Summer 2016

Quantum topology and me

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QUANTUM TOPOLOGY AND ME

by

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A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

August 2016

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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ACKNOWLEDGEMENTS

First and foremost, I thank Charlie Frohman for his unwaivering support of my mathematical career both academically and personally. Your insight and patience made undergraduate and graduate school a more pleasant experience. I would also like to thank Cody Armond for his willingness to discuss obscure math questions at the drop of a dime. Thank you for being there when I needed you the most. Thank you Thomas Kindred for keeping me honest and never letting me get away with circular logic. Our math discussions made me a mature mathematician. Thank you Matthew Drury for working through the early years with me. You have always been my academic role model. Thank you to the professors at Indiana University and the University of Iowa who supported me throughout graduate school. Specifically, thank you Ben Cooper for teaching me that you can only answer a question once you know how to ask it. Thank you Oguz Durumeric, Maggy Tomova, Gerhard Strohmer, and Frauke Bleher for being great instructors and always being available for questions and discussion. On a personal note, thank you to all of my friends and fellow graduate students who helped me relax and not go crazy. Thank you Greg Farnsworth for being a great friend. Finally, thank you April Loya. Your compassion and love have made me realize that all of this has been worth it. I look forward to our life together.

ABSTRACT

This thesis has four chapters. After a brief introduction in Chapter 1, the AJ-conjecture is introduced in Chapter 2. The AJ-conjecture for a knot $K \subset S^3$ relates the A-polynomial and the colored Jones polynomial of K. If K satisfies the AJ-conjecture, sufficient conditions on K are given for the $(r, 2)$-cable knot C to also satisfy the AJ-conjecture. If a reduced alternating diagram of K has η_+ positive crossings and η_- negative crossings, then C will satisfy the AJ-conjecture when $(r + 4\eta_-)(r - 4\eta_+) > 0$ and the conditions of Theorem 2.2.1 are satisfied. Chapter 3 is about quantum curves and their relation to the AJ conjecture. The variables l and m of the A-polynomial are quantized to operators that act on holomorphic functions. Motivated by a heuristic definition of the Jones polynomial from quantum physics, an annihilator of the Chern-Simons section of the Chern-Simons line bundle is found. For torus knots, it is shown that the annihilator matches with that of the colored Jones polynomial. In Chapter 4, a tangle functor is defined using semicyclic representations of the quantum group $U_q(sl_2)$. The semicyclic representations are deformations of the standard representation used to define Kashaev's invariant for a knot K in S^3. It is shown that at certain roots of unity the semicyclic tangle functor recovers Kashaev's invariant.

PUBLIC ABSTRACT

This thesis is an accumulation of work I completed in a subject called quantum topology while at the University of Iowa. One of the main things quantum topologists do is find ways to show that knots are different. Think of a knot as a closed loop in space. We've all tied our shoes. If you glue the ends together, this is an example of a knot. Knots can be used to model small particle interaction. Understanding how particles interact is important in physics. If mathematicians can tell knots apart, then physicists can tell when particles are interacting differently. Chapter 1 is an introduction to my area of quantum topology. Chapter 2 and 3 relate two polynomials that can tell knots apart. Chapter 4 is devoted to defining a new polynomial that turns out to recover an already famous polynomial.

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CHAPTER 1
INTRODUCTION

1.1 Motivation

This thesis serves as a record of my encounters with quantum topology during my time at the University of Iowa. Quantum topology began with the discovery of the Jones polynomial which is a link invariant for knots in the three sphere. In fact, many topologists define quantum topology as anything related to the Jones polynomial. Although the Jones polynomial was originally defined by Vaughn Jones under the setting of $\mathbb{C}^*$-algebras, its application to three-manifold theory has been abundantly fruitful.

The quantum counterpart to the Jones polynomial is the colored Jones polynomial. In this case the word quantum means the colored Jones polynomial is a sequence of Laurant polynomials $J(n) \in \mathbb{Z}[q^{\pm 1}]$ such that $J(1) = 1$ and $J(2)$ is the classical Jones polynomial. Each $J(n)$ is a link invariant which gives an infinite number of link invariants. The colored Jones polynomial has been the focus of the majority of my research.

The colored Jones polynomial can be defined using seemingly different areas of mathematics. This thesis will focus on three specific formulations. The first will be skein theoretic and involve the Kauffman bracket. The second will come from the area of physics where a certain partition function heuristically defined by a Feynman path integral is related to the colored Jones polynomial. Finally, using representations of quantum groups, the colored Jones polynomial can be defined using what is known as an R-matrix. Although these are not the only ways to define the colored Jones polynomial, they are the most commonly studied in quantum topology.

1.2 Overview

Chapter 2 is devoted to the AJ-conjecture. To state the conjecture, the A polynomial must be defined. The A polynomial of a knot K embedded in S^3 is determined by the fundamental group of the boundary torus of the knot exterior. In order to define the A polynomial, the representation and character varieties are introduced. The Jones polynomial and colored Jones polynomial of a knot will be defined using the Kauffman bracket skein algebra. This will be followed by a discussion of the quantum torus which will allow for the statement of the AJ-conjecture. After formulating the conjecture, a proof that certain cables of certain alternating knots satisfy the AJ-conjecture will be given.

Chapter 3 explores the Witten path integral that can be taken as a heuristic definition of the colored Jones polynomial. A brief discussion of Chern-Simons theory and quantum curves will be followed by an analytic proof that torus knots satisfy an AJ like conjecture. The result will be related to the algebraic proof that torus knots satisfy the AJ-conjecture .

In Chapter 4, tangle functors for semicyclic representations of $U_q(sl_2)$ will be discussed. The colored Jones polynomial is defined using what is simply called *the standard representation*. If this representation is perturbed, then one would expect to get new quantum invariants. However, it is shown that under the perturbed representation, no new information is found but instead the colored Jones polynomial is recovered at 2^{nth} roots of unity where n is an odd counting number. The method used in Chapter 4 to calculate the colored Jones polynomial is similar to methods used in the past. However, the way this invariant is calculated has certain notational advantages that will be discussed.

CHAPTER 2
THE AJ CONJECTURE

2.1 Preliminaries

The AJ-conjecture [11] is a proposed relationship between two different invariants of a knot K in S^3, the A-polynomial and the colored Jones polynomial. The A-polynomial is determined by the fundamental group of the knot complement, while the colored Jones polynomial is a sequence of Laurent polynomials $J_K(n) \in \mathbb{Z}[t, t^{-1}]$ that have no apparent connection to classical knot invariants. The relationship posited by the AJ-conjecture allows us to extract information about one invariant from the other. For example, if the AJ-conjecture is true, then the fact that the A-polynomial recognizes the unknot implies the colored Jones function does as well. To state the AJ-conjecture, some algebraic background will be discussed.

2.1.1 The Representation and Character Varieties

Let $G = \langle a_i \,|\, r_j \rangle$ be a finitely generated group with n generators and m relations. A representation $\rho : G \to SL_2(\mathbb{C})$ is a homomorphism determined by a choice of matrices $A_i \in SL_2(\mathbb{C})$ such that the image of each relation evaluates to the identity in $SL_2(\mathbb{C})$. Denote by $\text{Rep}(G) \subset \prod_{i=1}^{n} SL_2(\mathbb{C})$ the space of all representations of G under the relations r_j and embed $\text{Rep}(G)$ into $\mathbb{C}^{4n}$ by $\rho \mapsto (\rho(a_1), \rho(a_2), ..., \rho(a_n))$. Under this embedding, $\text{Rep}(G)$ is an algebraic variety called the **representation variety**. Specifically, $\text{Rep}(G)$ is cut out by $4m + n$ equations where $4m$ equations come from the m relations and n equations from the fact that $\det(\rho(a_i)) = 1$ for each

$1 \leq i \leq n$.

There is an action of $SL_2(\mathbb{C})$ on $\text{Rep}(G)$ by conjugation. The quotient of $\text{Rep}(G)$ by this action does not yield a Hausdorff space. To resolve this problem, identify representations of G that have the same character. A **character** of a representation $\rho \in \text{Rep}(G)$ is a homomorphism $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) = \text{tr}(\rho(g))$, for each $g \in G$, where $\text{tr}(\rho(g))$ is the trace of the matrix $\rho(g)$. The **character variety** of G, denoted $\chi(G)$, is the space of all characters of elements of $\text{Rep}(G)$. The variety $\chi(G)$ can be thought of as the categorical quotient $\text{Rep}(G) /\!\!/ SL_2(\mathbb{C})$ where elements of $\text{Rep}(G)$ with the same character have been identified. The fact that $\chi(G)$ is an algebraic variety does not follow as easily as with $\text{Rep}(G)$, but with some effort it can be shown [23].

2.1.2 The A-Polynomial

The A-polynomial is the defining polynomial of an algebraic curve in $\mathbb{C}^* \times \mathbb{C}^*$ where $\mathbb{C}^*$ are the nonzero complex numbers [4]. Let $K \subset S^3$ be a knot and $\mathcal{M}$ be the complement of a regular neighborhood of K. Then $\mathcal{M}$ is a compact manifold with boundary homeomorphic to a torus, $\partial\mathcal{M} = T$.

The fundamental group $\pi_1(T)$ is a free abelian group with two generators. Let $\{\mu, \lambda\}$ be the standard basis for $\pi_1(T)$. Consider the subset $\text{Rep}^\Delta(\pi_1(\mathcal{M}))$ of $\text{Rep}(\pi_1(\mathcal{M}))$ consisting of upper triangular $SL_2(\mathbb{C})$ representations. Set

$$\rho(\mu) = \begin{pmatrix} m & \star \\ 0 & m^{-1} \end{pmatrix} \qquad \text{and} \qquad \rho(\lambda) = \begin{pmatrix} l & \star \\ 0 & l^{-1} \end{pmatrix}$$

and let $\epsilon : \text{Rep}^\Delta(\pi_1(\mathcal{M})) \to \mathbb{C}^* \times \mathbb{C}^*$ be the eigenvalue map defined by $\epsilon(\rho) = (m, l)$.

Let Z be the Zariski closure of $\epsilon(\mathrm{Rep}^{\Delta}(\pi_1(\mathcal{M})))$ in $\mathbb{C}^* \times \mathbb{C}^*$. Each of the components of Z are one dimensional [7]. The components are hyper-surfaces and can be cut out by a single polynomial unique up to multiplication by a constant. The **A-polynomial**, $A_K(m, l)$, is the product of all such defining polynomials. The A-polynomial can be taken to have relatively prime integer coefficients and is well defined up to a unit. The abelian component of Z will have defining polynomial $l-1$ and thus the A-polynomial can be factored as $A_K(m, l) = (l-1)A'_K(m, l)$ [4].

2.1.3 The Colored Jones Function

Let L be a link in S^3. The colored Jones function is a quantum link invariant that assigns to each $n \in \mathbb{N}$ a Laurent polynomial $J_L(n) \in \mathbb{Z}[t^{\pm 1}]$. Like the classic Jones polynomial, $J_L(n)$ can be defined using the Kauffman bracket. The following discussion mostly follows [22] to define the colored Jones function.

2.1.3.1 The Kauffman Bracket Skein Module

Let $\mathcal{M}$ be an oriented 3-manifold. For a nonzero $A \in \mathbb{C}$, let $\mathcal{L}$ be the free $\mathbb{Z}[A, A^{-1}]$ module generated by isotopy classes of links in $\mathcal{M}$. The Kauffman bracket skein module of $\mathcal{M}$, denoted $K_A(\mathcal{M})$, is the quotient of $\mathcal{L}$ by the relations

where the links are identical outside of the dotted circles. The arcs represent strips of embedded annuli. In the case where arcs in the first picture come from the same annulus, it is assumed that the same side of the annulus is up.

To define the colored Jones polynomial of a knot $K \subset S^3$, the 3-manifold to consider is the solid torus; a tubular neighborhood of K. Let $\mathcal{M}$ be the solid torus. In this case, $\mathcal{M}$ is a cylinder over the annulus which gives an algebra structure on $K_A(\mathcal{M})$ with multiplication defined by stacking one cylinder on top of another. If z represents the framed meridian of $\mathcal{M}$, then $K_A(\mathcal{M}) = \mathbb{C}[A, A^{-1}][z]$ with z^n meaning n parallel copies of z. Define a basis $\{S_i(z)\}_{i\geq 0}$ for $K_A(\mathcal{M})$ recursively by

a. $S_0(z) = 1$

b. $S_1(z) = z$

c. $S_i(z) = zS_{i-1}(z) - S_{i-2}(z)$

These $S_i(z)$ are called the Chebyshev polynomials of the second kind.

It makes sense to apply these polynomials to a framed knot, K, in S^3. For each n, $S_n(K)$ is a linear combination of parallels of the knot and, in this way, can be thought of as an element of $K_A(\mathcal{M})$. After embedding $K_A(\mathcal{M})$ into $K_A(S^3)$, we can apply the Kauffman bracket, $\langle\ \rangle$, to $S_n(K)$ to get a polynomial in $A \in \mathbb{C}^*$. For each $n \in \mathbb{N}$, define the $n^{\mathbf{th}}$ **colored Jones polynomial** of K by

$$J_K(n) = (-1)^{n-1} \langle S_{n-1}(K) \rangle$$

The $(-1)^{n-1}$ factor is included as a normalization so that for the unknot, U,

$$J_U(n) = [n] = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

Extend this definition to all integers by $J_K(-n) = -J_K(n)$ and $J_K(0) = 0$ and make $J_K(n)$ a function of t by substituting $t^{-\frac{1}{2}} = A^2$.

2.1.4 The AJ Conjecture

To set up the AJ conjecture, we will follow [11] and use notation from [22]. Given a discrete function $f : \mathbb{N} \to \mathbb{C}[t^{\pm 1}]$, define operators M and L acting on f by

$$M(f)(n) = t^{2n} f(n) \quad \text{and} \quad L(f)(n) = f(n+1)$$

It can be seen that these operators satisfy $LM = t^2 ML$. Let

$$\mathcal{T} = \mathbb{C}[t^{\pm 1}] \left\langle M^{\pm 1}, L^{\pm 1} \right\rangle / (LM - t^2 ML)$$

This non-commutative ring, $\mathcal{T}$, is called the quantum torus.

If there exists a $P \in \mathcal{T}$ such that $P(f) = 0$, then P is called a *recurrence relation* for f. The **recurrence ideal** of a discrete function f is a left ideal A_f of $\mathcal{T}$ consisting of recurrence relations for f.

$$A_f = \{P \in \mathcal{T} | P(f) = 0\}$$

When $A_f \neq \{0\}$, the discrete function f is said to be **q-holonomic**. Also, denote by A_K the recurrence ideal of $J_K(n)$.

The ring $\mathcal{T}$ is not a principal ideal domain, so a non-trivial recurrence ideal is not guaranteed to have a single generator. Fix this by localizing at Laurent polynomials in M [11]. The process of localization works since $\mathcal{T}$ satisfies the Ore condition. The resulting ring, $\widetilde{\mathcal{T}}$, has a Euclidean algorithm and is therefore a principal ideal domain. Let $\mathbb{C}[t^{\pm 1}](M)$ be the fraction field of the polynomial ring $\mathbb{C}[t^{\pm 1}][M]$. Then,

$$\widetilde{\mathcal{T}} = \left\{ \sum_{i \in \mathbb{Z}} a_i(M) L^i \,\middle|\, a_i(M) \in \mathbb{C}[t^{\pm 1}](M),\, a_i(M) = 0 \text{ for almost every i} \right\}.$$

Extend the recurrence ideal A_K to an ideal $\widetilde{A_K}$ of $\widetilde{\mathcal{T}}$ by $\widetilde{A_f} = \widetilde{\mathcal{T}} A_K$ which can be done since $\mathcal{T}$ embeds as a subring of $\widetilde{\mathcal{T}}$. The extended ideal will then have a single generator

$$\widetilde{\alpha_K}(t, M, L) = \sum_{i=0}^{n} \alpha_i(M) L^i$$

This generator has only positive powers of M and L, and the degree of L is assumed to be minimal since it generates the ideal. The coefficients $\alpha_i(M)$ can be assumed to be co-prime and $\alpha_i(M) \in \mathbb{Z}[t^{\pm 1}, M]$. The operator $\widetilde{\alpha_K}$ is called the **recurrence polynomial** of K and is defined up to a factor of $\pm t^j M^k$ for $j, k \in \mathbb{Z}$. Garoufalidis and Le showed that for every knot K, $J_K(n)$ satisfies a nontrivial recurrence relation [12].

Let $f, g \in \mathbb{C}[M, L]$. Then f and g are said to be M*-essentially equivalent*, denoted $f \overset{M}{=} g$, if the quotient f/g does not depend on L. In other words, the functions f and g are equal up to a factor only depending on M. Let ϵ be the map were the substitution $t = -1$ is made.

The AJ Conjecture [11]: If $A_K(M, L)$ is the A-polynomial of a knot K and $\widetilde{\alpha_K}(t, M, L)$ is as above, then $\epsilon(\widetilde{\alpha_K}) \overset{M}{=} A_K(M, L)$

2.2 The AJ Conjecture for Cables of Alternating Knots

2.2.1 Cabling Formulas and the Resultant

Let $r, s \in \mathbb{Z}$ with greatest common divisor d. The (r, s)-cable, C, of a zero framed knot K is the link formed by taking d parallel copies of the $(\frac{r}{d}, \frac{s}{d})$ curve on the torus boundary of a tubular neighborhood of K. Homologically, one can think of this curve as $\frac{r}{d}$ times the meridian and $\frac{s}{d}$ times the longitude on the torus boundary. Notice that if r and s are relatively prime, C is a knot.

Let $A_K(M, L)$ be the A-polynomial of a knot K in S^3 and let C be the $(r, 2)$-cable knot of K. Write $A_C(M, L)$ in terms of $A_K(M, L)$ using the following cabling formula given by Ni and Zhang, c.f. [30]. Let

$$F_r(M, L) = \begin{cases} M^{2r}L + 1 & \text{if} \quad r > 0 \\ \\ L + M^{-2r} & \text{if} \quad r < 0 \end{cases}$$

Then,

$$A_C(M, L) = (L - 1)F_r(M, L)\mathrm{Res}_\lambda \left(\frac{A_K(M^2, \lambda)}{\lambda - 1}, \lambda^2 - L \right)$$

where Res_λ is the resultant defined below. Use the fact that the product of the resultants is the resultant of the product and $\mathrm{Res}_\lambda(\lambda - 1, \lambda^2 - 1) = L - 1$ to rewrite this formula as

$$A_C(M, L) = F_r(M, L)\mathrm{Res}_\lambda \left(A_K(M^2, \lambda), \lambda^2 - L \right) \tag{2.1}$$

Let $\mathbb{F}$ be a field. Let $f(x) = \sum_{i=0}^{n} f_i x^i$ and $g(x) = \sum_{i=0}^{m} g_i x^i$ be polynomials in $\mathbb{F}[x]$. The **resultant** of f and g is the determinant of the following $(m+n) \times (m+n)$

matrix.

$$\operatorname{Res}(f,g) = \begin{vmatrix} f_0 & 0 & \dots & 0 & g_0 & 0 & \dots & 0 \\ f_1 & f_0 & & 0 & g_1 & g_0 & & 0 \\ f_2 & f_1 & \ddots & 0 & g_2 & g_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n-1} & \ddots & f_0 & g_k & \vdots & \ddots & g_0 \\ 0 & f_n & \ddots & f_1 & g_{k+1} & g_k & \ddots & g_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & f_n & f_{n-1} & 0 & \dots & g_m & g_{m-1} \\ 0 & \dots & 0 & f_n & 0 & \dots & 0 & g_m \end{vmatrix}$$

The coefficients of $f(x)$ constitute the first m columns while the coefficients of $g(x)$ are in the last n columns.

There is also a formula relating the colored Jones polynomial of the cable to a subsequence of the colored Jones polynomial of K. Since we only consider the $(r, 2)$-cable knot C, the cabling formula given in [31] simplifies as

$$(M^r L + t^{-2r} M^{-r}) J_C(n) = J_K(2n + 1). \tag{2.2}$$

In the next section, a homogeneous annihilator $\widetilde{\beta}(t, M, L)$ of the odd subsequence $J_K(2n+1)$ is found. With this in hand, the above cabling formula implies that $\widetilde{\beta}(t, M, L)(M^r L + t^{-2r} M^{-r})$ annihilates $J_C(n)$.

2.2.2 The Annihilator

Lemma 2.2.1. *Let* $\widetilde{\alpha}_K(t, M, L) = \sum_{i=0}^{d} P_i(t, M) L^i$ *be a minimal degree homogeneous recurrence polynomial for* $J_K(n)$ *of degree* $d \geq 2$ *such that* $\widetilde{\alpha}_K(-1, M, L) \overset{M}{=} A_K(M, L)$.

If the matrix N defined later has nonzero determinant at $t = -1$, then $J_K(2n+1)$ has the homogeneous recurrence polynomial given by

$$\widetilde{\beta}(t, M, L) = \sum_{i=0}^{d} (-1)^i Q_i(t, M) L^i$$

with

$$Q_i(t, M) = \det(\mathrm{A_{i+1}})$$

where the A_{i+1} are matrices to be defined below.

Proof: The process is similar to the case of the figure eight knot [30], except here a homogeneous annihilator is considered. Since $\widetilde{\alpha}_K$ is a homogeneous annihilator, $\widetilde{\alpha}_K(t, M, L)J_K(n) = 0$. Since M acts on discrete functions as multiplication by t^{2n}, change M to t^{2n} yielding

$$\left(\sum_{i=0}^{d} P_i(t, t^{2n}) L^i\right) J_K(n) = \sum_{i=0}^{d} P_i(t, t^{2n}) J_K(n+i) = 0$$

For each $0 \leq j \leq d$, substitute $2n + j + 1$ for n yielding $d + 1$ equations of the form

$$\sum_{i=0}^{d} P_i(t, t^{2(2n+j+1)}) J_K(2n + j + 1 + i) = 0$$

A degree d homogeneous recurrence relation of $J_K(2n + 1)$ has the form

$$\left(\sum_{i=0}^{d} Q_i(t, t^{2n}) L^i\right) J_K(2n+1) = \sum_{i=0}^{d} Q_i(t, t^{2n}) J_K(2(n+i)+1) = 0$$

To construct $\widetilde{\beta}$ solve

$$\sum_{j=0}^{d} c_j \sum_{i=0}^{d} P_i(t, t^{4n+2j+2}) J_K(2n+i+j+1) - \sum_{i=0}^{d} Q_i(t, t^{2n}) J_K(2n+2i+1) = 0 \quad (2.3)$$

Setting the coefficients of each $J_K(m)$ equal to zero, where m satisfies $2n+1 \leq m \leq 2n+2d+1$, gives $2d+1$ equations in the $2d+2$ unknowns $c_0, c_1, ..., c_d, Q_1, Q_2, ...Q_d$.

Form a $(2d+1)\times(2d+2)$ matrix, C, where the columns are labeled by the unknowns in the order given above. In the following matrix, $P_i(r)$ is used as shorthand for $P_i(t, t^{4n+r})$.

$$
C = \begin{bmatrix}
P_0(2) & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
P_1(2) & P_0(4) & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
P_2(2) & P_1(4) & P_0(6) & \cdots & 0 & \cdots & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\
P_3(2) & P_2(4) & P_1(6) & \ddots & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
P_4(2) & P_3(4) & P_2(6) & \ddots & 0 & \cdots & 0 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\
P_5(2) & P_4(4) & P_3(6) & \ddots & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
P_{d-1}(2) & P_{d-2}(4) & P_{d-3}(6) & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
P_d(2) & P_{d-1}(4) & P_{d-2}(6) & \cdots & P_0(2k+2) & & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & P_d(4) & P_{d-1}(6) & \cdots & P_1(2k+2) & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & P_d(6) & \cdots & P_2(2k+2) & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & P_3(2k+2) & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & P_{d-1}(2k+2) & \ddots & P_k(2d) & P_{k-2}(2d+2) & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & P_d(2k+2) & \ddots & P_{k+1}(2d) & P_k(2d+2) & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \ddots & P_{k+2}(2d) & P_{k+1}(2d+2) & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & P_d(2d) & P_{d-1}(2d+2) & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & P_d(2d+2) & 0 & 0 & 0 & 0 & \cdots & 0 & -1
\end{bmatrix}
$$

If $< c_0, ..., c_d, Q_0, ..., Q_d >^T = \vec{X} \in \mathbb{R}^{2d+2}$ is the vector of unknowns, then (2.3) is equivalent to $C\vec{X} = \vec{0}$. Because C is not a square matrix, Cramer's rule can not be applied. However, a slight modification can be made to get a square matrix.

Let D be the $(2d+1)\times(2d+1)$ matrix obtained from C by removing the last column. Let $\vec{Y} \in \mathbb{R}^{2d+1}$ be the vector obtained from $\vec{X}$ by removing the last entry Q_d. Finally, let $\vec{b} =< 0, 0, ..., 0, Q_d >^T \in \mathbb{R}^{2d+1}$. It can be seen that the equation $C\vec{X} = \vec{0}$ is equivalent to $D\vec{Y} = \vec{b}$.

Assume $\det(D) \neq 0$, then by Cramer's rule

$$c_k = \frac{\det(D_{k+1})}{\det(D)} \qquad \text{and} \qquad Q_k = \frac{\det(D_{d+k+2})}{\det(D)}$$

where D_j is the same as the matrix D except the j-th column is replaced by $\vec{b}$. At this point it is convenient to make the choice $Q_d(t, M) = \det(D)$. In order to verify the AJ-conjecture later, it will be helpful to simplify these determinants. Let us first examine $\det(D)$. The last d columns of D have only one nonzero entry(that is, -1). Successive row expansion along each of the last d columns yields,

$$\det(D) = \det(X) \tag{2.4}$$

where X is a $(d+1) \times (d+1)$ matrix made up of the first $d+1$ columns of D with the odd rows removed except the $2d+1$ row. There is no factor of -1 in (2.4) since the column expansion always results in an even number of -1's. Now simplify a little further to get

$$\det(X) = P_d(t, t^{4n+2d}) P_d(t, t^{4n+2d+2}) \det(N)$$

where N is the upper left $(d-1) \times (d-1)$ block sub-matrix of X. It is the matrix N that will be important in verifying the AJ-conjecture.

Example: Let $d = 3$. Then,

$$D = \begin{pmatrix} P_0(2) & 0 & 0 & 0 & -1 & 0 & 0 \\ P_1(2) & P_0(4) & 0 & 0 & 0 & 0 & 0 \\ P_2(2) & P_1(4) & P_0(6) & 0 & 0 & -1 & 0 \\ P_3(2) & P_2(4) & P_1(6) & P_0(8) & 0 & 0 & 0 \\ 0 & P_3(4) & P_2(6) & P_1(8) & 0 & 0 & -1 \\ 0 & 0 & P_3(6) & P_2(8) & 0 & 0 & 0 \\ 0 & 0 & 0 & P_3(8) & 0 & 0 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} P_1(2) & P_0(4) \\ P_3(2) & P_2(4) \end{pmatrix}$$

Now simplify the determinants corresponding to the c_j for $0 \leq j \leq d$. For example, $c_0 = \frac{\det(D_1)}{\det(D)}$. Simplify $\det(D_1)$ by expanding along the last d columns to get a factor of 1 and then expand along the first column. Recall that the first column of D_1 is the vector $\vec{b}$ with only one nonzero entry, Q_d, in the last row. Therefore,

$$\det(D_1) = (-1)^d Q_d(t, M) \cdot \det(B_1)$$

where B_1 is the $(d+1, 1)$-cofactor of X. In general,

$$\det(D_j) = (-1)^{d+j-1} Q_d(t, M) \cdot \det(B_j)$$

where B_j is the $(d+1, j)$ cofactor of X and $1 \leq j \leq d+1$. This gives

$$c_j = (-1)^{d+j} \det(B_{j+1}) \quad \text{for} \quad 0 \leq j \leq d.$$

Finally, simplify the determinants corresponding to the Q_j for $0 \leq j \leq d-1$. For example, $\det(D_{d+2})$ simplifies by expanding along the last d columns, the first of which is the vector $\vec{b}$. This gives

$$\det(D_{d+2}) = -Q_d \cdot \det(A_1)$$

where A_1 is a $(d+1) \times (d+1)$ sub-matrix of D. The cofactor expansion implies that A_1 is formed from the first $d+1$ columns of D while removing all the odd rows except the first. Similarly,

$$\det(D_{d+j+2}) = -Q_d \cdot \det(A_j)$$

where A_j is a $(d+1) \times (d+1)$ sub-matrix of D. Form A_j from the first $d+1$ columns of D while removing all the odd rows except the $2j-1$ row.

Example: In the case $d = 3$,

$$A_1 = \begin{pmatrix} P_0(2) & 0 & 0 & 0 \\ P_1(2) & P_0(4) & 0 & 0 \\ P_3(2) & P_2(4) & P_1(6) & P_0(8) \\ 0 & 0 & P_3(6) & P_2(8) \end{pmatrix} \qquad B_1 = \begin{pmatrix} P_0(4) & 0 & 0 \\ P_2(4) & P_1(6) & P_0(8) \\ 0 & P_3(6) & P_2(8) \end{pmatrix}$$

$$A_2 = \begin{pmatrix} P_1(2) & P_0(4) & 0 & 0 \\ P_2(2) & P_1(4) & P_0(6) & 0 \\ P_3(2) & P_2(4) & P_1(6) & P_0(8) \\ 0 & 0 & P_3(6) & P_2(8) \end{pmatrix} \qquad B_2 = \begin{pmatrix} P_1(2) & 0 & 0 \\ P_3(2) & P_1(6) & P_0(8) \\ 0 & P_3(6) & P_2(8) \end{pmatrix}$$

$$A_3 = \begin{pmatrix} P_1(2) & P_0(4) & 0 & 0 \\ P_3(2) & P_2(4) & P_1(6) & P_0(8) \\ 0 & P_3(4) & P_2(6) & P_1(8) \\ 0 & 0 & P_3(6) & P_2(8) \end{pmatrix} \qquad B_3 = \begin{pmatrix} P_1(2) & P_0(4) & 0 \\ P_3(2) & P_2(4) & P_0(8) \\ 0 & 0 & P_3(8) \end{pmatrix}$$

To complete the proof of the lemma, it remains to show that

$$\widetilde{\beta}(t, M, L) = \sum_{i=0}^{d} (-1)^i Q_i(t, M) L^i$$

defines a nontrivial operator which can be accomplished with the following claim.

Claim: $\det(B_{j+1}(-1, M)) = P_j(-1, M^2) \cdot \det(N(-1, M))$ where N is the $(d-1) \times (d-1)$ matrix defined above.

Proof of Claim: Break the claim into two cases; d is even or d is odd. The case where d is odd is shown, the even case is analogous. All calculations are done at $t = -1$ and P_k is shorthand for $P_k(-1, M^2)$.

When d is odd, the matrix N is formed from the two vectors

$$\vec{v} = \begin{pmatrix} P_1 \\ P_3 \\ \vdots \\ P_{d-2} \\ P_d \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{w} = \begin{pmatrix} P_0 \\ P_2 \\ \vdots \\ P_{d-3} \\ P_{d-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $\vec{v}, \vec{w} \in \mathbb{R}^{d-1}$. For example, when $d = 7$

$$\vec{v} = \begin{pmatrix} P_1 \\ P_3 \\ P_5 \\ P_7 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{w} = \begin{pmatrix} P_0 \\ P_2 \\ P_4 \\ P_6 \\ 0 \\ 0 \end{pmatrix}$$

Let S be the $(d-1) \times (d-1)$ permutation matrix that shifts each vector entry by the permutation $(1\,2\,3\,4 \ldots d-1)$. Then in vector notation, N is the $(d-1) \times (d-1)$

matrix given by

$$N = \begin{pmatrix} \vec{v} & \vec{w} & \vec{Sv} & \vec{Sw} & \dots & \overrightarrow{S^{\frac{d-3}{2}}v} & \overrightarrow{S^{\frac{d-3}{2}}w} \end{pmatrix}.$$

Let $i : \mathbb{R}^{d-1} \to \mathbb{R}^d$ be inclusion. Then $i(\vec{v})$ is almost the same as $\vec{v}$ except there is an extra zero in the last row of $i(\vec{v})$. Let T be the permutation matrix that shifts each vector entry by the permutation $(1\,2\,3\,4\dots d)$. Form any of the matrices B_j by choosing d vectors (all but the j-th vector) from the following set of $d+1$ vectors;

$$\{i(\vec{v}), i(\vec{w}), \overrightarrow{Ti(v)}, \overrightarrow{Ti(w)}, \cdots, \overrightarrow{T^{\frac{d-1}{2}}i(v)}, \overrightarrow{T^{\frac{d-1}{2}}i(w)}\}$$

For example,

$$B_{d+1} = \begin{pmatrix} i(\vec{v}) & i(\vec{w}) & \overrightarrow{Ti(v)} & \overrightarrow{Ti(w)} & \dots & \overrightarrow{T^{\frac{d-3}{2}}i(v)} & \overrightarrow{T^{\frac{d-3}{2}}i(w)} & \overrightarrow{T^{\frac{d-1}{2}}i(v)} \end{pmatrix}.$$

Notice that $\det(B_{d+1}) = P_d \cdot \det(N)$. This is clear since B_{d+1} is a block matrix. To show the claim for the rest of the B_j, consider the equation

$$B_{d+1} \cdot \vec{X} = \overrightarrow{T^{\frac{d-1}{2}}i(w)}$$

This equation has the solution

$$\vec{X} = \frac{1}{P_d} \begin{pmatrix} P_0 \\ -P_1 \\ \vdots \\ -P_{d-2} \\ P_{d-1} \end{pmatrix}$$

The fact that $\vec{X}$ is the solution to this equation follows from the successive shifting of the vectors $i(\vec{v})$ and $i(\vec{w})$. Notice that because $\widetilde{\alpha_K}$ was a minimal degree annihilator, $P_0, P_d \neq 0$, implying $\vec{X}$ is defined and nonzero.

By Cramer's rule,

$$\frac{P_i}{P_d} = \frac{\det(B_{d+1})_{i+1}}{\det(B_{d+1})}$$

where $(B_{d+1})_{i+1}$ is the matrix B_{d+1} with the $i+1$ column replaced by $T^{\frac{d-1}{2}}i(\vec{w})$. Note that placing the vector $T^{\frac{d-1}{2}}i(\vec{w})$ in the $i+1$ column and then exchanging appropriate columns cancels the negative signs from the solution $\vec{X}$. After this replacement and shifting of columns, the matrix B_{i+1} is obtained. Therefore,

$$\frac{P_i}{P_d} = \frac{\det(B_{i+1})}{\det(B_{d+1})} = \frac{\det(B_{i+1})}{P_d \cdot \det(N)}$$

Conclude that $\det(B_{i+1}) = P_i \cdot \det(N)$ for $0 \leq i \leq d$ which proves the claim. □

For general d, an important observation is

$$\det(A_i) = \sum_{j+k=2i-1} (-1)^{k+i-1} P_k(2j) \cdot \det(B_j)$$

where $0 \leq k \leq d$ and $1 \leq j \leq d+1$. This can be calculated directly by expanding along the extra row included in A_i that B_i does not contain.

The proof of Lemma 2.2.1 can be completed by combining the above observation and the previous claim.

$$(-1)^i Q_i(-1, M) = \det(A_{i+1}) = \sum_{j+k=2i+1} (-1)^{k+i} P_k \cdot P_{j-1} \cdot \det(N)$$

where $0 \leq k \leq d$ and $1 \leq j \leq d+1$. Now shift the j indexing by 1 to get

$$\det(A_{i+1}) = \sum_{j+k=2i} (-1)^{k+i} P_k \cdot P_j \cdot \det(N)$$

where $0 \leq k \leq d$ and $0 \leq j \leq d$. But then,

$$(-1)^i Q_i(-1, M) = (-1)^{2i} \sum_{j+k=2i} (-1)^k P_k \cdot P_j \cdot \det(N). \tag{2.5}$$

The fact that $\widetilde{\alpha}_K(t,M,L) = \sum_{i=0}^{d} P_i(t,M)L^i$ is a minimal degree annihilator implies that $P_0(-1,M)$ and $P_d(-1,M)$ are nonzero. Since the condition $\det(N(-1,M)) \neq 0$ has been assumed,

$$Q_0(-1,M) = \sum_{j+k=0} (-1)^k P_k(-1,M^2) \cdot P_j(-1,M^2) \cdot \det(N(-1,M))$$

$$= P_0(-1,M^2) \cdot P_0(-1,M^2) \cdot \det(N(-1,M)) \neq 0$$

and

$$Q_d(-1,M) = \sum_{j+k=2d} (-1)^k P_k(-1,M^2) \cdot P_j(-1,M^2) \cdot \det(N(-1,M))$$

$$= P_d(-1,M^2) \cdot P_d(-1,M^2) \cdot \det(N(-1,M)) \neq 0.$$

Therefore, the operator $\widetilde{\beta}(t,M,L) = \sum_{i=0}^{d} (-1)^i Q_i(t,M)L^i$ is a nontrivial degree d annihilator of the odd sequence $J_K(2n+1)$ completing the proof of Lemma 2.2.1. □

Until now, is has been assumed that $\det(N) \neq 0$ when evaluated at $t = -1$. It would be nice if this determinant was nonzero in general. Generically though, there are cases where it is clearly zero. For example, assume that a knot K has associated annihilator

$$\widetilde{\alpha_K}(t,M,L) = \sum_{i=0}^{r} P_{2i}(t,M) \cdot L^{2i}$$

for some positive integer r. This annihilator has only even power of L and thus each column of N corresponding to the odd powers of L is the zero column. Then clearly $\det(N) = 0$. Therefore, knots where $A_K(M,L)$ has at least one nonzero odd degree L coefficient should be considered to apply Lemma 2.2.1.

Remark: For twist knots K_m, Anh Tran uses skein theory to show that a variant of N has a nonzero determinant, [32]. In fact, he shows that under a suitable basis, the columns of N are linearly independent. It should be noted that other authors have encountered this matrix obstruction while proving the AJ-conjecture for cables of knots.

2.2.3 Verifying the AJ conjecture

Let K be a knot that satisfies the AJ-conjecture with homogeneous annihilator $\widetilde{\alpha_K}(t, M, L)$. Let C be the $(r, 2)$ cable knot of K. In the previous section, we found the annihilator $\widetilde{\beta}$ of the colored Jones function $J_K(2n+1)$. From the cabling formula (2.2) for the colored Jones function, we have an annihilator of $J_C(n)$ given by

$$\widetilde{\beta}(t, M, L)(M^r L + t^{-2r} M^{-r})$$

Recall (2.1) the A-polynomial of the $(r, 2)$-cable is given in terms of the resultant.

$$A_C(M, L) = F_r(M, L)\mathrm{Res}_\lambda\left(A_K(M^2, \lambda), \lambda^2 - L\right)$$

where $F_r(M, L) = M^{2r}L + 1$ if $r > 0$ and $L + M^{-2r}$ if $r < 0$.

Therefore, to verify the AJ-conjecture it must be shown that

$$\widetilde{\beta}(-1, M, L) = R(M)\mathrm{Res}_\lambda\left(A_K(M^2, \lambda), \lambda^2 - L\right)$$

where $R(M)$ is some function of M.

Lemma 2.2.2. *Let $P(L, M) = \sum_{i=0}^{d} P_i(M)L^i$ be a degree d polynomial. Then,*

$$Res_\lambda(P(M^2, \lambda), \lambda^2 - L) = \sum_{i=0}^{d}\left(\sum_{k+j=2i}(-1)^k P_k(M^2)P_j(M^2)\right)L^i$$

Proof: The proof uses induction on the L degree of $P(L,M)$ and the fact that

$$\mathrm{Res}_\lambda(P(M^2,\lambda),\lambda^2-L)=P(M^2,\sqrt{L})\cdot P(M^2,-\sqrt{L}).$$

When $d=2$, calculate the determinant of the resultant matrix to get

$$\mathrm{Res}_\lambda(P(M^2,\lambda),\lambda^2-L)=P_2{}^2(M^2)L^2+(2P_0(M^2)P_2(M^2)-P_1{}^2(M^2))L+P_0{}^2(M^2).$$

Now assume that the statement holds for degree less than d and let $P(M,L)$ be a degree d polynomial in L. Then

$$\begin{aligned}
&\mathrm{Res}_\lambda(P(M^2,\lambda),\lambda^2-L)=P(M^2,\sqrt{L})\cdot P(M^2,-\sqrt{L})\\
&=(-1)^dP_d{}^2L^d+P_d\sqrt{L^d}\left(\sum_{i=0}^{d-1}(-1)^dP_i\sqrt{L^i}+P_i(-\sqrt{L})^i\right)+\sum_{i=0}^{d-1}P_i\sqrt{L}^i\cdot\sum_{i=0}^{d-1}P_i\sqrt{-L}^i\\
&=(-1)^dP_d{}^2L^d+2(-1)^dP_d\sqrt{L^d}\left(\sum_{\substack{i=0\\ i\equiv d \bmod 2}}^{d-1}P_i\sqrt{L^i}\right)+\mathrm{Res}_\lambda(\sum_{i=0}^{d-1}P_i\lambda^i,\lambda^2-L)\\
&=(-1)^dP_d{}^2L^d+2\left(\sum_{\substack{i=0\\ i\equiv d \bmod 2}}^{d-1}(-1)^dP_dP_i\sqrt{L^{i+d}}\right)+\sum_{i=0}^{d-1}\left(\sum_{k+j=2i}(-1)^kP_kP_j\right)L^i\\
&=\sum_{i=0}^{d}\left(\sum_{k+j=2i}(-1)^kP_kP_j\right)L^i\quad\square
\end{aligned}$$

Now use the set up from the previous section to finish verifying the AJ-conjecture. Recall (2.5),

$$(-1)^iQ_i(-1,M)=(-1)^{2i}\sum_{j+k=2i}(-1)^kP_k\cdot P_j\cdot\det(N).$$

Therefore, by Lemma 2.2.2,

$$\widetilde{\beta}(-1,M,L)=\sum_{i=0}^{d}(-1)^iQ_i(t,M)L^i$$

$$= \sum_{i=0}^{d} (-1)^{2i} \sum_{j+k=2i} (-1)^k P_k \cdot P_j \cdot \det(N) \cdot L^i$$

$$= \det(N) \sum_{i=0}^{d} \sum_{j+k=2i} (-1)^k P_k \cdot P_j \cdot L^i$$

$$= \det(N) \cdot \mathrm{Res}_\lambda(\widetilde{\alpha_K}(-1, M^2, \lambda), \lambda^2 - L)$$

$$= \det(N) \cdot \mathrm{Res}_\lambda \left(R(M) \cdot A_K(M^2, \lambda), \lambda^2 - L\right)$$

$$= \det(N) \cdot R^2(M) \cdot \mathrm{Res}_\lambda \left(A_K(M^2, \lambda), \lambda^2 - L\right)$$

The $R(M)$ in the last two expressions arises because K satisfies the AJ-conjecture. Therefore, it has been shown that this annihilator of the cable knot C evaluated at $t = -1$ is M-essentially equivalent to the A-polynomial $A_C(L, M)$ when $\det(N) \neq 0$. The only thing left to show is that there does not exist an annihilator of lower L degree than $\widetilde{\beta}(t, M, L)(M^r L + t^{-2r} M^{-r})$. To this end, it is sufficient to put some restrictions on the A-polynomial and then use following lemmas and propositions.

Lemma 2.2.3. *If $P(M, L) \in \mathbb{C}[M, L]$ is an irreducible polynomial that contains only even powers of M, then $P(M^2, L)$ is irreducible over $\mathbb{C}[M, L]$.*

Proof: Let $u = M^2$. Since $P(M, L)$ contains only even powers of M, $P(u, L) \in \mathbb{C}[M, L]$. Assume for contradiction that $P(u^2, L)$ is reducible in $\mathbb{C}[M, L]$. By symmetry, if $h(M, L)$ is a factor of $P(u^2, L)$ then so is $h(-M, L)$. Since $P(u, L)$ is irreducible, $P(u^2, L) = h(M, L)h(-M, L)$ where $h(M, L)$ is irreducible in $\mathbb{C}[M, L]$. If $h(M, L)$ contains a term of odd degree in M, let d be the smallest of those degrees. Then $P(u^2, L)$ contains an term of degree d in u, a contradiction. This means that all the terms in $h(M,L)$ are of even degree in M. Therefore, $P(u,L) = h(u,L)h(-u,L)$ is a polynomial factorization, contradicting the assumption that $P(M,L)$ is irreducible.

Lemma 2.2.4 (Tran [32]). *Let $P(M,L) \in \mathbb{C}[M,L]$ be an irreducible polynomial with $P(M,L) \neq P(M,-L)$. Then,*

$$R_K(M,L) = Res_\lambda(P(M,\lambda), \lambda^2 - L) \in \mathbb{C}[M,L]$$

is irreducible and has L degree equal to that of $P(M,L)$.

Proposition 2.2.1 (Tran [33]). *Suppose K is a non-trivial alternating knot. Then the annihilator of the odd sequence $J_K(2n+1)$ has L-degree greater than 1.*

Proposition 2.2.2 (Tran [33]). *For any non-trivial annihilator $\widetilde{\delta}(t,M,L)$ of the odd sequence $J_K(2n+1)$, $\epsilon(\widetilde{\delta})$ is divisible by $L-1$.*

Proposition 2.2.3 (Tran [32]). *Suppose K is a non-trivial knot with reduced alternating diagram D. Let $\widetilde{\delta}(t,M,L)$ be the minimal degree annihilator of $J_K(2n+1)$ and $\widetilde{\Delta}_C(t,M,L)$ be the minimal degree annihilator of $J_C(n)$. Then for odd integers r with $(r+4\eta_-)(r-4\eta_+) > 0$, $\widetilde{\Delta}_C(t,M,L) = \widetilde{\delta}(t,M,L)(L + t^{-2r}M^{-2r})$.*

Theorem 2.2.1. *Let K be a knot with a reduced alternating diagram D that has η_+ positive crossings and η_- negative crossings. Let C be the $(r,2)$-cable of K. Assume the following properties;*

i. K satisfies the AJ-conjecture and $J_K(n)$ has a minimal degree d homogeneous annihilator $\alpha_K(t,M,L)$ with $\widetilde{\alpha}_K(-1,M,L) \overset{M}{=} A_K(M,L)$, where $d \geq 2$.

ii. The A' polynomial, $A'_K(M,L)$ of K is irreducible and $A'_K(M,L) \neq A'_K(M,-L)$.

iii. The matrix $N(-1,M)$ described in Section 2.2.2 has nonzero determinant.

Then for odd integers r with $(r+4\eta_-)(r-4\eta_+) > 0$ *the cable knot* C *satisfies the* AJ*-conjecture.*

Proof: The proof is similar to that of Theorem 1 given by Anh Tran [32]. Conditions (i) and (iii) allow for the application of Lemma 2.2.1 to find a non-trivial operator $\widetilde{\beta}(t,M,L)$ such that $\widetilde{\beta}(t,M,L)J_K(2n+1) = 0$. From the discussion above, $\epsilon(\widetilde{\beta}) \overset{M}{=} (L-1)R_K(M^2,L)$ where $R_K(M^2,L) = \mathrm{Res}(A'_K(M^2,\lambda), \lambda^2 - L)$. Let $\widetilde{\delta}(t,M,L)$ be the minimal degree annihilator of $J_K(2n+1)$. Then $\epsilon(\widetilde{\delta})$ left divides $\epsilon(\widetilde{\beta})$. Since K is alternating Propositions 2.2.1 and 2.2.2 imply $\epsilon(\widetilde{\delta})$ has L degree ≥ 2 and is divisible by $L-1$. It is well known that the A-polynomial has only even powers of M. Therefore, the resultant, $R_K(M^2,L)$, is irreducible over $\mathbb{C}[M,L]$ by condition (ii) and Lemmas 2.2.3 and 2.2.4. Since $\frac{\epsilon(\widetilde{\delta})}{L-1}$ has L degree at least 1 and $R_K(M^2,L)$ is irreducible, conclude that $\frac{\epsilon(\widetilde{\delta})}{L-1} \overset{M}{=} R_K(M^2,L)$. Let $\widetilde{\Delta}_C(t,M,L)$ be the minimal degree annihilator of $J_C(n)$. All that is left to show is $A_C(M,L) \overset{M}{=} \epsilon(\widetilde{\Delta}_C)$. Since $(r+4\eta_-)(r-4\eta_+) > 0$, Proposition 2.2.3 implies that $\widetilde{\Delta}_C = \widetilde{\delta}(L + M^{-2r})$. By these remarks and the cabling formula (2.1) for A-polynomials,

$$\begin{aligned}
A_C(M,L) &= (L-1)R_K(M^2,L)(L+M^{-2r}) \\
&\overset{M}{=} \epsilon(\widetilde{\delta})(L+M^{-2r}) \\
&= \epsilon(\widetilde{\Delta}_C) \qquad \square
\end{aligned}$$

2.2.4 Examples

To apply Theorem 2.2.1, first find a knot with reduced alternating diagram that satisfies the AJ-conjecture. Certain two-bridge knots will meet this requirement.

In [24], conditions were given in order for a knot to satisfy the AJ conjecture. Specifically, all two-bridge knots for which the $\mathrm{SL}_2(\mathbb{C})$ character variety has exactly two irreducible components satisfy the AJ conjecture. We will verify the AJ conjecture for $(r, 2)$-cables of some of these two-bridge knots. Some two-bridge knots are also torus knots or twist knots for which the AJ conjecture has been verified for their cables [31, 32, 34], so we will exclude them from the results even though Theorem 2.2.1 does apply.

The knot 6_2 The A-polynomial of the knot 6_2 (without the $L-1$ factor) is irreducible and is given by

$$A'_{6_2}(M,L) = -L^5M^{26} + (3M^{18} + 2M^{24} + M^{26} - 5M^{22} - 5M^{20} - 2M^{28} + M^{30})L^4 + (-3M^{10} + M^{28} - 12M^{18} + 3M^{14} + 8M^{12} + M^{24} - 13M^{16} - 3M^{26} + 5M^{22} + 3M^{20})L^3 + (8M^{18} + 3M^{10} - 12M^{12} + M^2 - 13M^{14} - 3M^4 + 3M^{16} - 3M^{20} + 5M^8 + M^6)L^2 + (M^4 + 3M^{12} - 2M^2 - 5M^{10} + 1 - 5M^8 + 2M^6)L - M^4$$

The only condition to verify in order to apply Theorem 4.1 is $\det(N) \neq 0$. Since the degree of $A_{6_2}(M, L)$ is 6, N will be a 5×5 matrix. Below, the shorthand $P_i = P_i(M^2)$ is used.

$$N = \begin{pmatrix} P_1 & P_0 & 0 & 0 & 0 \\ P_3 & P_2 & P_1 & P_0 & 0 \\ P_5 & P_4 & P_3 & P_2 & P_1 \\ 0 & P_6 & P_5 & P_4 & P_3 \\ 0 & 0 & 0 & P_6 & P_5 \end{pmatrix}$$

With the aid of Maple the determinant of N is found using the coefficients of

$A_{6_2}(M, L)$. The determinant of $N(-1, M)$ will be a nonzero multiple of the below polynomial and therefore still nonzero.

$$3264\,M^{80} - 1392\,M^{72} + 1209\,M^{76} + 527\,M^{68} - 100\,M^{64} - 9211\,M^{84} - 1113\,M^{88} + 32739\,M^{92} - 24591\,M^{96} + 100161\,M^{104} - 74138\,M^{100} + 139618\,M^{108} - 273161\,M^{112} - 226146\,M^{116} + 583714\,M^{120} + 369251\,M^{124} - 641890\,M^{132} - 1048610\,M^{128} + 1537826\,M^{136} + 1105168\,M^{140} - 1786793\,M^{144} - 1591546\,M^{148} + 1591546\,M^{152} + 1786793\,M^{156} - 1105168\,M^{160} - 1537826\,M^{164} + 641890\,M^{168} + 1048610\,M^{172} - 369251\,M^{176} + 24591\,M^{204} - 32739\,M^{208} + 1113\,M^{212} + 9211\,M^{216} - 1209\,M^{224} + 1392\,M^{228} - 527\,M^{232} + 100\,M^{236} - 3264\,M^{220} - 8\,M^{240} + 74138\,M^{200} - 583714\,M^{180} + 226146\,M^{184} + 273161\,M^{188} - 139618\,M^{192} - 100161\,M^{196} + 8\,M^{60}$$

The knot 6_2 has a reduced alternating diagram with four positive and two negative crossings. Therefore, when $(r-16)(r+8) > 0$, the $(r, 2)$-cable knot of 6_2 will satisfy the AJ conjecture.

Follow the same process to show that the $(r, 2)$-cable knots of some two-bridge knots satisfies the AJ conjecture when r is in the proper range. It would be too cumbersome and take up to much space to list all of the determinants. However, it has been verified with Maple that each relevant determinant is non-zero. Below is a table of some two-bridge knots whose cables satisfy the AJ conjecture by Theorem 2.2.1. The needed A-polynomials were gathered from KnotInfo.

Rolfsen Notation	Two-Bridge Notation
6_2	$\mathfrak{b}(11,3)$
6_3	$\mathfrak{b}(13,5)$
7_3	$\mathfrak{b}(13,3)$
7_5	$\mathfrak{b}(17,7)$
7_6	$\mathfrak{b}(19,7)$
8_2	$\mathfrak{b}(17,3)$
8_3	$\mathfrak{b}(17,13)$
8_4	$\mathfrak{b}(19,5)$
8_6	$\mathfrak{b}(23,7)$
8_7	$\mathfrak{b}(23,5)$
8_8	$\mathfrak{b}(25,9)$
8_9	$\mathfrak{b}(25,7)$
8_{12}	$\mathfrak{b}(29,12)$
8_{13}	$\mathfrak{b}(29,11)$
8_{14}	$\mathfrak{b}(31,12)$
9_3	$\mathfrak{b}(19,13)$

Rolfsen Notation	Two-Bridge Noation
9_4	$\mathfrak{b}(19,13)$
9_5	$\mathfrak{b}(21,5)$
9_7	$\mathfrak{b}(29,13)$
9_8	$\mathfrak{b}(31,11)$
9_9	$\mathfrak{b}(31,9)$
9_{13}	$\mathfrak{b}(37,27)$
9_{14}	$\mathfrak{b}(37,14)$
9_{15}	$\mathfrak{b}(39,16)$
9_{17}	$\mathfrak{b}(39,14)$
9_{18}	$\mathfrak{b}(41,17)$
9_{19}	$\mathfrak{b}(41,16)$
9_{20}	$\mathfrak{b}(41,15)$
9_{21}	$\mathfrak{b}(43,18)$
9_{26}	$\mathfrak{b}(47,18)$
9_{27}	$\mathfrak{b}(49,19)$

CHAPTER 3
QUANTUM $\hat{A}$ CURVES

3.1 Preliminaries

This chapter is motivated by Witten's work on quantum field theory and the Jones polynomial [37]. He showed that a partition function defined using a Feynman path integral satisfies the same skein relations as the Jones polynomial. In what follows, this partition function, at level N, is loosely taken as the definition of the N^{th} colored Jones polynomial. Then through the process of quantization, the A-polynomial is promoted to an operator that is a recurrence relation for this version of the colored Jones polynomial for torus knots.

3.1.1 The Chern-Simons Line Bundle

The goal of this section is to define a bundle over the character variety of the torus boundary of a 3-manifold. [20]

Let M be a 3-manifold with torus boundary T. Let $\{\mu, \lambda\}$ be the standard basis for $\pi_1(T)$ and denote by $\chi(T)$ (respectively $\chi(M)$) the character variety $\chi(\pi_1(T))$ (respectively $\chi(\pi_1(M))$). Define a map $v : \mathrm{Hom}(\pi_1(T), \mathbb{C}) \to \chi(T)$ by $v(f) = (\alpha \mapsto e^{2\pi i f(\alpha)})$. This is a 2:1 branch covering map with covering group $G \cong \mathbb{Z} \oplus \mathbb{Z} \rtimes \mathbb{Z}_2$ which has presentation,

$$G = \langle x, y, b \,|\, xy - yx = bxbx = byby = b^2 = 1 \rangle.$$

Send each $f \in \mathrm{Hom}(\pi_1(T), \mathbb{C})$ to the pair $(f(\mu), f(\lambda)) \in \mathbb{C} \times \mathbb{C}$. With this identification, the action of G on $\mathrm{Hom}(\pi_1(T), \mathbb{C})$ is

$$x(z, w) = (z + 1, w), \quad y(z, w) = (z, w + 1), \quad b(z, w) = (-z, -w).$$

Extend the this action to the trivial bundle $\mathrm{Hom}(\pi_1(T), \mathbb{C}) \times \mathbb{C}^*$, where $\mathbb{C}^*$ are the nonzero complex numbers, by

$$x(z, w, \zeta) = (z+1, w, \zeta e^{2\pi i w}),\ y(z, w, \zeta) = (z, w+1, \zeta e^{-2\pi i z}),\ b(z, w, \zeta) = (-z, -w, \zeta).$$

Define the Chern-Simons line bundle over the character variety $\chi(T)$ as the quotient bundle

$$CS(T) = \mathrm{Hom}(\pi_1(T), \mathbb{C}) \times \mathbb{C}^*/G.$$

As explained in [20], although the action has been defined by a fixed basis of $\pi_1(T)$, the action only depends on the orientation of T. Therefore, for the remainder of the paper, elements of $CS(T)$ are written $[z, w, \zeta]$ with the assumption of a fixed standard basis $\{\mu, \lambda\}$.

The Chern-Simons section is a map $CS_M : \chi(M) \to CS(T)$.

$$CS_M : \rho \mapsto \left[z, w, e^{2\pi i cs(\rho)}\right] \tag{3.1}$$

where $cs(\rho)$ is the Chern-Simons invariant associated to the representation ρ.

The following theorem shows how to calculate the change in the Chern-Simons invariant along a path of representations.

Theorem 3.1.1 (Kirk, Klassen [20])**.** *Let M denote an oriented 3-dimensional manifold whose boundary $\partial M = T$ consists of a 2-dimensional torus. Let $\{\mu, \lambda\}$ denote* *an oriented basis for $\pi_1(T)$. Let $\rho(t) : \pi_1(M) \to SL_2((C))$, $t \in [0,1]$, be a path of representations where $(z(t), w(t))$ denote a lift of $\rho(t)_{|\pi_1(T)}$ to $\mathbb{C}^2$. Suppose*

$$CS_M(\rho(t)) = [z(t), w(t), cs(z(t))]$$

for all t. Then,

$$cs(z(1)) \cdot cs(z(0))^{-1} = e^{2\pi i \int_0^1 z(t)w'(t) - z'(t)w(t)dt}$$

and if $z(0)$ corresponds to the trivial representation, $cs(z(0)) = 1$.

Assuming a path of representations is followed, the formula from Theorem 3.1.1 can be rewritten as

$$cs(z) = cs(z_0) \cdot e^{2\pi i \int_{z_o}^{z} zdw - wdz} \tag{3.2}$$

which gives a local expression of the Chern-Simons section as a function of z in a neighborhood of z_0.

3.1.2 The A Polynomial for Torus Knots

Denote by $T(a,b)$ the (a,b)-torus knot. The A-polynomials of torus knots can be found in [4] and are given by

$$A_{T(p,q)}(m,l) = \begin{cases} (l-1)(lm^{2b}+1) & : p = 2, b > 2 \\ (l-1)(l^2m^{2ab}-1) & : a, b > 2 \end{cases}$$

The A-polynomial gives a parameterization of the representation space. Lift each component of the zero locus of the A-polynomial to a curve in $\mathbb{C} \times \mathbb{C}$ by using logarithmic coordinates. Specifically, let $m = e^{2\pi i z}$ and $-l = e^{2\pi i w}$. Using the principal branch of log, the A-curves of the (a,b)-torus knots are cut out by

$$A_{T(a,b)}(z,w) = \begin{cases} w(w+2bz) & : a=2, b>2 \\ w(w+abz)(w+\frac{1}{2}+abz) & : a,b>2 \end{cases} \tag{3.3}$$

3.2 Quantum Curves

3.2.1 A Quantization of the A-Polynomial

The A-polynomial, $A(z,w)$, cuts out a Lagrangian subvariety of $\mathbb{C}\times\mathbb{C}$ endowed with the symplectic form

$$2\pi ih\, dz \wedge dw. \tag{3.4}$$

The A-curve is the phase space of analytically continued Chern-Simons theory [15] with a classical state being a $SL_2(\mathbb{C})$ representation up to trace equivalence. The goal is to promote the A-curve to an operator $\hat{A}(q,M,L)$ that will annihilate $cs^N(z)$ (3.1) for some operators M, L quantizing m, l. This is reminiscent of the AJ-conjecture [11] where the recurrence relation, $\widetilde{\alpha}_K(q,M,L)$, of the colored Jones function is expected to semi-classically limit to the A-polynomial (c.f Section 2.1.4). In that setting, the operators M and L are elements of a ring called the quantum torus and satisfy the relation $LM = q^2ML$. In the case at hand, it will be shown that the following operators acting on holomorphic functions lead to the same non-commutativity relation.

$$M = e^{2\pi iz} \quad \text{and} \quad L = e^{h\frac{d}{dz}+2\pi iw} \quad (q = e^{\frac{\pi i}{h}}) \tag{3.5}$$

Notice that as $h \to 0$, $M \to e^{2\pi iz} = m$ and $L \to e^{2\pi iw} = -l$. Here $h = \frac{1}{N}$ where $N \in \mathbb{N}$. In this sense, the operators are a coherent quantization of the classical coordinates. It will be seen that in the case of torus knots, the annihilator of certain power of the Chern-Simons section naturally limits to the geometric factor of the A-polynomial.

3.2.2 The Operator L

The action of $M = e^{2\pi i z}$ on holomorphic functions of z is by multiplication. The action of $L = e^{h\frac{d}{dz}+2\pi i w(z)}$ needs more clarification.

Lemma 3.2.1. *The operator* $L = e^{h\frac{d}{dz}+2\pi i w(z)}$ *acts on holomorphic functions of* z *as*

$$L(f(z)) = f(z+h)\exp\left(\frac{2\pi i}{h}\int_z^{z+h} w(u)du\right).$$

Proof: If $g(z,t) = e^{t\left(h\frac{d}{dz}+2\pi i w(z)\right)}f(z)$, then $g(z,t)$ satisfies the partial differential equation

$$\frac{\partial g}{\partial t} - h\frac{\partial g}{\partial z} = 2\pi i w \cdot g$$

with boundary condition $g(z,0) = f(z)$.

Let $z(t) = z - ht$ and $G(t) = g(z(t),t)$. With this substitution, the above PDE can be written as the ODE $G'(t) = 2\pi i w(z(t)) \cdot G(t)$ where $G(0) = f(z)$. The solution is

$$G(t) = f(z)\exp\left(2\pi i\int_0^t w(z(s))ds\right).$$

Replacing z with $z + th$ and setting $t = 1$ yields

$$g(z,1) = f(z+h)\exp\left(2\pi i\int_0^1 w(z+(1-s)h)ds\right).$$

Now substitute $u = z + (1-s)h$ to conclude

$$L(f(z)) = g(z,1) = f(z+h)\exp\left(\frac{2\pi i}{h}\int_z^{z+h} w(u)du\right). \qquad \square$$

Corollary 3.2.0.1. *The operators M and L acting on holomorphic functions of z satisfy the relation $LM = q^2 ML$ where $q = e^{\frac{\pi i}{h}}$.*

Proof: Let $f(z)$ be a holomorphic function over $\mathbb{C}$. By Lemma 3.2.1 and the definition of M,

$$LM(f(z)) = L(e^{2\pi i z} f(z)) = e^{2\pi i(z+h)} f(z+h) \exp\left(\frac{2\pi i}{h}\int_z^{z+h} w(u)du\right)$$

$$= q^2 ML(f(z)) \quad \square$$

3.3 The $\hat{A}$ Curve of Torus Knots

Let $A \subset \mathbb{C}\times\mathbb{C}$ be the zero locus of the A-polynomial in logarithmic coordinates $(z, w) \in \mathbb{C}\times\mathbb{C}$. There is a projection map $\pi : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ onto the first factor given by $\pi(z, w) = z$. There are two kinds of "singular points" on A, those where A is genuinely singular as an algebraic curve and those where the projection map restricted to A is not a local submersion. Away from the finite set of singular points there is a unique tangent vector $\widetilde{\frac{d}{dz}}$ such that $\pi_*\left(\widetilde{\frac{d}{dz}}\right) = \frac{d}{dz}$. Define an operator L acting on local holomorphic sections of the Chern-Simons line bundle over A by

$$L = e^{h\widetilde{\frac{d}{dz}} + 2\pi i w(z)}. \tag{3.6}$$

where $w(z)$ is a local parameterization of the A-curve. This operator acts locally on holomorphic sections of the Chern-Simons bundle in the same way that L from Lemma 3.2.1 acts on holomorphic functions of z.

3.3.1 T(2,b) knots

The A-polynomial of the $(2, b)$-torus knot (3.3) has two factors. The factor w corresponds to the abelian component of the character variety while $w + 2bz$ defines the *geometric* component denoted for now by A_g. On the geometric component, there is a local expression $w(z) = -2bz$ which defines a plane curve with no singular points. Fix a point $z_0 \in A_g$. From Theorem 3.1.1 there is a local expression for the Chern-Simons section given by

$$cs(z) = cs(z_0) \cdot e^{2\pi i \int_{z_o}^{z} zdw - wdz}$$

where the integral is assumed to be over a path from z_0 to z contained in A_g. Let $h = \frac{1}{N}$ with $N \in \mathbb{N}$ and consider $cs^{\frac{1}{h}}(z)$ as a section of the N-fold tensor power of the bundle $CS(T)$ defined in 3.1.1. The natural number N represents the level of quantization.

Lemma 3.3.1. *The operator* $L = e^{h\frac{d}{dz} + 2\pi i w(z)}$ *acts on* $cs^{\frac{1}{h}}(z)$ *as*

$$L\left(cs^{\frac{1}{h}}(z)\right) = cs^{\frac{1}{h}}(z) \exp\left(\frac{2\pi i}{h} \int_{z}^{z+h} zdw\right)$$

Proof: The lemma follows from a direct application of Lemma 3.2.1.

Theorem 3.3.1. *On the component of the A-curve parameterized by* $w(z) = -2bz$ (3.3)*, the section* $cs^{\frac{1}{h}}(z)$ *of the* N^{th} *tensor power bundle* $CS^N(T)$*, where* T *is the boundary of the* $T(2, b)$ *knot complement in* S^3*, is annihilated by the operator*

$$\hat{A} = 1 - q^{2b} M^{2b} L \tag{3.7}$$

Proof: Recall that $h = \frac{1}{N}$ and $q = e^{\pi i h}$. The parameterization $w(z) = -2bz$ gives $zdw = -2bz\,dz$. By Lemma 3.3.1,

$$\begin{aligned} L\left(cs^{\frac{1}{h}}(z)\right) &= cs^{\frac{1}{h}}(z)\exp\left(\frac{2\pi i}{h}\int_z^{z+h} zdw\right) \\ &= cs^{\frac{1}{h}}(z)\exp\left(-\frac{2\pi i b}{h}(2zh+h^2)\right) \\ &= cs^{\frac{1}{h}}(z)(e^{2\pi i z})^{-2b}(e^{\pi i h})^{-2b} \\ &= q^{-2b}M^{-2b}(cs^{\frac{1}{h}}(z)) \end{aligned}$$

Therefore, $(q^{-2b}M^{-2b} - L)cs^{\frac{1}{h}}(z) = 0$. Multiplying on the left by $q^{2b}M^{2b}$ gives the desired result. □

Corollary 3.3.1.1. $\hat{A}\big|_{q=-1} = \frac{A_{T(2,b)}(m,l)}{l-1}$

Proof: When $q = -1$, $L = -l$. Therefore, $\hat{A}\big|_{q=-1} = lm^{2b} + 1$. □

3.3.2 T(a,b) knots

The A-curve of $T(a, b)$ torus knots has two geometric components. Denote by (A_1, w_1) the component corresponding to the factor $w + \frac{1}{2} + abz$ and (A_2, w_2) the component corresponding to $w + abz$ from (3.3). The operator L (3.6) changes depending on the parameterization. Let L_i be the operator defined by the parameterization $w_i(z)$, for $i = 1, 2$. With this notation, the operators satisfy $L_1 = -L_2$. Denote by $cs_i(z)$ the Chern-Simons section over the component (A_i, w_i).

Theorem 3.3.2. *Over the (A_i, w_i)-component of the A-curve parameterized by $w_i(z)$ the section $cs_i^{\frac{1}{h}}(z)$ of the N^{th} tensor power bundle $CS^N(T)$, where T is the boundary* *of the $T(a,b)$ knot complement in S^3, is annihilated by the operator*

$$\hat{A}_i = 1 - q^{ab} M^{ab} L_i \tag{3.8}$$

Proof: The proof that $cs_i^{\frac{1}{h}}(z)$ is annihilated by $\hat{A}_i$ on either component is almost identical to that of Theorem 3.3.1 since $zdw_i = -abz$. □

Corollary 3.3.2.1. *The operator $\hat{A}_1 \hat{A}_2 = (1 - q^{ab} M^{ab} L_1)(1 - q^{ab} M^{ab} L_2)$ annihilates the section $cs^{\frac{1}{h}}(z)$ defined over both geometric components of the A curve.*

Proof: It must be shown that $\hat{A}_1 \hat{A}_2$ annihilates both $cs_1^{\frac{1}{h}}(z)$ and $cs_2^{\frac{1}{h}}(z)$. By Theorem 3.3.2,

$$(1 - q^{ab} M^{ab} L_1)(1 - q^{ab} M^{ab} L_2) cs_1^{\frac{1}{h}}(z) = (1 - q^{ab} M^{ab} L_1) 2 cs_1^{\frac{1}{h}}(z) = 0$$

and

$$(1 - q^{ab} M^{ab} L_1)(1 - q^{ab} M^{ab} L_2) cs_2^{\frac{1}{h}}(z) = 0$$

which completes the proof. □

Corollary 3.3.2.2. $(\hat{A}_1 \circ \hat{A}_2)\big|_{q=-1} = \frac{A_{T(a,b)}(m,l)}{l-1}$

Proof: $\hat{A}_1 \circ \hat{A}_2 = (1 - q^{ab} M^{ab} L_1)(1 - q^{ab} M^{ab} L_2) = (1 - q^{ab} M^{ab} L_1)(1 + q^{ab} M^{ab} L_1)$ If $q = -1$, then without loss of generality, $L_1 = -l$. After this replacement, the A-polynomial of $T(a,b)$ (without the $(l-1)$ factor) is recovered. □

Remark: The factor $(l-1)$ of the A-polynomial corresponds to $w = 0$. In this case, $cs(z) = 1$ and $L(f(z)) = f(z+h)$. Therefore, the operator $L - 1$ annihilates $cs^{\frac{1}{h}}(z)$.

3.4 Conclusions and Discussion

As mentioned in the introduction, the motivation for finding an annihilator of the Chern-Simons section (3.1) stems from a relationship between the Witten path integral [37] and the Jones polynomial.

3.4.1 Chern-Simons Theory

Let M be a compact oriented 3-manifold with a single torus boundary and consider the principal $\mathrm{SL}_2(\mathbb{C})$-bundle, P, over M. Let A be an $sl_2(\mathbb{C})$-valued one form on M and define the *Chern-Simons action* on A by

$$cs(A) = \frac{t}{8\pi}\int_M Tr\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) + \frac{\widetilde{t}}{8\pi}\int_M Tr\left(\overline{A} \wedge d\overline{A} + \frac{2}{3}\overline{A} \wedge \overline{A} \wedge \overline{A}\right)$$

where Tr denotes the trace and $t = N + is, \bar{t} = N - is$ are *coupling constants*. The integer N is called the level and $s \in \mathbb{R}(\text{or } i\mathbb{R})$ is introduced to ensure the action behaves consistently under a change of orientation on M [15]. Using this action, define the following partition function by means of the Feynman path integral.

$$Z(M) = \int_{\mathcal{A}} e^{ics(A)}\mathcal{D}A \tag{3.9}$$

The partition function is not rigorously defined since the measure, $\mathcal{D}A$, is postulated on the (infinite dimensional) space, $\mathcal{A}$, of connections on M. Proceeding heuristically, it was shown that in the case of compact gauge group, (3.9) satisfies the same skein relation as the colored Jones polynomial [37].

To make a more concrete connection between the partition function and the current paper, it is prudent to discuss quantum perturbation theory. The analytically continued partition function can be written as a finite sum over contributions from different critical points,

$$Z(M,h,\widetilde{h}) = \sum_{\alpha,\widetilde{\alpha}} n_{\alpha,\widetilde{\alpha}} Z^{\alpha}(M,h)\overline{Z}^{\widetilde{\alpha}}(M,\widetilde{h}) \tag{3.10}$$

where $h = \frac{1}{t}$ and $\widetilde{h} = \frac{1}{\bar{t}}$, $n_{\alpha,\widetilde{\alpha}} \in \mathbb{Z}$, and $\alpha, \widetilde{\alpha}$ label the local branch and conjugate branch of the A-curve. In the limit, $h \to 0$, the *holomorphic blocks*, $Z_{\alpha}(M,h)$, have asymptotic expansion given by [8],

$$Z^{\alpha}(M,h) \sim \exp\left(\frac{1}{h}S_0^{\alpha} - \frac{1}{2}\delta^{\alpha}\log h + \sum_{n=1}^{\infty} S_n^{\alpha} h^{n-1}\right) \tag{3.11}$$

The leading order term, $S_0^{\alpha}(z) = 2\int_{A(m,l)=0}^{z} w(z)dz$ is the value of the classical Chern-Simons section on the α^{th} branch of the A-curve. It is related to (3.2) by an application of integration by parts. Given a classical state, (i.e. a flat connection A) there is an associated flat bundle, E_A, over M. The term δ^{α} is the following difference in the dimension of the cohomology groups of the bundle [8] and is therefore locally constant.

$$\delta^{\alpha} = dim(H^1(M,E_A) - dim(H^0(M,E_A))$$

The term $S_1^{\alpha}(z) = \frac{1}{2}\log(T(z))$ is the twisted Reidemeister torsion with coefficients in the adjoint representation[9]. The goal is to extend the defining polynomial of the A-curve to an operator, $\hat{A}(q,M,L) = \sum_{i=0}^{\infty} a_i(q,M)L^i$, that annihilates the partition function (3.10). The equation

$$\hat{A}Z = 0 \tag{3.12}$$

leads to an infinite hierarchy of difference equations that can be solved recursively given the initial condition $S_0^{\alpha}(z)$ [8]. In the case of torus knots, the Reidemeister torsion is locally constant along the geometric component of the A-curve. Therefore, all the higher order terms in the asymptotic expansion (3.11) are left constant under the action of L. Thus the local partition function and the local operator $\hat{A}$ are completely determined by the Chern-Simons section.

3.4.2 Matching Results

The AJ-conjecture was discussed in Chapter 2 and asserts that $\widetilde{\alpha}_K(-1, M, L) = R(M)A_K(M, L)$ where $R(M)$ is some rational function of M, and $A_K(M, L)$ is the A-polynomial of K. In the case of torus knots, the AJ-conjecture has been verified [32].

For $T(2, b)$ torus knots, the colored Jones polynomial is annihilated by the operator $\widetilde{\alpha}(q, M, L) = c_2L^2 + c_1L + c_0$ where

$$c_2 = q^2M^2 - q^{-2}M^{-2}$$

$$c_1 = q^{-2b}\left(q^{-4b}M^{-2b}(q^2M^2 - q^{-2}M^{-2}) - (q^6M^2 - q^{-6}M^{-2})\right)$$

$$c_0 = -q^{-4b}M^{-2b}(q^6M^2 - q^{-6}M^{-2}).$$

This annihilator can be factored as

$$\widetilde{\alpha}(q, M, L) = \left((q^2M^2 - q^{-2}M^{-2})L - (q^6M^2 - q^{-6}M^{-6})q^{-2b}\right)\left(L + q^{-2b}M^{-2b}\right)$$

Recall the annihilator $\hat{A}(q, M, L) = 1 + q^{2b}M^{2b}L$ from Theorem 3.3.1. It can be seen that these annihilators satisfy

$$\widetilde{\alpha}(q, M, L) = \left((q^2M^2 - q^{-2}M^{-2})L - (q^6M^2 - q^{-6}M^{-6})q^{-2b}\right)q^{2b}M^{2b}\hat{A}(q, M, L)$$

and

$$\widetilde{\alpha}(-1, M, L) = (M^2 - M^{-2})(L - 1)(L + M^{-2b}) = (M^2 - M^{-2})(L - 1)\hat{A}(-1, M, L).$$

In the case of $T(a,b)$ torus knots, the colored Jones polynomial is annihilated by the operator $c_3L^3 + c_2L^2 + c_1L + c_0$ where

$$c_3 = q^2(q^{2(a+b)}M^{a+b} + q^{-2(a+b)}M^{-(a+b)}) - q^{-2}(q^{2(a-b)}M^{a-b} + q^{-2(a-b)}M^{-(a-b)})$$

$$c_2 = -q^{-2ab}\left(q^2(q^{4(a+b)}M^{a+b} + q^{-4(a+b)}M^{-(a+b)}) + q^{-2}(q^{4(a-b)}M^{a-b} + q^{-4(a-b)}M^{-(a-b)}\right)$$

$$c_1 = -q^{-8ab}M^{-2ab}c_3$$

$$c_0 = -q^{-4ab}M^{-2ab}c_2.$$

This operator can also be factored where one of the factors matches with the $\hat{A}$ curve from Theorem 3.3.2.

CHAPTER 4
TANGLE FUNCTORS FOR SEMICYCLIC REPRESENTATIONS

4.1 Preliminaries

Throughout this chapter $q = e^{\pi \mathbf{i}/N}$ where $N \geq 3$ is an odd counting number. The quantum integer l, denoted $[l]$ is defined as,

$$[l] = \frac{q^l - q^{-l}}{q - q^{-1}}.$$

The quantum factorial is defined recursively by $[0]! = 1$, and $[n]! = [n][n-1]!$. The quantum binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

The quantum binomial theorem states that if $AB = q^2 BA$ then,

$$(A+B)^n = \sum_{k=0}^{n} q^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} A^k B^{n-k}.$$

The Weyl algebra W_q is a Hopf algebra generated by E, K and K^{-1} with relation $KE = q^2 EK$, antipode, $S(K) = K^{-1}$, $S(E) = -EK^{-1}$, counit given by $\epsilon(K) = 1$, $\epsilon(E) = 0$, and comultiplication $\Delta(K) = K \otimes K$ and $\Delta(E) = E \otimes K + 1 \otimes E$.

4.1.1 Cyclic Representations of the Weyl Algebra

Let V be the finite dimensional vector space over the complex numbers with basis v_i where $i \in \{0, 1, \ldots, N-1\}$. Choose a nonzero complex number a. Let $M_{N,N}(\mathbb{C})$ denote $N \times N$ matrices with complex coefficients, identified with $End(V)$

via the choice of basis. Define a representation of W_q by,

$$\rho_a : W_q \to M_{N,N}(\mathbb{C}),$$

by $\rho_a(K)v_i = q^{2i}v_i$, and $\rho_a(E)v_i = v_{i+1}$ when $i < N-1$ and $\rho_a(E)v_{N-1} = av_0$.

Cyclic representations of the Weyl algebra have been studied by Kashaev [19], Baseilhac and Benedetti [2] and Bonahon et al [3].

4.1.2 A Version of $U_q(sl_2)$

Definition 4.1. Let $q \neq 0, \pm 1$ be a complex number. The algebra $U_q(sl_2)$ is generated by K, K^{-1}, E and F with relations,

$$KE = q^2EK, \quad KF = q^{-2}FK, \quad \text{and } EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{4.1}$$

The counit, antipode and comultiplication extend as,

$$\epsilon(K) = 1,\ \epsilon(E) = 0,\ \epsilon(F) = 0 \tag{4.2}$$

$$S(K) = K^{-1},\ S(E) = -EK^{-1},\ S(F) = -KF \tag{4.3}$$

$$\Delta(K) = K \otimes K,\ \Delta(E) = E \otimes K + 1 \otimes E,\ \Delta(F) = F \otimes 1 + K^{-1} \otimes F. \tag{4.4}$$

It is important to emphasize that only the cases $q = e^{\pi \mathbf{i}/N}$ where $N \geq 3$ is an odd counting number are being considered.

The following equations and proposition will be useful in showing that the R-matrix, defined in Section 4.2, satisfies the quantum Yang-Baxter equation. By induction

$$EF^l = F^lE + [l]F^{l-1}\frac{q^{1-l}K - q^{l-1}K^{-1}}{q - q^{-1}}.$$

Also

$$q^{\frac{H\otimes H}{2}}(E\otimes 1)q^{-\frac{H\otimes H}{2}} = E\otimes K \tag{4.5}$$

$$q^{\frac{H\otimes H}{2}}(1\otimes E)q^{-\frac{H\otimes H}{2}} = K\otimes E \tag{4.6}$$

$$q^{\frac{H\otimes H}{2}}(F\otimes 1)q^{-\frac{H\otimes H}{2}} = F\otimes K^{-1} \tag{4.7}$$

$$q^{\frac{H\otimes H}{2}}(1\otimes F)q^{-\frac{H\otimes H}{2}} = K^{-1}\otimes F \tag{4.8}$$

where $K = q^H$.

Proposition 4.1.1.

$$(\Delta\otimes Id)(q^{\frac{H\otimes H}{2}}) = q^{\frac{\Delta(H)\otimes H}{2}} \quad \textit{and} \quad (Id\otimes\Delta)(q^{\frac{H\otimes H}{2}}) = q^{\frac{H\otimes\Delta(H)}{2}}.$$

Proof. The fact that $\Delta(K) = K\otimes K$ forces $\Delta(H) = H\otimes Id + Id\otimes H$. Using induction, it can be shown that

$$\Delta(H^n) = \sum_{k=0}^{n}\binom{n}{k}H^k\otimes H^{n-k}.$$

Then,

$$\begin{aligned}
(\Delta\otimes Id)\exp\left(\frac{h}{4}H\otimes H\right) &= (\Delta\otimes Id)\left(\sum_{n=0}^{\infty}\frac{h^n}{4^n n!}H^n\otimes H^n\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{h^n}{4^n n!}\binom{n}{k}H^k\otimes H^{n-k}\otimes H^n\\
&= \sum_{n=0}^{\infty}\left(\frac{h^n}{4^n n!}Id\otimes H^n\otimes H^n\right)\left(\sum_{k=0}^{n}\binom{n}{k}H^k\otimes H^{-k}\otimes Id\right)\\
&= \sum_{n=0}^{\infty}\left(\frac{h^n}{4^n n!}Id\otimes H^n\otimes H^n\right)\left(H\otimes H^{-1}\otimes Id + Id\otimes Id\otimes Id\right)^n\\
&= \sum_{n=0}^{\infty}\frac{h^n}{4^n n!}\left(Id\otimes H\otimes H + H\otimes Id\otimes H\right)^n\\
&= \exp\left(\frac{h}{4}\Delta(H)\otimes Id\right).
\end{aligned}$$

□

4.1.3 Extending Cyclic Representations of the Weyl Algebra to $U_q(sl_2)$.

The cyclic representations of W_q defined in Section 4.1.1 be extended to N representations of $U_q(sl_2)$ as follows. Let V be the vector space over the complex numbers with basis v_k where $k \in \{0, 1, \ldots, N-1\}$. Fix $i \in \{0, 1, ..., N-1\}$ and $a \in \mathbb{C} - \{0\}$. Let $\rho_{a,i}(K)v_k = q^{1-N+2(k-i)}v_k$, and $\rho_{a,i}(E)v_k = v_{k+1}$ when $i < N-1$ and $\rho_{a,i}(E)v_{N-1} = av_0$. Notice that $\rho_{a,i}(E)^N = a \cdot I_N$ and $\rho_{a,i}(K)^N = I_N$ where I_N is the $N \times N$ identity matrix.

We find a representation $\rho_{a,i}$ of F such that $\rho_{a,i}(F)^N = 0$. Under this condition, the equation $\rho_{a,i}(EF - FE) = \rho_{a,i}(\frac{K-K^{-1}}{q-q^{-1}})$ leads to the following solution for $\rho_{a,i}(F)$.

Denote by $\overline{j}$ the equivalence class of $j \mod N$. Define

$$\rho_{a,i}(F)v_{\overline{i+k}} = \begin{cases} \sum_{j=0}^{\overline{k-1}} -[2(k-j)+N-1]v_{\overline{i+k-1}} & : \overline{i+k} \neq \overline{0} \\ \\ \sum_{j=0}^{\overline{k-1}} -\frac{[2(k-j)+N-1]}{a}v_{\overline{i+k-1}} & : \overline{i+k} = \overline{0} \end{cases}$$

The notation $\overline{j}$ means the remainder on division by N. Equivalently, $\rho_{a,i}(F)$ can be expressed using the sum to product formula,

$$\rho_{a,i}(F)v_{\overline{i+k}} = \begin{cases} [k][N-k]v_{\overline{i+k-1}} & : \overline{i+k} \neq \overline{0} \\ \\ \frac{[k][N-k]}{a}v_{\overline{i+k-1}} & : \overline{i+k} = \overline{0} \end{cases}$$

The definition of these N representations leads to the following relations.

$$\rho_{a,i}(F)v_j = \rho_{a,i+k}(F)v_{\overline{j+k}} \tag{4.9}$$

$$\rho_{a,i}(K)v_j = \rho_{a,i+k}(K)v_{\overline{j+k}} \tag{4.10}$$

$$\rho_{a,i}(E)v_j = \rho_{a,i+k}(E)v_j \quad \forall k \tag{4.11}$$

In fact, all N of the representations are isomorphic via conjugation by $\rho_{a,i}(E)$.

$$\rho_{a,i}(E)^j \rho_{a,i}(F) \rho_{a,i}(E)^{-j} = \rho_{a,\overline{i+j}}(F) \quad (4.12)$$

$$\rho_{a,i}(E)^j \rho_{a,i}(K) \rho_{a,i}(E)^{-j} = \rho_{a,\overline{i+j}}(K) \quad (4.13)$$

These representations are a subclass of those studied in [14] and are called semicyclic representations. Since there is only one semicyclic representation up to isomorphism, the representation that is most convenient will be used.

4.1.4 The Standard Irreducible Representation

One of the main goals of this chapter is to compare the semicyclic and standard irreducible representations of $U_q(sl_2)$. Let ρ_0 denote the standard N-dimensional irreducible representation of $U_q(sl_2)$. Fix a basis $\{v_0, v_1, ..., v_{N-1}\}$ of a vector space V, then $\rho_0(E), \rho_0(F)$, and $\rho_0(K)$ act on this basis as follows. $\rho_0(E)v_i = v_{i+1}$ for $i \neq N-1$ and $\rho_0(E)v_{N-1} = 0$, $\rho_0(F)v_i = [i][N-i]v_{i-1}$, $\rho_0(K)v_i = q^{1-N+2i}v_i$. The standard representation, ρ_0, is similar to the semicyclic representation $\rho_{a,0}$. In fact, the only difference is $\rho_0(E)v_{N-1} = 0$ while $\rho_{a,0}(E)v_{N-1} = av_0$.

4.2 The R-matrix

In the case of the standard irreducible representation ρ_0, the R-matrix

$$R = q^{\frac{H \otimes H}{2}} \sum_{l=0}^{N-1} \frac{(q-q^{-1})^l}{[l]!} q^{\frac{l(l-1)}{2}} E^l \otimes F^l,$$

satisfies $R \Delta R^{-1} = \Delta'$ where Δ' is the flipped comultiplication. This equation also holds for subrepresentations of tensor powers of the two dimensional irreducible representation. This formula first appeared in Kirby and Melvin [21], only they were working in the quotient of $U_q(sl_2)$ by $E^N = F^N = 0$, and $K^{2N} = 1$. This formula for the R-matrix also appears in the work of Ohtsuki where he notices that the condition $K^{2N} = 1$ is superfluous. It is also used in the unfolded version of the quantum group, where a generator H is added with $q^H = K$. In the following proposition, it will be shown that the image of R in the representations $\rho_{a,i}$ conjugates the image of comulitplication to the image of flipped comultiplication.

Proposition 4.2.1. *For all i, $\rho_{a,i}(R)$ and $Z \in U_q(sl_2)$, satisfies*

$$(\rho_{a,i} \otimes \rho_{a,i})(R\Delta(Z)R^{-1}) = (\rho_{a,i} \otimes \rho_{a,i})(\Delta'(Z)). \tag{4.14}$$

Proof. In the following proof, the notation $\rho_{a,i}$ is suppressed but instead E, F, K and R are used to mean their image under the representation $\rho_{a,i}$. Since $F^N = 0$ the proof that $R\Delta(F) - \Delta'(F)R = 0$ and $R\Delta(K) - \Delta'(K)R = 0$ is the same as the proof appearing in [21]. All that is left to show is $R\Delta(E) - \Delta'(E)R = 0$.

To simplify notation, write

$$R = q^{\frac{H\otimes H}{2}} \sum_{l=0}^{N-1} c_l E^l \otimes F^l.$$

Expanding,

$$R\Delta(E) - \Delta'(E)R = 0$$

using equation 4.4 we get,

$$q^{\frac{H\otimes H}{2}}(\sum_{l=0}^{N-1} c_l E^l \otimes F^l)(E \otimes K + 1 \otimes E) - (K \otimes E + E \otimes 1)q^{\frac{H\otimes H}{2}}(\sum_{l=0}^{N-1} c_l E^l \otimes F^l) = 0.$$

Using the commutation relations for $q^{\frac{H\otimes H}{2}}$ from equation 4.5, move it to the front of the second term and factor it out to get,

$$q^{\frac{H\otimes H}{2}}\left((\sum_{l=0}^{N-1} c_l E^l\otimes F^l)(E\otimes K+1\otimes E)-(1\otimes E+E\otimes K^{-1})(\sum_{l=0}^{N-1} c_l E^l\otimes F^l)\right)=0.$$

Cancel $q^{\frac{H\otimes H}{2}}$, distribute and then collect in powers of E, to get,

$$\sum_{l=0}^{N-1} c_l E^{l+1}\otimes(F^l K-K^{-1}F^l)+\sum_{l=0}^{N-1} c_l E^l\otimes(F^l E-EF^l)=0. \tag{4.15}$$

There is exactly one term with E^N appearing in it,

$$c_{N-1}E^N\otimes(F^{N-1}K-K^{-1}F^{N-1}).$$

All other terms will cancel as in the proof in [21] so we only need to show this term is zero.

Under all representations $\rho_{a,i}$, F has rank $N-1$, so F^{N-1} has rank 1. Specifically v_{i-1} spans the cokernel of F^{N-1}, and its kernel is spanned by all other v_j. Applying $F^{N-1}K-K^{-1}F^{N-1}$ to v_{i-1} under $\rho_{a,i}$ gives

$$\begin{aligned}
F^{N-1}K-K^{-1}F^{N-1}v_{i-1} &= F^{N-1}K-q^{2N-2}F^{N-1}K^{-1}v_{i-1}\\
&= F^{N-1}(K-q^{2N-2}K^{-1})v_{i-1}\\
&= F^{N-1}(q^{-1-N}-q^{N-1})v_{i-1}\\
&= 0 \qquad \square
\end{aligned}$$

Remark. Now, the standard inductive computation of the R-matrix can be made, noting that $c_N=0$. Going back to Equation 4.15, commute the E past F^l in the second term, and K^{-1} past F^l in the first, and renumber the first term so that E^{l+1} become E^l. We get,

$$\sum_{m=0}^{N-1} c_{m-1}E^m\otimes(F^{m-1}K-q^{2(m-1)}F^{m-1})-\sum_{m=0}^{N-1} c_m E^m\otimes([m]F^{m-1}\frac{q^{1-m}K-q^{m-1}K^{-1}}{q-q^{-1}})=0.$$

Collecting in powers of E we get the recursive formula,

$$c_m = \frac{(q-q^{-1})}{[m]} q^{m-1} c_{m-1}.$$

Setting $c_0 = 1$ we get the standard formula for the R-matrix given above.

Recall, a Universal R-matrix satisfies $(\Delta \otimes Id)(R) = R_{13}R_{23}$, and $(Id \otimes \Delta)(R) = R_{13}R_{12}$. The first of these two equations holds in semicyclic representations because it is true when $E^N = 0 = F^N$, and $F^N = 0$ suffices. This is because the only powers of E and F in the formula that exceed $N-1$ are in fact powers of F, for instance see [21] for the proof. The second equation does not hold in the semicyclic representations. To see why, apply Proposition 4.1.1 to the left hand side $(Id \otimes \Delta)(R)$.

$$(Id \otimes \Delta)(q^{\frac{H\otimes H}{2}} \sum_{l=0}^{N-1} c_l E^l \otimes F^l) = q^{\frac{H\otimes\Delta(H)}{2}} \sum_{l=0}^{N-1} E^l \otimes \Delta(F)^l.$$

Notice that the highest power of E or F that appears is $N-1$. Since the equation is true when $E^N = F^N = 0$ in the algebra, it means that the coefficients of $R_{13}R_{12}$ all monomials where the powers of E and F are less than or equal to $N-1$ agree with the answer above. For the right hand side, we have

$$R_{13}R_{12} = q^{\frac{H\otimes 1\otimes H}{2}} (\sum_{m=0}^{N-1} c_m E^m \otimes 1 \otimes F^m) q^{\frac{H\otimes H\otimes 1}{2}} (\sum_{n=0}^{N-1} c_n E^n \otimes F^n \otimes 1)$$

Commuting $q^{\frac{H\otimes H\otimes 1}{2}}$ to the front yields,

$$q^{\frac{H\otimes\Delta(H)}{2}} (\sum_{m=0}^{N-1} c_m E^m \otimes K^{-m} \otimes F^m)(\sum_{n=0}^{N-1} c_n E^n \otimes F^n \otimes 1).$$

Now cancel the exponentiated $H\otimes\Delta(H)$ from both sides. We know that all the terms where $m+n\leq N-1$ cancel with the left hand side so the remainder is,

$$\sum_{m+n\geq N} c_m c_n E^{m+n}\otimes K^{-m}F^n\otimes F^m.$$

The matrices E^k where k ranges from N to $2N-2$ are linearly independent. Hence each of the parts of the sum where $m+n=k$ needs to be zero. Letting $k=N$ we get,

$$\sum_{m=1}^{N-1} c_m c_{n-m} E^N\otimes K^{-m}F^{N-m}\otimes F^m.$$

Consider $v_0\otimes v_{i+1}\otimes v_{i+1}$. The values of this vector under each term of the sum are linearly independent, so the sum is nonzero and the relation $(Id\otimes\Delta)(R)=R_{13}R_{12}$ does not hold under the semicyclic representations.

However, the R-matrix still satisfies the Yang-Baxter equation, as long as we are evaluating in the representations $\rho_{a,i}$.

Let $\sigma : A\otimes B\to B\otimes A$ be the flip $\sigma(Z\otimes W)=W\otimes Z$. Let $P : A\otimes B\otimes C$ denote its extension $P(Z\otimes W\otimes X)=(W\otimes Z\otimes X)$. If $R=\sum_i s_i\otimes t_i$ then

$$R_{12}=\sum_i s_i\otimes t_i\otimes 1,\ \ R_{13}=\sum_i s_i\otimes 1\otimes t_i,\ \ R_{23}=\sum_i 1\otimes s_i\otimes t_i.$$

Theorem 4.2.1. *Suppose $\rho_{a,i} : U_q(sl_2)\to M_{N,N}(\mathbb{C})$ is any of the semicyclic representations of $U_q(sl_2)$. Then*

$$\rho_{a,i}\otimes\rho_{a,i}\otimes\rho_{a,i}(R_{12}R_{13}R_{23}-R_{23}R_{13}R_{12})=0.$$

Proof. The proof follows [21]. It really only depends on the fact that if we evaluate in $\rho\otimes\phi : U_q(sl_2)\otimes U_q(sl_2)\to End(V)\otimes M_{N,N}(\mathbb{C})$, where $\rho : U_q(sl_2)\to End(V)$ is an arbitrary representation, and ϕ is a semicyclic representation where the kernel of F contains v_i, then

$$(\rho \otimes \phi)(R\Delta - \Delta' R) = 0,$$

and

$$(\rho \otimes \phi \otimes \psi)(\Delta \otimes Id(R) - R_{13}R_{23}) = 0,$$

when in addition ψ is semicyclic.

To simplify notation we suppress the representation from the formulas.

$$R_{12}R_{13}R_{23} = R_{12}(\Delta \otimes 1)(R) = (\Delta' \otimes 1)(R)R_{12} =$$

$$P \circ (\Delta \otimes 1)(R)R_{12} = P \circ (R_{13}R_{23})R_{12} = R_{23}R_{13}R_{12}. \quad \square$$

4.3 The Tangle Functor

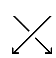
In this section, a tangle functor is defined for $(1,1)$-tangles that have been colored with the extended semicyclic representations defined in Section 4.1.3. Let $e_0, e_1, ..., e_{N-1}$ be a basis for an N-dimensional vector space V over $\mathbb{C}$. For $v \in V$ and $\phi \in V^*$ define the cup, cap, and crossing operators as follows.

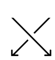
 $\rightarrow (1 \mapsto e_i \otimes e^i)$

 $\rightarrow (1 \mapsto e^i \otimes K^{-1}(e_i)$

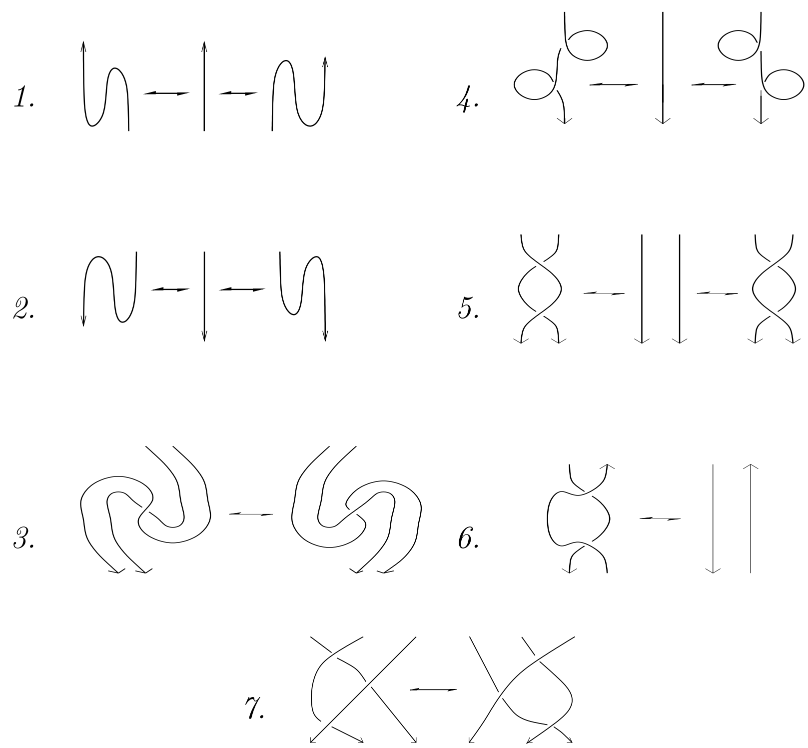
 $\rightarrow (\phi \otimes v \mapsto \phi(v))$

 $\rightarrow (v \otimes \phi \mapsto \phi(Kv))$

 $\rightarrow (e_i \otimes e_j \mapsto \check{R}(e_i \otimes e_j))$

 $\rightarrow (e_i \otimes e_j \mapsto \check{R}^{-1}(e_i \otimes e_j))$

Theorem 4.3.1 (Turaev [25, 36])**.** *Two oriented framed tangle diagrams express the same iosotopic tangle if and only if the two diagrams are related by a finite sequence of Turaev moves shown below.*



It remains to show that each of these moves are satisfied using the cap, cup, and crossing definitions given above under specified semicyclic representations. If so, then a $(1,1)$ tangle functor has been defined.

Lemma 4.3.1. *Let* $q = e^{\frac{\pi i}{N}}$ *and* $f_q(l) = \frac{(q-q^{-1})^l}{[l]!} q^{\frac{l(l-1)}{2}}$. *Then,*

$$f_{q^{-1}}(a-b) f_q(b) = (-1)^{b-a} \frac{(q-q^{-1})^a}{[a-b]![b]!} q^{\frac{a-a^2}{2}} q^{b(a-1)}$$

Also,

$$\sum_{b=0}^{a} f_{q^{-1}}(a-b) f_q(b) = 0 = \sum_{b=0}^{a} f_q(a-b) f_{q^{-1}}(b)$$

when $0 < a < N$.

Proposition 4.3.1. *The operator $\rho_{a,k}(\check{R})$ for $k \in \{0, 1, ..., N-1\}$ along with the corresponding cup and cap operators arising from $\rho_{a,k}(K)$ define a $(1,1)$-tangle functor.*

Proof: In the following proof, assume that $k = \frac{N+1}{2}$ in which case $K(e_i) = q^{2i}e_i$ and $q^{\frac{H\otimes H}{2}}(e_i \otimes e_j) = q^{2ij}(e_i \otimes e_j)$. Since all the semicyclic representations are isomorphic, we only need to check for one k. Also, suppress the representation $\rho_{a,k}$ and write E, F, K and $\check{R}$ for shorthand. Notice that 1 and 2 follow directly from the definition of the cap and cup operators. Also, 5 is clearly true since $\check{R} \circ \check{R}^{-1}(e_i \otimes e_j) = e_i \otimes e_j$ and $\check{R}^{-1} \circ \check{R}(e_i \otimes e_j) = e_i \otimes e_j$. Since $\check{R}$ satisfies the braid relation, 7 is satisfied. It remains to verify 3, 4, and 6. To simplify notation, use the modified Einstein notation where there is a sum over each variable that appears at least twice.

Left hand side of 3: Here, the arrows are going up so the input to the map is the tensor product of dual vectors $e^i \otimes e^j$. While passing from the bottom to the top of the diagram, there are two untwisted cups then an application of $\check{R}$ at the crossing, then two untwisted caps.

$$\begin{aligned}
e^i \otimes e^j &\overset{cups}{\mapsto} e^i \otimes e^j \otimes e_k \otimes e_p \otimes e^p \otimes e^k \\
&\overset{\check{R}}{\mapsto} e^i \otimes e^j \otimes \check{R}(e_k \otimes e_p) \otimes e^p \otimes e^k \\
&= e^i \otimes e^j \otimes q^{2(p-r)(k+r)} f_q(r) F^r e_p \otimes E^r e_k \otimes e^p \otimes e^k \\
&\overset{caps}{\mapsto} q^{2(p-r)(k+r)} f_q(r) (F^r)^j_p (E^r)^i_k e^p \otimes e^k \ (k = i-r, p = j+r) \\
&= q^{2ij} f_q(r) (F^r)^j_{j+r} (E^r)^i_{i-r} e^{j+r} \otimes e^{i-r}
\end{aligned}$$

Right hand side of 3: As in the case of the left hand side, we calculate the action of the map on $e^i \otimes e^j$. The only difference is that the cup and cap operators involve the action of K.

$$\begin{aligned}
e^i \otimes e^j &\overset{cups}{\mapsto} q^{-2(p+k)} e^p \otimes e^k \otimes e_k \otimes e_p \otimes e^i \otimes e^j \\
&\overset{\check{R}}{\mapsto} q^{-2(p+k)} e^p \otimes e^k \otimes \check{R}(e_k \otimes e_p) \otimes e^i \otimes e^j \\
&= q^{-2(p+k)+2(p-r)(k+r)} f_q(r) e^p \otimes e^k \otimes F^r e_p \otimes E^r e_k \otimes e^i \otimes e^j \\
&\overset{caps}{\mapsto} q^{-2(p+k)+2(p-r)(k+r)+2(i+j)} f_q(r) (E^r)^i_k (F^r)^j_p e^p \otimes e^k \ (k = i-r, p = j+r) \\
&= q^{2ij} f_q(r) (F^r)^j_{j+r} (E^r)^i_{i-r} e^{j+r} \otimes e^{i-r}
\end{aligned}$$

Left hand side of 4:

$$\begin{aligned}
e_i &\overset{cup}{\mapsto} q^{-2k} e^k \otimes e_k \otimes e_i \\
&\overset{\check{R}}{\mapsto} q^{-2k} e^k \otimes \check{R}(e_k \otimes e_i) \\
&= q^{-2k} q^{-2(i-r)(k+r)} f_q(r) e^k \otimes F^r e_i \otimes E^r e_k \\
&\overset{cap}{\mapsto} q^{-2k} q^{2(i-r)(k+r)} f_q(r) (F^r)^k_i E^r e_k \quad (k = i-r) \\
&\overset{cup}{\mapsto} q^{-2k+2(i-r)(k+r)} f_q(r) (F^r)^k_i E^r e_k \otimes e_j \otimes e^j \\
&\overset{\check{R}^{-1}}{\mapsto} q^{-2k+2(i-r)(k+r)} f_q(r) (F^r)^k_i \check{R}^{-1}(E^r e_k \otimes e_j) \otimes e^j \\
&= q^{-2k+2(i-r)(k+r)-2j(k+r)} f_q(r) f_{q^{-1}}(s) (F^r)^k_i E^s e_j \otimes F^s E^r e_k \otimes e^j \\
&\overset{cap}{\mapsto} q^{2j-2k+2(i-r)(k+r)-2j(k+r)} f_q(r) f_{q^{-1}}(s) (F^r)^k_i (F^s E^r)^j_k E^s e_j \ (j = i-s) \\
&= q^{2(i-s)-2(i-r)+2i(i-r)-2i(i-s)} f_q(r) f_{q^{-1}}(s) (F^{r+s} E^{r+s})^i_i e_i
\end{aligned}$$

Since $\check{R}$ is an intertwiner, Schur's lemma says the linear map associated to the $(1,1)$ tangle is a multiple of the identity. Therefore, we can choose any input $0 \leq i \leq N-1$ to determine the map. Set $i = \frac{N+1}{2}$, then $F^r e_i = 0$ for $r > 0$. This means $r = 0$ and $i = k$. Then $F^s E^r e_k = F^s e_k = 0$ for $s > 0$. Since both $r, s = 0$, the left hand side of 4 is equal to the identity map. The right hand side of 4 is a similar calculation.

Left hand side of 6: By the same methods as above, it can be shown that

$$e^i \otimes e_j \quad \mapsto \quad \sum_{r,s=0}^{N-1} q^{-2(r+s)(i-r)+2s} f_{q^{-1}}(r) f_q(s) (E^{r+s})^i_{i-r-s} (F^{r+s})^{j-r-s}_j e^{i-r-s} \otimes e_{j-r-s}$$

Let $a = r+s$ and $b = s$. Notice that when $a \geq N$, $F^a = 0$ so those terms do not contribute.

$$\begin{aligned} e^i \otimes e_j \quad &\mapsto \quad \sum_{b=0}^{a} \sum_{a=0}^{N-1} q^{-2a(i-a+b)+2b} f_{q^{-1}}(a-b) f_q(b) (E^a)^i_{i-a} (F^a)^{j-a}_j e^{i-a} \otimes e_{j-a} \\ &= \sum_{a=0}^{N-1} q^{-2a(i-a)} (E^a)^i_{i-a} (F^a)^{j-a}_j \left(\sum_{b=0}^{a} q^{-2b(1-a)} f_{q^{-1}}(a-b) f_q(b) \right) e^{i-a} \otimes e_{j-a} \end{aligned}$$

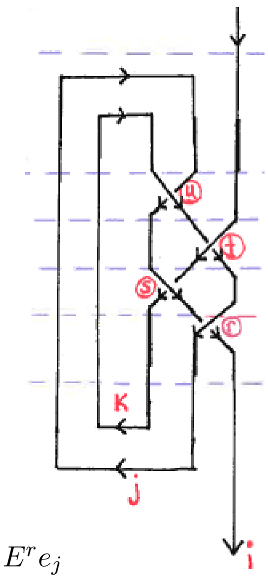
To finish the proof, it remains to show that when $a \neq 0$, then $\sum_{b=0}^{a} q^{-2b(1-a)} f_{q^{-1}}(a - b) f_q(b) = 0$. Note that when $a = 0$, $\sum_{b=0}^{a} q^{-2b(1-a)} f_{q^{-1}}(a - b) f_q(b) = 1$. By lemma 4.3.1, when $a \neq 0$,

$$\begin{aligned} \sum_{b=0}^{a} q^{-2b(1-a)} f_{q^{-1}}(a-b) f_q(b) &= \sum_{b=0}^{a} (-1)^{b-a} \frac{(q-q^{-1})^a}{[a-b]![b]!} q^{\frac{a-a^2}{2}} q^{-b(a-1)} \\ &= (-1)^a (q-q^{-1})^a q^{\frac{a-a^2}{2}} \sum_{b=0}^{a} (-1)^b \frac{q^{-b(a-1)}}{[a-b]![b]!} \\ &= (-1)^a q^{a-a^2} \sum_{b=0}^{a} f_q(a-b) f_{q^{-1}}(b) \\ &= 0 \end{aligned}$$

This concludes the proof of the proposition. □

4.4 The Invariant of the Figure Eight Knot

For notational ease, the modified Einstein notation is used where there is a sum over each variable that appears at least twice. To calculate the invariant, restrict to the representation $\rho_{a,\frac{N+1}{2}}$. As in the proof of Proposition 4.3.1 the invariant is calculated by starting at the bottom of the tangle with input vector e_i and applying the proper operator as we pass through cups, caps, and crossings. Since the number of caps and cups are the same, the final sum will only be over the variables r, s, t and u indicated at each crossing. Recall that the twisted cup acts as $1 \mapsto e^j \otimes K^{-1}(e_j)$ and the twisted cap as $e^j \otimes e_k \mapsto K(e_k)e^j(e_k)$. Under $\rho_{a,\frac{N+1}{2}}$, $K^{\pm 1}(e_j) = q^{\pm 2j}e_j$ and $q^{\frac{H\otimes H}{2}}(e_j \otimes e_k) = q^{2jk}e_j \otimes e_k$.



$$e_i \overset{cups}{\mapsto} q^{-2(j+k)}e^j \otimes e^k \otimes e_k \otimes e_j \otimes e_i$$

$$\overset{\check{R}}{\mapsto} q^{-2(j+k)}q^{2(i-r)(j+r)}f_q(r)e^j \otimes e^k \otimes e_k \otimes F^r e_i \otimes E^r e_j$$

$$\overset{\check{R}^{-1}}{\mapsto} q^{-2(j+k)}q^{2(i-r)(j+r)}q^{-2(i-r)k}f_q(r)f_{q^{-1}}(s)e^j \otimes e^k \otimes E^s F^r e_i \otimes F^s e_k \otimes E^r e_j$$

$$\overset{\check{R}}{\mapsto} q^{-2(j+k)}q^{2(i-r)(j+r)}q^{-2(i-r)k}q^{2(j+r-t)(k-s+t)}f_q(r)f_{q^{-1}}(s)f_q(t)$$

$$e^j \otimes e^k \otimes E^s F^r e_i \otimes F^t E^r e_j \otimes E^t F^s e_k$$

$$\overset{\check{R}^{-1}}{\mapsto} \overbrace{q^{-2(j+k)+2(i-r)(j+r)-2(i-r)k+2(j+r-t)(k-s+t)-2(i-r+s-u)(j+r-t+u)}}^{f(i,j,k,r,s,t,u)} f_q(r)f_{q^{-1}}(s)f_q(t)f_{q^{-1}}(u)$$

$$e^j \otimes e^k \otimes E^u F^t E^r e_j \otimes F^u E^s F^r e_i \otimes E^t F^s e_k$$

$$\overset{caps}{\mapsto} q^{f(i,j,k,r,s,t,u)}(E^u F^t E^r)^k_j (F^u E^s F^r)^j_i (E^t F^s)^i_k e_{k-s+t} \ (j = i-r+s-u, k = j+r-t+u)$$

$$\mapsto q^{f(i,j(i,r,s,t,u),k(i,r,s,t,u),r,s,t,u)}(E^t F^s E^u F^t E^r F^u E^s F^r)^i_i e_i$$

Remark: In the last step, the exponent of q, $f(i,j,k,r,s,t,u)$ is replaced with $f(i, j(i, r, s, t, u), k(i, r, s, t, u), r, s, t, u)$. This is due to the linear relations $j = j(i, r, s, t, u)$ and $k = k(i, r, s, t, u)$ coming from the caps. This will be the case for every knot. Later it will be shown that this invariant is the same as Kashaev's invariant so we do not find a closed form for the final solution.

4.5 Recovering Kashaev's Invariant

Given a $(1, 1)$-tangle K denote the invariant of K coming from the semicyclic representations with $E^N = a$ by $T_{a,N}(K)$. It will now be shown that $T_{a,N}(K)$ evaluated at a $2N^{th}$ root of unity is equivalent to Kashaev's invariant evaluated at a $2N^{th}$ root of unity for any knot.

In the prior section, $T_{a,N}(K)$ was calculated for the firgure eight knot. That calculation shows that $T_{a,N}(K)$ for any knot will be a sum of balanced words in E and F with coefficient functions of q to a power.

Let E be defined as before. A more general version of F is given by

$$(F)^i_j = \begin{cases} f_i & : i + 1 \equiv j \bmod N \text{ and } i \neq 0 \\ \frac{f_o}{a} & : i = 0 \text{ and } j = N - 1 \\ 0 & : \text{else} \end{cases}$$

where $f_i \in \mathbb{C}[q^{\pm 1}]$ for $0 \leq i \leq N-1$. For example, when $N = 5$,

$$F = \begin{pmatrix} 0 & f_1 & 0 & 0 & 0 \\ 0 & 0 & f_2 & 0 & 0 \\ 0 & 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & 0 & f_4 \\ \frac{f_0}{a} & 0 & 0 & 0 & 0 \end{pmatrix}$$

Proposition 4.5.1. *Let $E^{x_1}F^{x_2}\cdots F^{x_{n-2}}E^{x_{n-1}}F^{x_n}$ be a word in the matrices defined above where the total degree of E equals the total degree of F (i.e. $\sum x_{odd} = \sum x_{even}$). Then $E^{x_1}F^{x_2}\cdots F^{x_{n-2}}E^{x_{n-1}}F^{x_n}$ is a diagonal matrix. Furthermore, the entries of are elements of $\mathbb{C}[q^{\pm 1}]$.*

Proof: The proof will be by induction on the length of a word w where the total degree of E and F are equal. For the base case, consider the word E^mF^m of length 2. For a fixed integer N where $0 < m < N$,

$$(E^mF^m)^i_j = \begin{cases} \prod_{k=i}^{\overline{i+m-1}} f_{\overline{N-k}} & : i = j \\ 0 & : \text{else} \end{cases}$$

Also, when $m = 0$, $w = id$. Therefore, for any $0 \leq m < N$, the word w satisfies the proposition.

Now assume any word of length less than n satisfies the proposition and let $w = E^{x_1}F^{x_2}\cdots F^{x_{n-2}}E^{x_{n-1}}F^{x_n}$ be a word of length n. There are two cases.

Case 1: $x_n < x_{n-1}$. In this case, decompose w into the words

$a = E^{x_1}F^{x_2}\cdots F^{x_{n-2}}E^{x_{n-1}-x_n}$ and $b = E^{x_n}F^{x_n}$.

Case 2: $x_n > x_{n-1}$. In this case, it will be necessary to use the following generalized rule to commute the $E^{x_{n-1}}$ past the F^{x_n}.

$$E^cF^d = \sum_{r=0}^{c}\left(\frac{[c]![d]!}{[r]![c-r]![d-r]!}F^{d-r}E^{c-r}\prod_{k=0}^{r-1}[K;c-d-k]\right), c < d$$

where $[K;t] = \frac{Kq^t - K^{-1}q^{-t}}{q-q^{-1}}$.

It is convenient to shorthand the above formula as

$$E^cF^d = \sum_{r=0}^{c} F^{d-r}E^{c-r}D(c,d,r), c < d$$

where D is a diagonal matrix comprised entirely of Laurent polynomials in q with no a or $\frac{1}{a}$ appearing.

Applying this formula to the last two terms in the word w gives

$$\begin{aligned} w &= E^{x_1}F^{x_2}\cdots F^{x_{n-2}}E^{x_{n-1}}F^{x_n} \\ &= E^{x_1}F^{x_2}\cdots F^{x_{n-2}}\left(\sum_{r=0}^{x_{n-1}} F^{x_n-r}E^{x_{n-1}-r}D(x_{n-1},x_n,r)\right) \\ &= \sum_{r=0}^{x_{n-1}} E^{x_1}F^{x_2}\cdots F^{x_{n-2}+x_n-r}E^{x_{n-1}-r}D(x_{n-1},x_n,r) \end{aligned}$$

The result is a word of length less than n which by induction involves no $a's$. This completes the proof. □

The center of $U_q(sl_2)$ is generated by the standard quadratic Casimir defined by $C = EF + \frac{q^{-1}K+qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2}$. If the Casimir is evaluated at the standard irreducible representation, ρ_0, of $U_q(sl_2)$ then it will clearly not involve any $a's$ since $E^N = 0$. The prior proposition shows that the Casimir will not involve any $a's$ when evaluated at the semicyclic irreducible representations discussed in this chapter.

Proposition 4.5.2. *Let $w = E^{x_1}F^{x_2}\cdots F^{x_{n-2}}E^{x_{n-1}}F^{x_n}$ be a balanced word in E and F. (i.e. $\sum x_{odd} = \sum x_{even}$). Then w can be written as a product of the terms $(C - [q^{2\alpha_i}K]_q)$ where $[q^rK]_q = \frac{q^rK - q^{-r}K^{-1}}{(q-q^{-1})^2}$ and $\alpha_i \in \{0, 1, ..., N-1\}$.*

Proof: The proof is similar to that of Proposition 4.5.1 and will be by induction on the length of the word w. Let w be a word of length 2. Then $w = E^mF^m$ or $w = F^mE^m$. By induction and using the formula for the Casimir, it can be shown that $E^mF^m = \prod_{i=1}^{m}(C - [q^{-2(m-i)}K]_{q^{-1}})$ and $F^mE^m = \prod_{i=1}^{m}(C - [q^{2(m-i)}K]_q)$. With this in hand, the same induction argument from Proposition 4.5.1 will show that any balanced word can be written in terms of the Casimir and $[q^{2\alpha_i}K]_q$. □

Theorem 4.5.1. *The invariant $T_{a,N}(K)$ evaluated at a $2N$th root of unity is equivalent to Kashaev's invariant, $< K >_N$, evaluated at a $2N$th root of unity for any $(1,1)$-tangle K.*

Proof: In this proof, the representation $\rho_{a,0}$ is used to calculate $T_{a,N}(K)$. Kashaev's invariant is calculated in the same way as $T_{a,N}(K)$ except evaluated at the standard irreducible representation ρ_0. By definition, we have $\rho_{a,0}(K) = \rho_0(K)$. Therefore, the cup and cap operators are identical under either representation. Also, $q^{\frac{H\otimes H}{2}}$ acts the same under either representation. By Proposition 4.5.2, we can complete the proof by checking that EF or FE evaluates the same under both representations. Notice that $\rho_{a,0}(F) = \rho_0(F)$ and the only difference between $\rho_{a,0}(E)$ and $\rho_0(E)$ is their action on the basis vector v_{N-1}. Thus, it is enough to check the action of FE on v_{N-1} under each representation. But, under both representations $FEv_{N-1} = 0$. □

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