Spring 2017

Averages of fractional exponential sums weighted by Maass forms

Huan Qin
University of Iowa

AVERAGES OF FRACTIONAL EXPONENTIAL SUMS WEIGHTED BY MAASS FORMS

by

Huan Qin

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

May 2017

Thesis Supervisor: Professor Yangbo Ye

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Huan Qin

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2017 graduation.

Thesis Committee: ______________________________
Prof. Yangbo Ye, Thesis Supervisor

____
Prof. Philip Kutzko

____
Prof. Muthu Krishnamurthy

____
Prof. Gerhard Strohmer

____
Prof. Mark McKee

ABSTRACT

The purpose of this study is to investigate the oscillatory behavior of the fractional exponential sum weighted by certain automorphic forms for $GL_2 \times GL_3$ case. Automorphic forms are complex-values functions defined on some topological groups which satisfy a number of applicable properties. One nice property that all automorphic forms admit is the existence of Fourier series expansions, which allows us to study the properties of automorphic forms by investigating their corresponding Fourier coefficients. The Maass forms is one family of the classical automorphic forms, which is the major focus in this study.

Let f be a fixed Maass form for $SL_3(\mathbb{Z})$ with Fourier coefficients $A_f(m,n)$. Also, let $\{g_j\}$ be an orthonormal basis of the space of the Maass cusp form for $SL_2(\mathbb{Z})$ with corresponding Laplacian eigenvalues $1/4+k_j^2$, $k_j>0$. For real $\alpha \neq 0$ and $\beta > 0$, we considered the asymptotics for the sum in the following form

$$\mathcal{S}_X(f \times g_j, \alpha, \beta) = \sum_{n=1}^{\infty} A_f(m,n)\lambda_{g_j}(n)e(\alpha n^{\beta})\phi\left(\frac{n}{X}\right), \tag{1}$$

where ϕ is a smooth function with compactly support, $\lambda_{g_j}(n)$ denotes the n-th Fourier coefficient of g_j, and X is a real parameter that tends to infinity. Also, $e(x) = e^{2\pi i x}$ throughout this thesis.

We proved a bound of the weighted average sum of (1) over all Laplacian eigenvalues, which is better than the trivial bound obtained by the classical Rankin-Selberg method. In this case, we allowed the form be vary so that we can obtain

information about their oscillatory behaviors in a different aspect. Similar to the proofs of the subconvexity bounds for Rankin-Selberg L-functions for $GL_2 \times GL_3$ case, the method we used in this study includes several sophisticated techniques such as weighted first and second derivative test, Kutznetsov trace formula, and Voronoi summation formula.

PUBLIC ABSTRACT

During the last half-century, the theory of automorphic forms has become a major focus in the development of the modern number theory. Automorphic forms are functions from some topological groups to the complex plane, which have many applications to different aspects in Mathematics. Because automorphic forms have Fourier expansions. We can study its properties by studying the corresponding Fourier coefficients. Taking the weighted sums of these Fourier coefficients against various exponential functions will case a rise of resonance. We call this type of sum as a resonance sum. Resonance is a physical phenomenon that occurs between two interactive vibrating systems. Fixing one of these two vibrating systems, we may control the second one to detect the resonance frequencies of the first system, and thus obtain its oscillation spectrum. The most classical example of this is the Fourier series expansion of a periodic function, which is the resonance sum for GL_1 case. This study is to learn the property of a resonance sum in a higher dimensional space.

TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

This thesis is to study the average of fractional exponential sums weighted by Maass forms for $SL_2(\mathbb{Z})$ and $SL_3(\mathbb{Z})$. In the first section of this chapter, the historical development in the subconvexity bound problem of automorphic L-function will be brushed up on, which also serves as an motivation of this research. Moreover, basic settings of the whole problem including the definitions and notations that will be used throughout the whole thesis, as well as the main result will be introduced in the next section.

This work is largely inspired by Sarnark [30], Liu-Ye [18], Li [17], and McKee-Sun-Ye [20], which are all important works for finding asymptotic bounds for certain type of Rankin-Selberg L-functions. It is worth of mentioning several sophisticated techniques that I have applied to this study that were adapted from those papers. In each of the fours sections in Chapter 2, the derivative tests, the Kutznetsov trace formula, the Voronoi's summation formula, and the stationary phase argument will be introduced. In Chapter 3, we provide a proof of out main result.

1.1 History Development

The first and most famous L-function has been given Riemann's name. The Riemann-Zeta function is defined by the generalized harmonic series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s = \sigma + it$ and $\sigma > 1$. Even though we name this function after Riemann's name, it was studied even before Euler. The start of the branch of analytic number theory has been considered as Euler's discovery of connections between $\zeta(s)$ and the prime numbers, which is described in Euler product formula:

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Euler published his results in his widely know *Introductio*, and it was this work that Riemann went on from.

Riemann stated that $\zeta(s)$ has a meromorphic continuation to the whole complex plane. There is only one singularity, which is a simple pole with residue 1 at $s = 1$. Moreover, this analytic continuation satisfied the following functional equation

$$\zeta(1-s) = \pi^{-s} 2^{s-2} \Gamma(s) \frac{\cos(\pi s)}{2} \zeta(s),$$

for all complex numbers, provided valid gamma functions. These results are in Riemann's only number theory publication "Über die Anzahl der Primzahlen unter einer gegebenen Größe" ("On the number of primes that are less than a given quantity") in 1859. It was his way of thanking the Berlin Academy for appointing him a corresponding member. This paper was great, not only because Riemann brought the famous Riemann hypothesis in it. Before this paper, no one would, at first glance, expect to connect the natural numbers, which represents the discrete and the disconnected aspect of mathematics to complex analysis, which has to do with the continuous representations and treatments.

We know that $\zeta(s)$ has trivial zeros at $-2n$ for $n \geq 1$, which are all negative even integers. However, the trivial zeros are not the only values for which $\zeta(s)$ is zero. The others zeros are called non-trivial zeros. Here comes the widely known Riemann Hypothesis:

$$\text{All non-trivial zeros of } \zeta(s) \text{ are on the critical line: } Re(s) = \frac{1}{2}.$$

The hypothesis in Riemann's words:

"and it is likely that all roots" have real part 1/2: "Of course, it would be desirable to have a rigorous proof of this; in the meantime, after a few perfunctory vain attempts, I temporarily put aside looking for one, for it seemed unnecessary for the next objective of my investigation." Even though Riemann did not continue to work on his own hypothesis, many accomplished mathematicians such as Stieltjes, Hardy, and Ramanujan, have tried to prove it.

Lindelöf hypothesis is one consequence of the Riemann hypothesis investigaing the rate of growth of $\zeta(s)$ on the critical line, which states that for any $\varepsilon > 0$,

$$\zeta\left(\frac{1}{2} + it\right) \ll (|t| + 1)^{\varepsilon}$$

as $t \to \infty$. This result was proved by Finnish mathematician Ernst Leonard Lindelöf in 1908. This hypothesis says that: for any $\varepsilon > 0$, as t tends to infinity, the number of zeros of $\zeta(s)$ with real part be at least $\frac{1}{2} + \varepsilon$ and imaginary part be between t and $t+1$ is $o(\log t)$. The Riemann hypothesis implies that there are no zeros at all in this region and thus implies the Lindelöf hypothesis. One may think that the Lindelöf hypothesis should be easier to establish, but most number theorists think that the Riemann hypothesis must be solved first. The point is that the Riemann hypothesis is constructed in a more natural mathematical way.

In the general theory of L-functions, there are some quite significant features such as "conductor" and of "primitivity" do not show in the fundamental Riemann-Zeta Function. So, we need to consider one kind of more generalized L-function, which is called the Dirichlet L-function and has been defined as the Dirichlet series associated to Dirichlet characters χ as

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Just like the case in $\zeta(s)$, the above Dirichlet series and the corresponding Euler product are absolutely convergent for $Re(s) > 1$; the Dirichlet L-function has analytic continuation to the whole complex plane and satisfies a functional equation; it is also conjectured to obey the generalized Riemann hypothesis.

Here, we can follow Iwaniec-Kowalski [8] to provide a definition for L-functions: we say that $L(s,f)$ is an L-function if we have the following data and conditions.

(1) A Dirichlet series with Euler product of degree $d \geq 1$,

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_1(p)}{p^s}\right)^{-1} \cdots \left(1 - \frac{\alpha_d(p)}{p^s}\right)^{-1}$$

with $\lambda_f(1) = 1$, $\lambda_f(n) \in \mathbb{C}$. Of course, the series and Euler products must be absolutely convergent for $Re(s) > 1$. The $\alpha_i(p)$, $1 \leq i \leq d$, are called the local roots or local parameters of $L(s,f)$ at p with the property that for all p, $|\alpha_i(p)| < p$.

(2) A gamma factor

$$\gamma(s,f) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s+\kappa_j}{2}\right)$$

where the numbers $\kappa_j \in \mathbb{C}$ are called the local parameters of $L(s,f)$ at infinity. We assume these numbers are either real or show up in conjugate pairs. In addition, $Re(k_j) > -1$, which means that $\gamma(s,f)$ has no zero in $\mathbb{C}$ and no pole for $Re(s) \geq 1$.

(3) An integer $q(f) \geq 1$, called the conductor of $L(s,f)$, such that $\alpha_i(p) \neq 0$ $(1 \leq i \leq d)$ for $p \nmid q(f)$ and such prime p is said to be unramified.

As a generalization of Riemann-zeta function and the Dirichlet L-function, Langlands considered a large family of L-functions defined as the following form

$$L(s,\pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}$$

for $Re(s) > 1$, where π is the automorphic cuspidal representations of $GL_n(\mathbb{A}_\mathbb{Q})$ (the general linear group GL_n over the adele ring of $\mathbb{Q}$), $\lambda_\pi(n)$ is the Fourier coefficient of π, with nice family structures. The nice relationship among this family of L-functions is called "Langlands Functoriality Conjecture", which means the principle that sometimes we can use existing L-functions to build a new on by taking operations on their Fourier coefficients $\lambda_\pi(n)$, such as raising to the squared power as $\lambda_\pi(n)^2$ or $\lambda_\pi(n^2)$. However, it is not known whether or not the resulting representation is again an automorphic one, although this is suspected to be true and is known in a few select cases. That is why this remains as a conjecture.

The main conjectures in the Langlands Program predict the existence of a correspondence between analytic number theory and algebraic number theory. Specifically,

$$L(s,\pi) \longleftrightarrow L(s,\rho),$$

which means every L-function arising from an automorphic cuspidal representation of $GL_n(\mathbb{A}_\mathbb{Q})$ is equal to an Artin L-function arising from a finite dimensional representations of Galois group of a number field. The analytic properties of the left hand side has been well studied. So, the Langlands program has been considered as an effective tool for finding a non-abelian class field theory.

The Grand Riemann Hypothesis refers to the following conjectural statement about zeros of the family of L-functions defined above:

$$\text{All non-trivial zeros of } L(s,\pi) \text{ are on the critical line: } Re(s) = \frac{1}{2}.$$

There is no need to say we believe that every such L-function satisfies the Grand Riemann Hypothesis. However, proving this even for one L-function would be an far-reaching achievement in the history of human beings.

For this generalized case, we also want to investigate the size of an L-function at its central point. Thus, we expect the generalized Lindelöf hypothesis to follow from the generalized(grand) Riemann hypothesis as in the Riemann-zeta function case. However, this remains as an open question. For any $\varepsilon > 0$,

$$L\left(\frac{1}{2} + it, \pi\right) \ll (1 + |t| + Q_\pi)^\varepsilon$$

where Q_π is a conductor of the representation π. By Applying the Phragmén-Lindelöf Principle in complex analysis, we can obtain a trivial bound of $L\left(\frac{1}{2} + it, \pi\right)$ as follows:

$$L\left(\frac{1}{2} + it, \pi\right) \ll (1 + |t| + Q_\pi)^{\frac{1}{4}+\varepsilon}.$$

This trivial (upper) bound is also called the convexity bound. The achievable goal is to prove an upper bound for $L\left(\frac{1}{2} + it, \pi\right)$ that better than the convexity bound. And this problem is called the subconvexity problem of automorphic L-functions

Here are some known results. The recent breakthrough was obtained by Xiaoqing Li [17]. In her paper published in *Annals of Mathematics*, she proved a subconvexity bound for the L-function in $GL_2(\mathbb{Z}) \times GL_3(\mathbb{Z})$ case.The result has been improved by my advisor YangboYe with two colleges Mark McKee and Haiwei Sun [20] in their paper published in *Transactions of the AMS*.

This present research is to study the oscillatory behavior of Fourier coefficients $\lambda_\pi(n)$. My goal is to find an upper bound in average for a sum related to the subconvexity bound of automorphic L-functions. Here are the basic settings and the main results.

1.2 Basic Settings and the Main Results

Let f be a fixed Maass form for $SL_3(\mathbb{Z})$ with Fourier coefficients $A_f(m,n)$. Let g be a Maass cusp form for $SL_2(\mathbb{Z})$ with Laplacian eigenvalue $1/4+k^2$, $k>0$. Let $\{g_j\}$ be an othogonormal basis of the space of Maass cup forms with Laplacian eigenvalue $1/4+k_j^2$, $k_j>0$. Denote the n-th Fourier coefficient of g_j by $\lambda_{g_j}(n)$, normalized by $<g_j,g_j>=1$. Then, g_j has the Fourier-Whittaker expansion

$$g_j(z) = (y\cosh \pi k_j)^{\frac{1}{2}} \sum_{n\neq 0} \lambda_{g_j}(n) K_{ik_j}(2\pi|n|y) e(nx).$$

Here, K_{ik_j} is the Bessel K-function, $z=x+iy$, and $e(x)=e^{2\pi i x}$. Since each g_j is normalized by $||g_j||=1$, the leading coefficient $\lambda_{g_j}(1)$ is no longer equal to 1. By Hoffsein-Lockhat, we can use a new normalization of g_j with $\lambda_{g_j}(1)=1$ after a $O(k_j^\epsilon)$

discrepancy.

We consider the following resonance sum:

$$\mathcal{S}_X(f \times g, \alpha, \beta) = \sum_{n>0} A_f(1,n)\lambda_g(n)e(\alpha n^\beta)\phi\left(\frac{n}{X}\right) \tag{1.1}$$

where $\alpha \neq 0$, $\beta > 0$, $\phi \in C_c^\infty((1,2))$, X is a parameter tends to infinity. Motivated by wokrs in Liu-Ye [18], Ren-Ye [27], Li [17] and Salazar-Ye [29], we will provide a non-trivial bound for the weighted summation of $\mathcal{S}_X(f \times g_j)$ over $K - L \leq k_j \leq K + L$ when $\beta = 1/3$. Specifically, we have the following theorem:

Theorem 1.1. *Suppose $\alpha \neq 0$ is fixed. Let $\{g_j\}$ be an orthonormal basis of Hecke-Maass eigenforms with Laplace eigenvalues $1/4 + k_j^2$, $k_j > 0$. Let K be a parameter tending towards infinity with $K^\varepsilon \leq L \leq K^{1-\varepsilon}$ and $LK \geq \sqrt{X}$. Then, for $\beta = 1/3$,*

$$\sum_{K-L \leq k_j \leq K+L} e^{-\frac{(k_j-L)^2}{K^2}} \mathcal{S}_X\left(f \times g_j, \alpha, \frac{1}{3}\right) \ll LK^{1+\varepsilon}X^{\frac{1}{2}+\varepsilon} + LK^\varepsilon X^{1+\varepsilon}. \tag{1.2}$$

Note that under the assumption $LK \geq \sqrt{X}$, the above summation (1.2) is dominated by $LK^{1+\varepsilon}X^{\frac{1}{2}+\varepsilon}$. Otherwise, it is dominated by $LK^\varepsilon X^{1+\varepsilon}$. Since by Weyl's law, there are about $L^{1+\varepsilon}K$ terms in the summation, Rankin-Selberg method yields the trivial bound $L^{1+\varepsilon}KX^{1+\epsilon}$.

This problem was first brought by Iwaniec-Luo-Sarnak [9],they studied a resonance sum with f being a holomorphic cusp form in $SL_2(\mathbb{Z})$, $\beta = 1$, and $\alpha = \pm 2\sqrt{q}$ ($q \in \mathbb{Z}_+$). Followed their works, Ren-Ye [23] and Sun [32] investigated the resonance behavior of automorphic forms for certain α and β in $GL_2(\mathbb{Z})$. They were able to obtain the main term and thus asymptotic upper bound of the corresponding sum. Later, in Sun-Wu [33], the authors considered the case when f is a Maass cusp form for $SL_2(\mathbb{Z})$, similar asymptotic results can be obtained. As a crucial tool for future works in $GL_3(\mathbb{Z})$ cases, Ren-Ye [24] proved the asymptotic Voronoi's summation formula for $SL_3(\mathbb{Z})$ and discussed its applications and the duality property.

CHAPTER 2
TECHNIQUES

It is worth mentioning several sophisticated techniques that I have applied to this present research. For dealing with the exponential integral by either showing it is negligible or obtaining the asymptotic expansions, the second derivative test in Huxley[6] is used to bound the integral; a weighted first derivative test from MacKee-Sun-Ye [20] with more strength than the usual first derivative test in Huxley [6] is used to deal with the instances when the phase function is infinitely differentiable. The Kuznetsov trace formula is one of the various methods that can be used to develop the spectral decompositions of sums of Kloosterman sums. After applying the Kuznetsov trace formula and manipulating the order of terms, the desired sum can be rewritten as a separate spectral part and a geometric part. In particular, the integral on the spectral side represents the continuous spectrum of the Laplace operator with the divisor function being the Fourier coefficient of Eisenstein Series. Another very important and widely used technique in modern analytic number theory is the Voronoi's summation formula. It is used to convert certain sums of arithmetic terms into sums over integrals. An asymptotic expansion of Voronoi's summations formula for Maass forms in $SL_3(\mathbb{Z})$ proved by Ren-Ye [24] is used in this research. Finally, the stationary phase argument technique is used to control the growth of these obtained integrals.

2.1 Derivative Tests

Consider one kind of integrals named exponential integrals of the following form

$$\int_{\alpha}^{\beta} g(x)e(f(x))\ dx, \tag{2.1}$$

where x, $f(x)$, and $g(x)$ are all real, $f'(x)$ changes signs at some point between α and β. Here, the function $f(x)$ is called the *exponent* and the $g(x)$ is called the *weight*. We will need to use the second derivative test lemma in Huxley[6] to obtain its asymptotic expansion. The lemma is state here in its entirety.

Lemma 2.1. *(Huxley, [6]) Let $f(x)$ be real and twice differentiable on the open interval (α, β) with $f''(x) \geq \lambda > 0$ on (α, β). Let $g(x)$ be real, and V be the total variation of $g(x)$ on the closed interval $[\alpha, \beta]$ plus the maximum modulus of $g(x)$ on $[\alpha, \beta]$. Then*

$$\left| \int_{\alpha}^{\beta} g(x)e(f(x))\ dx \right| \leq \frac{4V}{\sqrt{(\pi\lambda)}}.$$

While in some other cases, we need to show the exponential integrals (2.1) are negligible. The weighted first derivative test lemma in Huxley [6] was widely used to accomplish this task.

Lemma 2.2. *(Huxley, [6]) Let $f(x)$ be a real function, three times continuously differentiable for $\alpha \leq x \leq \beta$, and let $g(x)$ be a real function, twice continously differentiable* *for $\alpha \le x \le \beta$. Suppose that there are positive parameters M, N, T, U with $M \ge \beta - \alpha$, and positive constants C, such that, for $\alpha \le x \le \beta$,*

$$\left|f^{(r)}(x)\right| \le C_r T/M^r, \qquad \left|g^{(s)}(x)\right| \le C_s U/N^s,$$

for $r = 2, 3$, and $s = 0, 1, 2$. If $f'(x)$ and $f''(x)$ do not change sign on the interval $[\alpha, \beta]$, then we have

$$\begin{aligned} I = \int_\alpha^\beta g(x)e(f(x))\,dx &= \frac{g(\beta)e(f(\beta))}{2\pi i f'(\beta)} - \frac{g(\alpha)e(f(\alpha))}{2\pi i f'(\alpha)} \\ &+ \mathcal{O}\left(\frac{TU}{M^2}\left(1 + \frac{M}{N} + \frac{M^2}{N^2}\frac{\min|f'(x)|}{T/M}\right)\frac{1}{\min|f'(x)|^3}\right) \end{aligned}$$

The implies constants are constructed from the constants C_r.

In McKee-Sun-Ye [20], the authors proved a stronger version of the above Lemma 2.2 by including more boundary terms and smaller error terms. Here is the weighted first derivative test theorem in McKee-Sun-Ye[20]:

Lemma 2.3. *(McKee, Sun and Ye, [20]) Let $\sigma(t)$ be a real-valued function, $n+2$ times continuously differentiable for $a \le t \le b$, and let $g(t)$ be a real-valued function $n+1$ times continuously differentiable for $a \le t \le b$. Suppose that there are positive parameters M, N, T, U with $M \ge b - a$, and positive constants C_r such that for $a \le t \le b$,*

$$|\sigma^{(r)}(t)| \le C_r \frac{T}{M^r}, \quad |g^{(s)}(t)| \le C_s \frac{U}{N^s},$$

for $r = 2, \ldots, n+2$ and $s = 0, \ldots, n+1$. If $\sigma'(t)$ and $\sigma''(t)$ do not change signs on *the interval* $[a,b]$*, then we have*

$$\begin{aligned}\int_a^b e(\sigma(t))g(t)dt &= \left[e(\sigma(t))\sum_{i=1}^{n} H_i(t)\right]_a^b \\ &+\mathcal{O}\left(\frac{M}{N}\sum_{j=1}^{[n/2]}\frac{UT^j}{\min|\sigma'|^{n+j+1}M^{2j}}\sum_{t=j}^{n-j}\frac{1}{N^{n-j-t}M^t}\right) \\ &+\mathcal{O}\left(\left(\frac{M}{N}+1\right)\frac{U}{N^n\min|\sigma'|^{n+1}}\right) \\ &+\mathcal{O}\left(\sum_{j=1}^{n}\frac{UT^j}{\min|\sigma'|^{n+j+1}M^{2j}}\sum_{t=0}^{n-j}\frac{1}{N^{n-j-t}M^t}\right),\end{aligned}$$

where

$$H_1(t)=\frac{g(t)}{2\pi i\sigma'(t)},\quad H_i(t)=-\frac{H'_{i-1}(t)}{2\pi i\sigma'(t)}$$

for $i=2,\ldots,n$*.*

Besides, we need to use the following lemma (theorem 1.2 from Salazar-Ye [29]) to deal with an oscillatory integral of the form

$$W_{K,L}(x)=\int_{\mathbb{R}} e\left(\frac{tK}{L}+\frac{x}{2\pi}\cosh\left(\frac{\pi t}{L}\right)\right)(h(u)(uL+K))^{\wedge}(t)dt,$$

where $h(r)$ is assumed to be an even analytic function on the strip $|Im(r)|\leq 1/2+\varepsilon$ and $h(r)\ll r^{-2-\delta}$ for some $\delta>0$ as $r\to\infty$. Also, assume $h(r)\geq 0$. Thus, $h(r)$ is a Schwartz function on $\mathbb{R}$ and is negligible outside $(-L^{\varepsilon},L^{\varepsilon})$.

Lemma 2.4. *(Salazar-Ye, [29]) Suppose* $K^{\varepsilon}\leq L\leq K^{1-\varepsilon}$*.*

(i) If $|x|<LK^{1-\varepsilon/2}$*, then* $W_{K,L}\ll_M K^{-M}$ *for any* $M>0$*.*

(ii) If $|x| \geq LK^{1-\varepsilon/2}$*, then*

$$W_{K,L}(x) = \sum_{\nu=0}^{n} \widetilde{W}_\nu(x) + \mathcal{O}\left(\frac{KL^{2n+5}}{|x|^{n+2}}\right) + \mathcal{O}\left(\frac{KL^{2n+2}}{|x|^{n+1}}\right). \tag{2.2}$$

Here

$$\begin{aligned} \widetilde{W}_\nu(x) &= \frac{i^\nu(2\nu-1)!!(1+i)}{sgn(x)^\nu \pi^{2\nu+1/2}} \frac{KL^{2\nu+1}}{(4K^2+x^2)^{\nu/2+1/4}} H_{2\nu}(\gamma) \\ &\times \; e\left(\frac{sgn(x)}{2\pi}\sqrt{4K^2+x^2} - \frac{K}{\pi}\sinh^{-1}\left(\frac{2K}{x}\right)\right), \end{aligned} \tag{2.3}$$

where $H_{2\nu}(\gamma)$ *is defined as*

$$\begin{aligned} H_{2\nu}(\gamma) &= \frac{H^{(2\nu)(\gamma)}}{(2\nu)!} + \sum_{l=0}^{2\nu-1} \frac{H^{(l)(\gamma)}}{l!} \sum_{k=1}^{2\nu-l} \frac{2^k C_{2\nu,l,k}}{\sigma^{(2)}(\gamma)} \\ &\times \sum_{\substack{3 \leq n_1,\ldots,n_k \leq 2n+3 \\ n_1+\cdots+n_k = 2\nu-l+2k}} \frac{\sigma^{(n_1)}(\gamma)\cdots\sigma^{(n_k)}(\gamma)}{n_1!\cdots n_k!}, \end{aligned} \tag{2.4}$$

with

$$\gamma = -\frac{\eta_1 L}{\pi}\sinh^{-1}\left(\frac{Kc}{2\pi\sqrt{t^3X}}\right). \tag{2.5}$$

$$H(z) = \left(h(u)\left(u\frac{L}{K}+1\right)\right)^{\wedge}(z), \tag{2.6}$$

$$\sigma(z) = \frac{zK}{L} + \frac{2\eta_1\sqrt{t^3X}}{c}\cosh\left(\frac{\pi z}{L}\right) \tag{2.7}$$

and $C_{2\nu,l,k}$ are constants.

Thoughout this thesis, ε is any arbitrarily small positive number. Its value may be different on each occurrence.

2.2 Kutznetsov Trace Formula

Use the same notation as in the basic settings in section 1.2, and let $\{g_j\}$ be an othonormal basis of the space of Maass cup forms with Laplacian eigenvalue $1/4 + k_j^2$, $k_j > 0$. Denote the n-th Fourier coefficient of g_j by $\lambda_{g_j}(n)$, normalized by $< g_j, g_j >= 1$. The Fourier expansion of g_j is of the form

$$g_j(z) = (y \cosh \pi k_j)^{\frac{1}{2}} \sum_{n \neq 0} \lambda_{g_j}(n) K_{ik_j}(2\pi|n|y)e(nx).$$

This normalization for g_j is very important. In fact, the Kuznetsov formula will not be valid for Maass form case if $\{g_j\}$ is not a orthonormal base.

The Kuznetsov trace formula is stated as follows:

Lemma 2.5. *(Kuznetsov, [13]) Let $h(r)$ satisfies the following conditions*

- *$h(r)$ is even.*
- *$h(r)$ is holomorphic in the strip $|Im(r)| \leq \frac{1}{2} + \varepsilon$.*
- *$h(r) \ll (|r| + 1)^{-2-\delta}$ in the strip.*

Then, for any n, $m \geq 1$,

$$\sum_{g_j} h(k_j)\lambda_{g_j}(m)\overline{\lambda_{g_j}(n)} + \frac{1}{\pi}\int_{\mathbb{R}} \tau_{ir}(n)\tau_{ir}(m)h(r)\frac{1}{|\zeta(1+2ir)|^2}dr \tag{2.8}$$

$$= \frac{\delta_{n,m}}{\pi^2}\int_{\mathbb{R}} \tanh(\pi r)h(r)dr \tag{2.9}$$

$$+\frac{2i}{\pi}\sum_{c\geq 1}\frac{S(n,m;c)}{c}\int_{\mathbb{R}} J_{2ir}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{h(r)r}{\cosh(\pi r)}dr, \tag{2.10}$$

where $\tau_\nu(n) = \sum_{ab=|n|}(a/b)^\nu$ *and* $S(n,m;c)$ *is the classical Kloomsterman sum.*

The left-hand side (2.8) of the formula contains the spectral information, and the right-hand side (2.9), (2.10) represent the geometric aspect. Moreover, the integral in term (2.8) represents the continuous spectrum of the Laplacian operate. And the divisor function $\tau_{ir}(n)$ is the Fourier coefficient of corresponding Eisenstein series.

The following

$$|\zeta(1+2ir)| \gg \frac{1}{\log(2+|r|)}$$

is called a bound of de la Vallée Poussin. Because of this bound, the integral on the left-hand side (2.8) converges absolutely. Since $\tau_{ir}(n) \ll n$ and $h(r) \ll (|r|+1)^{-2-\delta}$, the integral on the right-hand side in term (2.9) is also convergent.

The Kloomsterman sums $S(n,m;c)$ is one very important type of arithmetic function in analytic number theory because it links with spectral theory of automorphic forms. The classical Kloomsterman sum is defined by

$$S(n,m;c) = \sideset{}{^\star}\sum_{x \ (mod\ c)} e\left(\frac{nx+m\bar{x}}{c}\right),$$

for integers n, m, and $c \geq 1$. The $\star$ here means the summation is taken over x such that $x\bar{x} \equiv 1(mod\ c)$. The Kloomsterman sums satisfy nice symmetric properties:

$$S(n, m; c) = S(m, n; c)$$

and

$$\text{if } (n', c) = 1,\ S(nn', m; c) = S(n, mn'; c).$$

Also, it satisfy a nice multiplicative property:

$$\text{if } (c, d) = 1,\ S(n, m; cd) = S(n\bar{c}, m\bar{c}; d)S(n\bar{d}, m\bar{d}; c).$$

Kloomsterman sums will make appearances in the proof of the main theorem. They has close connection with the spectral theory of automorphic forms.

2.3 Voronoi's Summation Formula

This section is to present an important tool, the Voronoi's summation formula for GL_3 case. Follow Goldfield -Li [5], McKee-Sun-Ye [20], and Ren-Ye [27], consider a Maass form f for $SL_3(\mathbb{Z})$ of type $\nu = (\nu_1, \nu_2)$. Then, we can define the Langland's parameters for f as

$$\mu_f(1) = \nu_1 + 2\nu_2 - 1, \quad \mu_f(2) = \nu_1 - \nu_2, \quad \mu_f(3) = 1 - 2\nu_1 - \nu_2.$$

Moreover, f has the following Fourier-Whittaker expansion:

$$f(z) = \sum_{\gamma \in U_2(\mathbb{Z}) \backslash SL_2(\mathbb{Z})} \sum_{m_1 \geq 1} \sum_{m_2 \neq 0} \frac{A_f(m_1, m_2)}{m_1 |m_2|} W_J \left(M \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z, \nu, \psi_{1,1} \right),$$

where $U_2 = \left\{ \begin{pmatrix} \gamma & * \\ 0 & 1 \end{pmatrix} \right\}$, W_J is the Jacquet-Whittaker function, $M = \mathrm{diag}(m_1|m_2|, m_1, 1)$, $\psi_{1,1}$ is a fixed character on the abelianization of the standard unipotent upper triangular subgroup of $SL_3(\mathbb{Z})$, and $A_f(m_1, m_2)$ are Fourier coefficients of f.

Let $\psi(x) \in C_c^\infty(0, \infty)$ and denote $\tilde{\psi}(s)$ be the Mellin transform as

$$\tilde{\psi}(s) = \int_0^\infty \psi(x) x^{s-1} dx.$$

For $k = 0$ or 1, define

$$\Psi_k(x) = \int_{Re(s)=\sigma} (\pi^3 x)^{-s} \prod_{j=1}^{3} \frac{\Gamma\left(\frac{1+s+2k+\mu_f(j)}{2}\right)}{\Gamma\left(-\frac{s+\mu_f(j)}{2}\right)} \tilde{\psi}(-s-k) ds,$$

where

$$\sigma > \max(-1 - Re(\mu_f(1)) - 2k, -1 - Re(\mu_f(2)) - 2k, -1 - Re(\mu_f(3)) - 2k).$$

and denote

$$\Psi_{0,1}^0(x) = \Psi_0(x) + \frac{1}{i\pi^3 x} \Psi_1(x), \qquad \Psi_{0,1}^1(x) = \Psi_0(x) - \frac{1}{i\pi^3 x} \Psi_1(x).$$

The Voronoi's summation formula is stated in the following lemma.

Lemma 2.6. *(Miller-Schmit, [22]) Let $A_f(m,n)$ be the Fourier coefficients of the Maass cusp form f for $SL_3(\mathbb{Z})$. Suppose that $\psi \in C_c^\infty(0,\infty)$. Let c,d be integers such that $c \geq 1$, $(c,d)=1$ and $d\bar{d} \equiv 1(mod\ c)$. Then*

$$\begin{aligned} &\sum_{n>0} A_f(m,n) e\left(\frac{n\bar{d}}{c}\right)\psi(n) \\ = &\frac{c\pi^{-5/2}}{4i}\sum_{n_1|cm}\sum_{n_2>0}\frac{A_f(n_2,n_1)}{n_1n_2}S\left(md,n_2;\frac{mc}{n_1}\right)\Psi_{0,1}^0\left(\frac{n_2n_1^2}{c^3m}\right) \\ + &\frac{c\pi^{-5/2}}{4i}\sum_{n_1|cm}\sum_{n_2>0}\frac{A_f(n_1,n_2)}{n_1n_2}S\left(md,-n_2;\frac{mc}{n_1}\right)\Psi_{0,1}^1\left(\frac{n_2n_1^2}{c^3m}\right), \end{aligned}$$

where again $S(a,b;r)$ is the classical Kloomsterman sum.

Since $\Psi_1(x)/x$ has very similar asymptotic behavior as Ψ_0, we only need to deal the case with Ψ_0. The asymptotics of $\Psi_0(x)$ are included in the following lemma.

Lemma 2.7. *(Ren-Ye, [27]) Suppose that ψ is a fixed smooth function of compact support on $[X,2X]$ where $X>0$. Then for $x>0$, $xX \gg 1$, $K \geq 1$, we have*

$$\begin{aligned} \Psi_0(x) &= 2\pi^3 xi \int_0^\infty \psi(y)\sum_{j=1}^K \frac{c_j\cos(6\pi(xy)^{1/3}) + d_j \sin(6\pi(xy)^{1/3})}{(xy)^{j/3}}dy \\ &\quad + \mathcal{O}\left((xX)^{\frac{2-K}{3}}\right), \end{aligned}$$

where c_j and d_j are constants with $c_1 = 0$ and $d_1 = -2/\sqrt{3\pi}$.

2.4 Stationary Phase Argument

Similar to the weighted first derivative test we mentioned in section 2.1, in McKee-Sun-Ye [20], the authors improved the traditional weighted stationary phase integral lemma in Huxley [6]. We stated both statements in their entirety below for comparison.

Lemma 2.8. *(Huxley, [6]) Let $f(x)$ be a real function, four times continuously differentiable for $\alpha \leq x \leq \beta$, and let $g(x)$ be a real function, three times continuously differentiable for $\alpha \leq x \leq \beta$. Suppose that there are positive parameters M, N, T, U, with*

$$M \geq \beta - \alpha, \qquad N \geq M/\sqrt{T},$$

and positive constants C_r such that, for $\alpha \leq x \leq \beta$,

$$\left|f^{(r)}(x)\right| \leq C_r T/M^r, \qquad \left|g^{(s)}(x)\right| \leq C_s U/N^s,$$

for $r = 2, 3, 4$ and $s = 0, 1, 2, 3,$ and

$$f''(x) \geq T/C_2 M^2.$$

Suppose also that $f'(x)$ changes sign from negative to positive at a point $x = \gamma$ with

$\alpha < \gamma < \beta$. If T is sufficiently large in terms of the constants C_r, then we have

$$
\begin{aligned}
&\int_\alpha^\beta g(x)e(f(x))dx \\
&= \frac{g(\gamma)e(f(\gamma)+1/8)}{\sqrt{f''(\gamma)}} + \frac{g(\beta)e(f(\beta))}{2\pi i f'(\beta)} - \frac{g(\alpha)e(f(\alpha))}{2\pi i f'(\alpha)} \\
&+\mathcal{O}\left(\frac{M^4U}{T^2}\left(1+\frac{M}{N}\right)^2\left(\frac{1}{(\gamma-\alpha)^3}+\frac{1}{(\beta-\gamma)^3}\right)\right) \\
&+\mathcal{O}\left(\frac{MU}{T^{3/2}}\left(1+\frac{M}{N}\right)^2\right).
\end{aligned}
$$

The implies constants are constructed from the constants C_r.

Lemma 2.9. *(McKee-Sun-Ye, [20]) Let $\sigma(t)$ be a real-valued function $2n+3$ times continuously differentiable for $a \le t \le b$, and let $g(t)$ be a real-valued function $2n+1$ times continuously differentiable for $a \le t \le b$. Suppose $\sigma'(t)$ changes signs only at $t = \gamma$, from negative to positive, with $a < \gamma < b$. Define $H_k(t)$ by*

$$
H_1(t) = \frac{g(t)}{2\pi i \sigma'(t)}, \quad H_i(t) = -\frac{H'_{i-1}(t)}{2\pi i \sigma'(t)}
$$

for $i = 2, \dots, n$. Suppose that there are positive parameters M, N, T, U with $M > b-a$ and positive constants C_r such that for $a \le t \le b$,

$$
\left|\sigma^{(r)}(t)\right| \le C_r \frac{T}{M^r}, \quad \sigma''(t) \ge \frac{T}{C_2 M^2}, \quad \left|g^{(s)}(t)\right| \le C_s \frac{U}{N^s}
$$

for $r = 2, \ldots, 2n+3$ *and* $s = 0, \ldots, 2n+1$. *Define*

$$\Delta = \min\left\{ \frac{\log 2}{C_2}, \frac{1}{C_2^2 \max\limits_{2 \le k \le 2n+3} \{C_k\}} \right\}.$$

If T *is sufficiently large such that* $T^{\frac{1}{2n+3}}\Delta > 1$, *we have for* $n \geq 2$ *that*

$$\begin{aligned}
&\int_a^b g(t) e(\sigma(t)) dt \\
= \quad & \frac{e(\sigma(\gamma)+1/8)}{\sqrt{\sigma''(\gamma)}} \left(g(\gamma) + \sum_{j=1}^{n} \varpi_{2j} \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \right) + \left[e(\sigma(t)) \cdot \sum_{i=1}^{n+1} H_i(t) \right]_a^b \\
& + \mathcal{O}\left(\frac{UM^{2n+5}}{T^{n+2} N^{n+2}} \left(\frac{1}{(\gamma-a)^{n+2}} + \frac{1}{(b-\gamma)^{n+2}} \right) \right) \\
& + \mathcal{O}\left(\frac{UM^{2n+4}}{T^{n+2}} \left(\frac{1}{(\gamma-a)^{2n+3}} + \frac{1}{(b-\gamma)^{2n+3}} \right) \right) \\
& + \mathcal{O}\left(\frac{UM^{2n+4}}{T^{n+2} N^{2n}} \left(\frac{1}{(\gamma-a)^{3}} + \frac{1}{(b-\gamma)^{3}} \right) \right) \\
& + \mathcal{O}\left(\frac{U}{T^{n+1}} \left(\frac{M^{2n+2}}{N^{2n+1}} + M \right) \right).
\end{aligned}$$

Here

$$\begin{aligned}
\varpi_{2\nu} &= \eta_{2\nu} + \sum_{\ell=0}^{2\nu-1} \eta_\ell \sum_{k=1}^{2\nu-\ell} \frac{C_{2\nu,\ell,k}}{\lambda_2^k} \sum_{\substack{3 \le n_1, \ldots, n_k \le 2n+3 \\ n_1 + \cdots + n_k = 2\nu - \ell + 2k}} \lambda_{n_1} \cdots \lambda_{n_k}, \\
\eta_\ell &= \frac{g^{(\ell)}(\gamma)}{\ell!} \quad \text{for } 0 \le \ell \le 2n, \\
\lambda_\ell &= \frac{\sigma^{(\ell)}(\gamma)}{\ell!} \quad \text{for } 2 \le \ell \le 2n+2,
\end{aligned}$$

and $C_{2\nu,\ell,k}$ *are absolute constants.*

CHAPTER 3
BOUNDS IN THE AVERAGE SUMS

In this chapter, we will prove the main result Theorem 1.1.

3.1 Weighted Summation

Suppose $K^{\varepsilon} \leq L = K^{u} \leq K^{1-\varepsilon}$ and $K \to \infty$. Let $h(r)$ satisfies the conditions presented in Lemma 2.5, i.e. $h(r)$ is an even holomorphic function in the strip $|Im(r)| \leq 1/2 + \varepsilon$ and $h(r) \ll (|r|+1)^{-N}$ for arbitrary $N > 0$. We will estimate the bound for the weighted summation of $\mathcal{S}_X(f \times g_j, \alpha, \beta)$ in (1.1), which is of the form:

$$\sum_{g_j} \left[h\left(\frac{k_j - K}{L}\right) + h\left(-\frac{k_j + K}{L}\right) \right] \mathcal{S}_X(f \times g_j, \alpha, \beta). \tag{3.1}$$

First, we rewrite the $\mathcal{S}_X(f \times g_j, \alpha, \beta)$ with the right-hand side of its definition in (1.1). Thus, the weighted summation goes to

$$\sum_{n} A_f(n,1) e(\alpha n^{\beta}) \phi\left(\frac{n}{X}\right) \sum_{g_j} \left[h\left(\frac{k_j - K}{L}\right) + h\left(-\frac{k_j + K}{L}\right) \right] \lambda_{g_j}(n). \tag{3.2}$$

With the defined normalization $\bar{\lambda}_{g_j}(1) = 1$, we then apply the Kuznetsov trace formula to the inner sum of the above expression (3.2):

$$\begin{aligned} &\sum_{g_j}\left[h\left(\frac{k_j-K}{L}\right)+h\left(-\frac{k_j+K}{L}\right)\right]\lambda_{g_j}(n)\bar{\lambda}_{g_j}(1) \\ =\ &\frac{\delta_{1,n}}{\pi^2}\int_{\mathbb{R}}\tanh(\pi r)\left[h\left(\frac{r-K}{L}\right)+h\left(-\frac{r+K}{L}\right)\right]rdr \qquad (3.3)\\ &-\frac{1}{\pi}\int_{\mathbb{R}}d_{ir}(n)d_{ir}(1)\left[h\left(\frac{r-K}{L}\right)+h\left(-\frac{r+K}{L}\right)\right]\frac{1}{|\zeta(1+2ir)|^2}dr \qquad (3.4)\\ &+\int_{\mathbb{R}}J_{2ir}\left(\frac{4\pi\sqrt{n}}{c}\right)\left[h\left(\frac{r-K}{L}\right)+h\left(-\frac{r+K}{L}\right)\right]\frac{r}{\cosh(\pi r)}dr \\ &\times 2i\pi^{-1}\sum_{c\geq 1}\frac{S(n,1;c)}{c}, \qquad (3.5)\end{aligned}$$

where $d_\nu(n)=\sum_{ab=|n|}(a/b)^\nu$ is the division function, $\zeta(s)$ is the Riemann zeta function, and $S(n,m;c)$ is the classical Kloosterman sum.

For term (3.4), we break the integral into two:

$$\begin{aligned} &-\frac{1}{\pi}\int_{\mathbb{R}}d_{ir}(n)d_{ir}(1)h\left(\frac{r-K}{L}\right)\frac{1}{|\zeta(1+2ir)|^2}dr \qquad (3.6)\\ &-\frac{1}{\pi}\int_{\mathbb{R}}d_{ir}(n)d_{ir}(1)h\left(-\frac{r+K}{L}\right)\frac{1}{|\zeta(1+2ir)|^2}dr \qquad (3.7)\end{aligned}$$

For the first integral (3.6), let $u=\frac{r-K}{L}$; and for the second one (3.7), let $u=-\frac{r+K}{L}$. By $d_{ir}=\bar{d}_{ir}$ and $\zeta(\bar{s})=\bar{\zeta}(s)$, the term (3.4) goes to

$$-\frac{2L}{\pi}\int_{\mathbb{R}}d_{i(uL+K)}(n)d_{i(uL+K)}(1)\frac{h(u)}{|\zeta(1+2i(uL+K))|^2}du.$$

Note that $|\zeta(1+2ir)|\geq c\log^{-2/3}(2+|r|)$ for some $c>0$. Term (3.4) contributes to the weighted summation (3.1) as

$$-\frac{2L}{\pi}\int_{\mathbb{R}}\sum_{n} A_f(n,1)\phi\left(\frac{n}{X}\right) d_{i(uL+K)}(n)e(\alpha n^{\beta}) \times\frac{h(u)}{|\zeta(1+2i(uL+K))|^2}du, \tag{3.8}$$

which is asymptotically bounded by $LK^{\varepsilon}X^{1+\varepsilon}$ by the Rankin-Selberg methods.

Now, we turn to the diagonal term (3.3). Since it is nonzero only if $n = 1$, this diagonal term contributes to the sum (3.1) as

$$\begin{aligned} \mathcal{D} \;:=\; & A_f(1,1)e(\alpha)\phi\left(\frac{1}{X}\right)\frac{1}{\pi^2} \\ & \times\int_{\mathbb{R}}\tanh(\pi r)\left[h\left(\frac{r-K}{L}\right)+h\left(-\frac{r+K}{L}\right)\right]rdr. \end{aligned} \tag{3.9}$$

However, $\mathcal{D} = 0$ since $\phi\left(\frac{1}{X}\right) = 0$, as $X \to \infty$.

For finding the asymptotics for the off-diagonal term (3.5), we follow Liu-Ye

[18] closely. Let $x = \frac{4\pi\sqrt{n}}{c}$ and define

$$\begin{aligned} V_{K,L}(x) &:= \int_{\mathbb{R}} J_{2ir}(x)\left[h\left(\frac{r-K}{L}\right)+h\left(-\frac{r+K}{L}\right)\right]\frac{r}{\cosh(\pi r)}dr \\ &= \frac{1}{2}\int_{\mathbb{R}} J_{2ir}(x)\left[h\left(\frac{r-K}{L}\right)+h\left(-\frac{r+K}{L}\right)\right]\frac{r}{\cosh(\pi r)}dr \\ &\quad -\frac{1}{2}\int_{\mathbb{R}} J_{-2ir}(x)\left[h\left(\frac{r-K}{L}\right)+h\left(-\frac{r+K}{L}\right)\right]\frac{r}{\cosh(\pi r)}dr \\ &= \frac{1}{2}\int_{\mathbb{R}} [J_{2ir}(x)-J_{-2ir}(x)] \\ &\quad \times\left[h\left(\frac{r-K}{L}\right)+h\left(-\frac{r+K}{L}\right)\right]\frac{r}{\cosh(\pi r)}dr \\ &= \frac{1}{2}\int_{\mathbb{R}} \frac{J_{2ir}(x)-J_{-2ir}(x)}{\sinh(x)} \\ &\quad \times\left[h\left(\frac{r-K}{L}\right)+h\left(-\frac{r+K}{L}\right)\right]\tanh(\pi r) r dr. \end{aligned} \tag{3.10}$$

The assumptions on $h(r)$ are equivalent to state that the $h(r)$ is negligible for r outside $(K-L^{1+\varepsilon}, K+L^{1+\varepsilon})$ and also for r outside $(-K-L^{1+\varepsilon}, -K+L^{1+\varepsilon})$. Since the h functions on the right-hand side of (3.10) isolate r to $\pm K$, we can remove $\tanh(\pi r)$ by getting a negligible $\mathcal{O}(K^{-N})$ for any $N>0$. By Parseval's theorem, (3.10) goes to

$$\begin{aligned} V_{K,L}(x) &= \frac{1}{2}\int_{\mathbb{R}}\left(\frac{J_{2ir}(x)-J_{-2ir}(x)}{\sinh(x)}\right)^{\wedge}(y) \\ &\quad \times\left[h\left(\frac{r-K}{L}\right)|r|+h\left(-\frac{r+K}{L}\right)|r|\right]^{\wedge}(-y)dy \\ &\quad +\mathcal{O}(K^{-N}). \end{aligned} \tag{3.11}$$

Then, we apply the following formula in Bateman[1] to (3.11)

$$\left(\frac{J_{2ir}(x) - J_{-2ir}(x)}{\sinh(x)}\right)^{\wedge}(y) = -i\cos(x\cosh(\pi y)).$$

With a variable change from $-y$ to y in (3.11), we have

$$\begin{aligned} V_{K,L}(x) &= \frac{1}{2i}\int_{\mathbb{R}} \cos(x\cosh(\pi y)) \\ &\times \left[h\left(\frac{r-K}{L}\right)|r| + h\left(-\frac{r+K}{L}\right)|r|\right]^{\wedge}(y)dy \\ &+\mathcal{O}(K^{-N}). \end{aligned} \tag{3.12}$$

Then, we write out the Fourier transforms of $[h+h]$ term in (3.12) explicitly.

$$\begin{aligned} &\left[h\left(\frac{r-K}{L}\right)|r| + h\left(-\frac{r+K}{L}\right)|r|\right]^{\wedge}(y) \\ =& \int_{\mathbb{R}} h\left(\frac{r-K}{L}\right)|r|e(ry)dr + \int_{\mathbb{R}} h\left(-\frac{r+K}{L}\right)|r|e(ry)dr \\ =& \int_{\mathbb{R}} h(u)|uL+K|e\left((uL+K)y\right)Ldu \\ &+\int_{\mathbb{R}} h(u)|uL+k|e\left(-(uL+K)y\right)(-L)du \\ =& \ e(-Ky)L(h(u)|uL+K|)^{\wedge}(-Ly) \\ &+e(Ky)L(h(u)|uL+K|)^{\wedge}(Ly), \end{aligned}$$

where we did variable changes by letting $u = \pm\frac{r-K}{L}$. Since $L \leq K^{1-\varepsilon}$ for some $\varepsilon > 0$

and $h(u)$ isolates u to $\mathcal{O}(1)$, we may drop the absolute value. Thus, (3.12) goes to

$$\begin{aligned} V_{K,L}(x) &= \frac{1}{2i}\int_{\mathbb{R}} \cos(x\cosh(\pi y)) \\ &\quad \times[e(-Ky)L\,(h(u)(uL+K))^\wedge\,(-Ly) \\ &\quad +e(Ky)L\,(h(u)(uL+K))^\wedge\,(Ly)]dy \\ &\quad +\mathcal{O}(K^{-N}). \end{aligned}$$

Letting $y = t/L$,

$$\begin{aligned} V_{K,L}(x) &= \frac{1}{2i}\int_{\mathbb{R}} \cos\left(x\cosh\left(\frac{\pi t}{L}\right)\right) \\ &\quad \times\left[e\left(-\frac{Kt}{L}\right)(h(u)(uL+K))^\wedge(-t)\right. \\ &\quad \left.+e\left(\frac{Kt}{L}\right)(h(u)(uL+K))^\wedge(t)\right]dt \\ &= -i\int_{\mathbb{R}} \cos\left(x\cosh\left(\frac{\pi t}{L}\right)\right) \\ &\quad \times e\left(\frac{tK}{L}\right)(h(u)(uL+K))^\wedge(t)dt \\ &= \frac{1}{2i}(W_{K,L}(x)+W_{K,L}(-x)), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} W_{K,L}(x) &= \int_{\mathbb{R}} e\left(\frac{tk}{L}+\frac{x}{2\pi}\cosh\left(\frac{\pi t}{L}\right)\right)(h(u)(uL+K))^\wedge(t)dt \\ &:= \int_{\mathbb{R}} e(\sigma(t))g(t)dt. \end{aligned} \tag{3.14}$$

3.2 Application of a Voronoi's Summation Formula

Let η_1, η_2 and η_3 be independently either 1 or -1. By Lemma 2.4 in section 2.1, we only need to consider the case when $|x| \geq LK^{1-\varepsilon/2}$ in (3.14). Recall that we have assumed $x = \frac{4\pi\sqrt{n}}{c}$ when we define $V_{K,L}$ (3.10) in the previous section. Combine these two restrictions for x, we get $c \leq 4\pi\sqrt{n}/(LK^{1-\varepsilon/2})$. Moreover, since $\phi \in C_c^\infty(1,2)$ in (3.2),

$$1 \leq \frac{n}{X} \leq 2 \implies c \leq \frac{4\sqrt{2X}\pi}{LK^{1-\varepsilon/2}}.$$

By opening up the Kloosterman sum, a typical term in (3.5) is of the following form

$$\begin{aligned} R_\nu^{(\eta_1)} &= KL^{2\nu+1} \sum_{1 \leq c \leq \frac{4\sqrt{2X}}{LK^{1-\varepsilon/2}}} c^{\nu-\frac{1}{2}} \sum_{\substack{z \mod c \\ (z,c)=1}} e\left(\frac{\bar{z}}{c}\right) \\ &\quad \times \sum_n A_f(1,n) e(\alpha n^\beta) \phi\left(\frac{n}{X}\right) \\ &\quad \times e\left(\frac{nz}{c}\right) (c^2K^2 + 4\pi^2 n)^{-\frac{\nu}{2}-\frac{1}{4}} \\ &\quad \times H_{2\nu}\left(-\frac{\eta_1 L}{\pi} \sinh^{-1}\left(\frac{Kc}{2\pi\sqrt{n}}\right)\right) \\ &\quad \times e\left(\frac{\eta_1}{\pi c}\sqrt{c^2K^2 + 4\pi^2 n} - \frac{\eta_1 K}{\pi} \sinh^{-1}\left(\frac{Kc}{2\pi\sqrt{n}}\right)\right). \end{aligned} \tag{3.15}$$

Next, we apply the Voronoi's summation formula for GL_3 case we presented in Lemma 2.6 to the summation over n in (3.15) with

$$\begin{aligned} \psi(y) &= e(\alpha y^\beta) \phi\left(\frac{y}{X}\right) (c^2K^2 + 4\pi^2 y)^{-\frac{\nu}{2}-\frac{1}{4}} \\ &\quad \times H_{2\nu}\left(-\frac{\eta_1 L}{\pi} \sinh^{-1}\left(\frac{Kc}{2\pi\sqrt{y}}\right)\right) \\ &\quad \times e\left(\frac{\eta_1}{\pi c}\sqrt{c^2K^2 + 4\pi^2 y} - \frac{\eta_1 K}{\pi} \sinh^{-1}\left(\frac{Kc}{2\pi\sqrt{y}}\right)\right). \end{aligned} \tag{3.16}$$

This summation goes to

$$
\begin{aligned}
& \sum_n A_f(1,n) e\left(\frac{nz}{c}\right)\psi(n) \\
= \quad & \frac{c\pi^{-5/2}}{4i}\sum_{n_1|c}\sum_{n_2>0}\frac{A_f(n_2,n_1)}{n_1n_2}S\left(\bar{z},n_2;\frac{c}{n_1}\right)\Psi^0_{0,1}\left(\frac{n_1^2n_2}{c^3}\right) \\
& +\frac{c\pi^{-5/2}}{4i}\sum_{n_1|c}\sum_{n_2>0}\frac{A_f(n_1,n_2)}{n_1n_2}S\left(\bar{z},-n_2;\frac{c}{n_1}\right)\Psi^1_{0,1}\left(\frac{n_1^2n_2}{c^3}\right),
\end{aligned} \tag{3.17}
$$

where for $k=0,1$,

$$
\Psi^k_{0,1}(x)=\Psi_0(x)+(-1)^k\frac{1}{x\pi^3 i}\Psi_1(x).
$$

And by Lemma 2.7 in section 2.3, for any fixed $r\geq 1$, $x=n_2n_1^2/c^3$, and $xX\gg 1$,

$$
\begin{aligned}
\Psi_0(x) \quad = \quad & 2\pi^3 xi\int_0^\infty \psi(y)\sum_{j=1}^r\frac{c_j\cos(6\pi(xy)^{1/3})+d_j\sin(6\pi(xy)^{1/3})}{(xy)^{j/3}}dy \\
& +\mathcal{O}\left((xX)^{\frac{2-r}{3}}\right) \\
= \quad & \pi^3\sum_{j=1}^r x^{1-\frac{j}{3}}\int_0^\infty \psi(y)\left[(ic_j+d_j)e(3(xy)^{\frac{1}{3}})+(ic_j-d_j)e(-3(xy)^{\frac{1}{3}})\right]\frac{dy}{y^{j/3}} \\
& +\mathcal{O}\left((xX)^{\frac{2-r}{3}}\right) \\
= \quad & \pi^3\sum_{j=1}^r x^{1-\frac{j}{3}}\int_0^\infty e(\alpha y^\beta)\phi\left(\frac{y}{X}\right)\left(c^2K^2+4\pi^2 y\right)^{-\frac{\nu}{2}-\frac{1}{4}} \\
& \times H_{2\nu}\left(-\frac{\eta_1 L}{\pi}\sinh^{-1}\left(\frac{Kc}{2\pi\sqrt{y}}\right)\right) \\
& \times e\left(\frac{\eta_1}{\pi c}\sqrt{c^2K^2+4\pi^2 y}-\frac{\eta_1 K}{\pi}\sinh^{-1}\left(\frac{Kc}{2\pi\sqrt{y}}\right)\right) \\
& \times\left[(ic_j+d_j)e(3(xy)^{\frac{1}{3}})+(ic_j-d_j)e(-3(xy)^{\frac{1}{3}})\right]\frac{dy}{y^{j/3}} \\
& +\mathcal{O}\left((xX)^{\frac{2-r}{3}}\right)
\end{aligned} \tag{3.18}
$$

Recall that c_j and d_j are constants depending on the Langlands parameters with $c_1 = 0$ and $d_1 = -2/\sqrt{3\pi}$. Note that by doing change of variables, $x^{-1}\Psi_1(x)$ has similar asymptotic to Ψ_0, so we only need to deal with Ψ_0. Let $a_j = ic_j + d_j$, $b_j = ic_j - d_j$, and set $y = t^3X$ with $dy = 3t^2Xdt$. For $x = n_2n_1^2/c^3$, (3.18) goes to

$$\begin{aligned}
\Psi_0(x) &= 3\pi^3 \sum_{j=1}^{r} x^{1-\frac{j}{3}} X^{-\frac{\nu}{2}-\frac{j}{3}+\frac{3}{4}} \\
&\quad \times \int_1^{2^{1/3}} \phi(t^3) \left(\left(\frac{cK}{\sqrt{X}} \right)^2 + 4\pi^2 t^3 \right)^{-\frac{\nu}{2}-\frac{1}{4}} \\
&\quad \times H_{2\nu} \left(-\frac{\eta_1 L}{\pi} \sinh^{-1} \left(\frac{Kc}{2\pi\sqrt{t^3X}} \right) \right) \\
&\quad \times e \left(\frac{\eta_1\sqrt{X}}{\pi c} \sqrt{\left(\frac{cK}{\sqrt{X}} \right)^2 + 4\pi^2 t^3} - \frac{\eta_1 K}{\pi} \sinh^{-1} \left(\frac{Kc}{2\pi\sqrt{t^3X}} \right) \right) \\
&\quad \times e(\alpha X^{\beta} t^{3\beta}) \left[a_j e(3(xt^3X)^{\frac{1}{3}}) + b_j e(-3(xt^3X)^{\frac{1}{3}}) \right] t^{2-j} dt \\
&\quad + \mathcal{O}\left((xX)^{\frac{2-r}{3}} \right) \\
&= 3\pi^3 \sum_{j=1}^{r} x^{1-\frac{j}{3}} X^{-\frac{\nu}{2}-\frac{j}{3}+\frac{3}{4}} \left[a_j I_j^{\eta_1\eta_3+} + b_j I_j^{\eta_1\eta_3-} \right] \\
&\quad + \mathcal{O}\left((xX)^{\frac{2-r}{3}} \right).
\end{aligned} \tag{3.19}$$

Here, we define

$$
\begin{aligned}
I_j^{\eta_1\eta_2\eta_3}(t) &:= \int_1^{2^{1/3}} t^{2-j}\phi(t^3)\left(\left(\frac{cK}{\sqrt{X}}\right)^2+4\pi^2t^3\right)^{-\frac{\nu}{2}-\frac{1}{4}} \\
&\quad\times H_{2\nu}\left(-\frac{\eta_1 L}{\pi}\sinh^{-1}\left(\frac{Kc}{2\pi\sqrt{t^3X}}\right)\right) \\
&\quad\times e\left(\frac{\eta_1\sqrt{X}}{\pi c}\sqrt{\left(\frac{cK}{\sqrt{X}}\right)^2+4\pi^2t^3}-\frac{\eta_1 K}{\pi}\sinh^{-1}\left(\frac{Kc}{2\pi\sqrt{t^3X}}\right)\right) \\
&\quad\times e\left(\eta_3|\alpha|X^\beta t^{3\beta}+3\eta_2(xX)^{1/3}t\right)dt. \\
&= \int_1^{2^{1/3}} G_j(t)e(\theta(t))dt,
\end{aligned}
\tag{3.20}
$$

where

$$
G_j(t) := t^{2-j}\phi(t^3)\left(\left(\frac{cK}{\sqrt{X}}\right)^2+4\pi^2t^3\right)^{-\frac{\nu}{2}-\frac{1}{4}} H_{2\nu}(\gamma), \tag{3.21}
$$

and

$$
\begin{aligned}
\theta(t) &= \eta_3|\alpha|X^\beta t^{3\beta}+3\eta_2(xX)^{1/3}t \\
&\quad+\frac{\eta_1\sqrt{X}}{\pi c}\sqrt{\left(\frac{cK}{\sqrt{X}}\right)^2+4\pi^2t^3}-\frac{\eta_1 K}{\pi}\sinh^{-1}\left(\frac{Kc}{2\pi\sqrt{t^3X}}\right).
\end{aligned}
\tag{3.22}
$$

Recall that $H_{2\nu}(\gamma)$ is defined as (2.4) in Lemma 2.4 and $x = n_2n_1^2/c^3$ is from (3.18).

3.3 Application of Derivative Tests

In order to apply derivative tests to the integrals in (3.20), we need to bound derivatives of $G_j(t)$ and $\theta(t)$ in order to satisfy assumptions of derivative tests. We will deal with $G_j(t)$ and $\theta(t)$ separately under the assumption in the main result in Theorem 1.1, which is $LK \geq \sqrt{X}$.

First, we consider $G_j(t)$. Since $1 \leq t \leq 2^{1/3}$, $(t^{2-j})^{(s)} \ll 1$ and $(\phi(t^3))^{(s)} \ll 1$. Moreover, since $c \leq \frac{4\sqrt{2X}\pi}{LK^{1-\varepsilon/2}}$,

$$\frac{cK}{\sqrt{X}} \leq \frac{4\sqrt{2}\pi K^{\frac{\varepsilon}{2}}}{L} \leq 4\sqrt{2}\pi.$$

Thus,

$$\Big(\frac{cK}{\sqrt{X}}\Big)^2 + 4\pi^2 t^3 \asymp \frac{1}{X},$$

which implies

$$\left(\left(\Big(\frac{cK}{\sqrt{X}}\Big)^2 + 4\pi^2 t^3\right)^{-\frac{\nu}{2}-\frac{1}{4}}\right)^{(s)} \ll X^{\frac{\nu}{2}+\frac{1}{4}} \ll 1,$$

for $s \geq 0$.

Now, we compute the derivatives of $(H_{2\nu}(\gamma))^{(s)}$. The first term in $H_{2\nu}(\gamma)$ defined in (2.4), $H^{(2\nu)}(\gamma)/(2\nu)!$ is negligible since $H(t)$ function defined in (2.6) is of rapid decay, i.e. $\frac{d^s}{dz^s}H^{(l)}(z) \ll 1$. Also the term γ defined in (2.5) has the first derivative:

$$\gamma' = 3\eta_1 \frac{LKc}{\sqrt{X}} \frac{1}{2\pi t\sqrt{\Big(\frac{cK}{\sqrt{X}}\Big)^2 + 4\pi^2 t^3}} \asymp \frac{LKc}{\sqrt{X}},$$

and the higher derivatives of γ are of the same size. Thus, by the chain rule,

$$\frac{d^s}{dt^s}H^{(l)}(\gamma) \ll \Big(\frac{LKc}{\sqrt{X}}\Big)^s.$$

Then, in $H_{2\nu}(\gamma)$ defined in (2.4), there is a sum of products in $H_{2\nu}(\gamma)$ of the form

$$\sum_{l=0}^{2\nu-1} \frac{H^{(l)}(\gamma)}{l!} \sum_{k=1}^{2\nu-l} \frac{2^k C_{2\nu,l,k}}{\sigma^{(2)}(\gamma)} \sum_{\substack{3\leq n_1,\ldots,n_k \leq 2n+3 \\ n_1+\cdots+n_k=2\nu-l+2k}} \frac{\sigma^{(n_1)}(\gamma)\cdots\sigma^{(n_k)}(\gamma)}{n_1!\cdots n_k!},$$

which can be treated as $H^{(l)}\sigma^{(n_1)}\cdots\sigma^{(n_k)}(\sigma^{(2)})^{-k}$ by neglecting constant terms, where $n_i \geq 3$, $k \geq 1$, and

$$\sigma(z) = \frac{zK}{L} + \frac{2\eta_1\sqrt{t^3X}}{c}\cosh\left(\frac{\pi z}{L}\right),$$

as defined in (2.7) in Section 2.1. For the derivative of the above sum of products, we start from the derivatives of $\sigma(t)$. For $r \geq 1$,

$$\sigma^{(r)}(z) = \begin{cases} \frac{2\eta_1\sqrt{t^3X}}{c}\left(\frac{\pi}{L}\right)^r \cosh\left(\frac{\pi z}{L}\right) & r \text{ is even;} \\ \\ \frac{2\eta_1\sqrt{t^3X}}{c}\left(\frac{\pi}{L}\right)^r \sinh\left(\frac{\pi z}{L}\right) & r \text{ is odd.} \end{cases} \tag{3.23}$$

Evaluate (3.23) at γ in (2.5) yields,

$$\sigma^{(r)}(\gamma) = \begin{cases} \frac{\eta_1\sqrt{X}}{\pi c}\left(\frac{\pi}{L}\right)^r \sqrt{\left(\frac{cK}{\sqrt{X}}\right)^2 + 4\pi^2 t^3} \asymp \frac{\sqrt{X}}{cL^r} & r \text{ is even;} \\ \\ -\frac{K}{\pi}\left(\frac{\pi}{L}\right)^r \asymp \frac{K}{L^r} & r \text{ is odd.} \end{cases} \tag{3.24}$$

Then, $\frac{cK}{\sqrt{X}} \leq 4\sqrt{2}\pi$, which is equivalent as $\frac{\sqrt{X}}{cL^r} \geq \frac{K}{L^r}$ implies

$$\sigma^{(r)}(\gamma) \ll \frac{\sqrt{X}}{cL^r}. \tag{3.25}$$

Hence, each derivative of $\sigma^{(n_i)}(\gamma)$ produces L^{-1}, as does each derivative of $(\sigma^{(2)}(\gamma))^{-k}$, and by the chain rule we get the additional factor $LKc/\sqrt{X}$. Specifically, since $n_1 + n_2 + \cdots n_k = 2\nu - l + 2k$ with each $n_i \geq 3$ as we defined in (2.4),

$$\frac{d^{r_1}}{dt^{r_1}}[\sigma^{n_1}(\gamma)\cdots\sigma^{(n_k)}(\gamma)] \ll \frac{X^{\frac{k}{2}}}{c^k L^{2\nu-l+2k+r_1}}\left(\frac{LKc}{\sqrt{X}}\right)^{r_1}$$

and

$$\frac{d^{r_2}}{dt^{r_2}}[(\sigma^{(2)}(\gamma))^{-k}] \ll \frac{c^k L^{2k-r_2}}{X^{\frac{k}{2}}}\left(\frac{LKc}{\sqrt{X}}\right)^{r_2},$$

which implies

$$\frac{d^r}{dt^r}\left[\frac{\sigma^{n_1}(\gamma)\cdots\sigma^{(n_k)}(\gamma)}{(\sigma^{(2)}(\gamma))^k}\right] \ll \frac{1}{L^{2\nu-l}}\left(\frac{Kc}{\sqrt{X}}\right)^r \ll \frac{1}{L^{2\nu-l}}$$

for $r, r_1, r_2 \geq 0$. So, under the assumptions $K^\varepsilon \leq L = K^\mu \leq K^{1-\varepsilon}$ and $s \geq 0$, the desired derivatives of the sum of products term in $H_{2\nu}(\gamma)$ defined in (2.4) is bounded:

$$\frac{d^s}{dt^s}H^{(l)}\sigma^{(n_1)}\cdots\sigma^{(n_k)}(\sigma^{(2)})^{-k} \ll (LKc/\sqrt{X})^s,$$

which implies

$$\frac{d^s}{dt^s}H_{2\nu}(\gamma) \ll \left(\frac{LKc}{\sqrt{X}}\right)^s \ll K^{\mu s}.$$

Therefore,

$$G_j^{(s)}(t) \ll K^{\mu s}. \tag{3.26}$$

For the phase term $\theta(t)$, its first derivative with $x = n_2 n_1^2/c^3$ is computed as follows:

$$\begin{aligned}\theta'(t) &= 3\eta_3|\alpha|\beta X^{\beta} t^{3\beta-1} + 3\eta_2 \left(\frac{n_2 n_1^2}{c^3} X\right)^{\frac{1}{3}} \\ &\quad + \frac{3\eta_1\sqrt{X}}{2\pi c}\sqrt{\left(\frac{Kc}{t\sqrt{X}}\right)^2 + 4\pi^2 t}. \end{aligned} \tag{3.27}$$

Suppose $\beta = \frac{1}{3}$ and set $\theta'(t) = 0$, we obtain

$$\sqrt{\left(\frac{Kc}{t\sqrt{X}}\right)^2 + 4\pi^2 t} = -\frac{2}{3}\pi c\eta_1 \left(\eta_3|\alpha| + 3\eta_2 \frac{n_2^{\frac{1}{3}} n_1^{\frac{2}{3}}}{c}\right) X^{-\frac{1}{6}}. \tag{3.28}$$

Recall that we have $\frac{cK}{\sqrt{X}} \leq 4\sqrt{2}\pi$ and $1 \leq t \leq 2^{1/3}$. Thus, the left-hand side of (3.28) is of the size $\mathcal{O}(1)$. This means we will get stationary points if the long coefficient of $X^{-1/6}$ on the right-hand side of (3.28) is very close to $X^{1/6}$. Thus, we can divide the case into three:

- case 1: $\frac{|\alpha|}{3}c + n_2^{\frac{1}{3}} n_1^{\frac{2}{3}} \asymp_+ X^{\frac{1}{6}}$;
- case 2: $\frac{|\alpha|}{3}c - n_2^{\frac{1}{3}} n_1^{\frac{2}{3}} \asymp_+ X^{\frac{1}{6}}$;
- case 3: $-\frac{|\alpha|}{3}c + n_2^{\frac{1}{3}} n_1^{\frac{2}{3}} \asymp_+ X^{\frac{1}{6}}$.

In other words, if we apply the weighted first derivative test stated in Lemma 2.3 to the integrals $I_j^{\eta_1\eta_2\eta_3}(t)$ in (3.20), the integrals will be negligible for all cases except the above three. Recall the bound of derivatives of the θ term in (3.22), $\theta^{(r)}(t) \asymp \sqrt{X}/c$ for $r \geq 2$. Also put the bound we obtained for the derivatives of G_j term in (3.26) into consideration, we may choose $T = \sqrt{X}/c$, $M = 1$, $U = 1$ and $N = K^{-\mu}$ in Lemma 2.3. Also check that when $1 \leq t \leq 2^{1/3}$, $M = 1 \geq 2^{1/3} - 1$. Then, three error terms showed up by applying the weighted first derivative test in Lemma 2.3 to $I_j^{\eta_1\eta_2\eta_3}(t)$ are

$$\mathcal{O}\left(K^{\mu}\sum_{j=1}^{[n/2]}\left(\frac{c}{\sqrt{X}}\right)^{n+1}\sum_{t=j}^{n-j}K^{\mu(n-j-t)}\right)$$
$$+\mathcal{O}\left((K^{\mu}+1)K^{n\mu}\left(\frac{c}{\sqrt{X}}\right)^{n+1}\right)$$
$$+\mathcal{O}\left(\sum_{j=1}^{n}\left(\frac{c}{\sqrt{X}}\right)^{n+1}\sum_{t=0}^{n-j}K^{\mu(n-j-t)}\right).$$

Therefore,

$$I_j^{\eta_1\eta_2\eta_3}(t) \ll \frac{c}{\sqrt{X}}.$$

So, what we left to do is to deal with the three exceptional cases defined above.

Recall that $c \leq 4\sqrt{2X}/LK^{1-\varepsilon/2}$ and $\theta''(t) \asymp \sqrt{X}/c$ are results we already obtained. Then, apply the second derivative test stated in Lemma 2.1 to $I_j^{\eta_1\eta_2\eta_3}(t)$, we have

$$I_j^{\eta_1\eta_2\eta_3}(t) \ll \frac{c^{1/2}}{X^{1/4}} \ll \frac{1}{\sqrt{LK^{1-\varepsilon/2}}},$$

which is a better bound than the trivial one $\mathcal{O}(1)$. Therefore, plug in all obtained bounds back to (3.19) with $x = n_2 n_1^2/c^3$ yields

$$\Psi_0\left(\frac{n_2 n_1^2}{c^3}\right) \ll \sum_{j=1}^{r}\left(\frac{n_2 n_1^2}{c^3}\right)^{1-\frac{j}{3}} c^{\frac{1}{2}} X^{-\frac{\nu}{2}+\frac{1}{2}-\frac{j}{3}} + \mathcal{O}\left(\left(\frac{n_2 n_1^2}{c^3} X\right)^{\frac{2-r}{3}}\right). \tag{3.29}$$

In case 1: $\frac{|\alpha|}{3}c + n_2^{\frac{1}{3}} n_1^{\frac{2}{3}} \asymp_+ X^{\frac{1}{6}}$, we have $c \ll X^{1/6}$ and $n_2 n_1^2 \ll X^{1/2}$. Then,

$$\frac{n_2 n_1^2}{c^3} X \gg n_2 n_1^2 X^{\frac{1}{2}} \gg X^{\frac{1}{2}}.$$

So, the summation among j in (3.29) is convergent. By picking up the first term, we have

$$\Psi_0\left(\frac{n_2 n_1^2}{c^3}\right) \ll \left(\frac{n_2 n_1^2}{c^3}\right)^{\frac{2}{3}} c^{\frac{1}{2}} X^{-\frac{\nu}{2}+\frac{1}{6}} \ll c^{-\frac{3}{2}} X^{-\frac{\nu}{2}+\frac{1}{2}}$$

Then, plug in all obtained bounds back to $R_\nu^{(\eta_1)}$ in (3.15) at the very beginning of section 3.2, we have

$$R_\nu^{(\eta_1)} \ll KL^{2\nu+1} \sum_{1 \le c \le \frac{4\sqrt{2X}}{LK^{1-\epsilon}}} c^{\nu+\frac{1}{2}} \sum_{\substack{z (\mathrm{mod}\ c) \\ (z,c)=1}} e\left(\frac{\bar{z}}{c}\right) \times \sum_{\substack{n_1 | c,\ n_2 > 0 \\ n_2 n_1^2 \le X^{1/2}}} \frac{A_f(n_2, n_1)}{n_1 n_2} S\left(\bar{z}, n_2; \frac{c}{n_1}\right) c^{-\frac{3}{2}} X^{-\frac{\nu}{2}+\frac{1}{2}}.$$

Note that the formulas (4.22) in Li [17] and (6.18) in Mckee-Sun-Ye [20] states that

$$\begin{aligned}
\sum_{\substack{0\le z<c\\(z,c)=1}} e\left(\frac{z}{c}\right) S\left(z,n_2;\frac{c}{n_1}\right) &= \sum_{\substack{u(\bmod cn_1^{-1})\\ u\bar{u}\equiv 1(\bmod cn_1^{-1})}} S(0,1+un_1;c)e\left(\frac{n_2\bar{u}}{cn_1^{-1}}\right)\\
&= \sum_{\substack{u(\bmod cn_1^{-1})\\ u\bar{u}\equiv 1(\bmod cn_1^{-1})}} e\left(\frac{n_2n_1\bar{u}}{c}\right)\sum_{\substack{z(\bmod c)\\(z,c)=1}} e\left(\frac{(1+un_1)z}{c}\right)\\
&\ll \frac{mc^{1+\varepsilon}}{n_1}.
\end{aligned}$$

Also note that by assuming $LK \ge \sqrt{X}$, we have $c \le 4\sqrt{2X}/LK^{1-\varepsilon/2} \ll K^{\varepsilon/2}$. And by the classical result that $\sum_{n_1,n_2}\frac{A_f(n_2,n_1)}{(n_2n_1^2)^s}$ converges absolutely for $s>1$, we have

$$\begin{aligned}
\sum_{\substack{n_1|c,\ n_2>0\\ n_2n_1^2\le\sqrt{X}}} \frac{|A_f(n_2,n_1)|}{n_2n_1^2} &\le \sum_{\substack{n_1,n_2\\ n_2n_1^2\le\sqrt{X}}} \frac{|A_f(n_2,n_1)|}{n_2n_1^2}\\
&\ll (\sqrt{X})^{\epsilon}.
\end{aligned}$$

Hence,

$$\begin{aligned}
R_\nu^{(\eta_1)} &\ll KL^{2\nu+1}\sum_{c\le K^{\varepsilon/2}} c^{\nu+\frac{1}{2}} \sum_{\substack{n_1,n_2\\ n_2n_1^2\le X^{1/2}}} \frac{|A_f(n_2,n_1)|}{n_2n_1^2} c^{1+\varepsilon}\\
&\quad \times c^{-\frac{3}{2}} X^{-\frac{\nu}{2}+\frac{1}{2}}\\
&\ll KL^{2\nu+1}\sum_{c\le K^{\varepsilon/2}} c^{\nu+\frac{1}{2}} c^{1+\varepsilon} c^{-\frac{3}{2}} X^{-\frac{\nu}{2}+\frac{1}{2}}\\
&\ll KL^{2\nu+1}X^{-\frac{\nu}{2}+\frac{1}{2}}(K^{\varepsilon})^{1+\nu+\varepsilon}\\
&\ll LK^{1+\varepsilon}X^{\frac{1}{2}+\varepsilon}\left(\frac{L^2K^{\varepsilon}}{\sqrt{X}}\right)^{\nu}
\end{aligned}$$

Under the assumption that $LK \geq \sqrt{X}$,

$$\sum_{\nu} R_{\nu}^{(\eta_1)} \ll LK^{1+\varepsilon} X^{\frac{1}{2}+\varepsilon}.$$

Combine with the bound for the continuous spectrum term (3.8) that we have obtained in section 2.1, which is $LK^{\varepsilon} X^{1+\varepsilon}$, we have

$$\sum_{K-L \leq k_j \leq K+L} S_X(f \times g_j, \alpha, \frac{1}{3}) \ll LK^{1+\varepsilon} X^{\frac{1}{2}+\varepsilon} + LK^{\varepsilon} X^{1+\varepsilon}.$$

For case 2: $\frac{|\alpha|}{3} c - n_2^{\frac{1}{3}} n_1^{\frac{2}{3}} \asymp_+ X^{\frac{1}{6}}$ and case 3: $-\frac{|\alpha|}{3} c + n_2^{\frac{1}{3}} n_1^{\frac{2}{3}} \asymp_+ X^{\frac{1}{6}}$, we have

$$c \in \frac{3 n_2^{\frac{1}{3}} n_1^{\frac{2}{3}}}{|\alpha|} + \mathcal{O}\left(X^{1/6}\right),$$

which implies

$$n_2^{\frac{1}{3}} n_1^{\frac{2}{3}} \in \frac{|\alpha| c}{3} + \mathcal{O}(X^{1/6}).$$

Thus, if $c \ll X^{1/6}$ and $n_2^{1/3} n_1^{2/3} \ll X^{1/6}$, we go back to case 1. Otherwise, for $c \not\ll X^{1/6}$, we have

$$\frac{n_2^{\frac{1}{3}} n_1^{\frac{2}{3}}}{c} \in \frac{|\alpha|}{3} + \mathcal{O}\left(\frac{X^{\frac{1}{6}}}{c}\right) \asymp 1.$$

Thus,

$$X \ll \frac{n_2 n_1^2}{c^3} X.$$

So, the summation among j in (3.29) is convergent for both cases and we can only look at the first term in this summation:

$$\Psi_0\left(\frac{n_2n_1^2}{c^3}\right) \ll \left(\frac{n_2n_1^2}{c^3}\right)^{\frac{2}{3}} c^{\frac{1}{2}}X^{-\frac{\nu}{2}+\frac{1}{6}}$$
$$\ll c^{\frac{1}{2}}X^{-\frac{\nu}{2}+\frac{1}{6}}$$

However, when we plug in the bound back to $R_\nu^{(\eta_1)}$ in (3.15), we found that there would be no c in $X^{1/6} \not\ll c \ll K^\varepsilon$ under the assumption that $LK \geq \sqrt{X}$. So, we do not need to consider these two cases until we remove the assumption.

REFERENCES

[1] H. Bateman, Tables of Integral Transforms, *McGraw-Hill*, New York, 1954.

[2] K. Czarnecki, Resonance sums for Rankin–Selberg products of $SL_m(\mathbb{Z})$ Maass cusp forms, *Journal of Number Theory*, **63** (2016), 359-374.

[3] A. M. Ernvall-Hytönen. On certain exponential sums related to $GL(3)$ cusp forms, *Comptes Rendus Mathematique*, **348** (1-2) (2010), 5-8.

[4] A. M. Ernvall-Hytönen, J. Jääsaari, and E. V. Vesalainen. Resonances and Ω-results for exponential sums related to Maass forms for $SL(n, Z)$, *Journal of Number Theory*, **153** (2015), 135-157.

[5] D. Goldfeld and X. Li, Voronoi formulas on $GL(n)$, *International Mathematics Research Notices*, (2006), DOI 10.1093/imrn/rnm144. MR2418857(2009b:11077)

[6] M. N. Huxley, Area, lattice points, and exponential sums, London Mathematical Society Monographs, New Series 13, *The Clarendon Press, Oxford University Press*, New York, 1996.

[7] H. Iwaniec, Spectral Methods of Automorphic Forms (2nd edition), Graduate Studies in Mathematics, Volume 53, *American Mathematical Society Providence*, Rhode Island, 2002.

[8] H. Iwaniec and E. Kowalski, Analytic number theory, American Mathematical Society Colloquium Publications, Volume 53, *American Mathematical Society Providence*, Rhode Island, 2004.

[9] H. Iwaniec, W. Z. Luo, and P. Sarnak, Low lying zeros of families of L-functions, *Publications Mathématiques de I'IHÉS*, **91** (2000), 55-131.

[10] J. Kaczorowski and A. Perelli, On the structure of the Selberg class VI: non-linear twists, *Acta Arithmetica*, **116** (2005), 315-341.

[11] H. Kim, Functoriality for the exterior square of GL_4 and symmetric fourth of GL_2, *Journal of the American Mathematical Society*, **16** (2002), 139-183.

[12] H. Kim and P. Sarnak, Refined estimates towards the Ramanujan and Selberg conjectures, appendix to [11].

[13] N. V. Kuznetsov, Petersson's conjecture for cusp forms of weight zero and Linnik's conjecture. Sums of Kloosterman sums, *Mathematics of the USSR-Sbornik*, **39** (1981), 299-342.

[14] Y. K. Lau, J. Liu, and Y. Ye, A new bound $k^{2/3+\varepsilon}$ for Rankin-Selberg L-functions for Hecke congruence subgroups, *International Mathematics Research Papers*, (2006) 178, Article ID 35090.

[15] Y. K. Lau, J. Liu, and Y. Ye, Subconvexity bounds for Rankin-Selberg L-functions for congruence subgroups, *Journal of Number Theory*, **121** (2006), 204-223.

[16] D. Laugwitz, translated by A. Shenitzer, Bernhard Riemann 1826-1866. Turning points in the conception of Mathematics, *Birkhäuser*, Boston, 2008.

[17] X. Li, Bounds for $GL(2) \times GL(3)$ L-functions and $GL(3)$ L-functions. *Annals of Mathematics*, **173** (2011), 301-336.

[18] J. Liu and Y. Ye, Subconvexity for Rankin-Selberg L-functions of Maass forms, *Geometric and Functional Analysis*, **12** (2002) 1296-1323.

[19] J. Liu and Y. Ye, Petersson and Kuznetsov trace formulas, AMS IP Studies in Advanced Mathematics. Volume 37, *American Mathematical Society Providence*, Rhode Island, 2006.

[20] M. McKee, H. Sun, and Y. Ye, Improved subconvexity bounds for $GL(2) \times GL(3)$ and $GL(3)$ L-functions by weighted stationary phase, *Transactions of the American Mathematical Society*, DOI 10.1090/tran/7159

[21] P. Michel, Analytic number theory and families of automorphic L-functions, in: Automorphic Forms and Applications, IAS/Park City Mathematics Series, volume 12, edited by P. Sarnak and F. Shahidi, *American Mathematical Society and Institute for Advanced Study* (2007), 179-295

[22] S. D. Miller and W. Schmid, Automorphic distributions, L-functions, and Voronoi summation for GL(3), *Annals of Mathematics*, **164** (2006), no.2, 423-488, DOI 10.4007/annals.2006.164.423. MR2247965 (2007j:11065)

[23] X. Ren and Y. Ye, Resonance between automorphic forms and exponential functions, *Science China Mathematics*, **53** (2010), 2463-2472.

[24] X. Ren and Y. Ye, Asymptotic Voronoi's summation formulas and their duality for $SL_3(\mathbb{Z})$, "Arithmetic in Shangri-La" -Proceedings of the 6th China-Japan Seminar on Number Theory held in Shanghai Jiaotong University, August 15-17, 2011, edited by S. Kanemitsu, H. Li, and J. Liu, Series on Number Theory and its Applications, Volume 9, *World Scientific*, Singapore, 2012.

[25] X. Ren and Y. Ye, Sums of Fourier coefficients of a Maass form for $SL_3(\mathbb{X})$ twisted by exponential functions. *Forum Math*, **26** (2014), 221-238.

[26] X. Ren and Y. Ye, Resonance and rapid decay of exponential sums of Fourier coefficients of a Maass form for $GL_m(\mathbb{Z})$, *Science China Mathematics*, **58** (2015), 1-20.

[27] X. Ren and Y. Ye, Resonance of automorphic forms for GL(3), *Transactions of the American Mathematical Society*, **367** (2015), 2137-2157.

[28] X. Ren and Y. Ye, Hyper-Kloosterman sums of different moduli and their applications to automorphic forms for $SL_m(\mathbb{Z})$, *Taiwanese Journal of Mathematics*, **20** (2016).

[29] N. Salazar and Y. Ye, Spectral square moments of a resonance sum for Maass forms, *Frontiers of Mathematics in China* (2017), DOI 10.1007/s11464-016-0621-0.

[30] P. Sarnak, Estimates for Rankin-Selberg L-functions and quantum unique ergodicity, *Journal of Functional Analysis* **184** (2001), 419-453.

[31] P. Savala, Computing the Laplace eigenvalue and level for Maass cusp forms, *Journal of Number Theory*, **173** (2017), 1-22.

[32] Q. Sun, On cusp form coefficients in nonlinear exponential sums, *The Quarterly Journal of Mathematics*, **61** (2010), 363-372.

[33] Q. Sun and Y. Wu, Exponential sums involving Maass forms, *Frontiers of Mathematics in China*, **9** (2014), 1349 - 1366.