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# Algebras of cross sections

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# ALGEBRAS OF CROSS SECTIONS

by

Erin Griesenauer

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

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Thesis Supervisor: Professor Paul Muhly

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the August 2016 graduation.

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# ABSTRACT

In a sense, my research begins with the study of polynomial functions of tuples of matrices. The polynomial functions that I investigate are called *matrix concomitants*. Observe that $GL_n(\mathbb{C})$ acts on $d$-tuples of $n \times n$ matrices by conjugating each entry. That is, if $\mathfrak{z} = (Z_1, \ldots, Z_d) \in M_n(\mathbb{C})^d$ and if $g \in GL_n(\mathbb{C})$, then $g\mathfrak{z}g^{-1} := (gZ_1g^{-1}, \ldots, gZ_2g^{-1})$. Matrix concomitants are the functions $f : M_n(\mathbb{C})^d \to M_n(\mathbb{C})$ that intertwine this action with the action of $GL_n(\mathbb{C})$ on $M_n(\mathbb{C})$ by conjugation, i.e. functions with the property that $f(g\mathfrak{z}g^{-1}) = gf(\mathfrak{z})g^{-1}$ for $\mathfrak{z} \in M_n(\mathbb{C})^d$ and $g \in GL_n(\mathbb{C})$. Such functions arise naturally from the $n$-dimensional representations of the free algebra. They are of great importance in pure algebra, primarily in invariant theory and in the study of polynomial identity rings.

I am interested in the algebra of *holomorphic matrix concomitants*, i.e. the holomorphic functions that the polynomial concomitants generate. This algebra is denoted by $Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^{GL_n(\mathbb{C})}$. We have shown that holomorphic matrix concomitants can by studied in terms of a naturally occurring holomorphic matrix bundle. Before describing this matrix bundle, I must introduce some terminology. Call $\mathfrak{z} \in M_n(\mathbb{C})^d$ an *irreducible point* if the components of $\mathfrak{z}$ generate $M_n(\mathbb{C})$. Let $\mathcal{V}(d,n)$ denote the set of irreducible points. The projective linear group $PGL_n(\mathbb{C})$ acts freely and properly on $\mathcal{V}(d,n)$. Consequently, $\mathcal{V}(d,n)$ is the bundle space of a holomorphic principal $PGL_n(\mathbb{C})$-bundle over the orbit space $\mathcal{V}(d,n)/PGL_n(\mathbb{C})$. In [Pro74, Theorem 5.10] Procesi identified the quotient space $Q_0(d,n) = \mathcal{V}(d,n)/PGL_n(\mathbb{C})$ with a subset of the smooth points of an abstract algebraic variety. Denote the quotient map by $\pi_0$. We study the principal $PGL_n(\mathbb{C})$-bundle $\mathfrak{B}(d,n) = (\mathcal{V}(d,n), \pi_0, Q_0(d,n))$ as well as the associated $M_n(\mathbb{C})$-fibre bundle, $\mathfrak{M}(d,n)$.

Our first main result is that the algebra of concomitants $Hol(\mathcal{V}(d,n), M_n(\mathbb{C}))^{GL_n(\mathbb{C})}$ is isomorphic to $\Gamma_h(Q_0(d,n), \mathfrak{M}(d,n))$, the holomorphic cross sections of $\mathfrak{M}(d,n)$. The problem we then face is that while $\Gamma_h(Q_0(d,n), \mathfrak{M}(d,n))$ and $\Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ are topological algebras, they do not have any evident natural norm structures. More precisely, if $X$ is a compact subset of $Q_0(d,n)$, then $\Gamma_c(X, \mathfrak{M}(d,n))$ does not have an evident norm structure. However, the algebra of cross sections *does* have norm structures. Each *reduction* of $\mathfrak{B}(d,n)$ to a principal $PU_n(\mathbb{C})$-bundle yields a $C^*$-algebra structure on $\Gamma_c(X, \mathfrak{M}(d,n))$. To indicate that $\mathfrak{M}(d,n)$ has been given a $*$-structure, we shall denote it by $\mathfrak{M}^*(d,n)$. An isomorphic copy of $\Gamma_h(X, \mathfrak{M}(d,n))$ sits inside of $\Gamma_c(X, \mathfrak{M}^*(d,n))$, but the image depends on the choice of reduction and will consist of non-holomorphic sections. Nevertheless, once a $*$-structure is fixed and $X$ is chosen to be the Shilov boundary $\partial\mathcal{D}$ of a domain $\mathcal{D}$ so that $\overline{\mathcal{D}} \subset Q_0(d,n)$ is compact, we get an interesting Banach subalgebra of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ by taking the closure of $\Gamma_h(\partial\mathcal{D}, \mathfrak{M}(d,n))$, denoted $\mathbb{S}(\mathcal{D}; d, n)$.

Our second main result identifies the boundary representations of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ for the non self-adjoint subalgebra $\mathbb{S}(\mathcal{D}; d, n)$: every point in the *Choquet* boundary of $\mathcal{D}$ determines a boundary representation for $\mathbb{S}(\mathcal{D}; d, n)$. From this we conclude that $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ is the $C^*$-envelope of $\mathbb{S}(\mathcal{D}; d, n)$ in the sense of Arveson [Arv69]. Our third main result shows that $\mathbb{S}(\mathcal{D}; d, n)$ is an *Azumaya algebra*, assuming some natural, mild conditions on $\mathcal{D}$. This dissertation also includes extensions of these results to more general holomorphic $M_n(\mathbb{C})$-fibre bundles.

## PUBLIC ABSTRACT

My research studies algebras of holomorphic functions from $d$-tuples of $n \times n$- matrices, $M_n(\mathbb{C})^d$, to $M_n(\mathbb{C})$. In particular, I study the holomorphic functions that can be approximated by *polynomial matrix concomitants*, that is polynomial maps from $M_n(\mathbb{C})^d$ to $M_n(\mathbb{C})$ that satisfy the relationship

$$f(g^{-1}\mathfrak{z}g) = g^{-1}f(\mathfrak{z})g$$

for every $\mathfrak{z} \in M_n(\mathbb{C})^d$ and $g \in GL_n(\mathbb{C})$. In a sense, these are the polynomial maps that "remember" the structure of the $d$-tuple $\mathfrak{z}$.

My first result is that these holomorphic matrix concomitants can be identified with holomorphic cross sections of certain matrix bundles. A holomorphic matrix bundle is a fibred space in which every fibre is $M_n(\mathbb{C})$ and the fibres are glued together in such a way that the total space has a holomorphic structure.

Once the identification between holomorphic cross sections and holomorphic concomitants is established, the structure of the matrix bundle is used to endow the algebra of continuous cross sections with a $C^*$-algebra structure. Then we study the subalgebra of cross sections that can be approximated by polynomial concomitants. By identifying the matrix concomitants with cross sections, we are able to prove interesting results about these algebras.

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# CHAPTER 1
# INTRODUCTION AND MOTIVATION

This chapter provides motivation and context for this work, as well as a summary of the chapters and main results that follow. It is intended that this work will be approachable for a first- or second-year graduate student who is familiar with the basic theory of Banach algebras and $C^*$-algebras, such as that provided in the first chapter of [Dix77].

The following work is inspired to a great extent by the study of noncommutative functions. For $n \in \mathbb{Z}^+$, each element in the free algebra $\mathbb{C}\langle X_1, X_2, \ldots, X_d\rangle$ can be "evaluated" on $d$-tuple of $n \times n$ matrices, $\mathfrak{z} = (Z_1, Z_2, \ldots, Z_d) \in M_n(\mathbb{C})^d$, by replacing the indeterminate $X_i$ with the matrix $Z_i$. In this way, $\mathbb{C}\langle X_1, X_2, \ldots, X_d\rangle$ can be represented as a subalgebra of $Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))$, the polynomial maps from $M_n(\mathbb{C})^d$ to $M_n(\mathbb{C})$. For $a \in \mathbb{C}\langle X_1, X_2, \ldots, X_d\rangle$, denote the associated polynomial function by $\hat{a}_n$. The subalgebra of $Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))$ that consists of $\hat{a}_n$ for $a \in \mathbb{C}\langle X_1, X_2, \ldots, X_d\rangle$ is called the *algebra of $d$ generic $n \times n$ matrices*. It is natural to ask, then, how can you determine if a function in $Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))$ is in $\mathbb{G}_0(d, n)$. This was studied by Taylor in [Tay72] and [Tay73], and has recently seen a revival that was spurred by the work of Voiculescu in [Voi04].

In particular, our work is inspired by the recent book by Kaliuzhnyi-Verbovetskyi and Vinnikov [KVV14]. They provide an answer to the question of identifying elements of $\mathbb{G}_0(d, n)$ in their Theorem 6.1: if $\{f_n\}$ is a sequence with each $f_n \in Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))$ and $\sup \deg(f_n) < \infty$, then there exists an $a \in \mathbb{C}\langle X_1, X_2, \ldots, X_d\rangle$ so that $f_n = \hat{a}_n$ for each $n$ if and only if

1. The sequence $\{f_n\}$ respects direct sums, i.e. for $\mathfrak{z} \in M_k(\mathbb{C})^d$ and $\mathfrak{w} \in M_l(\mathbb{C})^d$,

$$f_{k+l}\left(\begin{bmatrix} \mathfrak{z} & 0 \\ 0 & \mathfrak{w} \end{bmatrix}\right) = \begin{bmatrix} f_k(\mathfrak{z}) & 0 \\ 0 & f_l(\mathfrak{w}) \end{bmatrix}$$

and

2. Each $f_n$ respects similarities, i.e. for $\mathfrak{z} \in M_n(\mathbb{C})^d$ and $s \in GL_n(\mathbb{C})$,

$$f_n(s^{-1}\mathfrak{z}s) = s^{-1}f_n(\mathfrak{z})s \tag{1.1}$$

The intertwining property (1.1) is called the concomitant relation, and polynomial functions from $M_n(\mathbb{C})^d$ to $M_n(\mathbb{C})$ which satisfy (1.1) are called *polynomial matrix concomitants*. Holomorphic matrix concomitants, then, are functions from $M_n(\mathbb{C})^d$ to $M_n(\mathbb{C})$ that can be approximated by polynomial matrix concomitants. These are studied, for instance, in [Lum97]. Satisfying this concomitant relation is one of the defining features of noncommutative functions, and one of the goals of this work is to illustrate that these noncommutative functions can be realized and studied as cross sections of matrix bundles.

To define the bundles that we will study, define an *irreducible point* to be a tuple $\mathfrak{z} = (Z_1, Z_2, \ldots, Z_d) \in M_n(\mathbb{C})^d$ whose entries generate $M_n(\mathbb{C})$ as an algebra. The collection of irreducible points in $M_n(\mathbb{C})^d$ is denoted by $\mathcal{V}(d, n)$. The space $\mathcal{V}(d, n)$ is the total space of a holomorphic principal $PGL_n(\mathbb{C})$-bundle, which we will denote by $\mathfrak{B}(d, n)$. The bundle $\mathfrak{B}(d, n)$ is important in geometric invariant theory and has been studied extensively. For instance, see the work of Procesi in [Pro74]. We will be interested in the associated fibre bundle $\mathfrak{M}(d, n) := \mathfrak{B}(d, n)[M_n(\mathbb{C})]$. Our first main result is Theorem 4, which asserts in particular that there is a bijection between the holomorphic matrix comcomitants defined on $\mathcal{V}(d,n)$ and the holomorphic cross sections of the bundle $\mathfrak{M}(d,n)$. Now we are ready to summarize the chapters and main results of this work.

The rest of this thesis is organized as follows: Chapter 2 outlines the necessary background from bundle theory. Chapter 3 delves deeper into bundle theory, with a specific focus on cross sections. This chapter includes Theorem 4, which was mentioned above and asserts that there is a way to identify concomitants with cross sections. Chapter 4 contains background information about matrix concomitants from geometric invariant theory and introduces Procesi's bundle $\mathfrak{B}(d,n)$. Chapter 5 describes the algebraic structure of the collection of continuous cross sections $\Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$. This chapter also discusses two methods for endowing the algebra of continuous cross sections $\Gamma_c(X, \mathfrak{M}(d,n))$ and the subalgebra of holomorphic cross sections $\Gamma_h(X, \mathfrak{M}(d,n))$ with Banach algebra structures, where $X \subset Q_0(d,n)$ is compact. The second method that is described is inspired by a theorem of Tomiyama and Takesaki that asserts that the algebra of continuous cross sections for any $M_n(\mathbb{C})$-fibre bundle over a compact Hausdorff space whose structure group is $PU_n(\mathbb{C})$ is an $n$-homogeneous $C^*$-algebra [TT61, Theorem 8]. So we reduce $\mathfrak{B}(d,n)$ to a principal $PU_n(\mathbb{C})$-bundle and consider the associated $M_n(\mathbb{C})$-fibre bundle, denoted $\mathfrak{M}^*(d,n)$. This imposes an $C^*$-structure on the algebra of continuous cross sections. In the process of reduction the holomorphic structure of the bundle $\mathfrak{B}(d,n)$ is lost, and so it no longer makes sense to discuss the algebra of holomorphic cross sections. However, we may consider the subalgebra of $\Gamma_c(X, \mathfrak{M}^*(d,n))$ which is the image of the holomorphic cross sections under the process of reduction. More specifically, choosing $X$ to be the Shilov boundary of a domain

$\mathcal{D}$ such that $\overline{\mathcal{D}}$ is a compact subset of $Q_0(d,n)$, we will define the *tracial function algebra* $\mathbb{S}(\mathcal{D};d,n) \subset \Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$. The tracial function algebra is closely related to the algebra of holomorphic cross sections and is the principal object of study in Chapters 6 and 7. Chapter 6 explores how Arveson's theory of subalgebras of $C^*$-algebras applies to $\mathbb{S}(\mathcal{D};d,n)$ as a closed, non-self-adjoint subalgebra of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$. The main result of this chapter is Theorem 20 which describes the boundary representations of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ for $\mathbb{S}(\mathcal{D};d,n)$. Chapter 7 introduces the theory of polynomial identity algebras (or PI algebras), which are generalizations of commutative algebras. The matrix algebra $M_n(\mathbb{C})$ is a PI algebra, and this property is inherited by the algebra $\mathbb{S}(\mathcal{D};d,n)$. The main result of Chapter 7, Theorem 28, asserts that $\mathbb{S}(\mathcal{D};d,n)$ is an *Azumaya algebra*.

Our results can be generalized further. Both Theorem 20 and Theorem 28 hold when the bundle $\mathfrak{B}(d,n)$ is replaced by a bundle whose total space is a so-called *$n$-variety*, rather than $\mathcal{V}(d,n)$. An $n$-variety is a subset of $\mathcal{V}(d,n)$ and is a noncommutative analogue of an algebraic variety. These generalizations are motivated by a paper of Reichstein and Vonessesn [RV07]. For an $n$-variety $\mathcal{W}$, the analogue of a coordinate ring is the *polynomial identity coordinate ring* of $\mathcal{W}$, which is the quotient of $\mathbb{G}_0(d,n)$ by the ideal consisting of elements which are identically zero on $\mathcal{W}$. These $n$-varieties are important for the study of PI algebras because Theorem 6.4 in [RV07] asserts that every finitely generated prime $\mathbb{C}$-algebra of PI-degree $n$ is isomorphic to the PI coordinate ring of some irreducible $n$-variety.

# CHAPTER 2
# BACKGROUND ON BUNDLE THEORY

In the chapters that follow, we will investigate algebras of cross sections of certain matrix bundles. This chapter serves to provide background information about bundles that will be necessary. Much of this information can be found in greater detail in the books by Steenrod [Ste51] and Husemoller [Hus94].

## 2.1 The Category of Bundles

A *bundle* is a triple $\mathfrak{X} = (X, p, B)$, where $X$ and $B$ are two topological spaces and $p : X \to B$ is a continuous map. $X$ is called the *total space* or the *bundle space*, $B$ is called the *base space*, and $p$ is called the *projection* of the bundle. For each $b \in B$, the space $p^{-1}(b)$ is called the *fibre* over $b$. It is useful to think of a bundle as the collection of fibres $p^{-1}(b)$, which are glued together by the topology of the total space $X$.

For example, if $B$ and $F$ are two topological spaces, then the projection onto the first coordinate $\mathrm{proj}_1 : B \times F \to B$ defines a bundle whose total space is $B \times F$ and whose base space is $B$. In this case, for each $b \in B$, the fibre over $b$ is $\{b\} \times F \subset B \times F$ which is homeomorphic to $F$. The bundle $(B \times F, \mathrm{proj}_1, F)$ is called the *product bundle* over $B$ with fibre $F$.

Suppose that $\mathfrak{X} = (X, p, B)$ and $\mathfrak{Y} = (Y, q, C)$ are two bundles. A *bundle morphism* from $\mathfrak{X}$ to $\mathfrak{Y}$ is a pair of continuous maps $u : X \to Y$ and $v : B \to C$ such that $q \circ u = v \circ p$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[v]{} & C \end{array}$$

With these definitions, the *category of bundles* is the category whose objects are bundles and morphisms are bundle morphisms. A *bundle isomorphism* from $\mathfrak{X}$ to $\mathfrak{Y}$ is a morphism $(u, v)$ that is invertible in the sense that there is a morphism $(u', v')$ from $\mathfrak{Y}$ to $\mathfrak{X}$ so that $u' \circ u = id_X$, $u \circ u' = id_Y$, $v' \circ v = id_B$, and $v \circ v' = id_C$.

Often, we will be interested in the subcategory of bundles that share a common base space $B$. A bundle morphism $(u, v)$ from a bundle $(X, p, B)$ to a bundle $(Y, q, B)$ in which $v = id_B$ is called a *$B$-morphism* or a *bundle morphism over $B$*. An *equivalence of bundles* is a $B$-morphism which is an isomorphism. A bundle $(X, p, B)$ which is equivalent to a product bundle $(B \times F, \text{proj}_1, B)$ is called a *trivial bundle.* A bundle $\mathfrak{X} = (X, p, B)$ is *locally trivial* if each $b \in B$ has a neighborhood $U$ so that the restriction $\mathfrak{X}|U = (p^{-1}(U), p|p^{-1}(U), U)$ is trivial. Locally trivial bundles are discussed in more detail in Section 2.3.

## 2.2 Principal Bundles

We will be interested in bundles with more structure. Let's start by looking at bundles that arise from the action of a topological group $G$ on a topological space $X$. A topological group $G$ acts continuously on (the right of) a topological space $X$ if there is a continuous map $\phi : X \times G \to X$ that satisfies the following axioms:

1. For $x \in X$ and $g, h \in G$, we have $\phi(x, gh) = \phi(\phi(x, g), h)$.

2. If $e \in G$ is the identity and $x \in X$, then $\phi(x, e) = x$.

We will denote $\phi(x, g)$ as $x \cdot g$ or $xg$. If such a map exists, then $X$ is called a *$G$-space*. We will often be interested in actions that are *effective*, meaning that if $x \cdot g = x$ for every $x \in X$, then $g = e$.

Given a topological group $G$ acting continuously on a space $X$, we can use it to build a bundle. Let $X/G$ be the collection of $G$-orbits in $X$, and let $\pi : X \to X/G$ by the map that takes an element $x \in X$ to its orbit in $X/G$. The orbit of $x$ will be denoted $[x]_G$, or simply $[x]$ if the group $G$ is understood. The map $\pi$ is a quotient map which is open. Endow $X/G$ with the quotient topology. Then $(X, \pi, X/G)$ is a bundle. A bundle $\mathfrak{X} = (X, p, B)$ is called a *$G$-bundle* if there is an action of $G$ on $X$ so that $\mathfrak{X}$ is isomorphic to the bundle $(X, \pi, X/G)$.

In general, the orbit space $X/G$ can be pathological. For example, it need not be Hausdorff even if both $X$ and $G$ are Hausdorff. As a specific example, consider the action of the general linear group $GL_2(\mathbb{C})$ on $M_2(\mathbb{C})$ by conjugation. That is, for $g \in GL_2(\mathbb{C})$ and $A \in M_2(\mathbb{C})$, define $A \cdot g := g^{-1}Ag$. The orbit of $A$ under this action is the conjugacy class of $A$, and it is well-known that two matrices are in the same orbit if and only if they have the same Jordan canonical form. This gives a very nice set-theoretic description of the orbit space $M_2(\mathbb{C})/GL_2(\mathbb{C})$, but we cannot put a Hausdorff topology on the orbit space due to the existence of non-closed orbits in $M_2(\mathbb{C})$. For example, fix $\lambda \in \mathbb{C}$ and let

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \text{ and } B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

The matrices $A$ and $B$ are in different Jordan canonical forms, so they belong to distinct orbits. However, for any $\varepsilon \neq 0$, we have

$$A \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & \varepsilon \\ 0 & \lambda \end{bmatrix}$$

belongs to the orbit of $A$. Letting $\varepsilon \to 0$, we see that $B$ is in the closure of the orbit of $A$. This implies that in the orbit space $M_2(\mathbb{C})/GL_2(\mathbb{C})$, the equivalence classes $[A]$ and $[B]$ are distinct but every neighborhood of $[A]$ also contains $[B]$, i.e. the orbit space is not Hausdorff. The action of $GL_n(\mathbb{C})$ on $M_n(\mathbb{C})$ will appear again later as we will consider matrix bundles that are defined using this action.

To guarantee that the orbit space $X/G$ is "nice," we will restrict our attention to *principal* $G$-spaces. The space $X$ is called a principal $G$-space if $G$ acts freely and properly on $X$. An action is *free* if, for $x \in X$ and $g \in G$, $x \cdot g = x$ implies $g = e$ where $e$ is the identity in $G$. This is stronger than the assumption that $G$ acts effectively on $X$.

Assume that the action of $G$ is free, and consider the set

$$\mathcal{R} = \{(x, y) \in X \times X : y = x \cdot g \text{ for some } g \in G\}.$$

So $\mathcal{R}$ is the orbit equivalence relation of the action. Endow $\mathcal{R} \subset X \times X$ with the subspace topology. Since the action is free, there is a well-defined function $\tau : \mathcal{R} \to G$ so that $x \cdot \tau(x, y) = y$ for each $(x, y) \in \mathcal{R}$. The function $\tau$ is called the translation map, and the action of $G$ is *proper* if the translation map is continuous.

If $X$ is a locally compact set and $G$ is a locally compact group, then the action of $G$ on $X$ is proper if and only if "compact sets are wandering." That is, the action is proper if for every compact set $K \subset X$, the set

$$\{g \in G : K \cdot g \cap K \neq \emptyset\}$$

is precompact, i.e. has compact closure in $G$. In particular, from this characterization it is clear that the action of a compact group on a locally compact set is always proper.

A $G$-bundle $(X, p, B)$ is called a *principal $G$-bundle* whenever $X$ is a principal $G$-space. In this case, the fibre above each $b \in B$ is homeomorphic to $G$. This is because for $x \in p^{-1}(b)$ the map $\phi_b : G \to p^{-1}(b)$ defined by $\phi(g) = x \cdot g$ is continuous and has a continous inverse defined by $\phi_b^{-1}(y) = \tau(x, y)$ for $y \in \pi^{-1}(b)$ [Hus94, Prop. 4.2.6]. In general, we will not distinguish between a principal $G$-bundle $(X, p, B)$ and the isomorphic principal $G$-bundle $(X, \pi, X/G)$.

In some cases, the homeomorphisms $\phi_b$ between the fibres $\pi^{-1}(b)$ and the group $G$ can be pieced together locally, in the sense that, under certain conditions, for each $[x] \in X/G$ there is an open neighborhood $U$ of $[x]$ and a homeomorphism $\phi_U : U \times G \to \pi^{-1}(U)$. That is, the bundle $(X, \pi, X/G)$ is locally trivial. For instance, Gleason proved in [Gle50] that if $G$ is a compact Lie group acting on a completely regular space $X$, then the bundle $(X, \pi, X/G)$ is locally trivial. More generally, it was shown by Palais in [Pal61, Prop. 2.2.1] that if $X$ is a differentiable manifold and $G$ is a Lie group that acts differentiably on $X$, then $(X, \pi, X/G)$ is a locally trivial bundle. Further, in this case the maps $\phi_U : U \times G \to \pi^{-1}(U)$ are diffeomorphisms [Pal61, Remark 2.2.3].

### 2.3 Locally Trivial Bundles

The bundles that we will consider arise from the action of a Lie group on complex manifolds, and so they will be locally trivial in the sense of the previous paragraph. Because locally trivial bundles will be a primary focus, we will now formalize the definition of a locally trivial bundle and describe some of their properties, following [Ste51]. Notice that the first three parts of this definition match the definition of a bundle given above, with the additional assumption that the projection is surjective.

**Definition 1.** *A coordinate bundle is a collection of the following things:*

1. *A topological space $X$, the total or bundle space.*
2. *A topological space $B$, the base space.*
3. *A surjective continuous map, $p : X \to B$, the projection.*
4. *A space $F$ called the fibre, such that $p^{-1}(b)$ is homeomorphic to $F$ for every $b \in B$.*
5. *An effective topological transformation group $G$ of $F$ called the* structure group *of the bundle.*
6. *An open cover of $B$, denoted $\mathcal{U}$. The elements of $\mathcal{U}$ are called* coordinate neighborhoods *and $\mathcal{U}$ is called a* coordinate atlas *for the bundle.*
7. *For each $U \in \mathcal{U}$, a homeomorphism $\phi_U : U \times F \to p^{-1}(U)$ which respects the fibre structure of $p^{-1}(U)$, i.e. satisfying $p \circ \phi_U(b, f) = b$. The maps $\phi_U$ are called* coordinate functions.

8. *For $U, V \in \mathcal{U}$, $\phi_U$ and $\phi_V$ must be compatible in the following sense: Let $b \in U \cap V$ and define $\phi_{U,b} : F \to p^{-1}(b)$ by $\phi_{U,b}(f) = \phi_U(b, f)$. Define $\phi_{V,b}$ analogously. Then we require that the homeomorphism $\phi_{V,b}^{-1} \circ \phi_{U,b} : F \to F$ corresponds to action by an element of $G$. Further, if we define $g_{U,V} : U \cap V \to G$ so that $g_{U,V}(b)$ is the element in $G$ whose action corresponds to the map $\phi_{U,b}^{-1} \circ \phi_{V,b}$, then the map $g_{U,V}$ is required to be continuous. Note that the map $g_{U,V}$ is well-defined since $G$ acts effectively. Identifying elements in $G$ with the mappings on $F$ given by their action, we can write $g_{U,V}(b) = \phi_{U,b}^{-1} \circ \phi_{V,b}$. The maps $g_{U,V}$ are called* coordinate transformations.

A bundle that can be given a (not necessarily unique) coordinate bundle structure is called a *locally trivial bundle*. The coordinate transformations of a locally trivial bundle have some properties that follow immediately from the definition. For example, for a coordinate neighborhood $U \in \mathcal{U}$ and a point $b \in U$, we have $g_{U,U}(b) = e$ the identity in $G$. For $U, V \in \mathcal{U}$ and $b \in U \cap V$, $g_{U,V}(b) = g_{V,U}(b)^{-1}$. And for $U, V, W \in \mathcal{U}$ and $b \in U \cap V \cap W$,

$$g_{U,V}(b) g_{V,W}(b) = g_{U,W}(b) \tag{2.1}$$

Equation 2.1 is called the cocycle condition on coordinate transformations.

In the case of a locally trivial principal $G$-bundle, we can say more. If $(X, p, B)$ is a principal $G$-bundle, so that $B = X/G$ and $p$ is the quotient map from $X$ to $X/G$, then the coordinate functions satisfy the relation

$$\phi_U(b, g) \cdot h = \phi_U(b, gh)$$

for $b \in U$ and $g, h \in G$. This is because, over a coordinate neighborhood $U$, $(X, p, X/G)$ is locally isomorphic to the product bundle $(U \times G, \mathrm{proj}_1, U)$. The space $U \times G$ is a principal $G$-space, where the action is right translation of the second coordinate, i.e. $(b, g) \cdot h = (b, gh)$.

Two coordinate bundles $\mathfrak{X}$ and $\mathfrak{X}'$ are defined to be *strongly equivalent* if they have the same bundle space, base space, fibre, and group, and their coordinate functions $\{\phi_U : U \in \mathcal{U}\}$ and $\{\psi_V : V \in \mathcal{U}'\}$ are compatible in the sense of Item 8 in Definition 1 above. Put another way, two coordinate bundles with the same bundle space, the same base space, the same fibre and the same group are called strongly equivalent if the union of the two collections of coordinate functions again forms a collection of coordinate functions.

A *fibre bundle*, in the sense of Steenrod, is an equivalence class of coordinate bundles under this strong equivalence relation [Ste51]. Equivalently, a fibre bundle $\mathfrak{X} = (X, p, B)$ can be thought of as a coordinate bundle whose collection of coordinate functions is “maximal” in the sense that any map $\psi : V \times F \to p^{-1}(V)$ which is compatible with each $\phi_U$, $U \in \mathcal{U}$, is already a coordinate function for $\mathfrak{X}$ [Gra58]. We will not distinguish between a coordinate bundle and the corresponding fibre bundle. This is justified by the fact that a coordinate bundle has a unique maximal coordinate atlas. This follows from the fact that if two local structures of the same bundle $\mathfrak{X}$ are given by coordinate atlas $\mathcal{U}$ (resp. coordinate atlas $\mathcal{V}$) and coordinate functions $\{\phi_U\}_{U \in \mathcal{U}}$ (resp. coordinate functions $\{\psi_V\}_{V \in \mathcal{V}}$), then the coordinate functions $\{\phi_U\}_{U \in \mathcal{U}}$ and $\{\psi_V\}_{V \in \mathcal{V}}$ are compatible [Hus94, Prop V.2.5].

It is a useful and well-known theorem that given a collection $\{g_{U,V}\}$ of coordinate transformations with values in a topological group $G$ and defined over a space $B$, and given a space $F$ on which $G$ acts effectively, we can build a coordinate bundle $\mathfrak{X}$ with base space

$B$, fibre $F$, structure group $G$ and coordinate transformations $\{g_{U,V}\}$ [Ste51, Theorem 3.2]. In fact, this construction is unique up to equivalence of bundles. This means that a locally trivial bundle is uniquely determined up to isomorphism by its fibre and its coordinate transformations.

To see that, given a fibre and coordinate functions, there is a corresponding bundle, suppose that we are given the coordinate transformations $\{g_{U,V}\}$ that take values in a group that acts effectively on the space $F$. The bundle with these coordinate transformations is built as follows: First, note that the coordinate atlas $\mathcal{U}$ is implicit in the definition of the coordinate transformations. Define

$$\widetilde{X} = \{(b, f, U) \in B \times F \times \mathcal{U} \mid b \in U\}.$$

Define an equivalence relation on $\widetilde{X}$ by $(b_1, f_1, U) \sim (b_2, f_2, V)$ whenever $b_1 = b_2$ and $f_1 = g_{U,V}(b_1) \cdot f_2$. Then the bundle with the specified coordinate transformations and fibre is the bundle whose total space is the quotient space $X = \widetilde{X}/\sim$ and whose projection map is defined by $p([b, f, U]) = b$.

## 2.4 Associated Fibre Bundles

Suppose that we have a principal $G$-bundle, $\mathfrak{X} = (X, p, B)$, along with a topological space $F$ on which $G$ acts continuously on the left. Further suppose that the action of $G$ on $F$ is effective. In general, it is assumed that $G$ acts on the right of the total space $X$ and on the left of the fibre $F$. We can build the a bundle with base space $B$, structure group $G$ and fibre $F$, called the *fibre bundle associated to* $\mathfrak{X}$ *with fibre* $F$ and denoted by $\mathfrak{X}[F]$. If $\mathfrak{X}$ is a locally trivial bundle, the associated fibre bundle is the bundle whose coordinate transformations are the same as the coordinate transformations of the bundle $\mathfrak{X}$ but whose fibre is $F$. In this case, we can describe $\mathfrak{X}[F]$ using the construction in the previous paragraph. More generally, the bundle $\mathfrak{X}[F]$ is constructed in the following way: Define an action of $G$ on $X \times F$ by $(x, f) \cdot g := (x \cdot g, g^{-1} \cdot f)$. The total space of $\mathfrak{X}[F]$ is the orbit space $X \times_G F := (X \times F)/G$. Elements of $X \times_G F$ will be written as equivalence classes $[x, f]_G$, or $[x, f]$ when $G$ is understood, with $x \in X$ and $f \in F$. So $X \times_G F$ is the space $X \times F$ modulo the equivalence relation $\sim$, where $(x_1, f_1) \sim (x_2, f_2)$ if there exists $g \in G$ so that $(x_2, f_2) = (x_1, f_1) \cdot g$. It is useful to note that this equivalence implies that $[x \cdot g, f] = [x, g \cdot f]$.

The projection map of the bundle $\mathfrak{X}[F]$ is the map $\hat{p}$ defined by $\hat{p}([x, f]) := p(x)$. This is well-defined because if $[x_1, f_1] = [x_2, f_2]$, then there is $g \in G$ so that $(x_2, f_2) = (x_1, f_1) \cdot g = (x_1 \cdot g, g^{-1} \cdot f_1)$. So $x_1$ and $x_2$ are in the same $G$-orbit in $X$. This implies that $p(x_1) = p(x_2)$, and hence $\hat{p}([x_1, f_1]) = \hat{p}([x_2, p_2])$. The base space of $\mathfrak{X}[F]$ will be $B$, the same as the base space of the original bundle $\mathfrak{X}$.

As mentioned above, if $\mathfrak{X}$ is a locally trivial bundle, then $\mathfrak{X}[F]$ will be locally trivial as well. Further, the two bundles $\mathfrak{X}$ and $\mathfrak{X}[F]$ will have the same coordinate transformations. The following lemma describes a relationship between the coordinate functions of $\mathfrak{X}$ and the coordinate functions of $\mathfrak{X}[F]$ that will be useful for computations in the future.

**Lemma 2.** *Let $\mathfrak{X} = (X, p, B)$ be a locally trivial principal $G$-bundle with coordinate atlas $\mathcal{U}$, coordinate functions $\{\phi_U : U \times G \to p^{-1}(U)\}$ and coordinate transformations $\{g_{U,V} : U \cap V \to G\}$. Suppose $F$ is a left $G$-space and build the associated fibre bundle $\mathfrak{X}[F] = (X \times_G F, \hat{p}, B)$.*

*Then $\mathfrak{X}[F]$ is locally trivial with coordinate atlas $\mathcal{U}$, coordinate transformations $\{g_{U,V}\}$ and coordinate functions $\{\hat{\phi}_U\}$, where $\hat{\phi}_U$ is defined by $\hat{\phi}_U(b,f) = [\phi_U(b,e), f]$.*

*Proof.* Again, the coordinate atlas of $\mathfrak{X}[F]$ is the same as the coordinate atlas of $\mathfrak{X}$, that is $\mathcal{U}$. Fix $U \in \mathcal{U}$. To show that $\hat{\phi}_U$ is a coordinate function, we must show that $\hat{\phi}_U$ maps $U \times F$ into $\hat{p}^{-1}(U)$, that $\hat{\phi}_U$ is a homeomorphism, and that the coordinate transformations defined by the $\hat{\phi}_U$ are the same as the original coordinate transformations $g_{U,V}$. The first statement is true, since for $b \in U$ and $f \in F$, we have

$$\hat{p} \circ \hat{\phi}_U(b,f) = \hat{p}([\phi_U(b,e), f]) = p(\phi_U(b,e)) = b.$$

To show that $\hat{\phi}_U$ is a homeomorphism, we will start by showing that $\hat{\phi}_U$ is continuous. The map $b \mapsto (b,e) \mapsto \phi_U(b,e)$ is continuous. Also, the quotient map from $X \times F$ to $X \times_G F$ is continuous. Hence the map $(b,f) \mapsto (\phi_U(b,e), f) \mapsto [\phi_U(b,e), f]$ is continuous, i.e. $\hat{\phi}_U$ is continuous.

Next we will construct the inverse of $\hat{\phi}_U$. Define $\psi_U : \hat{p}^{-1}(U) \to U \times F$ by $\psi_U([x,f]) = (p(x), \phi^{-1}_{U,p(x)}(x) \cdot f)$. To see that $\phi_U$ is well-defined, suppose $[x_1, f_1] = [x_2, f_2]$ in $\hat{p}^{-1}(U)$. Then there is $g \in G$ so that $x_2 = x_1 \cdot g$ and $f_2 = g^{-1} \cdot f_1$. This implies that $x_1$ and $x_2$ are in the same fibre in $X$, so $p(x_1) = p(x_2)$. Say $p(x_1) = b \in U$. Further, recall that $\phi^{-1}_{U,b} : p^{-1}(b) \to G$ is a homeomorphism. So

$$\phi^{-1}_{U,b}(x_2) \cdot f_2 = \phi^{-1}_{U,b}(x_1 \cdot g) \cdot (g^{-1} \cdot f_1) = \phi^{-1}_{U,b}(x_1 \cdot gg^{-1}) \cdot f_1 = \phi^{-1}_{U,b}(x_1) \cdot f_1.$$

Therefore $\psi_U([x_2, f_2]) = (b, \phi^{-1}_{U,b}(x_2) \cdot f_2) = (b, \phi^{-1}_{U,b}(x_1) \cdot f_1) = \psi_U([x_1, f_1])$ and hence $\psi_U$

is well-defined. Further, $\psi_U$ is continuous since the map $(x, f) \mapsto (p(x), \phi^{-1}_{i,p(x)}(x) \cdot f)$ is a continuous function on $X \times F$ which is constant on orbits.

To show that the $\hat{\phi}_U$ are homeomorphisms, all that remains is to show that $\psi_U$ and $\hat{\phi}_U$ are inverses, i.e. $\psi_U^{-1} \circ \hat{\phi}_U$ is the identity on $U \times F$ and $\hat{\phi}_U \circ \psi_U$ is the identity on $\hat{p}^{-1}(U)$. This is confirmed by the following computations.

$$
\begin{aligned}
\psi_U^{-1} \circ \hat{\phi}_U(b, f) &= \psi_U([\phi_U(b, e), f]) \\
&= (p(\phi_U(b, e)), \phi^{-1}_{U,p(\phi_U(b,e))}(\phi_U(b, e)) \cdot f) = (b, \phi^{-1}_{U,b}(\phi_U(b, e)) \cdot f) \\
&= (b, e \cdot f) = (b, f)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\hat{\phi}_U \circ \psi_U([x, f]) &= \hat{\phi}_U(p(x), \phi^{-1}_{U,p(x)}(x) \cdot f) \\
&= [\phi_U(p(x), e), \phi^{-1}_{U,p(x)}(x) \cdot f] = [\phi_U(p(x), e) \cdot \phi^{-1}_{U,p(x)}(x), f] \\
&= [\phi_U(p(x), \phi^{-1}_{U,p(x)}(x)), f] = [x, f]
\end{aligned}
$$

Hence $\hat{\phi}_U : U \times F \to \hat{p}^{-1}(U)$ is a homeomorphism, and the maps $\{\hat{\phi}_U\}_{U \in \mathcal{U}}$ are coordinate functions.

To check that the coordinate transformations determined by the coordinate functions $\hat{\phi}_U$ are the same as the original coordinate transformations, suppose $b \in U \cap V$ and let $\hat{g}_{U,V}$

denote the coordinate transformations determined by the maps $\hat{\phi}_U$. Fix $f \in F$. Then

$$\begin{aligned}\hat{g}_{U,V}(b) \cdot f = \hat{\phi}_{U,b}^{-1} \circ \hat{\phi}_V(b, f) &= \hat{\phi}_{U,b}^{-1}([\phi_V(b, e), f]) \\ &= \hat{\phi}_{U,b}^{-1}([\phi_U(b, e) \cdot g_{U,V}(b), f]) = \hat{\phi}_{U,b}^{-1}([\phi_U(b, e), g_{U,V}(b) \cdot f]) \\ &= \hat{\phi}_{U,b}^{-1}(\hat{\phi}_U(b, g_{U,V}(b) \cdot f)) = \hat{\phi}_{U,b}^{-1}(\hat{\phi}_{U,b}(g_{U,V}(b) \cdot f)) = g_{U,V}(b) \cdot f\end{aligned}$$

So $\hat{g}_{U,V}(b) \cdot f = g_{U,V}(b) \cdot f$ for every $f \in F$. Since $G$ acts effectively on $F$, this implies that $\hat{g}_{U,V}(b) = g_{U,V}(b)$. Thus the coordinate coordinate functions of the associated fibre bundle $\mathfrak{X}[F]$ are related to the coordinate functions of the original principal bundle $\mathfrak{X}$ by the equation $\hat{\phi}_U(b, f) = [\phi_U(b, e), f]$. □

### 2.5 Holomorphic Fibre Bundles

We have defined bundles in the category of topological spaces and continuous maps. To emphasize this, we refer to these as topological bundles. We will want to consider bundles in other categories. For example, in the last paragraph of Section 2.2 a result of Palais was mentioned: If the bundle space $X$ is a differentiable manifold, being acted on by a Lie group, then the coordinate functions are diffeomorphisms [Pal61]. We now want to look at bundles whose fibre, total space, and base space are complex manifolds and whose coordinate functions are biholomorphisms. Such bundles were considered by Grauert in [Gra58], though he worked in the more general case of complex spaces. Complex spaces are a generalization of complex manifolds that accommodate singularities. More basically, a complex manifold is a manifold that is locally homoemorphic to open sets in $\mathbb{C}^m$ for some $m$ and so that the transition maps are holomorphic. For what follows it will be sufficient to consider only complex manifolds.

If $M$ is a complex manifold, then a function $f : M \to \mathbb{C}$ is holomorphic at a point $p \in M$ if there is a coordinate neighborhood $(U, \varphi)$ of $p$, where $\varphi$ is a homeomorphism from $U$ to an open set in $\mathbb{C}^m$, such that $f \circ \varphi_U^{-1} : \mathbb{C}^m \to \mathbb{C}$ is holomorphic. If $M$ and $N$ are two complex manifolds, the a map $f : M \to N$ is holomorphic at a point $p \in M$ if there are coordinate neighborhoods $(U, \varphi)$ of $p$ and $(V, \psi)$ of $f(p)$ so that $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1}$ is a holomorphic map.

A topological fibre bundle in which the base space and fibre are complex manifolds, the total space is Hausdorff, and the structure group is a complex Lie group is a *holomorphic fibre bundle* if the coordinate transformations $g_{U,V} : U \cap V \to G$ are holomorphic. This induces a complex structure on the total space under which the coordinate functions $\phi_U : U \times F \to p^{-1}(U)$ are biholomorphic equivalences and with respect to which the projection map is holomorphic. Every holomorphic fibre bundle is also a continuous or topological fibre bundle. We will be concerned with holomorphic fibre bundles and studying the interaction of their topological structure with their holomorphic structure.

# CHAPTER 3
# CROSS SECTIONS

## 3.1 Cross Sections of Fibre Bundles

In a topological bundle $\mathfrak{X} = (X, p, B)$, a cross section is a continuous map $s : B \to X$ that sends each point $b \in B$ to an element in the fibre over $b$, i.e. a map with the property that $p \circ s = id_B$. The family of continuous cross sections of the bundle $\mathfrak{X}$ will be denoted by $\Gamma_c(B, \mathfrak{X})$. In the case that $\mathfrak{X}$ is a holomorphic fibre bundle, we also want to consider the family of holomorphic cross sections, i.e. holomorphic maps from the base space $B$ to the total space $X$ that send each point $b \in B$ to a point in the fibre over $b$. The holomorphic cross sections will be denoted $\Gamma_h(B, \mathfrak{X})$.

In the case of topological fibre bundles associated to principal bundles, Husemoller proves the following theorem in [Hus94, Theorem 4.8.1].

**Theorem 3.** *Suppose that $\mathfrak{X} = (X, p, B)$ is a principal $G$-bundle, and that $F$ is a topological space on which $G$ acts effectively on the left. Then the cross sections of the associated fibre bundle $\mathfrak{X}[F]$ are in bijective correspondence with the maps $f : X \to F$ that satisfy the relation*

$$f(x \cdot g) = g^{-1} \cdot f(x) \tag{3.1}$$

*for each $x \in X$ and $g \in G$.*

This correspondence takes $f : X \to F$ to the section $s_f \in \Gamma_c(B, \mathfrak{X}[F])$ defined by

$$s_f(p(x)) = [x, f(x)] \tag{3.2}$$

and it takes a section $s$ to the function $f_s : X \to F$ defined by the relation

$$s(p(x)) = [x, f_s(x)]. \tag{3.3}$$

In classical invariant theory, a function satisfying (3) is called an $F$-valued $G$-concomitant on $X$. The following proof is taken from Husemoller, and is included because we will use features of this proof to show that, when $\mathfrak{X}$ has is a holomorphic bundle, the bijection $f \leftrightarrow s_f$ induces a bijection between the holomorphic cross sections of $\mathfrak{X}[F]$ and the holomorphic functions from $X$ to $F$ satisfying the concomitant relation. Let $\Psi$ denote the correspondence that takes a concomitant $f$ to the section $s_f$.

*Proof.* Suppose that $f : X \to F$ is a continuous function such that $f(x \cdot g) = g^{-1} \cdot f(x)$ for all $x \in X$ and $g \in G$. Define $s_f : B \to X \times_G F$ by $s_f([x]) := [x, f(x)]$. We need to show that $s_f$ is a well-defined, continuous section of $\mathfrak{X}$. To see that $s_f$ is well-defined, suppose that $x_1, x_2 \in X$ are such that $[x_1] = [x_2]$. This implies that $x_1$ and $x_2$ are in the same orbit, so there exists $g \in G$ so that $x_1 \cdot g = x_2$. This implies that

$$\begin{aligned} s_f([x_2]) = [x_2, f(x_2)] &= [x_1 \cdot g, f(x_1 \cdot g)] \\ &= [x_1 \cdot g, g^{-1} \cdot f(x_1)] = [x_1 \cdot gg^{-1}, f(x_1)] \\ &= [x_1, f(x_1)] = s_f([x_1]). \end{aligned} \tag{3.4}$$

So $s_f$ is well-defined.

To see the $s_f$ is continuous, consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{(\cdot, f(\cdot))} & X \times F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[s_f]{} & X \times_G F \end{array}$$

The map $x \mapsto (x, f(x)) \mapsto [x, f(x)]$ from $X$ to $X \times_G F$ is continuous. Further, this map is constant on orbits since, as was shown in (3.4), $[x \cdot g, f(x \cdot g)] = [x, f(x)]$ for each $x \in X$ and $g \in G$. This map induces the map $s_f$ from $B$ to $X \times_G F$, so $s_f$ is a continuous map [Mun75, Theorem 22.2]. The map $s_f$ is a section since $\hat{p}(s_f([x])) = \hat{p}([x, f(x)]) = p(x) = [x]$, where $\hat{p}$ is the projection map for the fibre bundle $\mathfrak{X}[F]$.

Now suppose that $s : B \to X \times_G F$ is a section and define $f_s : X \to F$ by the relation $s([x]) = [x, f_s(x)]$. We need to show that $f_s$ is continuous and satisfies (3). For the second condition, let $x \in X$ and $g \in G$. Then $s([x \cdot g]) = s([x])$. This implies that $[x \cdot g, f_s(x \cdot g)] = [x, f_s(x)]$. Also, $[x \cdot g, f_s(x \cdot g)] = [x, g \cdot f_s(x \cdot g)]$. Therefore $g \cdot f_s(x \cdot g) = f_s(x)$, i.e. $f_s(x \cdot g) = g^{-1} \cdot f_s(x)$.

To see that $f_s$ is continuous, fix $x_0 \in X$ and let $y_0 = f_s(x_0) \in F$. Then $s([x_0]) = [x_0, y_0] \in X \times_G F$. Let $W \subseteq F$ be an open neighborhood of $y_0$. We will find an open neighborhood $V'$ of $x_0$ so that $f_s(V') \subseteq W$. In what follows, let $\pi : X \times F \to X \times_G F$ denote the orbit map. By continuity of the action of $G$ on $F$, there exists an open neighborhood $W'$ of $y_0$ and an open neighborhood $N$ of $e \in G$ so that $N \cdot W' \subseteq W$. Since $X$ is a principal $G$-space, the translation map $\tau : \mathcal{R} \to G$ is continuous, so there is an open neighborhood $V$ of $x_0$ so that $\tau((V \times V) \cap \mathcal{R}) \subseteq N$. Also, $s : B \to X \times_G F$ is continuous, so there is an open neighborhood $U$ of $[x_0]$ so that $s(U) \subseteq \pi(V \times W')$. Consider $V' = V \cap p^{-1}(U)$. Then we have $S(U) \subseteq \pi(V' \times W')$ and $p(V') = U$.

We are ready to show that $f_s(V') \subseteq W$. Let $x \in V'$. Then $s([x]) = [x', y']$ for some $x' \in V'$ and $y' \in W'$. But $[x', y'] = [x \cdot \tau(x, x'), y'] = [x, \tau(x, x') \cdot y']$, so $f_s(x) = \tau(x, x') \cdot y' \in N \cdot W' \subseteq W$. It is clear that the assignments $s \mapsto f_s$ and $f \mapsto s_f$ are inverses, so this completes the proof. □

We will prove that the correspondence between concomitants and cross sections carries over to the holomorphic category.

**Theorem 4.** *Suppose that $\mathfrak{X} = (X, p, B)$ is a holomorphic principal $G$-bundle, where $X$ and $B$ are complex manifolds and $G$ is a Lie group. Suppose that $F$ is a complex manifold on which $G$ acts holomorphically. Consider the fibre bundle associated to $\mathfrak{X}$ with fibre $F$, $\mathfrak{X}[F]$. Under the correspondence above, the holomorphic cross sections of $\mathfrak{X}[F]$ are in bijective correspondence with the holomorphic concomitants, i.e. the holomorphic maps $f : X \to F$ which satisfy $f(x \cdot g) = g^{-1} \cdot f(x)$ for all $x \in X$ and $g \in G$.*

*Proof.* First, consider the case where $\mathfrak{X}$ is a trivial bundle. In this case, there is a single coordinate function $\phi_B : B \times G \to X$, and a single coordinate transformation $g_{B,B} \equiv e$. This implies that $\mathfrak{X}[F]$ will also have a single coordinate function $\hat{\phi}_B : B \times F \to X \times_G F$ defined by $\hat{\phi}_B(b, y) = [\phi_B(b, e), y]$ for $b \in B$ and $y \in F$. Further, $\mathfrak{X}[F]$ also has the same unique coordinate function $g_{B,B} \equiv e$. Giving $X \times_G F$ the complex structure induced by this bundle, we have that $\hat{\phi}_B$ is a biholomorphism and $\hat{p}$ is holomorphic.

Suppose that $f : X \to F$ is a holomorphic map satisfying (3). Then $s_f : B \to X \times_G F$ is holomorphic if and only if $\hat{\phi}_B^{-1} \circ s_f$ is holomorphic. Fix $x \in X$ and let $b = p(x) \in B$. Since

$x$ is in the fibre over $b$, there exists $g \in G$ so that $\phi_B(b, g) = x$. Then

$$\hat{\phi}_B^{-1} \circ s_f(b) = \hat{\phi}_B^{-1}([x, f(x)]) = \hat{\phi}_B^{-1}([\phi_B(b, g), f(x)])$$
$$= \hat{\phi}_B^{-1}([\phi_B(b, e), g \cdot f(x)]) = \hat{\phi}_B^{-1}(\hat{\phi}_B(b, f(x \cdot g^{-1})) = (b, f(x \cdot g^{-1}))$$

So we need to show that the map $b \mapsto f(x \cdot g^{-1})$ is holomorphic. Note that $x = \phi_B(b, g) = \phi_B(b, eg) = \phi_B(b, e) \cdot g$ so $x \cdot g^{-1} = \phi_B(b, e)$. So the composition of maps

$$b \mapsto (b, e) \mapsto \phi_B(b, e) \mapsto f(\phi_B(b, e)) = f(x \cdot g^{-1})$$

is a composition of holomorphic maps.

On the other hand, suppose that $s : B \to X \times_G F$ is a holomorphic cross section. We want to show that $f_s : X \to F$ is holomorphic. This is equivalent to showing that $f_s \circ \phi_B : B \times G \to F$ is holomorphic.

Since $\mathfrak{X}[F]$ is a trivial bundle, the fact that $s$ is a holomorphic cross section implies that there is some holomorphic function $f : B \to F$ so that $\hat{\phi}_B^{-1} \circ s(b) = (b, f(b))$. So

$$(b, f(b)) = \hat{\phi}_B^{-1}(s(b)) = \hat{\phi}_B^{-1}([x, f_s(x)]).$$

Applying $\hat{\phi}_B$, we get $\hat{\phi}_B(b, f(b)) = [x, f_s(x)]$. Let's rewrite the left-hand side:

$$\hat{\phi}_B(b, f(b)) = [\phi_B(b, e), f(b)] = [\phi_B(b, e) \cdot g, g \cdot f(b)] = [\phi_B(b, g), g \cdot f(b)]$$

So $[\phi_B(b,g), g \cdot f(b)] = [x, f_s(x)] = [\phi_B(b,g), f_s \circ \phi_B(b,g)]$. Hence $f_s \circ \phi_B(b,g) = g \cdot f(b)$. Since $f$ is a holomorphic map and the action of $G$ on $F$ is holomorphic, it follows that $f_s \circ \phi_B$ is holomorphic as well.

Now we will generalize to the case that $\mathfrak{X}$ is locally trivial. Suppose that $\mathfrak{X}$ is locally trivial with a coordinate atlas $\mathcal{U}$, coordinate functions $\{\phi_U\}$, and coordinate transformations $\{g_{U,V}\}$. We have seen that $\mathfrak{X}[F]$ is also locally trivial with the same coordinate neighborhoods and coordinate transformations, and with coordinate functions $\{\hat{\phi}_U\}$ defined by $\hat{\phi}_U(b,f) = [\phi_U(b,e), f]$.

Suppose that $f : X \to F$ is a holomorphic concomitant, and define the section $s_f : B \to X \times_G F$ as in (3.2). Fix $b \in B$. We want to show that $s_f$ is holomorphic on a neighborhood of $b$. There is some $U \in \mathcal{U}$ so that $b \in U$. By the same argument as above, $s_f|U$ is holomorphic. Hence $s_f$ is holomorphic at each $b \in B$, so $s_f$ is holomorphic on $B$. Similarly, if $s : B \to X \times_G F$ is a holomorphic cross section and we define $f_s : X \to F$ as in (3.3), then for each $x \in X$ there is some neighborhood $p^{-1}(U)$ of $x$ so that $f_s|p^{-1}(U)$ is holomorphic. Here $U \in \mathcal{U}$ and $p(x) \in U$. Hence $f_s$ will be holomorphic on $X$. □

Later it will be shown that, in the case that $F$ is an algebra, $\Gamma_c(B, X \times_G F)$ has an algebraic structure and the bijection $\Psi$ is an isomorphism.

### 3.2 Alternative View of Cross Sections

There is another way of describing cross sections of locally trivial bundles that takes advantage of the local structure. Let $\mathfrak{X} = (X, p, B)$ be a locally trivial bundle with respect to a coordinate atlas $\mathcal{U}$, with fibre $F$, coordinate functions $\{\phi_U\}$, and coordinate transformations

$\{g_{U,V}\}$. Then a cross section $s \in \Gamma_c(B, \mathfrak{X})$ can be identified with a family of maps $\{s_U : U \to F\}_{U \in \mathcal{U}}$ that have the property that

$$s_U(b) = g_{U,V}(b) \cdot s_V(b) \tag{3.5}$$

for all $b \in U \cap V$.

This identification is accomplished as follows: For $s \in \Gamma_c(B, X)$ and $U \in \mathcal{U}$, define $s_U(b) := \text{proj}_2\left(\phi_U^{-1}(s(b))\right) = \phi_{U,b}^{-1}(s(b))$ for each $b \in U$. Here $\text{proj}_2 : U \times F \to F$ is projection onto the second component. It must be shown that the family $\{s_U\}$ satisfy (3.5). That is, for $U, V \in \mathcal{U}$ and $b \in U \cap V$, we need to show that $\phi_{U,b}^{-1}(s(b)) = g_{U,V}(b) \cdot \phi_{V,b}^{-1}(s(b))$. This follows immediately from the definition $g_{U,V}(b) := \phi_{U,b}^{-1} \circ \phi_{V,b}$.

To go the other way, suppose that $\{s_U : U \to F\}$ is a family of functions that satisfy (3.5). Define $s : B \to X$ by $s(b) := \phi_U(b, s_U(b))$ for $b \in U$. To see that $s$ is well-defined, suppose $b \in U \cap V$. Then we have $s_U(b) = \phi_{U,b}^{-1} \circ \phi_{V,b}(s_V(b))$. It follows that $\phi_U(b, s_U(b)) = \phi_V(b, s_V(b))$. Since $p(\phi_U(b, s_U(b))) = b$, we also have that $s$ is a cross section.

It is clear that if $\mathfrak{X}$ is a holomorphic fibre bundle, then a cross section $s$ is holomorphic if and only if each of the functions $s_U$ is holomorphic.

### 3.3 Reductions to Subgroups

Given a group $G$ and a closed subgroup $K$ of $G$, we would like to investigate the relation between principal $G$-bundles and principal $K$-bundles. In particular, we will look at *reductions* of principal $G$ bundles to principal $K$-bundles. Before introducing the terminology, I want to emphasize that the process of reduction is a continuous process that does not in general respect holomorphic structures. That is, reducing a holomorphic principal $G$-bundle does not yield a holomorphic bundle. Let $\mathfrak{X} = (X, \pi, B)$ be a principal $G$-bundle and $K$ a closed subgroup of $G$. A reduction of $\mathfrak{X}$ to $K$ is a principal $K$-bundle $\mathfrak{Y} = (Y, q, B)$ together with a map $f : Y \to X$ that is a homeomorphism onto the closed subset $f(Y)$ and satisfies the relation $f(y \cdot k) = f(y) \cdot k$ for $y \in Y$ and $k \in K$. In what follows, we will identify the bundle space $Y$ with its image $f(Y)$ and consider $Y$ a subset of $X$.

It is not always the case that a reduction exists. In fact, Theorem 6.2.3 in [Hus94] asserts that, with $G, K$, and $\mathfrak{X}$ as above, $\mathfrak{X}$ has a reduction to a principal $K$-bundle if and only if the fibre bundle $\mathfrak{X}[G/K]$ has a cross section. To use this theorem, we need a way to know when cross sections exist. We will need the following definition: A space $Y$ is called *solid* if, for any normal space $B$, any closed subset $A \subseteq B$, and any map $f : A \to Y$, there is a map $\hat{f} : B \to Y$ that extends $f$. For example, every contractible space is solid. Theorem 12.2 in [Ste51] states that, if $\mathfrak{X}$ is a fibre bundle whose base space $B$ is normal and paracompact and whose fibre $F$ is solid, then any cross section $s$ defined on a closed subset $A$ of $B$ can be extended to a cross section defined on all of $B$. In particular, letting $A = \emptyset$, it follows that $\mathfrak{X}$ has a cross section. Combining these theorems, we get the following corollary.

**Corollary 5.** *Suppose $\mathfrak{X} = (X, p, B)$ is a principal $G$-bundle whose base space $B$ is normal and paracompact. Further suppose that $K \leq G$ is a closed subgroup and that $G/K$ is solid. Then $\mathfrak{X}$ has a reduction to a principal $K$-bundle.*

In particular, later we will consider principal $PGL_n(\mathbb{C})$-bundles, where $PGL_n(\mathbb{C})$ is the projective linear group. The projective unitary group $PU_n(\mathbb{C})$ is a closed subgroup of $PGL_n(\mathbb{C})$ that has the property that $PGL_n(\mathbb{C})/PU_n(\mathbb{C})$ is contractible, by polar de-composition. Thus every principal $PGL_n(\mathbb{C})$-bundle will have a reduction to a principal $PU_n(\mathbb{C})$-bundle.

In the case that $\mathfrak{X}$ is a locally trivial bundle, any reduction of $\mathfrak{X}$ will also be locally trivial and the local information of $\mathfrak{X}$ is related to the local information of its reductions in the following way:

**Theorem 6.** *[Hus94, Theorem 6.4.1] Let $\mathfrak{X} = (X, p, B)$ be a principal $G$-bundle with coordinate atlas $\mathcal{U}$ and coordinate transformations $\{g_{U,V}\}$. Then $\mathfrak{X}$ has a reduction to a principal $K$-bundle if and only if there is a family of maps $\Lambda = \{\lambda_U : U \to G\}_{U\in\mathcal{U}}$ so that the coordinate transformations $h_{U,V}(b) := \lambda_U(b)^{-1} g_{U,V}(b)\lambda_V(b)$ have values in $K$ whenever $b \in U \cap V$.*

The coordinate transformations $\{h_{U,V}\}$ will define the reduction $\mathfrak{Y} = (Y, q, B)$.

The following result ([Hus94, Theorem 6.3.1]) gives a useful relationship between the cross sections of fibre bundles associated to a principal $G$-bundle $\mathfrak{X}$ and those of fibre bundles associated to its reduction $\mathfrak{Y}$. Since the main results in the following chapters deal with algebras of cross sections of reduced bundles, this theorem will guarantee that our results are independent of which particular reduction is chosen. Suppose that $\mathfrak{X} = (X, p, B)$ is a locally trivial principal $G$-bundle and $\mathfrak{Y} = (Y, q, B)$ is a reduction of $\mathfrak{X}$ to a principal $K$-bundle given by a family $\Lambda$. Further suppose that $F$ is a $G$-space and hence also an $K$-space. We can consider the fibre bundles $\mathfrak{X}[F]$ and $\mathfrak{Y}[F]$.

**Theorem 7.** *The family $\Lambda$ gives a bijection between $\Gamma_c(B, \mathfrak{X}[F])$ and $\Gamma_c(B, \mathfrak{Y}[F])$. Locally, this bijection takes the cross section $s = \{s_U\} \in \Gamma_c(B, \mathfrak{X}[F])$ to the section $\Lambda s = \{(\Lambda s)_U\} \in$*

$\Gamma_c(b, \mathfrak{Y}[F])$, *where* $(\Lambda s)_U$ *is defined by*

$$(\Lambda s)_U(b) := \lambda_U(b)^{-1} \cdot s_U(b).$$

To show that $\Lambda s$ defines a cross section, it must be shown that the family $\{(\Lambda s)_U\}$ satisfies equation 3.5. That is, it must be shown that $(\Lambda s)_U(b) = h_{U,V}(b) \cdot (\Lambda s)_V(b)$ for $b \in U \cap V$. This is shown by the following computations. Throughout, recall that the maps $\lambda_U$ and $g_{U,V}$ take values in $G$ while the map $s_U$ takes values in $F$.

$$\begin{aligned}(\Lambda s)_U(b) &= \lambda_U(b)^{-1} \cdot s_U(b) = \lambda_U(b)^{-1} \cdot (g_{U,V}(b) \cdot s_V(b)) \\ &= \big(\lambda_U(b)^{-1} g_{U,V}(b)\big) \cdot s_V(b) = \big(\lambda_U(b)^{-1} g_{U,V}(b) \lambda_V(b)\big) \cdot \lambda_V(b)^{-1} \cdot s_V(b) \\ &= h_{U,V}(b) \cdot (\Lambda s)_V(b)\end{aligned}$$

So $\Lambda s$ defines a cross section. Further, it is evident that this process has an inverse. Namely, if $t = \{t_U\} \in \Gamma_c(B, \mathfrak{Y}[F])$ then define $\Lambda^{-1} t = \{(\Lambda^{-1} t)_U\} \in \Gamma_c(B, \mathfrak{X}[F])$ by $(\Lambda^{-1} t)(b) := \lambda_U(b) \cdot t_U(b)$.

# CHAPTER 4
# BACKGROUND ON GEOMETRIC INVARIANT THEORY

## 4.1 Matrix Concomitants

The general linear group, $GL_n(\mathbb{C})$, acts on $M_n(\mathbb{C})$ by conjugation. In what follows, we will usually write this action on the right. That is, for $A \in M_n(\mathbb{C})$ and $g \in GL_n(\mathbb{C})$,

$$A \cdot g := g^{-1}Ag$$

Under this action the center of $GL_n(\mathbb{C})$ acts trivially, i.e. if $g$ is in the center of $GL_n(\mathbb{C})$ and $A \in M_n(\mathbb{C})$, then $A \cdot g = A$. Therefore, this action should really be considered as an action of the quotient of $GL_n(\mathbb{C})$ by its center, that is the projective linear group $PGL_n(\mathbb{C})$. Throughout this chapter, $PGL_n(\mathbb{C})$ will be denoted by $G$. Elements of $PGL_n(\mathbb{C})$ will be identified with their representatives in $GL_n(\mathbb{C})$. So for $A \in M_n(\mathbb{C})$ and $g \in PGL_n(\mathbb{C})$, we will write $A \cdot g = g^{-1}Ag$. This identification should cause no confusion, since by definition the action of $g \in PGL_n(\mathbb{C})$ on $M_n(\mathbb{C})$ is given by conjugation by any of its representatives in $GL_n(\mathbb{C})$.

Note that we can also consider the *left* action of $G$ on $M_n(\mathbb{C})$ by conjugation. This action is defined by $g \cdot A = gAg^{-1}$, for $A \in M_n(\mathbb{C})$ and $g \in G$. We will primarily be interested in the right action, but later we will use the left action of $G$ on $M_n(\mathbb{C})$ to build $M_n(\mathbb{C})$-fibre bundles from principal $G$-bundles.

The action of an element of $G$ on $M_n(\mathbb{C})$ is an automorphism of $M_n(\mathbb{C})$, and every automorphism of $M_n(\mathbb{C})$ arises from such an action. In some cases we will be interested in automorphisms of $M_n(\mathbb{C})$ that respect the $*$-structure of $M_n(\mathbb{C})$. These form a subgroup of all the automorphisms, and are given by conjugation by unitary matrices. Again, elements in the center act trivially, so we will consider the action of the quotient group of unitary matrices by its center, the projective unitary group $PU_n(\mathbb{C})$. We will denote $PU_n(\mathbb{C})$ by $K$. The group $K$ is a closed, compact subgroup of the group $G$.

The action of $G$ on $M_n(\mathbb{C})$ can be used to define an action of $G$ on $d$-tuples of $n \times n$ matrices. For a $d$-tuple $\mathfrak{z} = (Z_1, \ldots, Z_d) \in M_n(\mathbb{C})^d$ and $g \in G$, define

$$\mathfrak{z} \cdot g = g^{-1}\mathfrak{z}g = (g^{-1}Z_1 g, \ldots, g^{-1}Z_d g). \tag{4.1}$$

Throughout this thesis, $d$ and $n$ will be assumed to be at least 2 when discussing $d$-tuples of $n \times n$ matrices.

Let $\mathcal{D} \subset M_n(\mathbb{C})^d$ be a domain that is invariant under this action. A function $f : \mathcal{D} \to M_n(\mathbb{C})$ that intertwines the action of $G$ on $\mathcal{D}$ with its action on $M_n(\mathbb{C})$ is called a *matrix concomitant*. That is, $f$ is a matrix concomitant if

$$f(\mathfrak{z} \cdot g) = f(\mathfrak{z}) \cdot g$$

for all $\mathfrak{z} \in \mathcal{D}$ and $g \in G$. Under the identification of $G$ with $GL_n(\mathbb{C})$, this concomitant relation becomes $f(g^{-1}\mathfrak{z}g) = g^{-1}f(\mathfrak{z})g$. Considering the *left* action of $G$ on $M_n(\mathbb{C})$, this relation can be rewritten as $f(\mathfrak{z} \cdot g) = g^{-1} \cdot f(\mathfrak{z})$, so we see that this matches the concomitant relation defined in Section 3.1. Other authors call these covariant maps or intertwining maps.

In particular, we are interested in algebras of holomorphic matrix concomitants. The collection of all holomorphic matrix concomitants defined on the domain $\mathcal{D}$ will be denoted $Hol(\mathcal{D}, M_n(\mathbb{C}))^G$. To understand holomorphic matrix concomitants better, we will first study polynomial matrix concomitants. In general, a polynomial from $M_n(\mathbb{C})^d$ to $M_n(\mathbb{C})$ is a $n \times n$ matrix whose entries are polynomials in the $dn^2$ entries of the tuples in $M_n(\mathbb{C})^d$. Consider the algebra of polynomials $\mathbb{C}[x_{ij}^k]$ in the $dn^2$ indeterminates $x_{ij}^k$, with $1 \leq k \leq d$ and $1 \leq i, j \leq n$. Form the matrix algebra of $\mathbb{C}[x_{ij}^k]$, $M_n\left(\mathbb{C}[x_{ij}^k]\right)$. By evaluating the indeterminates $x_{ij}^k$ at complex numbers, we may view elements of $M_n\left(\mathbb{C}[x_{ij}^k]\right)$ as polynomial maps from $M_n(\mathbb{C})^d$ to $M_n(\mathbb{C})$. Here, the indeterminate $x_{ij}^k$ is evaluated at the $(i,j)$ component of $Z_k$ for each $\mathfrak{z} = (Z_1, Z_2, \ldots, Z_d) \in M_n(\mathbb{C})^d$. For $p \in M_n\left(\mathbb{C}[x_{ij}^k]\right)$, denote the associated polynomial map from $M_n(\mathbb{C})^d$ to $M_n(\mathbb{C})$ by $\hat{p}$. It is important to make a distinction between polynomials in $M_n\left(\mathbb{C}[x_{ij}^k]\right)$ and polynomial functions. The algebra of polynomial functions from $M_n(\mathbb{C})^d$ to $M_n(\mathbb{C})$ will be denoted by $Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))$, and the algebra of polynomial matrix concomitants will be denoted $Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$. This is a subalgebra of the algebra of continuous matrix concomitants, $C(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$. Let's start by understanding two algebras of polynomial concomitants, the algebra of generic matrices and the trace algebra.

For $k = 1, \ldots, d$, define $\mathcal{Z}_k \in Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))$ by $\mathcal{Z}_k(\mathfrak{z}) = Z_k$, where $\mathfrak{z} = (Z_1, \ldots Z_d) \in M_n(\mathbb{C})^d$. That is, $\mathcal{Z}_k$ is the $k^{th}$ matrix coordinate function. In terms of the algebra $M_n(\mathbb{C}[x_{ij}^k])$, $\mathcal{Z}_k$ is the polynomial function that arises from evaluating the matrix of polynomials $X_k = [x_{ij}^k]_{ij}$ at points in $M_n(\mathbb{C})^d$. It is clear from the definition of the action of $G$ on $M_n(\mathbb{C})^d$ that each matrix coordinate function $\mathcal{Z}_k$ is a polynomial matrix concomitant. The *algebra of generic matrices*, denoted $\mathbb{G}_0(d,n)$, is the unital subalgebra of $Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))$ generated by $\mathcal{Z}_1, \ldots, \mathcal{Z}_d$. Thus $\mathbb{G}_0(d,n)$ is isomorphic to the subalgebra of $M_n\left(\mathbb{C}[x_{ij}^k]\right)$ generated by $X_1, X_2, \ldots, X_d$. Since matrix concomitants form an algebra under pointwise sums and products, elements in $\mathbb{G}_0(d,n)$ are polynomial matrix concomitants. Other characterizations of $\mathbb{G}_0(d,n)$ and more information about the properties of $\mathbb{G}_0(d,n)$ are described in [Pro73] and [MR01].

The algebra of generic matrices is a very small subalgebra of $Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$ and by itself does not give much insight into the algebra of polynomial matrix concomitants. Another algebra of polynomial matrix concomitants is the *algebra of polynomial invariants*, denoted $\mathbb{I}_0(d,n)$. This algebra is the subalgebra of $Pol(M_n(\mathbb{C})^d, \mathbb{C})$ consisting of polynomial functions that are invariant under the action of $G$. That is, $\mathbb{I}_0(d,n)$ consists of all polynomial functions $p : M_n(\mathbb{C})^d \to \mathbb{C}$ so that $p(\mathfrak{z} \cdot g) = p(\mathfrak{z})$ for all $\mathfrak{z} \in M_n(\mathbb{C})^d$ and $g \in G$. An invariant polynomial $p \in \mathbb{I}_0(d,n)$ can be identified with the matrix-valued function $\mathfrak{z} \mapsto p(\mathfrak{z})I_n$. Under this identification, every $p \in \mathbb{I}_0(d,n)$ is a polynomial matrix concomitant since $p(\mathfrak{z})$ is a diagonal matrix and hence $p(\mathfrak{z}) \cdot g = p(\mathfrak{z}) = p(\mathfrak{z} \cdot g)$ for each $g \in G$. The subalgebra of $Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))$ that is generated by the algebra of generic matrices together with the algebra of invariants is called the *trace algebra*, and is denoted $\mathbb{S}_0(d,n)$. Again, since $\mathbb{S}_0(d,n)$ is generated by polynomial matrix concomitants it is clear that $\mathbb{S}_0(d,n)$ consists of matrix concomitants, i.e. $\mathbb{S}_0(d,n) \subseteq Pol(M_n^d, M_n)^G$.

Much work has been done by algebraists working in geometric invariant theory to study the algebras $\mathbb{G}_0(d,n)$ and $\mathbb{S}_0(d,n)$. In [Pro76, Theorem 2.1], Procesi proved that $\mathbb{S}_0(d,n)$ is *precisely* the set of all polynomial matrix concomitants. That is, $\mathbb{S}_0(d,n) = Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$. This fact gives rise to the following theorem that every holomorphic matrix concomitant can be approximated by elements in $\mathbb{S}_0(d,n)$.

**Theorem 8.** *The algebra* $Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$ *is the closure of* $\mathbb{S}_0(d,n)$ *in the topology of uniform convergence on compact subsets of* $M_n(\mathbb{C})^d$*.*

*Proof.* The key fact that will be used in this proof is Weyl's unitarian trick, which, in this situation, means that any polynomial function on $M_n(\mathbb{C})^d$ that is invariant under the action of $K$ is also invariant under the action of $G$. This is because $K$ is the maximal compact subgroup of $G$, and $G$ is a reductive group, so $K$ is Zariski dense in $G$ [Pro07, page 224].

Given $f \in Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$, choose a sequence $\{p_i\}$ of polynomial functions in $Pol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))$ that converges to $f$ uniformly on compact subsets of $M_n(\mathbb{C})^d$. We will use this sequence to construct a new sequence $\{\tilde{p}_i\}$ of polynomial matrix concomitants that also converges to $f$ uniformly on compact subsets of $M_n(\mathbb{C})^d$. Define

$$\tilde{p}_i(\mathfrak{z}) := \int_K p_i(\mathfrak{z} \cdot k) \cdot k^{-1} dk,$$

where $dk$ denotes the Haar measure on $K$.

To see that $\tilde{p}_i \longrightarrow f$, let $X \subset M_n(\mathbb{C})^d$ be a compact set. Since $K$ is a compact group, $\tilde{X} = \cup_{k \in K}(X \cdot k)$ is also compact. So, without loss of generality, we can assume that $X$ is invariant under the action of $K$. Note that, since $f$ is a matrix concomitant, for each $\mathfrak{z} \in M_n(\mathbb{C})^d$,

$$\int_K f(\mathfrak{z} \cdot k) \cdot k^{-1} dk = \int_K f(\mathfrak{z}) \cdot (kk^{-1}) dk = \int_K f(\mathfrak{z}) dk = f(\mathfrak{z}).$$

So, for $\mathfrak{z} \in X$,

$$\begin{aligned}\|\tilde{p}_i(\mathfrak{z}) - f(\mathfrak{z})\| &= \left\| \int_K p_i(\mathfrak{z} \cdot k) \cdot k^{-1} dk - \int_K f(\mathfrak{z} \cdot k) \cdot k^{-1} dk \right\| \\ &= \left\| \int_K (p_i(\mathfrak{z} \cdot k) - f(\mathfrak{z} \cdot k)) \cdot k^{-1} dk \right\| \\ &\leq \int_K \left\| (p_i(\mathfrak{z} \cdot k) - f(\mathfrak{z} \cdot k)) \cdot k^{-1} \right\| dk\end{aligned}$$

Since $k \in K$ acts by conjugation by a unitary matrix, which will have norm 1, this integral is equal to $\int_K \|p_i(\mathfrak{z} \cdot k) - f(\mathfrak{z} \cdot k)\| \, dk$. Since $\mathfrak{z} \cdot k \in X$, this integral is bounded by $\int_K \sup_{x \in X} \|p_i(x) - f(x)\| dk$, which is equal to $\sup_{x \in X} \|p_i(x) - f(x)\|$. As $p_i \longrightarrow f$ uniformly on $X$, this supremum is bounded and approaches 0 as $i \longrightarrow \infty$. Thus the $\tilde{p}_i$ converge uniformly to $f$ on the set $X$. By construction, each $\tilde{p}_i$ is a polynomial function and satisfies the relation $\tilde{p}_i(\mathfrak{z} \cdot k) = \tilde{p}_i(\mathfrak{z}) \cdot k$ for each $k$ in $K$. Hence, by Weyl's unitarian trick, each $\tilde{p}_i$ is a polynomial matrix concomitant. This completes the proof. □

Procesi also proved in [Pro73, Corollary IV.6.1] that $\mathbb{I}_0(d, n)$ is the center of $\mathbb{S}_0(d, n)$. Since $G$ is reductive, $\mathbb{I}_0(d, n)$ is finitely generated by Hilbert's Finiteness Theorem [Hil90]. In [Pro76, Theorem 3.4], Procesi gives explicity a set of generators for $\mathbb{I}_0(d, n)$: the trace maps $\mathfrak{z} \mapsto \operatorname{tr}(Z_{i_1} Z_{i_2} \cdots Z_{i_s})$ where $i_k \in \{1, \ldots, d\}$ for each $k$ and $s \leq 2^n - 1$. Further, he proved that $\mathbb{S}_0(d, n)$ is finitely generated as a module over $\mathbb{I}_0(d, n)$, and one collection of generators consists of the maps $\mathfrak{z} \mapsto Z_{i_1} Z_{i_2} \cdots Z_{i_s}$ for $s \leq 2^n - 2$.

Let $Q(d, n)$ denote the spectrum or maximal ideal space of $\mathbb{I}_0(d, n)$. Since $\mathbb{I}_0(d, n)$ is a subalgebra of $Pol(M_n(\mathbb{C})^d, \mathbb{C})$, there is a natural map from the maximal ideal space of $Pol(M_n(\mathbb{C})^d, \mathbb{C})$ to maximal ideals in $\mathbb{I}_0(d, n)$, and hence to $Q(d, n)$. This map is given by intersecting a maximal ideal $M$ in $Pol(M_n(\mathbb{C})^d, \mathbb{C})$ with $\mathbb{I}_0(d,n)$. By the weak Nullstellensatz, the maximal ideal space of $Pol(M_n(\mathbb{C})^d, \mathbb{C})$ can be identified with $M_n(\mathbb{C})^d$. Ultimately, then, we get a map $\pi : M_n(\mathbb{C})^d \to Q(d,n)$. Considering $Q(d,n)$ as an abstract affine variety defined over $\mathbb{C}$, the map $\pi$ is regular.

Because $\mathbb{I}_0(d,n)$ is finitely generated, $Q(d,n)$ can be viewed as a subset of $\mathbb{C}^p$ for some $p$, and an explicit formula for $\pi$ can be given in this case. To see this, let $f_1, \ldots, f_p$ generate $\mathbb{I}_0(d,n)$. Then we can define a map $\alpha : \mathbb{C}[X_1, X_2, \ldots, X_p] \to \mathbb{I}_0(d,n)$ by defining $\alpha(X_i) = f_i$ and extending to a homomorphism on $\mathbb{C}[X_1, X_2, \ldots, X_p]$. The homomorphism $\alpha$ is onto $\mathbb{I}_0(d,n)$ and $\ker \alpha$ is an ideal in $\mathbb{C}[X_1, X_2, \ldots, X_p]$. Let $\mathcal{I}$ denote this ideal. Then $\mathcal{I}$ is the ideal associated to some algebraic variety $V \subset \mathbb{C}^p$. Since $\mathbb{I}_0(d,n) \cong \mathbb{C}[X_1, X_2, \ldots, X_p]/\mathcal{I}$, their maximal ideal spaces can be identified. By Hilbert's Nullstellensatz, the maximal ideal space of $\mathbb{C}[X_1, X_2, \ldots, X_p]/\mathcal{I}$ can be identified with the variety $V$. Hence $Q(d,n)$ can be identified with an algebra variety in $\mathbb{C}^p$, where $p$ is the number of generators chosen.

Let $\varphi_1, \varphi_2, \ldots, \varphi_q \in \mathbb{C}[X_1, X_2, \ldots, X_p]$ be the polynomials that define the variety $V$. Since $\mathcal{I} = \ker \alpha$, this means that $\alpha(\varphi_i) = 0 \in \mathbb{I}_0(d,n)$ for each $i$, i.e. $\varphi_i(f_1, f_2, \ldots, f_p) = 0$. Thus $\varphi_i(f_1(\mathfrak{z}), f_2(\mathfrak{z}), \ldots, f_p(\mathfrak{z})) = 0$ for every $\mathfrak{z} \in M_n(\mathbb{C})^d$ and $i \in \{1, 2, \ldots, q\}$. It follows that the points in $V$ are exactly the points of the form $(f_1(\mathfrak{z}), f_2(\mathfrak{z}), \ldots, f_p(\mathfrak{z})) \in \mathbb{C}^p$. So, in this concrete situation, the map $\pi : M_n(\mathbb{C})^d \to Q(d,n)$ is given by $\pi(\mathfrak{z}) = (f_1(\mathfrak{z}), f_2(\mathfrak{z}), \ldots, f_p(\mathfrak{z}))$. Note, however, that the generators $f_1, f_2, \ldots, f_p$ are not unique and different choices of generators will give different realizations of $Q(d,n)$.

### 4.2 The Irreducible Points and the Bundle $\mathfrak{B}(d, n)$

The points in $M_n(\mathbb{C})^d$ can be viewed as the $n$-dimensional representations of the free algebra on $d$ generators, $\mathbb{C}\langle X_1, X_2, \ldots, X_d\rangle$. The representation associated to $\mathfrak{z} = (Z_1, \ldots, Z_d) \in M_n(\mathbb{C})^d$ is "evaluation" of $a \in \mathbb{C}\langle X_1, X_2, \ldots, X_d\rangle$ at the point $\mathfrak{z}$ given by replacing the indeterminate $X_i$ with the matrix $Z_i$. For $\mathfrak{z} \in M_n(\mathbb{C})^d$, denote the associated representation as $\pi_\mathfrak{z}$. Then for each $a \in \mathbb{C}\langle X_1, \ldots, X_d\rangle$, we can define a map $\hat{a}_n : M_n(\mathbb{C})^d \to M_n(\mathbb{C})$ by $\hat{a}_n(\mathfrak{z}) := \pi_\mathfrak{z}(a)$. Note that the map $\hat{a}_n$ does depend on the size of matrices being considered. Another characterization of the algebra of generic matrices is

$$\mathbb{G}_0(d, n) = \{\hat{a}_n : a \in \mathbb{C}\langle X_1, \ldots, X_d\rangle\}$$

It what follows, we will not distinguish between an element $a \in \mathbb{C}\langle X_1, \ldots X_d\rangle$ and its image $\hat{a} \in \mathbb{G}_0(d, n)$ even though this assignment is not injective.

We would like to study the similarity classes of representations, which are in correspondence with the $G$-orbits of $M_n(\mathbb{C})^d$. The trouble is that some of these orbits are not closed. For $\mathfrak{z} \in M_n(\mathbb{C})^d$, let $\mathfrak{z}G$ denote the orbit of $\mathfrak{z}$ under the action of $G$. The following results come from Artin [Art69, 12.6] and Procesi [Pro74, Theorem 4.1]. We will call a point $\mathfrak{z}$ *semisimple* if the image of $\pi_\mathfrak{z}$ is a semisimple subalgebra of $M_n(\mathbb{C})$. A point $\mathfrak{z}$ is semisimple if and only if $\mathfrak{z}G$ is closed. Further, each closure of an orbit in $M_n(\mathbb{C})^d$ contains a unique closed orbit. That is, for $\mathfrak{z} \in M_n(\mathbb{C})^d$, the closed set $\overline{\mathfrak{z}G}$ contains a unique closed orbit $\mathfrak{z}_sG$, where $\mathfrak{z}_s$ is a semisimple point. $\mathfrak{z}_s$ is called the semisimplication of $\mathfrak{z}$. If $\mathfrak{z}$ and $\mathfrak{w}$ are two semisimple points, then $\pi_\mathfrak{z} = \pi_\mathfrak{w}$ if and only if $\mathfrak{z}G = \mathfrak{w}G$. Because each prbit closure contains a unique closed orbit, $Q(d,n)$ may be thought of as parameterizing either the closed orbits or the orbit closures in $M_n(\mathbb{C})^d$.

To simplify the picture, we restrict our attention to the irreducible points in $M_n(\mathbb{C})^d$. A tuple $\mathfrak{z} = (Z_1, Z_2, \ldots, Z_d)$ is called *irreducible* if its entries generate $M_n(\mathbb{C})$ as an algebra. By Burnside's Theorem for matrix algebras, a tuple $\mathfrak{z}$ is irreducible if and only if its entries have no common invariant subspace. Call the collection of irreducible points $\mathcal{V}(d,n)$. If $\mathfrak{z}$ is an irreducible point in $M_n(\mathbb{C})^d$, then the representation $\pi_{\mathfrak{z}}$ is semisimple since its image is all of $M_n(\mathbb{C})$. Further, $\mathfrak{z}$ is an irreducible point if and only if $\pi_{\mathfrak{z}}$ is an irreducible representation.

The set $\mathcal{V}(d,n)$ is a $G$-invariant, Zariski open (and therefore dense) subset of $M_n(\mathbb{C})^d$. For example, it is well known that the irreducible points in $M_2(\mathbb{C})^2$ are the pairs $(Z_1, Z_2)$ such that the commutator $[Z_1, Z_2] = Z_1Z_2 - Z_2Z_1$ is invertible. Lacking an explicit reference for this characterization, here is a simple proof: We may assume without loss of generality that $Z_1$ is in Jordan canonical form and that $Z_1$ either has distinct eigenvalues or is the Jordan cell, whose upper right-hand entry is 1 and other entries are 0. If $Z_1$ has distinct eigenvalues $\lambda_1$ and $\lambda_2$, the we may write

$$[Z_1, Z_2] = \left[\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} w & x \\ y & z \end{bmatrix}\right] = \begin{bmatrix} 0 & (\lambda_1 - \lambda_2)x \\ (\lambda_2 - \lambda_1)y & 0 \end{bmatrix}.$$

On the other hand, if $Z_1$ is the Jordan cell, then

$$[Z_1, Z_2] = \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} w & x \\ y & z \end{bmatrix}\right] = \begin{bmatrix} -y & w - z \\ 0 & y \end{bmatrix}.$$

In either case, it is clear the $[Z_1, Z_2]$ is invertible if and only if $Z_1$ and $Z_2$ have no common invariant subspace. Thus $(Z_1, Z_2) \mapsto \det([Z_1, Z_2])$ is a polynomial in $\mathbb{I}_0(2,2)$ whose zero set is $M_2(\mathbb{C})^2 \setminus \mathcal{V}(2,2)$, and from this characterization of $\mathcal{V}(2,2)$ it is clear that $\mathcal{V}(2,2)$ is $G$-invariant and Zariski open in $M_2(\mathbb{C})^2$.

For general $n$ and $d$, note $G$ acts holomorphically, freely, and properly on $\mathcal{V}(d,n)$, and hence $\mathcal{V}(d,n)$ is the total space of a holomorphic principal $G$-bundle. This assertion follows from proof of Proposition 2.2.1 in [Pal61]. The base space of the bundle associated to the action of $G$ on $\mathcal{V}(d,n)$ is the orbit space $\mathcal{V}(d,n)/G$. This orbit space is hard to understand, but a fundamental theorem of Procesi [Pro74, Theorem 5.10] gives a nice description of the base space. Let $\pi : M_n(\mathbb{C})^d \to Q(d,n)$ be the map defined at the end of section 4.1. Denote the restriction of $\pi$ to $\mathcal{V}(d,n)$ by $\pi_0$, and denote the image of $\pi_0$ by $Q_0(d,n)$. Since every irreducible point is semisimple and hence has a closed orbit, the points in $Q_0(d,n)$ parametrize exactly the $G$-orbits in $\mathcal{V}(d,n)$. So the base space of the principal $G$-bundle defined by the action of $G$ on $\mathcal{V}(d,n)$ can be identified with $Q_0(d,n)$. Further, Procesi proves that $Q_0(d,n)$ is an open subset of the smooth points of $Q(d,n)$. The bundle $\mathfrak{B}(d,n) = (\mathcal{V}(d,n), \pi_0, Q_0(d,n))$ will be of fundamental importance in the chapters that follow.

For clarification, let's consider the case when $d = n = 2$. Much of what follows is computed by LeBruyn in Section 1.2 of his book [LB08]. The algebra of invariants $\mathbb{I}_0(2,2)$ is generated by the five functions $\mathfrak{z} \mapsto \operatorname{tr}(Z_1), \mathfrak{z} \mapsto \operatorname{tr}(Z_2), \mathfrak{z} \mapsto \operatorname{tr}(Z_1^2), \mathfrak{z} \mapsto \operatorname{tr}(Z_1 Z_2)$, and $\mathfrak{z} \mapsto \operatorname{tr}(Z_2^2)$. This list of generators is not unique. For instance, replacing $\mathfrak{z} \mapsto \operatorname{tr}(Z_i^2)$ with $\mathfrak{z} \mapsto \det(Z_i)$ for $i = 1$ or $2$ yields another collection of five maps that generate $\mathbb{I}_0(2,2)$. In either case, the five generators are algebraically independent. The spectrum $Q(2,2)$ can be identified with $\mathbb{C}^5$, the image of $M_2(\mathbb{C})^2$ under these five generators.

In this case, the algebra $\mathbb{S}_0(2,2)$ is a free module over $\mathbb{I}_0(d,n)$ and is generated by the four maps $\mathfrak{z} \mapsto I_2, \mathfrak{z} \mapsto Z_1, \mathfrak{z} \mapsto Z_2$, and $\mathfrak{z} \mapsto Z_1 Z_2$. The irreducible points $\mathcal{V}(2,2)$ are the pairs $(Z_1, Z_2)$ so that $\det(Z_1 Z_2 - Z_2 Z_1) \neq 0$, and the set $Q_0(2,2)$ is

$$Q_0(2,2) = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : z_5^2 - z_1 z_3 z_5 + z_1^2 z_4 + z_3^2 z_2 - 4 z_2 z_4 \neq 0\}.$$

This is an open set in $\mathbb{C}^5$ whose complement is an affine hypersurface.

### 4.3 Concomitants and Cross Sections

We will consider $\mathfrak{M}(d,n)$, the $M_n(\mathbb{C})$-fibre bundle associated to $\mathfrak{B}(d,n)$. This is a special case of the construction described in Section 2.4. Here, we consider $G$ acting on the *left* of the fibre $M_n(\mathbb{C})$ by conjugation by elements in $GL_n(\mathbb{C})$. Define an action of $G$ on the *right* of $\mathcal{V}(d,n) \times M_n(\mathbb{C})$ by $(\mathfrak{z}, A) \cdot g = (\mathfrak{z} \cdot g, g^{-1} \cdot A) = (g^{-1}\mathfrak{z} g, g^{-1} A g)$. The bundle space of $\mathfrak{M}(d,n)$ is orbit space of this action, $\mathcal{V}(d,n) \times_G M_n(\mathbb{C})$. The projection map of the bundle $\mathfrak{M}(d,n)$ is $\hat{\pi}_0 : \mathcal{V}(d,n) \times_G M_n(\mathbb{C})$ defined by $\hat{\pi}_0([\mathfrak{z}, A]) = \pi_0(\mathfrak{z}) = [\mathfrak{z}]$. The base space of $\mathfrak{M}(d,n)$ is $Q_0(d,n)$. Since $\mathfrak{B}(d,n)$ is a locally trivial holomorphic bundle, $\mathfrak{M}(d,n)$ is also a locally trivial holomorphic bundle by Lemma 2.

The reason $\mathfrak{M}(d,n)$ is of interest is because, by Theorem 3, the continuous cross sections $\Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ are in bijection with the continuous matrix concomitants $C(\mathcal{V}(d,n), M_n(\mathbb{C}))^G$. Recall that the bijection $\Psi$ identifies a concomitant $f$ with a cross section $s$ if and only if $s$ and $f$ satisfy the equation

$$s([\mathfrak{z}]) = [\mathfrak{z}, f(\mathfrak{z})]$$

for each irreducible point $\mathfrak{z} \in \mathcal{V}(d,n)$. Theorem 4 asserts that this bijection identifies the set of holomorphic cross section $\Gamma_h(Q_0(d,n), \mathfrak{M}(d,n))$ with the holomorphic matrix concomitants $Hol(\mathcal{V}(d,n), M_n(\mathbb{C}))^G$.

In the case that $d$ and $n$ are not both 2, the bijection $\Psi : Hol(\mathcal{V}(d,n), M_n(\mathbb{C}))^G \to \Gamma_h(Q_0(d,n), \mathfrak{M}(d,n))$ is in fact a bijection from the holomorphic matrix concomitants defined on all of $M_n(\mathbb{C})^d$ to the holomorphic cross sections of $\mathfrak{M}(d,n)$. This is a consequence of the following theorem.

**Theorem 9.** *If $d > 2$ or $n > 2$, then every holomorphic concomitant from $\mathcal{V}(d,n)$ to $M_n(\mathbb{C})$ admits a unique extension to a holomorphic concomitant defined on all of $M_n(\mathbb{C})^d$. That is, $Hol(\mathcal{V}(d,n), M_n(\mathbb{C}))^G = Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$.*

The proof of this theorem is due to Zinovy Reichstein and may be found in [GMS15, Theorem 1.2]. The argument rests on calculating the dimension of complement of $\mathcal{V}(d,n)$ in $M_n(\mathbb{C})^d$. Reichstein's calculations show that, as long as either $d > 2$ or $n > 2$, the dimension of the complement $M_n(\mathbb{C})^d \setminus \mathcal{V}(d,n)$ is at most $dn^2 - 2$. Consequently, by [Gun90, Theorem K.1], every function that is holomorphic on $\mathcal{V}(d,n)$ extends uniquely to a function that is holomorphic on all of $M_n(\mathbb{C})^d$. More over, if $f : \mathcal{V}(d,n) \to M_n(\mathbb{C})$ is a holomorphic matrix concomitant, then its extension $\tilde{f}$ is a matrix comcomitant as well. This is because, for each $g \in G$, the holomorphic map $\mathfrak{z} \mapsto \left(\tilde{f}(\mathfrak{z} \cdot g) - \tilde{f}(\mathfrak{z}) \cdot g\right)$ is zero on $\mathcal{V}(d,n)$, which is a dense subset of $M_n(\mathbb{C})^d$.

The above theorem is not true if $d = n = 2$. The domain $\mathcal{V}(2,2)$ is a domain of holomorphy and there are elements in $Hol(\mathcal{V}(2,2), M_2(\mathbb{C}))^G$ that do not extend to $M_2(\mathbb{C})^2$. Since $\mathcal{V}(2,2) = \{(Z_1, Z_2) : \det([Z_1, Z_2]) \neq 0\}$, and since $(Z_1, Z_2) \mapsto \det([Z_1, Z_2])$ is clearly invariant under the action of $G$, the function

$$f(\mathfrak{z}) = \frac{1}{\det(Z_1Z_2 - Z_2Z_1)} I_2$$

is a holomorphic matrix concomitant that cannot be extended to any point $\mathfrak{z} \in M_2(\mathbb{C})^2$ which is not irreducible.

# CHAPTER 5
# THE ALGEBRAS I STUDY

In what follows, I will investigate several algebras that are related to the principal $G$-bundle $\mathfrak{B}(d,n) = (\mathcal{V}(d,n), \pi_0, Q_0(d,n))$ and the associated $M_n(\mathbb{C})$-fibre bundle $\mathfrak{M}(d,n) = (\mathcal{V}(d,n) \times_G M_n(\mathbb{C}), \hat{\pi}_0, Q_0(d,n))$. My goal is to study algebras whose elements are cross sections of subbundles of $\mathfrak{M}(d,n)$. Let $X \subset Q_0(d,n)$ be a compact set, and consider the restriction of $\mathfrak{M}(d,n)$ over $X$, i.e. the bundle $\mathfrak{M}(d,n)|X = (\hat{\pi}_0^{-1}(X), \hat{\pi}_0, X)$. When it is obvious that we are considering a restriction, I will simply denote $\mathfrak{M}(d,n)|X$ by $\mathfrak{M}(d,n)$. In particular, I would like to study the algebra of holomorphic cross sections $\Gamma_h(X, \mathfrak{M}(d,n))$ and the algebra of continuous cross sections $\Gamma_c(X, \mathfrak{M}(d,n))$. At this stage, $\Gamma_c(X, \mathfrak{M}(d,n))$ and $\Gamma_h(X, \mathfrak{M}(d,n))$ are only topological algebras. They currently have no norm or Banach algebra structure. In this chapter, we will explore a couple of methods for endowing them with Banach algebra structures.

Throughout this chapter, $G$ will continue to denote the projective linear group and $K$ will denote the projective unitary group. Further, $\mathcal{U}$ will denote a coordinate atlas for $\mathfrak{B}(d,n)$. $\mathcal{U}$ is also a coordinate atlas for $\mathfrak{M}(d,n)$ by Lemma 2. The coordinate functions of $\mathfrak{B}(d,n)$ will be denoted by $\psi_U : U \times G \to \pi_0^{-1}(U)$ and the coordinate functions of $\mathfrak{M}(d,n)$ will be $\phi_U : U \times M_n(\mathbb{C}) \to \pi^{-1}(U)$. The coordinate transformations of $\mathfrak{B}(d,n)$, and consequently also of $\mathfrak{M}(d,n)$, will be denoted by $g_{U,V} : U \cap V \to G$.

## 5.1 Operations on Cross Sections

It is well-known that $\Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ inherits an algebraic structure from the fibre $M_n(\mathbb{C})$. In this section, we will make this algebraic structure explicit and look at how it interacts with the variety of perspectives we have for cross sections. First, we will define operations on the fibre $\pi^{-1}(b)$ for each $b \in Q_0(d,n)$. The operations on cross sections will then be defined pointwise.

Fix $b \in Q_0(d,n)$ and let $U \in \mathcal{U}$ be a neighborhood of $b$. Recall that, since the bundle $\mathfrak{B}(d,n)$ is holomorphic, the fibre $\pi^{-1}(b)$ is biholomorphically equivalent to $M_n(\mathbb{C})$ via the mapping $\phi_{U,b} : M_n(\mathbb{C}) \to \pi^{-1}(b)$. The operations of addition, multiplication and scalar multiplication on $\pi^{-1}(b)$ will be defined so that $\phi_{U,b}$ is an isomorphism of $\mathbb{C}$-algebras. That is, for $\lambda \in \mathbb{C}$, $x, y \in \pi^{-1}(b)$, and $A = \phi_{U,b}^{-1}(x)$ and $B = \phi_{U,b}^{-1}(y)$, define

$$xy := \phi_{U,b}(AB) \tag{5.1}$$

$$x + y := \phi_{U,b}(A + B) \tag{5.2}$$

$$\lambda x := \phi_{U,b}(\lambda A) \tag{5.3}$$

We will show that these operations are well-defined. Suppose that $b \in U \cap V$ for $U, V \in \mathcal{U}$. Let $x, y \in \pi^{-1}(b)$ and let $A_U = \phi_{U,b}^{-1}(x)$ (respectively $A_V = \phi_{V,b}^{-1}(x)$) and $B_U = \phi_{U,b}^{-1}(y)$ (respectively $B_V = \phi_{V,b}^{-1}(y)$). Then, by definition, $\phi_{U,b}^{-1} \circ \phi_{V,b}(A_V) = A_U$ and $\phi_{U,b}^{-1} \circ \phi_{V,b}(B_V) = B_U$. That is, $g_{U,V}(b) \cdot A_V = A_U$ and $g_{U,V}(b) \cdot B_V = B_U$. The action of $g_{U,V}(b) \in G$ on $M_n(\mathbb{C})$ is an automorphism, so $g_{U,V}(b) \cdot (A_V B_V) = A_U B_U$, i.e. $\phi_{U,b}^{-1} \circ \phi_{V,b}(A_V B_V) = A_U B_U$. Hence $\phi_{V,b}(A_V B_V) = \phi_{U,b}(A_U B_U)$. So the multiplication in $\pi^{-1}(b)$ defined by (5.1) is well-defined. By similar arguments, the addition and scalar multiplication in $\pi^{-1}(b)$ are also well-defined. Note that since the operations in $M_n(\mathbb{C})$ are holomorphic and the maps $\phi_{U,b}$ are biholomorphic, the operations on $\pi^{-1}(b)$ are holomorphic as well.

Using these operations on the fibre $\pi^{-1}(b)$, define the operations of multiplication, addition, and scalar multiplication on the elements of $\Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ pointwise. Before discussing strategies for endowing algebras of cross sections with Banach and $C^*$-algebra structures, let's revisit the two other perspectives of cross sections and see how these perspectives relate to the operations of addition and multiplication that have been defined on cross sections.

By Theorem 3, the cross sections $\Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ can be identified with the algebra of concomitants $C(\mathcal{V}(d,n), M_n(\mathbb{C}))^G$, where $s \in \Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ and $f \in C(\mathcal{V}(d,n), M_n(\mathbb{C}))^G$ are identified if they satisfy the relation $s(\pi_0(\mathfrak{z})) = [\mathfrak{z}, f(\mathfrak{z})]$ for every $\mathfrak{z} \in \mathcal{V}(d,n)$. The concomitants $C(\mathcal{V}(d,n), M_n(\mathbb{C}))^G$ is a collection of functions that take values in $M_n(\mathbb{C})$, and so it is an algebra under pointwise addition and multiplication. The next theorem shows that the operations in $C(\mathcal{V}(d,n), M_n(\mathbb{C}))^G$ are consistent with the operations defined above on $\Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$. That is, the identification $\Psi$ that takes a concomitant $f$ to the corresponding cross section $s_f$ is an isomorphism of $\mathbb{C}$-algebras.

**Theorem 10.** *The map that sends $f \in C(\mathcal{V}(d,n), M_n(\mathbb{C}))^G$ to $s_f \in \Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ is an algebra isomorphism.*

*Proof.* Fix $b \in Q_0(d,n)$ and suppose that $U \in \mathcal{U}$ is a coordinate neighborhood of $b$. Recall from Lemma 2 that the two bundles $\mathfrak{M}(d,n)$ and $\mathfrak{B}(d,n)$ have the same coordinate transformations $g_{U,V}$ and the coordinate functions $\phi_U$ of $\mathfrak{M}(d,n)$ are related to the coordinate functions $\psi_U$ of $\mathfrak{B}(d,n)$ by the equation $\phi_U(b,f) = [\psi_U(b,e), f]$. Here, $e$ denotes the identity in $G$. To prove the theorem, we will need a lemma.

**Lemma 11.** *Given the local data of $\mathfrak{B}(d,n)$ and $\mathfrak{M}(d,n)$ as described above, with $b \in U \subset Q_0(d,n)$ and $\mathfrak{z} \in \pi_0^{-1}(b)$, the maps $\phi_{U,b} : M_n(\mathbb{C}) \to \pi^{-1}(b)$ and $\phi_{U,b}^{-1} : \pi^{-1}(b) \to M_n(\mathbb{C})$ are given by the formulas*

$$\phi_{U,b}(A) = [\psi_{U,b}(e), A]$$

*and*

$$\phi_{U,b}^{-1}([\mathfrak{z}, A]) = \left(\psi_{U,b}^{-1}(\mathfrak{z})\right)^{-1} \cdot A$$

The first equation is clear from the definition $\phi_{U,b}(A) := \phi_U(b, A)$. To prove the second equation, let $\psi_{U,b}^{-1}(\mathfrak{z}) = g \in G$. Then $\mathfrak{z} = \psi_U(b,g) = \psi_U(b,e) \cdot g$, and hence $[\mathfrak{z}, A] = [\psi_U(b,e) \cdot g, A] = [\psi_U(b,e), g^{-1} \cdot A]$. That is $[\mathfrak{z}, A] = [\psi_{U,b}(e), \left(\psi_{U,b}^{-1}(\mathfrak{z})\right)^{-1} \cdot A] = \phi_{U,b}\left(\left(\psi_{U,b}^{-1}(\mathfrak{z})\right)^{-1} \cdot A\right)$ and the lemma follows.

Now let's return to the proof of the theorem. Fix $b \in Q_0(d,n)$ and let $U$ be a coordinate neighborhood of $b$. Fix $\mathfrak{z} \in \pi_0^{-1}(b)$, and $f, g \in C(\mathcal{V}(d,n), M_n(\mathbb{C}))^G$. Then

$$\begin{aligned} s_f(b)s_g(b) &= [\mathfrak{z}, f(\mathfrak{z})][\mathfrak{z}, g(\mathfrak{z})] = \phi_{U,b}\left(\phi_{U,b}^{-1}([\mathfrak{z}, f(\mathfrak{z})])\phi_{U,b}^{-1}([\mathfrak{z}, g(\mathfrak{z})])\right) \\ &= \phi_{U,b}\left((\psi_{U,b}^{-1}(\mathfrak{z})^{-1} \cdot f(\mathfrak{z}))(\psi_{U,b}^{-1}(\mathfrak{z})^{-1} \cdot g(\mathfrak{z}))\right) = \phi_{U,b}\left(\left(\psi_{U,b}^{-1}(\mathfrak{z})\right)^{-1} \cdot f(\mathfrak{z})g(\mathfrak{z})\right) \\ &= \phi_{U,b}(\phi_{U,b}^{-1}([\mathfrak{z}, f(\mathfrak{z})g(\mathfrak{z})])) = [\mathfrak{z}, f(\mathfrak{z})g(\mathfrak{z})] = [\mathfrak{z}, (fg)(\mathfrak{z})] = s_{fg}(b) \end{aligned}$$

The key fact used in this computation is that the action of $\psi_{U,b}^{-1}(\mathfrak{z})^{-1} \in G$ on elements in $M_n(\mathbb{C})$ is an automorphism. So this action respects multiplication, addition and multiplication by scalars. Hence the above argument can be modified to show that $s_{f+g}(b) = s_f(b) + s_g(b)$ and $s_{\lambda f}(b) = \lambda s_f(b)$. The proof of the theorem is complete. □

A third perspective is to view a cross section $s$ as a family of functions $\{s_U : U \to M_n(\mathbb{C})\}_{U\in\mathcal{U}}$ which are compatible in a certain way. That is, a cross section $s : Q_0(d,n) \to \mathcal{V} \times_G M_n(\mathbb{C})$ can be viewed as a family of maps $\{s_U : U \to M_n\}_{U\in\mathcal{U}}$ which satisfy the relationship $s_U(b) = g_{U,V}(b) \cdot s_V(b)$ for every $b \in U \cap V$. We can define operations on these families of functions pointwise. Since $s_U(b) := \phi^{-1}_{U,b}(s(b))$, these operations will be the same as those defined in (5.1).

These alternative views of cross sections and the operations on them give insight into the structure of $\Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$. For instance, it is clear that the algebra of continuous cross sections is unital. Thinking locally, the identity is the family of maps defined by $s_U \equiv I_n$. Now that we have defined an algebraic structure on $\Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$, we can prove a corollary to Theorem 9.

**Corollary 12.** *The bundle $\mathfrak{M}(d,n)$ is not a trivial bundle when either $d > 2$ or $n > 2$.*

If $\mathfrak{M}(d,n)$ were a trivial bundle, then it would be isomorphic to the product bundle $(G_0(d,n) \times M_n(\mathbb{C}), \mathrm{proj}_1, G_0(d,n))$. This would imply that the total space $\mathcal{V}(d,n) \times_G M_n(\mathbb{C})$ is biholomorphically equivalent to $Q_0(d,n) \times M_n(\mathbb{C})$. In this case, each holomorphic cross section could be identified with a holomorphic map from $Q_0(d,n)$ to $M_n(\mathbb{C})$, and this identification would be an isomorphism. Clearly the algebra of holomorphic maps from $Q_0(d,n)$ to $M_n(\mathbb{C})$ has zero divisors. Hence if $\mathfrak{M}(d,n)$ were trivial, then the algebra $\Gamma_h(Q_0(d,n), \mathfrak{M}(d,n))$ would have zero divisors.

However, $\Gamma_h(Q_0(d,n), \mathfrak{M}(d,n)) \cong Hol(\mathcal{V}(d,n), M_n(\mathbb{C}))^G$. Moreover, we have that every holomorphic concomitant defined on $\mathcal{V}(d,n)$ can be extended uniquely to a holomorphic concomitant on $M_n(\mathbb{C})^d$ by Theorem 9. I.e. $Hol(\mathcal{V}(d,n), M_n(\mathbb{C}))^G = Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$. Luminet proved in [Lum97, Proposition 4.4] that $Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))$ has no zero divisors. Consequently, the algebra of holomorphic cross section $\Gamma_h(Q_0(d,n), \mathfrak{M}(d,n))$ also has no zero divisors. Therefore $\mathfrak{M}(d,n)$ is not a trivial bundle whenever $(d,n) \neq (2,2)$.

In fact, $\mathfrak{M}(2,2)$ is also not a trivial bundle. The proof of this fact was communicated through private correspondence with Ben Williams and Zinovy Reichstein. The argument depends on computing the first homology group of $\mathcal{V}(2,2)$. As previously discussed, $\mathcal{V}(2,2)$ consists of all pairs $(Z_1, Z_2) \in M_2(\mathbb{C})^2$ such that $\det([Z_1, Z_2]) \neq 0$. Since the map $\mathfrak{z} \mapsto \det([Z_1, Z_2])$ is a polynomial function, the complement of $\mathcal{V}(2,2)$ in $M_2(\mathbb{C})^2 = \mathbb{C}^8$ is a hypersurface. By Corollary 1.4 on page 103 in [Dim92], this implies that $H_1(\mathcal{V}(2,2), \mathbb{Z}) = \mathbb{Z}^k$ where $k$ is the number of irreducible components of the hypersurface. The bundle $\mathfrak{M}(2,2)$ is trivial if and only if $\mathfrak{B}(2,2)$ is trivial. Thus if $\mathfrak{M}(2,2)$ were trivial, $\mathcal{V}(2,2)$ would be homeomorphic to $Q_0(d,n) \times PGL_2(\mathbb{C})$. This would imply that $H_1(\mathcal{V}(2,2), \mathbb{Z})$ would contain $H_1(PGL_2(\mathbb{C}), \mathbb{Z})$ as a summand. However $H_1(PGL_2(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, so this would contradict that $H_1(\mathcal{V}(2,2), \mathbb{Z}) = \mathbb{Z}^k$. Thus $\mathfrak{M}(2,2)$ is not trivial. The fact that $H_1(PGL_2(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ seems to be well-known, but we lack an explicit reference in the literature. For completeness, note the $PGL_2(\mathbb{C})$ is contractible to $PU_2(\mathbb{C})$. Tomiyama and Takesaki argue in the paragraph before the Remark on page 517 in [TT61] that the fundamental group of $PU_2(\mathbb{C})$ is $\mathbb{Z}/2\mathbb{Z}$. Since $PU_2(\mathbb{C})$ is connected and its fundamental group is abelian, it follows that the first integral homology group of $PU_2(\mathbb{C})$ is the same as its fundamental group, that is $H_1(PGL_2(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

### 5.2 Banach Algebra Structure

Let $X \subset Q_0(d, n)$ be a compact set and consider the algebras $\Gamma_h(X, \mathfrak{M}(d, n))$ and $\Gamma_c(X, \mathfrak{M}(d, n))$. Both $\Gamma_h(X, \mathfrak{M}(d, n))$ and $\Gamma_c(X, \mathfrak{M}(d, n))$ have natural structures as *topological* algebras. However, there is no evident intrinsic or natural *Banach algebra* structure on these algebras. In this section and the next we will explore various possibilities. One natural approach would be to show that $\mathfrak{M}(d, n)$ has the structure of a Banach bundle in the sense of [DG83] or a structure of a continuous field of Banach spaces in the sense of [DD63]. However, since the fibres are all finite dimensional of same dimension, we will begin simply by imposing Banach algebra structures on $\Gamma_h(X, \mathfrak{M}(d, n))$ and $\Gamma_c(X, \mathfrak{M}(d, n))$ in an *ad hoc* fashion, using the Banach algebra structure of the fibre $M_n(\mathbb{C})$.

Let $\mathcal{U}$ be a coordinate atlas for the restriction $\mathfrak{M}(d, n)|X$. Since $X$ is compact, we may assume that $\mathcal{U}$ is finite. For $s \in \Gamma_c(X, \mathfrak{M}(d, n))$, define

$$\|s\|_{\mathcal{U}} := \sup_{b \in X} \sup_{b \in U} \|s_U(b)\| \tag{5.4}$$

Here, $s_U : U \to M_n(\mathbb{C})$ is defined by $s_U(b) = \phi_{U,b}^{-1}(s(b))$ (see section 3.2). Since $s_U(b) \in M_n(\mathbb{C})$, the norm on the right-hand side is the operator space norm that arises from viewing $M_n(\mathbb{C})$ as operators on $\mathbb{C}^n$.

For each $s$ and $\mathcal{U}$, $\|s\|_{\mathcal{U}}$ is finite since $\mathcal{U}$ is finite. Further, $\|\cdot\|_{\mathcal{U}}$ is a norm: It is clear that $\|s\|_{\mathcal{U}} \geq 0$ and that $\|s\|_{\mathcal{U}} = 0$ if and only if $s \equiv 0$, where $s \equiv 0$ means that $s_U(b) = 0$ for every $U \in \mathcal{U}$ and every $b \in U$. Using the corresponding properties for the norm on $M_n(\mathbb{C})$, it is easy to prove that $\|\lambda s\|_{\mathcal{U}} = |\lambda|\|s\|_{\mathcal{U}}$, $\|s+t\|_{\mathcal{U}} \leq \|s\|_{\mathcal{U}} + \|t\|_{\mathcal{U}}$, and $\|st\|_{\mathcal{U}} \leq \|s\|_{\mathcal{U}}\|t\|_{\mathcal{U}}$ for $s, t \in \Gamma_c(X, \mathfrak{M})$ and $\lambda \in \mathbb{C}$. Since the identity in $\Gamma_c(X, \mathfrak{M}(d,n))$ is the cross section $e$ defined locally by $e_U(b) = I_n$ for each $b \in U$, we have $\|e\|_{\mathcal{U}} = 1$. Now observe that $\Gamma_c(X, \mathfrak{M}(d,n))$ is complete in the norm $\|\cdot\|_{\mathcal{U}}$. Indeed, a sequence of sections $\{s_n\}_{n\geq 1}$ that is Cauchy in this norm converges uniformly on $X$ to a section $s$ on $X$ which must be continuous. Hence $\Gamma_c(X, \mathfrak{M}(d,n))$ is a Banach algebra.

The norm $\|\cdot\|_{\mathcal{U}}$ depends on the choice of cover $\mathcal{U}$. The next theorem says that although different local trivializations give rise to different norms, all the norms are equivalent in the sense that the Banach algebras constructed are mutually isomorphic.

**Theorem 13.** *Suppose $X \subset Q_0(d,n)$ is compact, and let $\mathcal{U}$ and $\mathcal{V}$ be two finite coordinate atlases of $\mathfrak{M}(d,n)|X$. Then the map $\iota : (\Gamma_c(X, \mathfrak{M}(d,n)), \|\cdot\|_{\mathcal{U}}) \to (\Gamma_c(X, \mathfrak{M}(d,n)), \|\cdot\|_{\mathcal{V}})$ defined by $\iota(s) = s$ is an isomorphism of Banach algebras.*

*Proof.* It is clear that the map $\iota$ is an algebraic isomorphism. It remains to show that $\iota$ is bounded below. This will imply that $\iota^{-1}$ is also bounded below. To show this, we will prove that there exists an $\varepsilon > 0$ so that

$$\|s\|_{\mathcal{V}} \geq \varepsilon\|s\|_{\mathcal{U}}$$

for each cross section $s$.

Note that since $\mathfrak{M}(d,n)|X$ has a unique maximal atlas ([Hus94, Prop V.2.5]), $\mathcal{U}$ and $\mathcal{V}$ are compatible in the sense of Definition 1 in Chapter 2, so $\mathcal{U} \cup \mathcal{V}$ is also a coordinate atlas for $\mathfrak{M}(d,n)|X$. Since both $\mathcal{U}$ and $\mathcal{V}$ are finite, their union will be finite as well.

Fix $s \in \Gamma_c(X, \mathfrak{M}(d,n))$ and $b_0 \in X$. Suppose that the coordinate neighborhoods and coordinate functions of $b_0$ in $\mathcal{U}$, respectively in $\mathcal{V}$, are $(U_1, \phi_1), \ldots, (U_k, \phi_k)$, respectively $(V_1, \psi_1), \ldots, (V_l, \psi_l)$. Then for each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$, there exists $g_{U_i,V_j}(b_0) \in G$ so that $g_{U_i,V_j}(b_0) = \phi^{-1}_{U_i,b_0} \circ \psi^{-1}_{V_j,b_0}$ as automorphisms of $M_n(\mathbb{C})$. This implies that

$$s_{U_i}(b_0) = \phi^{-1}_{U_i,b_0}(s(b_0)) = \phi^{-1}_{U_i,b_0}\left(\phi_{V_j,b_0}(s_v(b_0))\right) = g_{U_i,V_j}(b_0) \cdot s_{V_j}(b_0)$$

It follows that

$$\sup_i \|s_{U_i}(b_0)\| \leq \sup_{i,j} \|g_{U_i,V_j}(b_0)\| \sup_j \|s_{V_j}(b_0)\|$$

Denote the supremum $\sup_{i,j} \|g_{U_i,V_j}(b_0)\|$ by $c_{b_0}$. It will be shown that the map $b \mapsto c_b$ is a continuous map on $X$.

The intersection $\tilde{U} = (\cap_i U_i) \cap (\cap_j V_j)$ is a nonempty, open neighborhood of $b_0$. On this neighborhood, for each $i$ and $j$, the coordinate transformation $g_{U_i,V_j}$ is continuous. So the map from $\tilde{U}$ to $\mathbb{R}$ defined by $b \mapsto \sup_{i,j} \|g_{U_i,V_j}(b)\|$ is continuous. Hence the map $b \mapsto c_b$ is continuous on the open neighborhood $\tilde{U}$ of $b_0$. Thus the map $b \mapsto c_b$ is continuous on $X$. Because $X$ is compact, $\sup_b c_b = M$ for some positive $M \in \mathbb{R}$ and we have

$$\|s\|_{\mathcal{U}} = \sup_{b\in X} \sup_{b\in U} \|s_U(b)\| \leq \sup_{b\in X} \left( c_b \sup_{b\in V} \|s_V(b)\| \right) \leq M \sup_{b\in X} \sup_{b\in V} \|s_V(b)\| = M\|s\|_{\mathcal{V}}$$

where $U$ denotes a coordinate neighborhood in $\mathcal{U}$ and $V$ denotes a coordinate neighborhood in $\mathcal{V}$. □

### 5.3 Reductions and $C^*$-structures

The preceding discussion shows that it is possible to give $\Gamma_c(X, \mathfrak{M}(d,n))$ many Banach algebra structures for any compact set $X \subset Q_0(d,n)$. Since the transition functions used to define these Banach structures take their values in $G$ rather than $K$, none of these norms is a $C^*$-norm. Indeed, there is no natural way to define an involution on $\mathfrak{M}(d,n)$. Nevertheless, it is also possible to endow $\Gamma_c(X, \mathfrak{M}(d,n))$ with a $C^*$-structure in such a way that $\Gamma_h(X, \mathfrak{M}(d,n))$ is a non-self-adjoint, closed subalgebra of $\Gamma_c(X, \mathfrak{M}(d,n))$.

It is a consequence of Corollary 5 in Chapter 3 that every principal $G$-bundle over a normal and paracompact space can be reduced to a principal $K$-bundle since $G/K$ is contractible. So we can reduce $\mathfrak{B}(d,n)$ to a principal $K$-bundle. In fact, there are many reductions and each reduction is given by a family of maps $\Lambda = \{\lambda_U : U \to G\}_{U \in \mathcal{U}}$ with the property that $\lambda_U(b)^{-1} g_{U,V}(b) \lambda_V(b) \in K$ for each $b \in U \cap V$ and $U, V \in \mathcal{U}$. Denote the reduction of $\mathfrak{B}(d,n)$ that is given by a family $\Lambda$ by $\mathfrak{C}(\Lambda; d,n)$. The coordinate transformations of $\mathfrak{C}(\Lambda; d,n)$ are given by $h_{U,V}(b) = \lambda_U(b)^{-1} g_{U,V}(b) \lambda_V(b)$. Denote the $M_n(\mathbb{C})$-fibre bundle associated to $\mathfrak{C}(\Lambda; d,n)$ by $\mathfrak{M}^*(\Lambda; d,n)$, or simply $\mathfrak{M}^*(d,n)$ when the reduction is unimportant. Let $\pi$ denote the projection map of the bundle $\mathfrak{M}^*(\Lambda; d,n)$ and note that $\mathfrak{M}^*(\Lambda; d,n) = (\mathcal{W} \times_K M_n(\mathbb{C}), \pi, Q_0(d,n))$, where $\mathcal{W}$ is the total space of $\mathfrak{C}(\Lambda; d,n)$. $\mathcal{W}$ is a closed, $K$-invariant subset of $\mathcal{V}(d,n)$. The fibre bundles $\mathfrak{M}(d,n)$ and $\mathfrak{M}^*(d,n)$ are isomorphic as bundles [Hus94, Theorem 6.3.1], and by Theorem 7 in Chapter 3, there is a bijection between $\Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ and $\Gamma_c(Q_0(d,n), \mathfrak{M}^*(\Lambda; d,n))$ given locally by the map that takes a family $\{s_U\} \in \Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ to the family $\{(\Lambda s)_U\} \in \Gamma_c(Q_0(d,n), \mathfrak{M}^*(\Lambda; d,n))$ defined by $(\Lambda s)_U(b) := \lambda_U(b)^{-1} \cdot s_U(b)$.

It is easy to see that this map $\Lambda$ respects the pointwise operations defined on cross sections: If $U$ is a coordinate neighborhood of $b \in Q_0(d,n)$ and $s, t \in \Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$, then

$$\begin{aligned}(\Lambda st)_U(b) &= \lambda_U(b)^{-1} \cdot (st)_U(b) = \lambda_U(b)^{-1} \cdot s_U(b) t_U(b) \\ &= \left(\lambda_U(b)^{-1} \cdot s_U(b)\right)\left(\lambda_U(b)^{-1} \cdot t_U(b)\right) = (\Lambda s)_U(b)(\Lambda t)_U(b).\end{aligned}$$

Again, the key fact used in the preceding computations is that $G$ acts on $M_n(\mathbb{C})$ as automorphisms, so the same argument can be applied to show that $\Lambda$ respects addition and scalar multiplication. Thus the bijection between $\Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ and $\Gamma_c(Q_0(d,n), \mathfrak{M}^*(\Lambda; d,n))$ defined by $s \mapsto \Lambda s$ is a $\mathbb{C}$-algebra isomorphism.

For any compact set $X \subset Q_0(d,n)$, the bundle $\mathfrak{M}^*(d,n)|X$ is a locally trivial bundle with fibre $M_n(\mathbb{C})$, structure group $K$, and a compact base space. By Theorem 8 of Tomiyama and Takesaki in [TT61], this implies that $\Gamma_c(X, \mathfrak{M}^*(d,n))$ is an $n$-homogeneous $C^*$-algebra. That is, $\Gamma_c(X, \mathfrak{M}^*(d,n))$ is a $C^*$-algebra with the property that every irreducible representation of $\Gamma_c(X, \mathfrak{M}^*(d,n))$ is $n$-dimensional. As in Section 5.1, in equation (5.1), we define the operations on $\pi^{-1}(b)$ for each $b \in X$ using the operations of the fibre $M_n(\mathbb{C})$. In $\mathfrak{M}^*(d,n)$, we can extend this idea and define an involution and norm on $\pi^{-1}(b)$ using the involution and norm of $M_n(\mathbb{C})$. Let $\mathcal{U}$ be a coordinate atlas of $\mathfrak{M}^*(d,n)|X$, and let the coordinate functions be $\{\zeta_U\}$. Fix $b \in X$ and let $U$ be a coordinate neighborhood of $b$. Fix $x \in \pi^{-1}(b)$.

Let $A = \zeta_{U,b}^{-1}(x) \in M_n(\mathbb{C})$. Then define

$$x^* := \zeta_{U,b}(A^*)$$

$$\|x\| := \|A\|$$

Because elements in $K$ act on $M_n(\mathbb{C})$ as $*$-automorphisms and $*$-automorphisms are isometric, computations similar to those in Section 5.1 show that this involution and norm are well-defined.

The operations, including involution, on $\Gamma_c(X, \mathfrak{M}^*(d,n))$ are defined pointwise. The norm of an element $s \in \Gamma_c(X, \mathfrak{M}^*(d,n))$ is defined by $\|s\| := \sup_{b \in X} \|s(b)\|$. Since $X$ is compact, this norm is always finite. It does not depend on the choice of cover $\mathcal{U}$. However, it does depend on the choice of a reduction from $\mathfrak{B}(d,n)$ to a principal $K$-bundle. Suppose that $\Lambda_1$ and $\Lambda_2$ are two families of maps that give reductions of $\mathfrak{B}(d,n)$ to principal $K$-bundles. Then the associated $M_n(\mathbb{C})$-fibre bundles $\mathfrak{M}^*(\Lambda_1; d,n)$ and $\mathfrak{M}^*(\Lambda_2; d,n)$ are isomorphic as topological bundles, as they are each isomorphic to $\mathfrak{M}(d,n)$. Further, the algebras of cross sections $\Gamma_c(X, \mathfrak{M}^*(\Lambda_1; d,n))$ and $\Gamma_c(X, \mathfrak{M}^*(\Lambda_2; d,n))$ are algebraically isomorphic, each being isomorphic to $\Gamma_c(X, \mathfrak{M}(d,n))$. This implies, by Theorem 7 in [TT61], that $\Gamma_c(X, \mathfrak{M} * (\Lambda_1; d,n))$ and $\Gamma_c(X, \mathfrak{M}^*(\Lambda_2; d,n))$ are in fact $*$-isomorphic.

**Theorem 14.** *For any compact subset $X \subset Q_0(d,n)$ and any choice of finite cover $\mathcal{U}$ of $X$, $\Gamma_c(X, \mathfrak{M}^*(\Lambda; d,n))$ and $\Gamma_c(X, \mathfrak{M}(d,n))$ are isomorphic as Banach algebras, where $\Gamma_c(X, \mathfrak{M}(d,n))$ is endowed with the norm associated to $\mathcal{U}$ in* (5.4).

*Proof.* The isomorphism will be given by the map $\Lambda : \Gamma_c(X, \mathfrak{M}(d,n)) \to \Gamma_c(X, \mathfrak{M}^*(\Lambda; d,n))$

that is induced by the reduction of $\mathfrak{B}(d,n)$ to a principal $K$-bundle. Recall that if $s \in \Gamma_c(X, \mathfrak{M}(d,n))$ is given locally by a family of maps $\{s_U : U \to M_n(\mathbb{C})\}$, then $\Lambda s \in \Gamma_c(X, \mathfrak{M}^*(d,n))$ is given locally by the family of maps $\{(\Lambda s)_U\}$ where $(\Lambda s)_U$ is defined by $(\Lambda s)_U(b) := \lambda_U(b)^{-1} \cdot s_U(b)$ for each $b \in U$. We have already seen that the map $\Lambda$ is an algebraic isomorphism. It must be shown that $\Lambda$ is continuous and has a continuous inverse.

Fix $s \in \Gamma_c(X, \mathfrak{M}(d,n))$ and $b \in X$. For each $U \in \mathcal{U}$ containing $b$, we have

$$\|(\Lambda s)(b)\| = \|(\Lambda s)_U(b)\| = \|\lambda_U(b)^{-1} \cdot s_U(b)\| \leq \|\lambda_U(b)^{-1}\| \|s_U(b)\| \|\lambda_U(b)\|. \tag{5.5}$$

Let $M$ denote $\sup_{b\in X} \max_{b\in U} (\|\lambda_U(b)^{-1}\| \|\lambda_U(b)\|)$. The number $M$ is finite since $\mathcal{U}$ is a finite cover and $X$ is a compact set. It follows from (5.5) by taking the supremum over $b \in X$ that $\|\Lambda s\| \leq M \|s\|_{\mathcal{U}}$. Hence $\Lambda$ is continuous.

On the other hand, $\Lambda^{-1} : \Gamma_c(X, \mathfrak{M}^*(\Lambda; d, n)) \to \Gamma_c(X, \mathfrak{M}(d,n))$ is given locally by the formula $(\Lambda^{-1}t)_U(b) = \lambda_U(b) \cdot t_U(b)$ for $t = \{t_U\} \in \Gamma_c(X, \mathfrak{M}^*(d,n))$. Arguing as in the preceding paragraph, it is clear that $\Lambda^{-1}$ is continuous. $\square$

**Remark 15.** *Each of the two types of Banach algebra norms we have put on $\Gamma_c(X, \mathfrak{M}(d,n))$ has it benefits and drawbacks. Although the norms defined in terms of covers are not $C^*$-norms, the isomorphism described in Theorem 13 preserves holomorphic cross sections. On the other hand, while two reductions give isomorphic $C^*$-algebra structures on $\Gamma_c(X, \mathfrak{M}^*(d,n))$, the isomorphisms involved rely on the family of continuous maps $\Lambda$ and need not map holomorphic sections to holomorphic sections. This may be thought of as a deficit. However, as we shall see later, Arveson's boundary theory helps to rectify it.*

There are many possible reductions of $\mathfrak{B}(d,n)$ to a principal $K$-bundle, and it is not clear if there is a natural or preferred reduction. However, a promising candidate is supplied by the Kempf-Ness Theorem. The bundle space of this reduction consists of the *hypernormal* points in $\mathcal{V}(d,n)$. A point $\mathfrak{z} = (Z_1, Z_2, \ldots, Z_d) \in M_n(\mathbb{C})^d$ is hypernormal if $\sum [Z_i, Z_i^*] = 0$. Observe that if $d = 1$, then a hypernormal point is a normal matrix. Denote the set of hypernormal points in $M_n(\mathbb{C})^d$ by $\mathcal{HN}(d,n)$. Theorem 2.11 in [LB08] asserts that there is a bijection between the closed $G$-orbits of $M_n(\mathbb{C})^d$ and the $K$-orbits of $\mathcal{HN}(d,n)$. That is, $Q(d,n)$ is in bijection with $\mathcal{HN}(d,n)/K$. It follows that $\mathcal{HN}(d,n) \cap \mathcal{V}(d,n)$ is the bundle space of a principal $K$-bundle, whose projection map is the restriction of $\pi_0$ to the hypernormal points in $\mathcal{V}(d,n)$ and whose base space is all of $Q_0(d,n)$. The bundle $\mathfrak{C} = (\mathcal{HN}(d,n) \cap \mathcal{V}(d,n), \pi_0, Q_0(d,n))$ is a reduction of $\mathfrak{B}(d,n)$ to a principal $K$-bundle, and $\mathfrak{C}$ seems to be the most natural choice for a reduction. However, the results that follow will be independent of the choice of reduction and the benefits of working with this particular reduction are not yet well understood.

## 5.4 The Tracial Function Algebra

Before we can define the algebra that we will actually study, we must recall some definitions from the study of *uniform algebras*. For more information, see e.g. [Sto71], [Gam69], and [Lei70]. Let $X$ be a locally compact set and let $\mathcal{F}$ be a family of $\mathbb{C}$-valued functions on $X$. The family $\mathcal{F}$ is a *function algebra* on $X$ if it is a subalgebra of $C_0(X)$ which strongly separates point, i.e. if $x, y \in X$ and $x \neq y$, then there exists $f \in \mathcal{F}$ so that $0 \neq f(x) \neq f(y)$. If $X$ is compact, $\mathcal{F}$ is assumed to contain the constants. A function algebra on a compact set which is uniformly closed is called a *uniform algebra*. Suppose that $\mathcal{F}$ is a function algebra on a space $X$. A subset $Y \subseteq X$ is called a *boundary* for $\mathcal{F}$ if, for each $f \in \mathcal{F}$, there is $y \in Y$ so that

$$|f(y)| = \sup_{x \in X} |f(x)|$$

The intersection of all closed boundaries for $\mathcal{F}$ is again a boundary. This intersection is the unique minimal closed boundary and is called the *Shilov boundary* of $X$ with respect to $\mathcal{F}$, and will be denoted $\partial X$.

We will be particularly interested in what Rickart calls *natural function algebras* [Ric79]. A function algebra $\mathcal{F}$ on a space $X$ is called a natural function algebra if every continuous homomorphism of $\mathcal{F}$ onto $\mathbb{C}$ is given by evaluation at some point in $X$, i.e. if $X$ is the spectrum of $\mathcal{F}$. For example, every commutative $C^*$-algebra is naturally considered as a function algebra over its spectrum via the Gelfand transform. The algebra $\mathbb{I}_0(d,n)$ is a natural function algebra over $Q(d,n)$.

Let $\mathcal{F}$ be a function algebra on a compact space $X$. Denote the dual space of $\mathcal{F}$ by $\mathcal{F}^*$ and define $T_{\mathcal{F}} \subset \mathcal{F}^*$ by $T_{\mathcal{F}} := \{L \in \mathcal{F}^* : L(1) = \|L\| = 1\}$. For each $x \in X$, we can define $\varphi_x \in \mathcal{F}^*$ by $\varphi_x(f) := f(x)$ for each $f \in \mathcal{F}$. Let $\partial_e X$ be the set of all $x \in X$ such that $\varphi_x$ is an extreme point of $T_{\mathcal{F}}$. The set $\partial_e X$ is called the *extreme* or *Choquet boundary* of $\mathcal{F}$. Theorem 4.5 in [Lei70] asserts that if $\mathcal{F}$ is a uniform algebra, then every element of $\mathcal{F}$ achieves its maximum modulus at some point in $\partial_e X$. In particular, this implies that $\partial_e X$ is a boundary of $X$ for $\mathcal{F}$. A point $x \in X$ is called a *strong boundary point* for $\mathcal{F}$ if, for every neighborhood $V$ of $x$, there exists $f \in \mathcal{F}$ so that $\|f\|_X = f(x) = 1$ and $|f(y)| < 1$ for every $y \in X \setminus V$. A strong boundary point $x$ is called a *peak point* for $\mathcal{F}$ if there exists $f \in \mathcal{F}$ so that $f(x) = 1$ and $|f(y)| < 1$ for every $y \in X,\ y \neq x$. If $X$ is a compact metric space, then the Choquet boundary $\partial_e X$ is exactly the set of peak points for $\mathcal{F}$ [Lei70, Theorem 4.22]. Further, the Choquet boundary $\partial_e X$ is dense in the Shilov boundary $\partial X$ by [Sto71, Corollary I.7.24].

Let $\mathcal{D} \subset Q_0(d,n)$ be a domain so that $\overline{\mathcal{D}}$ is a compact subset of $Q_0(d,n)$. Recall that for every matrix concomitant $f \in C(\mathcal{V}(d,n), M_n(\mathbb{C}))^G$ there is an corresponding cross section $s_f \in \Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ defined by $s_f([\mathfrak{z}]) = [\mathfrak{z}, f(\mathfrak{z})]$. We will study the cross sections on $\overline{\mathcal{D}}$ which correspond to elements in the algebra of invariants, $\mathbb{I}_0(d,n)$ and elements in the trace algebra, $\mathbb{S}_0(d,n)$.

Let $p \in \mathbb{I}_0(d,n)$ and note that for every $\mathfrak{z} \in \mathcal{V}(d,n)$ and every $g \in G$ we have $p(\mathfrak{z} \cdot g) = p(\mathfrak{z})$. That is, $p(\mathfrak{z}) = p(\mathfrak{w})$ whenever $\mathfrak{z}, \mathfrak{w} \in \mathcal{V}(d,n)$ are in the same orbit. So it is appropriate to think of $p$ as a function on the orbit space $\mathcal{V}(d,n)/G = Q_0(d,n)$. In this way we can identify $\mathbb{I}_0(d,n)$ with a subset of $C(\overline{\mathcal{D}})$. Now consider the set of cross sections $\{s_p : p \in \mathbb{I}_0(d,n)\}$. This is in bijection with $\mathbb{I}_0(d,n)$, and hence can also be considered as a subset of $C(\overline{\mathcal{D}})$. Let $\mathbb{I}(\mathcal{D}; d,n)$ denote the closure of $\{s_p : p \in \mathbb{I}_0(d,n)\}$ in $C(\overline{\mathcal{D}})$. Using the same identification of matrix concomitants and cross sections, we may consider $\mathbb{I}(\mathcal{D}; d,n)$ as a subset of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}(d,n))$.

$\mathbb{I}_0(d,n)$ contains the constant functions and separates the points of $Q(d,n)$, because $Q(d,n)$ is the spectrum of $\mathbb{I}_0(d,n)$. It follows that $\mathbb{I}(\mathcal{D}; d,n)$ is a uniform algebra on the set $\overline{\mathcal{D}}$. By definition, the elements of $\mathbb{I}(\mathcal{D}; d,n)$ consist of functions that are continuous on $\overline{\mathcal{D}}$ and holomorphic on $\mathcal{D}$. The set $\overline{\mathcal{D}}$ need not be the maximal ideal space of $\mathbb{I}(\mathcal{D};d,n)$. However, since $\mathbb{I}(\mathcal{D};d,n)$ is a function algebra over $\overline{\mathcal{D}}$, the set $\overline{\mathcal{D}}$ contains the Shilov boundary $\partial\mathcal{D}$ of $\mathcal{D}$ for $\mathbb{I}(\mathcal{D};d,n)$. The Choquet boundary of $\mathcal{D}$ with respect to $\mathbb{I}(\mathcal{D};d,n)$ will be denoted $\partial_e\mathcal{D}$.

For each finite coordinate atlas $\mathcal{U}$, the elements in $\Gamma_h(\overline{\mathcal{D}},\mathfrak{M}(d,n))$ achieve their maximums with respect to the norm $\|\cdot\|_{\mathcal{U}}$ on the Shilov boundary $\partial\mathcal{D}$. However, it need not be the case that this is true after reduction. That is, if $\Lambda$ is a reduction, then there may be elements in the image of $\Gamma_h(\overline{\mathcal{D}},\mathfrak{M}(d,n))$ under $\Lambda$ which do not take their maximum norms on $\partial\mathcal{D}$. This is precisely because $\Lambda$ takes holomorphic cross sections in $\Gamma_h(\overline{\mathcal{D}},\mathfrak{M}(d,n))$ to non-holomorphic, continuous cross sections in $\Gamma_c(\overline{\mathcal{D}},\mathfrak{M}^*(\Lambda;d,n))$. Therefore, we will focus directly on bundles over the set $\partial\mathcal{D}$.

Let $\mathfrak{C}(\Lambda;d,n)$ be a reduction of $\mathfrak{B}(d,n)$ to a principal $K$-bundle. Let $S$ be the collection of cross sections in $\Gamma_c(Q_0(d,n),\mathfrak{M}(d,n))$ that are associated to elements in $\mathbb{S}_0(d,n)$ under the identification of cross sections and concomitants. Let $\partial\mathcal{D}$ be as above. Then the closure of $\Lambda S$ in $\Gamma_c(\partial\mathcal{D},\mathfrak{M}^*(\Lambda;d,n))$ will be called the *tracial function algebra* of $\mathcal{D}$ determined by $\Lambda$, and will be denoted $\mathbb{S}(\mathcal{D},\Lambda;d,n)$, or $\mathbb{S}(\mathcal{D};d,n)$ when the reduction is allowed to be arbitrary. Since the bundle $\mathfrak{M}^*(\Lambda;d,n)$ is not holomorphic, $\mathbb{S}(\mathcal{D};d,n)$ is the closest analogue to holomorphic cross sections of $\mathfrak{M}^*(\Lambda;d,n)$ that we have. In the following chapters, we will prove several properties of the algebra $\mathbb{S}(\mathcal{D};d,n)$.

By way of example, consider the case when $n=1$. In this case, $G=K$ is the trivial group. $\mathcal{V}(d,n)$ and $Q_0(d,n)$ can both be identified with $\mathbb{C}^d\setminus\{0\}$. $\mathfrak{M}(d,1)$ is the trivial line bundle on $\mathbb{C}^d$, and since $G=K$, there is no reduction. For each domain $\mathcal{D}$, the algebras $\mathbb{I}(\mathcal{D};d,1)$ and $\mathbb{S}(\mathcal{D};d,1)$ are identified with $\mathcal{P}(\overline{\mathcal{D}})$, the sup-norm closure of the polynomial functions on $\mathbb{C}^d$ in $C(\overline{\mathcal{D}})$. This is a much studied algebra in complex analysis (see, e.g. [Sto07]). We believe that $\mathbb{S}(\mathcal{D}; d, n)$ is the natural generalization of $\mathcal{P}(\overline{\mathcal{D}})$ to noncommutative function theory.

# CHAPTER 6
# BOUNDARY REPRESENTATIONS

The algebra $\mathbb{S}(\mathcal{D}; d, n)$ is a closed subalgebra of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$, the algebra of continuous cross sections of $\mathfrak{M}^*(d,n)|\partial\mathcal{D}$. $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ is an $n$-homogeneous $C^*$-algebra by Theorem 8 in [TT61]. In this section, we will investigate the relationship between $\mathbb{S}(\mathcal{D}; d, n)$ and $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ using Arveson's theory of subalgebras of $C^*$-algebras [Arv69].

## 6.1 Definitions and Background

Before describing Arveson's theory of subalgebras of $C^*$-algebras, we need some definitions. The following can be found in Paulsen's book [Pau02]. Note that if $A$ is a $C^*$-algebra, then $M_n(A)$ is a $C^*$-algebra as well. The adjoint in $M_n(A)$ is defined by $[a_{ij}]^* = \left[a_{ji}^*\right]$. To define a norm on $M_n(A)$, consider $A$ as a subspace of $B(H)$ for some Hilbert space $H$. Then $M_n(A) \subseteq M_n(B(H))$, which can be identified with $B(H^n)$. $M_n(A)$ inherits the norm of $B(H^n)$. In fact, this norm does not depend on how $A$ is represented on Hilbert space since the norm of a $C^*$-algebra is unique. So for each $C^*$-algebra $A$, there is a sequence of associated $C^*$-algebras, $\{M_n(A), n \in \mathbb{Z}^+\}$.

If $S$ is a subspace of $A$, then $M_n(S)$ is a subspace of $M_n(A)$ and it will inherit the norm from $M_n(A)$. Let $\phi : S \to B$ be a linear map from $S$ to a $C^*$-algebra $B$. Then for each $n$, we can define a map $\phi^{(n)} : M_n(S) \to M_n(B)$ by $\phi^{(n)}([a_{ij}]) = [\phi(a_{ij})]$. The map $\phi$ is called *completely positive* if $\phi^{(n)}$ is positive for each $n \in \mathbb{Z}^+$, that is, if $\phi^{(n)}$ maps positive elements in $M_n(S)$ to positive elements in $M_n(B)$. Similarly, $\phi$ is called *completely isometric* if $\phi^{(n)}$ is isometric for each $n$, i.e. for each $n$, $\left\|\phi^{(n)}([a_{ij}])\right\| = \|[a_{ij}]\|$.

Now we are ready to describe Arveson's theory of subalgebras of $C^*$-algebras. The following definitions were introduced in [Arv69]. Let $B$ be a unital $C^*$-algebra and let $A$ be a norm-closed, non-self-adjoint subalgebra of $B$ which contains $1_B$ and generates $B$ as a $C^*$-algebra. An irreducible $C^*$-representation $\pi : B \to B(H_\pi)$ is called a *boundary representation of $B$ for $A$* if $\pi|A$ has a unique completely positive linear extension to $B$, namely $\pi$ itself. With the same conditions on $A$ and $B$, an ideal $I$ in $B$ is called a *boundary ideal* whenever the restriction to $A$ of the quotient map $q : B \to B/I$ is completely isometric. A boundary ideal which contains every other boundary ideal is call the *Shilov boundary ideal.* When [Arv69] was written, it was not known whether or not the Shilov boundary ideal always existed. However, it was proved that if there were "sufficiently many" boundary representations, the Shilov bounday ideal does exist and is equal to the intersection of the kernels of the boundary representations [Arv69, Theorem 2.2.3]. Today it is known that there are always sufficiently many boundary representations ([Arv11, Theorem 7.1] in the separable case and [DK15, Corollary 3.5] in the general case), so the Shilov boundary ideal for $A$ always exists and is equal to the intersection of the kernels of all boundary representations of $B$ for $A$. The quotient of $B$ by the Shilov boundary ideal is unique in a very strong sense [Arv69, Theorem 2.2.6]. This quotient is called the *$C^*$-envelope* of $A$.

**Definition 16.** *Suppose $A$ is a norm-closed subalgebra of a unital $C^*$-algebra $B$ that contains the unit of $B$ and generates $B$ as a $C^*$-algebra. An irreducible $C^*$-representation $\pi : B \to B(H_\pi)$ is called a* peaking representation *for $A$ if there is an integer $n \geq 1$ and a matrix $[a_{ij}] \in M_n(A)$ so that*

$$\|[\pi(a_{ij})]\| > \|[\sigma(a_{ij})]\|$$

*for every irreducible representation $\sigma$ of $B$ that is not unitarily equivalent to $\pi$.*

In this case, we say that $\pi$ peaks at $[a_{ij}]$. In [Arv11, Definition 7.1], Arveson defines the notion of a *peaking representation* in the context of operator systems, i.e., unital, closed, and self-adjoint subspaces of $C^*$-algebras. However, thanks to [Arv69, Proposition 1.2.8], if a representation is peaking in the sense of Definition 16, then it is a peaking representation with respect to the operator system generated by $A$, that is the norm-closure of $A + A^*$.

Every peaking representation is a boundary representation [Kle14, Remark 3.4]. This follows from [Kle14, Theorem 3.1], which asserts that for any element $[a_{ij}] \in M_n(A)$ there is a boundary representation $\pi_0$ of $B$ for $A$ such that

$$\|[a_{ij}]\| = \|[\pi_0(a_{ij})]\| . \tag{6.1}$$

If $\pi$ is a peaking representation which peaks at $[a_{ij}]$ and $\pi_0$ is the boundary representation satisfying (6.1), then we have $\|[a_{ij}]\| \geq \|[\pi(a_{ij})]\| \geq \|[\pi_0(a_{ij})]\|$. Since $\|[a_{ij}]\| = \|[\pi_0(a_{ij})]\|$, we have $\|[\pi(a_{ij})]\| = \|[\pi_0(a_{ij})]\|$ and hence $\pi$ is unitarily equivalent to $\pi_0$ by definition of a peaking representation. Thus $\pi$ is a boundary representation.

### 6.2 Boundary Representations of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ for $\mathbb{S}(\mathcal{D}; d, n)$

In this section, we will prove that each point in $\partial_e\mathcal{D}$, the Choquet boundary of $\mathcal{D}$, gives rise to a boundary representation of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ for $\mathbb{S}(\mathcal{D}; d, n)$. This is sufficient to show that the $C^*$-envelope of $\mathbb{S}(\mathcal{D}; d, n)$ is $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$. All of the following results are independent of the reduction $\Lambda$.

The $C^*$-algebra $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ is an $n$-homogeneous $C^*$-algebra and each irreducible representation of it is given, essentially, by evaluation at a unique point of $\partial\mathcal{D}$. In more detail, note that for $b \in Q_0(d,n)$, the fibre over $b$ is $\pi^{-1}(b) = \{[\mathfrak{z}, A] : \pi_0(\mathfrak{z}) = b, A \in M_n(\mathbb{C})\}$. So, once $\mathfrak{z}$ is chosen so that $\pi_0(\mathfrak{z}) = b$, the map $A \mapsto [\mathfrak{z}, A]$ is a unital $*$-homomorphism from $M_n(\mathbb{C})$ to $\pi^{-1}(b)$. Call this map $\rho$. Since $M_n(\mathbb{C})$ is simple, this map is injective. It is also surjective because if $[\mathfrak{w}, B] \in \pi^{-1}(b)$, then $\pi_0(\mathfrak{w}) = \pi_0(\mathfrak{z})$ which implies that $\mathfrak{w}$ and $\mathfrak{z}$ are in the same orbit. So there exists a unique $k \in K$ so that $\mathfrak{w} = \mathfrak{z}\cdot k$, and hence $[\mathfrak{w}, B] = [\mathfrak{z}\cdot k, B] = [\mathfrak{z}, k \cdot B] = \rho(k \cdot B)$. For each $b \in \partial\mathcal{D}$, write $ev_b$ for the $*$-homomorphism from $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ into $\pi^{-1}(b)$ defined by evaluating a cross section at $b$. Then, for each $b \in \partial\mathcal{D}$, $\rho^{-1} \circ ev_b$ is an irreducible representation of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$. Moreover, every irreducible representation of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ is unitarily equivalent to $\rho^{-1} \circ ev_b$ for a unique $b \in \partial\mathcal{D}$ by [Dix77, Corollary 10.4.4].

We are now ready to state our main result: for each $b \in \partial_e\mathcal{D}$ the map $ev_b$ is a boundary representation of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ for $\mathbb{S}(\mathcal{D}; d,n)$. Before we can prove this, we will need two lemmas to show that $\mathbb{S}(\mathcal{D}; d,n)$ is a norm-closed, proper subalgebra of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ which generates $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ as a $C^*$-algebra.

**Theorem 17.** *The center of* $\mathbb{S}(\mathcal{D}, \Lambda; d,n)$ *is* $\mathbb{I}(\mathcal{D}; d,n)$ *for any reduction* $\Lambda$.

*Proof.* This is similar to Procesi's result that $\mathbb{I}_0(d,n)$ is the center of $\mathbb{S}_0(d,n)$ [Pro73, Corollary IV.6.1]. Recall that for every concomitant $f \in C(\mathcal{V}(d,n), M_n(\mathbb{C})^d)^G$, there is an associated cross section $\Psi(f) = s_f \in \Gamma_c(Q_0(d,n), \mathfrak{M}(d,n))$ defined by $s_f([\mathfrak{z}]) = [\mathfrak{z}, f(\mathfrak{z})]$. It was shown in Theorem 10 that the correspondence $\Psi$ is an isomorphism. Recall that the subalgebra $\mathbb{I}(\mathcal{D}; d,n) \subset \Gamma_c(\partial\mathcal{D}, \mathfrak{M}(d,n))$ is the closure of the image $\Psi(\mathbb{I}_0(d,n))$ in $C(\overline{\mathcal{D}})$. The algebra $\mathbb{S}(\mathcal{D}; d,n)$ is defined similarly: Suppose that $\mathfrak{C}(\Lambda; d,n) = (\mathcal{W}, \pi_0, Q_0(d,n))$ is a reduction of $\mathfrak{B}(d,n)$ to a principal $K$-bundle. Let $S = \Psi(\mathbb{S}_0(d,n)) \subset \Gamma_c(\partial\mathcal{D}, \mathfrak{M}(d,n))$ be the collection of cross sections of that correspond to elements in $\mathbb{S}_0(d,n)$. Then $\mathbb{S}(\mathcal{D},\Lambda;d,n)$ is the closure of $\Lambda S$ in $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda;d,n))$.

Further, recall that, if $\mathcal{U}$ is a coordinate atlas of $\mathfrak{M}(d,n)|\partial\mathcal{D}$, then the isomorphism $\Lambda : \Gamma_c(\partial\mathcal{D}, \mathfrak{M}(d,n)) \to \Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda;d,n))$ is defined locally by

$$(\Lambda s)_U(b) = \lambda_U(b)^{-1} \cdot s_U(b)$$

for $s = \{s_U\}_{U\in\mathcal{U}} \in \Gamma_c(\partial\mathcal{D}, \mathfrak{M}(d,n))$, $b \in U$, and where $\lambda_U : U \to G$ are the local maps defining the reduction $\Lambda$. Notice that if $p : \partial\mathcal{D} \to \mathbb{C}$ is in $\mathbb{I}(\mathcal{D};d,n)$, then

$$\begin{aligned}(\Lambda p)_U(b) &= \lambda_U(b)^{-1} \cdot p_U(b) \\ &= \lambda_U(b)^{-1} \cdot p(b) \cdot I_n = \lambda_U(b)^{-1} p(b) I_n \lambda_U(b) \\ &= p(b) I_n = p_u(b)\end{aligned}$$

So $\mathbb{I}(\mathcal{D};d,n)$ is fixed under any reduction $\Lambda$ and we can view $\mathbb{I}(\mathcal{D};d,n)$ as a subset of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda;d,n))$. It is clear that $\mathbb{I}(\mathcal{D};d,n) \subset \mathbb{S}(\mathcal{D},\Lambda;d,n)$.

We want to determine the center of $\mathbb{S}(\mathcal{D};d,n)$. Let's first consider the center of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda;d,n))$, which we'll denote by $Z\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda;d,n))$. The identity of the algebra $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda;d,n))$ is the cross section $e$ defined by $e([\mathfrak{z}]) := [\mathfrak{z}, I_n]$. Every continuous function $c : \partial\mathcal{D} \to \mathbb{C}$ gives rise to a central element is $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda;d,n))$ defined by $(c \cdot e)([\mathfrak{z}]) = [\mathfrak{z}, c([\mathfrak{z}])I_n]$. This cross section will be denoted by $c \cdot e$. So, for example, $\mathbb{I}(\mathcal{D}; d, n)$ consists of central elements in $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$. Further, every element is $Z\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$ is given by an element in $C(\partial\mathcal{D})$ in this way. The identification $c \leftrightarrow c \cdot e$ is an isomorphism between $C(\partial\mathcal{D})$ and $Z\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$.

We will show that $\mathbb{I}(\mathcal{D}; d, n)$ is the center of $\mathbb{S}(\mathcal{D}, \Lambda; d, n)$ by constructing a conditional expectation from $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$ to its center which maps $\mathbb{S}(\mathcal{D}, \Lambda; d, n)$ onto $\mathbb{I}(\mathcal{D}; d, n)$. Let $\tau_0$ denote the normalized trace on $M_n(\mathbb{C})$, so that $\tau(I_n) = 1$. Define $\tau : \mathcal{W} \times_G M_n(\mathbb{C}) \to \mathbb{C}$ by $\tau([\mathfrak{z}, A]) := \tau_0(A)$. Here, $\mathcal{W} \subset \mathcal{V}(d, n)$ is the bundle space of the reduced bundle $\mathbb{C}(\Lambda; d, n)$. If $[\mathfrak{z}, A] = [\mathfrak{w}, B] \in \mathcal{W} \times_G M_n(\mathbb{C})$, then there exists $k \in K$ so that $\mathfrak{z} \cdot k = \mathfrak{w}$ and $k^{-1} \cdot A = B$. This implies that $\tau_0(B) = \tau_0(k^{-1} \cdot A) = \tau_0(k^{-1}Ak) = \tau_0(A)$. Hence $\tau$ is well-defined. Since $\tau_0$ is continuous, so is $\tau$.

Define $T : \Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n)) \to Z\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$ by

$$T(s) := (\tau \circ s) \cdot e.$$

If $c \cdot e$ is a central element is $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$, then $(\tau \circ (c \cdot e))([\mathfrak{z}]) = \tau([\mathfrak{z}, c([\mathfrak{z}])I_n]) = c([\mathfrak{z}])$. It follows that $c \cdot e$ is fixed by the map $T$.

Suppose that $s \in Z\mathbb{S}(\mathcal{D}, \Lambda; d, n)$. Recall that for $k = 1, \ldots, d$ the map $\mathcal{Z}_k : M_n(\mathbb{C})^d \to M_n(\mathbb{C})$ is the $k^{th}$ matrix coordinate function. Each $\mathcal{Z}_k$ is an element in $\mathbb{S}_0(d, n)$, so $s$ must commute with each of the sections $\Psi(\mathcal{Z}_k)$. The section $s$ is equal to $\Psi(f)$ for some $f \in C(\pi^{-1}(\partial\mathcal{D}), M_n(\mathbb{C}))^K$. Fix $\mathfrak{z} = (Z_1, Z_2, \ldots, Z_d) \in \mathcal{W}$. Then $(\Psi(f)\Psi(\mathcal{Z}_k))([\mathfrak{z}]) = [\mathfrak{z}, f(\mathfrak{z})\mathcal{Z}_k(\mathfrak{z})] = [\mathfrak{z}, f(\mathfrak{z})Z_k]$. So it must be that $f(\mathfrak{z})$ commutes with $Z_k$ for each $k$. But since $\mathfrak{z} \in \mathcal{W} \subset \mathcal{V}(d, n)$, the components $Z_1, Z_2, \ldots, Z_d$ generate $M_n(\mathbb{C})$ as an algebra and hence $f(\mathfrak{z})$ commutes with every element in $M_n(\mathbb{C})$. This is true for each $\mathfrak{z} \in \mathcal{W}$, so there is some continuous map $c : \mathcal{W} \to \mathbb{C}$ such that $f(\mathfrak{z}) = c(\mathfrak{z})I_n$. Since $f$ is a matrix concomitant, the map $c$ is invariant and so can be thought of as a function from $\mathcal{W}/K = \partial\mathcal{D}$ to $\mathbb{C}$. Thus $s = c \cdot e \in Z\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$.

That is, $Z\mathbb{S}(\mathcal{D}, \Lambda; d, n)$ is contained in $Z\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$. Moreover, each element in $Z\mathbb{S}(\mathcal{D}, \Lambda; d, n)$ is fixed by $T$. All that remains is to show that $T(\mathbb{S}(\mathcal{D}, \Lambda; d, n))$ is contained in $\mathbb{I}(\mathcal{D}; d, n)$. But this is true because if $p \in \mathbb{S}_0(d, n)$, then $T(s_p)([\mathfrak{z}]) = [\mathfrak{z}, \tau(s_p([\mathfrak{z}]))I_n] = [\mathfrak{z}, \tau([\mathfrak{z}, p(\mathfrak{z})])I_n] = [\mathfrak{z}, \tau_0(p(\mathfrak{z}))I_n]$. That is $T(s_p) = (\tau_0 \circ p) \cdot e$. Since $p$ is a polynomial matrix concomitant, $\tau_0 \circ p$ is a polynomial $G$-invariant function. That is $\tau_0 \circ p \in \mathbb{I}_0(d, n)$. Thus $T(s_p) \in \mathbb{I}(\mathcal{D}; d, n)$. This completes the proof that $Z\mathbb{S}(\mathcal{D}, \Lambda; d, n) = \mathbb{I}(\mathcal{D}; d, n)$. □

An important corollary of this theorem is

**Corollary 18.** $\mathbb{S}(\mathcal{D}, \Lambda; d, n)$ *is a proper subalgebra of* $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$.

This is a consequence of the fact that the center of $\mathbb{S}(\mathcal{D}, \Lambda; d, n)$ is a proper subset of the center of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$. In the proof of Theorem 17, it was shown that the center of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$ is isomorphic to $C(\partial D)$. There are continuous functions on $\partial D$ that cannot be approximated by polynomials, and hence $\mathbb{I}(\mathcal{D}; d, n) \subsetneq Z\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d, n))$.

It is clear that $e \in \mathbb{S}(\mathcal{D}; d, n)$ and by definition $\mathbb{S}(\mathcal{D}; d, n)$ is a norm-closed. In order to discuss boundary represenions of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d, n))$ for $\mathbb{S}(\mathcal{D}; d, n)$, we must also show that $\mathbb{S}(\mathcal{D}; d, n)$ generates $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d, n))$ as a $C^*$-algebra.

**Lemma 19.** $\mathbb{S}(\mathcal{D}; d, n)$ *generates* $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d, n))$ *as a* $C^*$*-algebra.*

To prove this lemma, we appeal to the general theory of continuous fields of $C^*$-algebras. This may seem heavy handed, but it works perfectly for our purposes. We first need to establish some vocabulary. Given a normed space $X$, a subset $Y \subset X$ is called *total* if the span of elements in $Y$ is dense in $X$. The following definitions come from [Dix77, Chapter 10]. Let $T$ be a topological space and $(E(t))_{t\in T}$ be a family of Banach spaces indexed over $T$. Then every element of $\prod_{t\in T} E(t)$ is called a *vector field*, i.e. a vector field is a function $x$ defined on $T$ so that $x(t) \in E(t)$ for each $t \in T$. In this setting, a *continuous field* of Banach spaces over $T$, denoted $\mathcal{E} = ((E(t))_{t\in T}, \Gamma)$, is family $(E(t))_{t\in T}$ of Banach spaces together with a set $\Gamma \subseteq \prod_{t\in T} E(t)$ of vector fields that satisfies the following properties:

1. $\Gamma$ is a complex linear subspace of $\prod_{t\in T} E(t)$,

2. For every $t \in T$, the set $\{x(t) : x \in \Gamma\}$ is dense in $E(t)$,

3. For every $x \in \Gamma$ the function $t \mapsto \|x(t)\|$ is continuous, and

4. $\Gamma$ is "closed" in the following sense: Let $x \in \prod_{t\in T} E(t)$ be a vector field. If, for every $t \in T$ and every $\varepsilon > 0$ there is a $x' \in \Gamma$ such that $\|x(t) - x'(t)\| < \varepsilon$ throughout some neighborhood of $t$, then $x \in \Gamma$.

If, in addition, each $E(t)$ is a $C^*$-algebra and $\Gamma$ is closed under pointwise multiplication and involution, then $\mathcal{E}$ is called a *continuous field of $C^*$-algebras* over $T$.

Notice that several examples of continuous fields of Banach spaces arise from the matrix bundles we have been studying. For example, letting $T = \partial\mathcal{D}$, $E(t) = \pi^{-1}(t) \cong M_n(\mathbb{C})$ and $\Gamma = \Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ gives rise to a continuous field of $C^*$-algebras over $\partial\mathcal{D}$. Similarly, with the same $T$ and $E(t)$, letting $\Gamma = \mathbb{S}(\partial\mathcal{D}; d, n)$ gives rise to continuous field of

Banach spaces over $\mathcal{D}$ which is not a continuous field of $C^*$-algebras because $\mathbb{S}(\partial\mathcal{D}; d, n)$ is not closed under adjoints.

Let $\mathcal{E} = ((E(t))_{t\in T}, \Gamma)$ be a continuous field of Banach spaces over $T$. Then a subset $\Lambda \subset \Gamma$ is said to be *total* if, for every $t \in T$, the set $\{x(t) : x \in \Lambda\}$ is total in $E(t)$. Lemma 19 is a consequence of the following proposition [Dix77, Prop 10.2.4]: Let $\mathcal{E} = ((E(t)), \Gamma)$ and $\mathcal{E}' = ((E'(t)), \Gamma')$ be two continuous fields of Banach spaces over $T$, and let $\Lambda$ be a total subset of $\Gamma$. For every $t \in T$, let $\varphi_t$ be an isometric isomorphism of $E(t)$ onto $E'(T)$. Then $\varphi = (\varphi_t)_{t\in T}$ is an isomorphism of $\mathcal{E}$ onto $\mathcal{E}'$ if and only if $\varphi(\Lambda) \subset \Gamma'$. Here, the family $(\varphi_t)$ is an isomorphism of the continuous fields of Banach spaces $\mathcal{E}$ and $\mathcal{E}'$ if and only if $\varphi$ transforms $\Gamma$ into $\Gamma'$.

In our context, let $\mathcal{E}$ be the continuous field of $C^*$-algebras over $\partial\mathcal{D}$ with $E(b) = \pi^{-1}(b) \cong M_n(\mathbb{C})$ the fibre over $b$ for each $b \in \partial\mathcal{D}$ and $\Gamma = \Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d, n))$. Let $\mathcal{E}'$ be the continuous field of $C^*$-algebras over $\partial\mathcal{D}$ with $E'(b) = E(b)$ for each $b \in \partial\mathcal{D}$ and $\Gamma' = C^*(\mathbb{S}(\mathcal{D}; d, n))$, the $C^*$-algebra generated by $\mathbb{S}(\mathcal{D}; d, n)$. For each $b$, the map $\varphi_b$ is the identity map on the fibre $\pi^{-1}(b)$. The total subset we consider is $\Lambda = \mathbb{S}(\mathcal{D}; d, n)$. It is clear that $\mathbb{S}(\mathcal{D}; d, n)$ is a total subset because, in particular, $\mathbb{S}(\mathcal{D}; d, n)$ contains the cross sections that are associated to the matrix coordinate maps $\mathcal{Z}_1, \ldots, \mathcal{Z}_d$ and for each point $\mathfrak{z} = (Z_1, Z_2, \ldots, Z_d) \in \mathcal{V}(d, n)$ the coordinates $Z_1, \ldots, Z_d$ generate $M_n(\mathbb{C})$ as an algebra. This implies that for each $b \in \partial\mathcal{D}$,

$$\{s(b) : s \in \mathbb{S}(\mathcal{D}; d, n)\} = \pi^{-1}(b).$$

For $s \in \Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ and $b \in \partial\mathcal{D}$, we have $(\varphi(s))(b) = id(s(b)) = s(b)$, so the map $\varphi$ is the identity map. Hence $\varphi(\mathbb{S}(\mathcal{D}; d, n)) \subset C^*(\mathbb{S}(\mathcal{D}; d, n))$, and so it follows by Proposition 10.2.4 in [Dix77] that $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n)) = C^*(\mathbb{S}(\mathcal{D}; d, n))$. Now we are ready to prove the main result of this chapter.

**Theorem 20.** *If $b \in \partial_e\mathcal{D}$, then $ev_b$ is a boundary representation of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ for $\mathbb{S}(\mathcal{D}; d, n)$.*

In fact, $ev_b$ is a peaking representation. Since $Q_0(d,n)$ is metrizable, so is $\overline{\mathcal{D}}$. Assume that $b \in \partial_e\mathcal{D}$. Then $b$ is a peak point for $\mathbb{I}(\mathcal{D}; d, n)$, i.e. there is a function $f \in \mathbb{I}(\mathcal{D}; d, n)$ so that $f(b) = 1$ and $|f(b')| < 1$ for all $b' \neq b$ in $\partial\mathcal{D}$. But then $\mathbb{I}(\mathcal{D}; d, n) \subset \mathbb{S}(\mathcal{D}; d, n)$ so $f$ can be identified with a $1 \times 1$ matrix over $\mathbb{S}(\mathcal{D}; d, n)$, and $ev_b$ peaks at $f$. Therefore $ev_b$ is a peaking representation, and hence a boundary representation, of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ for $\mathbb{S}(\mathcal{D}; d, n)$.

**Corollary 21.** *$\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(\Lambda; d,n))$ is the $C^*$-envelope of $\mathbb{S}(\mathcal{D}, \Lambda; d, n)$ for any reduction $\Lambda$.*

Any section $s \in \Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ in the kernel of $ev_b$ vanishes at $b$. So any section in $\bigcap_{b\in\partial_e\mathcal{D}} \ker ev_b$ vanishes on all of $\partial_e\mathcal{D}$. Since $\partial_e\mathcal{D}$ is dense in $\partial\mathcal{D}$ [Sto71, Theorem I.7.24], any such section is the zero section. Therefore the Shilov boundary ideal, which is equal to the intersection of the kernels of the boundary representations, contains only the zero section. Hence $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}^*(d,n))$ is the $C^*$-envelope of $\mathbb{S}(\mathcal{D}; d, n)$.

### 6.3 Generalizations of Theorem 20

The remainder of this chapter will address generalizations of Theorem 20 to other matrix bundles. The first concerns bundles whose total spaces are *n-varieties*, and is inspired by the work of Reichstein and Vonessen in [RV07]. An $n$-variety $\mathcal{W}$ is a closed $G$-invariant subvariety of $\mathcal{V}(d,n)$. In particular, $\mathcal{W} = \overline{\mathcal{W}} \cap \mathcal{V}(d,n)$, where $\overline{\mathcal{W}}$ is the Zariski closure of $\mathcal{W}$ in $M_n(\mathbb{C})^d$. If $\mathcal{W}$ is an $n$-variety, then $G$ acts freely and properly on $\mathcal{W}$ and hence $\mathcal{W}$ is the bundle space of a holomorphic principal $G$-bundle. Since this action on $\mathcal{W}$ is the restriction of the action of $G$ on $\mathcal{V}(d,n)$, the quotient map from $\mathcal{W}$ to the orbit space $\mathcal{W}/G$ is the restriction $\pi_0|\mathcal{W}$. The image of this map is a subset of $Q_0(d,n)$; denote it by $\mathcal{Q}$. The principal $G$-bundle determined by $\mathcal{W}$ is $\mathfrak{B}_{\mathcal{W}}(d,n) = (\mathcal{W}, \pi_0|\mathcal{W}, \mathcal{Q})$. This is the restriction of the bundle $\mathfrak{B}(d,n)$ over $\mathcal{Q}$. We want to study the cross sections of bundles related to $\mathfrak{B}_{\mathcal{W}}(d,n)$ in much the same way that we have studied those of bundles related to $\mathfrak{B}(d,n)$. Although the bundle $\mathfrak{B}_{\mathcal{W}}(d,n)$ is a subbundle of $\mathfrak{B}(d,n)$ and the arguments and theorems that follow are very similar to those in Section 6.2, there are important differences between the two cases. For instance, there may be reductions of the bundle $\mathfrak{B}_{\mathcal{W}}(d,n)$ which do not extend to reductions of the bundle $\mathfrak{B}(d,n)$.

Consider the $M_n(\mathbb{C})$-fibre bundle associated to $\mathfrak{B}_{\mathcal{W}}(d,n)$, which will be denoted by $\mathfrak{M}_{\mathcal{W}}(d,n)$. The algebra of cross sections $\Gamma_c(\mathcal{Q}, \mathfrak{M}_{\mathcal{W}}(d,n))$ is isomorphic to the algebra of matrix concomitants $C(\mathcal{W}, M_n(\mathbb{C}))^G$. We have seen that the algebra of invariants $\mathbb{I}_0(d,n)$ and the trace algebra $\mathbb{S}_0(d,n)$ are important for understanding the algebra of concomitants $C(\mathcal{V}(d,n), M_n(\mathbb{C}))^G$. When studying bundles related to $\mathfrak{B}_{\mathcal{W}}(d,n)$, we will consider quotients of these algebras. We must study quotients because, although each element of $\mathbb{S}_0(d,n)$ does define a function on $\mathcal{W}$ by restriction, this assignment may have a kernel. The following definitions are due to Reichstein and Vonessen generalize the concept of the coordinate ring of an affine variety to the context of $n$-varieties. Given an $n$-variety $\mathcal{W}$, define the ideal of

$\mathbb{S}_0(d,n)$ associated to $\mathcal{W}$ to be $\mathcal{I}(\mathcal{W}) := \{p \in \mathbb{S}_0(d,n) : p(\mathfrak{z}) = 0 \ \forall \mathfrak{z} \in \mathcal{W}\}$. Denote the quotient $\mathbb{S}_0(d,n)/\mathcal{I}(\mathcal{W})$ by $\mathbb{S}_0(\mathcal{W})$ and let $\mathbb{I}_0(\mathcal{W})$ be the center of $\mathbb{S}_0(\mathcal{W})$.

Let $D \subset \mathcal{Q}$ be a domain so that $\overline{D}$ is a compact subset of $\mathcal{Q}$. Let $\Lambda$ be a reduction of $\mathfrak{B}_{\mathcal{W}}(d,n)$ to a principal $K$-bundle, and denote the associated matrix bundle by $\mathfrak{M}^*_{\mathcal{W}}(\Lambda; d, n)$. The reduction $\Lambda$ induces an isomorphism from $\Gamma_c(\mathcal{Q}, \mathfrak{M}_{\mathcal{W}}(d,n))$ to $\Gamma_c(\mathcal{Q}, \mathfrak{M}^*_{\mathcal{W}}(\Lambda; d, n))$ which is also denoted by $\Lambda$.

Define $\mathbb{I}_{\mathcal{W}}(D; d, n)$ to be the closure of $\mathbb{I}_0(\mathcal{W})$ in $C(\overline{D})$. This makes sense because the elements of $\mathbb{I}_0(\mathcal{W})$ can be viewed as continuous functions on $\overline{D}$. Essentially this is a consequence of the fact that the points in $\mathcal{W}$ are irreducible points. In more detail, fix an element $p : \mathcal{W} \to M_n(\mathbb{C})$ in $\mathbb{I}_0(\mathcal{W})$ and fix a point $\mathfrak{z} = (Z_1, Z_2, \ldots, Z_d) \in \mathcal{W}$. The map $p$ is in the center of $\mathbb{S}_0(\mathcal{W})$, so in particular it must commute with the matrix coordinate functions $\mathcal{Z}_1 + \mathcal{I}(\mathcal{W}), \mathcal{Z}_2 + \mathcal{I}(\mathcal{W}), \ldots, \mathcal{Z}_d + \mathcal{I}(\mathcal{W})$. Evaluating at $\mathfrak{z}$, this implies that $p(\mathfrak{z})$ commutes with $Z_1, Z_2, \ldots, Z_d$. The matrices $Z_1, Z_2, \ldots, Z_d$ generate $M_n(\mathbb{C})$, and so $p(\mathfrak{z})$ commutes with every element in $M_n(\mathbb{C})$. This is true for every point $\mathfrak{z} \in \mathcal{W}$, so $p$ has the form $p(\mathfrak{z}) = \tilde{p}(\mathfrak{z}) I_n$ for $\tilde{p} : \mathcal{W} \to \mathbb{C}$. Since $p$ is a matrix concomitant, $\tilde{p}$ has the property that, for each $\mathfrak{z} \in \mathcal{W}$ and $g \in G$, $\tilde{p}(\mathfrak{z} \cdot g) = \tilde{p}(\mathfrak{z})$, i.e. $\tilde{p}$ is invariant. The map $\tilde{p}$ is constant on $G$-orbits, so it may be viewed as a function on the orbit space $\mathcal{W}/G = \mathcal{Q}$. Restricting this map to $\overline{D}$, we see that $\mathbb{I}_0(\mathcal{W})$ can be considered as a subalgebra of $\overline{D}$.

For each $\lambda \in \mathbb{C}$, $\mathbb{I}_0(\mathcal{W})$ contains the constant map $\mathfrak{z} \mapsto \lambda I_n$. Also, $\mathbb{I}_0(\mathcal{W})$ separates the points of $\overline{D} \subset Q_0(d,n)$ since $\mathbb{I}_0(d,n)$ separates the points of $Q_0(d,n)$. So $\mathbb{I}_{\mathcal{W}}(D; d, n)$ is a uniform algebra over $\overline{D}$. Let $\partial D$, respectively $\partial_e D$, denote the Shilov boundary, respectively the Choquet boundary, of $\overline{D}$ with respect to $\mathbb{I}_0(\mathcal{W})$.

Define $\mathbb{S}_{\mathcal{W}}(D,\Lambda;d,n)$ to be the closure of $\Lambda\mathbb{S}_0(\mathcal{W})$ in $\Gamma_c(\partial D,\mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n))$. This make sense as $\Gamma_c(\partial D,\mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n))$ is an $n$-homogeneous $C^*$-algebra by Theorem 8 in [TT61]. The identity element $e\in\Gamma_c(\partial D,\mathfrak{M}_{\mathcal{W}}(d,n))$ is the section defined by $[\mathfrak{z}]\mapsto[\mathfrak{z},I_n]$, from $\partial D\subset\mathcal{Q}$ to $\mathcal{W}\times_G M_n(\mathbb{C})$. The matrix concomitant that corresponds to this cross section is the constant function $f_e:\mathcal{W}\to M_n(\mathbb{C})$ defined by $f_e(\mathfrak{z})=I_n$. Clearly $f_e\in\mathbb{S}_0(\mathcal{W})$. It follows that the identity $\Lambda e=e$ of $\Gamma_c(\partial D,\mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n))$ is an element in $\mathbb{S}_{\mathcal{W}}(D,\Lambda;d,n)$.

Now we show that $\mathbb{S}_{\mathcal{W}}(D,\Lambda;d,n)$ generates $\Gamma_c(\partial D,\mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n))$ as a $C^*$-algebra. Arguing as in the proof of Lemma 19, it is sufficient to show that for each $b\in\partial D$ we have

$$\pi^{-1}(b)=\{s(b):s\in\mathbb{S}_{\mathcal{W}}(D,\Lambda;d,n)\} \tag{6.2}$$

To see this, consider the unreduced matrix bundle $\mathfrak{M}_{\mathcal{W}}(\partial D;d,n)$. Recall that the correspondence between the algebra of concomitants $C(\mathcal{W},M_n(\mathbb{C}))^G$ and the algebra of cross sections $\Gamma_c(\mathcal{Q},\mathfrak{M}_{\mathcal{W}}(d,n))$ takes a concomitant function $f$ to the section $s_f$ defined by $s_f([\mathfrak{z}])=[\mathfrak{z},f(\mathfrak{z})]$. Consider the image under this map of the matrix coordinate functions $\mathcal{Z}_1+\mathcal{I}(\mathcal{W}),\mathcal{Z}_2+\mathcal{I}(\mathcal{W}),\ldots,\mathcal{Z}_d+\mathcal{I}(\mathcal{W})\in\mathbb{S}_0(d,n)$. Fix $b\in\partial D$ and $\mathfrak{z}=(Z_1,Z_2,\ldots,Z_d)\in\pi_0^{-1}(b)$. Then we have $[\mathfrak{z},Z_1],[\mathfrak{z},Z_2],\ldots,[\mathfrak{z},Z_d]$ in the image of $b$ under the cross sections that are associated to functions in $\mathbb{S}_0(\mathcal{W})$. Since $\mathfrak{z}\in\mathcal{W}\subset\mathcal{V}(d,n)$, the entries $Z_1,Z_2,\ldots,Z_d$ generate $M_n(\mathbb{C})$ as an algebra. $\mathbb{S}_0(\mathcal{W})$ is an algebra under pointwise operations, so this implies that

$$\{s(b):s\in\mathbb{S}_0(\mathcal{W})\}=\pi^{-1}(b)$$

Since the reduction $\Lambda$ yields an isomorphism from $\Gamma_c(\partial D,\mathfrak{M}_{\mathcal{W}}(d,n))$ to $\Gamma_c(\partial D,\mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n))$

and $\Lambda\mathbb{S}_0(\mathcal{W}) \subset \mathbb{S}_{\mathcal{W}}(D;d,n)$, equation (6.2) follows.

So $\mathbb{S}_{\mathcal{W}}(D;d,n)$ is a norm-closed subalgebra of $\Gamma_c(\partial D, \mathfrak{M}_{\mathcal{W}}(\Lambda;d,n))$ which contains the identity and generates $\Gamma_c(\partial D, \mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n))$ as a $C^*$-algebra. It makes sense to discuss the boundary representations of $\Gamma_c(\partial D, \mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n))$ for $\mathbb{S}_{\mathcal{W}}(D,\Lambda;d,n)$, and we have the following analogue of Theorem 20.

**Theorem 22.** *Evaluation at each point in $\partial_e D$ determines a boundary representation of $\Gamma_c(\partial D, \mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n))$ for $\mathbb{S}_{\mathcal{W}}(D,\Lambda;d,n)$.*

As in section 6.2, every element $b \in \partial D$ determines an irreducible representation $ev_b$ of $\Gamma_c(\partial D, \mathfrak{M}^*_{\mathcal{W}}(d,n))$ and every irreducible representation is unitarily equivalent to such a representation by [Dix77, Corollary 10.4.4]. Further, as in the proof of Theorem 20, $\overline{D}$ is metrizable, so every point $b \in \partial_e D$ is a peak point for $\mathbb{I}_{\mathcal{W}}(D;d,n)$. Therefore $ev_b$ is a peaking representation, and hence a boundary representation, of $\Gamma_c(\partial D, \mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n))$ for $\mathbb{S}_{\mathcal{W}}(D,\Lambda;d,n)$.

**Corollary 23.** *The $C^*$-envelope of $\mathbb{S}_{\mathcal{W}}(D;d,n)$ is $\Gamma_c(\partial D, \mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n))$.*

This is because any section in the Shilov boundary ideal of $\Gamma_c(\partial D, \mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n))$ will be in the kernel of $ev_b$ for each $b \in \partial_e D$. Since $\partial_e D$ is dense in $\partial D$, this implies that the Shilov boundary ideal contains only the zero section. The corollary follows.

We can generalize these results further. A complex manifold $X$ is a *Stein manifold* if the following conditions hold:

1. $X$ is holomorphically convex, i.e. for each compact set $K \subset X$, the holomorphic convex hull of $K$, $\widehat{K} = \{x \in X : |f(x)| \leq \sup_{y \in K} |f(y)| \; \forall f$ holomorphic on $K\}$, is also

a compact subset of $X$.

2. $X$ is holomorphically separable, i.e. for any distinct points $x, y \in X$ there is a function $f$ which is holomorphic on $X$ such that $f(x) \neq f(y)$.

If $X$ is a Stein manifold and $K \subset X$ is a compact set which is holomorphically convex, i.e. $\widehat{K} = K$, then $K$ is called a *Stein compact set*. The benefit of working with a Stein manifold is that if $X$ is a Stein manifold, then the algebra of global holomorphic functions on $X$, denoted $\mathcal{O}_X$, is a function algebra over $X$ and $X$ is the spectrum of $\mathcal{O}_X$ [Ric79, Example 5.3]. Further, if $K \subset X$ is Stein compact, then $K$ is the spectrum of $A(K)$, the algebra of functions that are uniform limits of functions holomorphic on a neighborhood of $K$ [GR09, Corollary VII.A.7].

Suppose that $\mathfrak{B} = (X, \pi, B)$ is a holomorphic principal $K$-bundle in which $X$ is a complex manifold and $B$ is a Stein manifold. Consider the associated $M_n(\mathbb{C})$-fibre bundle, $\mathfrak{B}[M_n(\mathbb{C})] = (X \times_K M_n(\mathbb{C}), p, B)$. Let $D \subset B$ be a domain so that $\overline{D} \subset B$ is Stein compact. Further, assume that $\Gamma_h(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])$ is *sufficiently rich* in the sense that, for each $b \in \overline{D}$, the set $\{s(b) : s \in \Gamma_h(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])\}$ is dense in $p^{-1}(b)$. Note that the algebra of continuous cross sections $\Gamma_c(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])$ is a unital $n$-homogeneous $C^*$-algebra.

Let $A(D)$ denote the sup-norm closure of $Hol(D)$ in $C(\overline{D})$. Since we have assumed that $B$ is a Stein manifold, so $\mathcal{O}_B$ separates the points in $B$, it is easy to see that the algebra $A(D)$ is a uniform algebra over $\overline{D}$. Let $\partial D$, respectively $\partial_e D$, be the Shilov boundary, respectively the Choquet boundary, of $D$ for $A(D)$. Let $\mathcal{A}(D, \mathfrak{B}[M_n(\mathbb{C})])$ denote the norm-closure of $\Gamma_h(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])$ in $\Gamma_c(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])$. We will show that every point in $\partial_e D$ determines a boundary representation of $\Gamma_c(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])$ for $\mathcal{A}(D, \mathfrak{B}[M_n(\mathbb{C})])$. Before we can prove this result, we must show that $\mathcal{A}(D, \mathfrak{B}[M_n(\mathbb{C})])$ is a proper subalgebra of $\Gamma_c(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])$ that generates $\Gamma_c(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])$ as a $C^*$-algebra.

To see that $\mathcal{A}(D, \mathfrak{B}[M_n(\mathbb{C})])$ is a proper subalgebra, suppose that $f : \overline{D} \to \mathbb{C}$ is a continuous function on $\overline{D}$ which is not holomorphic on $D$. Then the cross section of $\mathfrak{B}[M_n(\mathbb{C})]$ that corresponds to $f$ will be a continuous cross section which is not holomorphic. Arguing as in Lemma 19, the fact that $\mathcal{A}(D, \mathfrak{B}[M_n(\mathbb{C})])$ generates $\Gamma_c(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])$ as a $C^*$-algebra follows from the assumption that $\Gamma_h(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])$ is sufficiently rich.

**Theorem 24.** *Every point of $\partial_e D$ determines a boundary representation of $\Gamma_c(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])$ for $\mathcal{A}(D, \mathfrak{B}[M_n(\mathbb{C})])$.*

Since $\overline{D}$ is a metrizable space, $b \in \partial_e D$ implies that $b$ is a peak point. So there exists $f \in A(D)$ so that $f(b) = 1$ and $\|f(b')\| < 1$ for $b \neq b'$. The function $f$ defines an invariant function on $\pi^{-1}(\overline{D})$, which has an associated cross section $\Psi(f) \in \mathcal{A}(D, \mathfrak{B}[M_n(\mathbb{C})])$. Thinking of $\Psi(f)$ as a $1 \times 1$ matrix over $\mathcal{A}(D, \mathfrak{B}[M_n(\mathbb{C})])$, we have that the representation $ev_b$ peaks at $\Psi(f)$. Therefore $ev_b$ is a peaking representation, and hence a boundary representation, of $\Gamma_c(\overline{D}, \mathfrak{B}[M_n(\mathbb{C})])$ for $\mathcal{A}(D, \mathfrak{B}[M_n(\mathbb{C})])$.

If we restrict our attention to cross sections over the Shilov boundary $\partial D$, rather than cross sections defined over $\overline{D}$, Theorem 24 has the following corollary.

**Corollary 25.** *The $C^*$-envelope of $\mathcal{A}(\partial D, \mathfrak{B}[M_n(\mathbb{C})])$ is $\Gamma_c(\partial D, \mathfrak{B}[M_n(\mathbb{C})])$.*

As in the proof of Corollary 21, this is a direct consequence of the fact that $\partial_e D$ is dense in $\partial D$.

# CHAPTER 7
# AZUMAYA ALGEBRAS

Let $X$ be a locally compact space and $\mathcal{A}$ a function algebra over $X$. For $K \subset X$ compact, the *$\mathcal{A}$-convex hull* of $K$ is

$$\widehat{K}_{\mathcal{A}} = \{x \in X : |f(x)| \leq \sup_{y \in K} |f(y)| \text{ for all } f \in \mathcal{A})\}$$

A subset $Y \subset X$ is called *$\mathcal{A}$-convex* if for every compact subset $K \subset Y$, one has $\widehat{K}_{\mathcal{A}} \subset Y$. A compact subset $K \subset X$ is $\mathcal{A}$-convex if $K = \widehat{K}_{\mathcal{A}}$. In the study of holomorphic functions in several complex variables, polynomially convex sets play an important role. These are subsets of $\mathbb{C}^k$ which are convex with respect to the algebra of polynomial functions. In what follows we will be concerned with sets $X \subset Q_0(d,n)$ which are $\mathbb{I}_0(d,n)$-convex. It will be shown that, for $\mathcal{D} \subset Q_0(d,n)$ with $\overline{\mathcal{D}} \subset Q_0(d,n)$ compact and $\mathbb{I}_0(d,n)$-convex, the algebra $\mathbb{S}(\mathcal{D}; d, n)$ is an *Azumaya algebra.*

**Definition 26.** *A unital algebra $A$ with center $Z(A)$ is called an* Azumaya algebra *in case*

1. *$A$ is projective as a right module over $Z(A)$, and*

2. *The map $\Theta : A \otimes_{Z(A)} A^{op} \to End(A_{Z(A)})$ defined by $\Theta(a \otimes b)(r) := arb$ is an isomorphism.*

This is one of many equivalent definitions. Later, we will discuss other characterizations of Azumaya algebras and see that they are algebraic versions of $n$-homogeneous $C^*$-algebras. Before we do that, though, we need some background on polynomial identity rings. More detailed treatments are found in [Row80], [Pro73], and in Chapter 13 of [MR01]. Assume that $A$ is a unital $\mathbb{C}$-algebra. $A$ is said to satisfy a polynomial in $k$ noncommuting variables $f(X_1, X_2, \ldots, X_k) \in \mathbb{C}\langle X_1, X_2, \ldots X_k\rangle$ if $f(a_1, a_2, \ldots, a_k) = 0$ for all $a_1, \ldots, a_k \in A$. In this case, $f$ is called a *polynomial identity* of $A$, and $A$ is called a *polynomial identity algebra*. For example, every commutative algebra is a polynomial identity algebra, since it satisfies the identity $f(X_1, X_2) = X_1X_2 - X_2X_1$. It is less obvious that every matrix algebra is also a polynomial identity algebra.

For a natural number $t$, define the $t^{th}$ standard polynomial to be

$$s_t = \sum_{\sigma \in S_t} (\operatorname{sgn} \sigma) X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(t)}$$

Here, $S_t$ is the symmetric group on $t$ elements. Note that $s_t \in \mathbb{C}\langle X_1, X_2, \ldots, X_t\rangle$, and the function obtained by evaluating $s_t$ at values in an $\mathbb{C}$-algebra $A$ is multilinear and alternating. Since $M_n(\mathbb{C})$ has dimension $n^2$ over $\mathbb{C}$, this implies that the standard polynomial $s_{n^2+1}$ is a polynomial identity for $M_n(\mathbb{C})$. This was first observed by Kolchin and Levi in [Lev49].

A natural question, then, is to determine the smallest $t$ so that $s_t$ is a polynomial identity for $M_n(\mathbb{C})$: The answer was first found by Amitsur and Levitski in [AL50]. They proved that $s_{2n}$ is an identity for $M_n(\mathbb{C})$, and $s_t$ is not an identity for $M_n(\mathbb{C})$ for any $t \leq 2n-1$. A simpler proof of this theorem, which relies on the Cayley-Hamilton Theorem, is given by Razmyslov in [Raz74].

Analogous to polynomial identities, we will also want to consider central polynomials. A polynomial $f(X_1, X_2 \ldots, X_k) \in \mathbb{C}\langle X_1, X_2, \ldots, X_k\rangle$ which always evaluates to a central element in $A$ but is not always zero is a *central polynomial* for $A$. Again, it is not clear *a priori* that there exist central polynomials for $M_n(\mathbb{C})$. However such polynomials do exist. For example, for $n = 2$, the Cayley-Hamilton Theorem asserts that, for $A \in M_2(\mathbb{C})$, we have $A^2 - \text{tr}(A)A + \det(A)I_2 = 0$. Letting $A = [X_1, X_2] = X_1X_2 - X_2X_1$, we see that $[X_1, X_2]^2 = -\det([X_1, X_2])I_2$ for $X_1, X_2 \in M_2(\mathbb{C})$. Hence the polynomial $p(X_1, X_2) = (X_1X_2 - X_2X_1)^2$ is a central polynomial for $M_2(\mathbb{C})$. Formanek, in [For72], and Razmyslov, in [Raz74], give examples of central polynomials for $M_n(\mathbb{C})$ for arbitrary $n$.

For a unital algebra $A$, the *Formanek center* of $A$, denoted by $F(A)$, is the subring of the center of $A$ obtained by evaluating all the central polynomials in $A$. We will use the Formanek center to prove certain algebras are Azumaya algebras.

A useful characterization of the central polynomials for $M_n(\mathbb{C})$ is given by Reichstein and Vonessen in [RV07, Definition 2.9]: An $d$-variable central polynomial for $M_n(\mathbb{C})$ is an element $p(X_1, X_2, \ldots, X_d) \in \mathbb{C}\langle X_1, X_2, \ldots, X_d\rangle$ which has constant coefficient zero and whose canonical image in $\mathbb{G}_0(d, n)$ is a nonzero central element. If $p$ is a central polynomial for $M_n(\mathbb{C})$, then $p$ will also be a central polynomial for $\mathbb{S}_0(d, n)$ and $\mathbb{S}(\mathcal{D}; d, n)$, since both are algebras of functions that take values in $M_n(\mathbb{C})$. Moreover, $p$ can be identified with an element in the Formanek center of $\mathbb{S}_0(d, n)$ or $\mathbb{S}(\mathcal{D}; d, n)$ by evaluating $p$ at the matrix coordinate functions, that is by identifying $p$ with $p(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_d)$.

Recall from Definition 26 that a unital $\mathbb{C}$-algebra $A$ is an Azumaya algebra over its center $Z(A)$ if $A$ is a projective right module over $Z(A)$ and the map $\Theta : A \otimes_{Z(A)} A^{op} \to End(A_{Z(A)})$ defined by $\Theta(a \otimes b)(r) = arb$ is an isomorphism. This is one of many equivalent definitions. In [Art69, Theorem 8.3], Artin proved that $A$ is an Azumaya algebra if and only if there is a natural number $n$ so that $A$ satisfies the polynomial identities of $M_n(\mathbb{C})$ and there are no unital homomorphisms of $A$ into $M_r(\mathbb{C})$ for $r < n$. That is, under the hypothesis that $A$ satisfies the identities of $M_n(\mathbb{C})$, this theorem asserts that $A$ is an Azumaya algebra if and only if each (algebraically) irreducible representation of $A$ is $n$-dimensional. Artin was inspired, in part, by Tomiyama and Takesaki's representation of an $n$-homogeneous $C^*$-algebra as the continuous cross sections of a matrix bundle in [TT61]. Thus, in one sense, the main result of this chapter may easily be anticipated, given that $\mathbb{S}(\mathcal{D}; d, n)$ is a subalgebra of the $n$-homogeneous $C^*$-algebra $\Gamma_c(\mathcal{D}, \mathfrak{M}^*(d, n))$. However, the proof is not obvious. Moreover, the theorem has consequences that are not immediately evident.

A third characterization of Azumaya algebras is provided by Procesi in [Pro73, Theorem VIII.2.1]. A unital $\mathbb{C}$-algebra $A$ is an Azumaya algebra if and only if $A$ satisfies the polynomial identities of $M_n(\mathbb{C})$ and $A = F(A)A$, where $F(A)A$ is the ideal generated by the Formanek center of $A$. We will use this third characterization to prove that, whenever $\mathcal{D} \subset Q_0(d, n)$ is a domain so that $\overline{\mathcal{D}} \subset Q_0(d, n)$ is compact and $\mathbb{I}_0(d, n)$-convex, $\mathbb{S}(\mathcal{D}; d, n)$ is an Azumaya algebra.

We will need the following special case of a lemma of Reichstein and Vonessen.

**Lemma 27.** *[RV07, Lemma 2.10] For each $\mathfrak{z} \in \mathcal{V}(d, n)$, there exists a central polynomial $p \in \mathbb{G}_0(d, n)$ such that $p(\mathfrak{z}) = I_n$.*

Now we are ready to prove our main theorem.

**Theorem 28.** *Suppose that $\mathcal{D} \subset Q_0(d, n)$ is chosen so that $\overline{\mathcal{D}}$ is the maximal ideal space of $\mathbb{I}(\mathcal{D}; d, n)$. Then the algebra $\mathbb{S}(\mathcal{D}; d, n)$ is a rank $n^2$ Azumaya algebra over $\mathbb{I}(\mathcal{D}; d, n)$.*

In this context, to say that $\mathbb{S}(\mathcal{D};d,n)$ has rank $n^2$ over its center meant that for every maximal 2-sided ideal $\mathfrak{m}$ of $\mathbb{S}(\mathcal{D};d,n)$, $\mathbb{S}(\mathcal{D};d,n)/\mathfrak{m} \cong M_n(\mathbb{C})$.

*Proof.* It is clear that $\mathbb{S}(\mathcal{D};d,n)$ satisfies the identities of the $n \times n$ matrices. Also, by Lemma 17 in Chapter 6, $\mathbb{I}(\mathcal{D};d,n)$ is the center of $\mathbb{S}(\mathcal{D};d,n)$. So given any point $b \in \overline{\mathcal{D}}$, we may choose $\mathfrak{z}$ in the bundle space of the reduction $\mathfrak{C}(d,n)$ so that $\pi(\mathfrak{z}) = b$. Since the bundle space of the reduced bundle is a closed subset of $\mathcal{V}(d,n)$, we have $\mathfrak{z} \in \mathcal{V}(d,n)$. Hence, by Lemma 27, there exists a $d$-variable central polynomial $p$ so that $p(\mathfrak{z}) = I_n$. Since a central polynomial is invariant, we may view $p$ as a function on $Q(d,n)$ that is 1 at $b$.

By the compactness of $\overline{\mathcal{D}}$, we may choose a finite number of central polynomials, $p_1, p_2, \ldots, p_N$, that have no common zero on $\overline{\mathcal{D}}$. Since $\overline{\mathcal{D}}$ is the maximal ideal space of $\mathbb{I}(\mathcal{D};d,n)$, every maximal ideal of $\mathbb{I}(\mathcal{D};d,n)$ consists of elements which evaluate to zero at some point $b \in \overline{\mathcal{D}}$. Therefore the fact that $p_1, p_2, \ldots, p_N$ have no common zero implies that the ideal generated by $p_1, p_2, \ldots, p_N$ is not contained in a maximal ideal. That is, the ideal generated by $p_1, p_2, \ldots, p_N$ is all of $\mathbb{I}(\mathcal{D};d,n)$. So $1 \in p_1\mathbb{I}(\mathcal{D};d,n) + p_2\mathbb{I}(\mathcal{D};d,n) + \cdots + p_n\mathbb{I}(\mathcal{D};d,n)$. Since each $p_i$ is in the Formanek center of $\mathbb{S}(\mathcal{D};d,n)$, it follows that $F(\mathbb{S}(\mathcal{D};d,n))\mathbb{S}(\mathcal{D};d,n) = \mathbb{S}(\mathcal{D};d,n)$. Thus, $\mathbb{S}(\mathcal{D};d,n)$ is an Azumaya algebra. □

A hypothesis that guarantees that $\overline{\mathcal{D}}$ is the maximal ideal space of $\mathbb{I}(\mathcal{D};d,n)$ is to choose $\mathcal{D}$ so that $\overline{\mathcal{D}}$ is $\mathbb{I}_0(d,n)$-convex. This follows from an argument similar to the proof of Proposition I.H8 in [GR09]: Every homomorphism of $\mathbb{I}(\mathcal{D};d,n)$ restricts to a homomorphism of $\mathbb{I}_0(d,n)$. On the other hand, since $\mathbb{I}_0(d,n)$ is dense in $\mathbb{I}(\mathcal{D};d,n)$, a homomorphism of $\mathbb{I}_0(d,n)$ will be correspond to a homomorphism of $\mathbb{I}(\mathcal{D};d,n)$ whenever it can be extended continuously to $\mathbb{I}(\mathcal{D};d,n)$. That is, the nonzero complex homomorphisms of $\mathbb{I}(\mathcal{D};d,n)$ correspond exactly with the nonzero complex homomorphisms of $\mathbb{I}_0(d,n)$ which have continuous extensions to $\mathbb{I}(\mathcal{D};d,n)$. A homomorphism $\psi : \mathbb{I}_0(d,n) \to \mathbb{C}$ can be extended continuously to $\mathbb{I}(\mathcal{D};d,n)$ if and only if $\psi$ satisfies the inequality

$$|\psi(p)| \leq \sup_{b\in\overline{\mathcal{D}}} |p(b)|$$

for every $p \in \mathbb{I}_0(d,n)$.

The nonzero complex homomorphisms of $\mathbb{I}_0(d,n)$ correspond to evaluations at points in the spectrum of $\mathbb{I}_0(d,n)$, which is $Q(d,n)$. So the nonzero complex homomorphisms of $\mathbb{I}(\mathcal{D};d,n)$ are given by evaluations at point in the set $\{x \in Q(d,n) : |p(x)| \leq \sup_{b\in\overline{\mathcal{D}}} |p(b)|\ \forall p \in \mathbb{I}_0(d,n)\}$, which is exactly the $\mathbb{I}_0(d,n)$-convex hull of $\overline{\mathcal{D}}$.

As mentioned above, this theorem may be easily anticipated given the similarity between Azumaya algebras and $n$-homogeneous $C^*$-algebras. However, this theorem has consequences that appear difficult to establish without it. For example,

**Corollary 29.** *If $\overline{\mathcal{D}}$ is $\mathbb{I}_0(d,n)$-convex, then there is a bijective correspondence between ideals $\mathfrak{a}$ of $\mathbb{I}(\mathcal{D};d,n)$ and ideals $\mathfrak{A}$ of $\mathbb{S}(\mathcal{D};d,n)$ given by $\mathfrak{a} \mapsto \mathfrak{a}\mathbb{S}(\mathcal{D};d,n)$ and $\mathfrak{A} \mapsto \mathfrak{A} \cap \mathbb{I}(\mathcal{D};d,n)$.*

This is an application of Corollary II.3.7 of [DI71], which is valid for any Azumaya algebra.

Theorem 28 can easily be generalized to bundles that are associated with $n$-varieties. As in Section 6.3, let $\mathcal{W} \subset \mathcal{V}(d,n)$ be an $n$-variety and let $\mathfrak{B}_{\mathcal{W}}(d,n) = (\mathcal{W}, \pi_0, \mathcal{Q})$ denote the principal $G$-bundle that arises from the action of $G$ on $\mathcal{W}$. Let $\Lambda$ be a reduction of $\mathfrak{B}_{\mathcal{W}}(d,n)$ to a principal $K$-bundle, and let $\mathfrak{C}_{\mathcal{W}}(\Lambda;d,n)$ denote the reduced bundle. $\mathfrak{M}^*_{\mathcal{W}}(\Lambda;d,n)$ will denote the associated $M_n(\mathbb{C})$-fibre bundle. Recall the following definitions. The algebra

$\mathbb{S}_0(\mathcal{W})$ is the quotient of $\mathbb{S}_0(d,n)$ by the elements that are identically zero on $\mathcal{W}$, and $\mathbb{I}_0(\mathcal{W})$ is the center of $\mathbb{S}_0(\mathcal{W})$. For a domain $D$ so that $\overline{D} \subset \mathcal{Q}$ is compact, we define $\mathbb{I}_{\mathcal{W}}(D; d, n)$ to be the closure of $\mathbb{I}_0(\mathcal{W})$ in $C(\overline{D})$ and $\mathbb{S}_{\mathcal{W}}(D, \Lambda; d, n)$ to be the closure of $\Lambda\mathbb{S}_0(\mathcal{W})$ in $\Gamma_c(\overline{D}, \mathfrak{M}^*_{\mathcal{W}}(\Lambda; d, n))$.

**Theorem 30.** *Suppose that $\overline{D}$ is the maximal ideal space of $\mathbb{I}_{\mathcal{W}}(D; d, n)$. Then $\mathbb{S}_{\mathcal{W}}(D, \Lambda; d, n)$ is an Azumaya algebra.*

*Proof.* The proof of this theorem is very similar to the proof of Theorem 28. Automatically, the elements of $\mathbb{S}_{\mathcal{W}}(D, \Lambda; d, n)$ satisfy the identities of the $n \times n$ matrices. Now fix $b \in \overline{D}$. Choose $\mathfrak{z} \in \mathcal{W}$ so that $\mathfrak{z}$ is in the fibre over $b$ in the reduced bundle $\mathfrak{C}_{\mathcal{W}}(\Lambda; d, n)$. We have $\mathfrak{z} \in \mathcal{V}(d, n)$, so again using the Lemma 27 there is a $d$-variable central polynomial $p \in \mathbb{G}_0(d, n)$ so that $p(\mathfrak{z}) = I_n$. The polynomial $p$ is a central polynomial for $\mathbb{S}_{\mathcal{W}}(D, \Lambda; d, n)$, and in fact $p$ may be viewed as an element of the Formanek center by identifying it with the evaluation $p\,(\mathcal{Z}_1 + \mathcal{I}(\mathcal{W}), \mathcal{Z}_2 + \mathcal{I}(\mathcal{W}), \ldots, \mathcal{Z}_d + \mathcal{I}(\mathcal{W}))$. Since $p$ is invariant, it may be viewed as a function on $\mathcal{Q}$ so that $p(b) = 1$.

$\overline{D}$ is compact, so choose a finite number of central polynomials $p_1, p_2, \ldots, p_N$ so that $p_1, p_2, \ldots, p_N$ have no common zero. Since $\overline{D}$ is assumed to be the spectrum of $\mathbb{I}_{\mathcal{W}}(D; d, n)$ this implies $1 \in p_1\mathbb{I}_{\mathcal{W}}(D; d, n) + p_2\mathbb{I}_{\mathcal{W}}(D; d, n) + \ldots p_N\mathbb{I}_{\mathcal{W}}(D; d, n)$. Hence $1 \in F(\mathbb{S}_{\mathcal{W}}(D, \Lambda; d, n))\mathbb{S}_{\mathcal{W}}(D, \Lambda; d, n)$, and so $\mathbb{S}_{\mathcal{W}}(D; d, n)$ is an Azumaya algebra over its center. $\square$

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