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On a free boundary problem for ideal, viscous and heat conducting gas flow

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ON A FREE BOUNDARY PROBLEM FOR IDEAL, VISCOUS AND HEAT CONDUCTING GAS FLOW

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the December 2016 graduation.

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ABSTRACT

We consider the flow of an ideal gas with internal friction and heat conduction in a layer between a fixed plane and an upper free boundary. We describe the top free surface as the graph of a time dependent function. This forces us to exclude breaking waves on the surface. For this and other reasons we need to confine ourselves to flow close to a motionless equilibrium state which is fairly easy to compute. The full equations of motion, in contrast to that, are quite difficult to solve. As we are close to an equilibrium, a linear system of equations can be used to approximate the behavior of the nonlinear system.

Analytic, strongly continuous semigroups defined on a suitable Banach space X are used to determine the behavior of the linear problem. A strongly continuous semigroup is a family of bounded linear operators $\{T(t)\}$ on X where $0 \leq t < \infty$ satisfying the following conditions.

1. $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$
2. $T(0) = E$, the identity mapping.
3. For each $x \in X$, $T(t)x$ is continuous in t on $[0, \infty)$.

Then there exists an operator A known as the infinitesimal generator of such that $T(t) = \exp(tA)$. Thus, an analytic semigroup can be viewed as a generalization of the exponential function.

Some estimates about the decay rates are derived using this theory. We then prove the existence of long term solutions for small initial values. It ought to be

emphasized that the decay is not an exponential one which engenders significant difficulties in the transition to nonlinear stability.

PUBLIC ABSTRACT

The subject of this thesis is about the motion of a compressible fluid with internal friction and heat conduction in a layer between a fixed lower boundary and a free upper boundary. We describe the top boundary as the graph of a function. This forces us to exclude breaking waves on the surface. For this and other reasons we need to confine ourselves to flow close to a motionless equilibrium state which is fairly easy to compute. We prove the existence of long term solutions for initial values close to the equilibrium. These solutions approach the equilibrium over time. This result is obtained by considering the linearization of the problem using the theory of analytic semigroups and deriving some estimates about the decay rates of some of the variables. It ought to be emphasized that the decay is not an exponential one which engenders significant difficulties in the transition to nonlinear stability.

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CHAPTER 1
INTRODUCTION

We consider the flow of an ideal heat conducting gas in a time-dependent region Ω^t in $\mathbb{R}^3$. We assume the equation of state of the gas is $p = c\rho\theta$, where p is the pressure, ρ is the density, θ is the temperature of the gas and c is a constant. Its internal energy density per unit mass is $e = c_2\theta$, where c_2 is another constant. We assume that Ω^t has two disjoint boundary components, one, $\Gamma_1^t = \{x \in \mathbb{R}^3 | x_3 = -h\}$ for $h > 0$, which does not move, while the other can be described by the graph of a time dependent function $\varphi(\overline{x}, t) > -h$ when $\overline{x} = (x_1, x_2) \in \mathbb{R}^2, t \in I$ by

$$\Gamma_2^t = \{x \in \mathbb{R}^3, t \in I | x_3 = \varphi(x_1, x_2, t)\}$$

where $I = [0, t] \cap \mathbb{R}$ with $t \in (0, \infty]$. Therefore the domains Ω^t can be described by

$$\Omega^t = \{x \in \mathbb{R}^3 | -h < x_3 < \varphi(x_1, x_2, t)\}$$

and $\Omega^t = \Omega_{\varphi(t)}$,where for a function $\phi : \mathbb{R}^2 \to (-h, \infty)$ we define

$$\Omega_\phi = \{x \in \mathbb{R}^3 | -h < x_3 < \phi(x_1, x_2)\}.$$

Throughout this thesis we will use the convention that domains described by a function will appear as a subscript on Ω whereas superscripts will be used only for numbers. Further assume there is a uniform gravity field with acceleration $-ge_3$ acting on this fluid. Let

$$\Omega = \{(x, t) | t \in I, x \in \Omega^t\}$$

In Ω the motion of this gas is described by the following equations.

$$\begin{aligned}
&\rho_t + \operatorname{div}(\rho v) = 0 \\
&\rho\left(v_t + v\cdot\nabla v\right) - \operatorname{div}\left(\mathbb{T}(v,p)\right) = -\rho g e_3 \\
&\rho c_2\left(\tfrac{\partial}{\partial t} + v\cdot\nabla\right)\theta = -c\theta\rho\operatorname{div}(v) + \operatorname{div}(\kappa\nabla\theta) + \psi
\end{aligned} \tag{1.1}$$

where the temperature $\theta = \theta(x,t)$, the density $\rho = \rho_(x,t)$ and the velocity $v = v(x,t)$ are now functions of x and t. Here we also use the notation

$$\mathbb{T}(v,p) = [-p\delta_{ij} + \mathbb{D}_{ij}(v)] \tag{1.2}$$

and

$$\psi = \sum_{i=1}^{n} \frac{1}{2}\mu\left(v_{ix_j} + v_{jx_i}\right)^2 + (\nu-\mu)\left(\operatorname{div}(v)\right)^2$$

where $\mathbb{D}_{ij}(v) = \mu\left(v_{ix_j} + v_{jx_i}\right) + (\nu-\mu)\,\delta_{ij}\operatorname{div}(v)$ is the stress tensor with viscosity constants μ and ν satisfying $\nu > \dfrac{\mu}{3} > 0$. The constant κ represents the heat conductivity, and we denote the outward unit normal to Ω^t on Γ_2^t by n. On the fixed bottom boundary Γ_1^t we have the velocity $v \equiv 0$ and temperature $\theta = \theta_e$. (This notation is related to the fact that this will also be the temperature of the equilibrium state we are considering.) On the top surface Γ_2^t we assume that there is no heat flow thus $\nabla\theta\cdot n = 0$ and

$$\mathbb{T}(v,p)\cdot n = -\mathbf{p}_0 n. \tag{1.3}$$

This means that if $T_t(x)$ represents the flow of the vector field $v(x,t)$ with

$$\frac{dT_t}{dt} = v(T_t(x),t)$$

and $T_0(x) = x$, then $x \in \Gamma_2^0$ exactly if $T_t(x) \in \Gamma_2^t$ for all $t \in I$. This implies the evolution equation

$$\varphi_t(\overline{x},t) = v_3(\overline{x},\varphi(\overline{x},t),t) - \sum_{k=1}^{2}\varphi_{x_k}(\overline{x},t)v_k(\overline{x},\varphi(\overline{x},t),t) \tag{1.4}$$

for φ. Finally we confine ourselves to functions and domains that are periodic in both x_1 and x_2. This means that with the vectors

$$\mathbf{k}_1 = (k_1,0,0) \text{ and } \mathbf{k}_2 = (0,k_2,0) \text{ for two numbers } k_1,k_2 > 0 \text{ and } i \in \{1,2\} \tag{1.5}$$

we have

$$v(x,t) = v(x+\mathbf{k}_i,t), \;\; \rho(x,t) = \rho(x+\mathbf{k}_i,t), \;\; \theta(x,t) = \theta(x+\mathbf{k}_i,t)$$

and

$$\varphi(x,t) = \varphi(x+\pi(\mathbf{k}_i),t),$$

where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection map $\pi(x) = \pi(x_1,x_2,x_3) = (x_1,x_2)$ for any $x = (x_1,x_2,x_3) \in \mathbb{R}^3$. In order to proceed we need to introduce more notation.

We consider sets Λ such that $\partial\Lambda$ is a C^1 surface and define the translation transformations $\Upsilon_{\mathbf{k_i}} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Upsilon_{\mathbf{k_i}}(x) = x + \mathbf{k}_i,$$

for the translation vectors $\mathbf{k}_i$ defined in equation (1.5). If Λ is a set such that $\Upsilon_{\mathbf{k_i}}(\Lambda) = \Lambda$ we also consider periodic functions f such that $f \circ \Upsilon_{\mathbf{k_i}} = f$. Define $\widetilde{\Lambda}$ as the bounded set

$$\widetilde{\Lambda} = \{x = (x_1,x_2,x_3) | x \in \Lambda, 0 \le x_1 \le k_1, 0 \le x_2 \le k_2\}.$$

Likewise, for $\Omega \subset \mathbb{R}^4$ we define

$$\widetilde{\Omega} = \{(x_1,x_2,x_3,t) | (x,t) \in \Omega, 0 \le x_1 \le k_1, 0 \le x_2 \le k_2\}.$$

Returning to subsets of $\mathbb{R}^3$, considering the case where Λ is the region between the plane $\{x_3 = -h\}$ and the graph of the function $\varphi = 0$, i.e., the set

$$S = \Omega_0 = \left\{x \in \mathbb{R}^3 : -h < x_3 < 0\right\},$$

and following the above convention with the set S we have

$$\widetilde{S} = \left\{x \in \mathbb{R}^3 : 0 \leq x_i \leq k_i \text{ for } i \in \{1,2\}, k_i \in \mathbb{R}^+ \text{ and } -h < x_3 < 0\right\},$$

which represents the periodic cell where the equilibrium functions are defined. The closure of S is given by $\overline{S}$ and we denote the disjoint boundaries on this periodic cell by $\partial_1 S = \{x \in S | x_3 = -h\}$, $\partial_2 S = \{x \in S | x_3 = 0\}$, $\partial_1 \widetilde{S} = \left\{x \in \widetilde{S} | x_3 = -h\right\}$ and $\partial_2 \widetilde{S} = \left\{x \in \widetilde{S} | x_3 = 0\right\}$.

If a function f is invariant under translation the restriction of the function to $\widetilde{\Lambda}$ determines f and the linear mapping $f \to f|_{\widetilde{\Lambda}}$ is one-to-one. If we know what a periodic function f does on $\widetilde{\Lambda}$ then we know what it does everywhere on Λ. Note that this is true for any set of this type that has the periodicity property.

Stationary Solution

Any time independent solution with velocity $v = 0$ and $\varphi(x_1, x_2, t) = 0$ must fulfill the equations

$$\begin{cases} \nabla p & = & -\rho g e_3 & \text{in S} \\ \operatorname{div}(\kappa \nabla \theta) & = & 0 & \text{in } S \\ -p \cdot n & = & -\mathbf{p}_0 n & \text{on } \partial_2 S \\ \dfrac{\partial \theta}{\partial n} & = & 0 & \text{on } \partial_2 S \end{cases} \tag{1.6}$$

First, consider the heat equation $\operatorname{div}(\kappa\nabla\theta) = 0$. This is uncoupled from the remainder of the system (1.6), and since θ is periodic in S we can use the strong maximum principle. This asserts that the maximum or minimum can only occur at an interior point or on Γ^t, thus $\theta = \theta_e$. Thus we have $\mathbf{p}_0 = c\theta_e\rho_0$, which determines $\rho_0 = \dfrac{\mathbf{p}_0}{c\theta_e}$. As we consider an ideal gas, then ρ_e and p_e are easily found to be

$$\rho_e(x_3) = \rho_0 \exp\left(\frac{-gx_3}{c\theta_e}\right) \quad \text{and} \quad p_e = \mathbf{p}_0 \exp\left(\frac{-gx_3}{c\theta_e}\right).$$

Then the equilibrium solutions are

$$v_e = 0, \;\; \theta_e = \theta_0, \;\; \rho_e = \rho_0 \exp\left(\frac{-gx_3}{c\theta_e}\right), \;\; p_e = \mathbf{p}_0 \exp\left(\frac{-gx_3}{c\theta_e}\right) \tag{1.7}$$

Given the height of the box h then the mass M in $\widetilde{S}$ is given by the expression

$$M(h) = k_1k_2 \int_{-h}^{0} \rho(x_3)dx_3. \tag{1.8}$$

Alternatively, given any mass M we can find an h such that there is a solution having that mass. Taking the derivative of equation (1.8) we have

$$M'(h) = k_1k_2\rho(-h) \geq k_1k_2\rho_0 > 0$$

is bounded below. As $h \to \infty$ we have $M(h) \to \infty$ so this is an invertible function with $M(0) = 0$. Therefore any mass bigger than zero corresponds to an h where equation (1.8) holds.

The methods applied here are inspired by those in [6], [7], [8] and [10].

Definition 1.1. *A domain $\mathbf{\Omega}^\omega \subset \mathbb{R}^4$ with the elements $(y,t)\,(y \in \mathbb{R}^3, t \in \mathbb{R})$ is called a regular flow domain for the time interval I_ω if the sets*

$$\left\{y \in \mathbb{R}^3 \mid (y,t) \in \overline{\mathbf{\Omega}}^\omega\right\}$$

are empty when $t \notin I_\omega$, while for $t \in I_\omega$ they are closures of bounded domains Ω^t such that the boundaries $\partial\Omega^t$ are a C^1-submanifolds of $\mathbb{R}^3$. Similarly, the spatial boundary

$$\left\{(y,t) \in \overline{\mathbf{\Omega}}^\omega \mid y \in \partial\Omega^t\right\}$$

of $\mathbf{\Omega}^\omega$ is a C^1-submanifold of $\mathbb{R}^4$ with boundary.

All flows we consider occur in regular flow domains.

Definition 1.2. *A quadruple $(\mathbf{\Omega}^\omega, v, \rho, \theta)$ solves problem (P1) if $\mathbf{\Omega}^\omega$ is a regular flow domain, if for all $\omega_0 \in I_\omega \cap \mathbb{R}$ we have $v \in \widetilde{W}_p^{2,1}\left(\{(y,t) \in \mathbf{\Omega}^\omega : t < \omega_0\}\right), v$, $\nabla v \in C^0\left(\overline{\{(y,t) \in \mathbf{\Omega}^\omega : t < \omega_0\}}\right)$, $\rho, \rho' \in \widetilde{W}_p^{1}\left(\{(y,t) \in \mathbf{\Omega}^\omega : t < \omega_0\}\right)$, $\theta \in \widetilde{W}_p^{2,1}\left(\{(y,t) \in \mathbf{\Omega}^\omega : t < \omega_0\}\right)$, if these functions fulfill (1.1) and (1.3) almost everywhere, and if $\mathbf{\Omega}^\omega$ moves with the flow of v.*

The main objective of this thesis is to prove the following theorem.

Theorem 1.3. *The equilibrium solution stated in (1.7) is stable in the following sense. There exists constants $\eta, \epsilon > 0$ and $C < \infty$ with the following properties. Let the initial state with initial domain Ω^0 be described by*

$$\begin{cases} \varphi_0 \in \widetilde{W}_p^{2-1/p}(\partial S) \quad \varphi_0 : \partial S \to (-h, \infty) \quad \Omega_{\varphi_0} = \Omega^0 \\ \rho_0 \in \widetilde{W}_p^{1}(\Omega^0) \qquad \rho_0 : \overline{\Omega}^0 \to (0,\infty) \\ \theta_0 \in \widetilde{W}_p^{1}(\Omega^0), \qquad \theta_0 : \overline{\Omega}^0 \to (0,\infty) \\ v_0 \in \widetilde{W}_p^{2-2/p}(\partial\Omega^0) \end{cases} \tag{1.9}$$

fulfilling the properties

$$\int_{\Omega^0} \rho_0 dy = M,$$

the compatibility condition $\mathbb{T}(v_0, p(\rho_0, \theta_0)) \cdot n_0 = -\mathbf{p}_0 n_0$ *and the smallness condition*

$$\mathcal{E}_1 = \|\varphi_0\|_{\widetilde{W}_p^{2-1/p}(\partial_2 S)} + \|\rho_0 - \rho_e\|_{\widetilde{W}_p^1(\Omega^0)} + \|\theta_0 - \theta_e\|_{\widetilde{W}_p^1(\Omega^0)} + \|v_0\|_{\widetilde{W}_p^{2-2/p}(\Omega^0)} \leq \eta. \quad (1.10)$$

Then for every $\omega \in [1, \infty]$ *there is a quadruple* $(\mathbf{\Omega}^\omega, v, \rho, \theta)$ *solving problem (P1), i.e.,*

$$\mathbf{\Omega}^\omega = \bigcup_{t \in I_\omega} \left\{(y,t) | y \in \Omega^t\right\}$$

is a regular flow domain with initial conditions

$$\Omega^0 = \Omega_{\varphi_0}, \;\; v(0) = v_0, \;\; \rho(0) = \rho_0, \;\; \theta(0) = \theta_0,$$

and these functions solve equations (1.1) and boundary conditions (1.3) almost everywhere, while $\mathbf{\Omega}^\omega$ *moves with the flow. In addition,*

$$\begin{gathered}\mathcal{E}_2\left(\mathbf{\Omega}^\omega, \tau_0, \rho_0, v_0\right) = \\ \sup_{(x,t) \in \mathbf{\Omega}^\omega} \left(|v(x,t)| + |\nabla v(x,t)|\right) + \sup_{0 \leq t \leq \omega - 1} \left\| |v'| + |\nabla^2 v| \right\|_{\widetilde{L}^p(\{(x,t)|(x,t) \in \Omega^\tau, \, t < \tau < t+1\})} \\ + \sup_{t \in I_\omega} \left(\|\rho(t) - \rho_e\|_{\widetilde{W}_p^1(\Omega^t)} + \|\theta(t) - \theta_e\|_{\widetilde{W}_p^1(\Omega^t)} + \|v(t)\|_{\widetilde{W}_p^{2-2/p}(\Omega^t)} \right) \leq \epsilon.\end{gathered} \quad (1.11)$$

These conditions determine $(\mathbf{\Omega}^\omega, v, \rho, \theta)$ *uniquely, and the solution fulfills the inequality*

$$\begin{aligned}&(1+t)\left(\|\rho(t) - \rho_e\|_{\widetilde{W}_p^1(\Omega^t)} + \|\theta(t)\|_{\widetilde{W}_p^1(\Omega^t)} + \|v(t)\|_{\widetilde{W}_p^{2-2/p}(\Omega^t)} \right) \\ &+ t^{4/3}\left(\|\rho(t) - \rho_e\|_{\widetilde{W}_p^{2/3}(\Omega^t)} + \|\theta(t)\|_{\widetilde{W}_p^{2/3}(\Omega^t)} + \|v(t)\|_{\widetilde{W}_p^{5/3}(\Omega^t)} \right) \leq C\mathcal{E}_1.\end{aligned} \quad (1.12)$$

There also exists a function

$$\varphi : \partial S \times I_\omega \to \mathbb{R}, \varphi \in L_\infty\left(I_\omega, \widetilde{W}_p^{2-1/p}(\partial S)\right) \cap C^1\left(I_\omega, \widetilde{W}_p^{1-1/p}(\partial S)\right)$$

such that $\varphi(0) = \varphi_0, \Omega^t = \Omega_{\varphi(t)}$ *and*

$$\|\varphi(t)\|_{\widetilde{W}_p^{2-1/p}(\partial S)} + t\|\varphi(t)\|_{\widetilde{W}_p^{1-1/p}(\partial S)} + t^{4/3}\|\varphi(t)\|_{\widetilde{W}_p^{2/3-1/p}(\partial S)} \leq C\mathcal{E}_1 \quad (1.13)$$

as well, and φ fulfills the equation (1.4), where the domain moves with the flow. Finally,

$$\int_{\widetilde{\Omega}^t} \rho(t)dy = M \quad \textit{for all } t \in I_\omega.$$

We begin to prove Theorem 1.3 by addressing the linearized system of equations

$$\begin{array}{ll}
\alpha_t + \operatorname{div}(\rho_e u) = 0 & \text{in} S \\
\rho_e u_t + c\nabla(\theta_e \alpha + \rho_e \tau) - \operatorname{div}(\mathbb{D}(u)) = -\alpha g e_3 & \text{in} S \\
c_2 \rho_e \tau_t = -c\theta_e \rho_e \operatorname{div}(u) + \operatorname{div}(\kappa \nabla \tau) & \text{in} S \\
[\mathbb{D}(u) - c\theta_e(\alpha + \rho_{ex_3}\beta)E - c\rho_e \tau E] \cdot e_3 = 0 & \text{on} \partial_2 S \\
\beta_t = u_3 & \text{on} \partial_2 S \\
\frac{\partial \tau}{\partial x_3} = 0 & \text{on} \partial_2 S \\
\tau = 0 & \text{on} \partial_1 S \\
u = 0 & \text{on} \partial_1 S,
\end{array} \tag{1.14}$$

by proving the estimate

$$\|u(t)\|_{\widetilde{L}_p} + t\|u(t)\|_{\widetilde{W}_p^2} + (1+t)(\|\alpha(t)\|_{\widetilde{W}_p^1} + \|\tau(t)\|_{\widetilde{W}_p^1}) + \|\beta(t)\|_{\widetilde{W}_p^{2-1/p}} + t\|\beta(t)\|_{\widetilde{W}_p^{1-1/p}}$$

$$\leq C\left(\|u_0\|_{\widetilde{L}_p} + \|\alpha_0\|_{\widetilde{W}_p^1} + \|\tau_0\|_{\widetilde{W}_p^2} + \|\beta_0\|_{\widetilde{W}_p^{2-1/p}}\right).$$

Remark. The definitions of the spaces $\widetilde{W}_p^k$ and $\widetilde{L}^p$ can be found in the following notation section, Section 1.1.

The proof of Theorem 1.3 will be obtained by a perturbation argument. For a similar problem involving incompressible flow see [5], and [4].

1.1 Notation

We use the standard Sobolev spaces W_p^k for open sets with integer values of k. When we have non-integer values of k, W_p^k is defined by suitable real interpolation spaces, for which the reader may refer to [11]. Define the function spaces on the periodic cells by

$$\widetilde{W}_p^k(\Lambda) = \left\{ f \in W_{p,loc}^k \left(\overline{\Lambda}\right) : f(x + \mathbf{k}_i) = f(x), \ \text{ for } \mathbf{k}_i \text{ as in equation (1.5)} \right\}$$

where $W_{p,loc}^k \left(\overline{\Lambda}\right)$ means that f is in the Sobolev space in a ball around every point in $\overline{\Lambda}$. Using the interpolation couple $\left\{\widetilde{W}^{k,p}, \widetilde{W}^{k+1,p}\right\}$ we can define the Sobolev space with fractional exponents $s \in (k, k+1)$ by

$$\widetilde{W}^{s,p}(S) = \left(\widetilde{W}^{k,p}, \widetilde{W}^{k+1,p}\right)_{s-k,p}. \tag{1.15}$$

Also on the periodic cells we define the closed sets

$$\widetilde{C}^k\left(\Lambda\right) = \left\{ f \in C^k\left(\Lambda\right) : f(x + \mathbf{k}_i) = f(x), \ \text{ for } \mathbf{k}_i \text{ as in equation (1.5)} \right\}$$

and

$$\widetilde{L}^p(\Lambda) = \left\{ f \in L_{loc}^p\left(\Lambda\right) : f(x + \mathbf{k}_i) = f(x), \ \text{ for } \mathbf{k}_i \text{ as in equation (1.5)} \right\}.$$

For these spaces we introduce the norms

$$\|f\|_{\widetilde{W}_p^k(\Lambda)} = \|f|_{\widetilde{\Lambda}}\|_{W_p^k(\widetilde{\Lambda})} \text{ and } \|f\|_{\widetilde{C}^k(\Lambda)} = \|f|_{\widetilde{\Lambda}}\|_{C^k(\widetilde{\Lambda})}.$$

CHAPTER 2
THE LINEARIZATION AND SOME OF ITS PROPERTIES

This chapter is devoted to studying the stability properties of the linearized system of equations (1.1). This will allow us to obtain non-linear stability later. First we derive the linearized system of equations about this specific equilibrium and then we study the decay properties of the solutions to the linearized system of equations.

2.1 Determining the Linearization

To obtain the linearized system of equations we consider solutions

$$(v, \rho, \theta, \varphi)(x, t, \epsilon)$$

which depend on a small parameter ϵ with $|\epsilon| << 1$ such that

$$(v, \rho, \theta, \varphi)(x, t, 0) = (0, \rho_e(x), \theta_e, 0).$$

We define the linearized variables by

$$u(x,t) = \frac{\partial v}{\partial \epsilon}\bigg|_{\epsilon=0}(x,t,0) \quad \alpha(x,t) = \frac{\partial \rho}{\partial \epsilon}\bigg|_{\epsilon=0}(x,t,0) \quad \tau(x,t) = \frac{\partial \theta}{\partial \epsilon}\bigg|_{\epsilon=0}(x,t,0)$$

$$\beta(x,t) = \frac{\partial \varphi}{\partial \epsilon}\bigg|_{\epsilon=0}(x,t,0) \qquad \text{and} \qquad U = [u, \alpha, \tau, \beta, \mathbf{d}].$$

The deformation $\mathbf{d}$ doesn't play a role in the estimates we find in Chapter 2, but will be needed for subsequent chapters.

Theorem 2.1. *Let $\delta > 0$. Assume that $v(x,t,\epsilon), \rho(x,t,\epsilon), \theta(x,t,\epsilon), \varphi(x,t,\epsilon)$ fulfill the equations (1.1) and $\varphi_t(x,t,\epsilon) = u_3 - \varphi_{x_1}u_1 - \varphi_{x_2}u_2$ for $\epsilon \in (-\delta, \delta)$, $t \in I, x \in \Omega_\epsilon^t$, that their initial values $v_0(x,\epsilon) = \epsilon\check{v}(x,t), \rho_0(x,\epsilon) = \rho_e(x) + \epsilon\check{\rho}(x,t), \theta_0(x,\epsilon) = \theta_e + \epsilon\check{\theta}(x,t)$*

for suitable vectors $\check{v}, \check{\rho}$ *and* $\check{\theta}$ *at* $t = 0$ *fulfill* $\int_{\widetilde{\Omega^t}} \rho dx = M$ *for all* ϵ*, that they all have three derivatives, and* $v(x,t,0) = 0, \rho(x,t,0) = \rho_e(x), \theta(x,t,0) = \theta_e, \varphi(x,t,0) = 0$*. Then the quantities* $u(x,t) = \left.\frac{\partial v}{\partial \epsilon}\right|_{\epsilon=0}(x,t,0), \alpha(x,t) = \left.\frac{\partial \rho}{\partial \epsilon}\right|_{\epsilon=0}(x,t,0), \tau(x,t) = \left.\frac{\partial \theta}{\partial \epsilon}\right|_{\epsilon=0}(x,t,0), \beta(x,t) = \left.\frac{\partial \varphi}{\partial \epsilon}\right|_{\epsilon=0}(x,t,0)$ *fulfill the system of equations*

$$\begin{gathered} \alpha_t + \operatorname{div}(\rho_e u) = 0 \\ \rho_e u_t + c\nabla\left(\theta_e \alpha + \rho_e \tau\right) - \operatorname{div}\left(\mathbb{D}(u)\right) = -\alpha g e_3 \\ \rho_e c_2 \tau_t = -c\theta_e \rho_e \operatorname{div}(u) + \operatorname{div}\left(\kappa \nabla \tau\right) \end{gathered} \tag{2.1}$$

and the boundary conditions

$$\begin{gathered} \beta_t = u_3, \\ \left(\mathbb{D}(u) - \left(c\theta_e\left(\alpha + \rho_{ex_3}\beta\right)E + c\rho_e \tau E\right)\right)\cdot n = 0 \text{ and } \tfrac{\partial \tau}{\partial x_3} = 0 \end{gathered} \tag{2.2}$$

on $\partial_2 S$ *and* $u = 0$ *and* $\tau = 0$ *on* $\partial_1 S$*. Here* $n = e_3$ *and in addition*

$$Q\left([u,\alpha,\tau,\beta,\mathbf{d}]\right) = \int_{\widetilde{S}} \alpha dx + \rho_e \int_{\partial_2 \widetilde{S}} \beta d\sigma = 0. \tag{2.3}$$

Proof. For a point x in the interior of S and any $\omega \in (0,\infty)$ we can find a sufficiently small value of ϵ so that $B_\epsilon(x) \subset \Omega^t$ for $t \in [0,\omega]$, as the deformation of the domain disappears as $\epsilon \to 0$. This allows us to directly differentiate our equations with respect to ϵ provided we are away from the boundary. First we do this for the conservation of mass equation $\rho_t + \operatorname{div}(\rho v) = 0$. We have

$$\begin{aligned} \left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0}\left(\rho_t + \operatorname{div}(\rho v)\right)| &= \left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0}\rho_t + \operatorname{div}\left(\left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0}\rho v\right) + \operatorname{div}\left(\rho \left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} v\right) \\ &= \alpha_t + \operatorname{div}\left(\rho_e u\right) \end{aligned}$$

resulting in the linearized version $\alpha_t + \operatorname{div}(\rho_e u) = 0$. Now consider the conservation of momentum equation $\rho\left(v_t + v\cdot\nabla v\right) - \operatorname{div}\left(\mathbb{T}(v,p)\right) + \rho g e_3 = 0$. The expression $v \cdot \nabla v$

contributes nothing to the linearization since it is quadratic in a variable going to zero. Thus,

$$\left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} \left(\rho(v_t + v\cdot\nabla v) - \operatorname{div}\left(\mathbb{T}(v,p)\right) + \rho g e_3\right)$$

$$= \rho v_{\epsilon t}\big|_{\epsilon=0} - \left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} \left(\operatorname{div}\left(-pE + \mathbb{D}(v)\right)\right) + \rho_e\big|_{\epsilon=0}\, g e_3$$

$$= \rho_e u_t + c\nabla\left(\theta_e \alpha + \rho_e \tau\right) - \operatorname{div}\left(\mathbb{D}(u)\right) + \alpha g e_3.$$

Next we address the evolution equation for the temperature

$$\rho c_2\left(\theta_t + v\cdot\nabla\theta\right) + c\theta\rho \operatorname{div}(v) - \operatorname{div}(\kappa\nabla\theta) - \psi = 0.$$

First note that the term $\psi = \sum_{i=1}^{3} \mu\left(v_{ix_j} + v_{jx_i}\right)^2 + (\nu-\mu)\left(\operatorname{div}(v)\right)^2$ does not appear in the linearization for the same reason as $v\cdot\nabla v$ in the previous equation. We can also remove the term $v\cdot\nabla\theta$ since $\theta|_{\epsilon=0} = \theta_e$ is constant. Thus,

$$\left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} \left(\rho c_2\left(\theta_t + v\cdot\nabla\theta\right) + c\theta\rho \operatorname{div}(v) - \operatorname{div}\left(\kappa\nabla\theta\right) - \psi\right)$$

$$= \rho_e c_2 \tau_t + c\theta_e\rho_e \operatorname{div}(u) - \operatorname{div}\left(\kappa\nabla\tau\right).$$

Finally, we compute the linearization for the boundary condition $\mathbb{T}(u,p)\cdot n = -\mathbf{p}_0 n$. For any function $G(x,t,\varphi,\epsilon)$ we have

$$\left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} \left(G(x_1,x_2,\varphi(x_1,x_2,t,\epsilon),t,\epsilon)\right)$$

$$= G_\epsilon\left(x_1,x_2,\varphi(x_1,x_2,t,0),t,0\right) + G_{x_3}\left(x_1,x_2,\varphi(x_1,x_2,t,0),t,0\right) \left.\frac{\partial\varphi}{\partial\epsilon}\right|_{\epsilon=0}.$$

When we describe the variables by

$$v = v\left(x_1,x_2,\varphi,t,\epsilon\right), p = p\left(\rho\left(x_1,x_2,\varphi,t,\epsilon\right), \theta\left(x_1,x_2,\varphi,t,\epsilon\right)\right), \text{ and } n = n(x_1,x_2,\epsilon),$$

then

$$\left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} \left[\mathbb{D}(v) - pE\right] = \mathbb{D}(u) - c\theta_e\left(\alpha + \rho_{ex_3}\frac{\partial\varphi}{\partial\epsilon}\right)E - c\rho_e\left(\frac{\partial\theta}{\partial\epsilon} + \frac{\partial\theta_e}{\partial x_3}\frac{\partial\varphi}{\partial\epsilon}\right)E$$

$$= \mathbb{D}(u) - c\theta_e\left(\alpha + \rho_{ex_3}\beta\right)E - c\rho_e\tau E$$

and

$$\begin{aligned}
0 &= \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} (\mathbb{T}(v,p)\cdot n + \mathbf{p}_0 n) \\
&= \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} (\mathbb{T}(v,p))\cdot e_3 + \mathbb{T}(0,\rho_e)\left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} n + \mathbf{p}_0 \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} n \\
&= \left(\mathbb{D}(u) - c\theta_e\left(\alpha + \rho_{ex_3}\beta\right)E - c\rho_e\tau E\right)\cdot n,
\end{aligned}$$

as $\mathbb{T}(0,\rho_e) = -\mathbf{p}_0$ on $\partial_2 S$. Thus, the linearization of the stress tensor on the boundary is

$$(\mathbb{D}(u) - c\theta_e(\alpha+\rho_{ex_3}\beta)E - c\rho_e\tau E)\cdot n = 0.$$

We also have the linearized boundary condition $u = 0$ on $\partial_1 S$. Lastly, we compute the linearization of the evolution equation

$$\left.\frac{\partial\varphi_t}{\partial\epsilon}\right|_{\epsilon=0} = \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0}(v_3 - \varphi_{x_1}v_1 - \varphi_{x_2}v_2) = v_{3\epsilon},$$

which reduces to $\beta_t = u_3$. The claim that

$$Q\left([u,\alpha,\tau,\beta,\mathbf{d}]\right) = \int_{\widetilde{S}}\alpha dx + \rho_e\int_{\partial_2\widetilde{S}}\beta d\sigma = 0$$

follows from the conservation of mass equation $\int_{\widetilde{\Omega^t}}\rho dx = M$. □

2.2 Some of the Properties of the Linearization

For future reference consider the operators obtained from the linearized system of equations (1.14)

$$\begin{aligned}
L_{11}(u,\alpha,\tau) &= -\rho_e^{-1}\operatorname{div}(\mathbb{D}(u)) + \rho_e^{-1}\nabla(c\theta_e\alpha + c\rho_e\tau) \\
L_{12}(u,\alpha,\tau) &= \rho_e^{-1}\alpha g e_3 \\
L_1(u,\alpha,\tau) &= L_{11} + L_{12} \qquad (2.4)\\
L_2(u,\alpha,\tau) &= \operatorname{div}(\rho_e u) \\
L_3(u,\alpha,\tau) &= cc_2^{-1}\theta_e\operatorname{div}(u) - \rho_e^{-1}c_2^{-1}\operatorname{div}(\kappa\nabla\tau)
\end{aligned}$$

together with the boundary condition

$$\mathbb{D}(u)\cdot n - c\theta_e(\alpha+\rho_{ex_3}\beta)En - c\rho_e\tau En = \mathrm{g}_1. \tag{2.5}$$

We define the operator

$$A = [L_1, L_2, L_3, -u_3, -u]^t\,. \tag{2.6}$$

For $1<p<\infty$ let us define two Banach spaces on S,

$$\widetilde{B} = \left\{[u,\alpha,\tau,\beta,\mathbf{d}] | u\in\widetilde{L}_p(S), \alpha\in\widetilde{W}_p^1(S), \tau\in\widetilde{L}_p(S), \mathbf{d}\in\widetilde{W}_p^2(S), \beta\in\widetilde{W}_p^{2-1/p}(\partial S)\right\}$$

and $B=\left\{U\in\widetilde{B}|Q(U)=0\right\}$, both with the norm

$$\|[u,\alpha,\tau,\beta,\mathbf{d}]\|_{\widetilde{B}} = \|u\|_{\widetilde{L}_p(S)} + \|\alpha\|_{\widetilde{W}_p^1(S)} + \|\tau\|_{\widetilde{L}_p(S)} + \|\mathbf{d}\|_{\widetilde{W}_p^2(S)} + \|\beta\|_{\widetilde{W}_p^{2-1/p}(\partial S)}.$$

Let

$$D(A) = \Big\{[u,\alpha,\tau,\beta,\mathbf{d}]\in\widetilde{B}|u\in\widetilde{W}_p^2, \tau\in\widetilde{W}_p^2, (\mathbb{D}(u)-c\theta_e\alpha-c\rho_e\tau)\cdot n$$
$$= \frac{\partial p_e}{\partial x_3}\beta n \,\mathrm{on}\,\partial S,\ \tau=0\,\mathrm{and}\,u=0\,\mathrm{on}\,\partial_1 S,\ \frac{\partial\tau}{\partial x_3}=0\,\mathrm{on}\,\partial_2 S\Big\}\cap B.$$

We can now write the system of equations 2.1 in the form $U(t)\in D(A)$, $U_t+AU=0$ for a function $U(t)$ defined on the interval I.

The following equation will be used many times throughout the remainder of this chapter.

Lemma 2.2. *Assume that* $u\in\widetilde{C}^2(\overline{S}), \alpha\in\widetilde{C}^1(\overline{S}), \tau\in\widetilde{C}^2(\overline{S})$. *Then*

$$\begin{aligned}\int_{\widetilde{S}}\theta_e\rho_e L_1(u,\alpha,\tau)\overline{u} + c\theta_e^2\rho_e^{-1}\overline{L_2(u,\alpha,\tau)}\alpha + \rho_e c_2\overline{L_3(u,\alpha,\tau)}\tau dx \\ = \int_{\partial_1\widetilde{S}\cap\partial_2\widetilde{S}}\left(c\theta_e^2\alpha\overline{u} + \theta_e\rho_e\tau\overline{u} - \theta_e\overline{u}\cdot\mathbb{D}(u) - \kappa(\nabla\overline{\tau})\tau\right)\cdot n d\sigma \\ + \int_{\widetilde{S}}\sum_{i=1}^{3}\sum_{j=1}^{3}\theta_e\mathbb{D}_{ij}(u)\overline{u}_{jx_i} + \kappa|\nabla\tau|^2 dx.\end{aligned} \tag{2.7}$$

Proof. Here we will use the divergence theorem to verify the desired equation. We begin examining the left hand side of equation (2.7), which we denote by I_1. Using equation (2.4) we have

$$I_1 = \int_{\widetilde{S}} -\theta_e \operatorname{div}(\mathbb{D}(u))\overline{u} + \theta_e \nabla(c\theta_e \alpha + c\rho_e \tau)\overline{u} + \theta_e \alpha g \overline{u}_3 + c\theta_e^2 \rho_e^{-1} \operatorname{div}(\rho_e \overline{u})\alpha$$
$$+ c\rho_e \theta_e \operatorname{div}(\overline{u})\tau - \kappa \operatorname{div}(\nabla \overline{\tau})\tau dx.$$

By the definitions of pressure and entropy density this again equals

$$I_1 = \int_{\widetilde{S}} - \operatorname{div}(\theta_e \mathbb{D}(u)\overline{u}) + \sum_{i=1}^{3}\sum_{j=1}^{3} \theta_e \mathbb{D}_{ij}(u)\overline{u}_{jx_i} + c\theta_e^2 \nabla(\alpha)\overline{u} + \theta_e \rho_e \alpha g \overline{u}_3$$

$$+ c\theta_e^2 \rho_e^{-1} \operatorname{div}(\rho_e \overline{u})\alpha + \theta_e \nabla(c\rho_e \tau)\overline{u} + c\rho_e \theta_e \operatorname{div}(\overline{u})\tau - \operatorname{div}(\kappa(\nabla \overline{\tau})\tau) + \kappa|\nabla \tau|^2 dx.$$

First note that

$$\nabla(c\theta_e \alpha) + \alpha g e_3 = \rho_e \rho_e^{-1} \nabla(c\theta_e \alpha) + \alpha g e_3 = \rho_e \nabla(c\theta_e \rho_e^{-1}\alpha) + c\theta_e \rho_e^{-1} \alpha \nabla(\rho_e) + \alpha g e_3.$$

Since we are only considering an ideal gas, we have in the x_3 direction

$$c\theta_e \rho_e^{-1} \nabla(\rho_e) + g e_3 = -\frac{\rho_e g e_3}{\rho_e} + g e_3 = 0.$$

This allows us to replace the terms $\nabla(c\theta_e \alpha) + \alpha g e_3$ with $\rho_e \nabla(c\theta_e \rho_e^{-1}\alpha)$.

Using the product rule,

$$\theta_e \left(\nabla(c\theta_e \alpha) \cdot \overline{u} + \alpha g \overline{u_3} + c\theta_e \rho_e^{-1} \operatorname{div}(\rho_e \overline{u})\alpha \right)$$

$$= \theta_e \left(\rho_e \nabla(c\theta_e \rho_e^{-1}\alpha)\overline{u} + c\theta_e \rho_e^{-1} \operatorname{div}(\rho_e \overline{u})\alpha \right)$$

$$= \theta_e \operatorname{div}(c\theta_e \alpha \overline{u}) = \operatorname{div}(c\theta_e^2 \alpha \overline{u})$$

and

$$\nabla(c\rho_e \tau)\overline{u} + c\rho_e \operatorname{div}(\overline{u})\tau = \operatorname{div}(c\rho_e \tau \overline{u}).$$

Thus,

$$I_1 = \int_{\widetilde{S}} - \operatorname{div}(\theta_e \mathbb{D}(u)\overline{u}) + \sum_{i=1}^{3}\sum_{j=1}^{3} \theta_e \mathbb{D}_{ij}(u)\overline{u}_{jx_i} + \operatorname{div}(c\theta_e^2 \alpha \overline{u}) + \operatorname{div}(c\theta_e \rho_e \tau \overline{u})$$

$$- \operatorname{div}(\kappa(\nabla\overline{\tau})\tau) + \kappa|\nabla\tau|^2 dx.$$

Now we complete the proof using the divergence theorem

$$I_1 = \int_{\partial_2 S} (c\theta_e^2 \alpha \overline{u} + c\theta_e \rho_e \tau \overline{u} - \theta_e \overline{u} \cdot \mathbb{D}(u) - \kappa(\nabla\overline{\tau})\tau) \cdot n d\sigma$$

$$+ \int_{\widetilde{S}} \sum_{i=1}^{3}\sum_{j=1}^{3} \theta_e \mathbb{D}_{ij}(u)\overline{u}_{jx_i} + \kappa|\nabla\tau|^2 dx.$$

$\square$

2.3 Verifying Ellipticity and the Lopatinsky Shapiro Condition

We need to show that the system of equations

$$L_1 (u, \alpha, \tau) + zu = F_1 \tag{2.8}$$

$$L_2 (u, \alpha, \tau) + z\alpha = F_2 \tag{2.9}$$

$$L_3 (u, \alpha, \tau) + z\tau = F_3 \tag{2.10}$$

is elliptic. For this and for the consideration of the boundary conditions we need to determine the principal part of the system. To this end we need to determine the weights t_j associated with the dependent variables and s_j with the equations. We fix the weights associated with the three components of u as $t_1 = t_2 = t_3 = 2$, and $t_4 = 1$ for α as well as $t_5 = 2$ for τ. Also we choose $s_1 = s_2 = s_3 = 0$ for the equations in (2.8), while $s_4 = -1$ for equation (2.9), finally $s_5 = 0$ for equation (2.10). Clearly no derivative of higher order than $s_i + t_j$ of the corresponding functions appears in the corresponding equations, the highest allowable derivatives for u and τ in (2.8) are the second derivatives, for α the first derivatives. For equation (2.9) only first derivatives of u and τ are allowed and only the function α itself. For the last equation we have the same status as in the first one. Omitting all lower derivatives gives us the principal part in non-divergence form

$$\widehat{L}_1(u,\alpha,\tau) = -\rho_e^{-1}\operatorname{div}(\mathbb{D}(u)) + \rho_e^{-1}c\theta_e\nabla\alpha$$

$$= -\rho_e^{-1}(\mu\Delta u + \nu\nabla\operatorname{div}(u)) + \rho_e^{-1}c\theta_e\nabla\alpha$$

$$\widehat{L}_2(u,\alpha,\tau) = \rho_e\operatorname{div}(u) + z\alpha$$

$$\widehat{L}_3(u,\alpha,\tau) = -\kappa\rho_e^{-1}e_\theta^{-1}\Delta\tau$$

2.3.1 Ellipticity

To check the ellipticity of the system we need to apply the operators to the functions $u = U\exp(i\xi\cdot x), \alpha = A\exp(i\xi\cdot x)$ and $\tau = T\exp(i\xi\cdot x)$. The linear mapping that takes $[U,A,T]\in\mathbb{C}^5$ to

$$\exp(-i\xi\cdot x)[L_1(u,\alpha,\tau), L_2(u,\alpha,\tau), L_3(u,\alpha,\tau)]\in\mathbb{C}^5$$

then has the matrix

$$\frac{1}{\rho_e}\begin{bmatrix} \mu(\xi_2^2+\xi_3^2)+\nu\xi_1^2 & (\nu-\mu)\xi_1\xi_2 & (\nu-\mu)\xi_1\xi_3 & ic\theta_e\xi_1 & 0 \\ (\nu-\mu)\xi_1\xi_2 & \mu(\xi_1^2+\xi_3^2)+\nu\xi_2^2 & (\nu-\mu)\xi_2\xi_3 & ic\theta_e\xi_2 & 0 \\ (\nu-\mu)\xi_1\xi_3 & (\nu-\mu)\xi_2\xi_3 & \mu(\xi_1^2+\xi_2^2)+\nu\xi_3^2 & ic\theta_e\xi_3 & 0 \\ \rho_e^2 i\xi_1 & \rho_e^2 i\xi_2 & \rho_e^2 i\xi_3 & \rho_e z & 0 \\ 0 & 0 & 0 & 0 & \frac{\kappa}{\theta_e}|\xi|^2 \end{bmatrix}.$$

The determinant is $\rho_e^{-3}\kappa e_\theta^{-1}|\xi|^2$ times that of

$$\begin{bmatrix} \mu(\xi_2^2+\xi_3^2)+\nu\xi_1^2 & (\nu-\mu)\xi_1\xi_2 & (\nu-\mu)\xi_1\xi_3 & ic\theta_e\xi_1 \\ (\nu-\mu)\xi_1\xi_2 & \mu(\xi_1^2+\xi_3^2)+\nu\xi_2^2 & (\nu-\mu)\xi_2\xi_3 & ic\theta_e\xi_2 \\ (\nu-\mu)\xi_1\xi_3 & (\nu-\mu)\xi_2\xi_3 & \mu(\xi_1^2+\xi_2^2)+\nu\xi_3^2 & ic\theta_e\xi_3 \\ i\xi_1 & i\xi_2 & i\xi_3 & \rho_e^{-1}z \end{bmatrix}.$$

Adding $i\xi_3(\nu-\mu)$ times the last row to the third one and $i\xi_2(\nu-\mu)$ times the last row to the second one and $i\xi_1(\nu-\mu)$ times the last row to the first one we find this matrix has the same determinant as

$$\begin{bmatrix} \mu|\xi|^2 & 0 & 0 & i\xi_1\left(c\theta_e+\rho_e^{-1}z(\nu-\mu)\right) \\ 0 & \mu|\xi|^2 & 0 & i\xi_2\left(c\theta_e+\rho_e^{-1}z(\nu-\mu)\right) \\ 0 & 0 & \mu|\xi|^2 & i\xi_3\left(c\theta_e+\rho_e^{-1}z(\nu-\mu)\right) \\ i\xi_1 & i\xi_2 & i\xi_3 & \rho_e^{-1}z \end{bmatrix}.$$

Now subtract $\mu^{-1}|\xi|^{-2}i\xi_1$ times the first row from the last one, $\mu^{-1}|\xi|^{-2}i\xi_2$ times the second row from the last one and $\mu^{-1}|\xi|^{-2}i\xi_3$ times the third row from the last one to obtain

$$\begin{bmatrix} \mu|\xi|^2 & 0 & 0 & i\xi_1\left(c\theta_e+\rho_e^{-1}z(\nu-\mu)\right) \\ 0 & \mu|\xi|^2 & 0 & i\xi_2\left(c\theta_e+\rho_e^{-1}z(\nu-\mu)\right) \\ 0 & 0 & \mu|\xi|^2 & i\xi_3\left(c\theta_e+\rho_e^{-1}z(\nu-\mu)\right) \\ 0 & 0 & 0 & \rho_e^{-1}z+\mu^{-1}\left(c\theta_e+\rho_e^{-1}z(\nu-\mu)\right) \end{bmatrix}.$$

The determinant of this matrix is $\mu^2|\xi|^6\rho_e^{-1}(c\theta_e+z\nu)$. The determinant of the entire matrix is therefore

$$\nu\rho_e^{-5}\kappa\theta_e^{-1}\mu^2\left(\theta z+\nu^{-1}c\theta_e\right)|\xi|^8.$$

The system is therefore uniformly elliptic on any domain on which $|\rho_e^{-1}z+\nu^{-1}p_\rho|$ is bounded away from zero, or pointwise where $z\neq c\nu^{-1}\theta_e$.

2.3.2 Boundary Conditions

Owing to the symmetry properties of the equation it is enough to verify this in case of the half-space $\mathbb{R}^3_+=\left\{x\in\mathbb{R}^3_+ \mid x_3>0\right\}$. We need to verify that with $\xi=(\xi_1,\xi_2)$ and $x'=(x_1,x_2)$ every solution $u=U(x_3)\exp(i\xi\cdot x'),\alpha=$

$A(x_3)\exp(i\xi \cdot x'), \tau = T(x_3)\exp(i\xi \cdot x')$ of the equations

$$\widehat{L}_1(u,\alpha,\tau) = 0, \qquad \widehat{L}_2(u,\alpha,\tau) = 0, \quad \widehat{L}_3(u,\alpha,\tau) = 0$$

$$\mathbb{T}(u,p)\cdot n = 0 \text{ and } \frac{\partial \tau}{\partial n} = 0 \qquad \text{on } \partial_2 S$$

$$u = 0 \text{ and } \tau = 0 \qquad \text{on } \partial_1 S$$

with zero boundary conditions on this set that is also decaying as $x_n \to \infty$ is actually zero. As we have two different kinds of boundary conditions, we need to verify it for both. Again, for reasons of symmetry we can confine ourselves to the case $\xi = (\xi_1, 0)$ with $\xi_1 > 0$. As the equation for the temperature splits off, we immediately obtain $\tau = 0$ for Dirichlet and Neumann boundary conditions. Now we draw conclusions that are true for any boundary condition. We have

$$0 = \rho_e L_1(u,\alpha,0) = -(\mu\Delta u + \nu\nabla \operatorname{div}(u)) + c\theta_e \nabla\alpha,$$

thus

$$-\Delta u + \nabla\left(\mu^{-1}(-\nu \operatorname{div}(u) + c\theta_e\alpha)\right) = 0,$$

and with

$$\widetilde{\alpha} = \mu^{-1}(-\nu \operatorname{div}(u) + c\theta_e\alpha) = \widetilde{A}(x_3)\exp(i\xi\cdot x')$$

we have

$$-\Delta u + \nabla\widetilde{\alpha} = 0$$

and from

$$0 = L_2(u,\alpha,0) = \rho_e \operatorname{div}(u) + z\alpha,$$

we have

$$\frac{z}{\rho_e}\alpha = -\operatorname{div}(u), \tag{2.11}$$

$$\frac{z}{\rho_e}\widetilde{\alpha} = \mu^{-1}\left(-\frac{z}{\rho_e}\nu\operatorname{div}(u) + \frac{z}{\rho_e}c\theta_e\alpha\right) = -\mu^{-1}\left(\frac{z}{\rho_e}\nu + c\theta_e\right)\operatorname{div}(u),$$

thus

$$-\operatorname{div}(u) = \frac{\mu z}{z\nu + \rho_e c\theta_e}\widetilde{\alpha}.$$

If we define

$$w = \frac{\mu z}{z\nu + \rho_e c\theta_e} = \frac{\mu z\left(\overline{z}\nu + \rho_e c\theta_e\right)}{\left|z\nu + \rho_e c\theta_e\right|^2} = \frac{\mu\nu\left|z\right|^2\left(1 + z\rho_e c\theta_e\right)}{\left|z\nu + \rho_e c\theta_e\right|^2},$$

then clearly $\operatorname{Re}(w) \geq 0$ and

$$-\Delta u + \nabla\widetilde{\alpha} = 0,\ -\operatorname{div}(u) = w\widetilde{\alpha}.$$

Taking the divergence of the first equation and using the second one we have

$$0 = \operatorname{div}\left[-\Delta u + \nabla\widetilde{\alpha}\right] = (w+1)\,\Delta\widetilde{\alpha},$$

thus $\Delta\widetilde{\alpha} = 0$. Therefore

$$\widetilde{A}''(x_3) = \xi_1^2\widetilde{A}(x_3)$$

and

$$\widetilde{A}(x_3) = C_1 e^{\xi_1 x_3} + C_2 e^{-\xi_1 x_3}.$$

As we only need to consider solutions with $\widetilde{A}(x_3) \to 0$ as $x_3 \to \infty$ we have

$$\widetilde{A}(x_3) = C_2 e^{-\xi_1 x_3},$$

and

$$\widetilde{\alpha}_{x_3} = -\xi_1\widetilde{\alpha}.$$

As

$$\Delta u_1 = \widetilde{\alpha}_{x_1}, \Delta u_3 = \widetilde{\alpha}_{x_3},$$

we have with $v = u_1 + iu_3$

$$\Delta v = \Delta (u_1 + iu_3) = \widetilde{\alpha}_{x_1} + i\widetilde{\alpha}_{x_3} = 0.$$

With $v(x) = C \exp(i\xi_1) V(x_3)$ we also have

$$\Delta V'' - \xi_1^2 V = 0,$$

thus also

$$V(x_3) = D_1 e^{\xi_1 x_3} + D_2 e^{-\xi_1 x_3},$$

$$V(x_3) = D_2 e^{-\xi_1 x_3}$$

and

$$v_{x_3} = -\xi_1 v$$

Therefore also

$$u_{3x_3} = \operatorname{div}(u) - u_{1x_1} = -w\widetilde{\alpha} - i\xi_1 u_1 = -w\widetilde{\alpha} - i\xi_1 v - \xi_1 u_3$$

and

$$u_{3x_3x_3} = -w\widetilde{\alpha}_{x_3} - i\xi_1 v_{x_3} - \xi_1 u_{3x_3}$$

$$= w\xi_1\widetilde{\alpha} + i\xi_1^2 v - \xi_1 (-w\widetilde{\alpha} - i\xi_1 v - \xi_1 u_3) = 2w\xi_1\widetilde{\alpha} + 2i\xi_1^2 v + \xi_1^2 u_3.$$

Now also

$$0 = -\Delta u_3 + \widetilde{\alpha}_{x_3} = -\left(2w\xi_1\widetilde{\alpha} + 2i\xi_1^2 v + \xi_1^2 u_3\right) + \xi_1^2 u_3 - \xi_1\widetilde{\alpha} =$$

$$-\xi_1 (2w+1)\widetilde{\alpha} - 2i\xi_1^2 v,$$

thus

$$v = -\frac{2w+1}{2i\xi_1}\widetilde{\alpha},$$

and

$$u_{3x_3} = -w\widetilde{\alpha} + \frac{2w+1}{2}\widetilde{\alpha} - \xi_1 u_3 = \frac{1}{2}\widetilde{\alpha} - \xi_1 u_3.$$

Also

$$u_{1x_3} = v_{x_3} - iu_{3x_3} = -\xi_1\left(-\frac{2w+1}{2i\xi_1}\widetilde{\alpha}\right) - i\left(\frac{1}{2}\widetilde{\alpha} - \xi_1 u_3\right) =$$
$$\frac{2w+1}{2i}\widetilde{\alpha} - \frac{i}{2}\widetilde{\alpha} + i\xi_1 u_3 = \left[\frac{2w+1}{2i} + \frac{1}{2i}\right]\widetilde{\alpha} + i\xi_1 u_3 = -i\left(1+w\right)\widetilde{\alpha} + i\xi_1 u_3.$$

Now we address the conditions for the top boundary. We have

$$u_{1x_3}\left(x_1, x_2, 0\right) = u_{3x_1}\left(x_1, x_2, 0\right) = i\xi_1 u_3\left(x_1, x_2, 0\right),$$

thus

$$i\xi_1 u_3\left(x_1, x_2, 0\right) = -i\left(1+w\right)\widetilde{\alpha}\left(x_1, x_2, 0\right) + i\xi_1 u_3\left(x_1, x_2, 0\right)$$

and $\widetilde{\alpha}\left(x_1, x_2, 0\right) = 0$ thus $\widetilde{\alpha} = 0$ and $v = 0$. For Dirichlet boundary conditions we have $u = 0$ on the boundary and therefore $v = 0$ on the boundary. Thus $v = 0$ everywhere and therefore $\widetilde{\alpha} = v = 0$. Thus also $\Delta u = 0$ and therefore

$$U_k\left(x_3\right) = C_k^1 e^{-\xi_1 x_3} + C_k^2 e^{\xi_1 x_3}$$

and

$$U_k\left(x_3\right) = C_k^1 e^{-\xi_1 x_3}.$$

For Dirichlet boundary conditions this implies $u = 0$. Otherwise, as $\widetilde{\alpha} = 0$ we have

$$\nu \operatorname{div}\left(u\right) = c\theta_e \alpha,$$

from the boundary conditions we also have

$$0 = -2\mu u_{3x_3}\left(0\right) + c\theta_e \alpha\left(0\right),$$

thus, also using $v = 0$,

$$0 = -2\mu u_{3x_3}(0) + \nu \operatorname{div}(u) = -2\mu u_{3x_3}(0) + \nu (u_{1x_1} + u_{3x_3})$$

$$= 2\mu\xi_1 u_3(0) + \xi_1\nu(iu_1 - u_3) = 2\mu\xi_1 u_3(0).$$

Thus $u_3 = u_1 = 0$. Also

$$0 = \mathbb{D}_{32}(u) = \exp(i\xi \cdot x')\,\mu U_2',$$

which implies $U_2'(0) = 0$ and therefore $u_2 = 0$, for Dirichlet boundary conditions we have $U_2(0) = 0$ and therefore $u_2 = 0$.

The remainder of this chapter is devoted to studying the operator A.

2.4 The Operator $A + zE$ is Injective

This section is devoted to showing the operator $A+zE$ for $z \neq 0$ and $Re(z) \geq 0$ is injective. This will be used in Section 2.6 to determine what values of z belong to the resolvent set of A. First we address the integral condition $Q(U) = 0$.

Lemma 2.3. *If* $U = [u, \alpha, \tau, \beta, \mathbf{d}] \in D(\widetilde{A})$, *then* $Q(\widetilde{A}U) = 0$.

Proof. Using (2.3) we have

$$Q(\widetilde{A}U) = \int_{\widetilde{S}} (-\operatorname{div}(\rho_e u))dx + \rho_e \int_{\partial_2\widetilde{S}} u_3 d\sigma.$$

By the divergence theorem we have

$$Q(\widetilde{A}U) = -\int_{\partial_2\widetilde{S}} (\rho_e u) \cdot n d\sigma + \rho_e \int_{\partial_2\widetilde{S}} u_3 d\sigma = 0.$$

Thus, $Q(\widetilde{A}U) = 0$. □

We also reformulate the condition $Q([u,\alpha,\tau,\beta,\mathbf{d}]) = 0$ in the language of Hilbert spaces. To do this, consider the Hilbert space

$$\mathfrak{h} = L^2(\widetilde{S}) \times L^2(\widetilde{S}) \times L^2(\widetilde{S}) \times L^2(\partial\widetilde{S}) \times L^2(\widetilde{S})$$

along with the scalar product $(\cdot,\cdot)_{\mathfrak{h}}$ defined by

$$\left([u,\alpha,\tau,\beta,\mathbf{d}],[\widetilde{u},\widetilde{\alpha},\widetilde{\tau},\widetilde{\beta},\widetilde{\mathbf{d}}]\right)_{\mathfrak{h}} = \int_{\widetilde{S}} \rho_e(x)\left(\overline{u}\widetilde{u} + \overline{\mathbf{d}}\widetilde{\mathbf{d}}\right) + \overline{\alpha}\widetilde{\alpha} + \overline{\tau}\widetilde{\tau}dx + \rho_e(h)\int_{\partial_2\widetilde{S}} \overline{\beta}\widetilde{\beta}d\sigma.$$

Then $Q([u,\alpha,\tau,\beta,\mathbf{d}]) = 0$ exactly if $[u,\alpha,\tau,\beta,\mathbf{d}]$ is orthogonal to the subspace $\mathcal{N}$ of $\mathfrak{h}$ spanned by the vector $[0,1,0,1,0]$.

Lemma 2.4. *Assume* $Re(z) \geq 0, u \in \widetilde{W}_p^2(S), \alpha \in \widetilde{W}_p^1(S), \tau \in \widetilde{W}_p^2(S), \beta \in \widetilde{W}_p^{2-1/p}(\partial S)$ *fulfill the system of equations*

$$L_1(u,\alpha,\tau) + zu = 0 \quad L_2(u,\alpha,\tau) + z\alpha = 0 \quad L_3(u,\alpha,\tau) + z\tau = 0$$

with the boundary conditions

$$\begin{aligned} &\beta_t = u_3, \\ &(\mathbb{D}(u) - (c\theta_e(\alpha + \rho_{ex_3}\beta)E + c\rho_e\tau E))\cdot n = 0 \text{ and } \tfrac{\partial\tau}{\partial x_3} = 0 \end{aligned} \tag{2.12}$$

on $\partial_2 S$*,* $u = 0$ *and* $\tau = 0$ *on* $\partial_1 S$ *and*

$$Q([u,\alpha,\tau,\beta,\mathbf{d}]) = \int_{\widetilde{S}} \alpha dx + \rho_e \int_{\partial_2\widetilde{S}} \beta d\sigma = 0.$$

Then $u = 0, \alpha = 0, \tau = 0, \beta = 0.$

Proof. As $(A+zE)U=0$ we use Lemma 2.2 to begin with the estimate

$$
\begin{aligned}
&0=\\
&\int_{\widetilde{S}} \theta_e\rho_e(L_1(u,\alpha,\tau)+zu)\overline{u}+c\theta_e^2\rho_e^{-1}\overline{(L_2(u,\alpha,\tau)+z\alpha)}\alpha+\rho_e c_2\overline{(L_3(u,\alpha,\tau)+z\tau)}\tau dx\\
&=\int_{\partial_2\widetilde{S}} \left(c\theta_e^2\alpha\overline{u}+c\theta_e\rho_e\tau\overline{u}-\theta_e\overline{u}\cdot\mathbb{D}(u)-\kappa(\nabla\overline{\tau})\tau\right)\cdot n d\sigma\\
&+\int_{\widetilde{S}}\sum_{i=1}^{3}\sum_{j=1}^{3}\theta_e\mathbb{D}_{ij}(u)\overline{u}_{jx_i}+c_2\kappa|\nabla\tau|^2+\theta_e\rho_e z|u|^2+c\theta_e^2\rho_e^{-1}\overline{z}|\alpha|^2\\
&\qquad +\rho_e c_2\overline{z}|\tau|^2 dx.
\end{aligned}
\tag{2.13}
$$

First note that $(\tau\nabla\overline{\tau})\cdot n=0$. Using this, as well as the boundary condition $u_3=z\beta$, the surface integral becomes

$$
\begin{aligned}
\int_{\partial_2\widetilde{S}}(\overline{u}\cdot[\mathbb{D}(\overline{u})-c\theta_e\alpha E-c\rho_e\tau E])n &= \int_{\partial_2\widetilde{S}}(-c\theta_e(\rho_{ex_3}\beta E))\overline{u}\cdot n ds\\
&=\int_{\partial_2\widetilde{S}}-c\theta_e(\rho_{ex_3})|\beta|^2\overline{z}ds=\int_{\partial_2\widetilde{S}}\rho_e g\overline{z}|\beta|^2 ds.
\end{aligned}
$$

Thus, equation (2.13) reduces to

$$
\begin{aligned}
&\int_{\partial_2\widetilde{S}} c\rho_0 c_2\rho_e g\overline{z}|\beta|^2 ds+\\
&\int_{\widetilde{S}}\sum_{i,j=1}^{3}\theta_e\mathbb{D}_{ij}(u)\overline{u}_{jx_i}+\theta_e\rho_e z|u|^2+c\theta_e^2\rho_e^{-1}\overline{z}|\alpha|^2+\kappa|\nabla\tau|^2+\rho_e c_2\overline{z}|\tau|^2 dx=0.
\end{aligned}
\tag{2.14}
$$

Looking at the term $\sum_{i,j=1}^{3}\mathbb{D}_{ij}(u)\overline{u}_{jx_i}$ we have

$$
\begin{aligned}
\sum_{i,j=1}^{3}\mathbb{D}_{ij}(u)\overline{u}_{jx_i} &= \sum_{i,j=1}^{3}\mu\left(u_{ix_j}+u_{jx_i}\right)\overline{u}_{jx_i}+(\nu-\mu)|\operatorname{div}(u)|^2\\
&= \frac{\mu}{2}\sum_{i,j=1}^{3}\left|u_{ix_j}+u_{jx_i}\right|^2+(\nu-\mu)|\operatorname{div}(u)|^2.
\end{aligned}
$$

Now we show that

$$\frac{\mu}{2}\sum_{i,j=1}^{3}\left|u_{ix_j}+u_{jx_i}\right|^2+(\nu-\mu)|\operatorname{div}(u)|^2\geq\zeta\sum_{i,j=1}^{3}\left|u_{ix_j}+u_{jx_i}\right|^2 \tag{2.15}$$

for some $\zeta > 0$, which implies that $\sum_{i,j=1}^{3}\theta_e\mathbb{D}_{ij}(u)\overline{u}_{jx_i} \geq 0$. If $\nu > \mu$ then from inequality (2.15) the expression $(\nu-\mu)|\operatorname{div}(u)|^2 \geq 0$ and the inequality is clearly true with $\zeta = \frac{\mu}{2}$. If $\nu \leq \mu$ then $\nu - \mu \leq 0$, and using $\operatorname{div}(u) = \sum_{i=1}^{3} 1u_{ix_i}$ we have by the Cauchy-Schwartz inequality that

$$|\operatorname{div}(u)| \leq \sqrt{3}\left(\sum_{i=1}^{3}|u_{ix_i}|^2\right)^{1/2}.$$

Then

$$\begin{aligned}&\frac{\mu}{2}\sum_{i,j=1}^{3}\left|u_{ix_j}+u_{jx_i}\right|^2+(\nu-\mu)|\operatorname{div}(u)|^2\\ \geq&\frac{\mu}{2}\sum_{i,j=1}^{3}\left|u_{ix_j}+u_{jx_i}\right|^2+3(\nu-\mu)\sum_{i=1}^{3}|u_{ix_i}|^2.\end{aligned} \tag{2.16}$$

Splitting up the first summation in the right-hand side of inequality (2.16) into two pieces, one where $i = j$ and one where $i \neq j$, we have

$$\begin{aligned}&\frac{\mu}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{3}\left|u_{ix_j}+u_{jx_i}\right|^2+2\mu\sum_{i=1}^{3}|u_{ix_i}|^2+3(\nu-\mu)\sum_{i=1}^{3}|u_{ix_i}|^2\\ &=\frac{\mu}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{3}\left|u_{ix_j}+u_{jx_i}\right|^2+(3\nu-\mu)\sum_{i=1}^{3}|u_{ix_i}|^2.\end{aligned}$$

Since $\mu > 0$ and $3\nu - \mu > 0$ we take $\zeta = \min\left\{\frac{\mu}{2}, 3\nu-\mu\right\} > 0$ and have verified inequality (2.15). Thus, each of the summands of equation (2.14) has non-negative real part, and it must be the case that the real part of each summand is zero. Then

$\nabla\tau = 0$, therefore τ is constant and must be zero as $\tau = 0$ on the bottom. Looking at the remaining terms involving the velocity u, we have

$$\theta_e \int_{\widetilde{S}} \sum_{i=1}^{3} \sum_{j=1}^{3} \mathbb{D}_{ij}(u)\overline{u}_{jx_i} + \theta_e \rho_e z|u|^2 dx = 0.$$

As $\frac{\mu}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \left|u_{ix_j} + u_{jx_i}\right|^2 + (3\nu - \mu) \sum_{i=1}^{3} |u_{ix_i}|^2 = 0$ we immediately find that $u_{ix_i} = 0$ and $u_{ix_j} + u_{jx_i} = 0$, so $u_{ix_j} = -u_{jx_i}$ for all $i, j \in [1, 3]$. Thus

$$u_{1x_2} = -u_{2x_1}, u_{2x_3} = -u_{3x_2}, u_{1x_3} = -u_{3x_2}.$$

Since $u_{ix_i} = 0$ we also have the second derivatives

$$u_{ix_ix_k} = u_{ix_kx_i} = -u_{kx_ix_i} = 0$$

for $i \in [1, 3]$. It remains to show that $u_{ix_jx_k} = 0$ for $i \neq j, j \neq k$ and $i \neq k$ when $i, j, k \in [1, 3]$.

$$u_{ix_jx_k} = -u_{jx_ix_k} = -u_{jx_kx_i} = u_{kx_jx_i} = u_{kx_ix_j} = -u_{ix_kx_j} = -u_{ix_jx_k},$$

which can only happen when $u_{ix_jx_k} = 0$ for all $i \neq j \neq k$ and $i, j, k \in [1, 3]$. Since the second derivatives of u are all zero and the first derivatives form an anti-symmetric matrix the velocity can be described by a translation and rotation of the fluid,

$$u = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 + h \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \Gamma_1^t, \tag{2.17}$$

which is zero at the bottom as $u \equiv 0$ in Γ_1^t. As $x = (0, 0, -h) \in \Gamma_1^t$ and $u = 0$ in Γ_1^t we conclude that the vector $b = 0$ in (2.17) as well. Now on Γ_1^t we have

$$u = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\omega_3 x_2 \\ \omega_3 x_1 \\ \omega_1 x_2 - \omega_2 x_1 \end{bmatrix} = 0.$$

Observe that $\omega_3 x_2 = 0 = \omega_3 x_1$, forces $\omega_3 = 0$. We also arrive at $\omega_1 = \omega_2 = 0$ since $\omega_1 x_2 - \omega_2 x_1 = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$ and x_1 and x_2 are linearly independent. Thus, $\omega = 0$. Therefore, $u = 0$. If $z \neq 0$, then since $u = \tau = 0$ we have $0 = L_2(0, \alpha, 0, \beta) + z\alpha = z\alpha$, thus $\alpha = 0$. Now using $u = \alpha = \tau = 0$ in the boundary condition we have $\rho_{ex_3} \beta E \cdot n = 0$ and thus $\alpha = 0$. If $z = 0$ it remains to show that both $\alpha = 0$ and $\beta = 0$. We have

$$L_1(0, \alpha, 0) = \nabla(c\theta_e \alpha) + \alpha g e_3 = c\theta_e \nabla\alpha + \alpha g e_3 = 0,$$

which gives us $\alpha_{x_1} = \alpha_{x_2} = 0$ and $\alpha_{x_3} = -\dfrac{g}{c\theta_e}\alpha$. Thus, $\alpha = 0$ or $\dfrac{\alpha_{x3}}{\alpha} = -\dfrac{g}{c\theta_e}$. Assume $\dfrac{\alpha_{x3}}{\alpha} = -\dfrac{g}{c\theta_e}$. Then $\alpha = \tilde{c}\exp\left(-\dfrac{g}{c\theta_e}x_3\right)$. On the boundary

$$0 = c\theta_e(\alpha + \rho_{ex_3}\beta) \cdot e_3 = c\theta_e(\alpha + \rho_{ex_3}\beta) \cdot e_3$$

so $\alpha = -\rho_{ex_3}\beta$ and $\beta_{x_1} = \beta_{x_2} = 0$. Here we note that α and β are of the same sign as $\rho_{ex_3} < 0$. Using this fact in the linearized conservation of mass equation (2.3) as $Q(U) = 0$ we have the sum

$$\int_{\widetilde{S}} \alpha dx + \rho_e \int_{\partial_2 S} \beta d\sigma = 0.$$

Each summand has the same sign (recall $\rho_{x_3} < 0$) both integrals must also be zero. As β is constant, $\beta = 0$ and $\alpha(x_1, x_2, 0) = 0$ gives $\alpha = 0$. $\square$

2.5 Preliminary Results

For $z \neq 0$ we solve the equation $L_2(u, \alpha, \tau) + zu = 0$ for u and elminate both α and τ to obtain a problem involving the operator $Lu = -\rho_e^{-1}\,\mathrm{div}(\mathbb{D}(u))$ and its corresponding boundary condition $\mathbb{D}(u) \cdot n = \mathrm{g}$. Then we can use the Existence and

Uniqueness results for the operator Lu and boundary condition $\mathbb{D}(u) \cdot n = \mathrm{g}$ given in [6] as a building block for gaining similar results about the entire operator A. For the convenience of the reader we will summarize those results here.

Lemma 2.5. *The equation $Lu + zu = f$, where $Lu = -\rho_e^{-1}(x)\,\mathrm{div}(\mathbb{D}(u))$ is elliptic, and the boundary condition $\mathbb{D}(u) \cdot n = \mathrm{g}$ is complementing.*

Proof. For the proof of this lemma see Lemma 8 in [6]. □

Lemma 2.6. *For $Re(z) > 0$ the equation $Lu + zu = f$ on the domain S with the boundary condition $\mathbb{D}(u) \cdot n = \mathrm{g}$ on $\partial_2 S$ and $u = 0$ on $\partial_1 S$ has at most one solution for each f and given boundary values.*

Proof. A proof of this is given in Lemma 9 in [6]. □

Lemma 2.7. *Assume $Re(z) > 0$. Then for every $f \in \widetilde{L}^p(S)$ and $\mathrm{g} \in \widetilde{W}_p^{1-1/p}(\partial_2 S)$ there is exactly one $u \in \widetilde{W}_p^2(S)$ such that $Lu + zu = f$ in S which also fulfills the boundary condition $\mathbb{D}(u) \cdot n = \mathrm{g}$.*

Lemma 2.6 guarantees that we have a unique solution of the differential equation provided that we are in the bounded domain S. Lemma 2.7 gives us uniqueness for sufficiently regular g with the correct boundary values. A proof of Lemma 2.7 can be found after Lemma 9 of [6] and for the proof of Lemma 2.8 see Lemma 10 in [6].

Lemma 2.8. *There exists constants C and M so that for $u \in \widetilde{W}_p^2(S)$ with boundary condition $u|_{\partial_1 S} = 0$ we have*

$$|z|\|u\|_{\widetilde{L}_p} + |z|^{1/2}\|u\|_{\widetilde{W}_p^1} + \|u\|_{\widetilde{W}_p^2} \leq C\|Lu + zu\|_{\widetilde{L}_p} + |z|^{1/2-1/2p}\|\mathbb{D}(u) \cdot n\|_{\widetilde{W}_p^{1-1/p}(\partial S)}$$

for $|z| > M$ *with* $Re(z) \geq 0$.

This lemma gives the first asymptotic estimate for sufficiently large z and will be used in proving a similar result for the operator A. Likewise we have the following lemma.

Lemma 2.9. *There exists constants* C *and* M *so that for* $u \in \widetilde{W}_p^2(S)$ *with boundary condition* $\tau|_{\partial_1 S} = 0$ *we have*

$$|z|\|\tau\|_{\widetilde{L}_p} + |z|^{1/2}\|\tau\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^2} \leq C\| - \Delta\tau + z\tau\|_{\widetilde{L}_p} + |z|^{1/2-1/2p}\left\|\frac{\partial\tau}{\partial n}\right\|_{\widetilde{W}_p^{1-1/p}}. \quad (2.18)$$

for $|z| > M$ *with* $Re(z) \geq 0$.

2.6 Estimates and Existence of the Resolvent

Now that we know the operator $AU + zU$ is injective for $z \neq 0$ and $Re(z) \geq 0$ we would like to find estimates and determine which nonzero values of z make the operator bijective. For this we need to find estimates and determine which nonzero values of z make this operator surjective. This can be broken up into three cases, depending on whether z is small, intermediate valued or quite large. First we prove the corresponding a-priori estimates.

Lemma 2.10. *There exists constants* C *and* M *so that for* $u \in \widetilde{W}_p^2(S)$ *and* $\tau \in \widetilde{W}_p^2(S)$ *with boundary conditions*

$$u = 0 \text{ and } \tau = 0 \quad \text{on } \partial_1 S$$

the inequality

$$|z|\|u\|_{\widetilde{L}_p} + |z|^{1/2}\|u\|_{\widetilde{W}_p^1} + \|u\|_{\widetilde{W}_p^2} + |z|\|\tau\|_{\widetilde{L}_p} + |z|^{1/2}\|\tau\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^2}$$

$$\leq C\left(\|Lu + zu\|_{\widetilde{L}_p} + \|-\Delta\tau + z\tau\|_{\widetilde{L}_p}\right)$$

$$+C\left(|z|^{1/2-1/2p}\|\mathbb{D}(u)\cdot n\|_{\widetilde{W}_p^{1-1/p}(\partial_2 S)} + |z|^{1/2-1/2p}\left\|\frac{\partial\tau}{\partial n}\right\|_{\widetilde{W}_p^{1-1/p}(\partial_2 S)}\right)$$

holds when $|z| > M$ *and* $Re(z) \geq 0$.

Proof. The proof of this simply combines the results of Lemmas 2.8 and 2.9. □

Theorem 2.11. *There exists constants* $C, M < \infty$, *such that if* $F_1, F_3 \in \widetilde{L}_p(S)$, $F_2 \in \widetilde{W}_p^1(S)$, $\mathrm{g}_1 \in \widetilde{W}_p^{1-1/p}(\partial_2 S)$, $Re(z) \geq 0$, $|z| > M$, *and if* (u, α, τ, β) *is a solution of*

$$L_1(u,\alpha,\tau) + zu = F_1, \quad L_2(u,\alpha,\tau) + z\alpha = F_2, \quad L_3(u,\alpha,\tau) + z\tau = F_3$$

with the boundary conditions

$$\tau = 0 \text{ and } u = 0 \quad \text{on } \partial_1 S$$

$$\frac{\partial\tau}{\partial n} = \mathrm{g}_2, \;\; (\mathbb{D}(u) - c\theta_e(\alpha + \rho_{ex_3}\beta)E - c\rho_e\tau E)\cdot n = \mathrm{g}_1, \;\text{ and } z\beta = u_3$$

on $\partial_2 S$ *then the following inequality holds*

$$\|u\|_{\widetilde{W}_p^2} + (|z|+1)(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{L}_p}) + \|\tau\|_{\widetilde{W}_p^2}$$

$$\leq C\left(\|F_1\|_{\widetilde{L}_p} + \|F_2\|_{\widetilde{W}_p^1} + \|F_3\|_{\widetilde{L}_p} + |z|^{1/2-1/2p}\left(\|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}\right)\right)$$

and the operator $A + zE : D(A) \to B$ *is bijective.*

Proof. First solve the second equation for α to obtain $\alpha = z^{-1}F_2 - z^{-1}\operatorname{div}(\rho_e u)$ and substitute it into the first equation. At the same time one can solve the second boundary condition for $\beta, \beta = z^{-1}u_3$ and substitute that into the first boundary condition. The resulting system of equations becomes

$$Lu + zu =$$

$$F_1 - \rho_e^{-1} c\nabla \cdot (\theta_e(z^{-1}F_2 - z^{-1}\operatorname{div}(\rho_e u)) + \rho_e\tau) - \rho_e^{-1}(z^{-1}F_2 - z^{-1}\operatorname{div}(\rho_e u))ge_3$$

$$-\Delta\tau + z\tau = \frac{F_3}{\kappa} - \frac{c\rho_e\theta_e}{\kappa}\operatorname{div}(u)$$

with boundary condition

$$\mathbb{D}(u) \cdot n = \mathrm{g}_1 + c(\theta_e(z^{-1}F_2 - z^{-1}\operatorname{div}(\rho_e u) - \rho_{ex_3}(z^{-1}u_3)) + \rho_e\tau) \cdot n.$$

Using Lemma 2.10 we get

$$|z|\|u\|_{\widetilde{L}_p} + |z|^{1/2}\|u\|_{\widetilde{W}_p^1} + \|u\|_{\widetilde{W}_p^2} + |z|\|\tau\|_{\widetilde{L}_p} + |z|^{1/2}\|\tau\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^2}$$

$$\leq C\Bigg(\|Lu + zu\|_{\widetilde{L}_p} + \| - \Delta\tau + z\tau\|_{\widetilde{L}_p}$$

$$+|z|^{1/2-1/2p}\left(\|\mathbb{D}(u) \cdot n\|_{\widetilde{W}_p^{1-1/p}(\partial_2 S)} + \left\|\frac{\partial\tau}{\partial n}\right\|_{\widetilde{W}_p^{1-1/p}(\partial_2 S)}\right)\Bigg)$$

$$\leq C\left(\|F_1\|_{\widetilde{L}_p} + |z|^{-1}\|F_2\|_{\widetilde{W}_p^1} + |z|^{-1}\|u\|_{\widetilde{W}_p^2} + \|\tau\|_{\widetilde{W}_p^1} + \|F_3\|_{\widetilde{L}_p} + \|u\|_{\widetilde{W}_p^1}\right)$$

$$+C|z|^{1/2-1/2p}\left(\|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|g\|_{\widetilde{W}_p^{1-1/p}} + |z|^{-1}\|F_2\|_{\widetilde{W}_p^{1-1/p}} + |z|^{-1}\|u\|_{\widetilde{W}_p^2} + \|\tau\|_{\widetilde{W}_p^1}\right).$$

Moving all terms involving u and τ to the left-hand side we have the inequality

$$|z|\|u\|_{\widetilde{L}_p} + |z|^{1/2}(1 - C|z|^{-1/2})\|u\|_{\widetilde{W}_p^1} + (1 - C|z|^{-1} - C|z|^{-1/2-1/2p})\|u\|_{\widetilde{W}_p^2}$$

$$+|z|\|\tau\|_{\widetilde{L}_p} + |z|^{1/2}(1 - C|z|^{-1/2} - C|z|^{-1/2p})\|\tau\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^2}$$

$$\leq C\left(\|F_1\|_{\widetilde{L}_p} + |z|^{-1}\|F_2\|_{\widetilde{W}_p^1} + \|F_3\|_{\widetilde{L}_p} + |z|^{1/2-1/2p}\left(\|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}\right)\right).$$

For sufficiently large z the terms

$$1 - C|z|^{-1/2} \geq \frac{1}{2}, \quad 1 - C|z|^{-1} - C|z|^{-1/2-1/2p} \geq \frac{1}{2}$$

and

$$1 - C|z|^{-1/2} - C|z|^{-1/2p} > \frac{1}{2}.$$

Thus,

$$\|u\|_{\widetilde{W}_p^2} + \|\tau\|_{\widetilde{W}_p^2} + (|z|+1)(\|u\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p})$$

$$\leq C\left(\|F_1\|_{\widetilde{L}_p} + \|F_2\|_{\widetilde{W}_p^1} + \|F_3\|_{\widetilde{L}_p} + |z|^{1/2-1/2p}\left(\|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}\right)\right).$$

Using the second equation, $\alpha = z^{-1}F_2 - z^{-1}\operatorname{div}(\rho_e u)$, we can now add $(|z|+1)\|\alpha\|_{\widetilde{W}_p^1}$ to the left-hand side as it is bounded by the terms $\|F_2\|_{\widetilde{W}_p^1}$ and $\|u\|_{\widetilde{W}_p^2}$. Therefore,

$$\|u\|_{\widetilde{W}_p^2} + \|\tau\|_{\widetilde{W}_p^2} + (|z|+1)(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{L}_p})$$

$$\leq C\left(\|F_1\|_{\widetilde{L}_p} + \|F_2\|_{\widetilde{W}_p^1} + \|F_3\|_{\widetilde{L}_p} + |z|^{1/2-1/2p}\left(\|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}\right)\right).$$

Using the boundary condition $z\beta = u_3$ we find $|z|\|\beta\|_{W_p^{2-1/p}} \leq \|u\|_{\widetilde{W}_p^2}$ and from the other equation we find the estimate

$$\|\beta\|_{\widetilde{W}_p^{1-1/p}} \leq C\left(\|u\|_{\widetilde{W}_p^2} + \|\alpha\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^1} + \|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}}\right).$$

Thus,

$$\|u\|_{\widetilde{W}_p^2} + \|\tau\|_{\widetilde{W}_p^2} + (|z|+1)(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{L}_p}) + \|\beta\|_{W_p^{1-1/p}} + |z|\|\beta\|_{W_p^{2-1/p}}$$

$$\leq C\left(\|F_1\|_{\widetilde{L}_p} + \|F_2\|_{\widetilde{W}_p^1} + \|F_3\|_{\widetilde{L}_p} + |z|^{1/2-1/2p}\left(\|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}\right)\right)$$

$$+C(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}).$$

Based on these estimates the reader will not have any difficulty proving the existence of the solution by means of the Banach Fixed Point Theorem. As Lemma 2.4 gives the injectivity of the operator $AU + zU$ we conclude the operator $AU + zU$ is bijective for large z. $\square$

Lemma 2.12. *There exists a C such that the following is true. Let $u \in \widetilde{W}_p^2(S)$, $F_2, \alpha \in \widetilde{W}_p^1(S)$, $F_1 \in \widetilde{L}_p(S)$, $\beta \in \widetilde{W}_p^{2-1/p}(\partial S)$. Assume $Re(z) \geq 0$ and*

$$\begin{aligned} -\Delta u + \nabla\alpha &= F_1 \\ \operatorname{div}(u) &= F_2 \end{aligned}$$

with boundary conditions

$$\begin{aligned} \mathbb{D}(u)\cdot n - \alpha n &= \beta n + \mathrm{g}_1 \\ z\beta - u_3 &= \mathrm{g}_3 \end{aligned}$$

on $\partial_2 S$ and the boundary condition

$$u = 0$$

on $\partial_1 S$. Then

$$\|u\|_{\widetilde{W}_p^2} + \|\alpha\|_{\widetilde{W}_p^1} + \|\beta\|_{\widetilde{W}_p^{1-1/p}}$$

$$\leq C\left(\|F_1\|_{\widetilde{L}_p} + \|F_2\|_{\widetilde{W}_p^1} + \|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_3\|_{\widetilde{W}_p^{2-1/p}} + \|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}\right).$$

Proof. We get the estimates for u and α by combining the results of Lemma 14 in [6] on $\partial_1 S$ and use the standard estimates for the Stokes equation on $\partial_2 S$. From the boundary conditions we get the estimates for the β similar to the previous case. By Agmon-Douglas-Nirenberg and the fact that these are complementing boundary conditions on both sides we can easily glue these two problems together and obtain the desired estimate. □

Theorem 2.13. *Let $u \in \widetilde{W}_p^2(S), \alpha \in \widetilde{W}_p^1(S), \tau \in \widetilde{W}_2^1(S), \beta \in \widetilde{W}_p^{2-1/p}(\partial_2 S)$. There exists a constant C and a $\delta > 0$, so that when $|z| < \delta$ and $Re(z) \geq 0$ solve the system of equations*

$$L_{11}(u,\alpha,\tau) = F_1, L_2(u,\alpha,\tau) = F_2, L_3(u,\alpha,\tau) = F_3$$

with boundary conditions

$$u = 0 \quad \textit{and} \quad \tau = 0 \quad \textit{on } \partial_1 S$$

and

$$\mathbb{D}(u)\cdot n - c\theta_e \alpha E n - c\rho_e \tau E n = c\theta_e \rho_{ex_3}\beta E n + \mathrm{g}_1,$$

$$\frac{\partial \tau}{\partial n} = \mathrm{g}_2 \textit{ and } z\beta - u_3 = \mathrm{g}_3$$

on $\partial_2 S$ then the following estimate holds.

$$\|u\|_{\widetilde{W}^2_p} + \|\alpha\|_{\widetilde{W}^1_p} + \|\tau\|_{\widetilde{W}^2_p} + \|\beta\|_{\widetilde{W}^{2-1/p}_p}$$

$$\leq C(\|F_1\|_{\widetilde{L}_p} + \|F_2\|_{\widetilde{W}^1_p} + \|F_3\|_{\widetilde{L}_p} + \|\mathrm{g}_1\|_{\widetilde{W}^{1-1/p}_p} + \|\mathrm{g}_2\|_{\widetilde{W}^{1-1/p}_p} + \|\mathrm{g}_3\|_{\widetilde{W}^{2-1/p}_p})$$

$$+C(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}).$$

Proof. Solve the second equation for $\mathrm{div}(u)$, $\mathrm{div}(u) = \rho_e^{-1}(F_2 - u\cdot\nabla\rho_e)$ so that we can substitute the term $\mathrm{div}(u)$ in both the first and third equation with the term $\rho_e^{-1}(F_2 - u\cdot\nabla\rho_e)$. In order to do this in the first equation we must first rewrite the term $\mathrm{div}(\mathbb{D}(u))$.

$$\mathrm{div}(\mathbb{D}(u)) = \left(\sum_{k=1}^{3}\frac{\partial \mathbb{D}_{k1}(u)}{\partial x_k}, \sum_{k=1}^{3}\frac{\partial \mathbb{D}_{k2}(u)}{\partial x_k}, \sum_{k=1}^{3}\frac{\partial \mathbb{D}_{k3}(u)}{\partial x_k}\right) = \mu\Delta u + \nu\nabla\,\mathrm{div}(u).$$

Substituting this into the first equation gives

$$-\mu\Delta u - \nu\nabla\,\mathrm{div}(u) + c\theta_e\nabla(\alpha) = \rho_e F_1 - c\nabla(\rho_e\tau).$$

Now $\mathrm{div}(u)$ can be removed from the first and third equation using the second one, resulting in the system of equations

$$\begin{aligned} -\Delta u + \mu^{-1}c\theta_e\nabla\alpha &= \mu^{-1}\rho_e F_1 + \frac{\nu}{\mu}\nabla(\rho_e^{-1}(F_2 - u\cdot\nabla\rho_e)) - c\nabla(\mu^{-1}\rho_e\tau) \\ -\Delta\tau &= \kappa^{-1}\rho_e c_2 F_3 - c\kappa^{-1}\theta_e(u\cdot\nabla\rho_e - F_2). \end{aligned} \tag{2.19}$$

For convenience, let $\widetilde{\alpha} = \mu^{-1} c\theta_e \alpha, \widetilde{\beta} = \mu^{-1} c\theta_e \rho_{ex_3}\beta$, and $\widetilde{z} = \mu(c\theta_e\rho_{ex_3})^{-1} z$. Now we can write the equations (2.19) as

$$\begin{aligned} -\Delta u + \nabla\widetilde{\alpha} &= \mu^{-1}\rho_e F_1 + \frac{\nu}{\mu}\nabla(\rho_e^{-1}(F_2 - u\cdot\nabla\rho_e)) - \nabla(\mu^{-1}\rho_e\tau) \\ -\Delta\tau &= \kappa^{-1}\rho_e c_2 F_3 - c\kappa^{-1}\theta_e(u\cdot\nabla\rho_e - F_2) \end{aligned} \tag{2.20}$$

and the boundary conditions become

$$\begin{aligned} \left[(u_{ix_j} + u_{jx_i})n_i\right]_j - \widetilde{\alpha}\cdot n - \widetilde{\beta} n &= -\frac{\nu-\mu}{\mu}\rho_e^{-1}(F_2 - u\cdot\nabla\rho_e)n + \mu^{-1}c\rho_e\tau + \frac{\mathrm{g}_1}{\mu} \\ \widetilde{z}\widetilde{\beta} - u_3 &= \mathrm{g}_3. \end{aligned} \tag{2.21}$$

Let

$$\begin{aligned} P_1(u,\tau) &= -\frac{\nu}{\mu}\nabla\left(\rho_e^{-1}u\cdot\nabla u\right) - \mu^{-1}\nabla\left(\rho_e\tau\right) \\ P_2(u) &= \rho_e^{-1}u\cdot\nabla\rho_e \\ P_3(u) &= -c\kappa^{-1}\theta_e u\cdot\nabla\rho_e \\ P_4(u,\tau) &= \tau n + \frac{\nu-\mu}{\mu}\left(\rho_e u\cdot\nabla\rho_e\right)n \end{aligned}$$

and

$$\begin{aligned} \widetilde{F}_1 &= \mu^{-1}\rho_e F_1 + \frac{\nu}{\mu}\nabla\left(\rho_e^{-1}F_2\right) \\ \widetilde{F}_2 &= \rho_e^{-1}F_2 \\ \widetilde{F}_3 &= \kappa^{-1}\rho_e c_2 F_3 + c\kappa^{-1}\theta_e F_2 \\ \widetilde{\mathrm{g}}_1 &= \mu^{-1}\mathrm{g}_1. \end{aligned}$$

Now we can view the system of equations (2.20) as

$$\begin{aligned} -\Delta u + \nabla\widetilde{\alpha} &= \widetilde{F}_1 + P_1(u,\tau) \\ \mathrm{div}(u) &= \widetilde{F}_2 + P_2(u) \\ -\Delta\tau &= \widetilde{F}_3 + P_3(u) \end{aligned}$$

and the boundary conditions (2.21) as

$$\mathbb{D}(u)\cdot n - \widetilde{\alpha} n - \widetilde{\beta} n = \widetilde{\mathrm{g}}_1 + P_4(u,\tau)$$

$$\widetilde{z}\widetilde{\beta} - u_3 = \mathrm{g}_3$$

The system of equations in Lemma 2.12 differs from this one only in the terms involving temperature and the deformation **d**. The estimate for the equation

$$-\Delta\tau = F_2 + P_2(u)$$

is equivalent to finding an estimate for the equation

$$-\Delta\tau + \tau = F_2 + P_2(u) + \tau.$$

Thus, by Theorem A.1 we find

$$\begin{aligned} \|\tau\|_{\widetilde{W}_p^2} &\leq C(\|\widetilde{F}_3 + P_3(u)\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}) \\ &\leq C(\|\widetilde{F}_3\|_{\widetilde{L}_p} + \|u\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}). \end{aligned}$$

If we combine this with the result in Lemma 2.12 we find

$$\|u\|_{\widetilde{W}_p^2} + \|\alpha\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^2} + \|\beta\|_{W_p^{1-1/p}} \leq$$

$$\leq C\Big(\|\widetilde{F}_1 + P_1(u,\tau)\|_{\widetilde{L}_p} + \|\widetilde{F}_2 + P_2(u)\|_{\widetilde{W}_p^1} + \|\widetilde{\mathrm{g}}_1 + P_4(u,\tau)\|_{\widetilde{W}_p^{1-1/p}}$$

$$+\|\mathrm{g}_3\|_{\widetilde{W}_p^{2-1/p}} + \|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p} + \|\widetilde{F}_3 + P_3(u)\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}\Big)$$

$$\leq C\Big(\|F_1\|_{\widetilde{L}_p} + \|F_2\|_{\widetilde{W}_p^1} + \|F_3\|_{\widetilde{L}_p} + \|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_3\|_{\widetilde{W}_p^{2-1/p}}$$

$$+\|u\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^1} + \|u\|_{\widetilde{W}_p^{1-1/p}} + \|\tau\|_{\widetilde{W}_p^{1-1/p}} + \|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}\Big)$$

$$\leq C\Big(\|F_1\|_{\widetilde{L}_p} + \|F_2\|_{\widetilde{W}_p^1} + \|F_3\|_{\widetilde{L}_p} + \|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_3\|_{\widetilde{W}_p^{2-1/p}}$$

$$+\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}\Big)$$

by interpolation. $\square$

In the next lemma we address the case where z is intermediate valued.

Lemma 2.14. *For* $0 < M_1, M_2 < \infty$ *there exists a constant* $C < \infty$*, such that if* $F_1 \in \widetilde{L}_p(S), F_2 \in \widetilde{W}_p^1(S), F_3 \in \widetilde{W}_p^1(S), \mathrm{g}_1 \in \widetilde{W}_p^{1-1/p}(\partial S), Re(z) \geq 0,$ *with* $0 < M_1 \leq |z| \leq M_2 < \infty$ *If* (u, α, τ, β) *is a solution of*

$$L_1(u,\alpha,\tau) + zu = F_1, \quad L_2(u,\alpha,\tau) + z\alpha = F_2, \quad L_3(u,\alpha,\tau) + z\tau = F_3$$

with the boundary conditions

$$u = 0 \quad \text{and} \quad \tau = 0 \text{ on } \partial_1 S$$

and

$$(\mathbb{D}(u) - c\theta_e(\alpha + \rho_{ex_3}\beta)E - c\rho_e\tau E) \cdot n = \mathrm{g}_1,$$

$$\frac{\partial \tau}{\partial n} = \mathrm{g}_2 \text{ and } z\beta = u_3$$

on $\partial_2 S$ *then the following inequality holds.*

$$\|u\|_{\widetilde{W}_p^2} + \|\alpha\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^2}$$

$$\leq C\left(\|F_1\|_{\widetilde{L}_p} + \|F_2\|_{\widetilde{W}_p^1} + \|F_3\|_{\widetilde{L}_p} + \|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}\right)$$

$$+C\left(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p}\right).$$

Proof. The system of equations for intermediate valued z,

$$\begin{aligned} L_1(u,\alpha,\tau) &= F_1 \\ L_2(u,\alpha,\tau) + z\alpha &= F_2 \\ L_3(u,\alpha,\tau) &= F_3, \end{aligned} \tag{2.22}$$

is elliptic in the sense of Agmon, Douglas, and Nirenberg with complementing boundary condition (2.5). The boundary condition $z\beta = u_3$ does not play a role in the estimate as $\|u_3\|_{\widetilde{W}_p^{1-1/p}} \le C\|u\|_{\widetilde{W_p^1}}$, which is a lower-order term. By interpolation we have the a-priori estimate

$$\|u\|_{\widetilde{W}_p^2} + \|\alpha\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^2}$$

$$\le C\left(\|F_1\|_{\widetilde{L}_p} + \|F_2\|_{\widetilde{W}_p^1} + \|F_3\|_{\widetilde{L}_p} + \|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}\right)$$

$$+\left(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p}\right).$$

Adding the term zu to the first equation of (2.22) and $z\tau$ to the third equation does not change the above estimate as we already have the terms $\|u\|_{\widetilde{L}_p}$ and $\|\tau\|_{\widetilde{L}_p}$ in it. □

Theorem 2.15. *There exists a constant C such that if $Re(z) \ge 0$, $U = [u, \alpha, \tau, \beta, \mathbf{d}] \in D(A)$ and $AU + zU = F = [F_1, F_2, F_3, \mathrm{g}_1, \mathrm{g}_3] \in B$ with boundary condition*

$$\mathbb{D}(u)\cdot n - c\theta_e\alpha En - c\rho_e\tau En = c\theta_e\rho_{ex_3}\beta En \ \text{ on } \partial_2 S$$

then

$$\|u\|_{\widetilde{W}_p^2} + (1+|z|)(\|\tau\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{W}_p^1} + \|u\|_{\widetilde{L}_p}) + \|\tau\|_{\widetilde{W}_p^2} + \|\beta\|_{\widetilde{W}_p^{1-1/p}} + |z|\|\beta\|_{\widetilde{W}_p^{2-1/p}}$$

$$\le C(\|F\|_{\widetilde{B}} + \|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}).$$

Proof. Separate the equation $AU + zU = F$ into $A_1U + zU = F + A_2U$, where $A_1 = [L_{11}, L_2, L_3, u_3, 0]$ and $A_2 = [\alpha g e_3, 0, 0, 0, 0]$. Then

$$\|A_2U\|_{\widetilde{B}} = \|\alpha g e_3\|_{\widetilde{L}_p} \le C\|\alpha\|_{\widetilde{L}_p} \tag{2.23}$$

Using Theorem 2.11, Theorem 2.13 and Lemma 2.14 we get the desired estimate. □

CHAPTER 3
RESOLVENT ESTIMATES AND ANALYTIC SEMIGROUP PROPERTIES

In addressing the question of stability of an equilibrium point we have considered the linearization of the equations of motion at a particular point so that we can then prove linear stability. Then nonlinear stability can be addressed through a perturbation argument. The conservation of mass forces the solution of the system (1.1) to remain on a submanifold of the entire solution space and is given by the functional

$$\mathfrak{Q} = \int_{\Omega_t} \rho(t) dy.$$

In Chapter 2 the estimate in Theorem 3.28 suggest that we cannot expect exponential decay for the solutions of our problem. To treat the nonlinear hyperbolic equation $\rho_t + \mathrm{div}(\rho v) = 0$ as a perturbation of the linearized version $\alpha_t + \mathrm{div}(\rho_e u) = 0$, we transform the equation into Lagrange coordinates. In Lagrange coordinates the flow of a fluid particle is described by the function T_t, and using this we defined all of the functions on smooth domain Ω. In particular the deformation $\mathbf{d}$ belonging to T_t is obtained by integrating u over t. Since we only know $\|u(t)\|_{\widetilde{W}_p^2} \leq C_{U_0} t^{-1}$ we cannot directly estimate $\|\mathbf{d}\|_{\widetilde{W}_p^2}$ as $t \to \infty$, as t^{-1} is not integrable over $[1, \infty)$. To obtain the desired estimate we have added $\mathbf{d}$ to the vector containing our unknown functions and apply analytic semigroup theory. In doing this we need to add the equation $\mathbf{d}'(y,t) = v\left(T_t(y,t), t\right) = u(y,t)$ to the system of equations (1.14). The solution vector is now $U = [u, \alpha, \tau, \beta, \mathbf{d}]$ and the linear system of equations along with the boundary conditions becomes

$$
\begin{array}{ccc}
\alpha_t + \operatorname{div}(\rho_e u) & = & 0 \\
\rho_e u_t + c\nabla(\theta_e\alpha + \rho_e\tau) - \operatorname{div}(\mathbb{D}(u)) & = & -\alpha g e_3 \\
\rho_e c_2 \tau_t & = & -c\theta_e\rho_e \operatorname{div}(u) + \operatorname{div}(\kappa\nabla\tau) \\
(\mathbb{D}(u) - c\theta_e(\alpha + \rho_{ex_3}\beta)E - c\rho_e\tau E)\cdot n & = & 0 \\
\beta_t = u_3 & \text{and} & \dfrac{\partial\tau}{\partial x_3} = 0 \ \text{ on } \ \partial_2 S \\
u = 0 & \text{and} & \tau = 0 \ \text{ on } \ \partial_1 S \\
\mathbf{d}_t & = & u.
\end{array}
\tag{3.1}
$$

In Euler coordinates the gas is contained within the region Ω^t at time $t \in I_\omega$. In this chapter we will use modified Lagrange coordinates in which the domain is transformed to the fixed set S, which is also the domain on which the linearized problem is defined. The problem we address is obtained from transforming the equations (3.1) posed on $\mathbf{\Omega}^\omega$, when $\omega < \infty$, to $S \times [0, \omega]$ through the mappings $\mathbf{T}(y,t) = T_t(y)$ for the nonlinear equation. We consider the transformations and transformed operators for a fixed time so that time can be removed from the situation and obtain the transformation $T : \overline{S} \to \mathbb{R}^3$ with $T - E_S$ belonging to $\widetilde{W}_p^2(S)$, where $y = T(x)$, and denote the region by $\Omega = T(S)$. In Chapter 5 we address why these transformations, and in particular the equations in Lagrange coordinates with the variables $u, \alpha, \tau, \beta, \mathbf{d}$ have the same periodicity properties in S. Only injective transformations T satisfying the inequalities

$$
\|T - E_S\|_{\widetilde{W}_p^2(S)} \le 1, \quad \det(\nabla T) \ge \frac{1}{2}, \quad \min_{x\in\partial S} T_3(x) > \frac{h}{2}
$$

will be addressed. We will see that the transformations themselves are not periodic, but the differences between them are periodic. Throughout this analysis we will need the function matrix

$$\mathcal{Z}_T = (\nabla T)^{-1} = \big(\nabla(T^{-1})\big) \circ T, (\mathcal{Z}_T)_{ij} = \big(\nabla(T^{-1})\big)_{ij} \circ T = \frac{\partial x_i}{\partial y_j}(x). \tag{3.2}$$

belonging to $\widetilde{W}_p^1(S)$, and for any function $f \in \widetilde{W}_p^1(\Omega)$ we have $(\nabla f)\circ T = \nabla(f \circ T)\mathcal{Z}_T$.

As well as estimating $\mathbf{d}$ we must also show that certain norms of $u(t), \alpha(t), \tau(t)$, and $\beta(t)$ decay faster than we found in Theorem 3.28 in Chapter 2. To describe the more rapid decay we introduce scales of spaces and norms for our function vectors as follows. Let $\mathfrak{B}^s = \Big\{[u, \alpha, \tau, \beta, \mathbf{d}]:$

$$u \in \widetilde{W}_p^{s-1}(S), \alpha \in \widetilde{W}_p^{s}(S), \tau \in \widetilde{W}_p^{s-1}(S), \beta \in \widetilde{W}_p^{s+1-1/p}(\partial_2 S), \mathbf{d} \in \widetilde{W}_p^{s+1}(S)\Big\}$$

with $p \in (9, \infty), s \in (1/p, 1]$. These are Banach spaces with the norms

$$\|[u, \alpha, \tau, \beta, \mathbf{d}]\|_{\mathfrak{B}^s} = \|u\|_{\widetilde{W}_p^{s-1}} + \|\alpha\|_{\widetilde{W}_p^{s}} + \|\tau\|_{\widetilde{W}_p^{s-1}} + \|\beta\|_{\widetilde{W}_p^{s+1-1/p}} + \|\mathbf{d}\|_{\widetilde{W}_p^{s+1}}.$$

Let

$$D^s = \Big\{[u, \alpha, \tau, \beta, \mathbf{d}] \in \mathfrak{B}^s | u \in \widetilde{W}_p^{s+1}(S), \tau \in \widetilde{W}_p^{s+1}(S)\Big\}$$

which are also Banach spaces with the norm

$$\|[u, \alpha, \tau, \beta, \mathbf{d}]\|_{D^s} = \|u\|_{\widetilde{W}_p^{s+1}} + \|\tau\|_{\widetilde{W}_p^{s+1}} + \|[u, \alpha, \tau, \beta, \mathbf{d}]\|_{\mathfrak{B}^s}.$$

The norm

$$\|[u, \alpha, \tau, \beta, \mathbf{d}]\|_{D^s_{-1}} = \|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^{s}} + \|\tau\|_{\widetilde{W}_p^{s+1}} + \|\beta\|_{\widetilde{W}_p^{s-1/p}} + \|\mathbf{d}\|_{\widetilde{W}_p^{s+1}}$$

will also be used. We will often want the vector U without the deformation $\mathbf{d}$ and therefore define the projector Π by

$$\Pi[u, \alpha, \tau, \beta, \mathbf{d}] = [u, \alpha, \tau, \beta, 0].$$

Let $\mathcal{P}^c$ be the orthogonal projection from $\mathfrak{h}$ to $[0,1,0,1,0]$ and $\mathcal{P} = E_{\mathfrak{h}} - \mathcal{P}^c$. As defined before we have

$$B = \mathcal{P}(\mathfrak{B}^1) = \left\{ U \in \mathfrak{B}^1 : \mathcal{P}^c(U) = 0 \right\}.$$

Then B contains all elements $U = [u, \alpha, \tau, \beta, \mathbf{d}]$ of $\mathfrak{B}^1$ for which $Q([u, \alpha, \tau, \beta, \mathbf{d}]) = 0$. For $\mathbf{U} \in \mathfrak{h}$ we have $\mathcal{P}^c\mathbf{U} \in D^1$ and

$$\|\mathcal{P}^c\mathbf{U}\|_{D^1} \leq C\|\mathbf{U}\|_{\mathfrak{h}}.$$

In order to state the result of this chapter precisely we define the operator $\mathcal{A}_{1T}$. For $u \in \widetilde{W}_p^2(S)$ let

$$\mathbb{D}_T(u) = \mu\left(\nabla u \mathcal{Z}_T + (\nabla u \mathcal{Z}_T)^\top\right) + (\nu - \mu)\left(\operatorname{tr}(\nabla u \mathcal{Z}_T)\right) E_3$$

which has the same viscosity coefficients μ and ν as before. Let

$$\begin{aligned}
\widetilde{\mathbb{T}}_T(u,\alpha) &= \mathbb{D}_T(u) - c\theta_e \alpha E_3, \\
(L_T)_k(u) &= -\frac{1}{\rho_e \circ T}\left((\mathbb{D}_T(u))_{kj}\right)_{x_q} (\mathcal{Z}_T)_{qj} \qquad\qquad k \in \{1,2,3\}, \\
\mathcal{L}_T(u,\alpha,\tau) &= L_T(u) + \frac{c}{\rho_e \circ T}\left(\operatorname{tr}\left(\nabla\left(\theta_e\alpha + (\rho_e \circ T)\tau\right)\mathcal{Z}_{T_t}\right)\right), \\
\mathcal{K}_T(u,\alpha,\tau) &= \frac{c\theta_e}{c_2}\operatorname{tr}\left(\nabla u \mathcal{Z}_{T_t}\right) - \frac{\kappa}{c_2\rho_{\mathbf{d}}}\operatorname{tr}\left(\nabla\left(\nabla\tau\mathcal{Z}_{T_t}\right)\mathcal{Z}_{T_t}\right).
\end{aligned} \tag{3.3}$$

Then on D^1 we define the operator $\mathfrak{U}_{1T}$ by

$$\mathfrak{U}_{1T}[u,\alpha,\tau,\beta,d] =$$

$$\left[\mathcal{L}_T(u,\alpha,\tau) + \frac{1}{\rho_e \circ T}\alpha g e_3, \operatorname{tr}\left(\nabla\left(\rho_e \circ T u\right)\mathcal{Z}_{T_t}\right), \mathcal{K}_T(u,\alpha,\tau), -u_3, -u\right]$$

Generally $\mathfrak{U}_{1T}(D^1 \cap B)$ is not contained in B, unless $T = E$, and thus we create a new operator

$$\mathcal{A}_{1T}U = \mathcal{P}\mathfrak{U}_{1T}$$

so that $\mathcal{A}_{1T}(D^1) \subset B$. For $U = [u, \alpha, \tau, \beta, \mathbf{d}]$ we define the boundary operator

$$\mathcal{B}_T(u, \alpha, \tau, \beta, \mathbf{d}) = \mathcal{B}_T(U) = \begin{bmatrix} \widetilde{\mathbb{T}}_T(u, \alpha) \cdot n_T - c\rho_e \tau n_T - \dfrac{\partial p_e}{\partial x_3} \beta n_T \\ \dfrac{\partial \tau}{\partial n_T} \end{bmatrix}$$

where $n_T(x)$ is the exterior unit normal to $\partial\Omega$ at $T(x)$. For $s \in (1/p, 1]$

$$D^s(\mathcal{A}_T) = \{U \in D^s \cap B : \mathcal{B}_T(U) = 0\}.$$

Let $\mathfrak{U}_T = \mathfrak{U}_{1T}|D^1(\mathcal{A}_T)$ and $\mathcal{A}_T = \mathcal{A}_{1T}|D^1(\mathcal{A}_T)$.

In this chapter stronger decay estimates for some norms of the vector $\mathbf{U}(t)$ are obtained and we conclude with the proof of the following theorem.

Theorem 3.1. *For $\in (1/p, 1]$ there exists $C < \infty, \eta > 0$ such that $\|T - E\|_{\widetilde{W}_p^2(S)} \leq \eta$, then $\mathcal{A}_{1T} : D^1(\mathcal{A}_T) \to B$ generates an analytic semigroup, and for $U_0 \in B, t > 0$ and $\mathbf{U}(t) = \exp(t\mathcal{A}_T)U_0$ we have the inequality*

$$\|\mathbf{U}(t)\|_{\mathfrak{B}_1} + t\|\mathbf{U}'(t)\|_{\mathfrak{B}_1} + t^2\|\Pi\mathbf{U}'(t)\|_{D^1_{-1}} + t^{2-s}\|\Pi\mathbf{U}(t)\|_{D^s_{-1}} \leq C\|U_0\|_{\mathfrak{B}_1}$$

In chapter 5 we will use that $\mathcal{A}_{T_\omega}$ generates an analytic semigroup and use this fact to study

$$\mathbf{U}' + \mathcal{A}_{T_\omega}\mathbf{U} = \mathbf{F} + \left(\mathcal{A}_{T_\omega} - \mathcal{A}_{T_1}\right)\mathbf{U}$$

with boundary conditions $\mathcal{B}_{T_\omega}(\mathbf{U}) = \mathbf{g} + \left(\mathcal{B}_{T_\omega} - \mathcal{B}_{T_t}\right)(\mathbf{U})$.

3.1 Preliminary Properties

This section is devoted to proving some a-priori estimates for elliptic systems as well as study continuity properties of different expressions occurring in the equations in preparation for the analysis in the next section. Here we consider $T, \widehat{T} : \overline{S} \to$

$\mathbb{R}^3, T = E_S, \widehat{T} - E_S \in \widetilde{W}_p^2(S)$ and assume that $\widehat{T}$ is injective and has the same properties as T.

Lemma 3.2. *There exists $C < \infty$ such that $\|\mathcal{Z}_T\|_{\widetilde{W}_p^1} \le C$ and $\|\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}}\|_{\widetilde{W}_p^1} \le C\|T - \widehat{T}\|_{\widetilde{W}_p^2}$.*

Proof. By definition $\|\mathcal{Z}_T\|_{\widetilde{W}_p^1} = \left\|(\nabla T)^{-1}\right\|_{\widetilde{W}_p^1} \le C\left(\|T - E\|_{\widetilde{W}_p^2} + \|E\|_{\widetilde{W}_p^2}\right) \le C$ and

$$\|\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}}\|_{\widetilde{W}_p^1} = \left\|(\nabla T)^{-1} - \left(\nabla \widehat{T}\right)^{-1}\right\|_{\widetilde{W}_p^1} \le C\|\nabla T - \nabla \widehat{T}\|_{\widetilde{W}_p^1}.$$

□

Lemma 3.3. *For $x \in \partial S$*

$$n_T(x) = \frac{e_3 \mathcal{Z}_T(x)}{|e_3 \mathcal{Z}_T(x)|}.$$

Proof. The boundary is given by $\partial\Omega = \left\{y \in \overline{\Omega} : (T^{-1})_3(y) = 0\right\}$ and an exterior normal to $\partial\Omega$ at $y \in \partial\Omega$ is given by the vector $\widetilde{n}(y) = \nabla(T^{-1})_3(y)$. Then $(\widetilde{n})_k = (\mathcal{Z}_T \circ T^{-1})_{3k}$ and $(\widetilde{n})_k \circ T(x) = (\mathcal{Z}_T)_{3k}$ for $x \in \partial S$. Thus, normalizing we get

$$n_T(x) = \frac{e_3 \mathcal{Z}_T(x)}{|e_3 \mathcal{Z}_T(x)|}.$$

□

Lemma 3.4. *There exists $C < \infty$ such that*

$$\|n_T\|_{\widetilde{W}_p^{1-1/p}(\partial_2 S)} + \|n_T\|_{\widetilde{W}_p^1(S)} \le C$$

and

$$\|n_T - n_{\widehat{T}}\|_{\widetilde{W}_p^{1-1/p}(\partial_2 S)} \le C\|n_T - n_{\widehat{T}}\|_{\widetilde{W}_p^1(S)} \le C\|T - \widehat{T}\|_{\widetilde{W}_p^2(S)}.$$

Proof. The proof of this is obtained by combining Lemma 3.3 and Lemma 3.2. □

Lemma 3.5. *Let $s \in [-1,1]$ and $s \neq 1/p - 1$. Then there exists a constant C such that for $f \in \widetilde{W}_p^s(S), g \in \widetilde{W}_p^1(S)$ we have*

$$\|fg\|_{\widetilde{W}_p^s} \leq C\|f\|_{\widetilde{W}_p^s}\|g\|_{\widetilde{W}_p^1}$$

Proof. This proof can be found in Lemma 2.3.4 of [2]. □

Lemma 3.6. *For $s \in [1,2]$ there exists a constant $C < \infty$ such that if $u \in \widetilde{W}_p^s(\partial_2 S)$ then*

$$\|\mathbb{D}_T(u)\|_{\widetilde{W}_p^{s-1}} \leq C\|u\|_{\widetilde{W}_p^s}$$

and

$$\|\mathbb{D}_T(u) - \mathbb{D}_{\widehat{T}}(u)\|_{\widetilde{W}_p^{s-1}} \leq C\|T - \widehat{T}\|_{\widetilde{W}_p^2}\|u\|_{\widetilde{W}_p^s}$$

Proof. The proof is identical to the proof of Lemma 3.5 from [2]. □

Lemma 3.7. *For $s \in [0,1], s \neq \dfrac{1}{p}$ there exists a constant $C < \infty$ such that if $u \in \widetilde{W}_p^2(S), \alpha \in \widetilde{W}_p^1(S), \tau \in \widetilde{W}_p^1(S)$ then*

$$\|L_T(u)\|_{\widetilde{W}_p^{s-1}} \leq C\|u\|_{\widetilde{W}_p^{s+1}}, \|\mathcal{L}_T(u,\alpha,\tau)\|_{\widetilde{W}_p^{s-1}} \leq C\left(\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^{s,1}} + \|\tau\|_{\widetilde{W}_p^s}\right),$$

$$\|L_T(u) - L_{\widehat{T}}(u)\|_{\widetilde{W}_p^{s-1}} \leq C\|T - \widehat{T}\|_{\widetilde{W}_p^2}\|u\|_{\widetilde{W}_p^{s+1}}$$

and

$$\|\mathcal{L}_T(u,\alpha,\tau) - \mathcal{L}_{\widehat{T}}(u,\alpha,\tau)\|_{\widetilde{W}_p^{s-1}} \leq C\|T - \widehat{T}\|_{\widetilde{W}_p^2}\left(\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^s} + \|\tau\|_{\widetilde{W}_p^s}\right)$$

Proof. By Lemma 3.6 in [2] we have $\|L_T(u)\|_{\widetilde{W}_p^{s-1}} \leq C\|u\|_{\widetilde{W}_p^{s+1}}$ and

$$\|L_T(u) - L_{\widehat{T}}(u)\|_{\widetilde{W}_p^{s-1}} \leq C\|T - \widehat{T}\|_{\widetilde{W}_p^2}\|u\|_{\widetilde{W}_p^{s+1}}.$$

Using the definition of $\mathcal{L}_T$ from definition 3.3 we have

$$\|\mathcal{L}_T(u,\alpha,\tau)\|_{\widetilde{W}_p^{s-1}} = \|L_T(u) + \nabla(c\theta_e\alpha)\mathcal{Z}_T + \nabla(c(\rho_e\circ T)\tau)\mathcal{Z}_T\|_{\widetilde{W}_p^{s-1}}$$

$$\leq C\left(\|L_T(u)\|_{\widetilde{W}_p^{s-1}} + \|\nabla(c\theta_e\alpha)\mathcal{Z}_T\|_{\widetilde{W}_p^{s-1}} + \|\nabla\left(c(\rho_e\circ T)\tau\right)\mathcal{Z}_T\|_{\widetilde{W}_p^{s-1}}\right).$$

By Lemma 3.2 and Lemma 3.5

$$\|\nabla(c\theta_e\alpha)\mathcal{Z}_T\|_{\widetilde{W}_p^{s-1}} \leq C\|\nabla(c\theta_e\alpha)\|_{\widetilde{W}_p^{s-1}}\|\mathcal{Z}_T\|_{\widetilde{W}_p^{1}} \leq C\|\alpha\|_{\widetilde{W}_p^{s}}$$

and similarly

$$\|\nabla\left(c(\rho_e\circ T)\tau\right)\mathcal{Z}_T\|_{\widetilde{W}_p^{s-1}} \leq C\|\tau\|_{\widetilde{W}_p^{s}}.$$

Thus, we arrive at

$$\|\mathcal{L}_T(u,\alpha,\tau)\|_{\widetilde{W}_p^{s-1}} \leq C\left(\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^{s}} + \|\tau\|_{\widetilde{W}_p^{s}}\right).$$

Also

$$\|\mathcal{L}_T(u,\alpha,\tau) - \mathcal{L}_{\widehat{T}}(u,\alpha,\tau)\|_{\widetilde{W}_p^{s-1}} \leq C\|L_T(u) - L_{\widehat{T}}(u)\|_{\widetilde{W}_p^{s-1}}$$

$$+C\left(\|\nabla\alpha(\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}})\|_{\widetilde{W}_p^{s-1}} + \|\nabla\tau(\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}})\|_{\widetilde{W}_p^{s-1}}\right)$$

$$\leq C\|L_T(u) - L_{\widehat{T}}(u)\|_{\widetilde{W}_p^{s-1}} + C\left(\|\nabla\alpha\|_{\widetilde{W}_p^{s-1}} + \|\nabla\tau\|_{\widetilde{W}_p^{s-1}}\right)\|\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}}\|_{\widetilde{W}_p^{1}}$$

$$\leq C\|L_T(u) - L_{\widehat{T}}(u)\|_{\widetilde{W}_p^{s-1}} + C\left(\|\nabla\alpha\|_{\widetilde{W}_p^{s-1}}\|T-\widehat{T}\|_{\widetilde{W}_p^{2}} + \|\nabla\tau\|_{\widetilde{W}_p^{s-1}}\|T-\widehat{T}\|_{\widetilde{W}_p^{2}}\right).$$

Thus,

$$\|\mathcal{L}_T(u,\alpha,\tau) - \mathcal{L}_{\widehat{T}}(u,\alpha,\tau)\|_{\widetilde{W}_p^{s-1}} \leq C\|T-\widehat{T}\|_{\widetilde{W}_p^{2}}\left(\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^{s}} + \|\tau\|_{\widetilde{W}_p^{s}}\right).$$

$\square$

Lemma 3.8. *For $s\in(1/p,1]$ there is a constant $C<\infty$ such that if $U\in D^1$ then*

$$\|\mathfrak{U}_{1T}U\|_{\mathfrak{B}^s} \leq C\|\Pi U\|_{D^s_{-1}}$$

and

$$\|\mathfrak{U}_{1T}U - \mathfrak{U}_{1\widehat{T}}U\|_{\mathfrak{B}^s} \le C\|T-\widehat{T}\|_{\widetilde{W}_p^2}\|\Pi U\|_{D_{-1}^s}$$

Proof. By definition of $\mathfrak{B}^s$

$$\|\mathfrak{U}_{1T}U\|_{\mathfrak{B}^s} = \|\mathcal{L}_T(u,\alpha,\tau)\|_{\widetilde{W}_p^{s-1}} + \|\operatorname{tr}\left(\nabla\left((\rho_e\circ T)u\right)\mathcal{Z}_T\right)\|_{\widetilde{W}_p^s} + \|u\cdot e_3\|_{\widetilde{W}_p^{s+1-1/p}}$$

$$+\|u\|_{\widetilde{W}_p^{s+1}} + \|-\kappa\operatorname{tr}\left(\nabla(\nabla\tau\mathcal{Z}_T)\mathcal{Z}_T\right) + c\rho_e\theta_e\operatorname{tr}(\nabla(u)\mathcal{Z}_T)\|_{\widetilde{W}_p^{s-1}}.$$

From Lemma 3.7 $\|\mathcal{L}_T(u,\alpha,\tau)\|_{\widetilde{W}_p^{s-1}} \le C\left(\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^s} + \|\tau\|_{\widetilde{W}_p^s}\right)$.

By Lemma 3.2

$$\|\operatorname{tr}\left(\nabla\left((\rho_e\circ T)u\right)\right)\mathcal{Z}_T\|_{\widetilde{W}_p^s} \le C\|\operatorname{tr}\left(\nabla\left((\rho_e\circ T)u\right)\right)\|_{\widetilde{W}_p^s}\|\mathcal{Z}_T\|_{\widetilde{W}_p^1} \le C\|u\|_{\widetilde{W}_p^{s-1}}$$

and

$$\|-\kappa\operatorname{tr}\left(\nabla\left(\nabla(\tau)\mathcal{Z}_T\right)\mathcal{Z}_T\right) + c\rho_e\theta_e\operatorname{tr}\left(\nabla(u)\mathcal{Z}_T\right)\|_{\widetilde{W}_p^s}$$

$$\le C\left(\|\kappa\Delta\tau\|_{\widetilde{W}_p^s}\|\mathcal{Z}_T\|_{\widetilde{W}_p^1} + \|c\rho_e\theta_e\operatorname{tr}\left(\nabla(u)\right)\|_{\widetilde{W}_p^s}\|\mathcal{Z}_T\|_{\widetilde{W}_p^1}\right)$$

$$\le C\left(\|\tau\|_{\widetilde{W}_p^{s-2}} + \|u\|_{\widetilde{W}_p^{s-1}}\right).$$

Also, $\|u\cdot e_3\|_{\widetilde{W}_p^{s+1-1/p}(\partial_2 S)} \le C\|u\|_{\widetilde{W}_p^{s+1}(S)}$. Using these we have

$$\begin{aligned}\|\mathfrak{U}_{1T}U\|_{\mathfrak{B}^s} &\le C\Big(\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^s} + \|\tau\|_{\widetilde{W}_p^s} + \|u\|_{\widetilde{W}_p^{s-1}} + \|\tau\|_{\widetilde{W}_p^{s-2}} \\ &+\|u\|_{\widetilde{W}_p^{s-1}} + \|\beta\|_{\widetilde{W}_p^{s-1/p}} + \|u\|_{\widetilde{W}_p^{s+1}}\Big) \le C\left(\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^s} + \|\tau\|_{\widetilde{W}_p^s}\right).\end{aligned} \tag{3.4}$$

Thus, $\|\mathfrak{U}_{1T}U\|_{\mathfrak{B}^s} \le C\|\Pi U\|_{D_{-1}^s}$. For the second inequality, also by definition of $\mathfrak{B}^s$, we have

$$\|\mathfrak{U}_{1T}U - \mathfrak{U}_{1\widehat{T}}U\|_{\mathfrak{B}^s} \le C\left(\|\mathcal{L}_T(u,\alpha,\tau) - \mathcal{L}_{\widehat{T}}(u,\alpha,\tau)\|_{\widetilde{W}_p^{s-1}} + \|\nabla u\left(\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}}\right)\|_{\widetilde{W}_p^s}\right)$$

$$+C\left(\|\Delta\tau\left(\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}}\right)\|_{\widetilde{W}_p^s} + \|\beta'\left(\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}}\right)\|_{\widetilde{W}_p^{s+1-1/p}} + \|u\left(\mathcal{Z}_T - \mathcal{Z}_{\widehat{T}}\right)\|_{\widetilde{W}_p^{s+1}}\right)$$

$$\le C\|T-\widehat{T}\|_{\widetilde{W}_p^2}\left(\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^s} + \|\tau\|_{\widetilde{W}_p^s} + \|\beta\|_{\widetilde{W}_p^{s-1/p}}\right)$$

$$= C\|T-\widehat{T}\|_{\widetilde{W}_p^2}\|\Pi U\|_{D_{-1}^s}.$$

$\square$

Lemma 3.9. *Let $s \in (1/p, 1]$. There is a constant $C < \infty$ such that if $U \in D^1$ then*

$$\|\mathcal{B}_T(U)\|_{\widetilde{W}_p^{s-1/p}} \leq C\|\Pi U\|_{D^s_{-1}}$$

and

$$\|\mathcal{B}_T(U) - \mathcal{B}_{\widehat{T}}(U)\|_{\widetilde{W}_p^{s-1/p}} \leq C\|T - \widehat{T}\|_{\widetilde{W}_p^2}\|\Pi U\|_{D^s_{-1}}.$$

Proof. Let $U = [u, \alpha, \tau, \beta, \mathbf{d}]$ and denote the first component of $\mathcal{B}_T(U)$ by $\mathcal{B}_T^1(U)$. Then

$$\|\mathcal{B}_T^1(U) - \mathcal{B}_{\widehat{T}}^1(U)\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}$$

$$= \left\| \widetilde{\mathbb{T}}_T(u,\alpha) \cdot n_T - c\rho_e \tau E n_T - \frac{\partial}{\partial x_3}\left(\mathbf{p}_0 \exp\left(\frac{-gx_3}{c\theta_e}\right)\right)\beta n_T \right.$$

$$\left. -\widetilde{\mathbb{T}}_{\widehat{T}}(u,\alpha) \cdot n_{\widehat{T}} + c\rho_e \tau E n_{\widehat{T}} + \frac{\partial}{\partial x_3}\left(\mathbf{p}_0 \exp\left(\frac{-gx_3}{c\theta_e}\right)\right)\beta n_{\widehat{T}} \right\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}$$

$$= \left\| \mathbb{D}_T(u) \cdot n_T - c\theta_e \alpha E n_T - c\rho_e \tau E n_T - \frac{\partial}{\partial x_3}\left(\mathbf{p}_0 \exp\left(\frac{-gx_3}{c\theta_e}\right)\right)\beta n_T \right.$$

$$\left. -\mathbb{D}_{\widehat{T}}(u) \cdot n_{\widehat{T}} + c\theta_e \alpha E n_{\widehat{T}} + c\rho_e \tau E n_{\widehat{T}} + \frac{\partial}{\partial x_3}\left(\mathbf{p}_0 \exp\left(\frac{-gx_3}{c\theta_e}\right)\right)\beta n_{\widehat{T}} \right\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}$$

$$\leq \left\|\mathbb{D}_T(u) \cdot n_T - \mathbb{D}_{\widehat{T}}(u) \cdot n_{\widehat{T}}\right\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)} + \left\|-c\theta_e \alpha E n_T + c\theta_e \alpha E n_{\widehat{T}}\right\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}$$

$$+ \left\|-c\rho_e \tau E n_T + c\rho_e \tau E n_{\widehat{T}}\right\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}$$

$$+ \left\| -\frac{\partial}{\partial x_3}\left(\mathbf{p}_0 \exp\left(\frac{-gx_3}{c\theta_e}\right)\right)\beta n_T + \frac{\partial}{\partial x_3}\left(\mathbf{p}_0 \exp\left(\frac{-gx_3}{c\theta_e}\right)\right)\beta n_{\widehat{T}} \right\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}$$

Using Theorem 4.7.1 in [11] we have

$$\|\mathbb{D}_T(u) \cdot n_T - \mathbb{D}_{\widehat{T}}(u) \cdot n_{\widehat{T}}\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)} \leq C\|\mathbb{D}_T(u) \cdot n_T - \mathbb{D}_{\widehat{T}}(u) \cdot n_{\widehat{T}}\|_{\widetilde{W}_p^s(S)}.$$

By Lemma 3.5 and Lemma 3.6

$$C\|\mathbb{D}_T(u)\cdot n_T-\mathbb{D}_{\widehat{T}}(u)\cdot n_T+\mathbb{D}_{\widehat{T}}(u)\cdot n_T-\mathbb{D}_{\widehat{T}}(u)\cdot n_{\widehat{T}}\|_{\widetilde{W}_p^s(S)}$$

$$\leq C\left(\|\left(\mathbb{D}_T(u)-\mathbb{D}_{\widehat{T}}(u)\right)\cdot n_T\|_{\widetilde{W}_p^s(S)}+\|\mathbb{D}_{\widehat{T}}(u)\cdot\left(n_T-n_{\widehat{T}}\right)\|_{\widetilde{W}_p^s(S)}\right)$$

$$\leq C\left(\|\mathbb{D}_T(u)-\mathbb{D}_{\widehat{T}}(u)\|_{\widetilde{W}_p^s(S)}\|n_T\|_{\widetilde{W}_p^1(S)}+\|\mathbb{D}_{\widehat{T}}(u)\|_{\widetilde{W}_p^s(S)}\|n_T-n_{\widehat{T}}\|_{\widetilde{W}_p^1(S)}\right).$$

Also by Lemma 3.4 and Lemma 3.5 we have

$$\left\|-c\theta_e\alpha En_T+c\theta_e\alpha En_{\widehat{T}}\right\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}\leq\|\alpha\|_{\widetilde{W}_p^2}\left\|n_T-n_{\widehat{T}}\right\|_{\widetilde{W}_p^{1-1/p}(\partial_2 S)}$$

$$\leq\|\alpha\|_{\widetilde{W}_p^2}\left\|n_T-n_{\widehat{T}}\right\|_{\widetilde{W}_p^{1-1/p}(\partial_2 S)}\leq\|\alpha\|_{\widetilde{W}_p^2}\left\|T-\widehat{T}\right\|_{\widetilde{W}_p^2(\partial_2 S)}$$

and

$$\left\|-c\rho_e\tau En_T+c\rho_e\tau En_{\widehat{T}}\right\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}\leq\|\tau\|_{\widetilde{W}_p^2}\left\|n_T-n_{\widehat{T}}\right\|_{\widetilde{W}_p^{1-1/p}(\partial_S)}$$

$$\leq\left\|\tau\|_{\widetilde{W}_p^2}\|n_T-n_{\widehat{T}}\right\|_{\widetilde{W}_p^{1-1/p}(\partial_2 S)}\leq\|\tau\|_{\widetilde{W}_p^2}\left\|T-\widehat{T}\right\|_{\widetilde{W}_p^2(\partial_2 S)}$$

By Lemma 3.5

$$\|\beta(n_T-n_{\widehat{T}})\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}\leq C\|\beta\|_{\widetilde{W}_p^s(S)}\|n_T-n_{\widehat{T}}\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}$$

Using Theorem 4.7.1(b) in [11] there exists an extension $\widetilde{\beta}\in\widetilde{W}_p^s(S)$ where $\widetilde{\beta}|_{\partial S}=\beta$ and

$$\|\widetilde{\beta}\|_{\widetilde{W}_p^s(S)}\leq C\|\beta\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}.$$

Then

$$\|\beta\|_{\widetilde{W}_p^s(S)}\|n_T-n_{\widehat{T}}\|_{\widetilde{W}_p^{s-1/p}(\partial_2 S)}\leq\|\widetilde{\beta}\|_{\widetilde{W}_p^{s-1/p}(S)}+\|T-\widehat{T}\|_{\widetilde{W}_p^2(S)}.$$

Thus,

$$\|\mathcal{B}_T^1(U)-\mathcal{B}_{\widehat{T}}^1(U)\|_{\widetilde{W}_p^{s-1/p}}$$

$$\leq C\|T-\widehat{T}\|_{\widetilde{W}_p^2}\left(\|u\|_{\widetilde{W}_p^{s+1}}+\|\alpha\|_{\widetilde{W}_p^s}+\|\tau\|_{\widetilde{W}_p^{s+1}}+\|\widetilde{\beta}\|_{\widetilde{W}_p^{s-1/p}}\right)$$

$$\leq C\|T-\widehat{T}\|_{\widetilde{W}_p^2}\|\Pi U\|_{D_{-1}^s}.$$

The first inequality follows from this one by taking $\widehat{T}=E$. In a similar manner we find the estimate for the boundary condition $\mathcal{B}_T^2(U)$. Adding the estimates for $\mathcal{B}_T^1(U)$ and $\mathcal{B}_T^2(U)$ together we get the desired results. □

Lemma 3.10. *Given $s\in(1/p,1]$ there exists a constant C such that if $U\in D^1$ then*

$$\|\mathcal{P}^c U\|_{D^1}\leq C\|U\|_{\mathfrak{B}^s}$$

Proof. Let $I=[0,1,0,1,0]$. Then the functional given by $U\to(U,I)_{\mathfrak{h}}$ is bounded on $\mathfrak{B}^s=\widetilde{W}_p^{s-1}(S)\times\widetilde{W}_p^{s}(S)\times\widetilde{W}_p^{s}(S)\times\widetilde{W}_p^{s+1-1/p}(\partial S)\times\widetilde{W}_p^{s+1}(S)$. Then

$$\|\mathcal{P}^c U\|_{D^1}\leq C\left|([0,\alpha,0,\beta,0],I)_{\mathfrak{h}}\right|\leq C\|[0,\alpha,0,\beta,0]\|_{L^1}\|I\|_{L^\infty}\leq C\|U\|_{\mathfrak{B}^s}.$$

□

Lemma 3.11. *If $U\in D^1$, then*

$$(\mathfrak{A}_{1E}U,[0,1,0,1,0])_{\mathfrak{h}}=0\quad\textit{and}\quad\|\mathcal{P}^c\mathfrak{A}_{1E}U\|_{D^1}=0.$$

Proof. By Lemma 2.3

$$(\mathfrak{A}_{1E}U,I)_{\mathfrak{h}}=\int_{\widetilde{S}}-\operatorname{div}(\rho_e u)dx+\rho_e\int_{\partial S}u\cdot e_3 d\sigma=0,$$

and $\mathcal{P}^c\mathfrak{A}_{1E}U=0$. □

Lemma 3.12. *Let $s\in(1/p,1]$. There is a constant $C<\infty$ such that for $U\in D^1$ then*

$$\|\mathcal{A}_{1T}U-\mathfrak{A}_{1T}U\|_{\mathfrak{B}^s}\leq C\|T-E\|_{\widetilde{W}_p^2}\|\Pi U\|_{D_{-1}^s}.$$

Proof. By definition of $\mathcal{A}_{1T}$,

$$\|\mathcal{A}_{1T}U - \mathfrak{U}_{1T}U\|_{\mathfrak{B}^s} = \|\mathcal{P}\mathfrak{U}_{1T}U - \mathfrak{U}_{1T}U\|_{\mathfrak{B}^s}$$

$$= \|E_{\mathfrak{h}}\mathfrak{U}_{1T}U - \mathcal{P}^C\mathfrak{U}_{1T}U - \mathfrak{U}_{1T}U\|_{\mathfrak{B}^s} = \|\mathcal{P}^C\mathfrak{U}_{1T}U\|_{\mathfrak{B}^s}$$

$$\leq \|\mathcal{P}^C\left(\mathfrak{U}_{1T}U - \mathfrak{U}_{1E}U\right)\|_{\mathfrak{B}^s} + \|\mathcal{P}^C\mathfrak{U}_{1E}U\|_{\mathfrak{B}^s}.$$

By Lemma 3.10

$$\|\mathcal{P}^C\left(\mathfrak{U}_{1T}U - \mathfrak{U}_{1E}U\right)\|_{\mathfrak{B}^s} \leq C\|\mathfrak{U}_{1T}U - \mathfrak{U}_{1E}U\|_{\mathfrak{B}^s}$$

and by Lemma 3.8

$$\|\mathfrak{U}_{1T}U - \mathfrak{U}_{1E}U\|_{\mathfrak{B}^s} \leq C\|T - E\|_{\widetilde{W}_p^2}\|\Pi U\|_{D^s_{-1}}.$$

Thus,

$$\|\mathcal{A}_{1T}U - \mathfrak{U}_{1T}U\|_{\mathfrak{B}^s} \leq C\|T - E\|_{\widetilde{W}_p^2}\|\Pi U\|_{D^s_{-1}}.$$

□

3.2 A-Priori Estimates for Elliptic Systems

First we consider differential operators on the set $\mathcal{C} = (0, 2\pi)^2 \times (0, 1)$ and $\mathcal{C}_2 = (0, 2\pi)^2$. Let $\sigma, \sigma_1 \notin \left\{n + \dfrac{1}{p} : n \in \mathbb{Z}\right\}$, $\sigma \geq \sigma_1$ be real numbers and let $S_k^{\sigma,p}$ for $k \in \{1, 2, 3\}$ be spaces of vector functions $u : \mathcal{C} \to \mathbb{C}^{m_1}$ for $k \in \{1, 2\}$ and $u : \mathcal{C}_2 \to \mathbb{C}^{m_2}$ for $k = 3$. These spaces are defined by

$$S_k^{\sigma,p} = \widetilde{W}_p^{\sigma+a^k_{m_1}}(\mathcal{C}) \times \cdots \times \widetilde{W}_p^{\sigma+a^k_1}(\mathcal{C}) \quad (k \in \{1, 2\})$$

and

$$S_3^{\sigma,p} = \widetilde{W}_p^{\sigma-\frac{1}{p}+a^3_1}(\mathcal{C}_2) \times \cdots \times \widetilde{W}_p^{\sigma-\frac{1}{p}+a^3_{m_2}}(\mathcal{C}_2)$$

with integers $a^k_n \geq 0$. Let $\mathfrak{L}$ be a differential operator with constant coefficients and assume that $\mathfrak{L} : S_1^{\sigma,p} \to S_2^{\sigma,p}$ and $\mathfrak{L}_\partial : S_1^{\sigma,p} \to S_3^{\sigma,p}$ are a bounded linear mappings.

Lemma 3.13. *Assume that for every* $\sigma \geq 0, \sigma \notin \left\{ n + \frac{1}{p} : n \in \mathbb{Z} \right\}$ *there exists a constant* C_σ *so that*

$$\|u\|_{S_1^{\sigma,p}} \leq C_\sigma \left(\|\mathfrak{L}(u)\|_{S_2^{\sigma,p}} + \|\mathfrak{L}_\partial(u)\|_{S_3^{\sigma,p}} \right),$$

and for $f \in S_2^{\sigma,p}, h \in S_3^{\sigma,p}$ *there exists a* $u \in S_1^{\sigma,p}$ *so that* $\mathfrak{L}(u) = f$ *and* $\mathfrak{L}_\partial(u) = h$. *For* $\sigma = \sigma_1 \in (1/p - 1, 0]$ *assume that* $u \in S_1^{\sigma_1,p}, \mathfrak{L}(u) = 0$ *and* $\mathfrak{L}_\partial(u) = 0$ *then* $u = 0$. *Then there is a constant* C_{σ_1} *so that when* $f \in S_2^{\sigma_1,p}, h \in S_3^{\sigma_1,p}$ *there is a* $u \in s_1^{\sigma_1,p}$ *with* $\mathfrak{L}(u) = f$ *and* $\mathfrak{L}_\partial(u) = h$ *fulfill the inequality*

$$\|u\|_{S_1^{\sigma_1,p}} \leq C_{\sigma_1} \left(\|f\|_{S_2^{\sigma_1,p}} + \|h\|_{S_3^{\sigma_1,p}} \right).$$

Proof. A proof of this lemma is given in Lemma 3.12 of [2]. □

Lemma 3.14. *If* $s > 1/p$ *and* $s - 1/p \notin \mathbb{Z}$, *then there exists a constant* C *satisfying the following. If* $f_1 \in \widetilde{W}_p^{s-1}(\mathcal{C})$, $f_2 \in \widetilde{W}_p^{s}(\mathcal{C})$, $f_3 \in \widetilde{W}_p^{s-1}(\mathcal{C})$, $h_1 \in \widetilde{W}_p^{s+1-1/p}(\mathcal{C}_2)$, $h_2 = (h_{21}, h_{22})^\top \in \widetilde{W}_p^{s-1/p}(\mathcal{C}_2)$, *then there exists exactly one solution* $u \in \widetilde{W}_p^{s+1}(\mathcal{C})$, $\alpha \in \widetilde{W}_p^{s}(\mathcal{C}), \tau \in \widetilde{W}_p^{s}(\mathcal{C})$ *of the equations*

$$-\Delta u + \nabla\alpha + \nabla\tau + u = f_1, \quad \operatorname{div}(u) = f_2, \quad -\operatorname{div}(u) + \Delta\tau = f_3$$

in $\mathcal{C}$ *with the boundary conditions* $u \cdot e_3 = h_1, e_k \cdot \left([\nabla u] + [\nabla u]^t\right) \cdot e_3 = (h_{21}, h_{22}, 0)^\top \cdot e_k$ *for* $k = \{1, 2\}$ *on* $\widetilde{\partial} C$ *on* $x_3 = 0$ *and* $\left([\nabla u] + [\nabla u]^t\right) \cdot e_3 - \alpha e_3 - \tau e_3 = 0$ *at* $x = 1$, *and these functions fulfill the inequality*

$$\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^{s}} + \|\tau\|_{\widetilde{W}_p^{s}}$$

$$\leq C \left(\|f_1\|_{\widetilde{W}_p^{s-1}} + \|f_2\|_{\widetilde{W}_p^{s}} + \|f_3\|_{\widetilde{W}_p^{s}} + \|h_1\|_{\widetilde{W}_p^{s+1-1/p}} + \|h_2\|_{\widetilde{W}_p^{s-1/p}} \right).$$

Proof. Since $\mathrm{div}(u)$ is lower order in the third equation it can be removed and the third equation becomes

$$\Delta\tau = \widetilde{f}_3, \quad \widetilde{f}_3 = f_3 + \mathrm{div}(u) \in \widetilde{W}^s_p.$$

Now this equation can be treated separately from the others. First since $\widetilde{f}_3 \in \widetilde{W}^s_p$ we can view this equation as

$$\Delta\tau = \mathrm{div}(g),$$

where $\mathrm{div}(g) = \widetilde{f}_3$ and $g \in L^p$. By Theorem A.1

$$\|\tau\|_{\widetilde{W}^{s+1}_p} \leq C\|f_3\|_{\widetilde{W}^{s-1}_p} + \|h_1\|_{\widetilde{W}^{s+1/p}_p} + \|h_2\|_{\widetilde{W}^{s+1-1/p}_p}. \tag{3.5}$$

Similarly, we can isolate the principal parts of the other equations.

$$\begin{gathered} -\Delta u + \nabla\alpha = \widetilde{f}_1 \\ \widetilde{f}_1 = f_1 - \nabla\tau - u \in \widetilde{W}^{s-1}_p \\ \mathrm{div}(u) = f_2 \end{gathered} \tag{3.6}$$

Then Lemma 3.13 in [2] we have the following estimate for the equations (3.6).

$$\|u\|_{\widetilde{W}^{s+1}_p} + \|\alpha\|_{\widetilde{W}^s_p} \leq C\left(\|f_1\|_{\widetilde{W}^{s-1}_p} + \|f_2\|_{\widetilde{W}^s_p} + \|h_1\|_{\widetilde{W}^{s+1-1/p}_p} + \|h_2\|_{\widetilde{W}^{s-1/p}_p}\right). \tag{3.7}$$

Putting the estimates (3.5) and (3.7) together we get the desired estimate

$$\begin{gathered} \|u\|_{\widetilde{W}^{s+1}_p} + \|\alpha\|_{\widetilde{W}^s_p} + \|\tau\|_{\widetilde{W}^{s+1}_p} \\ \leq C\left(\|f_1\|_{\widetilde{W}^{s-1}_p} + \|f_2\|_{\widetilde{W}^s_p} + \|f_3\|_{\widetilde{W}^{s-1}_p} + \|h_1\|_{\widetilde{W}^{s+1-1/p}_p} + \|h_2\|_{\widetilde{W}^{s-1/p}_p}\right). \end{gathered}$$

□

Lemma 3.15. *If $s > 1/p$ and $s - 1/p \notin \mathbb{Z}$, then there exists a constant C satisfying the following. If $f_1 \in \widetilde{W}^{s-1}_p(\mathcal{C})$, $f_2 \in \widetilde{W}^s_p(\mathcal{C})$, $f_3 \in \widetilde{W}^{s-1}_p(\mathcal{C})$, $h_1 \in \widetilde{W}^{s+1-1/p}_p(\mathcal{C}_2)$,*

$h_2 = (h_{21}, h_{22})^\top \in \widetilde{W}_p^{s-1/p}(\mathcal{C}_2)$, *then there exists exactly one solution* $u \in \widetilde{W}_p^{s+1}(\mathcal{C})$, $\alpha \in \widetilde{W}_p^s(\mathcal{C}), \tau \in \widetilde{W}_p^s(\mathcal{C})$ *of the equations*

$$-\Delta u + \nabla\alpha + \nabla\tau + u = f_1, \quad \operatorname{div}(u) = f_2, \quad -\operatorname{div}(u) + \Delta\tau = f_3$$

in $\mathcal{C}$ *with the boundary conditions* $\left([\nabla u] + [\nabla u]^t\right) \cdot e_3 - \alpha e_3 - \tau e_3 = 0$ *at* $x = 0$ *and* $u \cdot e_3 = h_1, e_k \cdot \left([\nabla u] + [\nabla u]^t\right) \cdot e_3 = (h_{21}, h_{22}, 0)^\top \cdot e_k$ *for* $k \in \{1, 2\}$ *on* $\widetilde{\partial} C$ *on* $x_3 = 1$, *and these functions fulfill the inequality*

$$\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^s} + \|\tau\|_{\widetilde{W}_p^s}$$

$$\leq C \left(\|f_1\|_{\widetilde{W}_p^{s-1}} + \|f_2\|_{\widetilde{W}_p^s} + \|f_3\|_{\widetilde{W}_p^s} + \|h_1\|_{\widetilde{W}_p^{s+1-1/p}} + \|h_2\|_{\widetilde{W}_p^{s-1/p}} \right).$$

Proof. The proof is identical to Lemma 3.14, but the boundary conditions are reversed. □

Lemma 3.16. *Let* $s \in (1/p, 1]$. *Then there exists a constant* C *such that if* $f_1 \in \widetilde{W}_p^{s-1}(S)$, $f_2 \in \widetilde{W}_p^s(S)$, $f_3 \in \widetilde{W}_p^{s-1}(S)$, $h_1 \in \widetilde{W}_p^{s+1-1/p}(\partial S)$, $h_2 \in \widetilde{W}_p^{s+1}(S), u \in \widetilde{W}_p^{s+1}(S)$, $\alpha \in \widetilde{W}_p^s(S), \tau \in \widetilde{W}_p^s(S)$ *and*

$$-\Delta u + \nabla\alpha + \nabla\tau = f_1, \quad \operatorname{div}(u) = f_2, \quad -\operatorname{div}(u) + \Delta\tau = f_3$$

in S *and* $u \cdot e_3 = h_1, \urcorner \cdot ([\nabla u] + [\nabla u]^t) \cdot e_3 = h_2 \cdot \urcorner$ *for every tangential vector field* $\urcorner$. *Then*

$$\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^s} + \|\tau\|_{\widetilde{W}_p^{s+1}}$$

$$\leq C \Big(\|f_1\|_{\widetilde{W}_p^{s-1}} + \|f_2\|_{\widetilde{W}_p^s} + \|f_3\|_{\widetilde{W}_p^s} + \|h_1\|_{\widetilde{W}_p^{s+1-1/p}} + \|h_2\|_{\widetilde{W}_p^{s-1/p}}$$

$$+ \|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} \Big).$$

If $s < s_1 \leq 1$ *and* $f_1 \in \widetilde{W}_p^{s_1-1}(S), f_2 \in \widetilde{W}_p^{s_1}(S), f_3 \in \widetilde{W}_p^{s_1+1}(S), h_1 \in \widetilde{W}_p^{s_1+1-1/p}(\partial S)$, $h_2 \in \widetilde{W}_p^{s_1+1}(S)$, *then also* $u \in \widetilde{W}_p^{s_1+1}(S), \alpha \in \widetilde{W}_p^{s_1}(S), \tau \in \widetilde{W}_p^{s_1}(S)$.

Proof. Combining Lemma 3.14 and 3.15 we can glue the two boundary conditions from these lemmas together to get the boundary condition

$$u \cdot e_3 = h_1, \quad e_k \cdot \left([\nabla u] + [\nabla u]^t\right) \cdot e_3 = \widetilde{h}_2 \cdot e_k.$$

on both sides. The estimate is obtained localizing the estimate from Lemma 3.14 and Lemma 3.15. □

3.3 Resolvent Estimates II

Lemma 3.17. *There is a constant C such that if $\operatorname{Re}(z) \geq 0, z \neq 0$ then $\mathfrak{U}_E + zE : D^1(\mathcal{A}_{\mathcal{E}}) \to B$ is bijective, and if $U \in D^1(\mathcal{A}_E)$ we have the inequality*

$$|z|\|U\|_{\mathfrak{B}^1} + \|\Pi U\|_{D^1_{-1}} \leq C\|\mathfrak{U}_E U + zU\|_{\mathfrak{B}^1} + \|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}$$

Proof. Let $U = [u, \alpha, \tau, \beta, \mathbf{d}]$ and $\mathfrak{U}_E U + zU = F = [F_1, F_2, F_3, \mathbf{g}_1, \mathbf{g}_3]$. With the operator A and the domain of definition $D(A)$ defined in Section 2.2 we have $[u, \alpha, \tau, \beta, \mathbf{d}] \in D(A)$. The equation $\mathfrak{U}_E U + zU = F$ is equivalent to the system of equations

$$A[u, \alpha, \tau, \beta, \mathbf{d}] + z[u, \alpha, \tau, \beta, \mathbf{d}] = [F_1, F_2, F_3, \mathbf{g}_1, \mathbf{g}_3],$$

$\mathfrak{U}_E$ maps $D^1(\mathcal{A}_E)$ into B by Lemma 2.3. Lemma 2.15 gives the estimate

$$\|u\|_{\widetilde{W}^2_p} + |z|\|u\|_{\widetilde{L}_p} + (1 + |z|)\left(\|\alpha\|_{\widetilde{W}^1_p} + \|\tau\|_{\widetilde{W}^1_p}\right) + \|\beta\|_{\widetilde{W}^{1-1/p}_p} + |z|\|\beta\|_{\widetilde{W}^{2-1/p}_p}$$

$$\leq C\left(\|F_1\|_{\widetilde{L}_p} + \|F_2\|_{\widetilde{W}^1_p} + \|F_3\|_{\widetilde{W}^1_p} + \|\mathbf{g}_1\|_{\widetilde{W}^{2-1/p}_p} + \|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}\right).$$

The estimate for $\mathbf{d}$ follows by using the equation $-u + z\mathbf{d} = \mathbf{g}_3$. Since $A + zE$ is surjective we have $\mathfrak{U}_E + zE$ is surjective. □

In studying the operator $\mathfrak{U}_T$ we need to consider right boundary values for $\mathfrak{U}_{1E}$.

Lemma 3.18. *There exists a number C and a linear operator $\mathcal{R}(\mathbf{g}) = [u, 0, \tau, 0, 0]$ and $\mathcal{B}_E(\mathcal{R}(\mathbf{g})) = \mathbf{g}$ for $\mathbf{g} \in \widetilde{W}_p^{1-1/p}(\partial_2 S)$ and*

$$\|\mathcal{R}(\mathbf{g})\|_{D^1} \leq C\|\mathbf{g}\|_{\widetilde{W}_p^{1-1/p}}$$

Proof. Theorem A.1 gives the existence of a unique solution $\tau \in \widetilde{W}_p^2$ of $-\Delta\tau + \tau = 0$ with boundary conditions $\tau = 0$ on $\partial_1 S$ and $\dfrac{\partial \tau}{\partial n} = \mathrm{g}_2$ on $\partial_2 S$ which fulfills the the estimate

$$\|\tau\|_{\widetilde{W}_p^2} \leq C\|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}.$$

From Lemma 9 in [6] there exists $u \in \widetilde{W}_p^2$ solving the equation $-\operatorname{div}(\mathbb{D}_E(u)) + u = 0$ with the boundary values $\mathbb{D}_E(u) \cdot n = \mathrm{g}$ on $\partial_2 S$ which fulfills the estimate

$$\|u\|_{\widetilde{W}_p^2} \leq C\|\mathrm{g}\|_{\widetilde{W}_p^{1-1/p}}.$$

Defining $\mathcal{R}(\mathbf{g}) = [u, 0, \tau, 0, 0]$ where $\mathbf{g} = \begin{bmatrix} \mathrm{g}_1 \\ \mathrm{g}_2 \end{bmatrix}$ we have

$$\mathrm{g}_1 = \mathbb{D}_E(u) \cdot n - c\rho_e \tau E n.$$

Let $\mathrm{g} = \mathrm{g}_1 + c\rho_e \tau E n$. Then on $\partial_2 S$. Thus,

$$\|\mathcal{R}(\mathbf{g})\|_{D^1} \leq C\|u\|_{\widetilde{W}_p^2} + \|\tau\|_{\widetilde{W}_p^2} \leq C\left(\|\mathrm{g}\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}\right)$$

$$\leq C\left(\|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\tau\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}\right)$$

$$\leq C\left(\|\mathrm{g}_1\|_{\widetilde{W}_p^{1-1/p}} + \|\mathrm{g}_2\|_{\widetilde{W}_p^{1-1/p}}\right).$$

Using a slight abuse of notation we write $\mathbf{g} \in \widetilde{W}_p^{1-1/p}(\partial_2 S)$ as both $\mathrm{g}_1 \in \widetilde{W}_p^{1-1/p}(\partial_2 S)$ and $\mathrm{g}_2 \in \widetilde{W}_p^{1-1/p}(\partial_2 S)$. Thus,

$$\|\mathcal{R}(\mathbf{g})\|_{D^1} \leq C\|\mathbf{g}\|_{\widetilde{W}_p^{1-1/p}}.$$

□

Lemma 3.19. *For $r_0 > 0$, there exists a C with the following properties. If $z \in \mathbb{C}$, $0 < |z| < r_0$ and $\mathrm{Re}(z) > 0, U \in B \cap D^1$, we have*

$$|z|\|U\|_{\mathfrak{B}^1} + \|\Pi U\|_{D_1^{-1}} \leq C\left(\|\mathfrak{U}_{1E}U + zU\|_{\mathfrak{B}^1} + \|\mathcal{B}_E(U)\|_{\widetilde{W}_p^{1-1/p}}\right)$$

$$+C\left(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}\right).$$

Proof. Let $\mathcal{B}_E(U) = \mathbf{g} = \begin{bmatrix} \mathrm{g}_1 \\ 0 \end{bmatrix}$ and $U_0 = \mathcal{R}(\mathbf{g})$. Then $U - U_0 \in D^1(\mathcal{A}_E)$. Then from Lemma 3.17 and Lemma 3.18 we have

$$|z|\|U - U_0\|_{\mathfrak{B}^1} + \|\Pi(U - U_0)\|_{D^1_{-1}} \leq C\|\mathfrak{U}_{1E}(U - U_0) + z(U - U_0)\|_{\mathfrak{B}^1}$$

$$\leq C\left(\|\mathfrak{U}_{1E}U + zU\|_{\mathfrak{B}^1} + \|\mathfrak{U}_{1E}U_0\|_{\mathfrak{B}^1} + |z|\|U_0\|_{\mathfrak{B}^1}\right)$$

$$\leq C\left(\|\mathfrak{U}_{1E}U + zU\|_{\mathfrak{B}^1} + \|\mathbf{g}\|_{\widetilde{W}_p^{1-1/p}} + \|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}\right).$$

□

Lemma 3.20. *Let $s \in (1/p, 1]$. Then there exists a constant $C < \infty$ such that if $U \in D^s \cap B$, then*

$$\|\Pi U\|_{D^s_{-1}} \leq C\left(\|\Pi\mathfrak{U}_{1E}U\|_{\mathfrak{B}^s} + \|\mathcal{B}_E(U)\|_{\widetilde{W}_p^{s-1/p}}\right)$$

$$+C\left(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}\right).$$

Proof. Let $\mathfrak{U}_{1E}U = F = [F_1, F_2, F_3, \mathbf{g}_1, \mathbf{g}_3], U = [u, \alpha, \tau, \beta, \mathbf{d}]$ and $\mathcal{B}_E(U) = \mathrm{g}$.

$$-\Delta u + c\nabla\left(\mu^{-1}(\theta_e\alpha + \rho_e\tau)\right) = \mu^{-1}\rho_e F_1 + \frac{\nu}{\mu}\nabla\left(\rho_e^{-1}(F_2 - u\cdot\nabla\rho_e)\right)$$

$$\mathrm{div}(u) = -\rho_F e^{-1}F_2 - \rho_e^{-1}\nabla\rho_e\cdot u$$

$$-\Delta\tau = \kappa^{-1}\rho_e c_2 F_3 - c\kappa^{-1}\theta_e\left(u\cdot\nabla\rho_e - F_2\right)$$

And the boundary condition $u_3 = \mathbf{g}_1$. Looking at the tangential part of $\mathcal{B}_E^1(U) = \mathrm{g}_1$ we have

$$\urcorner\cdot(\mathbb{D}(u) - pE)\cdot n = \mu^{-1}\urcorner\cdot\mathbb{D}(u)\cdot n = \mu^{-1}\urcorner\cdot\widetilde{\mathbb{T}}_E(u,\alpha,\tau)\cdot n = \mu^{-1}\urcorner\cdot\mathcal{B}_E(U) = \mu^{-1}\mathrm{g}_1\cdot\urcorner.$$

From Lemma 3.15 we have

$$\|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^s} + \|\tau\|_{\widetilde{W}_p^s} \leq C\left(\|\Pi F\|_{\mathfrak{B}^s} + \|\mathrm{g}\|_{\widetilde{W}_p^{s-1/p}} + \|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p}\right).$$

From equation (2.23) in Theorem 2.15 and the boundary condition

$$c\theta_e\rho_{ex_3}\beta E = n\cdot\widetilde{\mathbb{T}}_E(u,\alpha,\tau)\cdot n - gE\cdot n$$

we have

$$\|\Pi U\|_{D_{-1}^s} = \|u\|_{\widetilde{W}_p^{s+1}} + \|\alpha\|_{\widetilde{W}_p^s} + \|\tau\|_{\widetilde{W}_p^s} + \|\beta\|_{\widetilde{W}_p^{s-1/p}}$$

$$\leq C\left(\|\Pi F\|_{\mathfrak{B}^s} + \|\mathrm{g}\|_{\widetilde{W}_p^{s-1/p}} + \|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}\right).$$

If $\Pi F \in \mathfrak{B}^1$ and $\mathrm{g} \in \widetilde{W}_p^{1-1/p}$, then $u \in \widetilde{W}_p^2, \alpha \in \widetilde{W}_p^1, \tau \in \widetilde{W}_p^1$, and $\beta \in \widetilde{W}_p^{1-1/p}$. Using this along with Lemma 2.4 we get that (u,α,τ,β) is unique. We can remove $\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}$ from the right-hand side of the above inequality and arrive at the desired claim. □

Lemma 3.21. *For $r_0 > 0$ there exists numbers $\eta > 0, C < \infty$ such that if $\|T - E\|_{\widetilde{W}_p^2(S)} \leq \eta$, $U \in D^1 \cap B$ and $\mathrm{Re}(z) \geq 0, z \neq 0$, then*

$$|z|\|U\|_{\mathfrak{B}^1} + \|\Pi U\|_{D_{-1}^1} \leq C\left(\|\mathfrak{U}_{1T}U + zU\|_{\mathfrak{B}^1} + (1+|z|)^{1/2-1/2p}\|\mathcal{B}_T(U)\|_{\widetilde{W}_p^{1-1/p}}\right)$$

$$+C\left(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}\right).$$

Proof. Let $\mathfrak{U}_{1T}U + zU = F, \mathcal{B}_T(U) = \mathbf{g}$. Then

$$\mathfrak{U}_{1E}U + zU = (\mathfrak{U}_{1E} - \mathfrak{U}_{1T})\, U + F.$$

By Lemma 3.8

$$\|\mathfrak{U}_{1E}U + zU\|_{\mathfrak{B}^1} \leq C\eta \|\Pi U\|_{D^1_{-1}} + \|F\|_{\mathfrak{B}^1}.$$

From Lemma 3.9

$$\|\mathcal{B}_E(U)\|_{\widetilde{W}_p^{1-1/p}} \leq \|\mathcal{B}_T(U)\|_{\widetilde{W}_p^{1-1/p}} + \|\mathcal{B}_E(U) - \mathcal{B}_T(U)\|_{\widetilde{W}_p^{1-1/p}} \leq \|\mathbf{g}\|_{\widetilde{W}_p^{1-1/p}} + C\eta \|\Pi U\|_{D^1_{-1}}.$$

If $r_0 < \infty$ then by Lemma 3.19 there is a constant C so that when $|z| < r_0$ we have

$$|z|\|U\|_{\mathfrak{B}^1} + \|\Pi U\|_{D^1_{-1}} \leq C \left(\|\mathbf{g}\|_{\widetilde{W}_p^{1-1/p}} + \eta \|\Pi U\|_{D^1_{-1}} + \|F\|_{\mathfrak{B}^1} + \eta \|\Pi U\|_{D^1_{-1}} \right).$$

□

Lemma 3.22. *There exists numbers $\eta > 0, C < \infty$ such that if $\|T - E\|_{\widetilde{W}_p^2(S)} \leq \eta$, $U \in D^1 \cap B$ and $\mathrm{Re}(z) \geq 0, z \neq 0$, then*

$$|z|\|U\|_{\mathfrak{B}^1} + \|\Pi U\|_{D^1_{-1}} \leq C \left(\|\mathcal{A}_{1T}U + zU\|_{\mathfrak{B}^1} + (1 + |z|)^{1/2 - 1/2p} \|\mathcal{B}_T(U)\|_{\widetilde{W}_p^{1-1/p}} \right)$$

$$+ C \left(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p} \right).$$

Proof. Let $\mathcal{A}_{1T}U + zU = F$ and $\mathcal{B}_T(U) = \mathbf{g}$. For $|z| \leq r_0$ we have by Lemma 3.21

$$\mathfrak{U}_{1T}U + zU = F + \mathfrak{U}_{1T}U - A_{1T}U.$$

Taking $U \in B$ and using Lemma 3.12 we have

$$|z|\|U\|_{\mathfrak{B}^1} + \|\Pi U\|_{D^1_{-1}}$$

$$\leq C \left(\|F\|_{\mathfrak{B}^1} + (1 + |z|)^{1/2 - 1/2p} \|\mathbf{g}\|_{\widetilde{W}_p^{1-1/p}} + \|(\mathcal{A}_{1T} - \mathfrak{U}_{1T})U\|_{\mathfrak{B}^1} \right)$$

$$\leq C\|F\|_{\mathfrak{B}^1} + C(1 + |z|)^{1/2 - 1/2p} \|\mathbf{g}\|_{\widetilde{W}_p^{1-1/p}} + C\eta \|\Pi U\|_{D^1_{-1}} + C\|\mathbf{g}\|_{L_1(\partial S)}.$$

Taking $\eta > 0$ to be sufficiently small we get the desired result.

For large z we can easily prove the estimate by the same method as was previously done in Theorem 2.11 from Chapter 2.

$\square$

Lemma 3.23. *If $s \in (1/p, 1]$, then there exists numbers $\eta > 0$, $C < \infty$ such that if $\|T - E\|_{\widetilde{W}^2_p(S)} \leq \eta$, $U \in D^1(\mathcal{A}_T)$, then*

$$\|\Pi U\|_{D^s_{-1}} \leq C\|\Pi \mathcal{A}_T U\|_{\mathfrak{B}^s} + C\left(\|u\|_{\widetilde{L}_p} + \|\alpha\|_{\widetilde{L}_p} + \|\tau\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{L}_p}\right).$$

Proof. By Lemma 3.9

$$\|\mathcal{B}_E(U)\|_{\widetilde{W}^{s-1/p}_p} = \|\mathcal{B}_E(U) - \mathcal{B}_T(U)\|_{\widetilde{W}^{s-1/p}_p} \leq C\eta\|\Pi U\|_{D^s_{-1}}. \tag{3.8}$$

Since $\in D^1(\mathcal{A}_T)$, we have $\mathcal{P}^c U = 0$ and $\mathcal{B}_T(U) = 0$. Then using Lemma 3.12

$$\|\mathfrak{U}_{1T}U - \mathcal{A}_{1T}U\|_{\mathfrak{B}^s} \leq C\|T - E\|_{\widetilde{W}^2_p}\|\Pi U\|_{D^s_{-1}} \leq C\eta\|\Pi U\|_{D^s_{-1}}$$

by hypothesis. Also, using Lemma 3.8

$$\|\mathfrak{U}_{1E}U - \mathfrak{U}_{1T}U\|_{\mathfrak{B}^s} \leq C\|E - T\|_{\widetilde{W}^2_p}\|\Pi U\|_{D^s_{-1}} \leq C\eta\|\Pi U\|_{D^s_{-1}}.$$

Thus,

$$\begin{aligned}\|\Pi\mathfrak{U}_{1E}U\|_{\mathfrak{B}^s} &\leq \|\mathfrak{U}_{1E}U - \mathfrak{U}_{1T}U\|_{\mathfrak{B}^s} + \|\mathfrak{U}_{1T}U - \mathcal{A}_{1T}U\|_{\mathfrak{B}^s} + \|\Pi\mathcal{A}_{1T}U\|_{\mathfrak{B}^s} \\ &\leq C\eta\|\Pi U\|_{D^s_{-1}} + \|\Pi\mathcal{A}_{1T}U\|_{\mathfrak{B}^s}.\end{aligned}$$

From Lemma 3.20 we have

$$\|\Pi U\|_{D^s_{-1}} \leq C\left(\|\Pi\mathfrak{U}_{1E}U\|_{\mathfrak{B}^s} + \|\mathcal{B}_E(U)\|_{\widetilde{W}^{s-1/p}_p}\right) \tag{3.9}$$

and by equation (3.8) and equation (3.9) this becomes

$$\|\Pi U\|_{D^s_{-1}} \le C\eta\|\Pi U\|_{D^s_{-1}} + \|\Pi\mathcal{A}_{1T}U\|_{\mathfrak{B}^s} + \|\mathcal{B}_E(U)\|_{\widetilde{W}_p^{s-1/p}}$$

$$\le C\eta\|\Pi U\|_{D^s_{-1}} + \|\Pi\mathcal{A}_{1T}U\|_{\mathfrak{B}^s}$$

for a different C. Taking $\eta > 0$ to be sufficiently small we have

$$\|\Pi U\|_{D^s_{-1}} \le C\|\Pi\mathcal{A}_T U\|_{\mathfrak{B}^s}.$$

$\square$

3.4 The Operator $\mathcal{A}_T$ Generates an Analytic Semigroup

This section is devoted to showing that the operator $\mathcal{A}_T$ generates a strongly continuous one parameter semigroup of linear operators on a Banach Space. First we show that $\mathcal{A}_T$ satisfies the following five conditions. Following the strategy in [6] these conditions imply, using Theorem 2.1 in Part 2 in [2] that $\mathcal{A}_T$ generates an analytic seimgroup.

Condition 1: $D^1(\mathcal{A}_T)$ is dense in B.

Condition 2: $\mathcal{A}_T(D^1(\mathcal{A}_T)) \subset B$.

Condition 3: $\exists C \ni \forall U \in D^1(\mathcal{A}_T), z \in \mathbb{C}$ with $Re(z) \ge 0$, then $|z|\|U\|_B \le \|\mathcal{A}_T U + zU\|_B$ (3.10)

Condition 4: $\mathcal{A}_T$ is a closed operator

Condition 5: $\forall z \in \mathbb{C}\backslash 0$ with $Re(z) \ge 0$ the range of $\mathcal{A}_T U + zU$ is B.

The first three conditions are verified in the next three lemmas. First we prove Condition 2 and Condition 3 before proving Condition 1.

Lemma 3.24. *Let $z \neq 0$ and $F, U \in B$ and $\mathcal{A}_{1T}U + zU = F$. Then $U \in D^1(\mathcal{A}_T)$ exactly if $F \in B$.*

Proof. By Lemma 2.3, $zQ(U) = Q(F)$, so the condition $Q(U) = 0$ is equivalent to $Q(F) = 0$. $\square$

Lemma 3.25. *There exists a constant $C < \infty$ such that for $Re(z) \geq 0$ and $U \in D^1(\mathcal{A}_T)$ we have*

$$\|u\|_{\widetilde{W}_p^2} + \|\alpha\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^2} + \|\beta\|_{\widetilde{W}_p^{1-1/p}} + |z|\|U\|_B \leq C\|\mathcal{A}_T U + zU\|_B$$

Proof. If Lemma 3.25 is true for all such z with $z \neq 0$ and a constant independent of z then it is also true for $z = 0$. Now we prove this for $z \neq 0$. Suppose our estimate does not hold. Then we can find a sequence U_k with $0 \neq U_k \in D^1(\mathcal{A}_T)$ and a sequence $z_k \neq 0$ with $Re(z_k) \geq 0$ such that

$$\frac{\|u_k\|_{\widetilde{W}_p^2} + \|\alpha_k\|_{\widetilde{W}_p^1} + \|\tau_k\|_{\widetilde{W}_p^2} + \|\beta_k\|_{\widetilde{W}_p^{1-1p}} + |z_k|\|U_k\|_B}{\|\mathcal{A}_T U_k + z_k U_k\|_B} \to \infty.$$

By Theorem 3.22 we can multiply U_k by a suitable factor, unless $U_k = 0$, to make

$$\|u_k\|_{\widetilde{L}_p} + \|\alpha_k\|_{\widetilde{L}_p} + \|\tau_k\|_{\widetilde{L}_p} + \|\beta_k\|_{\widetilde{L}_p} = 1.$$

Also, we have by Theorem 3.22 that

$$\|u_k\|_{\widetilde{W}_p^2} + \|\alpha_k\|_{\widetilde{W}_p^1} + \|\tau_k\|_{\widetilde{W}_p^2} + \|\beta_k\|_{\widetilde{W}_p^{1-1/p}} + |z_k|\|U_k\|_B \leq C(\|\mathcal{A}_T U_k + z_k U_k\|_B + 1).$$

In order for this estimate to hold, we must therefore have that

$$\|\mathcal{A}_T U_k + z_k U_k\|_B \to 0$$

and by Theorem 3.22 z_k is bounded. Now select a subsequence of U_k, also denoted U_k, so that $u_k, \alpha_k, \tau_k, \beta_k$ converge strongly in $\widetilde{L}_p$ of their domains of definition to

$u_\infty \in \widetilde{W}_p^2, \alpha_\infty \in \widetilde{W}_p^1, \tau_\infty \in \widetilde{W}_p^2, \beta_\infty \in \widetilde{W}_p^{1-1/p}$ and $z_k \to z_\infty$. These fulfill the equations

$$\rho_e^{-1}\left(\mu\Delta u_\infty + \nu\nabla\operatorname{div}(u_\infty)\right) - \nabla\left(c\theta_e\alpha + c\rho_e\tau\right) - \alpha_\infty g e_3 - z_\infty u_\infty$$

$$-\operatorname{div}\left(\rho_e u_\infty\right) + z_\infty\alpha_\infty = 0$$

$$-c\rho_e^{-2}c_2^{-1}\left(\theta_e\operatorname{div}(u_\infty) - \kappa\Delta\tau_\infty\right) - z_\infty\tau_\infty = 0$$

with the boundary conditions

$$\widetilde{\mathbb{T}}(u_\infty, \alpha_\infty, \tau_\infty)\cdot n = c\theta_e(\rho_{ex_3}\beta_\infty)n$$

$$z_\infty\beta_\infty = u_\infty\cdot n$$

By Lemma 2.4 $u_\infty, \alpha_\infty, \tau_\infty, \beta_\infty$ are all zero, while

$$\|u_\infty\|_{\widetilde{L}_p} + \|\alpha_\infty\|_{\widetilde{L}_p} + \|\tau_\infty\|_{\widetilde{L}_p} + \|\beta_\infty\|_{\widetilde{L}_p} = 1.$$

Thus we have a contradiction. □

Lemma 3.26. *There exists a number $\eta > 0$ such that if $\|T - E\|_{\widetilde{W}_p^2(S)} \leq \eta$, then $D^1(\mathcal{A}_T)$ is dense in B and the operator $\mathcal{A}_T + zE : D^1(\mathcal{A}_T) \to B$ is bijective for* $\operatorname{Re}(z) \geq 0, z \neq 0$.

Proof. Lemma 3.25 gives the injectivity of the operator, and by definition we have $\mathcal{A}_{1T} : D^1(\mathcal{A}_T) \to B$. Let $\mathbf{g} \in \widetilde{W}_p^{1-1/p}$ and $U_0 = \mathcal{R}(\mathrm{g})$, where $\mathcal{R}$ is defined in Lemma 3.18, so $U_0 \in B$ and $\mathcal{B}_E(U_0) = \mathbf{g}$. When $F \in B$ then we have $F - U_0 - \mathcal{A}_{1E}U_0 \in B$. By Theorem 2.11 there exists as solution $V \in D^1(\mathcal{A}_E)$ of the equation

$$V + \mathcal{A}_E V = V + \mathfrak{U}_E V = F - U_0 - \mathcal{A}_{1E}U_0.$$

Then $U = U_0 + V \in B$ and

$$U + \mathcal{A}_{1E}U = U_0 + \mathcal{A}_{1E}U_0 + V + \mathcal{A}_E V = U_0 + \mathcal{A}_{1E}U_0 + F - U_0 - \mathcal{A}_{1E}U_0 = F.$$

Also,

$$\mathcal{B}_E(U) = \mathcal{B}_E(U_0) + \mathcal{B}_E(V) = \mathbf{g} + 0 = \mathbf{g}.$$

Then there is exactly one $U \in D^1 \cap B$ such that $\mathcal{B}_E(U) = \mathbf{g}$ and $U + \mathcal{A}_{1E}U = F$, where $\mathbf{g} \in \widetilde{W}_p^{1-1/p}$ and $F \in B$. If $\eta > 0$ is taken to be sufficiently small, the solution of $U + \mathcal{A}_{1T}U = F$ with $\mathcal{B}_T(U) = 0$ where $U \in D^1 \cap B$ can be found by looking at the following equations

$$U + \mathcal{A}_{1E}U = \mathcal{A}_{1E}U - \mathcal{A}_{1T}U + F, \quad \mathcal{B}_E(U) = \mathcal{B}_E(U) - \mathcal{B}_T(U)$$

as well as using the Banach fixed point theorem. The claim for other z follows using the Neumann resolvent series. It remains to show that $D^1(\mathcal{A}_T)$ is dense in B. For this we define the operator $\mathcal{R}_T : \widetilde{W}_p^{1-1/p}(\partial S) \to B \cap D^1$ such that $\mathcal{B}_T(\mathcal{R}_T(\mathbf{g})) = \mathbf{g}$ and $\mathcal{R}_T(\mathbf{g}) = [u, 0, \tau, 0, 0]$, where the operator $\mathcal{R}$ is from Lemma 3.18. If $\eta > 0$ is small enough, then

$$\begin{aligned}
&\|\mathcal{R}\left(\mathbf{g} - \mathcal{B}_T^1(U_1) + \mathcal{B}_E^1(U_1)\right) - \mathcal{R}\left(\mathbf{g} - \mathcal{B}_T^1(U_2) + \mathcal{B}_E^1(U_2)\right)\|_{D^1} \\
&= \|\mathcal{R}\left(\mathcal{B}_T^1(U_2 - U_1) - \mathcal{B}_E^1(U_2 - U_1)\right)\|_{D^1} \\
&\leq C\|\mathcal{B}_T^1(U_2 - U_1) - \mathcal{B}_E^1(U_2 - U_1)\|_{\widetilde{W}_p^{1-1/p}} \leq C\|T - E\|_{\widetilde{W}_p^2}\|U_2 - U_1\|_{D^1},
\end{aligned}$$

where the last inequality follows from Lemma 3.9. From these, we can solve the equation

$$U = \mathcal{R}\left(\mathbf{g} - \mathcal{B}_T^1(U) + \mathcal{B}_E^1(U)\right)$$

using the Banach fixed point theorem. If $U = [u, \alpha, \tau, \beta, \mathbf{d}] \in B$, then we also have $[0, \alpha, 0, \beta, \mathbf{d}] \in B$. Take $\mathbf{g} = \mathcal{B}_T([0, \alpha, 0, \beta, \mathbf{d}])$ and $U_1 = [0, \alpha, 0, \beta, \mathbf{d}] - \mathcal{R}_T(\mathbf{g})$. Then $\mathcal{B}_T(U_1) = \mathcal{B}_T([0, \alpha, 0, \beta, \mathbf{d}] - \mathcal{R}_T(\mathbf{g})) = \mathbf{g} - \mathbf{g} = 0$, so $U_1 \in D(\mathcal{A}_T)$. Also,

$U_0 = U - U_1 = [u_0, 0, \tau_0, 0, 0] \in B$. Take sequences $u_k, \tau_k \in \widetilde{C}_0^\infty(S)$ converging to u_0, τ_0 in $\widetilde{L}_p$. Then $\mathcal{P}[u_k, 0, \tau_0, 0, 0] \to \mathcal{P}[u_0, 0, \tau_0, 0, 0] = [u_0, 0, \tau_0, 0, 0]$, which is also in $\widetilde{L}_p$. Now take $V_k = \mathcal{P}[u_k, 0, \tau_k, 0, 0] = [v_k, 0, \tau_k, 0, 0]$, so that $V_k \in B \cap D^1$ and $\mathcal{B}_T(V_k) = \mathcal{B}_T(\mathcal{P}[u_k, 0, \tau_k, 0, 0]) + \mathcal{B}_T(\mathcal{P}^c[u_k, 0, \tau_k, 0, 0]) = \mathcal{B}_T(\mathcal{P} + \mathcal{P}^c)[u_k, 0, \tau_k, 0, 0] = \mathcal{B}_T([u_k, 0, \tau_k, 0, 0]) = 0$. Thus, $V_k \in D(\mathcal{A}_T)$. As $\mathcal{P}^c[u_k, 0, \tau_k, 0, 0] \to \mathcal{P}^c[u_0, 0, \tau_0, 0, 0] = 0$ in D^1 we conclude $v_k \to u_0$ and $\tau_k \to \tau_0$ in $\widetilde{L}_p$. Therefore $U_1 + V_k \in D^1(\mathcal{A}_T)$ and $U_1 + V_k \to U_1 + [u_0, 0, \tau_0, 0, 0] = U_1 + U - U_1 = U \in \mathfrak{B}^1$ we conclude that $D^1(\mathcal{A}_T)$ is dense in B. $\square$

Condition 4 is easily verified as $\|U\|_B + \|\mathcal{A}_T U\|_B$ bounds all relevant quantities. Condition 5 is verified in the following lemma.

Lemma 3.27. *The operator $\mathcal{A}_T$ satisfies condition 5 of (3.10).*

Proof. By Lemma (3.26), all large z when $Re(z) \geq 0$ belongs to the resolvent set. Using Lemma (3.25), the resolvent set contains all $z \neq 0$ with $Re(z) \geq 0$. By the continuity argument, which asserts that any subset of a connected set which is open, closed, and nonempty must be the whole set, we conclude that the range of $\mathcal{A}_T U + zU$ must be all of B. $\square$

Theorem 3.28. *The operator $\mathcal{A}_T$ generates an analytic semigroup. Define*

$$U = [u(t), \alpha(t), \tau(t), \beta(t), \mathbf{d}(t)] = \exp(t\mathcal{A}_T)[u_0, \alpha_0, \tau_0, \beta_0, \mathbf{d}_0]$$

where $[u_0, \alpha_0, \tau_0, \beta_0, \mathbf{d}_0] \in S$ and $t > 0$. There exists C such that

$$t\|u\|_{\widetilde{W}_p^2} + t\|\alpha\|_{\widetilde{W}_p^1} + t\|\tau\|_{\widetilde{W}_p^2} + t\|\beta\|_{\widetilde{W}_p^{1-1/p}} + \|u(t)\|_{\widetilde{L}_p} + \|\alpha(t)\|_{\widetilde{W}_p^1} + \|\tau(t)\|_{\widetilde{L}_p}$$

$$+\|\mathbf{d}\|_{\widetilde{W}_p^2} + \|\beta(t)\|_{\widetilde{W}_p^{2-1/p}} \leq C\left(\|u_0\|_{\widetilde{L}_p} + \|\alpha_0\|_{\widetilde{W}_p^1} + \|\tau_0\|_{\widetilde{L}_p} + \|\mathbf{d}_0\|_{\widetilde{W}_p^2} + \|\beta_0\|_{\widetilde{W}_p^{2-1/p}}\right).$$

Proof. The operator $\mathcal{A}_T$ generates an analytic semigroup, which means all five conditions of (3.10) are satisfied. From Lemma 3.25 we have

$$\|u\|_{\widetilde{W}_p^2} + \|\alpha\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^2} + \|\beta\|_{\widetilde{W}_p^{1-1/p}} + |z|\|U\|_B \leq C\|\mathcal{A}_T U + zU\|_B. \tag{3.11}$$

The desired estimate follows from equation (3.11) by means of Theorem 2.1 in [2], which gives us the estimates

$$\|U\|_B \leq C \quad \text{and} \quad \|\mathcal{A}_T U\|_B \leq \frac{C}{|t|}.$$

Using the definition of Uin the inequality (3.11) we find the estimate

$$t\left(\|u\|_{\widetilde{W}_p^2} + \|\alpha\|_{\widetilde{W}_p^1} + \|\tau\|_{\widetilde{W}_p^2} + \|\beta\|_{\widetilde{W}_p^{1-1/p}}\right) + \|u(t)\|_{\widetilde{L}_p} + \alpha(t)\|_{\widetilde{W}_p^1} + \|\tau(t)\|_{\widetilde{L}_p} + \|\mathbf{d}\|_{\widetilde{W}_p^2}$$

$$+\|\beta(t)\|_{\widetilde{W}_p^{2-1/p}} \leq Ct\|\mathcal{A}_T e^{\mathcal{A}_T t} U_0\|_B + C\|e^{\mathcal{A}_T t} U_0\|_B$$

$$\leq Ct\frac{C}{t}\|U_0\|_B + C\|U_0\|_B \leq C\|U_0\|_B$$

for a different constant C. □

3.4.1 The Proof Theorem 3.1

Proof. We prove this using analytic semigroup theory we have as described in Section 3.4

$$\|U(t)\|_{\mathfrak{B}^1} + t\|U'(t)\|_{\mathfrak{B}^1} + t^2\|\mathcal{A}_T U'(t)\|_{\mathfrak{B}^1} \leq C\|U_0\|_{\mathfrak{B}^1}$$

By Lemma 3.23

$$\|\Pi U'(t)\|_{D^1_{-1}} \leq C\|\Pi \mathcal{A}_T U'(t)\|_{\mathfrak{B}^1} \leq C\|\mathcal{A}_T U'(t)\|_{\mathfrak{B}^1}.$$

The remaining term we need to estimate is $t^{2-s}\|\Pi U(t)\|_{D^1_{-1}}$. Using Lemma 3.23 again with $s = 1$ we have

$$\|\Pi \mathcal{A}_T U(t)\|_{D^1_{-1}} \le \|\mathcal{A}_T^2 U(t)\|_{\mathfrak{B}^1} \le Ct^{-2}\|U_0\|_{\mathfrak{B}^1}$$

and

$$\|\Pi \mathcal{A}_T U(t)\|_{\mathfrak{B}^1} \le \|\mathcal{A}_T U(t)\|_{\mathfrak{B}^1} \le Ct^{-1}\|U_0\|_{\mathfrak{B}^1}.$$

For $\widehat{U} = [\widehat{u}, \widehat{\alpha}, \widehat{\tau}, \widehat{\beta}, \widehat{\mathbf{d}}] \in D^1$ we have by definition of the spaces D^1, D^1_1, and $\mathfrak{B}^1$

$$\|[\widehat{u}, \widehat{\alpha}, \widehat{\tau}, 0, \widehat{\mathbf{d}}]\|_{\mathfrak{B}^s} \le C \min\left(\|[\widehat{u}, \widehat{\alpha}, \widehat{\tau}, 0, \widehat{\mathbf{d}}]\|_{D^1_{-1}}, \|[\widehat{u}, \widehat{\alpha}, \widehat{\tau}, 0, \widehat{\mathbf{d}}]\|_{\mathfrak{B}^1}\right).$$

For $s > 1/p$

$$\begin{aligned} \|[0,0,0,\widehat{\beta},0]\|_{\mathfrak{B}^s} = \|\widehat{\beta}\|_{\widetilde{W}_p^{s+1-1/p}} &\le C\|\widehat{\beta}\|^{1-s}_{\widetilde{W}_p^{1-1/p}}\|\widehat{\beta}\|^s_{\widetilde{W}_p^{2-1/p}} \\ &\le C\|[0,0,0,\widehat{\beta},0]\|^{1-s}_{D^1_{-1}}\|[0,0,0,\widehat{\beta},0]\|^s_{\mathfrak{B}^1}. \end{aligned}$$

Thus, for $\widehat{U} \in D^1$

$$\|\widehat{U}\|_{\mathfrak{B}^1} \le C\|\widehat{U}\|^{1-s}_{D^1_{-1}}\|\widehat{U}\|^s_{\mathfrak{B}^1}.$$

Then

$$\|\Pi U(t)\|_{D^s_{-1}} \le \|\Pi \mathcal{A}_T U(t)\|_{\mathfrak{B}^s} \le Ct^{s-2}\|U_0\|_{\mathfrak{B}^1}.$$

$\square$

CHAPTER 4
ADDITIONAL A-PRIORI ESTIMATES

In this chapter we consider the case where the transformation T is no longer static, and we replace it with a family of transformations $\mathbf{T} : \overline{S} \times [0,\omega] \to \mathbb{R}^3$ defined by

$$\mathbf{T}(x,t) = T_t(x) \,\text{for}\, x \in S \,\text{and}\, t \in I_\omega, \tag{4.1}$$

where $I_\omega = [0,\omega] \cap \mathbb{R}$ for some $\omega \in [0,\infty)$.

Also define the deformation $\mathbf{d}(x,t) = \mathbf{T}(x,t) - x$ for $x \in S$ and $t \in I_\omega$, which satisfies the equation

$$\mathbf{d}' = u. \tag{4.2}$$

The boundary conditions $u = 0$ and $\tau = 0$ on $\partial_1 S$ are tacitly assumed throughout this chapter.

4.1 Time Dependent Norms and Spaces

Definition 4.1. *For the transformations* $\mathbf{T}$ *we introduce the space*

$$\mathbf{B}^\omega = C^0\left([0,\omega], \widetilde{W}_p^2(S)\right) \cap C^{2/3}\left([0,\omega], C^1(\overline{S})\right).$$

and the weighted norm

$$\|\mathbf{T}\|_{\mathbf{B}^\omega} = \sup_{t\in[0,\omega-1]} \left[\|\mathbf{T}\|_{C^0\left([t,t+1],\widetilde{W}_p^2\right)} + (1+t)^{1/3}\|\mathbf{T}\|_{C^{2/3}([t,t+1],C^1)}\right].$$

Let $B_1^\omega = L_p\left((0,\omega),\mathfrak{B}^1\right), B_2^\omega = \widetilde{W}_p^{1-1/p,1/2-1/2p}\left(\partial_2 S \times (0,\omega)\right)$ *and*

$\mathfrak{M}^\omega = \left\{\mathbf{U} = [u,\alpha,\tau,\beta,\mathbf{d}] : \mathbf{U} \in L_p\left((0,\omega), D^1\right), \mathbf{U}' \in L_p\left((0,\omega),\mathfrak{B}^1\right)\right\}.$

Let

$$\|\mathbf{F}\|_{B_1^\omega} = \sup_{t\in[0,\omega-1]} (1+t)^{4/3} \left[\int_t^{t+1} \|\mathbf{F}(r)\|_{\mathfrak{B}^1}^p dr\right]^{1/p}$$
$$\|\mathbf{g}\|_{B_2^\omega} = \sup_{t\in[0,\omega-1]} (1+t)^{4/3} \|\mathbf{g}\|_{\widetilde{W}_p^{1-1/p,1/2-1/2p}(\partial S\times[t,t+1])}.$$

Define the norm

$$\|\mathbf{U}\|_{\mathfrak{M}_1^\omega} = \sup_{t\in[0,\omega-1]} \left[\int_t^{t+1} \|\mathbf{U}'(r)\|_{\mathfrak{B}^1}^p + \|\mathbf{U}(r)\|_{D^1}^p dr\right]^{1/p}$$

and the two seminorms

$$[\mathbf{U}]_1 = \sup_{t\in[0,\omega-1]} (1+t) \left[\int_t^{t+1} \|\mathbf{U}'(r)\|_{\mathfrak{B}^1}^p + \|\Pi\mathbf{U}(r)\|_{D_{-1}^1}^p dr\right]^{1/p}$$
$$[\mathbf{U}]_2 = \sup_{t\in[0,\omega-1]} (1+t)^{4/3} \left[\int_t^{t+1} \|\Pi\mathbf{U}'(r)\|_{\mathfrak{B}_{-1}^1}^p dr\right]^{1/p} + \sup_{t\in[0,\omega]} (1+t)^{4/3} \|\Pi\mathbf{U}(t)\|_{D_{-1}^{2/3}}.$$

Now we can describe the norm

$$\|\mathbf{U}\|_{\mathfrak{M}^\omega} = [\mathbf{U}]_1 + [\mathbf{U}]_2 + \|\mathbf{U}\|_{\mathfrak{M}_1^\omega}.$$

Having replaced the fixed transformation T from Chapter 3 by the time dependent transformation $\mathbf{T}$ and using the notation from equations (4.1) and (3.2) we define $\mathcal{Z}_{\mathbf{T}}(x,t) = \mathcal{Z}_{T_t}(x)$ for $t \in [0,\omega]$ and $x \in \overline{S}$ and analogously we have the operator $(\mathcal{A}_{1\mathbf{T}}\mathbf{U})(t) = \mathcal{A}_{1T_t}\mathbf{U}(t)$ for $t \in I_\omega$.

Theorem 4.2. *There exists numbers $\eta > 0$ and $C < \infty$ with the following properties. Let $\omega \in [1,\infty), \mathbf{T} \in \mathbf{B}^\omega$ and $\widetilde{\mathbf{T}}(x,t) = \mathbf{T}(x,\omega)$ for $x \in \overline{S}$ and $t \in [0,\omega]$. If*

$$\|\mathbf{T} - \widetilde{\mathbf{T}}\|_{\mathbf{B}^\omega} + \|\mathbf{T}(t) - E_S\|_{\widetilde{W}_p^2} \le \eta,$$

then for $\mathbf{F} \in B_1^\omega, \mathbf{g} \in B_2^\omega, U_0 \in S \cap D^{1-2/p}$ with $\mathcal{B}_{T_0}(U_0) = \mathbf{g}(0)$ there is exactly one function $\mathbf{U} \in \mathfrak{M}^\omega$ with $\mathbf{U}(t) \in S$ for $t \in I_\omega$ such that

$$\mathcal{B}_{\mathbf{T}}(U) = \mathbf{g} \tag{4.3}$$

$\mathbf{U}(0) = U_0$ *and*

$$\mathbf{U}' = -\mathcal{A}_{1\mathbf{T}}\mathbf{U} + \mathbf{F} \tag{4.4}$$

for almost all $t \in I_\omega$, *where the time derivative is meant in the sense of distributions. This function also satisfies the inequality*

$$\|\mathbf{U}\|_{\mathfrak{M}^\omega} \le C\left(\|\mathbf{F}\|_{B_1^\omega} + \|\mathbf{g}\|_{B_2^\omega} + \|U_0\|_{\mathfrak{B}^1} + \|U_0\|_{D^{1-2/p}}\right).$$

Note that the constant C is independent of ω.

4.2 Theorem Which Implies Theorem 4.2

In order to prove Theorem 4.2 we begin by proving the following theorem where $\mathbf{T}$ does not depend on time.

Theorem 4.3. *There exists numbers* $\eta > 0$ *and* $C < \infty$ *with the following properties. Assume that* $T \in \widetilde{W}_p^2(S)$ *and*

$$\|T - E_S\|_{\widetilde{W}_p^2} \le \eta,$$

and for $\omega \in [1, \infty)$ *let* $\mathbf{T}(x,t) = T(x)(x \in \overline{S}, t \in [0,\omega])$. *Then for* $\mathbf{F} \in B_1^\omega$, $\mathbf{g} \in B_2^\omega$, $U_0 \in S \cap D^{1-2/p}$ *with* $\mathcal{B}_T(U_0) = \mathbf{g}(0)$ *there is exactly one function* $\mathbf{U} \in \mathfrak{M}^\omega$ *with* $\mathbf{U}(t) \in S$ *for* $t \in [0,\omega]$ *which solves the initial boundary value problem given by the equations*

$$\mathbf{U}' = -\mathcal{A}_{1\mathbf{T}}\mathbf{U} + \mathbf{F},\ \mathcal{B}_{\mathbf{T}}(\mathbf{U}) = \mathbf{g},$$

which are equations (4.3) and (4.4), and $\mathbf{U}(0) = U_0$. *This function also satisfies the inequality*

$$\|\mathbf{U}\|_{\mathfrak{M}^\omega} \le C\left(\|\mathbf{F}\|_{B_1^\omega} + \|\mathbf{g}\|_{B_2^\omega} + \|U_0\|_{\mathfrak{B}^1} + \|U_0\|_{D^{1-2/p}}\right)$$

with a constant C independent of ω.

For the convenience of the reader the next lemma and two theorems are quoted from [8], which are also necessary for the proof of Theorem 4.2.

Lemma 4.4. *For every $\omega < \infty$ there exists a constant $C(\omega) < \infty$ such that for $\mathbf{U} \in \mathfrak{M}^\omega$,*

$$\|\mathbf{U}\|_{\mathfrak{M}_1^\omega} \leq \|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq C(\omega)\|\mathbf{U}\|_{\mathfrak{M}_1^\omega}.$$

Also $\mathfrak{M}^\omega$ is a Banach space with both norms.

Proof. A proof of this is given in Lemma 4 of [8]. □

Theorem 4.5. *There exists a number $\eta > 0$ and a constant $C(\omega)$ such that if $\|T_t - E_S\|_{\widetilde{W}_p^2} \leq \eta$, then the following is true. For $\mathbf{F} \in B_1^\omega$ with $\mathbf{F}(t) \in S$ for almost all $t, U_0 \in S \cap D^{1-2/p}$, and $\mathbf{g} \in \widetilde{W}_p^{1-1/p,1/2-1/2p}(\partial S \times (0,\omega))$ with $\mathbf{g}(0) = \mathcal{B}_{T_0}(U_0)$ there is exactly one solution $\mathbf{U} \in \mathfrak{M}^\omega$ of the equation*

$$\mathbf{U}' = -\mathcal{A}_{1\mathbf{T}}\mathbf{U} + \mathbf{F} \tag{4.5}$$

with the boundary condition

$$\mathcal{B}_{\mathbf{T}}(\mathbf{U}) = \mathbf{g} \tag{4.6}$$

and the initial value $\mathbf{U}(0) = U_0$. Also $\mathbf{U}(t) \in S$ for $t \in [0,\omega]$, and

$$\|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq C(\omega)\left(\|\mathbf{g}\|_{B_2^\omega} + \|\mathbf{F}\|_{B_1^\omega} + \|U_0\|_{\mathfrak{B}^1} + \|U_0\|_{D^{1-2/p}}\right).$$

Proof. The proof is completely analogous to the proof of Theorem 14 in [8]. □

Lemma 4.6. *There exists a constant $C < \infty$ such that for $t \in [0, \omega - 1]$,*

$$\|\mathcal{Z}_{\mathbf{T}} - \mathcal{Z}_{\widehat{\mathbf{T}}}\|_{\widetilde{W}_p^{1-1/p,1/2-1/2p}(\partial_2 S \times (t,t+1))} + \|n_{\mathbf{T}} - n_{\widehat{\mathbf{T}}}\|_{\widetilde{W}_p^{1-1/p,1/2-1/2p}(\partial_2 S \times (t,t+1))}$$

$$\leq C\|\mathbf{T} - \widehat{\mathbf{T}}\|_{\mathbf{B}^\omega}.$$

Proof. A proof of this is given in Lemma 10 of [8] □

Theorem 4.7. *There exists a constant $C < \infty$ independent of ω such that if $\mathbf{U} \in \mathfrak{M}^\omega$ then*

$$\|\mathfrak{U}_{1\mathbf{T}}\mathbf{U} - \mathfrak{U}_{1\widetilde{\mathbf{T}}}\mathbf{U}\|_{B_1^\omega} \leq C\|\mathbf{T} - \widetilde{\mathbf{T}}\|_{\mathbf{B}^\omega}\|\mathbf{U}\|_{\mathfrak{M}^\omega}.$$

Proof. A proof of this is given in Theorem 9 of [8]. □

Theorem 4.8. *There exists a constant C independent of ω such that for $\mathbf{U} \in \mathfrak{M}^\omega$,*

$$\|\mathcal{B}_{\mathbf{T}}(\mathbf{U}) - \mathcal{B}_{\widehat{\mathbf{T}}}(\mathbf{U})\|_{B_2^\omega} \leq C\|\mathbf{T} - \widehat{\mathbf{T}}\|_{\mathbf{B}^\omega}\|\mathbf{U}\|_{\mathfrak{M}^\omega}.$$

Proof. The proof of this theorem is obtained by a minor modification to the proof of Theorem 11 in [8]. □

We shall prove Theorem 4.3 using the theory of Section 5 and Section 6 of [8] and to this end we need to make a few definitions.

Definition 4.9. *Let $\omega \in [1, \infty)$ and $\mathfrak{I}$ be an arbitrary set. For any $\mathrm{f} : [0, \omega] \cap \mathbb{R} \to \mathfrak{I}$ we define $\mathrm{f}^t : [0, \omega - t] \cap \mathbb{R} \to \mathfrak{I}$ for $t \in [0, \omega] \cap \mathbb{R}$ by $\mathrm{f}^t(r) = \mathrm{f}(t + r)$ for $r \in [0, \omega - t] \cap \mathbb{R}$. Let $\mathcal{I}_2$ be an Banach space and with $\omega \in [1, \infty]$ we define*

$$\widehat{\mathcal{S}}_2^\omega = \{u : [0, \omega] \cap \mathbb{R} \to \mathcal{I}_2\} / \sim$$

where two functions are considered equivalent in the sense of $\sim$ if they agree almost everywhere. We assume $\mathcal{S}_2 \subset \widehat{\mathcal{S}}_2^1$ is a Banach space with the norm $\|\cdot\|_{\mathcal{S}_2}$. Further we define

$$\mathcal{S}_2^\omega = \left\{ \mathrm{f} \in \widehat{\mathcal{S}}_2^\omega | \mathrm{f}^t \in \mathcal{S}_2 \, for\, t \in [0, \omega - 1] \; and \sup_{0 \le t \le \omega - 1} \|\mathrm{f}^t|[0,1]\|_{\mathcal{S}_2} < \infty \right\}$$

and

$$\|\mathrm{f}\|_{\mathcal{S}_2^\omega} = \sup_{1 \le t \le \omega - 1} \|\mathrm{f}^t|[0,1]\|_{\mathcal{S}_2}.$$

For $\omega \in [0, \infty], a_2 \ge 0$ we define

$$\|\mathrm{f}\|_{S_2^{\omega, a_2}} = \sup_{0 \le t \le \omega - 1} (1+t)^{a_2} \|\mathrm{f}^t|[0,1]\|_{\mathcal{S}_2}$$

and

$$S_2^{\omega, a_2} = \left\{ \mathrm{f} \in \mathcal{S}_2^\omega : \|\mathrm{f}\|_{S_2^{\omega, a_2}} < \infty \right\}.$$

The estimate from Theorem 4.3 is obtained by combining Theorem 3.1 with Theorem 4.5 and applying the theory of Section 5 in [8].

4.2.1 The Proof of Theorem 4.3

Proof. The existence and uniqueness follow from Theorem 4.5 so only need to show the estimate. Denote

$$\mathcal{S}_2 = \left\{ (\mathbf{F}, \mathbf{g}) : \mathbf{F} \in B_1^1, \mathbf{g} \in B_2^1 \right\} = B_1^1 \times B_2^1$$

and $\mathcal{I}_2 = S \times \widetilde{L}_p(\partial S)$ with the norms

$$\|(\mathbf{F}, \mathbf{g})\|_{\mathcal{S}_2} = \|\mathbf{F}\|_{B_1^1} + \|\mathbf{g}\|_{B_2^1}, \quad \|(\mathbf{F}, \mathbf{g})(t)\|_{\mathcal{I}_2} = \|\mathbf{F}(t)\|_{\mathfrak{B}^1} + \|\mathbf{g}(t)\|_{\widetilde{L}_p}.$$

We define the linear bounded operator $\mathcal{C} : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathcal{I}_2$ by

$$\mathcal{C}\left(U, (\mathbf{F}, \mathbf{g})\right) = (0, \mathcal{B}_T(U) - \mathbf{g}).$$

If $U \in \mathcal{I}_1$ and $(\mathbf{F}, \mathbf{g}) \in \mathcal{S}_2$ then the mapping $t \mapsto (U, (\mathbf{F}, \mathbf{g})(t))$ is continuous from $[0, 1]$ to $\mathcal{I}_2$. For $(\mathbf{F}, \mathbf{g}) \in \mathcal{S}_2$ and $U_0 \in \mathcal{I}_1$ with $\mathcal{C}(U_0, (\mathbf{F}, \mathbf{g})) = 0$ we have by Theorem 4.5 a solution $\mathbf{U}$ on $[0, 1]$ of the equation $\mathbf{U}' = -\mathcal{A}_{1\mathbf{T}}\mathbf{U} + \mathbf{F}$ with initial value $\mathbf{U}(0) = U_0$ and boundary condition $\mathcal{B}_{\mathbf{T}}(\mathbf{U}) = \mathbf{g}$. By definition of $\mathcal{C}$ we have

$$\mathcal{C}\left(\mathbf{U}(t), (\mathbf{F}(t), \mathbf{g}(t))\right) = (0, \mathcal{B}_{\mathbf{T}}(\mathbf{U}(t)) - \mathbf{g}(t))$$

thus $\mathcal{C}\left(\mathbf{U}(t), (\mathbf{F}(t), \mathbf{g}(t))\right)$ remains a continuous function from $[0, 1]$ to $\mathcal{I}_2$. Let

$$\mathcal{D} = \{(U_0, (\mathbf{F}, \mathbf{g})) \in \mathcal{I}_1 \times \mathcal{S}_2 : \mathcal{C}(U_0, (\mathbf{F}, \mathbf{g}))(0) = 0\}$$

and define $\Gamma : \mathcal{D} \to \mathcal{S}_1$ by $\Gamma(U_0, (\mathbf{F}, \mathbf{g})) = \mathbf{U}$. There exists a constant C such that

$$\|\Gamma(U_0, (\mathbf{F}, \mathbf{g}))\|_{\mathcal{S}_1} = \|\mathbf{U}\|_{\mathcal{S}_1} \leq C\left(\|\mathbf{U}\|_{\mathcal{I}_1} + \|(\mathbf{F}, \mathbf{g})\|_{\mathcal{S}_2}\right),$$

which also has the property that for $(U_0, (\mathbf{F}, \mathbf{g})) \in \mathcal{D}$,

$$\Gamma(U_0, (\mathbf{F}, \mathbf{g}))(0) = \mathbf{U}(0) = U_0$$

and

$$\mathcal{C}\left(\Gamma(U_0, (\mathbf{F}, \mathbf{g}))(t), (\mathbf{F}, \mathbf{g})\right)(t) = 0$$

for $t \in [0, 1]$.

If $r \in (0, 1)$, then for $t \in [0, 1 - r]$ we have by definition 4.9

$$\Gamma\left(\Gamma(U_0, (\mathbf{F}, \mathbf{g}))(r), (\mathbf{F}, \mathbf{g})^r\right)(t) = \Gamma(U_0, (\mathbf{F}, \mathbf{g}))(t + r).$$

This condition is due to the uniqueness of the solution.

$\mathcal{S}_2$ has the property that if $\mathrm{f} \in \mathcal{S}_2, \varphi \in C^1([0,1])$ and $(\varphi \mathrm{f})(t) = \varphi(t)\mathrm{f}(t)$ then $\varphi\mathrm{f} \in S_2$ and $\|\varphi \mathrm{f}\|_{\mathcal{S}_2} \leq C_1 \|\varphi\|_{C^1} \|\mathrm{f}\|_{\mathcal{S}_2}$. If $(\mathbf{F}, \mathbf{g}) \in \mathcal{S}_2$, then $(\mathbf{F}, \mathbf{g})(t) \in \mathcal{I}_2$ for almost all t, and $\mathcal{S}_2^\omega = B_1^\omega \times B_2^\omega$ for $\omega \in [1, \infty)$. There is a constant $C \in (0, \infty)$ independent of ω such that

$$C^{-1}\left(\|\mathbf{F}\|_{B_1^\omega} + \|\mathbf{g}\|_{B_2^\omega}\right) \leq \|(\mathbf{F}, \mathbf{g})\|_{S_2^{;\omega,4/3}} \leq C\left(\|\mathbf{F}\|_{B_1^\omega} + \|\mathbf{g}\|_{B_2^\omega}\right).$$

When $(\mathbf{F}, \mathbf{g}) \in S_2^{\omega,4/3}$ we can extend $(\mathbf{F}, \mathbf{g})$ to infinity by defining functions $\widetilde{\mathbf{F}}, \widetilde{\mathbf{g}}$ and φ, where $\varphi \in C^\infty(\mathbb{R})$ with $0 \leq \varphi(t) \leq 1, \varphi'(t) \leq 0$ and $\varphi(t) = 1$ for $t \leq 0, \varphi(t) = 0$ for $t \geq 1$.

$$\widetilde{\mathbf{F}}(t) = \begin{cases} \mathbf{F}(t) & \text{for } t \leq \omega, \\ 0 & \text{for } t > \omega, \end{cases} \quad \text{and} \quad \widetilde{\mathbf{g}}(t) = \begin{cases} \mathbf{g}(t) & \text{for } t \leq \omega, \\ \mathbf{g}(2\omega - t)\varphi(t-\omega) & \text{for } \omega < t \leq 2\omega, \\ 0 & \text{for } t > 2\omega. \end{cases}$$

Then for $t \leq \omega$ we have $\left(\widetilde{\mathbf{F}}, \widetilde{\mathbf{g}}\right)(t) = (\mathbf{F}, \mathbf{g})(t)$ and by the definition 4.9 we have

$$\|(\widetilde{\mathbf{F}}, \widetilde{\mathbf{g}})\|_{S_2^{\infty,4/3}} \leq C\|(\mathbf{F}, \mathbf{g})\|_{S_2^{\omega,4/3}}.$$

Furthermore, let

$$\mathcal{S}_1 = \left\{\mathbf{U} \in \mathfrak{M}_1^1 : \mathbf{U}(t) \in S \ \text{for}\, t \in [0,1]\right\}, \quad \mathcal{I}_1 = S \cap D^{1-2/p}$$

with the norms

$$\|\mathbf{U}\|_{\mathcal{S}_1} = \|\mathbf{U}\|_{\mathfrak{M}_1^1}, \quad \|\mathbf{U}\|_{\mathcal{I}_1} = \|U\|_{\mathfrak{B}^1} + \|u\|_{\widetilde{W}_p^{2-2/p}} + \|\tau\|_{\widetilde{W}_p^{2-2/p}}.$$

Using the inequalities

$$\max_{0 \leq t \leq 1} \|\mathbf{U}(t)\|_{\mathfrak{B}^1} \leq C\left[\int_0^1 \left(\|\mathbf{U}'(r)\|_{\mathfrak{B}^1}^p + \|\mathbf{U}(r)\|_{\mathfrak{B}^1}^p\right) dr\right]^{1/p} \leq C\|\mathbf{U}\|_{\mathfrak{M}_1^1}$$

and

$$\|u(t)\|_{\widetilde{W}_p^{5/3}} \leq C\|u(t)\|_{\widetilde{W}_p^{2-2/p}} \leq C\|u\|_{\widetilde{W}_p^{2,1}(S\times(0,1))} \leq C\|\mathbf{U}\|_{\mathfrak{M}_1^\omega}$$

we find

$$\|\mathbf{U}(t)\|_{\mathcal{I}_1} = \|U(t)\|_{\mathfrak{B}^1} + \|u(t)\|_{\widetilde{W}_p^{2-2/p}} + \|\tau(t)\|_{\widetilde{W}_p^{2-2/p}}$$

where each summand on the right is bounded by $\|U\|_{\mathfrak{M}_1^1}$, thus

$$\|\mathbf{U}(t)\|_{\mathcal{I}_1} \leq C_1\|\mathbf{U}\|_{\mathcal{S}_1}.$$

Also, $\mathcal{S}_1$ has the property that if $\mathrm{f} \in \mathcal{S}_1, \varphi \in C^1([0,1])$ and $(\varphi\mathrm{f})(t) = \varphi(t)\mathrm{f}(t)$ then $\varphi\mathrm{f} \in \mathcal{S}_1$ and $\|\varphi\mathrm{f}\|_{\mathcal{S}_1} \leq C_1\|\varphi\|_{C^1}\|\mathrm{f}\|_{\mathcal{S}_1}$, so

$$\|\varphi\mathbf{U}\|_{\mathcal{S}_1} = \|\varphi\mathbf{U}\|_{\mathfrak{M}_1^1} \leq C_1\|\varphi\|_{C^1}\|\mathbf{U}\|_{\mathfrak{M}_1^1} \leq C_1\|\varphi\|_{C^1}\|\mathbf{U}\|_{S_1}.$$

Let $[\cdot]_*$ be a seminorm defined on $\mathcal{S}_1$ given by

$$[\mathbf{U}]_* = \left[\int_0^1 \|\Pi\mathbf{U}'(r)\|^p_{\mathfrak{B}^1_{-1}}\,dr\right]^{1/p} + \sup_{0\leq t\leq 1}\|\Pi\mathbf{U}(t)\|_{D^{2/3}_{-1}}.$$

Using Lemma 4.4 we have

$$[\mathbf{U}]_* \leq C\|\mathbf{U}\|_{\mathfrak{M}_1^1}, \tag{4.7}$$

and by Theorem 3.1 we have that when $U_0 \in \mathcal{I}_1$ and $\mathbf{U} = \Gamma(U_0, 0)$, then with $s = \dfrac{2}{3}$

$$t^{4/3}\left(\|\Pi\mathbf{U}'(t)\|_{D^1_{-1}} + \|\Pi\mathbf{U}(t)\|_{D^{2/3}_{-1}}\right) \leq C\|U_0\|_{\mathfrak{B}^1}$$

for $t \geq 1$. Therefore

$$(1+t)^{4/3}[\mathbf{U}^t]_* \leq C\left(\|U_0\|_{\mathfrak{B}^1} + \|u\|_{W_p^{2-2/p}}\right) \tag{4.8}$$

for $t \geq 1$, while for $t \leq 1$ follows directly from Theorem 4.5. Equations (4.7) and (4.8) verify the conditions necessary to use Theorem 17 in [8] which implies that for

$t \geq 0$, we have

$$
\begin{aligned}
&(1+t)^{4/3}\left[\Gamma\left(U_0,(\widetilde{\mathbf{F}},\widetilde{\mathbf{g}})^t\right)\right]_* \\
&\qquad \leq \quad C\left(\|U_0\|_{\mathcal{I}_1}+\|(\widetilde{\mathbf{F}},\widetilde{\mathbf{g}})\|_{S_2^{\infty,4/3}}\right) \\
&\qquad \leq \quad C\left(\|U_0\|_{\mathfrak{B}^1}+\|u_0\|_{\widetilde{W}_p^{2-2/p}}+\|\tau_0\|_{\widetilde{W}_p^{2-2/p}}+\|(\widetilde{\mathbf{F}},\widetilde{\mathbf{g}})\|_{S_2^{\infty,4/3}}\right) \\
&\qquad \leq \quad C\left(\|U_0\|_{\mathfrak{B}^1}+\|u_0\|_{\widetilde{W}_p^{2-2/p}}+\|\tau_0\|_{\widetilde{W}_p^{2-2/p}}+\|(\mathbf{F},\mathbf{g})\|_{S_2^{\infty,4/3}}\right) \\
&\qquad \leq \quad C\left(\|U_0\|_{\mathfrak{B}^1}+\|u_0\|_{\widetilde{W}_p^{2-2/p}}+\|\tau_0\|_{\widetilde{W}_p^{2-2/p}}+\|\mathbf{F}\|_{B_1^\omega}+\|\mathbf{g}\|_{B_2^\omega}\right).
\end{aligned}
$$

Thus $[\Gamma(U_0,(\mathbf{F},\mathbf{g})]_2 = [\Gamma(U_0,(\widetilde{\mathbf{F}},\widetilde{\mathbf{g}})]_2$

$$
\leq C\left(\|U_0\|_{\mathfrak{B}^1}+\|u_0\|_{\widetilde{W}_p^{2-2/p}}+\|\tau_0\|_{\widetilde{W}_p^{2-2/p}}+\|\mathbf{F}\|_{B_1^\omega}+\|\mathbf{g}\|_{B_2^\omega}\right),
$$

as for $t \leq \omega$ we have $\Gamma\left(U_0,(\mathbf{F},g)\right)(t) = \Gamma\left(U_0,(\widetilde{\mathbf{F}},\widetilde{g})\right)$. A completely analogous argument shows that if we replace the exponent $a_1 = 4/3$ with $a_1 = 1$ and

$$
[U]_* = \left[\int_0^1 \left(\|\mathbf{U}'(t)\|_{\mathfrak{B}^1}^p + \|\Pi\mathbf{U}(t)\|_{D_{-1}^1}^p\right) dr\right]^{1/p}
$$

that

$$
[\Gamma\left(U_0,(\mathbf{F},\mathbf{g})\right)]_1 \leq C\left(\|U_0\|_{\mathfrak{B}^1}+\|u_0\|_{\widetilde{W}_p^{2-2/p}}+\|\tau_0\|_{\widetilde{W}_p^{2-2/p}}+\|\mathbf{F}\|_{B_1^\omega}+\|\mathbf{g}\|_{B_2^\omega}\right)
$$

and when we replace $a_1 = 4/3$ with $a_1 = 0$ and $[\mathbf{U}]_* = \|\mathbf{U}\|_{\mathfrak{M}_1^1}$ that

$$
\|\Gamma\left(U_0,(\mathbf{F},\mathbf{g})\right)\|_{\mathfrak{M}_1^\omega} \leq C\left(\|U_0\|_{\mathfrak{B}^1}+\|u_0\|_{\widetilde{W}_p^{2-2/p}}+\|\tau_0\|_{\widetilde{W}_p^{2-2/p}}+\|\mathbf{F}\|_{B_1^\omega}+\|\mathbf{g}\|_{B_2^\omega}\right).
$$

Putting these inequalities together we conclude that if $\mathbf{F} \in B_1^\omega, g \in B_2^\omega$, and $U_0 \in S \cap D^{1-2/p}$ then for $\omega \in [1,\infty)$

$$
\|\mathbf{U}\|_{\mathfrak{M}^\omega} = \|\mathbf{U}\|_{\mathfrak{M}_1^\omega} + [\mathbf{U}]_1 + [\mathbf{U}]_2
$$

$$\leq C\left(\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} + \|\tau_0\|_{\widetilde{W}_p^{2-2/p}} + \|\mathbf{F}\|_{B_1^\omega} + \|\mathbf{g}\|_{B_2^\omega}\right)$$

with $\mathbf{U} = \Gamma\left(U_0, (\mathbf{F}, \mathbf{g})\right)$. □

Now we are ready to prove Theorem 3.2 in its full generality.

4.3 The Proof of Theorem 4.2

Proof. Theorem 4.5 implies the existence and uniqueness of the solution. To obtain the estimate we rewrite the equation

$$\mathbf{U}' = -\mathcal{A}_{1\mathbf{T}}\mathbf{U} + \mathbf{F}$$

using $\widetilde{\mathbf{T}}(x,t) = T_\omega(x)$, as

$$\mathbf{U}' = -\mathcal{A}_{1\widetilde{\mathbf{T}}}\mathbf{U} + \left(\mathcal{A}_{1\widetilde{\mathbf{T}}} - \mathcal{A}_{1\mathbf{T}}\right)\mathbf{U} + \mathbf{F}$$

and the boundary conditions as

$$\mathcal{B}_{\widetilde{\mathbf{T}}}(\mathbf{U}) = \mathcal{B}_{\widetilde{\mathbf{T}}}(\mathbf{U}) - \mathcal{B}_T(\mathbf{U}) + \mathbf{g}.$$

Using Theorem 4.3, we have

$$\|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq C\left(\|\mathbf{F}\|_{B_1^\omega} + \|\mathbf{g}\|_{B_2^\omega} + \|\left(\mathcal{A}_{1\widetilde{\mathbf{T}}} - \mathcal{A}_{1\mathbf{T}}\right)\mathbf{U}\|_{B_1^\omega} + \|\mathcal{B}_{\widetilde{\mathbf{T}}}(\mathbf{U}) - \mathcal{B}_{\mathbf{T}}(\mathbf{U})\|_{B_2^\omega}\right).$$

By Theorem 4.7 we have

$$\|\left(\mathcal{A}_{1\widetilde{\mathbf{T}}} - \mathcal{A}_{1\mathbf{T}}\right)\mathbf{U}\|_{B_1^\omega} \leq C\|\mathbf{T} - \widetilde{\mathbf{T}}\|_{\mathbf{B}^\omega}\|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq C\eta\|\mathbf{U}\|_{\mathfrak{M}^\omega},$$

while Theorem 4.8 implies

$$\|\mathcal{B}_{\widetilde{\mathbf{T}}}(\mathbf{U}) - \mathcal{B}_{\mathbf{T}}(\mathbf{U})\|_{B_2^\omega} \leq C\|\mathbf{T} - \widetilde{\mathbf{T}}\|_{\mathbf{B}^\omega}\|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq C\eta\|\mathbf{U}\|_{\mathfrak{M}^\omega}.$$

Combining these estimates we have

$$\|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq C\left(\|\mathbf{F}\|_{B_1^\omega} + \|\mathbf{g}\|_{B_2^\omega}\right) + C\eta\|U\|_{\mathfrak{M}^\omega}.$$

For $C\eta \leq \dfrac{1}{2}$ we have the desired result. $\square$

CHAPTER 5
CONCLUSION

This chapter proves asymptotic estimates with respect to perturbations. This includes the proof that for every initial value close to an equilibrium a solution of the equations of motion exist for all time and converges to that equilibrium. We study the flow in a fixed time interval $I_\omega = [0, \omega] \cap \mathbb{R}$ with $\omega < \infty$. The results are verified for Euler coordinates, but the proofs mainly use Lagrange coordinates.The boundary conditions $u = 0$ and $\tau = 0$ on $\partial_1 S$ are tacitly assumed throughout this chapter.

5.1 Transformation from Euler to Lagrange Coordinates

Definition 5.1. *Let*

$$X = \Big\{ d : \overline{S} \to \mathbb{R}^3 : d \in C^1(\overline{S}), d(x_1, x_2, -h) = 0,$$

$$d(x + \mathbf{k}_\iota) = d(x) \, for \, \iota \in \{1, 2\}, \max_{x \in S, |z|=1} |\nabla d(x) z| \leq \frac{1}{10} \Big\}$$

Define the transformation $T = d + E_{\overline{S}}$ with $d \in X$.

Lemma 5.2. *If $d \in X, T = d + E_S$ and $\Omega = T(S)$, then Ω is open, T is invertible with its inverse belonging to $C^1(\overline{S})$, and for $x_1, x_2 \in \overline{S}, y_1, y_2 \in \overline{\Omega}, z \in \mathbb{R}^3$ then we have the estimates*

$$|\nabla T(x_1) z - z| \leq \frac{|z|}{10}$$

$$|T(x_1) - x_1| \leq \frac{h}{10} \tag{5.1}$$

$$\frac{9}{10} |x_1 - x_2| \leq |T(x_1) - T(x_2)| \leq \frac{11}{10} |x_1 - x_2|$$

and

$$
\begin{aligned}
|\nabla T^{-1}(y_1)z - z| &\leq \frac{|z|}{9} \\
|T^{-1}(y_1) - y_1| &\leq \frac{h}{10}.
\end{aligned}
\tag{5.2}
$$

If $s \in \partial S$ *then* $T(s) = w \in \partial\Omega$.

Proof. First we derive the inequalities (5.1) using the definition of the transformation T as well as the inequality for ∇d in the definition of X.

$$
\begin{aligned}
|\nabla T(x_1)z - z| &= |\nabla d(x_1) + \nabla E_{\overline{S}}(x_1)z - z| = |\nabla d(x_1)z + z - z| \\
&\leq |\nabla d(x_1)|\,|z| \leq \frac{|z|}{10} \\
|T(x_1) - x_1| &= |d(x_1) + E_{\overline{S}}(x_1) - x_1| = |d(x_1) + x_1 - x_1| = |d(x_1)| \\
&\leq \int_{-h}^{0} |d_{x_3}(x_1, x_2, r)|\,dr \leq \frac{h}{10} \\
|T(x_1) - T(x_2)| &= |d(x_1) + E_{\overline{S}}(x_1) - d(x_2) - E_{\overline{S}}(x_2)| \\
&= |d(x_1) - d(x_2) + x_1 - x_2|\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
|x_1 - x_2| - |d(x_1) - d(x_2)| &\leq |d(x_1) - d(x_2) + x_1 - x_2| \\
&\leq |x_1 - x_2| + |d(x_1) - d(x_2)|\,.
\end{aligned}
\tag{5.3}
$$

From inequality 5.3 we have

$$
\frac{9}{10}\,|x_1 - x_2| \leq |T(x_1) - T(x_2)| \leq \frac{11}{10}\,|x_1 - x_2|\,. \tag{5.4}
$$

The Inverse Function Theorem tells us that T is a locally invertible C^1 transformation, so by equation (5.4) T is injective and surjective on its image. Since $T(x_1) = y_1$ we have $T^{-1}(y_1) = x_1$, so

$$
\left|T^{-1}(y_1) - y_1\right| = |x_1 - T(x_1)| \leq \frac{h}{10}.
$$

To get the first inequality of equation (5.2) we look at the series expansion for ∇T^{-1}.

$$\nabla T^{-1} = (\nabla T)^{-1} = (E - \nabla d)^{-1} = E - \nabla d + (\nabla d)^2 - (\nabla d)^3 + \cdots$$

From this we have

$$\|\nabla T^{-1}(y_1) - E\| \leq \|\nabla d\| \leq \frac{10}{9}\|\nabla d\| \leq \frac{1}{9}$$

with the operator norm for matrices. $\square$

In order to show the uniqueness and existence of the solution we transform the system of equations and the solution into Lagrange coordinates for $t \in [0,1]$.

Definition 5.3. *Given $\beta \in \widetilde{W}_p^{2-1/p}(\partial S)$ let $\widetilde{\beta}$ be the solution of $\Delta\widetilde{\beta} = 0$ with $\widetilde{\beta}\Big|_{x_3=0} = \beta$ and $\widetilde{\beta}\Big|_{x_3=0} = -h$. For $x \in \overline{S}$, let*

$$\mathcal{T}_\beta(x) = \Big(x_1, x_2, \widetilde{\beta}(x_1, x_2, x_3)\Big)\,.$$

Lemma 5.4. *There exists $\epsilon_1 \in (0,1]$ and $C < \infty$ so that if $\beta \in \widetilde{W}_p^{2-1/p}(\partial S)$ and $\|\beta\|_{\widetilde{W}_p^{2-1/p}(\partial S)} \leq \epsilon_1$, then $\mathcal{T}_\beta \in \widetilde{W}_p^2(S)$ and $\mathcal{T}_\beta - E_S \in X, \mathcal{T}_\beta(S) = \Omega_\beta$ and*

$$\|\mathcal{T}_\beta - E_S\|_{\widetilde{W}_p^2(S)} + \|\mathcal{T}_\beta^{-1} - E_{\Omega_\beta}\|_{\widetilde{W}_p^2(S_\beta)} \leq C\|\beta\|_{\widetilde{W}_p^{2-1/p}(\partial S)}.$$

Proof.

Since $\widetilde{\beta}$ is a solution to the Dirichlet problem, we have $\mathcal{T}_\beta \in \widetilde{W}_p^2$. When $x = (x_1, x_2, 0)$ we have $\left(\mathcal{T}_\beta(x)\right)_3 = \beta(x)$, thus $\mathcal{T}_\beta(x) \in \partial\Omega_\beta$ and

$$\|T_\beta - E_S\|_{\widetilde{W}_p^2(S)} \leq \|\widetilde{\beta} - x_n\|_{\widetilde{W}_p^2(S)} \leq C\|\beta\|_{\widetilde{W}_p^{2-1/p}(\partial S)}.$$

If ϵ is small enough T_β is obviously a diffeomorphism, so $T_\beta(S) = \Omega_\beta$ and

$$\|T_\beta - E_S\|_{\widetilde{W}_p^2(S)} + \|T_\beta^{-1} - E_{\Omega_\beta}\|_{\widetilde{W}_p^2(S_\beta)} \leq C\|\beta\|_{\widetilde{W}_p^{2-1/p}(\partial S)}.$$

Thus, when ϵ_1 is small enough

$$\mathcal{T}_\beta - E \in X.$$

$\square$

Lemma 5.5. *There is a constant C so that when (Ω^1, v, ρ) is a solution of the problem (P1) with the initial values $\Omega^0 = \Omega_{\varphi_0}, v(0) = v_0, \rho(0) = \rho_0$, and $\|\varphi_0\|_{\widetilde{W}_p^{2-1/p}(\partial\Omega)} \leq \epsilon_1$, $\mathcal{E}_2(\Omega^1, v, \rho) \leq 1$ then there exists a function $\mathbf{T} : \overline{S} \times [0,1] \to \mathbb{R}^3$ defined by $T_t(x) = \mathbf{T}(x,t)$ that form a family of injective transformations $T_t \in \widetilde{W}_p^2(S)$ with $T_t(S) = \Omega^t$ for $t \in [0,1]$ and solve the ordinary differential equation*

$$\begin{aligned}
\mathbf{T}'(x,t) &= v\left(\mathbf{T}(x,t),t\right) \\
\mathbf{T}(x,0) &= \mathcal{T}_{\varphi_0}(x)
\end{aligned}$$

Also,

$$\|T_t - E_S\|_{\widetilde{W}_p^2(S)} \leq C\left(\|\varphi_0\|_{\widetilde{W}_p^{2-1/p}(\mathbb{R}^2)} + \int_0^t \|v\|_{\widetilde{W}_p^2(\Omega^t)} dt\right).$$

Proof. By assumption $v, \nabla v \in C^0(\Omega^t)$. Therefore for every $x \in \Omega^0$ there is a solution $w(t)$ of $w'(t) = v\left(w(t),t\right)$ with $w(0) = T_{\varphi_0}(x)$ and $w(t) \in \Omega^t$ for $t \in [0,1]$. Define $T_t(x) = w(t)$. These transformations are injective due to the uniqueness of solutions to such a system of ordinary differential equations. As Ω^t moves with the flow, so $T_t(S) = \Omega^t$ for $t \in [0,1]$. Owing to standard results for Ordinary Differential Equations $\mathbf{T} \in C^1\left([0,1], C^1(\overline{S})\right)$. Differentiating the equation for $\mathbf{T}$ with respect to x yields

$$(\nabla\mathbf{T}(x,t))' = (\nabla T_t(x))' = \nabla w'(t) = \nabla\left(v(w(t),t)\right) = (\nabla v)\left(w(t),t\right)\nabla w(t)$$

$$(\nabla v)\left(w(t),t\right)\nabla T_t(x) = (\nabla v)\left(\mathbf{T}(x,t),t\right)\nabla T(x,t).$$

Now define $\mathbf{D}(x,t) = (\nabla v)(\mathbf{T}(x,t),t)$. Then $\mathbf{D} \in C^0\left(\overline{S} \times [0,1]\right) \cap L_p\left([0,1], \widetilde{W}_p^1(S)\right)$ and since $|\mathcal{E}_2(\Omega, v, \rho, \tau)| \leq 1$ then $|\mathbf{D}(x,t)| \leq 1$. Then for every three dimensional column vector z we have

$$(\nabla\mathbf{T}(x,t)z)' = (\nabla\mathbf{T}(x,t))' z = (\nabla v)(\mathbf{T}(x,t),t)\,\nabla\mathbf{T}(x,t)z = \mathbf{D}(x,t)\nabla\mathbf{T}(x,t)z.$$

Then there is a constant $C \in (0,\infty)$ so that for every $t_1, t_2 \in [0,1]$ we have

$$|\nabla\mathbf{T}(x,t_1)z| \leq C|\nabla\mathbf{T}(x,t_2)z|.$$

This implies that for $x \in \overline{S}$ and $t \in [0,1]$

$$|\nabla\mathbf{T}(x,t)| + |(\nabla\mathbf{T})^{-1}(x,t)| \leq C(|\nabla\mathbf{T}(x,0)| + |(\nabla\mathbf{T})^{-1}(x,0)|) \leq C$$

for another constant. The last inequality follows from Lemma 5.4. Also,

$$\nabla\mathbf{T}(x,t) = \nabla\mathcal{T}_{\varphi_0} + \int_0^t \mathbf{D}(x,\xi)\nabla\mathbf{T}(x,\xi)d\xi,$$

and we can view this as an integral equation in $C^0\left([0,1], \widetilde{W}_p^1(S)\right)$ as well as in $C^0\left([0,1], C^0(\overline{S})\right)$. Thus

$$\begin{aligned}\|T_t - E_S\|_{\widetilde{W}_p^2(S)} = \|\nabla\mathbf{T} - E_S\|_{\widetilde{W}_p^1(S)} &= \left\| |\nabla\mathbf{T} - E_S|_{t=0} + \int_0^t (\nabla\mathbf{T})' d\sigma \right\|_{\widetilde{W}_p^2(S)} \\ &\leq C\left(\|\mathcal{T}_{\varphi_0} - E_S\|_{\widetilde{W}_p^{2-1/p}(\partial S)} + \int_0^t \|\nabla v\|_{\widetilde{W}_p^1(S)} d\sigma \right) \\ &\leq C\left(\|\varphi\|_{\widetilde{W}_p^{2-1/p}(\partial S)} + \int_0^t \|v\|_{\widetilde{W}_p^2(S)} d\sigma \right).\end{aligned}$$

□

The Eulerian description of fluid flow describes the motion of the fluid through a particular region in space as time evolves, whereas the Lagrangian description of this motion is from a different perspective, where a region of the fluid is followed in space as time progresses. The transition between the two types of flow is described by the family of transformation functions $T(x,t) = T_t(x)$. The Eulerian and Lagrangian specifications of fluid flow are related by the following.

$$u(x,t) = v\left(T_t(x),t\right), \quad \alpha(x,t) = \rho(T_t(x),t) - \rho_e(T_t(x)), \quad \tau(x,t) = \theta\left(T_t(x),t\right) - \theta_e,$$

$$\mathbf{d}(x,t) = \mathbf{T}(x,t) - x, \quad \beta : \partial\Omega \times [0,\omega] \to \mathbb{R} \text{ by } \beta(x,t) = (T_t)_3(x).$$

These variables are collected in the vector $\mathbf{U} = [u, \alpha, \tau, \beta, \mathbf{d}]$. Here we assume the transformations are close to the identity, which is true if $\mathcal{E}_1\left(\varphi_0, \theta_0, \rho_0, v_0\right)$ and $\mathcal{E}_2\left(\Omega^1, v, \rho, \theta\right)$ are small. The solution to be constructed will satisfy

$$\begin{aligned} &\|T_t - E_S\|_{\widetilde{W}^2_p} \leq 1 \\ &T_t - E_S \in X \text{ for } t \in [0,\omega]. \end{aligned} \tag{5.5}$$

for longer time intervals than $[0,1]$. Now we begin the transformation process. We rewrite the equations in (1.1) using the change of variables $\varrho = \rho - \rho_e, \vartheta = \theta - \theta_e$. This way the system can be rearranged so that the linearized system, the equations in (1.14), can be identified and the remaining terms can be analyzed. For convenience we will often write the term $c\rho\theta$ as $p(\rho,\theta)$. Using the above change of variables the second equation of (1.1) becomes

$$\begin{aligned} \rho_t &= -\operatorname{div}(\rho v) = -\operatorname{div}\left((\rho_e + \varrho)v\right) = -\operatorname{div}(\rho_e v) - \operatorname{div}(\varrho v) \\ &= -\operatorname{div}(\rho_e v) - \varrho \operatorname{div}(v) - (v \cdot \nabla)\varrho \end{aligned}$$

and $\rho_t = (\varrho + \rho_e)_t = \varrho_t$. Thus,

$$\varrho_t + (v \cdot \nabla)\varrho = -\operatorname{div}(\rho_e v) - \varrho \operatorname{div}(v). \tag{5.6}$$

The first equation of (1.1) becomes

$$\begin{aligned} v_t + (v\cdot\nabla)v &= \rho^{-1}\operatorname{div}(\mathbb{D}(v)) - \rho^{-1}\nabla p(\rho,\theta) - ge_3 \\ &= (\rho_e+\varrho)^{-1}\operatorname{div}(\mathbb{D}(v)) - (\rho_e+\varrho)^{-1}\nabla\left[p(\rho_e+\varrho,\theta_e+\vartheta)\right] - ge_3. \end{aligned} \tag{5.7}$$

Now add and subtract $(\rho_e+\varrho)^{-1}\nabla p(\rho_e,\theta_e)$ so that the pressure terms can be simplified, giving

$$v_t + (v\cdot\nabla)v$$

$$= (\rho_e+\varrho)^{-1}\operatorname{div}(\mathbb{D}(v)) - (\rho_e+\varrho)^{-1}\nabla\left[p(\rho_e+\varrho,\theta_e+\vartheta) - p(\rho_e,\theta_e)\right] + \frac{\varrho}{\rho_e+\varrho}ge_3.$$

The third equation of (1.1) becomes

$$\begin{aligned} \theta_t + (v\cdot\nabla)\theta &= cc_2^{-1}\theta\operatorname{div}(v) + \rho^{-1}c_2^{-1}\operatorname{div}(\kappa\nabla\theta) + \rho^{-1}c_2^{-1}\psi \\ &= cc_2^{-1}(\theta_e+\vartheta)\operatorname{div}(v) + (\rho_e+\varrho)^{-1}c_2^{-1}\operatorname{div}(\kappa\nabla(\vartheta+\theta_e)) + (\rho_e+\varrho)^{-1}c_2^{-1}\psi. \end{aligned} \tag{5.8}$$

Now we compose each variable with the transformation $T_t(x)$ and by the chain rule we find the following material derivatives.

$$\begin{aligned} u' &= \frac{Dv}{Dt}(T_t(x),t) &&= v_t + (v\cdot\nabla)v, \\ \alpha' &= \frac{D\alpha}{Dt}(T_t(x),t) &&= \varrho_t + (v\cdot\nabla)\varrho, \\ \tau' &= \frac{D\tau}{Dt}(T_t(x),t) &&= \vartheta_t + (v\cdot\nabla)\vartheta. \end{aligned}$$

These equations in Lagrange coordinates are

$$\begin{aligned} u' &= -\rho_{\mathbf{d}}\left(\rho_{\mathbf{d}}+\alpha\right)^{-1}L_{T_t}(u) - \left(\rho_{\mathbf{d}}+\alpha\right)^{-1}\nabla\left[p\left(\rho_{\mathbf{d}}+\alpha,\theta_e+\tau\right) - p(\rho_{\mathbf{d}},\theta_e)\right]\mathcal{Z}_{T_t} \\ &\qquad -\alpha(\rho_{\mathbf{d}}+\alpha)^{-1})ge_3 \\ \alpha' &= -\operatorname{tr}\left(\nabla\left(\rho_{\mathbf{d}}u\right)\mathcal{Z}_{T_t}\right) - \alpha\operatorname{tr}\left(\nabla u\mathcal{Z}_{T_t}\right) \\ \tau' &= cc_2^{-1}\left(\theta_e+\tau\right)\operatorname{tr}\left(\nabla u\mathcal{Z}_{T_t}\right) + \left(\rho_{\mathbf{d}}+\alpha\right)^{-1}c_2^{-1}\kappa\operatorname{tr}\left(\nabla\left(\nabla\left(\tau\right)\mathcal{Z}_{T_t}\right)\mathcal{Z}_{T_t}\right) \\ &\quad + \frac{\mu}{2c_2\left(\rho_{\mathbf{d}}+\alpha\right)}\left(\nabla u\mathcal{Z}_{T_t} + \left(\nabla u\mathcal{Z}_{T_t}\right)^{\top}\right)^2 + \frac{\nu-\mu}{c_2\rho_{\mathbf{d}}\left(\rho_{\mathbf{d}}+\alpha\right)}\left(\operatorname{tr}\left(\nabla u\mathcal{Z}_{T_t}\right)\right)^2 \end{aligned} \tag{5.9}$$

where $\rho_{\mathbf{d}} = \rho_e \circ (\mathbf{d} + E_S)$.

Differentiating $\beta(x,t) = (T_t)_3(x)$ with respect to time gives

$$\beta' = u_3(x,t) \tag{5.10}$$

and of course also

$$\mathbf{d}' = u. \tag{5.11}$$

We want to get estimates for the terms in the Lagrange system that are not present in the linearized one. For this purpose we separate the linearized terms from the non-linear ones. In the first equation

$$\rho_{\mathbf{d}} (\rho_{\mathbf{d}} + \alpha)^{-1} L_{T_t}(u) = L_{T_t}(u) - \alpha (\rho_{\mathbf{d}} + \alpha)^{-1} L_{T_t}(u),$$

$$(\rho_{\mathbf{d}} + \alpha)^{-1} \operatorname{tr} (\nabla (p(\rho_{\mathbf{d}} + \alpha, \theta_e + \tau) - p(\rho_{\mathbf{d}}, \theta_e)) \mathcal{Z}_{T_t})$$

$$= \rho_{\mathbf{d}}^{-1} \operatorname{tr} (\nabla (p(\rho_e + \alpha, \theta_e + \tau) - p(\rho_{\mathbf{d}}, \theta_e)) \mathcal{Z}_{T_t})$$

$$-\alpha(\rho_{\mathbf{d}} + \alpha)^{-1} \rho_{\mathbf{d}}^{-1} \operatorname{tr} (\nabla (p(\rho_{\mathbf{d}} + \alpha, \theta_e + \tau) - p(\rho_{\mathbf{d}}, \theta_e)) \mathcal{Z}_{T_t})$$

and

$$\begin{aligned} \alpha (\rho_{\mathbf{d}} + \alpha)^{-1} g e_3 &= \alpha \rho_{\mathbf{d}}^{-1} g e_3 - \alpha^2 \rho_{\mathbf{d}} (\rho_{\mathbf{d}} + \alpha)^{-1} g e_3 \\ &= -\rho_{\mathbf{d}}^{-1} \alpha g e_3 - \frac{\alpha^2}{\rho_{\mathbf{d}} (\rho_{\mathbf{d}} + \alpha)} g e_3. \end{aligned}$$

In the third equation

$$\frac{\kappa}{c_2 (\rho_{\mathbf{d}} + \alpha)} \operatorname{tr} (\nabla (\nabla (\tau + \theta_e)) \mathcal{Z}_{T_t}) = \frac{\kappa}{c_2 \rho_{\mathbf{d}}} \operatorname{tr} (\nabla (\nabla \tau \mathcal{Z}_{T_t}) \mathcal{Z}_{T_t})$$

$$-\frac{\kappa\alpha}{c_2 \rho_{\mathbf{d}} (\rho_{\mathbf{d}} + \alpha)} \operatorname{tr} (\nabla (\nabla \tau \mathcal{Z}_{T_t}) \mathcal{Z}_{T_t}).$$

On the boundary, for $\Omega^t = T_t(S)$ let n be the outward unit normal to Ω and let $n_T(x) = n(T(x))$. Then transforming the boundary condition gives

$$[-c (\alpha + \rho_e \circ T_t) (\tau + \theta_e) E + \mathbb{D}_{T_t}(u)] \cdot n_{T_t} = -\mathbf{p}_0 n_{T_t}$$

Define $\widetilde{\mathbb{T}}(u,\alpha,\tau) = \mathbb{D}_T(u) - (c\theta_e\alpha + c\rho_e\tau)\,E$ and

$$\mathbb{B}_T(U) = \mathbb{B}_T(u,\alpha,\tau,\beta) = \widetilde{\mathbb{T}}(u,\alpha,\tau) - c\rho_e\tau E - c\theta_e\left(\rho_{ex_3}\beta\right)E.$$

Let $\mathcal{B}_T(U) = \mathcal{B}_T(u,\alpha,\tau,\beta,\mathbf{d}) = \begin{bmatrix} \mathbb{B}_T(u,\alpha,\tau,\beta,\mathbf{d}) \cdot n_T \\ 0 \end{bmatrix}$. For $U = [u,\alpha,\tau,\beta,\mathbf{d}]$ define

$$\mathbf{g}(U) = \begin{bmatrix} \left(c\left(-\theta_e\alpha - \rho_e\tau - \theta_e\rho_{ex_3}\beta + (\rho_e\beta+\alpha)(\theta_e+\tau)\right) - \mathbf{p}_0\right)\cdot n_{\mathbf{d}+E_S} \\ 0 \end{bmatrix}.$$

The boundary condition takes the form

$$\mathcal{B}_T(U) = \mathbf{g}(U). \tag{5.12}$$

The system of equations in Lagrange coordinates can now be written

$$\begin{aligned}
u' + L_{T_t}(u) + \rho_{\mathbf{d}}^{-1}\nabla\left(p(\rho_{\mathbf{d}}+\alpha,\theta_e+\tau) - p(\rho_{\mathbf{d}},\theta_e)\right)\mathcal{Z}_{T_t} + \alpha g e_3 \mathcal{Z}_{T_t} &= \frac{\alpha}{\rho_{\mathbf{d}}+\alpha} L_{T_t}(U) \\
+\frac{\alpha}{\rho_{\mathbf{d}}(\rho_{\mathbf{d}}+\alpha)}\operatorname{tr}\left(\nabla(p(\rho_{\mathbf{d}}+\alpha,\theta_e+\tau) - p(\rho_{\mathbf{d}},\theta_e)\right)\mathcal{Z}_{T_t} &+ \frac{\alpha^2}{\rho_{\mathbf{d}}(\rho_{\mathbf{d}}+\alpha)} g e_3 \\
\alpha' + \operatorname{tr}\left(\nabla\left(\rho_{\mathbf{d}} u\right)\mathcal{Z}_{T_t}\right) &= -\alpha\operatorname{tr}\left(\nabla u \mathcal{Z}_{T_t}\right). \\
\tau' - \frac{c\theta_e}{c_2}\operatorname{tr}\left(\nabla u\mathcal{Z}_{T_t}\right) - \frac{\kappa}{c_2\rho_{\mathbf{d}}}\operatorname{tr}\left(\nabla\left(\nabla\tau\mathcal{Z}_{T_t}\right)\mathcal{Z}_{T_t}\right) \\
= \frac{c\tau}{c_2}\operatorname{tr}\left(\nabla u\mathcal{Z}_{T_t}\right) + \frac{\alpha\kappa}{c_2\rho_{\mathbf{d}}\left(\rho_{\mathbf{d}}+\alpha\right)}\operatorname{tr}\left(\nabla\left(\nabla\tau\mathcal{Z}_{T_t}\right)\mathcal{Z}_{T_t}\right) \\
+\frac{\mu}{2c_2\left(\rho_{\mathbf{d}}+\alpha\right)}\left(\nabla u\mathcal{Z}_{T_t} + \left(\nabla u\mathcal{Z}_{T_t}\right)^\top\right)^2 + \frac{\nu-\mu}{c_2\left(\rho_{\mathbf{d}}+\alpha\right)}\left(\operatorname{tr}\left(\nabla u\mathcal{Z}_{T_t}\right)\right)^2 \\
\mathcal{B}_T(U) &= \mathbf{g}(U) \\
\beta' &= u_3 \\
\mathbf{d}' = \mathbf{T}' &= u.
\end{aligned} \tag{5.13}$$

This system is closed by the equation

$$\mathbf{T} = \mathbf{d} + E_S \tag{5.14}$$

The operator $\mathfrak{U}_{1T}$ is defined by

$$\mathfrak{U}_{1T}\left[u, \alpha, \tau, \beta, d\right] = \left[\mathcal{L}_T\left(u, \alpha, \tau\right) + \frac{1}{\rho_{\mathbf{d}}}\alpha g e_3, \operatorname{tr}\left(\nabla\left(\rho_{\mathbf{d}} u\right)\mathcal{Z}_{T_t}\right), \mathcal{K}_T\left(u, \alpha, \tau\right), -u_3, -u\right]$$

where $\mathcal{L}_T$ and $\mathcal{K}_T$ are defined in equation 3.3. Further we define the operator $\mathfrak{U}_{1\mathbf{T}}$ by $\mathfrak{U}_{1\mathbf{T}}(t) = \mathfrak{U}_{1T_t}$. Then the equations (5.13) and (5.14) can be combined into the form

$$\mathbf{U}' + \mathfrak{U}_{1\mathbf{T}}\mathbf{U} = F(\mathbf{U}) \tag{5.15}$$

where the function $\mathbf{U}(t) = [u(t), \alpha(t), \tau(t), \beta(t), \mathbf{d}(t)]$ and $F(U) = [F_1(U), F_2(U), F_3(U), 0, 0]$ for $U = [u, \alpha, \tau, \beta, \mathbf{d}]$. Here we define

$$F_2(U) = -\alpha \operatorname{tr}\left(\nabla u \mathcal{Z}_{\mathbf{d}+E_S}\right), \tag{5.16}$$

as well as $F_1(U) = F_{11}(U) + F_{12}(U)$, where

$$F_{11}(U) = \frac{\alpha}{\alpha + \rho_{\mathbf{d}}} L_{\mathbf{d}+E_S}\left(u(t)\right) \tag{5.17}$$

and

$$\begin{aligned} &F_{12}(U) \\ &\quad = \frac{\alpha}{\rho_{\mathbf{d}}\left(\rho_{\mathbf{d}} + \alpha\right)}\nabla\left(p\left(\rho_{\mathbf{d}} + \alpha, \theta_e + \tau\right) - p(\rho_{\mathbf{d}}, \theta_e)\right)\mathcal{Z}_{\mathbf{d}+E_S} + \frac{\alpha^2}{\rho_{\mathbf{d}}\left(\rho_{\mathbf{d}} + \alpha\right)} g e_3. \end{aligned} \tag{5.18}$$

We also define $F_3(U) = F_{31}(U) + F_{32}(U) + F_{32}(U)$ with

$$F_{31}(U) = c c_2^{-1} \tau \operatorname{tr}\left(\nabla u \mathcal{Z}_{\mathbf{d}+E_S}\right), \tag{5.19}$$

$$F_{32}(U) = -\frac{\alpha\kappa}{c_2 \rho_{\mathbf{d}}\left(\rho_{\mathbf{d}} + \alpha\right)} \operatorname{tr}\left(\nabla\left(\nabla\tau\mathcal{Z}_{\mathbf{d}+E_S}\right)\mathcal{Z}_{\mathbf{d}+E_S}\right) \tag{5.20}$$

and

$$F_{33}(U) = \frac{\mu}{2c_2\left(\rho_{\mathbf{d}}+\alpha\right)}\left(\nabla u \mathcal{Z}_{\mathbf{d}+E_S}+\left(\nabla u \mathcal{Z}_{\mathbf{d}+E_S}\right)^{\top}\right)^2+\frac{\nu-\mu}{c_2\left(\rho_{\mathbf{d}}+\alpha\right)}\left(\operatorname{tr}\left(\nabla u \mathcal{Z}_{\mathbf{d}+E_S}\right)\right)^2. \tag{5.21}$$

Definition 5.6. *Any function $\mathbf{U} \in \mathfrak{M}^{\omega}$ for some $\omega \geq 1$ is said to solve the problem (P2) if it fulfills equations (5.12), (5.14), and (5.15). The initial values needed for this problem in order for us to solve (P1) are*

$$\begin{aligned}
\mathbf{d}(x,0)=T_0(x)-x &= \mathcal{T}_{\varphi_0}(x)-x \\
\beta(x,0) &= \varphi_0 \\
u(x,0) &= v_0\left(T_0(x)\right) \\
\alpha(x,0) &= \rho_0\left(T_0(x)\right)-\rho_e\left(T_0(x)\right) \\
\tau(x,0) &= \theta_0\left(T_0(x)\right).
\end{aligned} \tag{5.22}$$

Theorem 5.7. *Suppose $\left(\Omega^1, v, \rho, \tau\right)$ solves the problem (P1) and $\Omega^0=\Omega_{\varphi_0}$ for $\varphi_0 \in \widetilde{W}_p^{2-1/p}(\partial S)$. Then there exists a number $\epsilon_2>0$ such that if*

$$E_* = \mathcal{E}_1\left(\varphi_0, \theta_0, \rho_0, v_0\right)+\mathcal{E}_2\left(\Omega^1, \tau, \rho, v\right) \leq \epsilon_2$$

then the solution $\mathbf{T}: S \times [0,1] \to \mathbb{R}^3$ of the equations

$$\mathbf{T}'(x,t) = v\left(\tau(x,t), t\right)$$

$$\mathbf{T}(x,0) = \mathcal{T}_{\varphi_0}$$

has the property that with $T_t(x)=\mathbf{T}(x,t)$ we have $T_t-E_{\overline{S}} \in X$ and $T_t(S)=\Omega^t$.

The vector $\mathbf{U}(t)=[u(t), \alpha(t), \tau(t), \beta(t), \mathbf{d}(t)]$ defined by

$$\begin{aligned}
u(x,t) &= v\left(T_t(x), t\right) \\
\alpha(x,t) &= \rho\left(T_t(x), t\right)-\rho_e\left(T_t(x)\right) \\
\tau(x,t) &= \theta\left(T_t(x), t\right)-\theta_e \\
\beta(x,t) &= \mathbf{T}_3(x) \\
\mathbf{d}(x,t) &= \mathbf{T}(x,t)-x
\end{aligned}$$

belongs to $\mathfrak{M}^1$ and solves problem (P2). In addition

$$\|\mathbf{U}\|_{\mathfrak{M}^1} \leq CE_*.$$

Proof. Using Lemma 5.5 we have the solution $\mathbf{T} : \overline{S} \times [0,1] \to \mathbb{R}^3$ of the above differential equation with the property that $T_t(x) = \mathbf{T}(x,t)$ forms a family of injective transformations $T_t \in \widetilde{W}_p^2(S)$ with $T_t(S) = \Omega^t$ for $t \in [0,1]$ and inequality (5.5) guarantees that $T_t - E_S \in X$. Since $\left(\Omega^1, v, \rho, \theta\right)$ solves (P1) we have $\mathbf{U} \in L_p\left((0,1), D^1\right)$ and $\mathbf{U}' \in L_p\left((0,1), \mathfrak{B}^1\right)$. We also have that $\left(\Omega^1, v, \rho, \theta\right)$ solves (P2) because equations (5.12), (5.14), and (5.15) are satisfied. To show the final inequality note that by equation (5.9) the time-derivatives of u, α, and τ are

$$\|u_t\|_{\widetilde{L}_p} + \|\alpha_t\|_{\widetilde{L}_p} + \|\tau_t\|_{\widetilde{L}_p} \leq CE_*$$

and by equations (5.13) and (5.14) the time-derivatives of β and are

$$\begin{aligned} \|\beta_t\|_{\widetilde{L}_p} &\leq CE_* \\ \|\mathbf{d}_t\|_{\widetilde{L}_p} &\leq CE_*. \end{aligned}$$

Thus

$$\|\mathbf{U}\|_{\mathfrak{M}^1} \leq CE_*.$$

□

Lemma 5.8. *If $T : S \to \mathbb{R}^3, T \in \widetilde{W}_p^2(S)$ has the property that $T - E_S \in X$, then for $\xi \in \partial_2 S$ there is a unique $\varphi(\xi)$ such that*

$$\varphi\left(T_1(x), T_2(x)\right) = T_3(x) \tag{5.23}$$

when $x \in \partial_2 S$. The function φ belongs to $\widetilde{W}_p^{2-1/p}(\partial S)$ and

$$\|\varphi\|_{\widetilde{W}_p^{2-1/p}(\partial S)} \leq C\|T - E_S\|_{\widetilde{W}_p^2(S)}.$$

Proof. Using Lemma 5.2 and the Implicit Function Theorem gives the existence of a $\varphi \in C^1(\partial S)$ and the estimate $\|\varphi\|_{C^1(\partial S)} \leq C\|T - E_S\|_{\widetilde{W}_p^2(S)}$ for $x \in \partial S$. Let $\widehat{T}(x) = (T_1(x), T_2(x), x_3)$. Then $\widehat{T} - E_S \in X$ and $\widehat{T} : S \to S$ is a diffeomorphism, as one easily sees. Now let

$$\widehat{\varphi}(x) = T_3 \circ T^{-1}(x) - x_3$$

for $x \in S$. Then $\widehat{\varphi} \in \widetilde{W}_p^2(S)$ and $\widehat{T}$ maps the upper boundary to itself, so $\widehat{\varphi}|_{\partial_2 S} = \varphi$. Also

$$\begin{aligned}
\|\widehat{\varphi}\|_{\widetilde{W}_p^2} &\leq C\|T_3 \circ \widehat{T}^{-1}(x) - x_3\|_{\widetilde{W}_p^2} \\
&\leq C\|T \circ \widehat{T}^{-1}(x) - E_S\|_{\widetilde{W}_p^2} \\
&\leq C\|T \circ \widehat{T}^{-1}(x) - \widehat{T} \circ \widehat{T}^{-1}\|_{\widetilde{W}_p^2} \\
&\leq C\|(T - \widehat{T}) \circ \widehat{T}^{-1}\|_{\widetilde{W}_p^2} \\
&\leq C\|T - \widehat{T}\|_{\widetilde{W}_p^2} \\
&\leq C\left(\|T - E_S\|_{\widetilde{W}_p^2} + \|\widehat{T} - E_S\|_{\widetilde{W}_p^2}\right) \\
&\leq C\|T - E_S\|_{\widetilde{W}_p^2}.
\end{aligned}$$

□

Theorem 5.9. *There exists numbers $\epsilon_3 > 0, C < \infty$ such that when $\mathbf{U} = [u, \alpha, \tau, \beta, \mathbf{d}] \in \mathfrak{M}^\omega$ for some $\omega \in [1, \infty)$ and $\|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq \epsilon_3$, then $\mathbf{d}(t) \in X, T_t(x) = \mathbf{d}(x,t) + x$ is a diffeomorphism from $\overline{S}$ to $T_t\left(\overline{S}\right), \|\beta(t)\|_{\widetilde{W}_p^{2-1/p}} \leq \epsilon_1$ and $\|\alpha(t)\|_{\widetilde{C}^0(\overline{S})} \leq \frac{\rho_e}{2}(0)$, $\|\tau(t)\|_{\widetilde{C}^0(\overline{S})} \leq \theta_e$ for $t \in I_\omega$. If in addition, $\beta(x,0) = \mathbf{d}_3(x,0)$ and $\mathbf{U}$ solves problem (P2), then*

$$\mathbf{\Omega}^\omega = \{(y,t) : 0 \leq t \leq \omega, y = T_t(x) \ \textit{for some}\, x \in S\}$$

is a regular flow domain and with

$$v(y,t) = u\left(T_t(x),t\right), \rho(y,t) = \alpha\left(T_t(x),t\right) + \rho_e\left(T_t(x)\right), \theta(y,t) = \tau\left(T_t(x),t\right) + \theta_e$$

the quadruple $(\mathbf{\Omega}^\omega, v, \rho, \tau)$ *solves problem (P1) and* $\mathcal{E}_2\left(\mathbf{\Omega}^\omega, v, \rho, \theta\right) \leq C\|\mathbf{U}\|_{\mathfrak{M}^\omega}$. *There exists a function* $\varphi \in L^\infty\left(I_\omega, \widetilde{W}_p^{2-1/p}\left(\partial_2 S\right)\right) \cap C^1\left(I_\omega, \widetilde{W}_p^{1-1/p}\left(\partial_2 S\right)\right)$ *such that* $\Omega^t = \Omega_{\varphi(t)}$ *and*

$$\|\varphi(t)\|_{\widetilde{W}_p^{2-1/p}(\partial S)} + t\|\varphi(t)\|_{\widetilde{W}_p^{1-1/p}(\partial S)} + t^{4/3}\|\varphi(t)\|_{\widetilde{W}_p^{2/3-1/p}(\partial S)} \leq C\|U\|_{\mathfrak{M}^\omega}$$

This function also fulfills equation $\varphi' = \dfrac{d}{dt}T_{t3}(x)$.

Proof. From the definition of $\mathbf{U} \in \mathfrak{M}^\omega$ we have the deformation $\mathbf{d} \in \widetilde{W}_p^2(S)$. Then by equation (5.5) $\mathbf{d} \in X$ as $\mathbf{T}(x,t) = T_t(x)$ and $T_t(x) \in X$. Using equation (5.14) we have $\mathbf{T}(x,t) = \mathbf{d}(x,t) + x$ is a diffeomorphism.

$$\|\beta(t)\|_{\widetilde{W}_p^{2-1/p}} = \|\mathbf{T}(x,t) - h\|_{\widetilde{W}_p^{2-1/p}} \leq C\|\mathbf{d}(x,t) + x\|_{\widetilde{W}_p^2} \leq \epsilon_1,$$

$$\|\alpha(t)\|_{C^0(\overline{S})} = \|\rho\left(T_t\left(x\right),t\right) - \rho_e\left(T_t(x)\right)\|_{C^0(\overline{S})}$$

If $\beta(x,0) + h = \mathbf{d}(x,0) + x$ and $\mathbf{U}$ solves (P2) , then $\mathbf{\Omega}^\omega$ as defined above satisfies the definition of a regular flow domain. As in Theorem 5.7 $u \in \widetilde{W}_p^2\left(S\right)$ is equivalent to $v \in \widetilde{W}_p^{2,1}\left(\mathbf{\Omega}^\omega\right)$, so $(\mathbf{\Omega}^\omega, v, \rho, \theta)$ solves (P1). Using Lemma 5.8 we have

$$\|\varphi(t)\|_{\widetilde{W}_p^{2-1/p}(\partial S)} + t\|\varphi(t)\|_{\widetilde{W}_p^{1-1/p}(\partial S)} + t^{4/3}\|\varphi(t)\|_{\widetilde{W}_p^{2/3-1/p}(\partial S)} \leq C\|\mathbf{U}\|_{\mathfrak{M}^\omega}.$$

To prove $\varphi' = \dfrac{d}{dt}T_3(x)$ differentiate equation (5.23) with respect to time with $\varphi(t)$ and $T_t(x)$ replacing the fixed φ and T. □

5.2 Continuity Properties of Non-Linear Terms

The Lipschitz continuity that we claim in all the following cases will only be proved in certain cases as the proofs are quite lengthy and work similarly in each case.

Lemma 5.10. *There exists a constant $C < \infty$ such that if $\mathbf{U} \in \mathfrak{M}^{\omega}$, $\|\mathbf{U}\|_{\mathfrak{M}^{\omega}} \leq \epsilon_3$ for some $\omega \in [1, \infty)$, then $F_{11}(\mathbf{U})$ is well-defined and*

$$\|F_{11}(\mathbf{U}_1) - F_{11}(\mathbf{U}_2)\|_{\widetilde{L}_p(S\times[0,1])} \leq C\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1}\right)\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}$$

$$(1+t)^{7/3}\|F_{11}(\mathbf{U})\|_{L_p(S\times[t,t+1])} \leq C\|\mathbf{U}\|^2_{\mathfrak{M}^{\omega}}$$

for $0 \leq t \leq \omega - 1$.

Proof. The first inequality is left to the reader. For the second inequality we have for almost all $r \in (t, t+1)$

$$\|F_{11}(\mathbf{U})(r)\|_{\widetilde{L}_p} = \left\|\frac{\alpha(r)}{\alpha(r)+\rho_{\mathbf{d}}}L_{\mathbf{d}+E_S}\left(u(r)\right)\right\|_{\widetilde{L}_p} \leq \left\|\frac{\alpha(r)}{\alpha(r)+\rho_{\mathbf{d}}}\right\|_{\widetilde{L}_\infty}\|L_{\mathbf{d}+E_S}\left(u(r)\right)\|_{\widetilde{L}_p}$$

$$\leq \|\alpha(r)\|_{\widetilde{L}_\infty}\|u(r)\|_{\widetilde{W}^2_p} \leq C(1+t)^{-4/3}\|\mathbf{U}\|_{\mathfrak{M}^{\omega}}\|u(r)\|_{\widetilde{W}^2_p}.$$

Integrating the p-th power of this inequality from t to $t+1$ we have

$$\int_t^{t+1}(1+t)^{7p/3}\|F_{11}(\mathbf{U})(r)\|^p_{\widetilde{L}_p}dr \leq C\int_t^{t+1}(1+t)^{7p/3}(1+t)^{-4p/3}\|\mathbf{U}\|^p_{\mathfrak{M}^{\omega}}\|u(r)\|^p_{\widetilde{W}^2_p}dr$$

$$= C\int_t^{t+1}(1+t)^p\|\mathbf{U}\|^p_{\mathfrak{M}^{\omega}}\|u(r)\|^p_{\widetilde{W}^2_p}dr \leq C\|\mathbf{U}\|^{2p}_{\mathfrak{M}^{\omega}}.$$

Thus,

$$(1+t)^{7/3}\|F_{11}(\mathbf{U})\|_{\widetilde{L}_p(S\times[t,t+1])} \leq C\|\mathbf{U}\|^2_{\mathfrak{M}^{\omega}}.$$

□

Lemma 5.11. *There exists a constant $C < \infty$ such that if $\mathbf{U} \in \mathfrak{M}^{\omega}$, $\|\mathbf{U}\|_{\mathfrak{M}^{\omega}} \leq \epsilon_3$ for some $\omega \in [1, \infty)$, then $F_{12}(\mathbf{U})$ is well-defined and*

$$\|F_{12}(\mathbf{U}_1) - F_{12}(\mathbf{U}_2)\|_{\widetilde{L}_p(S\times[0,1])} \leq C\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1}\right)\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}$$

$$\sup_{0\leq t\leq\omega-1}(1+t)^{4/3}\|F_{12}(\mathbf{U})\|_{\widetilde{L}_p(S\times[t,t+1])} \leq C\|\mathbf{U}\|^2_{\mathfrak{M}^{\omega}}.$$

Proof. The first inequality is left to the reader. For the second inequality we have for almost all $r \in (t, t+1)$

$$\|F_{12}(\mathbf{U})(r)\|_{\widetilde{L}_p} = \|\frac{\alpha(r)}{\rho_{\mathbf{d}}\left(\rho_{\mathbf{d}}+\alpha(r)\right)}\nabla\left(p\left(\rho_{\mathbf{d}}+\alpha(r),\theta_e+\tau(r)\right)-p(\rho_{\mathbf{d}},\theta_e)\right)\mathcal{Z}_{\mathbf{d}}$$

$$+\frac{(\alpha(r))^2}{\rho_{\mathbf{d}}\left(\rho_{\mathbf{d}}+\alpha(r)\right)}ge_3\|_{\widetilde{L}_p}$$

$$\leq C\left(\|\frac{\alpha(r)}{\rho_{\mathbf{d}}+\alpha(r)}\|_{\widetilde{L}_\infty}\|\alpha(r)\tau(r)+\rho_{\mathbf{d}}\tau(r)+\theta_e\alpha(r)\|_{\widetilde{W}^1_p}+\|\frac{(\alpha(r))^2}{\rho_{\mathbf{d}}+\alpha(r)}\|_{\widetilde{L}_\infty}\right)$$

$$\leq C\left(\|\alpha(r)\|_{\widetilde{L}_\infty}\left(\|\alpha(r)\tau(r)\|_{\widetilde{W}^1_p}+\|\theta_e\alpha(r)\|_{\widetilde{W}^1_p}+\|\rho_{\mathbf{d}}\tau(r)\|_{\widetilde{W}^1_p}\right)+\|\alpha(r)\|^2_{\widetilde{L}_\infty}\right)$$

$$\leq C\left(\|\alpha(r)\|_{\widetilde{L}_\infty}\left(\|\tau(r)\|_{\widetilde{W}^2_p}\|\alpha(r)\|_{\widetilde{L}_\infty}+\|\alpha(r)\|_{\widetilde{W}^2_p}\|\tau(r)\|_{\widetilde{L}_\infty}+\|\alpha(r)\|_{\widetilde{W}^1_p}+\|\tau(r)\|_{\widetilde{W}^1_p}\right)\right.$$

$$\left.+\|\alpha(r)\|^2_{\widetilde{L}_\infty}\right)$$

$$\leq C\left((1+t)^{-4/3}\|\mathbf{U}\|_{\mathfrak{M}^\omega}\left((1+t)^{-7/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}+(1+t)^{-1}\|\mathbf{U}\|_{\mathfrak{M}^\omega}\right)+(1+t)^{-8/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}\right)$$

$$\leq C\left((1+t)^{-11/3}\|\mathbf{U}\|^3_{\mathfrak{M}^\omega}+(1+t)^{-7/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}+(1+t)^{-8/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}\right)$$

$$\leq C(1+t)^{-4/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

Thus,

$$\sup_{0\leq t\leq\omega-1}(1+t)^{4/3}\|F_{12}(\mathbf{U})\|_{\widetilde{L}_p(S\times[t,t+1])}\leq C\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

□

Lemma 5.12. *There exists a constant $C<\infty$ such that if $\mathbf{U}\in\mathfrak{M}^\omega$, $\|\mathbf{U}\|_{\mathfrak{M}^\omega}\leq\epsilon_3$ for some $\omega\in[1,\infty)$, then $F_2(\mathbf{U})$ is well-defined and*

$$\|F_2(\mathbf{U}_1)-F_2(\mathbf{U}_2)\|_{\widetilde{W}^{1,0}_p(S\times[0,1])}\leq C\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1}+\|\mathbf{U}_2\|_{\mathfrak{M}^1}\right)\|\mathbf{U}_1-\mathbf{U}_2\|_{\mathfrak{M}^1}$$

$$\sup_{0\leq t\leq\omega-1}(1+t)^2\|F_2(\mathbf{U})\|_{\widetilde{W}^{1,0}_p(S\times[t,t+1])}\leq C\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

Proof. By Lemma 5 in [8] we estimate the first inequality.

$$\|F_2(\mathbf{U}_1)-F_2(\mathbf{U}_2)\|_{\widetilde{W}^{1,0}_p(S\times[t,t+1])}$$

$$= \|\alpha_1 \operatorname{tr}(\nabla u_1 \mathcal{Z}_{\mathbf{d}_1+E_S}) - \alpha_2 \operatorname{tr}(\nabla u_2 \mathcal{Z}_{\mathbf{d}_2+E_S})\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])}$$

$$\leq C \Big(\|(\alpha_1-\alpha_2)\operatorname{tr}(\nabla u_1 \mathcal{Z}_{\mathbf{d}_1+E_S})\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])}$$

$$+\|\alpha_2 \operatorname{tr}(\nabla (u_1-u_2) \mathcal{Z}_{\mathbf{d}_1+E_S})\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])}$$

$$+\|\alpha_2 \operatorname{tr}(\nabla u_2 (\mathcal{Z}_{\mathbf{d}_1+E_S} - \mathcal{Z}_{\mathbf{d}_2+E_S}))\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])}\Big)$$

$$\leq C \Big(\|(\alpha_1-\alpha_2)\|_{\widetilde{L}_\infty} \|u_1\|_{\widetilde{W}_p^2} + \|\alpha_2\|_{\widetilde{L}_\infty}\|u_1-u_2\|_{\widetilde{W}_p^2}$$

$$+\|\alpha_2\|_{\widetilde{L}_\infty}\|u_2\|_{\widetilde{W}_p^2}\|\mathbf{d}_1-\mathbf{d}_2\|_{\widetilde{W}_p^1}\Big)$$

$$\leq C \left(\|\mathbf{U}_1-\mathbf{U}_2\|_{\mathfrak{M}^1}\|\mathbf{U}_1\|_{\mathfrak{M}^1}^2 + \|\mathbf{U}_1\|_{\mathfrak{M}^1}\|\mathbf{U}_2\|_{\mathfrak{M}^1}\|\mathbf{U}_1-\mathbf{U}_2\|_{\mathfrak{M}^1}\right.$$

$$\left. +\|\mathbf{U}_2\|_{\mathfrak{M}^1}\|\mathbf{U}_1-\mathbf{U}_2\|_{\mathfrak{M}^1}\right)$$

$$\leq C \left(\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1}\right)\|\mathbf{U}_1-\mathbf{U}_2\|_{\mathfrak{M}^1}.$$

For the second inequality we have for almost all $r \in (t, t+1)$

$$\|F_2(\mathbf{U})(r)\|_{\widetilde{W}_p^1} = \|\alpha(r)\operatorname{tr}(\nabla u(r)\mathcal{Z}_{\mathbf{d}+E_S})\|_{\widetilde{W}_p^1} \leq C\|\alpha(r)\|_{\widetilde{W}_p^1}\|\operatorname{tr}(\nabla u(r))\|_{\widetilde{W}_p^1}$$

$$\leq C(1+t)^{-1}\|\mathbf{U}\|_{\mathfrak{M}^\omega}\|u(r)\|_{\widetilde{W}_p^2}.$$

Integrating the p-th power of this inequality from t to $t+1$ we have

$$\begin{aligned} \int_t^{t+1}(1+t)^p\|F_2(\mathbf{U})(r)\|_{\widetilde{W}_p^1}^p dr \quad &\leq \quad C\int_t^{t+1}(1+t)^p(1+t)^{-p}\|\mathbf{U}\|_{\mathfrak{M}^\omega}^p\|u(r)\|_{\widetilde{W}_p^2}^p dr \\ &\leq \quad C(1+t)^{-2p}\|\mathbf{U}\|_{\mathfrak{M}^\omega}^{2p}. \end{aligned}$$

Thus,

$$\sup_{0\leq t\leq \omega-1}(1+t)^2\|F_2(\mathbf{U})\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])} \leq C\|\mathbf{U}\|_{\mathfrak{M}^\omega}^2.$$

□

Lemma 5.13. *There exists a constant $C < \infty$ such that if $\mathbf{U} \in \mathfrak{M}^\omega$, $\|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq \epsilon_3$ for some $\omega \in [1,\infty)$, then $F_{31}(\mathbf{U})$ is well-defined and*

$$\|F_{31}(\mathbf{U}_1)-F_{31}(\mathbf{U}_2)\|_{\widetilde{W}_p^{1,0}(S\times[0,1])}\leq C\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1}+\|\mathbf{U}_2\|_{\mathfrak{M}^1}\right)\|\mathbf{U}_1-\mathbf{U}_2\|_{\mathfrak{M}^1}$$

$$\sup_{0\leq t\leq\omega-1}(1+t)^2\|F_{31}(\mathbf{U})\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])}\leq C\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

Proof. By Lemma 5 in [8] we estimate the first inequality.

$$\|F_{31}(\mathbf{U}_1)-F_{31}(\mathbf{U}_2)\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])}$$

$$\begin{aligned}
&= \left\|cc_2^{-1}\tau_1\,\mathrm{tr}\left(\nabla u_1\mathcal{Z}_{\mathbf{d}_1}\right)-cc_2^{-1}\tau_2\,\mathrm{tr}\left(\nabla u_2\mathcal{Z}_{\mathbf{d}_2}\right)\right\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])}\\
&\leq C\left(\left\|cc_2^{-1}\left(\tau_1-\tau_2\right)\mathrm{tr}\left(\nabla u_1\mathcal{Z}_{\mathbf{d}_1}\right)\right\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])}\right.\\
&\quad+\left\|cc_2^{-1}\tau_2\,\mathrm{tr}\left(\nabla\left(u_1-u_2\right)\mathcal{Z}_{\mathbf{d}_1}\right)\right\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])}\\
&\quad\left.+\left\|cc_2^{-1}\tau_2\,\mathrm{tr}\left(\nabla u_2\left(\mathcal{Z}_{\mathbf{d}_2}-\mathcal{Z}_{\mathbf{d}_1}\right)\right)\right\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])}\right)
\end{aligned}$$

$$\leq C\left(\|\tau_1-\tau_2\|_{\widetilde{L}_\infty}\|u_1\|_{\widetilde{W}_p^2}+\|\tau_2\|_{\widetilde{L}_\infty}\|u_2-u_1\|_{\widetilde{W}_p^2}+\|\tau_2\|_{\widetilde{L}_\infty}\|u_2\|_{\widetilde{W}_p^2}\|\mathbf{d}_2-\mathbf{d}_1\|_{\widetilde{W}_p^1}\right)$$

$$\leq C\left(\|\mathbf{U}_1-\mathbf{U}_2\|_{\mathfrak{M}^1}\|\mathbf{U}_1\|_{\mathfrak{M}^1}+\|\mathbf{U}_2\|_{\mathfrak{M}^1}\|\mathbf{U}_1-\mathbf{U}_2\|_{\mathfrak{M}^1}+\|\mathbf{U}_2\|^2_{\mathfrak{M}^1}\|\mathbf{U}_1-\mathbf{U}_2\|_{\mathfrak{M}^1}\right)$$

$$\leq C\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1}+\|\mathbf{U}_2\|_{\mathfrak{M}^1}\right)\|\mathbf{U}_1-\mathbf{U}_2\|_{\mathfrak{M}^1}.$$

For the second inequality we have for almost all $r\in(t,t+1)$

$$\|F_{31}(\mathbf{U})(r)\|_{\widetilde{W}_p^1}=\|cc_2^{-1}\tau(r)\,\mathrm{tr}\left(\nabla u(r)\mathcal{Z}_{T_t}\right)\|_{\widetilde{W}_p^1}\leq C\|\tau(r)\|_{\widetilde{W}_p^1}\|\,\mathrm{tr}\left(\nabla u(r)\mathcal{Z}_{T_t}\right)\|_{\widetilde{W}_p^1}$$

$$\leq C(1+t)^{-1}\|\mathbf{U}\|_{\mathfrak{M}^\omega}\|u(r)\|_{\widetilde{W}_p^2}.$$

Integrating the p-th power of this inequality from t to $t+1$ we have

$$\begin{aligned}
\int_t^{t+1}(1+t)^p\|F_{31}(\mathbf{U})(r)\|^p_{\widetilde{W}_p^1}dr &\leq C\int_t^{t+1}(1+t)^p(1+t)^{-p}\|\mathbf{U}\|^p_{\mathfrak{M}^\omega}\|u(r)\|^p_{\widetilde{W}_p^2}dr\\
&\leq C(1+t)^{-2p}\|\mathbf{U}\|^{2p}_{\mathfrak{M}^\omega}.
\end{aligned}$$

Thus,

$$\sup_{0\leq t\leq\omega-1}(1+t)^2\|F_{31}(\mathbf{U})\|_{\widetilde{W}_p^{1,0}(S\times[t,t+1])}\leq C\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

□

Lemma 5.14. *There exists a constant $C < \infty$ such that if $\mathbf{U} \in \mathfrak{M}^\omega$, $\|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq \epsilon_3$ for some $\omega \in [1, \infty)$, then $F_{32}(\mathbf{U})$ is well-defined and*

$$\|F_{32}(\mathbf{U}_1) - F_{32}(\mathbf{U}_2)\|_{\widetilde{L}_p(S\times[0,1])} \leq C\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1}\right)\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}$$

$$\sup_{0\leq t\leq \Omega-1}(1+t)^{7/3}\|F_{32}(\mathbf{U})\|_{\widetilde{L}_p(S\times[t,t+1])} \leq C\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

Proof. The first inequality is left to the reader. For the second inequality we have for almost all $r \in (t, t+1)$

$$\|F_{32}(\mathbf{U})(r)\|_{\widetilde{L}_p} = \left\|-\frac{\alpha(r)\kappa}{c_2\rho_{\mathbf{d}}\left(\rho_{\mathbf{d}} + \alpha(r)\right)}\operatorname{tr}\left(\nabla\left(\nabla\tau(r)\mathcal{Z}_{T_t}\right)\mathcal{Z}_{T_t}\right)\right\|_{\widetilde{L}_p}$$

$$\leq C\left\|\frac{\alpha(r)\kappa}{c_2\rho_{\mathbf{d}}\left(\rho_{\mathbf{d}} + \alpha(r)\right)}\right\|_{\widetilde{L}_\infty}\|\operatorname{tr}\left(\nabla\left(\nabla\tau(r)\right)\mathcal{Z}_{T_t}\right)\|_{\widetilde{L}_p}$$

$$\leq C\|\alpha(r)\|_{\widetilde{L}_\infty}\|\tau(r)\|_{\widetilde{W}^2_p} \leq C(1+t)^{-4/3}\|\mathbf{U}\|_{\mathfrak{M}^\omega}\|\tau(r)\|_{\widetilde{W}^2_p}.$$

Integrating the p-th power of this inequality from t to $t+1$ we have

$$\int_t^{t+1}(1+t)^{4p/3}\|F_{32}(\mathbf{U})(r)\|^p_{\widetilde{L}_p}\,dr$$

$$\leq C\int_t^{t+1}(1+t)^{4p/3}(1+t)^{-4p/3}\|\mathbf{U}\|^p_{\mathfrak{M}^\omega}\|\tau(r)\|^p_{\widetilde{W}^2_p}dr$$

$$\leq C(1+t)^{-p}\|\mathbf{U}\|^{2p}_{\mathfrak{M}^\omega}.$$

Thus,

$$\sup_{0\leq t\leq \omega-1}(1+t)^{7/3}\|F_{32}(\mathbf{U})\|_{\widetilde{L}_p(S\times[t,t+1])} \leq C\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}$$

□

Lemma 5.15. *There exists a constant $C < \infty$ such that if $\mathbf{U} \in \mathfrak{M}^\omega$, $\|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq \epsilon_3$ for some $\omega \in [1, \infty)$, then $F_2(\mathbf{U})$ is well-defined and*

$$\|F_{33}(\mathbf{U}_1)(r) - F_{33}(\mathbf{U}_2)(r)\|_{\widetilde{L}_p(S\times[0,1])} \leq C\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1}\right)\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}$$

$$\sup_{0\leq t\leq \omega-1}(1+t)^2\|F_{33}(\mathbf{U})\|_{\widetilde{L}_p(S\times[t,t+1])} \leq C\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

Proof. The first inequality is left to the reader. For the second inequality we have for almost all $r \in (t, t+1)$

$$\begin{aligned}
\|F_{33}(\mathbf{U})(r)\|_{\widetilde{L}_p} &= \left\| \frac{\mu}{2c_2(\rho_{\mathbf{d}} + \alpha(r))} \left(\nabla u(r)\mathcal{Z}_{T_t} + (\nabla u(r)\mathcal{Z}_{T_t})^\top\right)^2 \right. \\
&\quad \left. + \frac{\nu-\mu}{c_2(\rho_{\mathbf{d}} + \alpha(r))} \left(\operatorname{tr}(\nabla u(r)\mathcal{Z}_{T_t})\right)^2 \right\|_{\widetilde{L}_p} \\
&\le C \left(\left\| \frac{1}{\rho_{\mathbf{d}} + \alpha(r)} \right\|_{\widetilde{L}_\infty} \left\| \left(\nabla u(r)\mathcal{Z}_{T_t} + (\nabla u(r)\mathcal{Z}_{T_t})^\top\right)^2 \right\|_{\widetilde{L}_p} \right. \\
&\quad \left. + |\nu - \mu| \| \frac{1}{\rho_{\mathbf{d}} + \alpha(r)} \|_{\widetilde{L}_\infty} \| \left(\operatorname{tr}(\nabla u(r)\mathcal{Z}_{T_t})\right)^2 \|_{\widetilde{L}_p} \right) \\
&\le C \left(\|\nabla u(r)\mathcal{Z}_{T_t} + (\nabla u(r)\mathcal{Z}_{T_t})^\top \|^2_{\widetilde{L}_{2p}} + \| \operatorname{tr}(\nabla u(r)\mathcal{Z}_{T_t}) \|^2_{L_{2p}} \right) \\
&\le C \left(\left\|\nabla u(r)\mathcal{Z}_{T_t} + (\nabla u(r)\mathcal{Z}_{T_t})^\top \right\|^2_{\widetilde{W}^1_p} + \| \operatorname{tr}(\nabla u(r)\mathcal{Z}_{T_t}) \|^2_{\widetilde{W}^1_p} \right) \\
&\le C \left(\|u(r)\|^2_{\widetilde{W}^2_p} + \|u(r)\|^2_{\widetilde{W}^2_p} \right) \\
&\le \widetilde{C} \|u(r)\|^2_{\widetilde{W}^2_p}.
\end{aligned}$$

Integrating the p-th power of this inequality from t to $t+1$ we have

$$\int_t^{t+1} \|F_{33}(\mathbf{U})(r)\|^p_{\widetilde{L}_p} dr \le C \int_t^{t+1} \|u(r)\|^{2p}_{\widetilde{W}^2_p} dr \le C(1+t)^{-2p} \|\mathbf{U}\|^{2p}_{\mathfrak{M}^\omega}.$$

Thus,

$$(1+t)^2 \|F_{33}(\mathbf{U})\|_{\widetilde{L}_p(S\times[t,t+1])} \le C \|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

□

Lemma 5.16. *There exists a constant $C < \infty$ so that when $\mathbf{U}_k = [u_k, \alpha_k, \tau_k, \beta_k, d_k] \in \mathfrak{M}^\omega$, $\|\mathbf{U}_k\| \le \epsilon_3$ (k=1,2) for some $\omega \in [1, \infty)$, then $\mathbf{g}(\mathbf{U}_k)$ is well-defined and*

$$\begin{aligned}
\|\mathbf{g}(\mathbf{U}_1) - \mathbf{g}(\mathbf{U}_2)\|_{S^1_2} &\le C\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1}\right) \|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1} \\
\|\mathbf{g}(\mathbf{U}_1)\|_{S^\omega_2} &\le C \|\mathbf{U}_1\|^2_{\mathfrak{M}^\omega}
\end{aligned}$$

Proof. For ease of notation we write $\widetilde{W}_p^{1-1p,1/2-1/2p}$ for $\widetilde{W}_p^{1-1p,1/2-1/2p}\left(\partial_2 S\times(t,t+1)\right)$ and let $\mathfrak{g}\left(\alpha,\tau,\beta\right)=-c\theta_e\alpha-c\rho_e\tau-c\theta_e\rho_{ex3}\beta+c(\rho_e\beta+\alpha)(\theta_e+\tau)-\mathbf{p}_0$. Then $\mathfrak{g}:\left(-\frac{\rho_e(0)}{2},\infty\right)\times\mathbb{R}\times(-h,2h)\to\mathbb{R}$ is infinitely differentiable. The first inequality is obvious in this form. For the second inequality we have to worry a little about the weights. For $(\alpha_2,\tau_2,\beta_2)=0$ we have

$$\begin{aligned}\|\mathfrak{g}\left(\alpha_1,\tau_1,\beta_1\right)\|_{\widetilde{W}_p^{1-1/p,1/2-1/2p}} &= \|\mathfrak{g}\left(\alpha_1,\tau_1,\beta_1\right)-\mathfrak{g}\left(\alpha_2,\tau_2,\beta_2\right)\|_{\widetilde{W}_p^{1-1/p,1/2-1/2p}}\\ &\leq C\|\left(\alpha_1,\tau_1,\beta_1\right)\|^2_{\widetilde{W}_p^{1-1/p,1/2-1/2p}}\end{aligned}$$

by Lemma 10 in [10]. Now by definition of $[\mathbf{U}_1]_1$ we have

$$\begin{aligned}\|\mathfrak{g}\left(\alpha_1,\tau_1,\beta_1\right)\|_{\widetilde{W}_p^{1-1/p,1/2-1/2p}} &\leq C\left((1+t)^{-1}[\mathbf{U}_1]_1\right)^2\\ &\leq C(1+t)^{-2}\|\mathbf{U}_1\|^2_{\mathfrak{M}^\omega}.\end{aligned}$$

Also, $\|\mathfrak{g}(\mathbf{U}_1)\|_{\widetilde{W}_p^{1-1/p,1/2-1/2p}}$

$$\begin{aligned}&\leq \|\mathfrak{g}(\mathbf{U}_1)\cdot n_{T_1}\|_{\widetilde{W}_p^{1-1/p,1/2-1/2p}}\|n_{T_1}\|_{\widetilde{W}_p^{1-1/p,1/2-1/2p}}\\ &\leq (1+t)^{-2}\|\mathbf{U}_1\|^2_{\mathfrak{M}^\omega}\left(\sup_{t\leq r\leq t-1}\|n_{T_1}(r)\|_{\widetilde{W}_p^{1-1/p}}+\|n_{T_1}\|_{\widetilde{C}^{2/3}([t,t+1],C^0)}\right)\\ &\leq \widetilde{C}(1+t)^{-2}\|\mathbf{U}_1\|^2_{\mathfrak{M}^\omega}.\end{aligned}$$

□

5.3 Existence and Uniqueness of Local Solutions

In this section problem (P2) will be solved for $\omega=1$. As a consequence we will also have Theorem 1.3 for $\omega=1$ as well. The following theorem summarizes the results of the proceeding section.

Theorem 5.17. *There exists a constant C so that when $\mathbf{U}_k=[u_k,\alpha_k,\tau_k,\beta_k,\mathbf{d}_k]\in\mathfrak{M}^\omega$ and $\|\mathbf{U}_k\|_{\mathfrak{M}^\omega}\leq\epsilon_3$, then*

$$\|F(\mathbf{U}_1)-F(\mathbf{U}_2)\|_{S_1^\omega}\leq C\|\mathbf{U}_1-\mathbf{U}_2\|_{\mathfrak{M}^1}\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1}+\|\mathbf{U}_2\|_{\mathfrak{M}^1}\right)$$

and

$$\|\mathbf{g}(\mathbf{U}_1) - \mathbf{g}(\mathbf{U}_2)\|_{S_2^\omega} \leq C\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1} \left(\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1}\right).$$

If in addition $\mathbf{d}_3(x,t) = \beta(x,t)$*, then also*

$$\|F(\mathbf{U}_1)\|_{S_1^\omega} \leq C\|\mathbf{U}_1\|_{\mathfrak{M}^1})^{4/3}, \|\mathbf{g}(\mathbf{U}_1)\|_{S_2^\omega} \leq C\|\mathbf{U}_1\|_{\mathfrak{M}^1}^2.$$

Proof. This is simply a combination of Lemmas 5.10, 5.11, 5.12, 5.13, 5.14, 5.15, and 5.16. □

We want to find a unique small solution of Problem (P2) for sufficiently small initial values as the fixed point of a contraction. For the solution $\mathbf{U} = [u, \alpha, \tau, \beta, \mathbf{d}] \in \mathfrak{M}^1$ we have

$$\mathbf{T}(\mathbf{U})(x,t) = x + \mathbf{d}(x,0) + \int_0^t u(x,r)dr. \tag{5.24}$$

Lemma 5.18. *There exists a constant* C *so that whenever* $\mathbf{U}_1, \mathbf{U}_2 \in \mathfrak{M}^1$*, then*

$$\|\mathbf{T}(\mathbf{U}_1) - \mathbf{T}(\mathbf{U}_2)\|_{S^1} \leq C\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}.$$

Proof. Using equation 5.24 we have

$$\begin{aligned}
\|\mathbf{T}(\mathbf{U}_1) - \mathbf{T}(\mathbf{U}_2)\|_{S^1} &\leq C\left(\|\mathbf{T}(\mathbf{U}_1 - \mathbf{U}_2)\|_{C^0([0,1],\widetilde{W}_p^2)} + \|\mathbf{T}(\mathbf{U}_1 - \mathbf{U}_2)\|_{C^{2/3}([0,1],\widetilde{C}^1)}\right) \\
&\leq \left(C\max\|\mathbf{T}(\mathbf{U}_1)(x) - \mathbf{T}(\mathbf{U}_2)(x)\|_{W_p^2} + \max\|\frac{d}{dt}\left(\mathbf{T}(\mathbf{U}_1) - \mathbf{T}(\mathbf{U}_2)\right)(x)\|_{C^1}\right) \\
&\leq C\left(\max\|\mathbf{d}_1(x,t) - \mathbf{d}_2(x,t)\|_{\widetilde{W}_p^2} + \max\|u_1(x,t) - u_2(x,t)\|_{\widetilde{C}^1}\right) \\
&\leq C\left(\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1} + \|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}\right) \\
&\leq C\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}.
\end{aligned}$$

□

Theorem 5.19. *There exists numbers* $\epsilon_4 \in (0, \epsilon_3], \eta_3 > 0, C < \infty$ *so that whenever* $\mathbf{U}_0 = [u_0, \alpha_0, \tau_0, \beta_0, \mathbf{d}_0] \in \mathfrak{B}^1 \cap D^{1-2/p}$,

$$\mathcal{B}_{\mathbf{d}_0+E_{\overline{S}}}(\mathbf{U}_0) = \mathbf{g}(\mathbf{U}_0) \text{ and } \|\mathbf{U}_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} \leq \eta_3,$$

then there exists exactly one function $\mathbf{U} \in \mathfrak{M}^1$ *which solves Problem (P2),* $\mathbf{U}(0) = \mathbf{U}_0$, *and fulfills*

$$\|\mathbf{U}\|_{\mathfrak{M}^1} \leq \epsilon_4.$$

Proof. Let $\delta > 0$. Define $\widetilde{D}^\delta =$

$$\left\{ \mathbf{U} = [u, \alpha, \tau, \beta, \mathbf{d}] \in \mathfrak{M}^1 : \|\mathbf{U}\|_{\mathfrak{M}^1} \leq \delta, \alpha(0) = \alpha_0, \tau(0) = \tau_0, \beta(0) = \beta_0, \mathbf{d}(0) = \mathbf{d}_0 \right\}.$$

Note that this set may be empty, depending on U_0 and δ. Take $\epsilon_4^1 \in (0, \epsilon_e]$ be sufficiently small so that if $\mathbf{U} \in \widetilde{D}^{\epsilon_4^1}$. Then the transformation $\mathbf{T}(\mathbf{U})$ fulfills the conditions of Theorem 13 in [8] for $\omega = 1$. Using the previous lemma,

$$\|T(\mathbf{U}) - E_S\|_{S^1} = \|\mathbf{T}(\mathbf{U}) - \mathbf{T}(0)\|_{S^1} \leq C\|\mathbf{U}\|_{\mathfrak{M}^1}.$$

For $\mathbf{U} \in \widetilde{D}^{\epsilon_4^1}$ there exists a solution $\mathbf{V}$ of

$$\mathbf{V}' = -\mathfrak{U}_{1\mathbf{T}(\mathbf{U})}\mathbf{V} + F(\mathbf{U})$$

with initial values $\mathbf{V}(0) = U_0$ and having boundary conditions $\mathcal{B}_{\mathbf{T}(\mathbf{U})}(\mathbf{V}) = \mathbf{g}(\mathbf{U})$. By assumption, the compatibility condition is fulfilled by

$$\mathcal{B}_{\mathbf{T}(\mathbf{U})(U_0)} = \mathcal{B}_{\mathbf{d}_0+E_S}(U_0) = \mathbf{g}(U_0).$$

Now we can define a mapping $\mathfrak{C}$ from $\widetilde{D}^{\epsilon_4^1}$ to $\mathfrak{M}^1$ by $\mathfrak{C}(\mathbf{U}) = V$.

$\mathbf{U} \in \widetilde{D}^{\epsilon_4^1}$ solves the problem exactly when it is a fixed point of $\mathfrak{C}$ on this set. Using

Theorem 5.17 as well as Theorem 13 in [8] we have

$$\begin{aligned} \|\mathbf{V}\|_{\mathfrak{M}^1} &\leq C\left(\|F(\mathbf{U})\|_{S_1^1} + \|\mathrm{g}(\mathbf{U})\|_{S_2^1} + \|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}}\right) \\ &\leq C\left(\|\mathbf{U}\|^2_{\mathfrak{M}^1} + \|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}}\right) \end{aligned} \tag{5.25}$$

Let $\mathbf{U}_k \in \widetilde{D}^{\epsilon_4^1}$ for $k \in \{1,2\}$ and $\mathbf{V}_k = \mathfrak{C}(\mathbf{U}_k)$. Take $\mathbf{W} = \mathbf{V}_1 - \mathbf{V}_2$ with $\mathbf{W}(0) = 0$.

$$\begin{aligned} \mathbf{W}' = (\mathbf{V}_1 - \mathbf{V}_2)' &= \mathfrak{U}_{1\mathbf{T}(\mathbf{U}_1)}(\mathbf{V}_1) + F(\mathbf{U}_1) - \mathfrak{U}_{1\mathbf{T}(\mathbf{U}_2)}(\mathbf{V}_2) - F(\mathbf{U}_2) \\ &= \mathfrak{U}_{1\mathbf{T}(\mathbf{U}_1)}(\mathbf{V}_1 - \mathbf{V}_2) + \mathfrak{U}_{1\mathbf{T}(\mathbf{U}_1)}(\mathbf{V}_2) - \mathfrak{U}_{1\mathbf{T}(\mathbf{U}_2)}(\mathbf{V}_2) \\ &\quad + F(\mathbf{U}_1) - F(\mathbf{U}_2) \\ &= \mathfrak{U}_{1\mathbf{T}(\mathbf{U}_1)}(\mathbf{W}) + \mathfrak{U}_{1\mathbf{T}(\mathbf{U}_1)}(\mathbf{V}_2) - \mathfrak{U}_{1\mathbf{T}(\mathbf{U}_2)}(\mathbf{V}_2) \\ &\quad + F(\mathbf{U}_1) - F(\mathbf{U}_2) \\ \mathcal{B}_{\mathbf{T}(\mathbf{U}_1)}(\mathbf{W}) &= \mathcal{B}_{\mathbf{T}(\mathbf{U}_1)}(\mathbf{V}_1 - \mathbf{V}_2) \\ &= \mathcal{B}_{\mathbf{T}(\mathbf{U}_1)}(\mathbf{V}_1) - \mathcal{B}_{\mathbf{T}(\mathbf{U}_1)}(\mathbf{V}_2) + \mathcal{B}_{\mathbf{T}(\mathbf{U}_2)}(\mathbf{V}_2) - \mathcal{B}_{\mathbf{T}(\mathbf{U}_2)}(\mathbf{V}_2) \\ &= \mathrm{g}(\mathbf{U}_1) - \mathcal{B}_{\mathbf{T}(\mathbf{U}_1)}(\mathbf{V}_2) + \mathcal{B}_{\mathbf{T}(\mathbf{U}_2)}(\mathbf{V}_2) - \mathrm{g}(\mathbf{U}_2) \\ \mathcal{B}_{\mathbf{T}(\mathbf{U}_1)}(\mathbf{W}) &= \mathcal{B}_{\mathbf{T}(\mathbf{U}_1)}(\mathbf{V}_2) - \mathcal{B}_{\mathbf{T}(\mathbf{U}_2)}(\mathbf{V}_2) + \mathrm{g}(\mathbf{U}_1) - \mathrm{g}(\mathbf{U}_2). \end{aligned}$$

Using Theorem 5.17 as well as Theorem 9, 11 and 13 in [8] we have

$$\|\mathbf{W}\|_{\mathfrak{M}^1} \leq C\|\mathfrak{U}_{1\mathbf{T}(\mathbf{U}_1)}(\mathbf{V}_2) - \mathfrak{U}_{1\mathbf{T}(\mathbf{U}_2)}(\mathbf{V}_2) + F(\mathbf{U}_1) - F(\mathbf{U}_2)\|_{S_1^1}$$

$$+C\|\mathcal{B}_{\mathbf{T}(\mathbf{U}_2)}(\mathbf{V}_2) - \mathcal{B}_{\mathbf{T}(\mathbf{U}_1)}(\mathbf{V}_2) + \mathbf{g}(\mathbf{U}_1) - \mathbf{g}(\mathbf{U}_2)\|_{S_2^1}$$

$$\leq C\|\mathbf{T}(\mathbf{U}_1) - T(\mathbf{U}_2)\|_{S^1}\|\mathbf{V}_2\|_{\mathfrak{M}^1} + C\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1}\right).$$

By Lemma 5.18

$$\|\mathbf{W}\|_{\mathfrak{M}^1} \leq C\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}\left(\|\mathbf{V}_2\|_{\mathfrak{M}^1} + \|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1}\right)$$

$$\leq C\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}\left(\|\mathbf{U}_1\|^2_{\mathfrak{M}^1} + \|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} + \|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1}\right).$$

Thus,

$$\|\mathbf{W}\|_{\mathfrak{M}^1} \leq C\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1} + \|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}}\right). \tag{5.26}$$

For $\mathbf{U}_1, \mathbf{U}_2 \in \widetilde{D}^{\epsilon_4^1}$ equation (5.25) gives

$$\|\mathfrak{C}(\mathbf{U}_1)\|_{\mathfrak{M}^1} = \|\mathbf{V}_1\|_{\mathfrak{M}^1} \leq C\|\mathbf{U}\|^2_{\mathfrak{M}^1} + \|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}}$$

and equation 5.26 gives

$$\|\mathfrak{C}(\mathbf{U}_1) - \mathfrak{C}(\mathbf{U}_2)\|_{\mathfrak{M}^1} = \|\mathbf{V}_1 - \mathbf{V}_2\|_{\mathfrak{M}^1} = \|\mathbf{W}\|_{\mathfrak{M}^1}$$

$$\leq \quad C\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1} + \|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}}\right).$$

Take $\epsilon_4 = \dfrac{1}{4C}$. Then if $\mathbf{U}_1, \mathbf{U}_2 \in \widetilde{D}^{\epsilon_4}$

$$\begin{aligned}
\|\mathfrak{C}(\mathbf{U}_1)\|_{\mathfrak{M}^1} &\leq C\|\mathbf{U}\|^2_{\mathfrak{M}^1} + C\left(\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}}\right) \\
&\leq C\|\mathbf{U}_1\|_{\mathfrak{M}^1} + C\left(\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}}\right) \\
&\leq \frac{1}{4}\|\mathbf{U}_1\|_{\mathfrak{M}^1} + C\left(\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}}\right)
\end{aligned}$$

and then $\|\mathfrak{C}(\mathbf{U}_1) - \mathfrak{C}(\mathbf{U}_2)\|_{\mathfrak{M}^1}$

$$\begin{aligned}
&\leq C\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}\left(\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \|\mathbf{U}_2\|_{\mathfrak{M}^1} + \|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}}\right) \\
&\leq \frac{1}{2}\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1} + C\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}\left(\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}}\right).
\end{aligned}$$

Now we take $\eta_3 \leq \dfrac{\epsilon_4}{4C}$ so that by hypothesis

$$\begin{aligned}
\|\mathfrak{C}(\mathbf{U}_1)\|_{\mathfrak{M}^1} &\leq \frac{1}{4}\|\mathbf{U}_1\|_{\mathfrak{M}^1} + C\eta_3 \\
&\leq \frac{1}{4}\|\mathbf{U}_1\|_{\mathfrak{M}^1} + \frac{\epsilon_4}{4}
\end{aligned}$$

and

$$\begin{aligned}
\|\mathfrak{C}(\mathbf{U}_1) - \mathfrak{C}(\mathbf{U}_2)\|_{\mathfrak{M}^1} &\leq \frac{1}{2}\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1} + C\eta_3\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1} \\
&\leq \frac{1}{2}\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1} + \frac{\epsilon_4}{4}\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1} \\
&\leq \frac{1}{2}\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1} + \frac{1}{4}\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathfrak{M}^1}.
\end{aligned}$$

Thus $\mathfrak{C}$ is a contraction of $\widetilde{D}^{\epsilon_4}$. If η_3 is small enough, then we also have for $\mathbf{U}_0 = [0, \alpha_0, \tau_0, \beta_0, \mathbf{d}_0] \in \widetilde{D}^{\epsilon_4}$, and thus it contains only one fixed point. Also,

$$\frac{d}{dt}\left(\beta_{\mathbf{T}}(x) - \mathbf{T}_3(x)\right) = u_3 - u_3 = 0.$$

$\square$

As a consequence of Theorem 5.9 and since $\epsilon_4 \in (0, \epsilon_3]$ we have by Theorem 5.9 a solution of problem (P1) provided $\omega = 1$. When the compatibility condition required for Lemma 5.10 is fulfilled for the initial values

If $(\mathbf{\Omega}^\omega, v, \rho, \theta)$ solves problem (P1), then the function F defined by

$$F(t) = \int_{\Omega^t} \rho(y,t)dy = \int_{\widetilde{S}} \rho\left(T_t(x), t\right) \mathcal{J}_{T_t} dx$$

is absolutely continuous since $\rho\left(T_t(x), t\right) \in \widetilde{W}_p^1\left(S \times (0, \Omega)\right)$. Differentiating the last integral and using the equations

$$\rho'(y,t) + \operatorname{div}(\rho v)(y,t) = 0 \quad \text{and} \quad \mathcal{J}_{T_t}^{-1}\mathcal{J}'_{T_t} = \operatorname{div}(v)\left(T_t(x), t\right)$$

we have

$$F'(t) = \int_{\widetilde{S}} \frac{d}{dt}\left(\rho\left(T_t(x), t\right)\right) \mathcal{J}_{T_t} + \rho\left(T_t(x), t\right) \frac{d}{dt}\mathcal{J}_{T_t} dx$$

$$= \int_{\widetilde{S}} \left(\rho'\left(T_t(x), t\right)\right) \mathcal{J}_{T_t} + \rho\left(T_t(x), t\right) \operatorname{div}(v)\left(T_t(x), t\right) \mathcal{J}_{T_t} dx$$

$$= \int_{\widetilde{S}} \left(\rho_t\left(T_t(x), t\right)\right) \mathcal{J}_{T_t} + (v \cdot \nabla\rho)(T_t(x), t) + \rho\left(T_t(x), t\right) \operatorname{div}(v)\left(T_t(x), t\right) \mathcal{J}_{T_t} dx$$

$$= \int_{\widetilde{S}} \left(\rho_t + \operatorname{div}(\rho v)\right)\left(T_t(x), t\right) \mathcal{J}_{T_t} dx = 0.$$

Using this identity, we arrive at

$$\frac{d}{dt}\int_{\widetilde{\Omega}^t} \rho dy = 0.$$

5.4 Long-term Existence

Assume $\mathbf{U} : [0, \omega] \to \mathfrak{B}^1$ belongs to $\mathfrak{M}^\omega$ and solves problem (P2) as well as the equation $\beta(x,t) = T_3(x,t)$ for $x \in \overline{S}, t \in [0,\omega]$. Theorem 13 in [8] only applies to solutions that are perpendicular to $\mathcal{N}$, so we must split our solution into two parts $\mathbf{V}_1 = \mathcal{P}\mathbf{U}$ and $\mathbf{V}_2 = \mathcal{P}^c\mathbf{U}$, where $\mathbf{U} = \mathbf{V}_1 + \mathbf{V}_2$. Now we derive estimates for $\mathbf{V}_2$ using the conservation law.

Lemma 5.20. *There exists a constant $C < \infty$ such that if $\mathbf{U} = [u, \alpha, \tau, \beta, \mathbf{d}] \in \mathfrak{M}^\omega$ solves problem (P2), $\beta(x,0) = \mathbf{d}_3(x,0)$ and $\|\mathbf{U}\|_{\mathfrak{M}^\omega} \le \epsilon_3$, then*

$$\|\mathcal{P}^c\mathbf{U}\|_{\mathfrak{M}^\omega} \le C\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

Proof. We want to show that there is a constant C so that for every $U_\mathcal{N} \in \mathcal{N}$ and for $t \in [0, \omega - 1]$

$$\| \left(\mathbf{U}, U_\mathcal{N}\right)_\mathfrak{h} \|_{\widetilde{W}^1_p(t,t+1)} (1+t)^{4/3} \le C\|U_\mathcal{N}\|_\mathfrak{h} \|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

We do so using the basis vector $U_\mathcal{N}[0,1,0,1,0]$. We know that the total mass is conserved. In Lagrange coordinates we have

$$\begin{aligned}
0 &= \int_{\widetilde{S}} \left(\rho_e(x+\mathbf{d}) + \alpha\right) \mathcal{J}_{T_t} - \rho_e(x)dx \\
&= \int_{\widetilde{S}} \alpha(\mathcal{J}_{T_t} - 1)dx + \int_{\widetilde{S}} \rho_e(x+\mathbf{d})\mathcal{J}_{T_t}dx - \int_{\widetilde{S}} \rho_e dx + \int_{\widetilde{S}} \alpha dx \\
&= \int_{\widetilde{S}} \alpha(\mathcal{J}_{T_t} - 1)dx + \int_{\widetilde{\Omega}^t} \rho_e(x_1,x_2,x_3)dx - \int_{\widetilde{S}} \rho_e dx + \int_{\widetilde{S}} \alpha dx \\
&= \int_{\widetilde{S}} \alpha(\mathcal{J}_{T_t} - 1)dx + \int_{\widetilde{\mathbb{R}}^2} \int_{-h}^{\varphi(x_1,x_2,t)} \rho_e(x_1,x_2,x_3)dx_3dx_2dx_1 \\
&\quad - \int_{\widetilde{\mathbb{R}}^2} \int_{-h}^{0} \rho_e dx + \int_{\widetilde{S}} \alpha dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\widetilde{S}} \alpha(\mathcal{J}_{T_t}-1)dx + \int_{\widetilde{\mathbb{R}}^2}\int_0^{\varphi(x_1,x_2,t)} \rho_e(x_1,x_2,s)dsdx_1dx_2 + \int_{\widetilde{S}} \alpha dx \\
&= \int_{\widetilde{S}} \alpha(\mathcal{J}_{T_t}-1)dx + \int_{\widetilde{\mathbb{R}}^2}\int_0^{\beta(y_1,y_2,t)} \rho_e(y_1,y_2,s)ds\mathcal{J}_{\widetilde{T}_t}dy_1dy_2 + \int_{\widetilde{S}} \alpha dx \\
&= \int_{\widetilde{S}} \alpha(\mathcal{J}_{T_t}-1)dx + \int_{\widetilde{\mathbb{R}}^2}\int_0^{\beta(y_1,y_2,t)} \rho_e(y_1,y_2,s)(\mathcal{J}_{\widetilde{T}_t}-1)dsdy_1dy_2 + \int_{\widetilde{S}} \alpha dx \\
&\quad + \int_{\widetilde{\mathbb{R}}^2}\int_0^{\beta(y_1,y_2,t)} \rho_e(y_1,y_2,s)dsdy_1dy_2 \\
&= \int_{\widetilde{S}} \alpha(\mathcal{J}_{T_t}-1)dx + \int_{\widetilde{\mathbb{R}}^2}\int_0^{\beta(y_1,y_2,t)} \rho_e(y_1,y_2,s)(\mathcal{J}_{\widetilde{T}_t}-1)dsdy_1dy_2 + \int_{\widetilde{S}} \alpha dx \\
&\quad + \int_{\widetilde{\mathbb{R}}^2}\int_0^{\beta(y_1,y_2,t)} \rho_e(y_1,y_2,0)dsdy_1dy_2 \\
&\quad + \int_{\widetilde{\mathbb{R}}^2}\int_0^{\beta(y_1,y_2,t)} \rho_e(y_1,y_2,s)-\rho_e(y_1,y_2,0)dsdy_1dy_2.
\end{aligned}
$$

Where $\widetilde{R}^2$ is constructed according to the convention described in Section 1.1. Thus

$$
\begin{aligned}
0 &= \int_{\widetilde{S}} \alpha(\mathcal{J}_{T_t}-1)dx + \int_{\widetilde{\mathbb{R}}^2}\int_0^{\beta(y_1,y_2,t)} \rho_e(y_1,y_2,s)(\mathcal{J}_{\widetilde{T}_t}-1)dsdy_1dy_2 + \int_{\widetilde{S}} \alpha dx \\
&+ \rho_e(y_1,y_2,0)\int_{\widetilde{\mathbb{R}}^2} \beta(y_1,y_2,t)dy_1dy_2 \\
&+ \int_{\widetilde{\mathbb{R}}^2}\int_0^{\beta(y_1,y_2,t)} \rho_e(y_1,y_2,s)-\rho_e(y_1,y_2,0)dsdy_1dy_2.
\end{aligned}
$$

Looking at each integral more closely we have

$$
\left|\int_{\widetilde{S}} \alpha(\mathcal{J}_{T_t}-1)dx\right| | \leq C\|T_t - E_S\|_{\widetilde{C}^1}\|\alpha\|_{\widetilde{L}_p} \leq C\|\mathbf{d}\|_{\widetilde{C}^1}\|\alpha\|_{\widetilde{L}_p},
$$

$$
\left|\int_{\widetilde{\mathbb{R}}^2}\int_0^{\beta(y_1,y_2,t)} \rho_e(y_1,y_2,s)(\mathcal{J}_{\widetilde{T}_t}-1)dsdy_1dy_2\right|
$$

$$
\leq C\int_{\widetilde{\mathbb{R}}^2} \rho_e(y_1,y_2,\beta)\max_{x\in\mathbb{R}^2}|\beta(y_1,y_2,t)|(\mathcal{J}_{\widetilde{T}_t}-1)dsdy_1dy_2
$$

$$
\leq C\|\beta\|_{\widetilde{L}_p}\|T_t - E_{\widetilde{S}}\|_{\widetilde{C}^1} \leq C\|\beta\|_{\widetilde{L}_p}\|\mathbf{d}\|_{\widetilde{C}^1}
$$

$$\left| \int_{\widetilde{\mathbb{R}}^2} \int_0^{\beta(y_1,y_2,t)} \rho_e(y_1,y_2,s) - \rho_e(y_1,y_2,0) ds dy_1 dy_2 \right|$$

$$\leq C \max_{x\in\mathbb{R}^2} \int_{\widetilde{\mathbb{R}}^2} |\rho_e(y_1,y_2,\beta) - \rho_e(y_1,y_2,0)| \, ds dy_1 dy_2$$

$$\leq C\|\beta\|^2_{\widetilde{L}_\infty}.$$

Thus,

$$\left| \int_{\widetilde{S}} \alpha dx + \rho_0 \int_{\widetilde{\mathbb{R}}^2} \beta(y_1,y_2,t) dy_1 dy_2 \right|$$

$$\leq C \left(\|\mathbf{d}\|_{\widetilde{C}^1} \|\alpha\|_{\widetilde{L}_p} + \|\beta\|_{\widetilde{W}_p^{1-1/p}} \|\mathbf{d}\|_{\widetilde{C}^1} + \|\beta\|^2_{\widetilde{W}_p^{1-1/p}} \right)$$

$$\leq C \left(\|\mathbf{U}\|_{\mathfrak{M}^\omega} (1+t)^{-4/3} \|\mathbf{U}\|_{\mathfrak{M}^\omega} + \|\mathbf{U}\|_{\mathfrak{M}^\omega} (1+t)^{-4/3} \|\mathbf{U}\|_{\mathfrak{M}^\omega} + (1+t)^{-2} \|\mathbf{U}\|^2_{\mathfrak{M}^\omega} \right)$$

$$\leq C(1+t)^{-4/3} \|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

Now we look at the time derivatives. Here we will use $\mathcal{J}_{T_t}^{-1} \mathcal{J}'_{T_t} = \operatorname{div}(u)(T_t(x),t)$.

$$\left| \frac{d}{dt} \left(\int_{\widetilde{S}} \alpha(\mathcal{J}_{T_t} - 1) dx \right) \right| = \left| \int_{\widetilde{S}} \alpha_t (\mathcal{J}_{T_t} - 1) + \alpha \frac{d}{dt} (\mathcal{J}_{T_t} - 1) dx \right|$$

$$= \left| \int_{\widetilde{S}} \alpha_t (\mathcal{J}_{T_t} - 1) + \alpha \operatorname{div}(u)(T_t(x),t) \mathcal{J}_{T_t} dx \right|$$

$$\leq \|\alpha_t\|_{\widetilde{L}_p} \|T_t - E_{\widetilde{S}}\|_{\widetilde{C}^1} + \|\alpha\|_{\widetilde{L}_p} \|u\|_{\widetilde{W}_p^1}$$

$$\leq \|\alpha_t\|_{\widetilde{L}_p} \|\mathbf{d}\|_{\widetilde{C}^1} + \|\alpha\|_{\widetilde{L}_p} \|u\|_{\widetilde{W}_p^1},$$

$$\left| \frac{d}{dt} \left(\int_{\partial_2 \widetilde{S}} \int_0^{\beta(y_1,y_2,t)} \rho_e(y_1,y_2,s) (\mathcal{J}_{\widetilde{T}_t} - 1) ds dy_1 dy_2 \right) \right|$$

$$\leq \left| \int_{\partial_2 \widetilde{S}} \left(\rho_e(y_1,y_2,\beta) - \rho_e(y_1,y_2,0) \right) (\mathcal{J}_{T_t} - 1) dy_1 dy_2 \beta_t \right|$$

$$+ \left| \int_{\partial_2 \widetilde{S}} \left(|\rho_e(y_1,y_2,\beta) - \rho_e(y_1,y_2,0) \right) \operatorname{div}(u)(T_t(x),t) \mathcal{J}_{T_t} | \, dy_1 dy_2 \right|$$

$$\leq \|\beta\|_{\widetilde{L}_\infty} \|\beta_t\|_{\widetilde{L}_p} \|T_t - E_{\widetilde{S}}\|_{\widetilde{C}^1} + \|u\|_{\widetilde{W}_p^1} \|\beta\|_{\widetilde{L}_p}$$

$$\leq \|\beta\|_{\widetilde{L}_\infty}\|\beta_t\|_{\widetilde{L}_p}\|\mathbf{d}\|_{\widetilde{C}^1}+\|u\|_{\widetilde{W}_p^1}\|\beta\|_{\widetilde{L}_p},$$

$$\left|\frac{\partial}{\partial t}\left(\int_{\partial_2\widetilde{S}}\int_0^{\beta(y_1,y_2,t)}\rho_e(y_1,y_2,s)-\rho_e(y_1,y_2,0)\right)dsdy_1dy_2\right|$$

$$=\left|\int_{\partial_2\widetilde{S}}\left(\rho_e(y_1,y_2,\beta)-\rho_e(y_1,y_2,0)\right)\beta_t dy_1dy_2\right|$$

$$\leq \|\rho_e(y_1,y_2,\beta)-\rho_e(y_1,y_2,0)\|_{\widetilde{L}_2}\|\beta_t\|_{\widetilde{L}_2}\leq \|\beta\|_{\widetilde{W}_p^{1-1/p}}\|\beta_t\|_{\widetilde{W}_p^{1-1/p}}.$$

Thus,

$$\left|\frac{d}{dt}\left(\int_{\widetilde{S}}\alpha dx+\rho_0\int_{\widetilde{\mathbb{R}}^2}\beta(y_1,y_2,t)dy_1dy_2\right)\right|=$$

$$\left|\frac{d}{dt}\left(\int_{\widetilde{S}}\alpha(\mathcal{J}_{T_t}-1)dx+\int_{\partial_2\widetilde{S}}\int_0^{\beta(y_1,y_2,t)}\rho_e(y_1,y_2,s)(\mathcal{J}_{\widetilde{T}_t}-1)dsdy_1dy_2\right.\right.$$

$$\left.\left.+\int_{\partial_2\widetilde{S}}\int_0^{\beta(y_1,y_2,t)}\rho_e(y_1,y_2,s)-\rho_e(y_1,y_2,0)sdy_1dy_2\right|\right)$$

$$\leq \|\alpha_t\|_{\widetilde{L}_p}\|\mathbf{d}\|_{\widetilde{C}^1}+\|\alpha\|_{\widetilde{L}_p}\|u\|_{\widetilde{W}_p^1}+\|\beta\|_{\widetilde{L}_\infty}\|\beta_t\|_{\widetilde{L}_p}\|\mathbf{d}\|_{\widetilde{C}^1}$$

$$+\|u\|_{\widetilde{W}_p^1}\|\beta\|_{\widetilde{L}_p}+\|\beta\|_{\widetilde{W}_p^{1-1/p}}\|\beta_t\|_{\widetilde{W}_p^{1-1/p}}$$

$$\leq (1+t)^{-4/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}+(1+t)^{-8/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}$$

$$+(1+t)^{-7/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}+(1+t)^{-8/3}\|\mathbf{U}\|^3_{\mathfrak{M}^\omega}+(1+t)^{-2}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}$$

$$\leq C(1+t)^{-4/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

$$\begin{aligned}
\|\left(\mathbf{U},U_{\mathcal{N}}\right)_{\mathfrak{h}}\|_{\widetilde{W}_p^1(t,t+1)} &\leq \|\left(\mathbf{U},U_{\mathcal{N}}\right)_{\mathfrak{h}}\|_{C^1(t,t+1)}\\
&= \left|\left(\mathbf{U},U_{\mathcal{N}}\right)_{\mathfrak{h}}\right|+\left|\frac{d}{dt}\left(\mathbf{U},U_{\mathcal{N}}\right)_{\mathfrak{h}}\right|\\
&\leq C(1+t)^{-8/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}\\
&\leq C(1+t)^{-4/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}\\
&\leq C(1+t)^{-4/3}\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.
\end{aligned}$$

□

Lemma 5.21. *If* $\mathbf{U} \in \mathfrak{M}^\omega$ *solves problem (P2), then*

$$\|T_t - T_\omega\|_{\widetilde{W}_p^{5/3}} \leq C\|\mathbf{U}\|_{\mathfrak{M}^\omega}(1+t)^{-1/3}$$

and with $\widehat{\mathbf{T}}(x,t) = T_\omega(x)$ *then*

$$\|\mathbf{T} - \widehat{\mathbf{T}}\|_{S^\omega} \leq C\|U\|_{\mathfrak{M}^\omega}.$$

Proof.

$$\|T_t - T_\omega\|_{\widetilde{W}_p^{5/3}} = \|\mathbf{d}(t) - \mathbf{d}(\omega)\|_{\widetilde{W}_p^{5/3}} = \|\int_1^\omega u dr\|_{\widetilde{W}_p^{5/3}} \leq \int_1^\omega \|u\|_{\widetilde{W}_p^{5/3}} dr$$

$$\leq C\|\mathbf{U}\|_{\mathfrak{M}^\omega} \int_1^\omega (1+r)^{-4/3} dr \leq C\|\mathbf{U}\|_{\mathfrak{M}^\omega} \int_1^\infty (1+r)^{-4/3} dr \leq C\|\mathbf{U}\|_{\mathfrak{M}^\omega}(1+t)^{-1/3}.$$

By definition of S^ω

$$\|\mathbf{T} - \widehat{\mathbf{T}}\|_{S^\omega} \leq (1+t)^{1/3}\|T_t - T_\omega\|_{\widetilde{W}_p^{5/3}} \leq C\|\mathbf{U}\|_{\mathfrak{M}^\omega}.$$

□

Lemma 5.22. *Assume* $\mathbf{U}$ *is a solution of problem (P2),* $\beta(x,0) = \mathbf{d}_3(x,0)$ *for* $x \in \partial S, \|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq \epsilon_3,$ *and* $\mathbf{V}_2 = \mathcal{P}^c\mathbf{U}$*. Then*

$$\|\mathfrak{U}_{1\mathbf{T}}\mathbf{V}_2\|_{B_1^\omega} + \|\mathcal{B}_{\mathbf{T}}(\mathbf{V}_2)\|_{B_2^\omega} \leq C\|\mathbf{U}\|^2_{\mathfrak{M}^\omega}.$$

Proof. Let $\widehat{\mathbf{T}}(x,t) = T_\Omega(x)$. Using Theorem 11 in [7] we have

$$\begin{aligned} \|\mathcal{B}_{\mathbf{T}}(\mathbf{V}_2)\|_{S_2^\omega} &\leq \|\mathcal{B}_{\widehat{\mathbf{T}}}(\mathbf{V}_2)\|_{S_2^\omega} + \|\mathcal{B}_{\mathbf{T}}(\mathbf{V}_2) - \mathcal{B}_{\widehat{\mathbf{T}}}(\mathbf{V}_2)\|_{S_2^\omega} \\ &\leq \|\mathcal{B}_{\widehat{\mathbf{T}}}(\mathbf{V}_2)\|_{S_2^\omega} + C\|\mathbf{T} - \widehat{\mathbf{T}}\|_{S^\omega}\|\mathbf{V}_2\|_{\mathfrak{M}^\omega} \qquad (5.27) \\ &\leq \|\mathcal{B}_{\widehat{\mathbf{T}}}(\mathbf{V}_2)\|_{S_2^\omega} + C\|\mathbf{V}_2\|_{\mathfrak{M}^\omega}. \end{aligned}$$

Since $\mathbf{T}$ is independent of time we also have

$$\|\mathcal{B}_{\widehat{\mathbf{T}}}(\mathbf{V}_2)\|^p_{\widetilde{W}_p^{1-1/p,1/2-1/2p}(\partial S\times[t-1,t])} \leq$$
$$C\left(\int_{t-1}^t \|\mathcal{B}_{T_\omega}(\mathbf{V}_2)(r)\|^p_{\widetilde{W}_p^{1-1/p}} + \|\mathcal{B}_{T_\Omega}(\mathbf{V}_2')\|^p_{\widetilde{L}_p(\partial S)} dr\right).$$

By Lemma 3.9 we have

$$\|\mathcal{B}_{T_\omega}(\mathbf{V}_2)(r)\|^p_{\widetilde{W}_p^{1-1/p}} \leq C\|\Pi\mathbf{V}_2\|_{D^1_{-1}}$$

and

$$\|\mathcal{B}_{T_\omega}(\mathbf{V}_2')\|^p_{\widetilde{L}_p(\partial S)} \leq C\|\Pi\mathbf{V}_2'\|_{D^1_{-1}}.$$

Thus,

$$\|\mathcal{B}_{\widehat{\mathbf{T}}}(\mathbf{V}_2)\|^p_{\widetilde{W}_p^{1-1/p,1/2-1/2p}(\partial S\times[t-1,t])} \leq C\int_{t-1}^t \|\Pi\mathbf{V}_2(r)\|_{D^1_{-1}} + \|\Pi\mathbf{V}_2'(r)\|_{D^1_{-1}} dr.$$

Since $\mathbf{V}_2$ belongs to the finite dimensional space $\mathcal{N}\subset D^1$, the $D^{2/3}_{-1}$ norm is equivalent to the D^1_{-1} and $\mathfrak{B}^1_{-1}$ norms on this space. Thus,

$$\|\mathcal{B}_{\widehat{\mathbf{T}}}(\mathbf{V}_2)\|^p_{\widetilde{W}_p^{1-1/p,1/2-1/2p}(\partial S\times[t-1,t])} \leq C\int_{t-1}^t \|\Pi\mathbf{V}_2(r)\|_{D^{2/3}_{-1}} + \|\Pi\mathbf{V}_2'(r)\|_{\mathfrak{B}^1_{-1}} dr.$$

Multiplying this by $(1+t)^{4/3}$ and taking the supremum over $t\in[0,\omega-1]$ we have

$$\|\mathcal{B}_{\widehat{\mathbf{T}}}(\mathbf{V}_2)\|^p_{S_2^\omega} \leq C\|\mathbf{V}_2\|_{\mathfrak{M}^\omega}$$

Combining this with the inequality (5.27) we obtain

$$\|\mathcal{B}_{\mathbf{T}}(\mathbf{V}_2)\|_{S_2^\omega} \leq C\|\mathbf{V}_2\|_{\mathfrak{M}^\omega}.$$

By Lemma 3.8 we have

$$\|\mathfrak{U}_{1T_t}\mathbf{V}_2\|_{\mathfrak{B}^1} \leq C\|\Pi\mathbf{V}_2(r)\|_{D^1_{-1}}.$$

Using the fact that $\mathbf{V}_2$ belongs to $\mathcal{N}$ we obtain

$$\|\mathfrak{U}_{1T_t}\mathbf{V}_2\|_{\mathfrak{B}^1} \le C\|\Pi\mathbf{V}_2(r)\|_{D^{2/3}_{-1}}.$$

Integrating with respect to time

$$\int_{t-1}^{t} \|\mathfrak{U}_{1T_t}\mathbf{V}_2(s)\|^p_{\mathfrak{B}^1} ds \le C(1+t)^{-4/3}\|\mathbf{V}_2\|_{\mathfrak{M}^\omega}.$$

Since $\mathbf{V}_2 = \mathcal{P}^c\mathbf{U}$ we use Lemma 5.20 to obtain

$$\|\mathfrak{U}_{1\mathbf{T}}\mathbf{V}_2\|_{S_1^\omega} + \|\mathcal{B}_{\mathbf{T}}(\mathbf{V}_2)\|_{S_2^\omega} \le C\|\mathbf{V}_2\|_{\mathfrak{M}^\omega} \le C\|\mathfrak{U}\|^2_{\mathfrak{M}^\omega}.$$

□

5.5 A-Priori Estimate

Lemma 5.23. *There exists $C < \infty$ such that if $\mathbf{U} \in \mathfrak{M}^\omega$ is a solution of problem (P2), $\beta(x,0) = \mathbf{d}_3(x,0)$ for $x \in \partial S$, $\|\mathbf{U}\|_{\mathfrak{M}^\omega} \le \epsilon_3$, then*

$$\|\mathbf{U}\|_{\mathfrak{M}^\omega} \le C\|\mathbf{U}\|^{4/3}_{\mathfrak{M}^\omega} + C\left(\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} + \|\tau_0\|_{\widetilde{W}_p^{2-2/p}}\right).$$

Proof. First we take the projection of the equation

$$\mathbf{U}_t + \mathfrak{U}_{1\mathbf{T}}\mathbf{U} = \mathbf{F}(\mathbf{U})$$

to obtain

$$\mathbf{V}_{1t} + \mathcal{A}_{1\mathbf{T}}\mathbf{U} = \mathcal{P}\mathbf{F}(\mathbf{U}).$$

Thus,

$$\mathbf{V}_{1t} + \mathcal{A}_{1\mathbf{T}}\mathbf{V}_1 = \mathcal{P}\mathbf{F}(\mathbf{U}) - \mathcal{A}_{1\mathbf{T}}\mathbf{V}_2$$

with the boundary condition

$$\mathcal{B}_{\mathbf{T}}(\mathbf{V}_1) = \mathbf{g}(\mathbf{U}) - \mathcal{B}_{\mathbf{T}}(\mathbf{V}_2).$$

Since $\mathbf{V}_1$ is a sufficiently regular solution we automatically fulfill the compatibility condition for the initial values.

$$\begin{aligned}\|\mathcal{P}\mathbf{F}(\mathbf{U}) - \mathcal{A}_{1\mathbf{T}}\mathbf{V}_2\|_{S_1^\omega} &= \|\mathbf{V}_{1t} + \mathcal{A}_{1\mathbf{T}}\mathbf{U} - \mathcal{A}_{1\mathbf{T}}\mathbf{V}_2\|_{\mathfrak{B}_1^\omega} = \|\mathbf{V}_{1t} + \mathcal{A}_{1\mathbf{T}}\mathbf{V}_1\|_{\mathfrak{B}_1^\omega} \\ &\le \|\mathbf{V}_{1t} + \mathfrak{U}_{1\mathbf{T}}\mathbf{V}_1\|_{\mathfrak{B}_1^\omega} = \|\mathbf{U}_t + \mathfrak{U}_{1\mathbf{T}}\mathbf{U} - \mathbf{V}_{2t} - \mathfrak{U}_{1\mathbf{T}}\mathbf{V}_2\|_{S_1^\omega} \\ &= \|\mathbf{F}(\mathbf{U}) - \mathbf{V}_{2t} - \mathfrak{U}_{1\mathbf{T}}\mathbf{V}_2\|_{S_1^\omega} \le C\|\mathbf{F}(\mathbf{U}) - \mathfrak{U}_{1\mathbf{T}}\mathbf{V}_2\|_{S_1^\omega}\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{g}(\mathbf{U}) - \mathcal{B}_{\mathbf{T}}(\mathbf{V}_2)\|_{S_2^\omega} &= \|\mathcal{B}_{\mathbf{T}}(\mathbf{V}_1)\|_{S_2^\omega} = \|\mathcal{B}_{\mathbf{T}}(\mathbf{U}) - \mathcal{B}_{\mathbf{T}}(\mathbf{V}_2)\|_{S_2^\omega} \\ &\le C\left(\|\mathcal{B}_{\mathbf{T}}(\mathbf{U})\|_{S_2^\omega} + \|\mathcal{B}_{\mathbf{T}}(\mathbf{V}_2)\|_{S_2^\omega}\right).\end{aligned}$$

Using Theorem 5.17 and Lemma 5.23 we have

$$C\|\mathbf{F}(\mathbf{U}) - \mathfrak{U}_{1\mathbf{T}}\mathbf{V}_2\|_{S_1^\omega} \le$$

$$C\|\mathbf{F}(\mathbf{U})\|_{S_1^\omega} + C\|\mathfrak{U}_{1\mathbf{T}}\mathbf{V}_2\|_{S_1^\omega} \le C\|\mathbf{U}\|_{\mathfrak{M}^\omega}^{4/3} + C\|\mathbf{U}\|_{\mathfrak{M}^\omega}^2 \le C\|\mathbf{U}\|_{\mathfrak{M}^\omega}^{4/3}$$

and

$$\begin{aligned}\|\mathbf{g}(\mathbf{U}) - \mathcal{B}_{\mathbf{T}}(\mathbf{V}_2)\|_{S_2^\omega} &\le C\left(\|\mathcal{B}_{\mathbf{T}}(\mathbf{U})\|_{S_2^\omega} + \|\mathcal{B}_{\mathbf{T}}(\mathbf{V}_2)\|_{S_2^\omega}\right) \le C\left(\|\mathbf{U}\|_{\mathfrak{M}^\omega}^{4/3} + \|\mathbf{U}\|_{\mathfrak{M}^\omega}^2\right) \\ &\le C\|\mathbf{U}\|_{\mathfrak{M}^\omega}^{4/3}.\end{aligned}$$

For sufficiently small $\|\mathbf{U}\|_{\mathfrak{M}^\omega}$ then by Theorem 3 in [8] and Lemma 5.21 that

$$\|\mathbf{V}_1\|_{\mathfrak{M}^\omega} \le C\Big(\|\mathcal{P}\mathbf{F}(\mathbf{U}) - \mathcal{A}_{1\mathbf{T}}\mathbf{V}_2\|_{S_1^\omega} + \|\mathbf{g}(\mathbf{U}) - \mathcal{B}_{\mathbf{T}}(\mathbf{V}_2)\|_{S_2^\omega}$$

$$+\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} + \|\tau_0\|_{\widetilde{W}_p^{2-2/p}}\Big).$$

By Lemma 5.20

$$\|\mathbf{V}_2\|_{\mathfrak{M}^\omega} \le C\|\mathbf{U}\|_{\mathfrak{M}^\omega}^{4/3}.$$

Thus, putting these together we have

$$\|\mathbf{U}\|_{\mathfrak{M}^\omega} \le C\|\mathbf{U}\|_{\mathfrak{M}^\omega}^{4/3} + C\left(\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} + \|\tau_0\|_{\widetilde{W}_p^{2-2/p}}\right).$$

□

Theorem 5.24. *There exists numbers $\eta_4, \epsilon_3 > 0$ and $C < \infty$ such that if $\mathbf{U} \in \mathfrak{M}^\omega$ is a solution of (P2), $\beta(x,0) = \mathbf{d}_3(x,0)$, $\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} + \|\tau_0\|_{\widetilde{W}_p^{2-2/p}} \le \eta_4$ and $\|\mathbf{U}\|_{\mathfrak{M}^\omega} \le \epsilon_5$ then*

$$\|\mathbf{U}\|_{\mathfrak{M}^\omega} \le C\left(\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} + \|\tau_0\|_{\widetilde{W}_p^{2-2/p}}\right).$$

Proof. Choose $\epsilon_5 \in (0, \epsilon_4]$ so that $C_0\epsilon_5^{1/3} \le \frac{1}{2}$ and $2C_0\eta_4 \le \frac{\epsilon_5}{2}$. Then for any $\omega_1 \in [1,\omega]$ we have that if $\|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}} \le \epsilon_5$ and because $\|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}}^{4/3} = \|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}}\|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}}^{1/3}$ then

$$\begin{aligned}
\|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}} &\le C_0\left(\|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}}^{4/3} + \|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} + \|\tau_0\|_{\widetilde{W}_p^{2-2/p}}\right) \\
&\le C_0\left(\epsilon_5^{1/3}\|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}} + \eta_4\right) \\
&\le \frac{1}{2}\|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}} + C_0\eta_4.
\end{aligned}$$

Thus,

$$\|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}} \le 2C_0\eta_4 \le \frac{\epsilon_5}{2}.$$

In summary, we have that for all $\omega_1 \in [1,\omega]$ the inequality $\|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}} \le \epsilon_5$ implies $\|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}} \le \frac{\epsilon_5}{2}$. Let $\mho = \omega_1 \in [1,\omega] : \|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}} \le \epsilon_5$. The set A is both open and closed due to $\|\mathbf{U}\|_{\mathfrak{M}^{\omega_1}}$ depending continuously on ω_1. Also, by assumption we have $1 \in \mho$ and we conclude that $\mho = [1,\omega]$. □

Theorem 5.25. *There exists numbers $\eta_5 > 0$ and $C < \infty$ such that if*

$$U_0 = [u_0, \alpha_0, \tau_0, \beta_0, \mathbf{d}_0] \in \mathfrak{B}^1 \cap D^{1-2/p}, \beta(x,0) = \mathbf{d}_3(x,0)$$

for $x \in \partial S, \mathcal{B}_{\mathbf{d}_0+E_S}(U_0) = \mathbf{g}(U_0),$ *and* $\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} + \|\tau_0\|_{\widetilde{W}_p^{2-2/p}} \leq \eta_5,$ *then for any* $\omega \in [1,\infty)$ *there exists a unique function* $\mathbf{U} \in \mathfrak{M}^\omega$ *such that* $\|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq \epsilon_5$ *and* $\mathbf{U}(0) = U_0$ *solving problem (P2) and*

$$\|\mathbf{U}\|_{\mathfrak{M}^\omega} \leq C\left(\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} + \|\tau_0\|_{\widetilde{W}_p^{2-2/p}}\right).$$

Proof. When $\omega = 1$ this is a consequence of Theorem 5.19. Now we assume that we have a solution $\mathfrak{M}^{\omega_1}$ for $1 = \omega_1 \in [0,\omega]$. By Theorem 5.24

$$\begin{aligned}\|\mathbf{U}(t)\|_{\mathfrak{B}^1} + \|u(t)\|_{\widetilde{W}_p^{2-2/p}} + \|\tau(t)\|_{\widetilde{W}_p^{2-2/p}} &\leq C\|\mathbf{U}\|_{\mathfrak{M}^\omega}\\ &\leq C\left(\|U_0\|_{\mathfrak{B}^1} + \|u_0\|_{\widetilde{W}_p^{2-2/p}} + \|\tau_0\|_{\widetilde{W}_p^{2-2/p}}\right)\end{aligned} \tag{5.28}$$

For $t \in [0,\omega_1]$. We take $\eta_5 \in (0,\eta_4]$ small enough so that $C\eta_5 \leq \eta_3$, allowing us to continue our solution to $\omega_1 + 1$ by Theorem 5.19. Now Theorem 5.24 gives the estimate (5.28) for $\omega_1 + 1$ instead of ω_1. Thus we have a solution to Problem (P2) on the interval $[0,\omega]$ and Theorem 5.19 implies its uniqueness. □

5.6 The Proof of Theorem 1.3

Proof. Theorem 5.7 shows that there is triple $(\mathbf{\Omega}^\omega, v, \rho, \theta)$ which solves Problem (P1) for every $\omega \in [1,\infty]$ when equation (1.10) holds. When equation (1.11) holds Theorem 5.25 gives us exactly one solution in Lagrange coordinates as well as the estimate (1.12). Theorem 5.9 shows that Problem (P1) and (P2) are equivalent and verifies equation (1.13). Equation (1.4), which describes the deformation of the flow on the surface, follows from the definition of Problem (P2). □

APPENDIX

The purpose of this appendix is to prove the following theorem.

Theorem A.1. *For $s > 1/p$ and $s - 1/p \notin \mathbb{Z}$, there is a constant C such that if $f \in \widetilde{W}^{s-1,p}(S)$ and $g_1 \in \widetilde{W}^{s-1/p,p}(S), g_2 \in \widetilde{W}^{s+1-1/p,p}(S)$ then there exists a unique $\tau \in \widetilde{W}^{s+1,p}(S)$ such that $\tau|_{\Gamma_2} = g_2$ and for $\varphi \in \widetilde{C}^{\infty}(\bar{S})$ with $\varphi|_{\Gamma_2} = 0$ we have*

$$\int_{\widetilde{S}} \nabla\tau\nabla\varphi + \tau\varphi dx =< f, \varphi > + \int_{\widetilde{\Gamma}_1} \varphi g_1 dS.$$

The function τ fulfills the estimate

$$\|\tau\|_{\widetilde{W}^{s+1,p}(S)} \leq C\left(\|f\|_{\widetilde{W}^{s-1,p}(S)} + \|g_1\|_{\widetilde{W}^{s-1/p,p}(S)} + \|g_2\|_{\widetilde{W}^{s+1-1/p,p}(S)}\right).$$

Although we will apply this to complex valued functions, as the operator is purely real we can do it separately for both the real and imaginary parts. Therefore we will assume from now on that the functions are real valued.

Definition A.2. *The difference quotient for a measurable function u in the coordinate direction e_i of length $h \neq 0$ is the function*

$$\Delta_i^h u = \frac{u(x + he_i) - u(x)}{h}.$$

Lemma A.3. *Let $q > 1, \tau_0, \varphi \in \widetilde{W}^{1,q}(S)$, and $F \in \widetilde{W}^{-1,p}$. The function*

$$d(\epsilon) = \int_{\widetilde{S}} \left(1 + |\nabla(\tau_0 + \epsilon\varphi)|^2\right)^{q/2} - qF\left(\tau_0 + \epsilon\varphi\right)$$

is a differentiable function of $\epsilon \in (-1, 1)$. Also

$$d'(0) = \int_{\widetilde{S}} q\left(1 + |\nabla\tau_0|^2\right)^{q/2-1} \nabla\tau_0 \cdot \nabla\varphi dx - qF\left(\varphi\right).$$

Proof. Given $\mathfrak{h}(\epsilon, x) = (1 + |\nabla\tau_0(x) + \epsilon\nabla\varphi(x)|^2)^{q/2}$ we have

$$d(\epsilon) = \int_{\widetilde{S}} \mathfrak{h}(\epsilon, x)dx - qF(\tau_0) - \epsilon qF(\varphi).$$

For almost all x the function $\mathfrak{h}(\cdot, x)$ is a continuously differentiable function of ϵ and we have

$$\frac{\partial\mathfrak{h}}{\partial\epsilon}(\epsilon, x) = q\left(1 + |\nabla\tau_0 + \epsilon\nabla\varphi|^2\right)^{q/2-1} \nabla\left(\tau_0 + \epsilon\varphi\right) \cdot \nabla\varphi$$

and

$$\Delta_\epsilon^h \mathfrak{h}(\epsilon, x) \to \frac{\partial\mathfrak{h}}{\partial\epsilon} \text{ as } h \to 0 \text{ almost everywhere}$$

where $\Delta_\epsilon^h \mathfrak{h}(\epsilon, x)$ is the difference quotient with respect to ϵ. As

$$|\nabla(\tau_0 + \epsilon\varphi) \cdot \nabla\varphi| \leq |\nabla\varphi| \cdot |\nabla(\tau_0 + \epsilon\varphi)| < |\nabla\varphi| \cdot (1 + |\nabla(\tau_0 + \epsilon\varphi)|^2)^{1/2},$$

since $q > 1$ and $|\epsilon| < 1$ we see

$$\left|\frac{\partial\mathfrak{h}}{\partial\epsilon}(\epsilon, x)\right| \leq q\left(1 + |\nabla\tau_0 + \epsilon\nabla\varphi|^2\right)^{(q-1)/2} |\nabla\varphi| \leq q\left(1 + 2|\nabla\tau_0|^2 + 2|\nabla\varphi|^2\right)^{q/2}.$$

Since this inequality is now independent of ϵ we have a bound on the difference quotient. We have

$$|\Delta_\epsilon^h h| = \left|\frac{h(\epsilon + h, x) - (\epsilon, x)}{h}\right| \leq \frac{1}{h}\int_\epsilon^{\epsilon+h} \left|\frac{\partial\mathfrak{h}}{\partial\epsilon}\right| d\epsilon \leq q\left(1 + 2|\nabla\tau_0|^2 + 2|\nabla\varphi|^2\right)^{q/2},$$

and therefore

$$|\Delta_\epsilon^h \mathfrak{h}| \leq q\left(1 + 2|\nabla\tau_0|^2 + 2|\nabla\varphi|^2\right)^{q/2} \in \widetilde{L}_1(S).$$

By the Lebesgue Dominated Convergence Theorem

$$\int_{\widetilde{S}} \Delta_i^h d dx \to \int_{\widetilde{S}} \frac{\partial\mathfrak{h}}{\partial\epsilon} dx = \int_{\widetilde{S}} q(1 + |\nabla\tau_0 + \epsilon\nabla\varphi|^2)^{q/2-1} \nabla(\tau_0 + \epsilon\varphi) \cdot \nabla\varphi dx.$$

Thus, d is a differentiable function of ϵ and

$$d'(0) = \int_{\widetilde{S}} q \left(1 + |\nabla \tau_0|^2\right)^{q/2-1} \nabla \tau_0 \cdot \nabla \varphi dx - qF(\varphi)$$

for all $\varphi \in \widetilde{W}_q^1(S)$. $\square$

A.1 Existence

Lemma A.4. *Let $\mathcal{A}$ be a subspace of $\widetilde{W}^{1,q}$, $F : \mathcal{A} \to \mathbb{R}$ be a linear continuous functional with $\|F\|_{L(\mathcal{A},\mathbb{R})} \leq 1$ and*

$$I(\tau) = \int_V (1 + |\nabla \tau|^2)^{q/2} dx - qF(\tau).$$

Then I has a minimum on $\mathcal{A}$ there is and a constant C such that for all minimizing functions τ we have $\|\tau\|_{\widetilde{W}^{1,q}} \leq C$.

Proof. Since $(1 + |\xi|^2)^{q/2}$ is a convex function of ξ , the functional I is lower semicontinuous. Also $\mathcal{A}$ is closed with respect to weak convergence in $\widetilde{W}^{1,q}$. Let

$$\mathcal{I} = \inf_{\tau \in A} I(\tau).$$

Since $0 \in \mathcal{A}$ we see that $\mathcal{I} \leq I(0) = |S|$. Now we define the set

$$\mathcal{A}_0 = \{\tau \in \mathcal{A} | I(\tau) \leq |S|\}.$$

Then we will have

$$\mathcal{I} = \inf_{\tau \in \mathcal{A}_0} I(\tau)$$

as well. For every $\tau \in \mathcal{A}_0$ we have

$$\int_{\widetilde{S}} |\nabla \tau|^q dx \leq I(\tau) + qF(\tau) \leq |S| + q\|F\|_{L(\mathcal{A},\mathbb{R})} \|\tau\|_{\widetilde{W}^{1,q}} \leq |S| + q\|\tau\|_{\widetilde{W}^{1,q}}.$$

From this and the Poincare inequality we have $\|\tau\|^q_{\widetilde{W}^{1,q}} \leq C\left(1+\|\tau\|_{\widetilde{W}^{1,q}}\right)$. Thus, $\|\tau\|_{\widetilde{W}^{1,q}} \leq C$ independent of F. This shows that $\mathcal{A}_0$ is a weakly compact subset of $\widetilde{W}^{1,q}$. As I is lower semicontinuous with respect to weak convergence in $\widetilde{W}^{1,q}$, it attains its minimum at a point τ_0, which also minimizes I over $\mathcal{A}$. Since the norm is lower semicontinuous we get $\|\tau_0\|_{\widetilde{W}^{1,q}} \leq C$. □

Lemma A.5. *Let $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. There exists a constant C such that for every $F \in (\widetilde{W}^{1,q})^*$ there exists a vector function $f \in \widetilde{L}_p$ and a $D \in \mathbb{R}$ such that*

$$\|f\|_{\widetilde{L}_p} \leq C\|F\|_{(\widetilde{W}^{1,q})^*},$$

and for every $\varphi \in \widetilde{W}^{1,q}$ we have the equation

$$\int_{\widetilde{S}} f \cdot \nabla\varphi dx + D\int_{\widetilde{S}} \varphi dx = F(\varphi). \tag{A.29}$$

Proof. If $F = 0$ we chose $f = 0$ and $D = 0$. Otherwise, we normalize the functional by dividing F by its norm $\|F\|_{(\widetilde{W}^{1,q})^*}$. Thus, after renaming this new functional F we have $\|F\|_{(\widetilde{W}^{1,q})^*} = 1$. By Lemma A.4 we have the existence of a minimum for the functional

$$I(\tau) = \int_{\widetilde{S}} (1+|\nabla\tau|^2)^{q/2}dx - qF(\tau)$$

with $\tau \in \mathcal{A} = \{\tau \in \widetilde{W}^{1,q}(S) | \overline{\tau}_S = 0\}$, where $\overline{\tau}_S$ is the average of τ in S. If we take $\varphi \in \widetilde{W}^{1,q}(S)$ with $\overline{\varphi}_S = 0$, where $\overline{\varphi}_S$ is the average of φ in S, and real valued ϵ with $|\epsilon| < 1$ then $\tau_0 + \epsilon\varphi \in \mathcal{A}$ and the function $d(\epsilon) = I(\tau_0 + \epsilon\varphi)$ has its absolute minimum at $\epsilon = 0$. Putting $\tau_0 + \epsilon\varphi$ in the definition of I and using Lemma A.3 we have $d(\epsilon)$ is a differentiable function with

$$0 = d'(0) = \int_{\widetilde{S}} q\left(1+|\nabla\tau_0|^2\right)^{q/2-1} \nabla\tau_0 \cdot \nabla\varphi dx - qF(\varphi)$$

for $\varphi \in \mathcal{A}$. When we define $f = \left(1 + |\nabla \tau_0|^2\right)^{q/2-1} \nabla \tau_0$ then

$$|f| \leq \left| \left|1 + |\nabla \tau_0|^2\right|^{q/2-1} \nabla \tau_0 \right| \leq 2^{q/2-1}(1 + |\nabla \tau_0|)^{q-1}.$$

Thus, $|f|^p \leq 2^{p(q/2-1)}(1 + |\nabla \tau_0|)^{p(q-1)} = 2^{p(q/2-1)}(1 + |\nabla \tau_0|)^q$ and $\|f\|_{\widetilde{L}_p} \leq C$. When $\varphi \in \widetilde{W}^{1,q}$ is an arbitrary function, then take $\varphi - \overline{\varphi}_S \in \mathcal{A}$ and then

$$\int_{\widetilde{S}} f \cdot \nabla \varphi dx = \int_{\widetilde{S}} f \cdot \nabla(\varphi - \overline{\varphi}_S) dx = F(\varphi - \overline{\varphi}_S) = F(\varphi) - F(1)|S|^{-1} \int_{\widetilde{S}} \varphi dx.$$

Taking $D = F(1)|S|^{-1}$ we satisfy equation (1) and get the desired estimate, as it easily carries over to $\|F\|_{(\widetilde{W}^{1,q})^*} \neq 1$. $\square$

Lemma A.6. *Let $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. There exists a constant C such that for every $F \in \widetilde{W}^{-1,p}$ there exists a vector function $f \in \widetilde{L}_p$ such that for every $\varphi \in \widetilde{W}_0^{1,q}$ we have the equation*

$$\int_{\widetilde{S}} f \cdot \nabla \varphi dx = F(\varphi). \tag{A.30}$$

and

$$\|f\|_{\widetilde{L}_p} \leq C \|F\|_{\widetilde{W}^{-1,p}}.$$

Proof. As before we take $\|F\|_{\widetilde{W}^{-1,p}} = 1$. By Lemma A.4 we have the existence of a minimum for the functional

$$I(\tau) = \int_{\widetilde{S}} (1 + |\nabla \tau|^2)^{q/2} dx - qF(\tau).$$

With $\tau \in \mathcal{A} = \left\{ \tau \in \widetilde{W}_0^{1,q}(S) \right\}$. When we take $\varphi \in \widetilde{W}_0^{1,q}$ and real valued ϵ with $|\epsilon| < 1$ then $\tau_0 + \epsilon\varphi \in \mathcal{A}$ and the function $d(\epsilon) = I(\tau_0 + \epsilon\varphi)$ has its absolute minimum at $\epsilon = 0$. Putting $\tau_0 + \epsilon\varphi$ in the definition of I and using Lemma A.3 we have $d(\epsilon)$ is a differentiable function with

$$0 = d'(0) = \int_{\widetilde{S}} q\left(1 + \nabla\tau_0|^2\right)^{q/2-1} \nabla\tau_0 \cdot \nabla\varphi dx - qF(\varphi) \tag{A.31}$$

for $\varphi \in \mathcal{A}$. When we define $f = \left(1 + |\nabla\tau_0|^2\right)^{q/2-1} \nabla\tau_0$ then

$$|f| \leq 2^{q/2-1}(1 + |\nabla\tau_0|)^{q-1}.$$

Thus, $|f|^p \leq 2^{p(q/2-1)}(1 + |\nabla\tau_0|)^{p(q-1)} = 2^{p(q/2-1)}(1 + |\nabla\tau_0|)^q$ and $\|f\|_{\widetilde{L}_p} \leq C$. Solving equation (A.31) for $F(\varphi)$ and using the definition of f above we arrive at equation (A.30). $\square$

A.2 Regularity

Lemma A.7. *Let V_1 be a subspace of $\widetilde{W}^{1,p}(S)$ and V_2 be a subspace of $\widetilde{C}^\infty(\overline{S})$ having the property that when $\tau \in V_\mu$ for $\mu \in \{1, 2\}$ then $\tau(x + he_i) \in V_\mu$ for $i, \mu \in \{1, 2\}$ as well as having the property that there exists a C such that if $\tau \in V_1$ and*

$$\int_{\widetilde{S}} \nabla\tau\nabla\varphi + \tau\varphi dx = 0 \tag{A.32}$$

for all $\varphi \in V_2$ then $\|\tau\|_{\widetilde{W}^{1,p}} \leq C\|\tau\|_{\widetilde{L}_p}$. Then for all $\tau \in V_1$ fulfilling (A.32) we have $\tau \in \widetilde{C}^\infty(\overline{S})$.

Proof. If the assumption that $\tau \in V_\mu$ then $\tau(x + he_i) \in V_\mu$ is true, and the difference quotient

$$\Delta_i^h \tau = \frac{\tau(x + he_i) - \tau(x)}{h} \in V_\mu.$$

We also have for $\varphi \in V_2$ then $\varphi(x + he_i) \in V_2$ and the difference quotient

$$\Delta_i^h \varphi = \frac{\varphi(x + he_i) - \varphi(x)}{h} \in V_2.$$

Now we use a discrete version of integration by parts as follows. By the definition of the difference quotient for φ,

$$\int_{\widetilde{S}} \tau \Delta_i^h \varphi dx = \frac{1}{h}\left(\int_{\widetilde{S}} \tau(x)\varphi(x+he_i)dx - \int_{\widetilde{S}} \tau(x)\varphi(x)dx\right).$$

Let $y = x + he_i$. Then

$$\int_{\widetilde{S}} \tau(x)\varphi(x+he_i)dx = \int_{\widetilde{S}} \tau(y-he_i)\varphi(y)dy.$$

We rename y as x to obtain

$$\int_{\widetilde{S}} \tau \Delta_i^h \varphi dx = -\int_{\widetilde{S}} \frac{\tau(x-he_i)\varphi(x) - \tau(x)\varphi(x)}{-h} dx = -\int_{\widetilde{S}} \Delta_i^{-h}\tau\varphi dx.$$

Applying the same reasoning to the derivatives of the functions τ and φ we get

$$\int_{\widetilde{S}} \nabla\tau\nabla\Delta_i^h \varphi dx = -\int_{\widetilde{S}} \nabla\Delta_i^{-h}\tau\nabla\varphi dx.$$

If we apply the difference quotient to φ in (A.32) we have

$$\int_{V_\mu} \nabla\tau\nabla\Delta_i^h\varphi + \tau\Delta_i^h\varphi dx = -\int_{V_\mu} \nabla\Delta_i^{-h}\tau\nabla\varphi + \Delta_i^{-h}\tau\varphi dx = 0. \tag{A.33}$$

By Lemma 7.23 in [3] we have

$$\|\Delta_i^h\tau\|_{\widetilde{L}_p} \leq C\|\tau_{x_i}\|_{\widetilde{L}_p}.$$

Using this and applying the estimate $\|\tau\|_{\widetilde{W}^{1,p}} \leq C\|\tau\|_{\widetilde{L}_p}$ to the difference quotient we have

$$\|\Delta_i^h\tau\|_{\widetilde{W}^{1,p}} \leq C\|\Delta_i^h\tau\|_{\widetilde{L}_p} \leq C\|\tau_{x_i}\|_{\widetilde{L}_p} \leq C\|\tau\|_{\widetilde{W}^{1,p}}$$

for $i, \mu \in \{1,2\}$. So we have τ_{x_i} is at least in $\widetilde{W}^{1,p}$. In the weak sense we have $\tau_{x_3x_3} = -\tau_{x_1x_1} - \tau_{x_2x_2} + \tau \in \widetilde{L}_p$, so $\tau \in \widetilde{W}^{2,p}$. Now if we put φ_{x_i} in equation (A.33) we get τ_{x_i} fulfills the same condition, so $\tau_{x_i} \in \widetilde{W}^{2,p}$ and that means that $\tau \in \widetilde{W}^{3,p}$. Continuing in this fashion we have by induction we have $\tau \in \widetilde{W}^{k,p}$ for all integer values of k. □

A.3 A-Priori Estimates

For any point $x_0 \in \partial S$ there is a C^∞ domain $\mathcal{D}_{x_0} \subset S$ such that

$$B\left(\frac{2}{3}h, x_0\right) \cap \overline{S} \subset \overline{\mathcal{D}}_{x_0} \subset B\left(\frac{3}{4}h, x_0\right),$$

where $B(r, x_0)$ is the ball of radius r around x_0. On the domain $\mathcal{D}_{x_0}$ the following a-priori estimates hold.

Lemma A.8. *Let $f \in W^{-1,p}(\mathcal{D}_{x_0})$. If $\tau \in W^{1,p}(\mathcal{D}_{x_0})$ such that*

$$-\Delta\tau + \tau = f$$

$$\tau|_{\partial D} = 0$$

then the solution τ satisfies the estimate

$$\|\tau\|_{W^{1,p}(\mathcal{D}_{x_0})} \leq C\left(\|f\|_{W^{-1,p}(\mathcal{D}_{x_0})} + \|\tau\|_{L_p(\mathcal{D}_{x_0})}\right).$$

Proof. Let $\tau \in W^{1,p}(\mathcal{D}_{x_0})$ and $F = f - \tau$ Then by Lemma A.6 we can write this as $F = div(\widetilde{f})$. Now τ is a solution of the problem

$$-\Delta\tau = F$$

$$\tau|_{\Gamma_1} = 0$$

Then $F \in W^{-1,p}(S)$. By Theorem 2.1 in [1] we get the existence of a unique solution to

$$-\Delta\tau = div(\widetilde{f})$$

$$\tau|_{\Gamma_1} = 0$$

with $\widetilde{f} \in L_p(S)$ and that satisfies the estimate

$$\|\nabla \tau\|_{L_p(\mathcal{D}_{x_0})} \leq C \|\widetilde{f}\|_{L_p(\mathcal{D}_{x_0})}.$$

Using the Poincare inequality we obtain our claim. □

Lemma A.9. *Let $f \in (W^{1,q})^*(\mathcal{D}_{x_0})$. If $\tau \in W^{1,p}(\mathcal{D}_{x_0})$ and $\varphi \in W^{1,q}(\mathcal{D}_{x_0})$ such that*

$$\int_S \nabla \tau \nabla \varphi + \tau \varphi dx =< f, \varphi >$$

then the solution τ satisfies the estimate

$$\|\tau\|_{W^{1,p}(\mathcal{D}_{x_0})} \leq C \left(\|f\|_{(W^{1,q})^*(\mathcal{D}_{x_0})} + \|\tau\|_{L_p(\mathcal{D}_{x_0})} \right).$$

Proof. See Lemma 7 in [9]. □

A.4 Localization Argument

If we multiply the functions τ and f by a partition of unity we can extend the a-priori estimates from Lemma A.8 and Lemma A.9 on $\mathcal{D}_{x_0}$ to a-priori estimates on S through a standard localization argument.

Lemma A.10. *Let $\psi \in C^\infty(\overline{S})$ have compact support on $\overline{\mathcal{D}}_{x_0} \cup \partial\overline{S}$ with $\dfrac{\partial \psi}{\partial n} = 0$ on $\partial\overline{S} \backslash \mathcal{D}$, $\varphi \in W^{1,q}(S)$ and $\tau \in W^{1,p}(S)$. Then*

$$\int_S \nabla(\tau\psi)\nabla\varphi + \tau\psi\varphi dx - \int_S \nabla\tau\nabla(\psi\varphi) + \tau\varphi\psi dx = \int_S 2\tau\nabla\psi\nabla\varphi + \tau\Delta\psi\varphi dx$$

and we have the estimate

$$\|\tau\psi\|_{W^{1,p}} \leq C \left(\|f\|_{W^{-1,p}} + \|\tau\|_{L_p} \right)$$

when we have Dirichlet boundary condition and we have the estimate

$$\|\tau\psi\|_{W^{1,p}} \leq C\left(\|f\|_{(W^{1,q})^*} + \|\tau\|_{L_p}\right)$$

when we have Neumann boundary condition.

Proof. We have

$$\nabla(\tau\psi)\nabla\varphi + \tau\psi\varphi - \nabla\tau\nabla(\psi\varphi) - \tau\varphi\psi = \nabla(\tau\psi)\nabla\varphi - \nabla\tau\nabla(\psi\varphi)$$

$$= \nabla\tau\psi\nabla\varphi + \tau\nabla\psi\nabla\varphi - \nabla\tau\nabla\psi\varphi - \nabla\tau\psi\nabla\varphi$$

$$= \tau\nabla\psi\nabla\varphi - \nabla\tau\nabla\psi\varphi.$$

Then also

$$\begin{aligned}
\int_S \tau\nabla\psi\nabla\varphi - \nabla\tau\nabla\psi\varphi dx &= \int_S \tau\nabla\psi\nabla\varphi + \tau div(\nabla\psi\varphi)dx - \int_{\partial S}\tau\frac{\partial\psi}{\partial n}\varphi dS. \\
&= \int_S 2\tau\nabla\psi\nabla\varphi + \tau\Delta\psi\varphi dx \\
&\leq \|\tau\|_{L_p}\|\psi\|_{C^1}\|\varphi\|_{W^{1,q}} + \|\tau\|_{L_p}\|\psi\|_{C^2}\|\varphi\|_{L^q}.
\end{aligned}$$

Then

$$\int_S \nabla(\tau\psi)\nabla\varphi + \tau\psi\varphi - \nabla\tau\nabla(\psi\varphi) - \tau\varphi\psi dx$$

$$\leq \|\tau\|_{L_p}\|\psi\|_{C^1}\|\varphi\|_{W^{1,q}} + \|\tau\|_{L_p}\|\psi\|_{C^2}\|\varphi\|_{L^q}.$$

We can now estimate the term $\int_S \nabla(\tau\psi)\nabla\varphi + \tau\psi\varphi dx$ by

$$\left|\int_S \nabla(\tau\psi)\nabla\varphi + \tau\psi\varphi dx\right|$$

$$\leq C\left(\|\tau\|_{L_p}\|\psi\|_{C^1}\|\varphi\|_{W^{1,q}} + \|\tau\|_{L_p}\|\psi\|_{C^2}\|\varphi\|_{L_q} + \left|\int_S \nabla\tau\nabla(\psi\varphi) + \tau\varphi\psi dx\right|\right).$$

We can also remove the terms $\|\psi\|_{C^1}$ and $\|\psi\|_{C^2}$ since both are bounded by a constant, and using $\|\varphi\|_{L^q} \leq C\|\varphi\|_{W^{1,q}}$ we have

$$\left|\int_{\mathcal{D}_{x_0}} \nabla(\tau\psi)\nabla\varphi + \tau\psi\varphi dx\right| = \left|\int_S \nabla(\tau\psi)\nabla\varphi + \tau\psi\varphi dx\right|$$

$$\leq C\left(\|\tau\|_{L_p}\|\varphi\|_{W^{1,q}} + \left|\int_S \nabla\tau\nabla(\psi\varphi) + \tau\varphi\psi dx\right|\right)$$

since $\psi\varphi = 0$ outside of $\mathcal{D}_{x_0}$. Now we wish to estimate the last term

$$\left|\int_S \nabla\tau\nabla\left(\psi\varphi\right) + \tau\psi\varphi dx\right|.$$

When we have Dirichlet boundary conditions this becomes

$$\left|\int_S \nabla\tau\nabla\left(\psi\varphi\right) + \tau\psi\varphi dx\right| = |\langle f, \psi\varphi\rangle|$$

$$\leq C\|f\|_{W^{-1,p}}\|\psi\varphi\|_{W^{1,q}} \leq C\|f\|_{W^{-1,p}}\|\psi\|_{C^1}\|\varphi\|_{W^{1,q}}.$$

When we have Neumann boundary conditions this becomes

$$\left|\int_S \nabla\tau\nabla\left(\psi\varphi\right) + \tau\psi\varphi dx\right| = |\langle f, \psi\varphi\rangle|$$

$$\leq C\|f\|_{(W^{1,q})^*}\|\psi\varphi\|_{W^{1,q}} \leq C\|f\|_{(W^{1,q})^*}\|\psi\|_{C^1}\|\varphi\|_{W^{1,q}}.$$

Thus by Lemma A.8 $\|\tau\psi\|_{W^{1,p}} \leq C\left(\|f\|_{W^{-1,p}} + \|\tau\|_{L_p}\right)$ for the Dirichlet case and by Lemma A.9 $\|\tau\psi\|_{W^{1,p}} \leq C\left(\|f\|_{(W^{1,q})^*} + \|\tau\|_{L_p}\right)$ for the Neumann case. □

Lemma A.11. *Let $f \in \widetilde{W}^{-1,p}(S)$. If $\tau \in \widetilde{W}^{1,p}(S)$ such that*

$$\begin{aligned} -\Delta\tau + \tau = f \qquad & \textit{in } S \\ \tau = 0 \qquad & \textit{on } \Gamma_1 \cup \Gamma_2 \end{aligned}$$

then the solution τ satisfies the estimate

$$\|\tau\|_{\widetilde{W}^{1,p}(S)} \leq C\left(\|f\|_{\widetilde{W}^{-1,p}(S)} + \|\tau\|_{\widetilde{L}_p(S)}\right).$$

Proof. The desired estimate is an immediate consequence of Lemma A.10 using $\psi = 1$ in

$B\left(\frac{2}{3}h, x_0\right) \cap S$ and $\frac{\partial \psi}{\partial n} = 0$ on $B\left(\frac{3}{4}h, x_0\right)$. Since

$$\int_{B\left(\frac{2}{3}h,x_0\right)} \Delta(\tau\psi)dx \leq \|\tau\|_{W^{1,p}\left(B\left(\frac{2}{3}h,x_0\right)\right)}$$

we can find a finite collection of points x_0 so that

$$\bigcup B\left(\frac{2}{3}h, x_0\right)$$

covers the entire region S. □

Lemma A.12. *Let $f \in (\widetilde{W}^{1,q})^*(S)$. If $\varphi \in \widetilde{W}^{1,q}(S)$ and $\tau \in \widetilde{W}^{1,p}(S)$ such that*

$$\int_{\widetilde{S}} \nabla\tau\nabla\varphi + \tau\varphi dx = \int_{\widetilde{S}} f\varphi dx$$

then the solution τ satisfies the estimate

$$\|\tau\|_{\widetilde{W}^{1,p}(S)} \leq C\left(\|f\|_{(\widetilde{W}^{1,q})^*(S)} + \|\tau\|_{\widetilde{L}_p(S)}\right).$$

Proof. The proof of this is analogous to the proof of Lemma A.11. □

Now we want to remove the term $\|\tau\|_{\widetilde{L}_p(S)}$ from the right-hand side of the inequality in Lemma A.12.

Lemma A.13. *Let $f \in \widetilde{W}^{-1,p}(S)$ when we have zero Dirichlet boundary conditions and $f \in (\widetilde{W}^{1,q})^*(S)$ when we have Neumann boundary conditons. If $\tau \in \widetilde{W}^{1,p}(S)$ and $\varphi \in \widetilde{W}^{1,q}$ such that*

$$\int_{\widetilde{S}} \nabla\tau\nabla\varphi + \tau\varphi dx = \int_{\widetilde{S}} f\varphi dx$$

then in the Dirichlet boundary case the solution τ satisfies the estimate

$$\|\tau\|_{\widetilde{W}^{1,p}(S)} \leq \|f\|_{\widetilde{W}^{-1,p}(S)}$$

and in the Neumann boundary case the solution satisfies the estimate

$$\|\tau\|_{\widetilde{W}^{1,p}(S)} \le \|f\|_{\left(\widetilde{W}^{-1,p}(S)\right)^*}.$$

Proof. Suppose we cannot remove the term $\|\tau\|_{\widetilde{L}_p(S)}$. Then there exists a sequences $\tau_n \in \widetilde{W}^{1,p}(S) \cap \widetilde{W}_0^{1,p}(S)$ and $f_n \in \widetilde{W}^{-1,p}(S)$ for which $\|\tau_n\|_{\widetilde{L}_p(S)} = 1$ and $\|f_n\|_{\widetilde{L}_p(S)} \to 0$. Since $\|\tau\|_{\widetilde{W}^{s+1,p}} \le \|\tau\|_{\widetilde{L}_p}$ with homogenous boundary data we have $\|\tau_n\|_{\widetilde{W}^{1,p}(S)} \le C$. Thus we can assume τ_n converges weakly to $\tau \in \widetilde{W}^{1,p}(S)$ and strongly in $\widetilde{L}_p(S)$. For a fixed $\varphi \in \widetilde{W}_0^{1,p}(S)$,

$$\int_{\widetilde{S}} \nabla\tau_n \nabla\varphi + \tau_n \varphi dx = \int_{\widetilde{S}} f_n \varphi dx, \forall \varphi \in \widetilde{W}_0^{1,p}(S)$$

We see that

$$\lim_{n\to\infty} \int_{\widetilde{S}} f_n \varphi dx = 0$$

for all $\varphi \in \widetilde{W}_0^{1,p}(S)$ and thus τ is a weak solution of $-\Delta\tau + \tau = 0$. By Lemma A.7 we have $\tau \in \widetilde{C}^\infty$ and by the strong maximum principle we have $\tau \equiv 0$. This contradicts

$$\|\tau\|_{\widetilde{W}^{1,p}(S)} = \lim_{n\to\infty} \|\tau_n\|_{\widetilde{W}^{1,p}(S)} = 1.$$

Thus, we arrive at the estimate

$$\|\tau\|_{\widetilde{W}^{1,p}(S)} \le C\|f\|_{\widetilde{W}^{-1,p}(S)}$$

in the Dirichlet boundary case and in the Neumann boundary case we have

$$\|\tau\|_{\widetilde{W}^{1,p}(S)} \le C\|f\|_{(\widetilde{W}^{1,q})^*(S)}.$$

$\square$

Lemma A.14. *Let $f \in \widetilde{W}^{k,p}$. If $\tau \in \widetilde{W}^{k+2,p}$ for $k \geq 0$ such that τ is a strong solution of*

$$\int_{\widetilde{S}} \nabla\tau\nabla\varphi + \tau\varphi dx = \int_{\widetilde{S}} f\varphi dx$$

with boundary condition $\tau = 0$ or $\dfrac{\partial\tau}{\partial n} = 0$, then the solution τ satisfies the estimate

$$\|\tau\|_{\widetilde{W}^{k+2,p}} \leq C\|f\|_{\widetilde{W}^{k,p}}.$$

Proof. If we apply Lemma A.13 to τ_{x_μ} for $\mu \in \{1, 2\}$ we have $\tau_{x_\mu} \in \widetilde{W}^{k+2,p}$ in an analogous way as shown in Lemma A.7 since τ_{x_μ} fulfills the equation

$$-\Delta\tau_{x_\mu} + \tau_{x_\mu} = f_{x_\mu}$$

with boundary conditions $\tau_{x_\mu} = 0$ or $\dfrac{\partial\tau_{x_\mu}}{\partial n} = 0$. □

Theorem A.15. *If $f \in \widetilde{H}^k$ for all natural numbers k there exists a unique solution of*

$$-\Delta\tau + \tau = f$$

with boundary condition either

$$\begin{cases} \tau|_{\Gamma_1} = 0 \\ \dfrac{\partial\tau}{\partial n}|_{\Gamma_2} = 0 \end{cases} \quad or \quad \begin{cases} \tau_{\Gamma_2} = 0 \\ \dfrac{\partial\tau}{\partial n}|_{\Gamma_1} = 0 \end{cases}$$

Proof. First we assume f is of the form $f(x_3)e^{i\xi\overline{x}}$ and τ is of the form $\tau(x_3)e^{i\xi\overline{x}}$. Then the equation

$$-\Delta\tau + \tau = f \tag{A.34}$$

becomes

$$(-\tau_{x_3x_3} + |\xi|^2\tau + \tau)e^{i\xi\overline{x}} = fe^{i\xi\overline{x}}$$

where $\overline{x} = (x_1, x_2, 0)$ and $\xi = (\xi_1, \xi_2, 0)$. To show uniqueness we look at the homogeneous equation

$$(-\tau_{x_3x_3} + |\xi|^2\tau + \tau)e^{i\xi\overline{x}} = 0$$

Since $e^{ik\overline{x}}$ cannot be zero, we only need to multiply $-\tau_{x_3x_3} + |\xi|^2\tau + \tau$ by $\overline{\tau}$ and integrate by parts.

$$0 = \int_{-h}^{0} -\tau_{x_3x_3}\overline{\tau} + (1 + |\xi|)^2)|\tau|^2 dx_3 = -\overline{\tau}\tau_{x_3}|_{-h}^{0} + \int_{-h}^{0} |\tau_{x_3}|^2 + (1 + |\xi|)^2|\tau|^2 dx_3$$

In either the Dirichlet boundary case ($\tau = 0$) or Neumann boundary case $\left(\frac{\partial \tau}{\partial x_3} = 0\right)$, the boundary term vanishes and we are left with

$$0 = \int_{-h}^{0} |\tau_{x_3}|^2 + (1 + |\xi|^2)|\tau|^2 dx_3$$

and thus we must have that $\tau = 0$ as the kernel of the operator $\tau_{x_3x_3} + |\xi|^2\tau + \tau$ is finite dimensional. Since the homogeneous equation has only one solution we have that the inhomogeneous equation is solvable for all functions $f(x_3)e^{i\xi x}$. Also any linear combination of functions of functions $f(x_3)e^{i\xi x}$and $\tau(x_3)e^{i\xi x}$ makes equation (A.34) solvable, and consequently any finite Fourier polynomial of these functions makes equation (A.34) solvable. Any function in $\widetilde{H}^k$ for $0 < k < \infty$ can now be approximated by Fourier polynomials. $\square$

A.5 Interpolation

We define $G_D : \widetilde{C}^\infty(\overline{S}) \to \widetilde{W}^{2,p}(S)$ by $G_D(f) = \tau$ where τ is the solution of

$$-\Delta\tau + \tau = f \quad \text{in } S$$

$$\tau = 0 \qquad \text{on } \Gamma_i,\ i = 1, 2.$$

Since C^∞ is dense in $\widetilde{W}^{k,p}$ we can extend the map G_D to

$$G_D : \widetilde{W}^{k-1,p} \to \widetilde{W}^{k+1,p}$$

for negative values of k by Lemma 13 in the Dirichlet boundary case. In the Neumann boundary case we define $G_N : \widetilde{C}^\infty(\overline{S}) \to \widetilde{W}^{2,p}(\overline{S})$ by $G_N(f) = \tau$ to be the solution of

$$-\Delta\tau + \tau = f \quad \text{in } S$$

$$\frac{\partial\tau}{\partial n} = 0 \qquad \text{on } \Gamma_i,\ i = 1, 2.$$

Similar to the Dirichlet case the map G_N can be extended to the maps

$$G_N : (\widetilde{W}^{1,q})^* \to \widetilde{W}^{1,p} \quad \text{and} \quad G_N : \widetilde{W}^{k-1} \to \widetilde{W}^{k+1}.$$

Note that by Lemma A.5 one can easily see that $\widetilde{L}_p$ is dense in $(\widetilde{W}^{1,q})^*$.

Lemma A.16. *If $f \in \widetilde{W}^{s-1,p}(S), s \geq 1, s - 1/p \notin \mathbb{Z}$ we have*

$$\|G_D(f)\|_{\widetilde{W}^{s+1,p}(S)} \leq C\|f\|_{\widetilde{W}^{s-1,p}(S)}.$$

Proof. By straight forward interpolation the map

$$G_D : \left(\widetilde{W}^{-1,p}, \widetilde{L}_p\right)_{s,p} \to \left(\widetilde{W}^{1,p}, \widetilde{W}^{2,p}\right)_{s,p}$$

is also continuous and by definition 1.15 we have $G_D : \widetilde{W}^{s-1,p} \to \widetilde{W}^{s+1,p}$. Thus for $s \geq 0, s - 1/p \notin \mathbb{Z}$

$$\|G_D(f)\|_{\widetilde{W}^{s+1,p}(S)} \leq C\|f\|_{\widetilde{W}^{s-1,p}(S)} \tag{A.35}$$

□

Lemma A.17. *If* $f \in \widetilde{W}^{s-1,p}(S), s > 1/p, s - 1/p \notin \mathbb{Z}$ *we have*

$$\|G_D(f)\|_{\widetilde{W}^{s+1,p}(S)} + \|G_N(f)\|_{\widetilde{W}^{s+1,p}(S)} \leq C\|f\|_{\widetilde{W}^{s-1,p}(S)}.$$

Proof. The Dirichlet case has been treated in Lemma A.16. When $s \geq 1$ the Neuman case can be treated in the same manner as the Dirichlet case. It is unfortunately more difficult in the Neuman case when $s - 1 < 0$. Since the map $G_N : (\widetilde{W}^{1,q})^* \to \widetilde{W}^{1,p}$ is continuous and the map $G_N : (\widetilde{L}_q)^* \to \widetilde{W}^{2,p}$ is continuous we also have that the map

$$G_N : \left((\widetilde{W}^{1,q})^*, (\widetilde{L}_q)^*\right)_{s,p} \to \left(\widetilde{W}^{1,p}, \widetilde{W}^{2,p}\right)_{s,p}$$

is continuous. By definition 1.15 we have the interpolation spaces

$$\left(\widetilde{W}^{1,p}, \widetilde{W}^{2,p}\right)_{s,p} = \widetilde{W}^{s+1,p}$$

and by Theorem 1.33 in [11]

$$\left((\widetilde{W}^{1,q})^*, (\widetilde{L}_q)^*\right)_{s,p} = \left(\left(\widetilde{W}^{1,q}, \widetilde{L}_q\right)_{s,q}\right)^* = \left(\widetilde{W}^{1-s,q}\right)^*.$$

When $1 - s - \dfrac{1}{q} < 0$ then $\widetilde{W}^{1-s,q} = \widetilde{W}_0^{1-s,q}$ so

$$\left(\widetilde{W}_0^{1-s,q}\right)^* = \left(\widetilde{W}^{1-s,q}\right)^* = \widetilde{W}^{s-1,p}.$$

Therefore $G_N : \widetilde{W}^{s-1,p} \to \widetilde{W}^{s+1,p}$ is continuous for $s > \dfrac{1}{p}$. solves

$$-\Delta\tau + \tau = f \text{ in } S. \tag{A.36}$$

Thus, for $s + 1 > \dfrac{1}{p}, s - \dfrac{1}{p} \notin \mathbb{Z}$

$$\|G_N(f)\|_{\widetilde{W}^{s+1,p}(S)} + \|G_D(f)\|_{\widetilde{W}^{s+1,p}(S)} \leq C\|f\|_{\widetilde{W}^{s-1,p}(S)} \tag{A.37}$$

□

We have the following four boundary cases.

$$\begin{aligned} \tau &= 0 \quad \text{on } \Gamma_1 \\ \tau &= 0 \quad \text{on } \Gamma_2 \end{aligned} \qquad \text{(BC1)} \qquad \begin{aligned} \frac{\partial \tau}{\partial n} &= 0 \quad \text{on } \Gamma_1 \\ \tau &= 0 \quad \text{on } \Gamma_2 \end{aligned} \qquad \text{(BC3)}$$

$$\begin{aligned} \tau &= 0 \quad \text{on } \Gamma_1 \\ \frac{\partial \tau}{\partial n} &= 0 \quad \text{on } \Gamma_2 \end{aligned} \qquad \text{(BC2)} \qquad \begin{aligned} \frac{\partial \tau}{\partial n} &= 0 \quad \text{on } \Gamma_1 \\ \frac{\partial \tau}{\partial n} &= 0 \quad \text{on } \Gamma_2 \end{aligned} \qquad \text{(BC4)}$$

From the Existence Theorem, Theorem A.15, we have the solution τ of

$$-\Delta\tau + \tau = f$$

with boundary condition (BC1) or (BC4) exists for functions $f \in \widetilde{C}^\infty$ and fulfills the estimate

$$\|\tau\|_{\widetilde{W}^{s+1,p}(S)} \leq C\|f\|_{\widetilde{W}^{s-1,p}(S)}. \tag{A.38}$$

By approximation the solution also exists for every $f \in \widetilde{W}^{s-1,p}$ and fulfills the same estimate (A.38).

A.6 Gluing Together

Now we need to combine the two problems.

Theorem A.18. *Let $f \in \widetilde{W}^{s-1,p}(S)$ and $s > 1/p$ with $s - 1/p \notin \mathbb{Z}$. There is exactly one $\tau \in \widetilde{W}^{s+1,p}(S)$ and a constant C such that*

$$\begin{aligned} -\Delta\tau + \tau &= f \qquad \textit{in } S \\ \tau &= 0 \qquad \textit{on } \Gamma_1 \\ \frac{\partial \tau}{\partial n} &= 0 \qquad \textit{on } \Gamma_2. \end{aligned} \tag{A.39}$$

then we have the estimate

$$\|\tau\|_{\widetilde{W}^{s+1,p}} \leq C\|f\|_{\widetilde{W}^{s-1,p}}.$$

Proof. Assume $f \in \widetilde{C}^{\infty}$. By the Existence Theorem there exists a τ that solves (A.39). Then we can estimate the norm $\|\Delta(\tau\psi)\|_{\widetilde{W}^{s,p}}$ as follows. Let ψ be a smooth function with $\psi(x_3) = 1$ when $x_3 \leq -\frac{2}{3}h$ and $\psi(x_3) = 0$ when $x_3 \geq -\frac{1}{3}h$ and $s \geq -1$. Let $\psi_1 = \psi, \psi_2 = (1-\psi)$, $\tau_1 = \psi_1\tau$ and $\tau_2 = \psi_2\tau$. Then for $\mu \in \{1, 2\}$ we have

$$-\Delta\tau_\mu + \tau_\mu = -\Delta(\tau\psi_\mu) + \tau\psi_\mu$$

$$= -\Delta\tau\psi_\mu - 2\nabla\tau\nabla\psi_\mu - \tau\Delta\psi_\mu + \tau\psi_\mu = -\tau\Delta\psi_\mu - 2\nabla\tau\nabla\psi_\mu + f\psi_\mu.$$

for both boundary conditions (BC1) and (BC4). Then from Lemma A.17 we can estimate $\|\tau_\mu\|_{\widetilde{W}^{s+1,p}}$ for $\mu \in \{1, 2\}$ as follows.

$$\begin{aligned}
\|\tau_\mu\|_{\widetilde{W}^{s+1,p}} &\leq C\left(\|\tau\|_{\widetilde{W}^{s-1,p}} + \|\tau\|_{\widetilde{W}^{s,p}} + \|f\psi\|_{\widetilde{W}^{s-1,p}}\right) \\
&\leq C\left(\|\tau\|_{\widetilde{W}^{s,p}} + \|f\|_{\widetilde{W}^{s-1,p}}\|\psi\|_{\widetilde{C}^1}\right) \\
&\leq C\left(\|\tau\|_{\widetilde{W}^{s,p}} + \|f\|_{\widetilde{W}^{s-1,p}}\right)
\end{aligned}$$

since $\|\psi\|_{\widetilde{C}^1} \leq C$. As $\widetilde{C}^{\infty}$ is dense in $\widetilde{W}^{s-1,p}$ we can approximate $f \in \widetilde{C}^{\infty}$ by $f \in \widetilde{W}^{s-1,p}$. Using $\tau = \tau_1 + \tau_2$ we can add the two previous estimates and arrive at

$$\|\tau\|_{\widetilde{W}^{s+1,p}} \leq C\left(\|\tau_1\|_{\widetilde{W}^{s+1,p}} + \|\tau_2\|_{\widetilde{W}^{s+1,p}}\right) \leq C\left(\|\tau\|_{\widetilde{W}^{1,p}} + \|f\|_{\widetilde{W}^{s-1,p}}\right)$$

for $f \in \widetilde{C}^{\infty}$. Since $s + 1 > 1/p$ we know that the unit ball in $\widetilde{W}^{s+1,p}$ is a compact subset of the unit ball in $\widetilde{W}^{1,p}$, which allows us to remove $\|\tau\|_{\widetilde{W}^{1,p}}$ as we did before in Lemma A.13. Thus we arrive at the desired estimate

$$\|\tau\|_{\widetilde{W}^{s+1,p}} \leq C\|f\|_{\widetilde{W}^{s-1,p}}$$

for $f \in \widetilde{C}^{\infty}$. □

A.7 The Proof of Theorem A.1

Proof. By means of the theorem in Section 3.9.3 in [11] it is obvious that there is a constant C such that given g_1 and g_2 there exists a function w with

$$w|_{\Gamma_1} = g_1 \quad \text{and} \quad \frac{\partial w}{\partial n}|_{\Gamma_2} = g_2$$

and

$$\|w\|_{\widetilde{W}^{s,p}} \leq C \left(\|g_1\|_{\widetilde{W}^{s-1/p,p}} + \|g_2\|_{\widetilde{W}^{s+1-1/p,p}}\right). \tag{A.40}$$

Consider now the mixed boundary value problem

$$\begin{aligned} -\Delta(w+z) + w + z = f \quad & \text{in } S \\ w + z = g_1 \quad & \text{on } \Gamma_1 \\ \frac{\partial(w+z)}{\partial n} = g_2 \quad & \text{on } \Gamma_2. \end{aligned}$$

where w is a known function. Then the solution z fulfills

$$\begin{aligned} -\Delta z + z = f + \Delta w - w \quad & \text{in } S \\ z = 0 \quad & \text{on } \Gamma_1 \\ \frac{\partial z}{\partial n} = 0 \quad & \text{on } \Gamma_2. \end{aligned}$$

By Lemma A.18 we have the estimate

$$\|z\|_{\widetilde{W}^{s+1,p}(S)} \leq \left(\|f + \Delta w - w\|_{\widetilde{W}^{s-1,p}(S)}\right).$$

This shows that $w + z$ is a solution to

$$\begin{aligned} -\Delta\tau + \tau = f \quad & \text{in } S \\ \tau = g_1 \quad & \text{on } \Gamma_1 \\ \frac{\partial \tau}{\partial n} = g_2 \quad & \text{on } \Gamma_2. \end{aligned}$$

and using (A.40) we have

$$\|\tau\|_{\widetilde{W}^{s+1,p}(S)}$$
$$\leq C\left(\|\tau\|_{\widetilde{L}_p(S)} + \|f\|_{\widetilde{W}^{s-1,p}(S)} + \|g_1\|_{\widetilde{W}^{s-1/p,p}(\Gamma_1)} + \|g_2\|_{\widetilde{W}^{s+1-1/p,p}(\Gamma_2)}\right)$$

Similar to before we remove $\|\tau\|_{\widetilde{L}_p}$ and have the estimate

$$\|\tau\|_{\widetilde{W}^{s+1,p}(S)} \leq C\left(\|f\|_{\widetilde{W}^{s-1,p}(S)} + \|g_1\|_{\widetilde{W}^{s-1/p,p}(\Gamma_1)} + \|g_2\|_{\widetilde{W}^{s+1-1/p,p}(\Gamma_2)}\right). \tag{A.41}$$

□

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