Crg_{\mathbb{A}}$ explicitly, let $F : \mathcal{A} \to Crg_{\mathbb{A}}$ be a functor from a small category, and let $(P, q_a : P \to C_a) = \varprojlim_a C_a = \varprojlim UF$ be the limit of $\mathbb{A}$-bimodules. The limit of F is

$$\mathscr{C}(P)_0 = \varinjlim_{[h:H \to \mathscr{C}(P)] \mid h,\, q_a h \in Crg_{\mathbb{A}}} H$$

with canonical maps $(q_a p_0)_a$, where $(\mathscr{C}(P)_0, p_0) = (\mathcal{H}_0(P, (q_a)_a), p_0)$ is the construction of 5.2. □

Corollary 5.6. The cofree coring $\mathscr{C}(V)$ on every $\mathbb{A}$-bimodule V can be explicitly given by

$$\mathscr{C}(V) = \varinjlim_{f:U(G) \to V \mid G \in Crg_{\mathbb{A}};\, |G| \leq \{|\mathbb{A}|, \aleph_0\}} G$$

Proof. It is an immediate application for Proposition 3.5 and for the equations 5.1 and 5.2.

□

Remark 5.7. We note that Corollary 5.5 is also the main result of [49], and Corollary 5.6 was raised in [4]; our emphasis is to answer these questions in a constructive way, and to explicitly produce the cofree coring and limit of corings. ⊞

Remark 5.8. [2] Let H be a bialgebra over a field $\mathbb{K}$. Consider the forgetful functors

$$ {}_HF : CoAlg({}_H\mathcal{M}) \longrightarrow {}_H\mathcal{M},\ F_H : CoAlg(\mathcal{M}_H) \longrightarrow {}_H\mathcal{M}, $$
$$ {}^HF : CoAlg({}^H\mathcal{M}) \longrightarrow {}^H\mathcal{M},\ F^H : CoAlg(\mathcal{M}^H) \longrightarrow \mathcal{M}^H $$

We know that ${}_HF$, F_H, ${}^HF$, and F^H have right adjoints, say ${}_HG$, G_H, ${}^HG$, and G^H, respectively. From the Constructions 5.1 and 5.3, cofree coalgebras in $CoAlg({}_H\mathcal{M})$, $CoAlg({}_H\mathcal{M})$, $CoAlg({}^H\mathcal{M})$ and $CoAlg(\mathcal{M}^H)$ could be precisely given in terms of colimits as follows

$$ G_H(V) = \varinjlim_{[f:D\to V]\in\mathcal{M}_H,\, D\in f.g.CoAlg(\mathcal{M}_H)} D $$

,

$$ {}_HG(V) = \varinjlim_{[f:D\to V]\in{}_H\mathcal{M},\, D\in f.g.CoAlg({}_H\mathcal{M})} D $$

,

$$ {}^HG(V) = \varinjlim_{[f:D\to V]\in{}^H\mathcal{M},\, D\in f.g.CoAlg({}^H\mathcal{M})} D $$

,

$$ G^H(V) = \varinjlim_{[f:D\to V]\in\mathcal{M}^H,\, D\in fin.dim.CoAlg(\mathcal{M}^H)} D $$

. ⊞

CHAPTER 6
FUTURE DIRECTIONS

Our work is essentially involved with monoidal categories. Recently, monoidal categories play a pivotal role in developing lots of mathematical concepts. In particular, the theory of tensor categories has significantly been studied as a vital subject that suggests an relevant connections to representation theory, conformal field theory, topological quantum field theory, invariants of knots, topological manifolds, quantum groups, theory of Hopf algebras, infinite dimensional Lie algebras, Frobeniusity and separability, comodule theory, number theory, algebraic geometry, geometric group theory, combinatorics, etc [27].

Hopf algebras could be seen as algebraic structures coming from tensor categories with fiber functors. In fact, this identification makes the theory of Hopf algebras overlaps with the theory of tensor categories.

On the other hand, the most influential ingredients of geometric group theory are involved with concept such as, free constructions, free groups, topological group, rigidity, etc, which can be identified in terms of tensor categories and their universal properties.

Tensor categories also suggest a systematic formalism for combinatorics since many combinatorial properties are essentially concerned with Grothendiek rings of tensor categories.

Representation theory has many applications for tensor categories. For instance, the category $Vect_{\mathbb{K}}$ of finite dimensional $\mathbb{K}$-representations, the category $Rep(G)$

of finite dimensional $\mathbb{K}$-representations of a group G, and the category $Rep(\mathfrak{g})$ of finite dimensional representations of a Lie algebra $\mathfrak{g}$ can be viewed as tensor categories.

Topology and quantum field theory can be nicely reshaped in terms of monoidal categories. For instance, the category of *n-dimensional Topological Quantum Field Theories* over $\mathbb{K}$, denoted by $nTQFT_{\mathbb{K}}$, is simply the category whose objects are $nTQFT$s over $\mathbb{K}$ and whose morphisms are the monoidal natural transformations between them.

More interestingly, some valuable categories of topological quantum field theories can be identified as tensor categories.

Similarly, the other categories mentioned above are pertinently and nicely related to the approach suggested by tensor categories.

Therefore, our research interests have a compatible connection with many fields, and this simply comes from the flexible large role of that monoidal categories play. It turns out that I can confidently and largely work on various areas of interest since my research is germanely involved with the theory of monoidal categories, and hence the theory of tensor categories.

This chapter is completely devoted to give a little insight into some directions for our future work.

6.1 Cofree Objects in The Centralizer and The Center Categories [1]

More recently, the centralizer category of an object or morphism in a monoidal category and the center or the weak center of a monoidal category play a vibrant role in characterizing and identifying many of interesting categories. For instance, to show that two finite tensor categories are Morita equivalent, it suffices to show that their centers are equivalent as braided tensor categories [27, p. 222]. Another example is to show that a finite tensor category is group-theoretical, it is sufficient to show its center contains a Lagrangian subcategory [27, p. 313].

In addition, there is a special importance for the center of a finite tensor category in finding its Frobenius-Perron dimension. This comes from the fact that for any finite tensor category $\mathscr{C}$, we have $FPdim(Z(\mathscr{C})) = FPdim(\mathscr{C})^2$ [27, p. 168]. We refer to [35] for basics on centralizer categories while we refer to [59, p. 76] and [27, p. 162] for basics on center categories.

We are interested in investigating cofree objects in these important categories. Explicitly, the problem can be formulated as follows. Let $\mathcal{C}$ be a monoidal category. Fix an object X and a morphism $h : A \to B$ in $\mathcal{C}$. For any $\mathscr{A} \in \{\mathcal{Z}_h(\mathcal{C}), \mathcal{Z}_X(\mathcal{C}), \mathcal{Z}(\mathcal{C}), \mathcal{Z}_\omega(\mathcal{C})\}$, let $\mathscr{U}_{\mathscr{A}} : CoMon(\mathscr{A}) \to \mathscr{A}$ be the forgetful functor corresponding to $\mathscr{A}$. Does $\mathscr{U}_{\mathscr{A}}$ have a right adjoint?

We start our inspection by studying the cocompleteness in $\mathscr{A}$, and we give some answers for the question: under what conditions the colimits of objects in $\mathscr{A}$ can be obtained from can be obtained by the corresponding construction for objects in $\mathcal{C}$. The later implicitly implies that the forgetful functor $\mathscr{U}_{\mathscr{A}}$ is cocontinuous.

Next, we study some conditions that make the co-wellpoweredness of the category $\mathscr{A}$ can be inherited from $\mathcal{C}$.

We also show how the braiding forces the category $\mathscr{A}$ to inherit generators from its base category $\mathcal{C}$.

Finally, we apply the mechanism of the dual of Special Adjoint Functor Theorem for each case. Furthermore, we try to visualize some interesting consequences by studying the braid category. This work is a (2016) preprint, and it has already been submitted to a journal. It is also available at ArXiv.org: math.CT/1603.02386. http://arxiv.org/pdf/1603.02386v4.pdf [1].

6.2 Categories of Yetter-Drinfeld Modules

A bialgebra is simply an algebra in a category of coalgebras, or equivalently, a coalgebra in a category of algebras. The behavior of some substantial categories that have been recently introduced gives rise to produce a generalization for the notion of bialgebra. This is what so-called a *quasi-bialgebra*, which is introduced in [25]. For the basic notions of quasi-bialgebra and quasi-Hopf algebra, wee also refer to [19].

The significant development of more recent studies justifiably suggests the emphasis of categories whose each of their objects admits two structures of a module and a comodule over a quasi-bialgebra. The complicated way for constructing the objects of some important categories brings about not only considering more than one structure for each object, but also bridging structures under some compatibility condition. This reasonably motivated Yetter and Drinfeld to introduce the concept for monoidal categories of Yetter-Drinfeld modules over a Hopf algebra or a bialgebra.

More recently, categories of Yetter-Drinfeld modules have been extensively and attentively studied by lots of mathematicians. They do not only introduce a very influential exposition of lot of categories of interest, but they also offer precious ingredients for new generations of categories that could impact dramatically on category theory, algebra, topology, tensor categories, quantum groups, representation theory, etc.

Schauenburg in [56] has revised and categorized the monoidal categories of Yetter-Drinfeld modules over quasi-bialgebras. His genuine approach of characterizing and identifying these kinds of categories gives an instrumental role to allow us to show that the categories of comonoids in the monoidal categories of Yetter-Drinfeld modules over quasi-bialgebras have cofree objects. He classifies them into two kinds of categories: *the category of left Yetter-Drinfeld H-modules of the first kind* and *the category of left Yetter-Drinfeld H-modules of the second kind.*

Following Drinfeld [25], a $\mathbb{K}$-*quasi-bialgebra* is a tuple $(H, \Delta, \epsilon, \Phi)$, where H is a $\mathbb{K}$-algebra, Φ is an invertible element in $H \otimes H \otimes H$, and $\Delta : H \to H \otimes_{\mathbb{K}} H$ and $\epsilon : H \to \mathbb{K}$ are $\mathbb{K}$-algebra morphisms satisfying

$$(id_{_H} \otimes \Delta)\Delta(h) = \Phi(\Delta \otimes id_{_H})\Delta(h)\Phi^{-1},$$

$$(id_{_H} \otimes \epsilon)\Delta(h) = h \otimes 1_H,$$

$$(\epsilon \otimes id_{_H})\Delta(h) = 1_H \otimes h,$$

for all $h \in H$, and Φ has to be a normalized 3-cocycle, in the sense that

$$(1_H \otimes \Phi)(id_{_H} \otimes \Delta \otimes id_{_H})(\Phi)(\Phi \otimes 1_H) = (id_{_H} \otimes id_{_H} \otimes \Delta)(\Phi)(\Delta \otimes id_{_H} \otimes id_{_H})(\Phi),$$

$$(id_H \otimes \epsilon \otimes id_H)(\Phi) = 1_H \otimes 1_H \otimes 1_H.$$

The map Δ is called the *coproduct* or the *comultiplication*, ϵ the *counit* and Φ the *reassociator*.

Following [19], we denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by small letters, namely

$\Phi = \sum X^1 \otimes X^2 \otimes X^3 = \sum T^1 \otimes T^2 \otimes T^3 = \sum V^1 \otimes V^2 \otimes V^3 =$

$\Phi^{-1} = \sum x^1 \otimes x^2 \otimes x^3 = \sum t^1 \otimes t^2 \otimes t^3 = \sum v^1 \otimes v^2 \otimes v^3 =$

Following [19], a quasi-bialgebra H is called a *quasi-Hopf algebra* if there exists an anti-automorphism S of the algebra H and $\alpha, \beta \in H$ such that:

$$\sum S(h_1)\alpha h_2 = \epsilon(h)\alpha \text{ and } \sum h_1\beta S(h_2) = \epsilon(h)\beta$$

$$\sum X^1\beta S(X^2)\alpha X^3 = 1_H \text{ and } \sum S(x^1)\alpha x^2\beta S(x^3) = 1_H$$

for all $h \in H$.

Every $\mathbb{K}$-bialgebra is $\mathbb{K}$-quasi-bialgebra, but the converse is not true [33].

Let H be a $\mathbb{K}$-quasi-bialgebra with reassociator Φ. A left H-module M together with a left H-coaction

$$\lambda^M : M \to H \otimes M, \lambda^M(m) = \sum m_{-1} \otimes m_0$$

is called a *left Yetter-Drinfeld module* if the following equalities hold, for all $h \in H$ and $m \in M$:

$$\sum X^1 m_{-1} \otimes (X^2.m_0)_{-1} X^3 \otimes (X^2.m_0)_0 =$$

$$\sum X^1 (Y^1.m)_{-1_1} Y^2 \otimes X^2 (Y^1.m)_{-1_2} Y^3 \otimes X^3 (Y^1.m)_0$$

$$\sum \epsilon(m_{-1}) m_0 = m$$

$$\sum h_1 m_{-1} \otimes h_2.m_0 = \sum (h_1.m)_{-1} h_2 \otimes (h_1.m)_0$$

Following Schauenburg's terminology in [56], the category of left Yetter-Drinfeld H-modules (which is actually the one introduced in [42]) is called *the category of left Yetter-Drinfeld H-modules of the first kind* and denoted by ${}^H_H\mathcal{Y}_1\mathcal{D}$. Majid in [42] has shown that ${}^H_H\mathcal{YD}$ is a (prebraided) monoidal category.

Let ${}_{ad}(-) : {}_H\mathcal{M}_H \to {}_H\mathcal{M}$ be the functor defined as follows

$\forall P \in Ob({}_H\mathcal{M}_H), {}_ad(P) := P$ as $\mathbb{K}$-module, with the adjoint left H-module structure defined by $h.p := h_1 p S(h_2)$.

A *Yetter-Drinfeld module of the second kind* is a left H-module V equipped with an H-module map ${}^V\lambda' : V \to H \Diamond V$ satisfying $(\epsilon \otimes id_V)\lambda'_V = id_V$, and making

$$\begin{array}{ccc}
V & \xrightarrow{{}^V\lambda'} & H\Diamond V \\
 & & \downarrow{\scriptstyle \Delta\Diamond id_V} \\
\downarrow{\scriptstyle {}^V\lambda'} & & (H \otimes_{\mathbb{K}} H)\Diamond V \\
 & & \downarrow{\scriptstyle \Omega} \\
H\Diamond V & \xrightarrow[id_H \Diamond {}^V\lambda']{} & H\Diamond(H\Diamond V)
\end{array}$$

commute, where $\Diamond$ is defined as follows

$\forall P \in Ob({}_H\mathcal{M}_H), N \in Ob({}_H\mathcal{M}),\ P\Diamond N := {}_{ad}(P \otimes_{\mathbb{K}} N)$.

Following [56], we denote by ${}^H_H\mathcal{Y}_2\mathcal{D}$ the category of Yetter-Drinfeld H-modules of the second kind. The following theorem shows that ${}^H_H\mathcal{Y}_1\mathcal{D}$ and ${}^H_H\mathcal{Y}_2\mathcal{D}$ can be viewed as essentially the same as the weak center and the center of ${}_H\mathcal{M}$ respectively.

Theorem 6.1. *[56] Let H be a $\mathbb{K}$-quasi-bialgebra. Then we have category equivalences*

$${}^H_H\mathcal{Y}_2\mathcal{D} \cong {}^H_H\mathcal{M}^H_H \cong {}^H_H\mathcal{Y}_1\mathcal{D} \cong \mathcal{Z}_\omega({}_H\mathcal{M}) \cong \mathcal{Z}({}_H\mathcal{M}),$$

where ${}^H_H\mathcal{M}^H_H$ is the category of H-bicomodules in the monoidal category ${}_H\mathcal{M}_H$, and $\mathcal{Z}_\omega({}_H\mathcal{M}), \mathcal{Z}({}_H\mathcal{M})$ are the weak center and the center of ${}_H\mathcal{M}$ respectively.

For any $\mathscr{X} \in \{{}^H_H\mathcal{Y}_2\mathcal{D}, {}^H_H\mathcal{M}^H_H, {}^H_H\mathcal{Y}_1\mathcal{D}, \mathcal{Z}_\omega({}_H\mathcal{M}), \mathcal{Z}({}_H\mathcal{M})\}$, let $\mathscr{U}_\mathscr{X} : CoMon(\mathscr{X} \to \mathscr{X}$ be the corresponding forgetful functor. From Theorem 6.1, to show that each concrete category $(CoMon(\mathscr{X}, \mathscr{X})$, $\mathscr{X} \in \{{}^H_H\mathcal{Y}_2\mathcal{D}, {}^H_H\mathcal{M}^H_H, {}^H_H\mathcal{Y}_1\mathcal{D}, \mathcal{Z}_\omega({}_H\mathcal{M}), \mathcal{Z}({}_H\mathcal{M})\}$, has cofree objects, it suffices to show that $(CoMon(\mathscr{X}), \mathscr{X})$ has cofree objects, for some $\mathscr{X} \in \{{}^H_H\mathcal{Y}_2\mathcal{D}, {}^H_H\mathcal{M}^H_H, {}^H_H\mathcal{Y}_1\mathcal{D}, \mathcal{Z}_\omega({}_H\mathcal{M}), \mathcal{Z}({}_H\mathcal{M})\}$.
The most important theme in this process is to construct a generating set, for some $\mathscr{X} \in \{{}^H_H\mathcal{Y}_2\mathcal{D}, {}^H_H\mathcal{M}^H_H, {}^H_H\mathcal{Y}_1\mathcal{D}, \mathcal{Z}_\omega({}_H\mathcal{M}), \mathcal{Z}({}_H\mathcal{M})\}$. Due to the distinctive setting of the categories above, one might need a careful treatment for reassociator maps.

6.3 Universal Investigation for Endofunctors Categories

Let $\mathcal{A}$ be a category, and let $[\mathcal{A}, \mathcal{A}]$ be the category of endofunctors on $\mathcal{A}$. One interesting thing about $[\mathcal{A}, \mathcal{A}]$ is that the monoids in $[\mathcal{A}, \mathcal{A}]$ are precisely the monads on $\mathcal{A}$ and the comonoids in $[\mathcal{A}, \mathcal{A}]$ are precisely the comonads on $\mathcal{A}$. Monads and comonads play a substantial role in the theory of higher categories, and this simply comes from a very abstract categorical viewpoint which is "An n-category is an algebra for a certain monad on the category **RefGSet** of reflexive globular sets" [22, p. 9].

Following [39, p. 26], a *monad* on a category $\mathcal{A}$ can be defined as a monoid in the monoidal category $([\mathcal{A},\mathcal{A}],\circ,id_{\mathcal{A}})$ of endofunctors on $\mathcal{A}$. The following theorem emphasizes the importance of studying the monoidal category $([\mathcal{A},\mathcal{A}],\circ,id_{\mathcal{A}})$ and show that the notion of monads and comonads can be characterized by adjunctions.

Theorem 6.2. *[13, p. 1723] Let $\mathcal{L}:\mathcal{A}\to\mathcal{A}$ and $\mathcal{R}:\mathcal{A}\to\mathcal{A}$ be an adjoint pair of functors with $\mathcal{L}\dashv\mathcal{R}$.*

1. *$\mathcal{L}$ is a monad if and only if $\mathcal{R}$ is a comonad.*

2. *$\mathcal{L}$ is a comonad if and only if $\mathcal{R}$ is a monad.*

Now, let $\mathcal{A}$ be a monoidal category and $\mathcal{U}:CoMon(\mathcal{A})\to\mathcal{A}$ a forgetful functor. Suppose that $(CoMon(\mathcal{A}),\mathcal{U})$ has cofree objects and consider the forgetful functor $\mathcal{U}':[CoMon(\mathcal{A}),CoMon(\mathcal{A})]\to[CoMon(\mathcal{A}),\mathcal{A}]$ induced by $\mathcal{U}$. This gives rise to the following question: can the cofree objects in $([CoMon(\mathcal{A}),CoMon(\mathcal{A})],\mathcal{U}')$ be characterized by cofrre objects in $(\mathcal{A},\mathcal{U}')$? In light of Theorem 6.2, it makes sense to ask whether they can be described in terms of adjunctions. In deed, this is reasonably worth close attention.

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