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Dynamics of the energy critical nonlinear Schrödinger equation with inverse square potential

Kai Yang
University of Iowa

DYNAMICS OF THE ENERGY CRITICAL NONLINEAR SCHRÖDINGER EQUATION WITH INVERSE SQUARE POTENTIAL

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Kai Yang

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Thesis Supervisor: Professor Xiaoyi Zhang

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Kai Yang

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Applied Mathematical and Computational Sciences at the May 2017 graduation.

Thesis committee:

Xiaoyi Zhang, Thesis Supervisor

Tong Li

Jorgensen Palle

Gerhard Strohmer

Lihe Wang

To my loving family

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ABSTRACT

We consider the Cauchy problem for the focusing energy critical NLS with inverse square potential. The energy of the solution, which consists of the kinetic energy and potential energy, is conserved for all time. Due to the focusing nature, solution with arbitrary energy may exhibit various behaviors: it could exist globally and scatter like a free evolution, persist like a solitary wave, blow up at finite time, or even have mixed behaviors. Our goal in this thesis is to fully characterize the solution when the energy is below or at the level of the energy of the ground state solution W_a. Our main result contains two parts.

First, we prove that when the energy and kinetic energy of the initial data are less than those of the ground state solution, the solution exists globally and scatters.

Second, we show a rigidity result at the level of ground state solution. We prove that among all solutions with the same energy as the ground state solution, there are only two (up to symmetries) solutions W_a^+, W_a^- that are exponential close to W_a and serve as the threshold of scattering and blow-up. All solutions with the same energy will blow up both forward and backward in time if they go beyond the upper threshold W_a^+; all solutions with the same energy will scatter both forward and backward in time if they fall below the lower threshold W_a^-.

In the case of NLS with no potential, this type of results was first obtained by Kenig-Merle [26] and Duyckaerts-Merle [17]. However, as the potential has the same scaling as Δ, one can not expect to extend their results in a simple perturbative

way. We develop crucial spectral estimates for the operator $-\Delta + a/|x|^2$, we also rely heavily on the recent understanding of the operator $-\Delta + a/|x|^2$ in [28].

PUBLIC ABSTRACT

Nonlinear Schrödinger equation(NLS) with singular potential has applications in quantum mechanics and nonlinear optics. In this thesis, we study the focusing energy critical NLS with inverse square potential. The energy of the solution, which consists of kinetic energy and potential energy, is conserved for all time. The focusing nonlinearity represents the attractive force among all microscopic particles. Depending on the competition between the scattering effect from the linear term and the focusing effect from the nonlinearity, a solution with arbitrary energy may exhibit various behaviors: it could exist globally and scatter like a free evolution, persist like a solitary wave, blow up at finite time, or have mixed behaviors. We fully characterize the solution when the energy of the solution is below or at the level of the energy of the ground state solution W.

Our main result contains two parts. First, we prove that when the energy and kinetic energy of the initial data are less than those of the ground state solution, the solution exists globally and scatters. Second, we classify the dynamical behavior of the solution assuming that the energy of the solution equals the energy of W. Specifically, if the kinetic energy of the solution is greater than that of W, the solution either blows up at finite time or has mixed behaviors; if the kinetic energy of the solution is less, the solution either scatters or has mixed behaviors, if both kinetic energies are the same, the solution persists like a solitary wave.

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INTRODUCTION

1.1 Short review of the energy critical NLS

The Cauchy problem of the energy critical NLS in $\mathbb{R}^d (d \geq 3)$ can be formulated as

$$\begin{cases} (i\partial_t + \Delta)u = \pm |u|^{\frac{4}{d-2}} u \\ u(0,x) = u_0(x) \end{cases}, \tag{NLS}$$

where " $+$ " corresponds to the defocusing problem and " $-$ " corresponds to the focusing problem. Cazenave and Weissler [11] first developed the well-posedness theory of (NLS) with small initial data. The main result is that if $\|u_0\|_{\dot{H}^1}$ is sufficiently small, then $u(t,x)$ is a global solution which does not blow up in either time directions. In addition, $\|u\|_{S(\mathbb{R})} < \infty$ and the solution scatters in both time directions. The well-posedness theory of (NLS) with large initial data had been a long time open problem.

The defocusing problem: It was conjectured that all $\dot{H}^1$ initial data lead to global solutions with finite scattering norm($\|u\|_{S(\mathbb{R})} < \infty$). The breakthrough was first made by Bourgain [6, 7] when $d = 3, 4$ assuming u_0 is radial. When $d = 3$, Grillakis [20] also gave an alternative proof. Then, Tao [47] extended the result for all $d \geq 5$ still under the radial assumption. For arbitrary initial data, see Colliander–Keel–Staffilani–Takaoka–Tao [13] for $d = 3$, Ryckman–Visan [44] for $d = 4$, and Visan [50] for all $d \geq 5$.

The focusing problem: (a) Global well-posedness and scattering. The condition $\|u_0\|_{\dot{H}^1} < \infty$ is not sufficient to guarantee either the global existence or the finite scattering norm. Indeed, the ground state solution $W_0(x) = (1 + \frac{|x|^2}{d(d-2)})^{\frac{2-d}{2}}$ is a global solution to the focusing problem which does not scatter but blows up in both time direction. Hence, stronger assumption is required. Under the assumption u_0 is radial,

$$\|u_0\|_{\dot{H}^1} \leq \|W_0\|_{\dot{H}^1} \text{ and } E(u_0) < E(W_0), \tag{1.1}$$

where $E(u) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u(t)|^2 - \frac{1}{2^*}|u(t)|^{2^*} dx$. Kenig and Merle [26] first achieve the global well-posedness and scattering result when $d = 3, 4, 5$. Then Killip and Visan [30] solve the problem when $d \geq 5$ for non-radial initial data by assuming $\sup_{t \in I} \|u(t)\|_{\dot{H}^1} < \|W_0\|_{\dot{H}^1}$ for all t on the maximal life-span I of u. And Dodson [15] obtains the result when $d = 4$ by introducing the long time Strichartz estimate. The non-radial case when $d = 3$ is currently open.

(b) Dynamics of energy critical NLS. Due to the focusing nature, solution with arbitrary energy may exhibit various behaviors: it could exist globally and scatter like a free evolution, persist like a solitary wave, blow up at finite time, or even have mixed behaviors. Duyckaerts and Merle [17] first studied the dynamics of the focusing energy critical NLS in $\mathbb{R}^d (3 \leq d \leq 5)$. In short, assuming $E(u_0) = E(W_0)$, they fully characterize the solution u based on the relation between $\|u_0\|_{\dot{H}^1}$ and $\|W_0\|_{\dot{H}^1}$: if $\|u_0\|_{\dot{H}^1} > \|W_0\|_{\dot{H}^1}$, the solution either blows up at finite time or has mixed behaviors; if $\|u_0\|_{\dot{H}^1} = \|W_0\|_{\dot{H}^1}$, the solution persists like a solitary wave; if $\|u_0\|_{\dot{H}^1} < \|W_0\|_{\dot{H}^1}$, the solution either scatters or has mixed behaviors. The precise formulation is similar to Theorem 1.3 and 1.4. Then Li and Zhang [37] extend their result to $\mathbb{R}^d (d \geq 6)$ by using the decay property of the ground state solution W_0 in high dimensions. For similar results of the dynamics of other types of PDE, see [18, 38, 41] and the references therein.

1.2 Energy critical NLS with inverse square potential

In this thesis, we consider the initial value problem of the focusing nonlinear Schrödinger equation(NLS) with inverse square potential

$$\begin{cases} (i\partial_t - \mathcal{L}_a)u = -|u|^{\frac{4}{d-2}}u \\ u(0,x) = u_0(x) \in \dot{H}^1 \end{cases}, \tag{NLS$_a$}$$

where $\mathcal{L}_a = -\Delta + \frac{a}{|x|^2}$ and $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$. This problem is energy critical as the energy

$$E(u(t)) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u(t)|^2 + \frac{a}{2|x|^2}|u(t)|^2 - \frac{1}{2^*}|u(t)|^{2^*}\,dx$$

is invariant under the scaling $u(t,x) \mapsto \lambda^{\frac{2-d}{2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$. However, (NLS$_a$) breaks the translation in space symmetry. For non-radial problems, this is a major obstacle to overcome, see [29]. On the life-span of the solution, both the mass and energy are conserved

$$M(u(t)) = \int_{\mathbb{R}^d} |u(t,x)|^2\,dx = M(u_0), \quad E(u(t)) = E(u_0).$$

We also denote $\int \frac{1}{2}|\nabla u(t)|^2 + \frac{a}{2|x|^2}|u(t)|^2 = \frac{1}{2}\|u\|^2_{\dot{H}^1_a}$ as the *kinetic energy* and remaining part of $E(u)$ as the *potential energy.*

We begin with the proper definition of the solution to NLS$_a$.

Definition 1.1 (Solution, [29])**.** Let $t_0 \in I \subset \mathbb{R}$($I$ is an interval) and $u_0 \in \dot{H}^1$. A function $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a *solution* to NLS$_a$ if $u(t) \in C_t\dot{H}^1_a \cap S(J)$ for any compact

$J \subset I$ and obeys the Duhamel formula

$$u(t) = e^{-i(t-t_0)\mathcal{L}_a}u_0 + i\int_{t_0}^{t} e^{-i(t-s)\mathcal{L}_a}(|u(s)|^{\frac{4}{d-2}}u(s))\,ds, \tag{1.2}$$

for all $t \in I$. We refer I as the *life-span* of u. We say u is a maximal life-span *solution* if the solution can not be extended to any strictly larger interval. If $I = \mathbb{R}$, u is a *global solution.*

From the small data theory in Theorem 2.15, we know that if the initial data is sufficiently small in $\dot{H}^1_a$, the solution $u(t)$ of (NLS$_a$) is global and scatters. Global well-posedness and scattering result for large solutions was first obtained by Killip-Miao-Visan-Zhang-Zheng [29] for the energy critical defocusing problem with non-radial initial data in $\mathbb{R}^3$. Their work readily extends to $\mathbb{R}^d(3 \le d \le 6)$. In this thesis, we prove the global well-posedness and scattering of (NLS$_a$) with radial initial data in $\mathbb{R}^d(3 \le d \le 6)$ and characterize the dynamics of NLS$_a$ in the radial case.

Similar to the focusing energy critical NLS problem, scattering occurs when the initial data u_0 is below the threshold of the *ground state solution* [29]:

$$W_a(x) = [d(d-2)\beta^2]^{\frac{d-2}{4}}(\frac{|x|^{\beta-1}}{1+|x|^{2\beta}})^{\frac{d-2}{2}}, \tag{1.3}$$

where $0 > a > -(\frac{d-2}{2})^2$ and $\beta = \sqrt{1+\frac{4a}{(d-2)^2}}$. Clearly, $W_a(x)$ is radial and non-negative. In addition, $\mathcal{L}_a W_a = W_a^{\frac{d+2}{d-2}}$, so it is a global solution to (NLS$_a$) which does not scatter in either time direction.

Our main result consists of Theorem 1.1, 1.3 and 1.4.

Theorem 1.1 (Main Theorem I)**.** *Fix $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$. Let u be a solution of*

NLS_a with radial initial data u_0 such that

$$E(u_0) < E(W_{a\wedge 0}) \text{ and } \|u_0\|_{\dot{H}^1_a} \leq \|W_{a\wedge 0}\|_{\dot{H}^1_a}, \tag{1.4}$$

then u is a global solution and $\|u\|_{S(\mathbb{R})} < \infty$. In addition, the solution scatters in both time direction, i.e. there exists $u_\pm \in \dot{H}^1_a$ such that

$$\lim_{t\to\pm\infty} \|u - e^{-it\mathcal{L}_a}u_\pm\|_{\dot{H}^1_a} = 0.$$

Remark. From energy trapping(Lemma 2.11), we indeed have $\|u_0\|_{\dot{H}^1_a} < \|W_{a\wedge 0}\|_{\dot{H}^1_a}$. The requirement $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$ comes from the construction of the local solution(Theorem 2.14) which counts on equivalence of Sobolev norms(Lemma 2.1) and the fractional product rule(Lemma 2.2). The condition (1.4) comes from three aspects: (i) when $a = 0$, scattering was achieved under (1.1). (ii) when $0 > a > -(\frac{d-2}{2})^2$, the ground state solution W_a is well defined and plays the same role as W_0 in view of the sharp Gagliardo-Nirenberg inequality(Proposition 2.9); (iii) when $a > 0$, no ground state solution exists in this case but W_0 contributes to the sharp bound of Gagliardo-Nirenberg inequality (See Proposition 2.9).

The proof of Theorem 1.1 follows from the general outline in [26]. We also incorporate some developments of (NLS$_a$) discussed in [29]. We will prove this theorem in Chapter 2. In [29], the authors also classify the blow-up solutions and their statement reads as follows.

Proposition 1.2 (Blow-up of (NLS$_a$), [29])**.** *Fix $d \geq 3$ and $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$. Let $u_0 \in \dot{H}^1$ be such that $E(u_0) < E(W_{a\wedge 0})$ and $\|u_0\|_{\dot{H}^1_a} \geq \|W_{a\wedge 0}\|_{\dot{H}^1_a}$. Assume also* *that either $xu_0 \in L^2$ or $u_0 \in L^2$ is radial. Then the corresponding solution u to NLS_a blows up in finite time.*

Next, we state our result of the dynamics of NLS_a.

Theorem 1.3 (Main Theorem II). *Fix $0 > a > -(\frac{d-2}{2})^2 + (\frac{2(d-2)}{d+2})^2$. Let $3 \leq d \leq 6$. There exists radial solutions W^- and W^+ of NLS_a defined on $[0, \infty)$ such that $E(W_a) = E(W^-) = E(W^+)$ and $W^{\pm}(t) \to W_a$ in $\dot{H}^1_a$ as $t \to \infty$. W^- is a global in time solution with $\|W^-(t)\|_{\dot{H}^1_a} < \|W_a\|_{\dot{H}^1_a}$ and $\|W^-\|_{S(-\infty,0)} < \infty$; $\|W^+(t)\|_{\dot{H}^1_a} > \|W_a\|_{\dot{H}^1_a}$, if in addition $d = 6$, then W^+ blows up backward in finite time.*

When $0 > a > -(\frac{d-2}{2})^2 + (\frac{2(d-2)}{d+2})^2$, $W_a \in L^2(\mathbb{R}^6)$ but $W_a \notin L^2(\mathbb{R}^d)$ when $3 \leq d \leq 5$. Note that when $a = 0$, $W_0 \in L^2(\mathbb{R}^d)$ when $d \geq 5$.

Theorem 1.4 (Main Theorem III). *Fix $0 > a > -(\frac{d-2}{2})^2 + (\frac{2(d-2)}{d+2})^2$. Let u_0 be a radial function in $\dot{H}^1_a(\mathbb{R}^d)$ and $3 \leq d \leq 6$. Assume $E(u_0) = E(W_a)$. Then*

(a) If $\|u_0\|_{\dot{H}^1_a} < \|W_a\|_{\dot{H}^1_a}$, then u exists globally. In addition, either $\|u\|_{S(\mathbb{R})} < \infty$ or u equals W^- up to the symmetry of the equation.

(b) If $\|u_0\|_{\dot{H}^1_a} = \|W_a\|_{\dot{H}^1_a}$, then u equals W_a up to the symmetry of the equation.

(c) $\|u_0\|_{\dot{H}^1_a} > \|W_a\|_{\dot{H}^1_a}$ and $u_0 \in L^2$, then either u blows up at finite time or u equals W^+ up to the symmetry of the equation.

Remark. (i) When $E(u_0) = E(W_a)$, the assumption $\|u_0\|_{\dot{H}^1_a} < \|W_a\|_{\dot{H}^1_a}$ is independent of the choice of the initial time t_0. In addition, if $\|u_0\|_{\dot{H}^1_a} < \|W_a\|_{\dot{H}^1_a}$, the Sharp Gagliardo-Nirenberg inequality(Proposition 2.9) then yields that $\|u(t)\|_{\dot{H}^1_a} < \|W_a\|_{\dot{H}^1_a}$ for any t on the life-span of u. An analogous result holds if $\|u_0\|_{\dot{H}^1_a} > \|W_a\|_{\dot{H}^1_a}$.

(ii) The range of a shrinks in Theorem 1.3 and 1.4 compared with Theorem 1.1 for the following reasons: when $a > 0$, no ground solution exists by Proposition 2.9 in [29]; the modulation argument, spectrul theory of linearized operator $\mathcal{L}$, some Strichartz type estimates of NLS_a in later chapters require $W_a^{2p_c-1}, \mathcal{L}_a W_a^{p_c} \in L^{\frac{2d}{d+2}}$, and $\sqrt{\mathcal{L}_a} W_a^{p_c} \in L^2$, or equivalently, $a > -(\frac{d-2}{2})^2 + (\frac{2(d-2)}{d+2})^2$.

The rest of the thesis is organized as follows.

Chapter 2: Section 2.1 mainly reviews the some basic estimates and inequalities adapted to the operator $\mathcal{L}_a$ or $e^{-it\mathcal{L}_a}$, coercivity of energy and energy trapping. In Section 2.2, we prove the local well-posedness, stability, and local theory of NLS_a. Section 2.3 reviews the linear profile decomposition for the linear flow $e^{-it\mathcal{L}_a}$ associated to bounded radial $\dot{H}_a^1$ functions. Section 2.4 demonstrates that the failure of Theorem 1.1 guarantees the existence of the minimal blow-up solution on $[0, T^*)$ (Theorem 2.19). Lastly, in Section 2.5, we prove that both $T^* < \infty$ and $T^* = \infty$ lead to a contradiction. Thus, the expected contradiction yields Theorem 1.1.

Chapter 3: In Section 3.1, we prove the existence of solutions of NLS_a converging to W_a(up to symmetry) both in the sub-critical and super critical case. The result is based on the development of the modulation argument and variational characterization of W_a. Section 3.2 explores the spectral theory of the linearized operator $\mathcal{L}$ and some preliminary Strichartz type estimates. Section 3.3 concludes the proof of Theorem 1.3 and 1.4.

Chapter 4: A short review of my previous work and several related future research topics are discussed in this last chapter.

CHAPTER 2
GLOBAL WELL-POSEDNESS AND SCATTERING OF NLS WITH INVERSE SQUARE POTENTIAL

2.1 Preliminaries

2.1.1 Notation

We first summarize some notations for all chapters.

$\mathcal{L}_a = -\Delta + \frac{a}{\lvert x\rvert^2}$	$2^* = \frac{2d}{d-2}$
$X \lesssim Y$ or $X = O(Y)$ if $X \le CY$ for some constant $C > 0$	$X \lesssim_Z Y$ if $X \le C(Z)Y$ for some constant $C(Z) > 0$ depending on Z
$X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$	$\langle x\rangle = \sqrt{1+\lvert x\rvert^2}$
$\sigma = \frac{d-2}{2} - \sqrt{(\frac{d-2}{2})^2 + a}$	$\beta = \sqrt{1 + \frac{4a}{(d-2)^2}}$
$A \wedge B = \min\{A, B\}$	$A \vee B = \max\{A, B\}$
$\lVert f\rVert_p = \lVert f\rVert_{L_x^p(\mathbb{R}^d)}$	$\lVert f\rVert_{q,r(I)} = \lVert f\rVert_{L_t^q L_x^r(I\times\mathbb{R}^d)}$
$\lVert f\rVert_{\dot H_a^1} = \lVert\sqrt{\mathcal{L}_a} f\rVert_2$	$\lVert f\rVert_{H_a^1} = \lVert\sqrt{1+\mathcal{L}_a} f\rVert_2$
$\lVert f\rVert_{\dot H_a^{1,r}} = \lVert\sqrt{\mathcal{L}_a} f\rVert_r$	$\lVert f\rVert_{H_a^{1,r}} = \lVert\sqrt{1+\mathcal{L}_a} f\rVert_r$

Table 2.1: Notations for all chapters

The following table only contains notations for Chapter 2.

$\dot{X}^1(I) = L_t^{\frac{2(d+2)}{d-2}} \dot{H}^{1,\frac{2d(d+2)}{d^2+4}}(I \times \mathbb{R}^d)$	$X^0(I) = L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I \times \mathbb{R}^d)$
$\dot{X}_a^1(I) = L_t^{\frac{2(d+2)}{d-2}} \dot{H}_a^{1,\frac{2d(d+2)}{d^2+4}}(I \times \mathbb{R}^d)$	$X_a^1(I) = L_t^{\frac{2(d+2)}{d-2}} H_a^{1,\frac{2d(d+2)}{d^2+4}}(I \times \mathbb{R}^d)$
$S^0(I) = L_t^2 L_x^{2^*} \cap L_t^\infty L_x^2(I \times \mathbb{R}^d)$	$\dot{S}^1(I) = \{u : I \times \mathbb{R}^d :\ \nabla u \in S^0(I)\}$
$S(I) = L_{t,x}^{\frac{2(d+2)}{d-2}}(I \times \mathbb{R}^d)$	$\dot{S}_a^1(I) = \{u : I \times \mathbb{R}^d :\ \sqrt{\mathcal{L}_a} u \in S^0(I)\}$
$N^0(I) := L_t^2 L_x^{\frac{2d}{d+2}} + L_t^1 L_x^2(I \times \mathbb{R}^d)$	$u(t,x)_{[\lambda]} = \lambda^{\frac{2-d}{2}} u(t, \frac{x}{\lambda})$

Table 2.2: Notations for Chapter 2 only

2.1.2 Basic estimates and inequalities

Lemma 2.1 (Equivalence of Sobolev norms, [28]). *Fix $d \geq 3$, $a \geq -(\frac{d-2}{2})^2$, and $0 < s < 2$. If $1 < p < \infty$ satisfies $\frac{s+\sigma}{d} < \frac{1}{p} < \min\{1, \frac{d-\sigma}{d}\}$, then*

$$\left\|(-\Delta)^{\frac{s}{2}} f\right\|_{L^p} \lesssim_{d,p,s} \left\|\mathcal{L}_a^{\frac{s}{2}} f\right\|_{L^p} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d \setminus \{0\}). \tag{2.1}$$

If $\max\{\frac{s}{d}, \frac{\sigma}{d}\} < \frac{1}{p} < \min\{1, \frac{d-\sigma}{d}\}$, which ensures already that $1 < p < \infty$, then

$$\left\|\mathcal{L}_a^{\frac{s}{2}} f\right\|_{L^p} \lesssim_{d,p,s} \left\|(-\Delta)^{\frac{s}{2}} f\right\|_{L^p} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d \setminus \{0\}). \tag{2.2}$$

In particular, if $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$, then

$$\|\nabla f\|_{L^p(\mathbb{R}^d)} \sim \|\sqrt{\mathcal{L}_a} f\|_{L^p(\mathbb{R}^d)} \quad \text{for all} \quad \frac{2d}{d+2} \leq p \leq \frac{2d(d+2)}{d^2+4}.$$

Lemma 2.2 (Fractional product rule, [28]). *Fix $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$. Then for all $f, g \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ we have*

$$\|\sqrt{\mathcal{L}_a}(fg)\|_p \lesssim \|\sqrt{\mathcal{L}_a} f\|_{p_1} \|g\|_{p_2} + \|f\|_{q_1} \|\sqrt{\mathcal{L}_a} g\|_{q_2},$$

for any exponents satisfying $\frac{2d}{d+2} \leq p, p_1, p_2, q_1, q_2 \leq \frac{2d(d+2)}{d^2+4}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

Recall the Littlewood–Paley projections, Bernstein estimates and square function estimate introduced in [28]. Let

$$f_{\le N} = P^a_{\le N} f = \phi_N(\sqrt{\mathcal{L}_a}), \qquad f_N = P^a_N f = \psi_N(\sqrt{\mathcal{L}_a}\,).$$

where $\phi_N(r) = \phi(r/N)$, $\psi_N(r) = \phi_N(r) - \phi_{N/2}(r)$ and ϕ is a smooth cutoff function such that $\phi(r) = 1$ for $0 \le r \le 1$ and $\phi(r) = 0$ for $r \ge 2$.

Lemma 2.3 (Bernstein estimates, [28]). *For $1 < p \le q \le \infty$ when $a \ge 0$ or $r_0 < p \le q < r_0' = \frac{d}{\sigma}$ when $-(\frac{d-2}{2})^2 \le a < 0$, the following hold:*

(1) *The operators $P^a_{\le N}$ and P^a_N, are bounded on L^p.*

(2) *The operators $P^a_{\le N}$ and P^a_N, map L^p to L^q with norm $O(N^{\frac{d}{p}-\frac{d}{q}})$.*

(3) *For any $s \in \mathbb{R}$, $N^s \|P^a_N f\|_{L^p_x} \sim \|(\mathcal{L}_a)^{\frac{s}{2}} P^a_N f\|_{L^p_x}$.*

Lemma 2.4 (Square function, [28]). *Fix $0 \le s < 2$. For $1 < p < \infty$ when $a \ge 0$ or $r_0 < p < r_0' := \frac{d}{\sigma}$ when $-(\frac{d-2}{2})^2 \le a < 0$ and any $f \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$,*

$$\|_{L^p}(\sum_{N \in 2^{\mathbb{Z}}} N^{2s} |P^a_N f|^2)^{\frac{1}{2}}\|_p \sim \|\mathcal{L}_a^{\frac{s}{2}} f\|_p.$$

Lemma 2.5 (Refined Fatou, [8]). *Fix $1 \le p < \infty$ and let $\{f_n\}$ be a bounded sequence in $L^p(\mathbb{R}^d)$. If $f_n \to f$ almost everywhere, then*

$$\int_{\mathbb{R}^d} \big||f_n|^p - |f_n - f|^p - |f|^p\big| \, dx \to 0.$$

In particular, $\|f_n\|_p^p - \|f_n - f\|_p^p \to \|f\|_p^p$.

Strichartz estimates for the linear flow(propagator) $e^{-it\mathcal{L}_a}$ were proved by Burq, Planchon, Stalker, and Tahvildar-Zadeh in [9].

Theorem 2.6 (Strichartz estimates, [9]). *Fix $a > -(\frac{d-2}{2})^2$. The solution u to*

$$i\partial_t u - \mathcal{L}_a u = F$$

on an interval $I \ni t_0$ obeys

$$\|u\|_{q,r(I)} \lesssim \|u(t_0)\|_2 + \|F\|_{\tilde{q}',\tilde{r}'(I)}, \tag{2.3}$$

whenever $\frac{2}{q} + \frac{d}{r} = \frac{2}{\tilde{q}} + \frac{d}{\tilde{r}} = \frac{d}{2}$, $2 \leq q, \tilde{q} \leq \infty$, and $(q,\tilde{q}) \neq (2,2)$.

Note that the double endpoint Strichartz estimate is not available for NLS_a. We record next a local smoothing result for the linear propagator $e^{-it\mathcal{L}_a}$.

Lemma 2.7 (Local smoothing, [29]). *Fix $a > -(\frac{d-2}{2})^2$ and let $w = e^{-it\mathcal{L}_a} w_0$. Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{|\nabla w(t,x)|^2}{R\langle R^{-1}x\rangle^3} + \frac{|w(t,x)|^2}{R|x|^2}\,dx\,dt \lesssim \|w_0\|_2 \|\nabla w_0\|_2 + R^{-1}\|w_0\|_2^2, \tag{2.4}$$

$$\int_{\mathbb{R}} \int_{|x-z|\leq R} \tfrac{1}{R}|\nabla w(t,x)|^2\,dx\,dt \lesssim \|w_0\|_2 \|\nabla w_0\|_2 + R^{-1}\|w_0\|_2^2, \tag{2.5}$$

uniformly for $z \in \mathbb{R}^d$ and $R > 0$.

Consequently, we have the following estimate of the linear flow $e^{-it\mathcal{L}_a} w_0$ on compact domains. We remind the reader that the bound we found below is not the optimal one but just serves our purpose.

Corollary 2.8. *Denote $B_{T,R} = \{|t-\tau| \leq T, |x-z| \leq R\}$. Fix $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$ and let $w_0 \in \dot{H}^1_a$. Then*

$$\begin{aligned}\|\nabla e^{-it\mathcal{L}_a} w_0\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}(B_{T,R})} &\lesssim T^{\frac{(d-2)^2}{2(d+2)^2}} R^{\frac{d^3+4d-16}{2d^2+8d+8}} \|e^{-it\mathcal{L}_a} w_0\|_{S(\mathbb{R})}^{\frac{d-2}{d+2}} \|w_0\|_{\dot{H}^1_a}^{\frac{4}{d+2}} \\ &\quad + T^{\frac{(d-2)^2}{4(d+2)^2}} R^{\frac{d^3+d^2-12}{2(d+2)^2}} \|e^{-it\mathcal{L}_a} w_0\|_{S(\mathbb{R})}^{\frac{d-2}{2(d+2)}} \|w_0\|_{\dot{H}^1_a}^{\frac{d+6}{2(d+2)}}\end{aligned}$$

uniformly in w_0 and the parameters $R, T > 0$, $\tau \in \mathbb{R}$, and $z \in \mathbb{R}^d$.

Proof. We first estimate the low frequency. By Hölder inequality, equivalent of Sobolev norms, and Bernstain inequality,

$$\begin{aligned}\|\nabla e^{-it\mathcal{L}_a}P^a_{\le N}\ w_0\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}(B_{T,R})} &\lesssim NT^{\frac{d-2}{2(d+2)}}R^{\frac{d^2+4}{2(d+2)}}\|e^{-it\mathcal{L}_a}P^a_{\le N}\ w_0\|_{S(B_{T,R})}\\ &\lesssim NT^{\frac{d-2}{2(d+2)}}R^{\frac{d^2+4}{2(d+2)}}\|e^{-it\mathcal{L}_a}w_0\|_{S(\mathbb{R})}.\end{aligned}$$

For the high frequency, from Hölder inequality and Strichartz estimate, we get

$$\begin{aligned}&\|\nabla e^{-it\mathcal{L}_a}P^a_{>N}\ w_0\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}(B_{T,R})}\\ \lesssim\ & R^{\frac{d-2}{4}}\|\nabla e^{-it\mathcal{L}_a}P^a_{>N}\ w_0\|^{\frac{d-2}{4}}_{2,2(B_{T,R})}\|e^{-it\mathcal{L}_a}P^{\Omega}_{>N}\ w_0\|^{\frac{6-d}{4}}_{\dot{X}^1(B_{T,R})}\\ \lesssim\ & R^{\frac{d-2}{4}}\|\nabla e^{-it\mathcal{L}_a}P^a_{>N}\ w_0\|^{\frac{d-2}{4}}_{2,2(B_{T,R})}\|w_0\|^{\frac{6-d}{4}}_{\dot{H}^1_a}.\end{aligned}$$

Lemma 2.7 and Bernstein inequality yield that

$$\begin{aligned}\|\nabla e^{-it\mathcal{L}_a}P^a_{>N}\ w_0\|^2_{2,2(B_{T,R})} &\lesssim R\|P^a_{>N}w_0\|_{L^2_x}\|\nabla P^a_{>N}w_0\|_{L^2_x}+\|P^a_{>N}w_0\|^2_{L^2_x}\\ &\lesssim (RN^{-1}+N^{-2})\|w_0\|^2_{\dot{H}^1_a}.\end{aligned}$$

Hence,

$$\begin{aligned}\|\nabla e^{-it\mathcal{L}_a}P^a_{>N}\ w_0\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}(B_{T,R})} &\lesssim R^{\frac{d-2}{4}}(RN^{-1}+N^{-2})^{\frac{d-2}{8}}\|w_0\|_{\dot{H}^1_a}\\ &\lesssim (R^{\frac{3d-6}{8}}N^{\frac{2-d}{8}}+R^{\frac{d-2}{4}}N^{\frac{2-d}{4}})\|w_0\|_{\dot{H}^1_a}.\end{aligned}$$

The conclusion follows from optimizing the choice of N. □

2.1.3 Coercivity of energy and energy trapping

The fact that ground state solution contributes to the best Sobolev constant dates back to [1, 46]. The best Sobolev constant C_{GN} for the embedding

$\|f\|_{2^*} \leq C_{GN}\|f\|_{\dot{H}^1_a}$ was achieved in [29] when $a \neq 0$. When $a = 0$, this was proved independently by Aubin [1] and Talenti [46].

Proposition 2.9 (Sharp Gagliardo-Nirenberg inequality, [1, 29, 46]). *Fix $d \geq 3$ and $a > -(\frac{d-2}{2})^2$.*

(i) *If $-(\frac{d-2}{2})^2 < a < 0$, then*

$$\|f\|_{2^*} \leq \|W_a\|_{\frac{2d}{d-2}}\|W_a\|^{-1}_{\dot{H}^1_a}\|f\|_{\dot{H}^1_a}. \tag{2.6}$$

Moreover, equality holds if and only if $f(x) = \alpha W_a(\lambda x)$ for some $\alpha \in \mathbb{C}$ and some $\lambda > 0$.

(ii) *The inequality* (2.6) *is valid also when $a = 0$; however, equality now holds if and only if $f(x) = \alpha W_0(\lambda x + y)$ for some $\alpha \in \mathbb{C}$, some $y \in \mathbb{R}^d$, and some $\lambda > 0$.*

(iii) *If $a > 0$, then*

$$\|f\|_{2^*} \leq \|W_0\|_{\frac{2d}{d-2}}\|W_0\|^{-1}_{\dot{H}^1_a}\|f\|_{\dot{H}^1_a}. \tag{2.7}$$

In this case, equality never holds (for $f \not\equiv 0$); however, the constant in (2.7) cannot be improved.

For defocusing problems, one advantage is that all the energies are positive and comparable with the kinetic energy. For focusing problems, we impose proper conditions to guarantee such phenomenon. Once again, the condition involves the ground state solution W_a.

Lemma 2.10 (Coercivity of Energy). *Let $f \in \dot{H}^1_a$ such that $\|f\|_{\dot{H}^1_a} \leq \|W_{a\wedge 0}\|_{\dot{H}^1_a}$. Then*

$$(\frac{1}{2} - \frac{1}{2^*})\|f\|^2_{\dot{H}^1_a} \leq E(f) \leq \frac{1}{2}\|f\|^2_{\dot{H}^1_a}. \tag{2.8}$$

Proof. W.L.O.G, let us assume $\|f\|_{\dot H^1_a} \neq 0$. By the definition of energy, we clearly have $E(f) \leq \frac{1}{2}\|f\|^2_{\dot H^1_a}$. It remains to show $(\frac{1}{2} - \frac{1}{2^*})\|f\|^2_{\dot H^1_a} \leq E(f)$. Recall that $\|W_{a\wedge 0}\|^{2^*}_{2^*} = \|W_{a\wedge 0}\|^2_{\dot H^1_a}$ and $\|f\|_{\dot H^1_a} \leq \|W_{a\wedge 0}\|_{\dot H^1_a}$, by the Sharp Gagliardo-Nirenberg inequality $\|f\|_{2^*} \leq (\frac{\|W_{a\wedge 0}\|_{2^*}}{\|W_{a\wedge 0}\|_{\dot H^1_a}})\|f\|_{\dot H^1_a}$, we get

$$\begin{aligned} E(f) &= \|f\|^2_{\dot H^1_a}(\frac{1}{2} - \frac{1}{2^*}\frac{\|f\|^{2^*}_{2^*}}{\|f\|^2_{\dot H^1_a}}) \geq \|f\|^2_{\dot H^1_a}(\frac{1}{2} - \frac{1}{2^*}(\frac{\|W_{a\wedge 0}\|_{2^*}}{\|W_{a\wedge 0}\|_{\dot H^1_a}})^{2^*}\|f\|^{2^*-2}_{\dot H^1_a}) \\ &= \|f\|^2_{\dot H^1_a}(\frac{1}{2} - \frac{1}{2^*}(\frac{\|f\|_{\dot H^1_a}}{\|W_{a\wedge 0}\|_{\dot H^1_a}})^{2^*-2}) \geq (\frac{1}{2} - \frac{1}{2^*})\|f\|^2_{\dot H^1_a}. \end{aligned}$$

Hence, (2.8) holds. □

Lemma 2.11 (Energy trapping, [26]). *Let $u : I \times \mathbb{R}^d \longmapsto \mathbb{C}$ be a solution of NLS$_a$ with initial data u_0. If $\|u_0\|_{\dot H^1_a} \leq \|W_{a\wedge 0}\|_{\dot H^1_a}$ and $E(u) < (1-\delta_0)E(W_{a\wedge 0})$ for some small $\delta_0 > 0$, then there exists $\delta_1 > 0$(depending on δ_0 and d only) such that for any $t \in I$,*

$$\begin{cases} \|u(t)\|^2_{\dot H^1_a} \leq (1-\delta_1)\|W_{a\wedge 0}\|^2_{\dot H^1_a}, \\ \delta_1\|u(t)\|^2_{\dot H^1_a} \lesssim \|u(t)\|^2_{\dot H^1_a} - \|u(t)\|^{2^*}_{2^*} \\ \|u_0\|^2_{\dot H^1_a} \sim \|u(t)\|^2_{\dot H^1_a} \sim E(u(t)) \end{cases}. \tag{2.9}$$

Note that from the condition $\|u_0\|_{\dot H^1_a} \leq \|W_{a\wedge 0}\|_{\dot H^1_a}$, one can only obtain $0 \leq \|u(t)\|^2_{\dot H^1_a} - \|u(t)\|^{2^*}_{2^*}$ by Lemma 2.10. The assumption $E(u) < E(W_{a\wedge 0})$ is essential to the energy trapping argument.

2.2 Local well-posedness and stability of NLS$_a$

In dimension three, the local well-posedness and stability has been proved in [29] for the defocusing energy critical NLS$_a$. It is straightforward to extend the result to $\mathbb{R}^d(3 \leq d \leq 6)$, for the sake of completeness, we present the proof below. Indeed, when $3 \leq d \leq 5$, one can drop the assumption $u_0 \in L^2(\mathbb{R}^d)$ and employ the treatment in [26] instead. The local well-posedness below is in $H^1_a(\mathbb{R}^d)$, to withdraw the assumption $u_0 \in L^2(\mathbb{R}^d)$, we use the stability(Theorem 2.13) and density argument.

Proposition 2.12. *Fix $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$. Given $A \geq 0$, there exists $\eta = \eta(A)$ so that the following holds: Suppose $u_0 \in H^1_a$ obeys*

$$\|\sqrt{\mathcal{L}_a} u_0\|_2 \leq A \quad \textit{and} \quad \|e^{it\mathcal{L}_a} u_0\|_{S(I)} \leq \eta \tag{2.10}$$

for some time interval $I \ni 0$. Then there is a unique strong solution u to NLS_a on the time interval I such that

$$\|u\|_{S(I)} \lesssim \eta \quad \textit{and} \quad \|\sqrt{\mathcal{L}_a} u\|_{C_t L^2_x \cap X^0(I)} \lesssim A. \tag{2.11}$$

Proof. From the Fixed Point Theorem on a complete partial metric space by Matthews [40], it suffices to show that the solution map

$$\Phi_{u_0} : u \mapsto e^{-it\mathcal{L}_a} u_0 + i \int_0^t e^{-i(t-s)\mathcal{L}_a} |u(s)|^{\frac{4}{d-2}} u(s)\, ds \tag{2.12}$$

is a contraction on the Banach space

$$B_{m,n} = \{u \in C_t H^1_a \cap X^1_a(I) : \ \|\sqrt{\mathcal{L}_a} u\|_{X^0(I)} \leq m, \|u\|_{S(I)} \leq n\}$$

endowed with the partial metric

$$d(u, v) = \|u - v\|_{X^0(I)}.$$

We first prove that Φ_{u_0} maps $B_{m,n}$ to $B_{m,n}$. In the following computation, the constant C only depends on d and a only, and C might be occupied. Fix $u \in B_{m,n}$, from Strichartz estimate and Lemma 2.2, we get

$$\|\sqrt{\mathcal{L}_a}\Phi_{u_0}(u)\|_{C_tL_x^2\cap X^0(I)} \le CA + C\|\sqrt{\mathcal{L}_a}(|u|^{\frac{4}{d-2}}u)\|_{2,\frac{2d}{d+2}(I)} \le CA + C\eta^{\frac{4}{d-2}}m;$$

$$\|\Phi_{u_0}(u)\|_{S(I)} \le \|e^{-it\mathcal{L}_a}u_0\|_{S(I)} + C\eta^{\frac{4}{d-2}}\|\sqrt{\mathcal{L}_a}u\|_{X^0(I)} \le \eta + C\eta^{\frac{4}{d-2}}m;$$

$$\|\Phi_{u_0}(u)\|_{C_tL_x^2\cap X^0(I)} \le C\|u_0\|_2 + C\eta^{\frac{4}{d-2}}\|u\|_{X^0(I)} < \infty.$$

By taking $m = 2CA$, $n = \eta(1 + 2C^2A)$, and $\eta < \min\{1, (2C)^{\frac{2-d}{4}}\}$,

$$\|\sqrt{\mathcal{L}_a}\Phi_{u_0}(u)\|_{C_tL_x^2\cap X^0(I)} \le m, \qquad \|\Phi_{u_0}(u)\|_{S(I)} \le n.$$

Note that if $d < 6$, one can pick $n = 2\eta$ by choosing η small enough.

It remains to show that Φ_{u_0} is a contraction with respect to the metric on $B_{m,n}$, then the solution is unique by the Banach fixed point theorem. Indeed, let $u, v \in B_{m,n}$, similarly, we have

$$\begin{aligned}
\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{X^0(I)} &\lesssim \||u|^{\frac{4}{d-2}}u - |v|^{\frac{4}{d-2}}v\|_{2,\frac{2d}{d+2}(I)} \\
&\lesssim (\|u\|_{S(I)}^{\frac{4}{d-2}} + \|v\|_{S(I)}^{\frac{4}{d-2}})\|u - v\|_{X^0(I)} \\
&\lesssim_A \eta^{\frac{4}{d-2}}\|u - v\|_{X^0(I)}.
\end{aligned}$$

By choosing η sufficiently small, we get $d(\Phi_{u_0}(u), \Phi_{u_0}(v)) \le \frac{1}{2}d(u, v)$.

Therefore, Φ_{u_0} is a contraction from $B_{m,n}$ to $B_{m,n}$. □

For NLS, the stability holds for $d \ge 7$ [31, 48] by using the exotic Strichartz estimate which relies on the dispersive estimate. However, [42] showed that dispersive estimates fail when $a < 0$ for the propogator $e^{-it\mathcal{L}_a}$. This draws our discussion to $\mathbb{R}^d(3 \le d \le 6)$.

Theorem 2.13 (Stability, [29])**.** *Fix $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$. Let I be a compact time interval and let $\tilde{u}$ be an approximate solution to NLS$_a$ on $I \times \mathbb{R}^d$ in the sense that*

$$(i\partial_t - \mathcal{L}_a)\tilde{u} = -|\tilde{u}|^{\frac{4}{d-2}}\tilde{u} + e$$

for some function e. For some positive constants E and L, assume that

$$\|\tilde{u}\|_{L_t^\infty \dot{H}_a^1(I\times\mathbb{R}^d)} \leq E \quad \textit{and} \quad \|\tilde{u}\|_{S(I)} \leq L.$$

Let $t_0 \in I$ and let $u_0 \in \dot{H}_a^1$. Assume that

$$\|u_0 - \tilde{u}(t_0)\|_{\dot{H}_a^1} + \|\sqrt{\mathcal{L}_a} e\|_{N^0(I)} \leq \varepsilon \tag{2.13}$$

for some $0 < \varepsilon < \varepsilon_1 = \varepsilon_1(E, L)$. Then, there exists a unique strong solution $u : I \times \mathbb{R}^d \mapsto \mathbb{C}$ to NLS$_a$ with initial data u_0 at time $t = t_0$ satisfying

$$\begin{cases} \|\sqrt{\mathcal{L}_a}(u - \tilde{u})\|_{S^0(I)} \leq C(E, L)\,\varepsilon \\ \|\sqrt{\mathcal{L}_a} u\|_{S^0(I)} \leq C(E, L) \end{cases}. \tag{2.14}$$

Proof. In the following proof, we merely focus on the forward in time case($t \geq t_0$). The backward in time case($t \leq t_0$) can be handled analogously. We first assume that $u_0 \in L^2(\mathbb{R}^d)$. The existence and uniqueness of a solution u to NLS$_a$ with initial data u_0 is guaranteed by Proposition 2.12. We will remove the assumption $u_0 \in L^2(\mathbb{R}^d)$ in the end.

Note that $\tilde{u}$ is the bounded in $\dot{X}^1(I)$ by first splitting the time interval I, using the continuity argument as proving (2.20), then gluing all pieces together. Let $\delta > 0$ be a small constant, we partition I into $C(L, \delta)$ adjacent intervals such that on each $J_k = [T_k, T_{k+1}]$,

$$\|\tilde{u}\|_{\dot{X}^1(J_k)} \leq \delta. \tag{2.15}$$

From Sobolev embedding, we get

$$\|\tilde{u}\|_{S(J_k)} \leq \|\tilde{u}\|_{\dot{X}^1(J_k)} \leq \delta. \tag{2.16}$$

We will prove the short time perturbation theory for NLS_a and the stability result follows from induction.

(Short time Perturbation) *On the interval J_k, we assume*

$$\begin{cases} \|u(T_k) - \tilde{u}(T_k)\|_{\dot{H}^1_a} + \|\sqrt{\mathcal{L}_a} e\|_{N^0(J_k)} \leq C(k)\varepsilon \\ \|\tilde{u}\|_{\dot{X}^1(J_k)} \leq \delta \end{cases}. \tag{2.17}$$

Denote $A(J_k) = \|\sqrt{\mathcal{L}_a}(|u|^{\frac{4}{d-2}}u - |\tilde{u}|^{\frac{4}{d-2}}\tilde{u})\|_{N^0(J_k)}$, then we have

$$\|u - \tilde{u}\|_{\dot{X}^1(J_k)} \leq C(k)\varepsilon, \quad A(J_k) \leq C(k)\varepsilon. \tag{2.18}$$

From Strichartz estimate, we get

$$\begin{aligned} \|u - \tilde{u}\|_{\dot{X}^1(J_k)} &\lesssim \|u(T_k) - \tilde{u}(T_k)\|_{\dot{H}^1_a} + A(J_k) + \|\sqrt{\mathcal{L}_a} e\|_{N^0(J_k)} \\ &\leq A(J_k) + C(k)\varepsilon. \end{aligned} \tag{2.19}$$

From Strichartz estimate, and (2.16), we have

$$\begin{aligned} A(J_k) &\leq C(\|u - \tilde{u}\|_{S(J_k)} + \|\tilde{u}\|_{S(J_k)})^{\frac{4}{d-2}} \|u - \tilde{u}\|_{\dot{X}^1(J_k)} \\ &\quad + C\|\tilde{u}\|_{\dot{X}^1(J_k)}(\|u - \tilde{u}\|_{S(J_k)} + \|\tilde{u}\|_{S(J_k)})^{\frac{6-d}{d-2}} \|u - \tilde{u}\|_{S(J_k)} \\ &\leq C(\delta)^{\frac{4}{d-2}} C(k)\varepsilon + C\delta \end{aligned}$$

By (2.17) and (2.19), we get

$$\begin{aligned} A(J_k) &\leq C[A(J_k) + C(k)\varepsilon + \delta]^{\frac{4}{d-2}}(A(J_k) + C(k)\varepsilon) \\ &\quad + C\delta[A(J_k) + C(k)\varepsilon + \delta]^{\frac{6-d}{d-2}}(A(J_k) + C(k)\varepsilon) \end{aligned}$$

By taking δ small enough and a continuity argument, we get

$$A(J_k) \le C(k)\varepsilon. \tag{2.20}$$

Together with (2.19), this yields (2.18).

By induction, it remains to verify that on J_{k+1},

$$\begin{cases} \|u(T_{k+1}) - \tilde{u}(T_{k+1})\|_{\dot{H}^1_a} + \|\sqrt{\mathcal{L}_a} e\|_{N^0(J_{k+1})} \le C(k+1)\varepsilon \\ \|\tilde{u}\|_{\dot{X}^1(J_{k+1})} \le \delta \end{cases}.$$

Note that $\|\sqrt{\mathcal{L}_a} e\|_{N^0(J_{k+1})} \le \varepsilon$ and $\|\tilde{u}\|_{\dot{X}^1(J_{k+1})} \le \delta$ follows from (2.13) and (2.15) directly. While from Strichartz estimate and (2.18) on each $J_m (0 \le m \le k)$, we get

$$\begin{aligned} &\|u(T_{k+1}) - \tilde{u}(T_{k+1})\|_{\dot{H}^1_a} \\ &\lesssim \|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}^1_a} + \|\sqrt{\mathcal{L}_a}(|u|^{\frac{4}{d-2}}u - |\tilde{u}|^{\frac{4}{d-2}}\tilde{u})\|_{N^0([T_0, T_{k+1}))} + \|\sqrt{\mathcal{L}_a} e\|_{N^1(J_{k+1})} \\ &\lesssim \varepsilon + \textstyle\sum_{m=0}^k A(J_m) \le C\varepsilon + \sum_{m=0}^k C(m)\varepsilon \le C(k+1)\varepsilon. \end{aligned}$$

Thus, the assumption on J_{k+1} is verified. Repeating the proof of (2.18), we obtain $\|u - \tilde{u}\|_{\dot{X}^1(J_{k+1})} \le C(k+1)\varepsilon$. Adding all estimates together, we have

$$\|u - \tilde{u}\|_{\dot{X}^1(I)} \lesssim \varepsilon \quad and \quad \|u\|_{\dot{X}^1(I)} \lesssim 1.$$

Another application of the Strichartz estimate proves (2.14).

Finally, we demonstrate the existence and uniqueness of u without the additional assumption $u_0 \in L^2$. We choose a sequence of functions $u_{n,0} \in H^1_a$ such that $\|u_{n,0} - u_0\|_{\dot{H}^1_a} \to 0$. Let u_n be the unique solution to NLS$_a$ with $u_n(t_0) = u_{n,0}$. Repeat above proof with $u = u_n$, $\tilde{u} = u_m$, and $e = 0$, we see that $\{u_n\}$ is a Cauchy sequence in $\dot{S}^1(I)$. Hence, u_n converges u in $\dot{S}^1_a(I)$ with $u(t_0) = u_0$. To see u solves NLS$_a$, we only need to prove $\Phi_{u_0}(u) = u$. Indeed, $\Phi_{u_0}(u_n) \to \Phi_{u_0}(u)$ in $\dot{S}^1(I)$ follows from Strichartz estimate, $\|u_{n,0} - u_0\|_{\dot{H}^1_a} \to 0$ and $u_n \to u$ in $\dot{S}^1_a(I)$. As $\Phi_{u_0}(u_n) = u_n$, the limit in $\dot{S}^1_a(I)$ must be unique. Hence, $\Phi_{u_0}(u) = u$ which shows that u is a solution to NLS_a with $u(t_0) = u_0$. $\square$

Theorem 2.14. *Fix $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$. Given $A \geq 0$, there exists $\eta = \eta(A)$ so that the following holds: Suppose $u_0 \in \dot{H}^1_a$ obeys*

$$\|\sqrt{\mathcal{L}_a} u_0\|_2 \leq A \quad \text{and} \quad \|e^{it\mathcal{L}_a} u_0\|_{S(I)} \leq \eta$$

for some time interval $I \ni 0$. Then there is a unique solution u to NLS_a on the time interval I such that

$$\|u\|_{S(I)} \lesssim \eta \quad \text{and} \quad \|\sqrt{\mathcal{L}_a} u\|_{C_t L^2_x \cap X^0(I)} \lesssim A.$$

Proof. Proposition 2.12 + Theorem 2.13. $\square$

By making obvious changes of the notations suited to our case, one can complete the proof of the local theory as shown in [10, 26]. For simplicity, we skip the proof.

Theorem 2.15 (Local theory, [10, 26])**.** *Given $u_0 \in \dot{H}^1_a$ and $t_0 \in \mathbb{R}$, there exists a unique maximal life-span solution $u : I \times \mathbb{R}^d \longmapsto \mathbb{C}$ to NLS_a with initial condition $u(t_0) = u_0$. In addition, we have:*

- ***Local existence:*** *I is an open interval such that $t_0 \in I$.*

- **`Scattering:`** *If u is a solution of NLS_a on $I \times \mathbb{R}^d$ with $[0,\infty) \subseteq I$ and $\|u\|_{S(I)} < \infty$, then there exists $u^+ \in \dot{H}^1_a$ such that $\lim_{t\to\infty} \|u(t) - e^{-it\mathcal{L}_a} u_+\|_{\dot{H}^1_a} = 0$. Similar result holds when $(-\infty, 0] \subseteq I$.*

- **`Small data theory:`** *There exists $\eta_0 > 0$ such that if $\|u_0\|_{\dot{H}^1_a} \leq \eta_0$, then the solution u to NLS_a is global in time with $\|u\|_{S(\mathbb{R})} \lesssim \|u_0\|_{\dot{H}^1_a}$ and scatters in both time directions.*

- **`Large data theory:`** *For large initial data u_0, there exists an open interval $I_0 \ni 0$ such that the hypothesis of local well-posedness is satisfied i.e., $\|e^{-it\mathcal{L}_a} u_0\|_{S(I_0)} \leq \eta$.*

- **`Blow-up criterion:`** *If $\sup I < \infty$, then $\|u\|_{S(I)} = \infty$. Also, a corresponding result holds for $\inf I > -\infty$.*

- **`Uniqueness:`** *If $\tilde{u} \in C_t \dot{H}^1(I_1)$ solves NLS_a with $\tilde{u}(t_0) = u$, then $I_1 \subset I$ and $\tilde{u}(t) = u(t)$ for all $t \in I_1$.*

- **`Continuity of the flow:`** *If $\tilde{u}$ solves NLS_a on I with initial data $\tilde{u}_0$ and $\sup_{t\in I} \|\tilde{u}(t)\|_{\dot{H}^1_a} + \|\tilde{u}\|_{S(I)} \leq A$ for some $A > 0$, then $\exists \varepsilon_0(A), C(A) > 0$ such that for any $u \in \dot{H}^1_a$ with $\|\tilde{u}_0 - u_0\|_{\dot{H}^1_a} = \varepsilon < \varepsilon_0$, the solution u of NLS_a with initial data u_0 is defined on I and satisfies $\|u\|_{S(I)} \leq C$ and $\sup_{t\in I} \|\tilde{u}(t) - u(t)\|_{\dot{H}^1_a} \leq C\varepsilon$.*

2.3 Linear profile decomposition

In this subsection, we recall the linear profile decomposition for $e^{-it\mathcal{L}_a}$ associated to a sequence of bounded radial functions in $\dot{H}^1_a$. In $\mathbb{R}^3$, the corresponding result in the non-radial case has been proved in [29]. From the refined Strichartz estimate(Lemma 2.17) for all $\mathbb{R}^d(d \geq 3)$, it is laborious to generalize their result to higher dimensions.

The liner profile decomposition gives us the insight that bounded functions in $\dot{H}^1_a$ can be decomposed into summation of linear profiles and a tail term. The linear profiles have decoupling properties(especially the energy decoupling), the tail term decays in certain Strichartz space norm, and different symmetries in the linear profiles are orthogonal. We will see the applications in Section 2.4 proving the existence of minimal blow-up solutions. This treatment was first introduced by Bahouri and Gérard [2] for energy critical NLW and Keraani [27] in the setting of NLS. Variants of linear profile decomposition for the Schrödinger equation were summarized in [31].

We remark some recent development and application of the linear profile decomposition: (1)obstacle problems(NLS on smooth exterior domains), see [33, 51]; (2)symplectic non-squeezing of NLS and Hartree equation, check [35, 52]. (3)global well-posedness and scattering of NLS$_a$, see [29].

In this thesis, we only use the linear profile decomposition for radial functions in $\dot{H}^1_a$. The formulation is simple compared with the non-radial case, and the major difference is the loss of the translation symmetry in the linear profile due to the radial assumption.

Theorem 2.16 (Radial $\dot{H}^1_a$ linear profile decomposition, [29])**.** *Let $\{f_n\}$ be a bounded sequence in $\dot{H}^1_a$. After passing to a subsequence, there exist $J^* \in \{0, 1, 2, \ldots, \infty\}$, non-zero profiles $\{\phi^j\}_{j=1}^{J^*} \subset \dot{H}^1$, $\{\lambda_n^j\}_{j=1}^{J^*} \subset (0, \infty)$, and $\{t_n^j\}_{j=1}^{J^*} \subset \mathbb{R}$ such that for each finite $0 \le J \le J^*$, we have the decomposition*

$$f_n = \sum_{j=1}^{J} \phi_n^j + w_n^J \quad \text{with} \quad \phi_n^j = (e^{-it_n^j \mathcal{L}_a} \phi^j)_{[\lambda_n^j]} \quad \text{and} \quad w_n^J \in \dot{H}^1_a$$

satisfying

$$\begin{aligned}
\lim_{J \to J^*} \limsup_{n \to \infty} \|e^{-it\mathcal{L}_a} w_n^J\|_{S(\mathbb{R})} = 0,\\
\lim_{n \to \infty} \{\|f_n\|_{\dot{H}^1_a}^2 - \sum_{j=1}^{J} \|\phi_n^j\|_{\dot{H}^1_a}^2 - \|w_n^J\|_{\dot{H}^1_a}^2\} = 0,\\
\lim_{n \to \infty} \{\|f_n\|_{2^*}^{2^*} - \sum_{j=1}^{J} \|\phi_n^j\|_{2^*}^{2^*} - \|w_n^J\|_{2^*}^{2^*}\} = 0.
\end{aligned}$$

Moreover, for all $j \ne k$ we have the asymptotic orthogonality property

$$|\tfrac{\lambda_n^j}{\lambda_n^k}| + |\tfrac{\lambda_n^k}{\lambda_n^j}| + \tfrac{|t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2|}{\lambda_n^j \lambda_n^k} \to \infty \quad as \quad n \to \infty. \tag{2.21}$$

In addition, we may assume that for each j either $t_n^j \equiv 0$ or $t_n^j \to \pm\infty$.

From the Refined Strichartz Estimate, it is routine to obtain an inverse Strichartz inequality which generates the linear profile decomposition by iteration. We refer the reader to [29, 31] for details.

Lemma 2.17 (Refined Strichartz Estimate)**.** *Let $f \in \dot{H}^1_a$ and $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$, then for all $d \ge 3$,*

$$\left\|e^{-it\mathcal{L}_a} f\right\|_{S(I)} \lesssim (\sup_{N \in 2^{\mathbb{Z}}} \left\|e^{-it\mathcal{L}_a} f_N\right\|_{S(I)})^{\frac{4}{d+2}} \cdot \|f\|_{\dot{H}^1_a}^{\frac{d-2}{d+2}}. \tag{2.22}$$

Proof. In the case $d=3$, one can refer to [29]. In the following, we focus on the case $d\geq 4$.

As $a>-(\frac{d-2}{2})^2+(\frac{d-2}{d+2})^2$, from the definition of σ, $\frac{2(d+2)}{d-2}<\frac{d}{\sigma}<\infty$. We pick $\varepsilon>0$ such that

$$\frac{2(d+2)}{d-2-\varepsilon}<\frac{d}{\sigma}\text{ and }\frac{2(d+2)}{d-2+\varepsilon}>(\frac{d}{\sigma})'. \tag{2.23}$$

Then the assumption in the Bernstein inequality is fulfilled. Together with Bernstein and Strichartz inequality, (2.23) yields that

$$\begin{cases} \left\|e^{-it\mathcal{L}_a}f_N\right\|_{\frac{2(d+2)}{d-2},\frac{2(d+2)}{d-2-\varepsilon}}\lesssim N^{\frac{d\varepsilon}{2(d+2)}}\left\|f_N\right\|_{\dot H^1_a} \\ \left\|e^{-it\mathcal{L}_a}f_N\right\|_{\frac{2(d+2)}{d-2},\frac{2(d+2)}{d-2+\varepsilon}}\lesssim N^{-\frac{d\varepsilon}{2(d+2)}}\left\|f_N\right\|_{\dot H^1_a}\end{cases}. \tag{2.24}$$

When $d\geq 4$, we have that $\frac{d+2}{3(d-2)}\leq 1$, thus $(\sum_n|a_n|)^{\frac{d+2}{3(d-2)}}\leq\sum_n|a_n|^{\frac{d+2}{3(d-2)}}$. From Littlewood-Paley's Theorem, Hölder inequality and (2.24), we get

$$\begin{aligned}
&\left\|e^{-it\mathcal{L}_a}f\right\|_{S(I)}^{\frac{2(d+2)}{d-2}}\lesssim\int(\sum_N\left|e^{-it\mathcal{L}_a}f_N\right|^2)^{\frac{d+2}{d-2}}dxdt\\
&\lesssim\int(\sum_{N_1}\left|e^{-it\mathcal{L}_a}f_{N_1}\right|^2)^{\frac{d+2}{3(d-2)}}\cdots(\sum_{N_3}\left|e^{-it\mathcal{L}_a}f_{N_3}\right|^2)^{\frac{d+2}{3(d-2)}}dxdt\\
&\lesssim\int\sum_{N_1\leq N_2\leq N_3}\left|e^{-it\mathcal{L}_a}f_{N_1}\right|^{\frac{2(d+2)}{3(d-2)}}\left|e^{-it\mathcal{L}_a}f_{N_2}\right|^{\frac{2(d+2)}{3(d-2)}}\left|e^{-it\mathcal{L}_a}f_{N_3}\right|^{\frac{2(d+2)}{3(d-2)}}dxdt\\
&\lesssim\sum_{N_1\leq N_2\leq N_3}\left\|e^{-it\mathcal{L}_a}f_{N_1}\right\|_{\frac{2(d+2)}{d-2},\frac{2(d+2)}{d-2-\varepsilon}}\left\|e^{-it\mathcal{L}_a}f_{N_1}\right\|_{S(I)}^{\frac{10-d}{3(d-2)}}\left\|e^{-it\mathcal{L}_a}f_{N_2}\right\|_{S(I)}^{\frac{2(d+2)}{3(d-2)}}\\
&\cdot\left\|e^{-it\mathcal{L}_a}f_{N_3}\right\|_{S(I)}^{\frac{10-d}{3(d-2)}}\left\|e^{-it\mathcal{L}_a}f_{N_3}\right\|_{\frac{2(d+2)}{d-2},\frac{2(d+2)}{d-2+\varepsilon}}\\
&\lesssim(\sup_{N\in 2^{\mathbb{Z}}}\left\|e^{-it\mathcal{L}_a}f_N\right\|_{S(I)})^{\frac{8}{d-2}}\sum_{N_1\leq N_3}[1+\log(\frac{N_3}{N_1})][\frac{N_1}{N_3}]^{\frac{d\varepsilon}{2(d+2)}}\left\|f_{N_1}\right\|_{\dot H^1_a}\left\|f_{N_3}\right\|_{\dot H^1_a}\\
&\lesssim(\sup_{N\in 2^{\mathbb{Z}}}\left\|e^{-it\mathcal{L}_a}f_N\right\|_{S(I)})^{\frac{8}{d-2}}\cdot\left\|f\right\|_{\dot H^1_a}^2.
\end{aligned}$$

The proof of this lemma is complete. □

2.4 Existence of Minimal Blow-up Solutions

Fix $E > 0$, we define

$$L(E) = \sup_u \{\|u\|_{S(I)}\},$$

where the supremum is taken for all solutions u to NLS$_a$ with maximal time interval I such that $E(u(t)) < E$ and $\|u(t)\|_{\dot{H}_a^1} \le \|W\|_{\dot{H}_a^1}$ for some $t \in I$.

From the Theorem 2.14 and coercivity of energy, there exists $\eta_0 > 0$ such that if $E(u_0) \le \eta_0$, then u exists globally and $\|u\|_{S(\mathbb{R})} \lesssim E(u_0)^{\frac{1}{2}}$. Indeed, from Strichartz estimate, the same argument also yields

$$\|u\|_{\dot{X}_a^1(\mathbb{R})} \lesssim E(u_0)^{\frac{1}{2}} \text{ for all } E(u_0) \le \eta_0. \tag{2.25}$$

Thus, there exists a **critical energy** E_c with $\eta_0 \le E_c$, such that, for any solution u to NLS$_a$ with $E(u) < E_c$ and $\|u_0\|_{\dot{H}_a^1} < \|W\|_{\dot{H}_a^1}$, then the maximal time interval $I = \mathbb{R}$ and $\|u\|_{S(\mathbb{R})} < \infty$. In other words,

$$L(E) = \infty \text{ if } E \ge E_c;\ L(E) < \infty \text{ if } E < E_c.$$

Note that from Theorem 2.15, if $\|u\|_{S(I)} < \infty$, then $I = \mathbb{R}$. We aim to show $E_c = E(W)$. By way of contradiction, we assume $E_c < E(W)$. Consequently, we have the following compactness result.

Lemma 2.18 (**Palais-Smale Condition**). *Let $u_n : I_n \times \mathbb{R}^d \to \mathbb{C}$ be a sequence of solutions to NLS$_a$ and $t_n \in I_n$. Suppose that*

$$\begin{cases} \|u_n(t_n)\|_{\dot{H}_a^1} < \|W\|_{\dot{H}_a^1} \\ \lim_{n\to\infty} E(u_n) = E_c < E(W) \\ \lim_{n\to\infty} \|u_n\|_{S(t\ge t_n)} = \lim_{n\to\infty} \|u_n\|_{S(t\le t_n)} = \infty \end{cases}, \tag{2.26}$$

then there exists $\{\lambda_n\} \subset \mathbb{R}^+$ *such that* $\{(u_n(t_n))_{[\lambda_n]}\}$ *is precompact in* $\dot{H}^1_a$.

Proof. By the time translation invariance, we may assume that $t_n \equiv 0$, thus

$$\lim_{n\to\infty} \|u_n\|_{S(t\geq 0)} = \lim_{n\to\infty} \|u_n\|_{S(t\leq 0)} = \infty. \tag{2.27}$$

Apply the linear profile decomposition on $\{u_{n,0}\}$, then after passing to a subsequence, there exist $J^* \in \{1, 2, \ldots, \infty\}$, non-zero profiles $\{\phi^j\}_{j=1}^{J^*} \subset \dot{H}^1(\mathbb{R}^d)$, $\{\lambda_n^j\}_{j=1}^{J^*} \subset (0, \infty)$, and $\{t_n^j\}_{j=1}^{J^*} \subset \mathbb{R}$ such that for each finite $1 \leq J \leq J^*$, we have the decomposition

$$u_{n,0} = \sum_{j=1}^{J} \phi_n^j + w_n^J \quad \text{with} \quad \phi_n^j = (e^{-it_n^j \mathcal{L}_a}\phi^j)_{[\lambda_n^j]} \text{ and } w_n^J \in \dot{H}^1_a \tag{2.28}$$

satisfying

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|e^{-it\mathcal{L}_a} w_n^J\|_{S(\mathbb{R})} = 0, \tag{2.29}$$

$$\lim_{n\to\infty} \{\|u_{n,0}\|^2_{\dot{H}^1_a} - \sum_{j=1}^{J} \|\phi_n^j\|^2_{\dot{H}^1_a} - \|w_n^J\|^2_{\dot{H}^1_a}\} = 0, \tag{2.30}$$

$$\lim_{n\to\infty} \{E(u_n) - \sum_{j=1}^{J} E(\phi_n^j) - E(w_n^J)\} = 0. \tag{2.31}$$

From coercivity of energy, all energies are non-negative. Thus, $\sum_{j=1}^{J} E(\phi_n^j) \leq E(u_n)$ for n large. Then, $\sup_j \overline{\lim}_{n\to\infty} E(\phi_n^j) \leq E_c$. We discuss two scenarios.

Scenario 1: $\sup_j \overline{\lim}_{n\to\infty} E(\phi_n^j) = E_c$.

When $J^* \geq 1$, from $\|u_{n,0}\|_{\dot{H}^1_a} \leq \|W\|_{\dot{H}^1_a}$, and (2.30), $\|\phi^j\|_{\dot{H}^1_a} \leq \|W\|_{\dot{H}^1_a}$ and $\|w_n^J\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}$ for n large. The coercivity of energy implies that $E(\phi_n^j)$ and $E(w_n^J)$ are all non-negative. Thus, $J^* = 1$ by (2.31). As $E(\phi_n^1) \to E_c$, coercivity of energy yields that $\|w_n^1\|_{\dot{H}^1_a} \to 0$. Up to a sub-sequence, $t_n^1 \to t^* \in \{-\infty, \infty, 0\}$. If $t^* = \infty$, from Strichartz estimate and Monotone convergence theorem,

$$\|e^{-it\mathcal{L}_a} u_{n,0}\|_{S(t\geq 0)} \lesssim \|e^{-it\mathcal{L}_a}\phi^1\|_{S(t\geq t_n^1)} + \|w_n^1\|_{\dot{H}^1_a} \to 0.$$

Theorem 2.14 implies that $\lim_{n\to\infty} \|u_n\|_{S(t\geq 0)} = 0$ which contradicts (2.27), thus $t^* \neq \infty$. Similarly, $t^* \neq -\infty$.

If $t^* = 0$, pick $\lambda_n = (\lambda_n^1)^{-1}$, then $(u_{n,0})_{[\lambda_n]} = e^{-it_n^1\mathcal{L}_a}\phi^1 + (w_n^1)_{[\lambda_n]}$. As $\|(w_n^1)_{[\lambda_n]}\|_{\dot{H}_a^1} = \|w_n^1\|_{\dot{H}_a^1} \to 0$ and $t_n^1 \to 0$, $(u_{n,0})_{[\lambda_n]} \to \phi^1$ in $\dot{H}_a^1$. Thus, $\{(u_n(0))_{[\lambda_n]}\}$ is precompact in $\dot{H}_a^1$.

It remains to exclude the following scenario.

Scenario 2: $\sup_j \overline{\lim}_{n\to\infty} E(\phi_n^j) < E_c - 2\delta$ for some $0 < \delta < \frac{E_c}{2}$.

For finite $J \leq J^*$,

$$E(\phi_n^j) \leq E_c - \delta \text{ for all } 1 \leq j \leq J. \tag{2.32}$$

From coercivity of energy, we also have

$$\|\phi_n^j\|_{\dot{H}_a^1} < \|W\|_{\dot{H}_a^1}, \quad 1 \leq j \leq J. \tag{2.33}$$

When $t_n^j \equiv 0$, define $v^j : I_j \times \mathbb{R}^d \to \mathbb{C}$ as the maximal life-span(I_j) solution to NLS$_a$ with initial data ϕ^j; when $t_n^j \to \pm\infty$, define $v^j : I_j \times \mathbb{R}^d \to \mathbb{C}$ as the maximal life-span(I_j) solution to NLS$_a$ which scatters to $e^{-it\mathcal{L}_a}\phi^j$ as $t \to \pm\infty$. Let $v_n^j = v^j((\lambda_n^j)^{-2}t + t_n^j, x)_{[\lambda_n^j]}$, then it is a maximal life-span$(I_n^j = \{t \in R, (\lambda_n^j)^{-2}t + t_n^j \in I_j\})$ solution. From above construction, we have

$$\lim_{n\to\infty} \|v_n^j(0) - \phi_n^j\|_{\dot{H}_a^1} = 0. \tag{2.34}$$

From $0 < E(\phi_n^j) < E_c - \delta$, (2.33), and the definition of E_c, v_n^j is a global solution with $\|v_n^j\|_{S(\mathbb{R})} \leq L(E_c - \delta) < \infty$. Strichartz estimate then yields that $\|v_n^j\|_{\dot{X}^1(\mathbb{R})} < \infty$. From density argument, given $\varepsilon > 0$, there exists $\psi_\varepsilon^j \in C_c^\infty$ such that

$$\|v^j - \psi_\varepsilon^j\|_{\dot{X}^1(\mathbb{R})} < \varepsilon.$$

From change of variable in time, we obtain

$$\|v_n^j(t,x) - \psi_\varepsilon^j((\lambda_n^j)^{-2}t + t_n^j, x)_{[\lambda_n^j]}\|_{\dot{X}^1(\mathbb{R})} < \varepsilon. \tag{2.35}$$

Next, we discuss several properties associated with v_n^j.

- Boundedness of v_n^j:

$$\|v_n^j\|_{\dot{X}^1(\mathbb{R})} \lesssim \begin{cases} E(\phi_n^j)^{\frac{1}{2}}, & \text{if } E(\phi_n^j) \le \eta_0 \\ 1, & \text{if } \eta_0 < E(\phi_n^j) \end{cases} \tag{2.36}$$

By (2.34), we get

$$E(v_n^j(0)) \lesssim E(\phi_n^j) \text{ for all } j \le J. \tag{2.37}$$

When $E(\phi_n^j) \le \eta_0$, from (2.25), we have $\|v_n^j\|_{\dot{X}^1(\mathbb{R})} \lesssim E(\phi_n^j)^{\frac{1}{2}}$; when $\eta_0 < E(\phi_n^j)$, by (2.32), we get $E(\phi_n^j) < E_c - \delta$. From the definition of E_c and Strichartz estimate, $\|v_n^j\|_{\dot{X}^1(\mathbb{R})} \lesssim 1$.

- Orthogonality of v_n^j and $v_n^k (j \ne k)$:

$$\|\nabla v_n^j \nabla v_n^k\|_{\frac{d+2}{d-2}, \frac{d(d+2)}{d^2+4}} + \|v_n^j v_n^k\|_{\frac{d+2}{d-2}, \frac{d+2}{d-2}} + \|\nabla v_n^j v_n^k\|_{\frac{d+2}{d-2}, \frac{d(d+2)}{d^2-d+2}} = o_n(1). \tag{2.38}$$

To ease notion, we define $T_n^j f(t,x) = f((\lambda_n^j)^{-2}t + t_n^j, x)_{[\lambda_n^j]}$

Indeed, by (2.35) and change of variable, we have

$$\begin{aligned} \|v_n^j v_n^k\|_{\frac{d+2}{d-2}, \frac{d+2}{d-2}} &\le \|v_n^j - T_n^j \psi_\varepsilon^j\|_{S(\mathbb{R})} \left\|v_n^k\right\|_{S(\mathbb{R})} + \|v_n^j\|_{S(\mathbb{R})} \|v_n^k - T_n^k \psi_\varepsilon^k\|_{S(\mathbb{R})} \\ &\quad + \|T_n^j \psi_\varepsilon^j T_n^k \psi_\varepsilon^k\|_{\frac{d+2}{d-2}, \frac{d+2}{d-2}} \\ &\lesssim \varepsilon + \|T_n^j \psi_\varepsilon^j T_n^k \psi_\varepsilon^k\|_{\frac{d+2}{d-2}, \frac{d+2}{d-2}}. \end{aligned}$$

In addition, if $\lambda_n^j(\lambda_n^k)^{-1}+(\lambda_n^j)^{-1}\lambda_n^k \to \infty$, by Hölder inequality(equip ψ_ε^j, ψ_ε^k with $L^\infty_{t,x}$ norm), we get

$$\begin{aligned}
&\|T_n^j\psi_\varepsilon^j T_n^k\psi_\varepsilon^k\|_{\frac{d+2}{d-2},\frac{d+2}{d-2}}\\
\leq\ & \min\{\|(T_n^k)^{-1}T_n^j\psi_\varepsilon^j\psi_\varepsilon^k\|_{\frac{d+2}{d-2},\frac{d+2}{d-2}}, \|\psi_\varepsilon^j(T_n^j)^{-1}T_n^k\psi_\varepsilon^k\|_{\frac{d+2}{d-2},\frac{d+2}{d-2}}\}\\
\leq\ & \min\{(\frac{\lambda_n^j}{\lambda_n^k})^{\frac{1}{2(d-2)}}, (\frac{\lambda_n^k}{\lambda_n^j})^{\frac{1}{2(d-2)}}\} = o_n(1).
\end{aligned}$$

Thus, we may assume $\lambda_n^j(\lambda_n^k)^{-1} \to \lambda_0 \in (0,\infty)$. (2.21) then yields that $\frac{|t_n^j(\lambda_n^j)^2-t_n^k(\lambda_n^k)^2|}{\lambda_n^j\lambda_n^k} \to \infty$. As $\lambda_n^j(\lambda_n^k)^{-1} \to \lambda_0$, $|t_n^j-(\lambda_n^k)^2(\lambda_n^j)^{-2}t_n^k| \to \infty$. This implies that

$$\begin{aligned}
\|T_n^j\psi_\varepsilon^j T_n^k\psi_\varepsilon^k\|_{\frac{d+2}{d-2},\frac{d+2}{d-2}} &\lesssim_{\lambda_0} \|\psi_\varepsilon^j((\lambda_0)^{-2}t+t_n^j-(\lambda_n^k)^2(\lambda_n^j)^{-2}t_n^k, \frac{x}{\lambda_0})\psi_\varepsilon^k\|_{\frac{d+2}{d-2},\frac{d+2}{d-2}}\\
&= o_n(1).
\end{aligned}$$

Similarly, other terms are $o_n(1)$.

We define an approximate solution u_n^J to NLS$_a$ by

$$u_n^J = \sum_{j=1}^J v_n^j + e^{-it\mathcal{L}_a}w_n^J. \tag{2.39}$$

We will prove the following three claims, which demonstrates that for large n and J, u_n^J is an close approximate solution to NLS$_a$ with finite $S(\mathbb{R})$ norm in the sense of Theorem 2.13. Then u_n is also a global solution with finite $S(\mathbb{R})$ norm which contradicts (2.27). Hence, we can exclude Scenario 2.

Claim 1: $\lim_{n\to\infty}\left\|u_n^J(0)-u_n(0)\right\|_{\dot H_a^1}=0$ for any J.

Claim 2: $\varlimsup\limits_{n\to\infty}\left\|u_n^J\right\|_{\dot X^1(\mathbb{R})} \lesssim_{E_c,\delta} 1$ uniformly in J.

Claim 3: $\lim_{J\to\infty} \overline{\lim}_{n\to\infty} \|(i\partial_t + \Delta_\Omega)u_n^J + |u_n^J|^{\frac{4}{d-2}} u_n^J\|_{\dot{N}^1(\mathbb{R})} = 0.$

To ease notion, we will skip writing $\mathbb{R}\times\mathbb{R}^d$ in the remaining part of this section.

Proof of Claim 1:

From the construction of $u_n(0)$ in (2.28) and u_n^J in (2.39), by (2.34), as $n \to \infty$, we have

$$\left\|u_n^J(0) - u_n(0)\right\|_{\dot{H}_a^1} \leq \sum_{j=1}^{J} \|v_n^j(0) - \phi_n^j\|_{\dot{H}_a^1} \to 0.$$

Proof of Claim 2:

From the linear profile decomposition, we see that $w_n^J \in \dot{H}_a^1$, thus it suffices to show

$$\overline{\lim_{n\to\infty}} \|\sum_{j=1}^{J} v_n^j\|_{\dot{X}^1(\mathbb{R})} \lesssim_{E_c,\delta} 1 \tag{2.40}$$

uniformly for finite $J \leq J^*$. While by (2.37) and (2.38),

$$\begin{aligned}
\|\sum_{j=1}^{J} v_n^j\|_{\dot{X}^1(\mathbb{R})}^2 &= \|(\sum_{j=1}^{J} \nabla v_n^j)^2\|_{\frac{d+2}{d-2}, \frac{d(d+2)}{d^2+4}} \\
&\lesssim \sum_{j=1}^{J} \|v_n^j\|_{\dot{X}^1(\mathbb{R})}^2 + C(J) \sum_{j\neq k} \|\nabla v_n^j \nabla v_n^k\|_{\frac{d+2}{d-2}, \frac{d(d+2)}{d^2+4}} \\
&\lesssim \sum_{j=1}^{J} E(v_n^j(0)) + o_n(1) \\
&\lesssim \sum_{j=1}^{J} E(\phi_n^j) + 1 \lesssim_{E_c,\delta} 1.
\end{aligned}$$

Proof of Claim 3:

Define $F(z) = -|z|^{\frac{4}{d-2}} z$, then

$$\begin{aligned}
(i\partial_t + \Delta_\Omega)u_n^J - F(u_n^J) &= \sum_{j=1}^{J} F(v_n^j) - F(u_n^J) \\
&= \sum_{j=1}^{J} F(v_n^j) - F(\sum_{j=1}^{J} v_n^j) + F(\sum_{j=1}^{J} v_n^j) - F(u_n^J).
\end{aligned}$$

Thus, it suffices to show

$$\lim_{J\to\infty}\overline{\lim_{n\to\infty}}\|\sum_{j=1}^{J}F(v_n^j)-F(\sum_{j=1}^{J}v_n^j)\|_{\dot N^1}=0, \tag{2.41}$$

and

$$\lim_{J\to\infty}\overline{\lim_{n\to\infty}}\left\|F(u_n^J-e^{-it\mathcal{L}_a}w_n^J)-F(u_n^J)\right\|_{\dot N^1}=0. \tag{2.42}$$

We first prove (2.41). Note that

$$|\nabla[\sum_{j=1}^{J}F(v_n^j)-F(\sum_{j=1}^{J}v_n^j)]|\lesssim_J\sum_{j\neq k}|v_n^k|^{\frac{4}{d-2}}|\nabla v_n^j|\ . \tag{2.43}$$

By Strichartz estimate and (2.38), we get

$$\begin{aligned}&\|\sum_{j=1}^{J}F(v_n^j)-F(\sum_{j=1}^{J}v_n^j)\|_{\dot N^1}\lesssim\sum_{j\neq k}\||\nabla v_n^j||v_n^k|^{\frac{4}{d-2}}\|_{2,\frac{2d}{d+2}}\\ \lesssim&\sum_{j\neq k}\left\|v_n^k\right\|_{S(\mathbb{R})}^{\frac{6-d}{d-2}}\|\nabla v_n^jv_n^k\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}}=o_n(1).\end{aligned}$$

Next, we prove (2.42). Note that

$$\begin{aligned}\left|\nabla(F(u_n^J-e^{-it\mathcal{L}_a}w_n^J)-F(u_n^J))\right|\ &\lesssim_J(|u_n^J|^{\frac{4}{d-2}}+|e^{-it\mathcal{L}_a}w_n^J|^{\frac{4}{d-2}})|\nabla e^{-it\mathcal{L}_a}w_n^J|\\ &\quad+(|e^{-it\mathcal{L}_a}w_n^J|^{\frac{4}{d-2}}+|e^{-it\mathcal{L}_a}w_n^J||u_n^J|^{\frac{6-d}{d-2}})|\nabla u_n^J|\end{aligned}$$

By Strichartz estimate and Claim 2, we have

$$\begin{aligned}&\left\|F(u_n^J-e^{-it\mathcal{L}_a}w_n^J)-F(u_n^J)\right\|_{\dot N^1}\\ \lesssim&\ \left\|u_n^J\right\|_{\dot X^1}(\left\|u_n^J\right\|_{S(\mathbb{R})}^{\frac{6-d}{d-2}}\left\|e^{-it\mathcal{L}_a}w_n^J\right\|_{S(\mathbb{R})}+\left\|e^{-it\mathcal{L}_a}w_n^J\right\|_{S(\mathbb{R})}^{\frac{4}{d-2}})\\ &+\left\|e^{-it\mathcal{L}_a}w_n^J\right\|_{\dot X^1}\left\|e^{-it\mathcal{L}_a}w_n^J\right\|_{S(\mathbb{R})}^{\frac{4}{d-2}}+\||u_n^J|^{\frac{4}{d-2}}(\nabla e^{-it\mathcal{L}_a}w_n^J)\|_{\dot N^1}\\ \lesssim&\ \left\|e^{-it\mathcal{L}_a}w_n^J\right\|_{S(\mathbb{R})}^{\frac{4}{d-2}}+\left\|e^{-it\mathcal{L}_a}w_n^J\right\|_{S(\mathbb{R})}+\||u_n^J|^{\frac{4}{d-2}}(\nabla e^{-it\mathcal{L}_a}w_n^J)\|_{\dot N^1}.\end{aligned}$$

By (2.29), it suffices to prove

$$\lim_{J\to\infty}\overline{\lim_{n\to\infty}}\||u_n^J|^{\frac{4}{d-2}}(\nabla e^{-it\mathcal{L}_a}w_n^J)\|_{\dot N^1}=0.$$

Indeed, by Claim 2,

$$\begin{aligned}\||u_n^J|^{\frac{4}{d-2}}|\nabla e^{-it\mathcal{L}_a}w_n^J|\|_{\dot N^1} &\lesssim \left\|u_n^J\right\|_{S(\mathbb{R})}^{\frac{6-d}{d-2}}\left\|u_n^J\nabla e^{-it\mathcal{L}_a}w_n^J\right\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}}\\ &\lesssim \|(\sum_{j=1}^J v_n^j+e^{-it\mathcal{L}_a}w_n^J)\nabla e^{-it\mathcal{L}_a}w_n^J\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}}.\end{aligned}$$

From equivalence of Sobolev norms, Strichartz estimate and (2.29),

$$\begin{aligned}\left\|e^{-it\mathcal{L}_a}w_n^J\nabla e^{-it\mathcal{L}_a}w_n^J\right\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}} &\lesssim \left\|e^{-it\mathcal{L}_a}w_n^J\right\|_{\dot X^1}\left\|e^{-it\mathcal{L}_a}w_n^J\right\|_{S(\mathbb{R})}\\ &\lesssim \left\|w_n^J\right\|_{H_0^1}\left\|e^{-it\mathcal{L}_a}w_n^J\right\|_{S(\mathbb{R})}\to 0.\end{aligned}$$

It remains to show

$$\lim_{J\to\infty}\overline{\lim_{n\to\infty}}\|(\sum_{j=1}^J v_n^j)\nabla e^{-it\mathcal{L}_a}w_n^J\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}}=0.$$

We need the following argument which can be proved same as Claim 2:

Given $\eta>0$, there exists $J'=J'(\eta)$ such that

$$\overline{\lim_{n\to\infty}}\|\sum_{j=J'}^J v_n^j\|_{\dot X^1}<\eta \qquad \text{uniformly in } J\ge J'.$$

Then,

$$\overline{\lim_{n\to\infty}}\|(\sum_{j=J'}^J v_n^j)\nabla e^{-it\mathcal{L}_a}w_n^J\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}}\lesssim\overline{\lim_{n\to\infty}}\|\sum_{j=J'}^J v_n^j\|_{S(\mathbb{R})}\left\|e^{-it\mathcal{L}_a}w_n^J\right\|_{\dot X^1}\lesssim\eta.$$

It is further reduced to prove

$$\lim_{J\to\infty}\overline{\lim_{n\to\infty}}\|v_n^j\nabla e^{-it\mathcal{L}_a}w_n^J\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}}=0,\qquad \text{for all } 1\le j\le J'. \tag{2.44}$$

We approximate v_n^j by $C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ function ψ_ε^j obeying (2.35) with support in $[-T,T]\times\{|x|\le R\}$. By Corollary 2.8 and (2.29), we have that

$$\begin{aligned}
&\|v_n^j \nabla e^{-it\mathcal{L}_a} w_n^J\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}}\\
\lesssim\ & \|v_n^j-\psi_\varepsilon^j\|_{S(\mathbb{R})}\left\|e^{-it\mathcal{L}_a}w_n^J\right\|_{\dot X^1}+\|\psi_\varepsilon^j\|_{L^\infty_{t,x}}\left\|\nabla e^{-it\mathcal{L}_a}w_n^J\right\|_{\frac{d+2}{d-2},\frac{d(d+2)}{d^2-d+2}}\\
\lesssim\ & \varepsilon+T^{\frac{(d-2)^2}{2(d+2)^2}}R^{\frac{d^3+4d-16}{2d^2+8d+8}}\|e^{-it\mathcal{L}_a}w_0\|_{S(\mathbb{R})}^{\frac{d-2}{d+2}}\|w_0\|_{\dot H^1_a}^{\frac{4}{d+2}}\\
&+T^{\frac{(d-2)^2}{4(d+2)^2}}R^{\frac{d^3+d^2-12}{2(d+2)^2}}\|e^{-it\mathcal{L}_a}w_0\|_{S(\mathbb{R})}^{\frac{d-2}{2(d+2)}}\|w_0\|_{\dot H^1_a}^{\frac{d+6}{2(d+2)}}.
\end{aligned}$$

By taking the limit and choosing ε small, we obtain (2.44). Hence, Claim 3 holds. The proof of this lemma is complete. □

From the Palais-Smale condition lemma, it is standard to derive the existence of the minimal blow-up solution.

Theorem 2.19 (Existence of minimal blow-up solution)**.** *Suppose Theorem 1.1 fails. Then there exist a critical energy* $0<E_c<E(W)$ *and a solution* $u:[0,T^*)\times\mathbb{R}^d\to\mathbb{C}$ *to* NLS$_a$ *with*

$$E(u(t))=E_c, \|u(t)\|_{\dot H^1_a}<\|W\|_{\dot H^1_a}, \quad \textit{and} \quad \|u\|_{S[0,T^*)}=\infty. \tag{2.45}$$

Moreover, there exists $\lambda(t):[0,T^*)\to\mathbb{R}^+$ *with* $\inf_{t\in[0,T^*)}\lambda(t)\ge 1$ *such that the set* $\{u(t)_{[\lambda(t)]}: t\in[0,T^*)\}$ *is precompact in* $\dot H^1_a$*. An analogous result holds backward in time.*

Proof. Assume Theorem 1.1 fails. From the definition of E_c, there exists a sequence of solutions $u_n: I_n\times\mathbb{R}^d\to\mathbb{C}$ to NLS$_a$ such that (2.26) holds. By picking $t_n\in I_n$ such that $\|u_n(t)\|_{S(t\ge t_n)}=\|u_n(t)\|_{S(t\le t_n)}$. From the time translation invariance, we may assume $t_n = 0$. By Lemma 2.18, there exists $\{\lambda_n\} \subset \mathbb{R}^+$ such that $\{u_n(0)_{[\lambda_n]}\}$ is precompact in $\dot{H}^1_a$, thus passing to a subsequence,

$$u_n(0)_{[\lambda_n]} \to u_0 \text{ in } \dot{H}^1_a.$$

Recall that in the proof of Lemma 2.18, only Scenario 1 remains valid. Hence,

$$E(u_0) = E_c < E(W), \quad \|u_0\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}. \tag{2.46}$$

Next, define $u : I \times \mathbb{R}^d \to \mathbb{C}$ with $u(0) = u_0$ as the maximal life-span solution to NLS_a, we obtain that

$$\|u(t)\|_{S(t\geq 0)} = \|u(t)\|_{S(t\leq 0)} = \infty,$$

from the stability(Theorem 2.13) and $\lim_{n\to\infty} S_{\geq t_n}(u_n) = \lim_{n\to\infty} S_{\leq t_n}(u_n) = \infty$ in (2.26). Since $\|u(t)\|_{S(I)} = \infty$, for any sequence $\{s_n\} \subset I$, $\|u(t)\|_{S(t\geq s_n)} = \|u(t)\|_{S(t\leq s_n)} = \infty$. Lemma 2.18 shows that $\{u(s_n)_{[\lambda_{s_n}]}\}$ is precompact in $\dot{H}^1_a$ for some sequence $\{\lambda_{s_n}\} \subset \mathbb{R}^+$.

Our remaining task is to show the existence of the function $\lambda(t)$. Fix $t \in [0, T^*)$, we define

$$\lambda(t) = \sup\{\lambda : \int_{|x|\leq \frac{1}{\lambda}} |\sqrt{\mathcal{L}_a} u(t)|^2 dx = E_c\}. \tag{2.47}$$

From the definition of energy, (2.46), and (2.47),

$$2E(u(t)) = 2E_c \leq \|u(t)\|^2_{\dot{H}^1_a}, \quad 0 < \lambda(t) < \infty. \tag{2.48}$$

As proven above, for any sequence $\{s_n\} \subset [0, T^*)$, there exists $\{\lambda_{s_n}\} \subset \mathbb{R}^+$ such that $u(s_n)_{[\lambda_{s_n}]} \to v$ in $\dot{H}^1_a$ up to a sub-sequence. Consequently,

$$\left\|u(s_n)_{[\lambda_{s_n}]}\right\|^2_{\dot{H}^1_a} = \|u(s_n)\|^2_{\dot{H}^1_a} \to \|v\|^2_{\dot{H}^1_a}. \tag{2.49}$$

We claim that there exists $C \geq 1$ such that

$$C^{-1}\lambda(s_n) \leq \lambda_{s_n} \leq C\lambda(s_n). \tag{2.50}$$

From changing of variables,

$$E_c = \int_{|x| \leq \frac{1}{\lambda(s_n)}} |\sqrt{\mathcal{L}_a} u(s_n)|^2 dx = \int_{|x| \leq \frac{\lambda_{s_n}}{\lambda(s_n)}} |\sqrt{\mathcal{L}_a} u(s_n)_{[\lambda_{s_n}]}|^2 dx.$$

If $\frac{\lambda_{s_n}}{\lambda(s_n)} \to \infty$, from (2.48), (2.49) and $u(s_n)_{[\lambda_{s_n}]} \to v$ in $\dot{H}^1_a$, we would have $\|v\|^2_{\dot{H}^1_a} = E_c \geq 2E_c$, which contradicts $E_c > 0$. If $\frac{\lambda_{s_n}}{\lambda(s_n)} \to 0$, as $u(s_n)_{[\lambda_{s_n}]} \to v$ in $\dot{H}^1_a$, we would get $E_c = 0$, which again contradicts $E_c > 0$. Hence, (2.50) holds.

Indeed, (2.50) and $u(s_n)_{[\lambda_{s_n}]} \to v$ in $\dot{H}^1_a$ imply that $u(s_n)_{[\lambda(s_n)]}$ converges strongly in $\dot{H}^1_a$. As $\{s_n\}$ is arbitrarily taken,

$$\{u(t)_{[\lambda(t)]} : t \in I\} \text{ is precompact in } \dot{H}^1_a. \tag{2.51}$$

In addition, (2.51) holds for a large class of functions $\lambda(t)$. One can extract one special class such that $\inf_{t\in[0,T^*)} \lambda(t) \geq 1$ following from the rescaling argument in [30]. $\square$

2.5 Proof of the Main Theorem

Suppose that Theorem 1.1 fails and let $u : [0, T^*) \times \mathbb{R}^d \to \mathbb{C}$ be a minimal blow-up solution proposed in Theorem 2.19. Together with $\lambda(t) \geq 1$ and the ArzeIà–Ascoli theorem, Theorem 2.19 yields that for any $\eta > 0$ there exists $C(\eta) > 0$ such that

$$\int_{|x|\geq C(\eta)} |\nabla u(t)|^2 + |u(t)|^{2^*} dx \leq \int_{|x|\geq \frac{C(\eta)}{\lambda(t)}} |\nabla u(t)|^2 + |u(t)|^{2^*} dx \leq \eta, \tag{2.52}$$

uniformly for $t \in [0, T^*)$. From equivalence of Sobolev norms and coercivity of energy,

$$\|\nabla u(t)\|_2^2 \sim E(u) = E_c \quad \text{uniformly for } t \in [0, T^*). \tag{2.53}$$

To exclude the existence of the minimal blow-up solution u as in Theorem 2.19, we split our discussion into the following two cases: $T^* = \infty$ and $T^* < \infty$. When $T^* = \infty$, we essentially use the energy trapping argument of the focusing problem and a weighted Virial argument. When $T^* < \infty$, we prove such minimal blow-up solution u will be in L^2 with 0 mass and contradicts the fact u is non-trivial. Both of the proofs are influenced by [26, 29].

Claim 2.20. *There are no solutions to NLS_a of the form given in Theorem 2.19 with $T^* = \infty$.*

Proof. By way of contradiction, we assume that $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{C}$ is such a solution. For some $R > 0$ to be chosen later, we define a weighted mass quantity

$$V_R(t) = \int_{\mathbb{R}^d} \phi_R(x) |u(t,x)|^2 \, dx,$$

where $\phi_R(x) = R^2 \phi(\frac{|x|^2}{R^2})$ and ϕ is a smooth radial cutoff function such that $\phi(x) = 0$ when $|x| \geq 2$ and $\phi(x) = |x|$ when $|x| \leq 1$. Taking partial derivatives on time, we have

$$\begin{aligned}
\partial_t V_R(t) &= 2\mathbf{Im} \int_{\mathbb{R}^d} \overline{u(t)}\, \nabla u(t) \cdot \nabla \phi_R dx; \\
\partial_{tt} V_R(t) &= 4\mathbf{Re} \int_{\mathbb{R}^d} (\phi_R)_{jk}(x) u_j(t) \bar{u}_k(t) \, dx - \tfrac{4}{d} \int_{\mathbb{R}^d} (\Delta \phi_R) |u(t)|^{2^*} \, dx \\
&\quad - \int_{\mathbb{R}^d} (\Delta\Delta\phi_R) |u(t)|^2 \, dx + 4a \int_{\mathbb{R}^d} \tfrac{x}{|x|^4} \nabla \phi_R |u(t)|^2 \, dx.
\end{aligned}$$

From Hölder inequality, Sobolev embedding, and (2.53), for any $t \in [0, \infty)$,

$$|\partial_t V_R(t)| \lesssim \|\phi'(\tfrac{|x|^2}{R^2})|x|\|_d \|u(t)\|_{2^*} \|\nabla u(t)\|_2 \lesssim_{E_c} R^2.$$

By Hardy inequality, Sobolev embedding, and (2.52) with $R = R(\eta)$ large enough, we obtain

$$\begin{aligned}
\partial_{tt} V_R(t) = 8 \int_{\mathbb{R}^d} |\nabla u(t)|^2 + \tfrac{a}{|x|^2} |u(t)|^2 - |u(t)|^{2^*} \, dx \\
+ O\Big(\int_{R \le |x| \le 2R} |\nabla u(t)|^2 + |u(t)|^{2^*} \, dx\Big) \\
\gtrsim \|\nabla u(t)\|_2^2 - \eta,
\end{aligned}$$

where $\int_{\mathbb{R}^d} |\nabla u(t)|^2 + \frac{a}{|x|^2} |u(t)|^2 - |u(t)|^{2^*} \, dx \gtrsim \|\nabla u(t)\|_2^2$ follows from the energy trapping (2.9) and equivalence of Sobolev norms. By taking η small enough depending on E_c and (2.53), we indeed have

$$\partial_{tt} V_R(t) \gtrsim_{E_c} 1, \quad \text{uniformly for } t \in [0, \infty).$$

Thus, from the fundamental theorem of calculus,

$$T \lesssim_{E_c} \left| \int_0^T \partial_{tt} V_R(t) dt \right| \lesssim_{E_c} R^2.$$

By taking $T \to \infty$, we get a contradiction and the proof of this claim is complete. □

Claim 2.21. *There are no solutions to NLS$_a$ of the form given in Theorem 2.19 with $T^* < \infty$.*

Proof. By way of contradiction, we assume that $u(t, x) : [0, T^*) \times \mathbb{R}^d \to \mathbb{C}$ is such a solution.

For some $R > 1$ to be chosen later, we define another weighted mass quantity

$$F_R(t) = \int_{\mathbb{R}^d} \phi(\frac{x}{R})|u(t)|^2 \, dx,$$

where ϕ is a smooth radial cutoff function such that $\phi(x) = 0$ when $|x| \geq 2$ and $\phi(x) = 1$ when $|x| \leq 1$. From taking partial derivatives on time, Hölder, Hardy inequality, and (2.53), we get

$$\begin{aligned} |\partial_t F_R(t)| &= \frac{2}{R}\left|\mathbf{Im}\int_{\mathbb{R}^d} \overline{u(t)}\,\nabla u(t) \cdot (\nabla\phi)(\frac{x}{R})dx\right| \\ &\lesssim \|\nabla u(t)\|_2^2 \lesssim E_c. \end{aligned}$$

The Mean Value Theorem yields that for any $t, T \in [0, T^*)$,

$$|F_R(t) - F_R(T)| \lesssim E_c|t - T|. \tag{2.54}$$

To conclude the proof, we need the following

$$\lim_{T\to T^*} F_R(T) = 0. \tag{2.55}$$

Indeed, for any $1 > r_0 > 0$, from changing of variables, Hölder inequality, and Sobolev embedding,

$$\begin{aligned} F_R(T) &= \int_{|x|\leq r_0} \phi(\frac{x}{R})|u(T)|^2 dx + \int_{|x|>r_0} \phi(\frac{x}{R})|u(T)|^2 dx \\ &\leq r_0^2\|u(T)\|_{2^*}^2 + \|\phi(\frac{x}{R})\|_{\frac{d}{2}}\|u(T)_{[\lambda(T)]}\|^2_{L_x^{2^*}(|x|>\lambda(T)r_0)} \\ &\lesssim r_0^2\|\nabla u(T)\|_2^2 + R^2\|\nabla u(T)_{[\lambda(T)]}\|^2_{L_x^2(|x|>\lambda(T)r_0)}. \end{aligned}$$

Note that $\lim_{T\to T^*}\lambda(T) = \infty$. If otherwise, there exists $\{t_n\} \subset [0, T^*)$ such that $\lambda(t_n) \to \lambda_0 \in \mathbb{R}^+$. From Lemma 2.18, $u(t_n)_{[\lambda(t_n)]} \to f$ in $\dot{H}^1$ up to a sub-sequence.

Hence, $u(t_n) \to f_{[\lambda_0^{-1}]}$ in $\dot{H}^1$. From the 2.14, u will be defined on a neighborhood of T^* which contradicts that $[0, T^*)$ is the maximal non-negative life-span of u.

As $\{u(t)_{[\lambda(t)]} : t \in [0, T^*)\}$ is relatively compact in $\dot{H}^1$, $\lim_{T \to T^*} \lambda(T) = \infty$ then yields that $R^2 \|\nabla u(T)_{[\lambda(T)]}\|^2_{2(|x| > \lambda(T) r_0)} \to 0$ for any fixed $r_0 > 0$ as $T \to T^*$. We get (2.55) by first taking $T \to T^*$ and then $r_0 \to 0$.

By taking $T \to T^*$ and $R \to \infty$ in (2.54) and using (2.55), we have

$$\int_{\mathbb{R}^d} |u(t)|^2 \, dx \lesssim E_c |t - T^*|.$$

Let $t \to T^*$, from conservation of mass, we get $u(t) = 0$ for all $t \in [0, T^*)$ which contradicts $E(u) = E_c > 0$. $\square$

CHAPTER 3
DYNAMICS OF NLS WITH INVERSE SQUARE POTENTIAL

To ease notion, we will use the following notations throughout this chapter: $p_c = \frac{d+2}{d-2}$, $W = W_a$, $\mathbf{d}(f) = |\|f\|^2_{\dot{H}^1_a} - \|W\|^2_{\dot{H}^1_a}|$, $f_{[\theta,\mu]} = e^{i\theta}\mu^{\frac{2-d}{2}} f(\frac{x}{\mu})$, and

$$\begin{aligned} S(I) &= L_t^{\frac{4(d+2)^2}{4\sigma-12d+2d\sigma+5d^2+4}} L_x^{\frac{-2d(d+2)^2}{4\sigma-8d+2d\sigma+3d^2-d^3+12}}, \\ Z(I) &= L_t^{\frac{4(d+2)^2}{4\sigma-12d+2d\sigma+5d^2+4}} L_x^{\frac{-2d(d+2)^2}{4\sigma-16d+2d\sigma+d^2-d^3+4}}, \\ N(I) &= L_t^{\frac{4(d^2-4)}{4\sigma-12d+2d\sigma+5d^2+4}} L_x^{\frac{-2d(d^2-4)}{4\sigma-8d+2d\sigma+d^2-d^3+20}}. \end{aligned}$$

The particular choice of these pamameters is to avoid the double end point Strichartz estimate. From Sobolev embedding and equivalence of Sobolev norms, we have

$$\|f\|_{S(I)} \lesssim \|\nabla f\|_{Z(I)} \sim \|\sqrt{\mathcal{L}_a} f\|_{Z(I)}, \quad \|\nabla f\|_{N(I)} \sim \|\sqrt{\mathcal{L}_a} f\|_{N(I)}.$$

3.1 Convergence to the ground state solution

In this section, our main task is to show that $u(t)$ converges to W exponentially fast as $t \to \infty$ up to symmetry both in the sub-critical case and super-critical case(Proposition 3.5 and 3.12). This result closely depends on the decay of $\mathbf{d}(u(t))$. One obvious connection is the variational characterization of W. One deep connection is the modulation argument developed in [17].

3.1.1 Variatoinal characterization of the ground state solution

The variational characterization of the ground state solution W_0 was first discussed in [1, 46]. Here, we present a different approach to get the variational characterization of W by the linear profile decomposition (Theorem 2.16) and the sharp Gagliardo-Nirenberg inequality.

Proposition 3.1 (Variational characterization of W)**.** *Assume that $f \in \dot{H}^1_a$ and $E(f) = E(W)$. Then*

$$\mathbf{d}(f) \to 0 \implies \inf_{\theta\in\mathbb{R},\mu>0} \|f_{[\theta,\mu]} - W\|_{\dot{H}^1_a} \to 0.$$

Proof. We argue by contradiction. Then $\exists\varepsilon_0 > 0$, $\forall\delta > 0$, $\exists f \in \dot{H}^1_a$ with $E(f) = E(W)$ and $\mathbf{d}(f) < \delta$, but $\inf_{\theta\in\mathbb{R},\mu>0} \|f_{[\theta,\mu]} - W\|_{\dot{H}^1_a} > \varepsilon_0$. Thus, one can pick a sequence of $\{\delta_n\} \subset \mathbb{R}^+$ and $\{f_n\} \subset \dot{H}^1_a$ with

$$\delta_n \to 0,\ E(f_n) = E(W),\ \mathbf{d}(f_n) < \delta_n, \tag{3.1}$$

while

$$\inf_{\theta\in\mathbb{R},\mu>0} \|(f_n)_{[\theta,\mu]} - W\|_{\dot{H}^1_a} > \varepsilon_0. \tag{3.2}$$

We define a functional $J(u) = \|u\|^{2^*}_{\dot{H}^1_a} \big/ \|u\|^{2^*}_{2^*}$ for all nontrivial $u \in \dot{H}^1_a$, and the Sharp Gagliardo-Nirenberg inequality(Proposition 2.9) shows that

$$J(W) = \inf_{u(\neq 0)\in\dot{H}^1_a} J(u).$$

From (3.1), and the linear profile decomposition, we get

$$\begin{aligned} J(W) &= \lim_{n\to\infty} J(f_n) = \lim_{J\to J^*}\lim_{n\to\infty} \frac{\left(\sum_{j=1}^J \|\phi^j\|^2_{\dot{H}^1_a} + \|r_n^J\|^2_{\dot{H}^1_a}\right)^{\frac{2^*}{2}}}{\sum_{j=1}^J \|\phi^j\|^{2^*}_{2^*}} \\ &\geq \lim_{J\to J^*}\lim_{n\to\infty} J(W) \frac{\left(\sum_{j=1}^J \|\phi^j\|^2_{\dot{H}^1_a} + \|r_n^J\|^2_{\dot{H}^1_a}\right)^{\frac{2^*}{2}}}{\sum_{j=1}^J \|\phi^j\|^{2^*}_{\dot{H}^1_a}}. \end{aligned}$$

Thus, $\sum_{j=1}^{J^*} \|\phi^j\|^{2^*}_{\dot{H}^1_a} \geq \left(\sum_{j=1}^{J^*} \|\phi^j\|^2_{\dot{H}^1_a} + \lim_{J\to J^*}\lim_{n\to\infty} \|r_n^J\|^2_{\dot{H}^1_a}\right)^{\frac{2^*}{2}}$ which implies that

$$J^* = 1 \text{ and } \lim_{J\to J^*}\lim_{n\to\infty} \|r_n^J\|^2_{\dot{H}^1_a} = \lim_{n\to\infty} \|f_n - \phi_n^1\|_{\dot{H}^1_a} = 0.$$

Hence, $J(\phi^1) = J(W)$. By the Sharp Gagliardo-Nirenberg inequality, $(\phi^1)_{[\theta,\mu]} = \lambda W$ for some $\theta \in \mathbb{R}$ and $\lambda, \mu > 0$. From $\mathbf{d}(f_n) \to 0$, we see that $\lambda = 1$ and $(\phi^1)_{[\theta,\mu]} = W$. Together with $\|f_n - \phi_n^1\|_{\dot{H}_a^1} \to 0$, this contradicts (3.2) and the proof is complete. $\square$

3.1.2 Modulation of threshold solutions

We first introduce a quadratic form $Q(g)$ based on the energy decomposition near W:

$$E(W+g) = E(W) + Q(g) + O(\|g\|^3_{\dot{H}_a^1}), \tag{3.3}$$

where $Q(g) = \frac{1}{2}\|g\|^2_{\dot{H}_a^1} - \frac{1}{2}\int W^{p_c-1}(p_c(\mathbf{Re}g)^2 + (\mathbf{Im}g)^2)$.

Let $W_1 = -\frac{d}{d\lambda}W_{[\lambda]}|_{\lambda=1} = \frac{d-2}{2}W + x\cdot\nabla W$. Direct computations show that W, iW, W_1 form three orthogonal directions in the real Hilbert space $\dot{H}_a^1$. Clearly, $Q(W) < 0$ and $Q(iW) = 0$ by definition. Note that $\mathcal{L}_a W = W^{p_c}$ implies $(\mathcal{L}_a - p_c W^{p_c-1})W_1 = 0$. Thus, $Q(W_1) = \langle(\mathcal{L}_a - p_c W^{p_c-1})W_1, W_1)\rangle = 0$. It turns out that those three directions are the only non-positive directions, and $Q(v)$ is positive definite when $v \perp H$, where

$$H = Span\{W, iW, W_1\}.$$

Namely,

Lemma 3.2 (Coercivity of Q)**.** *For all radial function $v \perp H$ in $\dot{H}_a^1$,*

$$\frac{2}{d+4}\|v\|^2_{\dot{H}_a^1} \le Q(v) \le \frac{1}{2}\|v\|^2_{\dot{H}_a^1}.$$

Proof. From the definition of Q, $Q(v) \le \frac{1}{2}\|v\|^2_{\dot{H}_a^1}$. And $\frac{2}{d+4}\|v\|^2_{\dot{H}_a^1} \le Q(v)$ follows from: assume v is a real valued function,

$$\|v\|^2_{\dot{H}^1_a} - p_c \int W^{p_c-1} v^2 \geq \frac{4}{d+4} \|v\|^2_{\dot{H}^1_a}, \quad \text{if } v \perp \{W, W_1\}; \tag{3.4}$$

$$\|v\|^2_{\dot{H}^1_a} - \int W^{p_c-1} v^2 \geq \frac{4}{d+2} \|v\|^2_{\dot{H}^1_a}, \quad \text{if } v \perp W. \tag{3.5}$$

Our method is inspired by Rey's argument in [43]. Let Π be the following projection of S^d(the unit sphere on $\mathbb{R}^{d+1}$) onto $\mathbb{R}^d$ with respect to the north pole:

$$\forall x = (x_1, \cdots, x_{d+1}) \in S^d, \qquad y_j = \frac{x_j}{f(x_{d+1})} (1 \leq j \leq d),$$

where $\Pi(x) = y \in \mathbb{R}^d$ and $f(x_{d+1}) = (1 + x_{d+1})^{\frac{\beta-1}{2\beta}} (1 - x_{d+1})^{\frac{\beta+1}{2\beta}}$. Denote $r = |y|$, we get

$$x_{d+1} = \frac{r^{2\beta} - 1}{r^{2\beta} + 1}, \qquad f(x_{d+1}) = \frac{2r^{\beta-1}}{r^{2\beta} + 1}.$$

To ease notion, we simply write $f(x_{d+1}) = f$, $f_{x_{d+1}} = \frac{\partial f}{\partial x_{d+1}}$, and $f_r = \frac{\partial f}{\partial r}$.

Write $u(x) = w(y) = v(y) f^{\frac{-d+2}{2}}$, then $W^{p_c-1} = \frac{d(d-2)\beta^2}{4} f^2$, and

$$\int_{S^d} |u(x)|^2 dx = \int |w(y)|^2 \beta f^d dy = \beta \int v^2 f^2 = \frac{4}{d(d-2)\beta} \int W^{p_c-1} v^2,$$

where βf^d is the area element from the determinant of the Jacobian, which can be derived by induction on d.

Note that $\nabla_{S^d} u(x) = \nabla u - (\nabla u \cdot x) x$ is the projection of ∇u on the tangent plane of x on S^d. Then

$$|\nabla_{S^d} u|^2 = \frac{|\nabla w|^2}{f^2} + (\nabla w \cdot y)^2 [(\frac{f_{x_{d+1}}}{f})^2 - (1 - \frac{x_{d+1} f_{x_{d+1}}}{f})^2].$$

From $w(y) = v(y)f^{\frac{-d+2}{2}}$, $|\nabla v|^2 = \frac{(\nabla v\cdot y)^2}{r^2}$($v$ is radial), we get

$$\begin{aligned}\frac{|\nabla w|^2}{f^2} &= f^{-d}|\nabla v|^2 + v^2(\frac{2-d}{2})^2 f^{-2-d}f_r^2 + (2-d)\frac{f^{-1-d}f_r}{r}v\nabla v\cdot y;\\ &(\nabla w\cdot y)^2[(\frac{f_{x_{d+1}}}{f})^2 - (1-\frac{x_{d+1}f_{x_{d+1}}}{f})^2]\\ &= \frac{(1-\beta^2)}{\beta^2}\left(\frac{|\nabla v|^2}{f^d} + v^2(\frac{2-d}{2})\frac{f_r^2}{f^{d+2}} + (2-d)\frac{f_r}{rf^{d+1}}v\nabla v\cdot y\right).\end{aligned}$$

By $a = (\frac{2-d}{2})^2(\beta^2-1)$, $W^{p_c-1} = \frac{d(d-2)\beta^2}{4}f^2$ and integration by parts, we have

$$\begin{aligned}&\int_{S^d}|\nabla_{S^d}u|^2dx\\ = &\int\frac{1}{\beta^2}\left(\frac{|\nabla v|^2}{f^d} + v^2(\frac{2-d}{2})^2\frac{f_r^2}{f^{d+2}} + (2-d)\frac{f_r}{rf^{d+1}}v\nabla v\cdot y\right)\beta f^d dy\\ = &\frac{1}{\beta}\int|\nabla v|^2dy + \frac{d-2}{2\beta}\int v^2\left(d\frac{f_r}{rf} + (\frac{f_r}{rf})'r + \frac{d-2}{2}\frac{f_r^2}{f^2}\right)dy\\ = &\frac{1}{\beta}\left(\|v\|^2_{\dot H^1_a} - a\int\frac{v^2}{r^2}dy\right) + \frac{d-2}{2\beta}\int v^2\left(d\frac{f_r}{rf} + (\frac{f_r}{rf})'r + \frac{d-2}{2}\frac{f_r^2}{f^2}\right)dy\\ = &\frac{1}{\beta}\left(\|v\|^2_{\dot H^1_a} - \frac{d(d-2)\beta^2}{4}\int f^2v^2\right)\\ = &\frac{1}{\beta}\left(\|v\|^2_{\dot H^1_a} - \int W^{p_c-1}v^2\right).\end{aligned}$$

Therefore,

$$\begin{cases}\int_{S^d}|u(x)|^2dx = \frac{4}{d(d-2)\beta}\int W^{p_c-1}v^2\\ \int_{S^d}|\nabla_{S^d}u|^2dx = \frac{1}{\beta}\left(\|v\|^2_{\dot H^1_a} - \int W^{p_c-1}v^2\right)\end{cases}. \tag{3.6}$$

Before we proceed, we recall the spectrum property of $-\Delta_{S^d}$ in [3]: the eigenvalues of $-\Delta_{S^d}$ are $\lambda_k = k(d+k-1)$ for all $k\ge 0$ with multiplicity n_k. In particular, we will use

$$\begin{cases}\lambda_0 = 1,\ n_0 = 1,\ u_{0,1} = 1\\ \lambda_1 = d,\ n_1 = d+1,\ u_{1,j} = x_j,\ 1\le j\le d+1\\ \lambda_2 = 2(d+1),\end{cases} \tag{3.7}$$

where $u_{k,j}$ is the corresponding eigenfunctions of λ_k.

Proof of (3.4)**:**

If $v \perp W, W_1$ in $\dot{H}^1_a(\mathbb{R}^d)$, then $v \perp \mathcal{L}_a W, \mathcal{L}_a W_1$ in $L^2(\mathbb{R}^d)$. From previous construction, we get

$$\begin{aligned}
\langle u, 1\rangle_{L^2(S^d)} &= \beta \int v(y) f^{\frac{d+2}{2}} dy = c_1 \int v(y)\mathcal{L}_a W = 0; \\
\langle u, x_{d+1}\rangle_{L^2(S^d)} &= \beta \int v(y) f^{\frac{d+2}{2}} \frac{r^{2\beta}-1}{r^{2\beta}+1} dy = c_2 \int v(y)\mathcal{L}_a W_1 = 0; \\
\langle u, x_j\rangle_{L^2(S^d)} &= \beta \int v(y) f^{\frac{d+4}{2}} y_j dy = 0,
\end{aligned}$$

where $\beta \int v(y) f^{\frac{d+4}{2}} y_k dy = 0$ follows from $v(y) f^{\frac{d+4}{2}}$ being radial. As $x_j (1 \le j \le d+1)$ are eigenfunctions of $\lambda_1 = d$, we have

$$\int_{S^d} |\nabla_{S^d} u|^2 dx \ge \lambda_2 \int_{S^d} |u|^2 dx.$$

Together with (3.6), this yields (3.4).

Proof of (3.5)**:**

If $v \perp W$ in $\dot{H}^1_a(\mathbb{R}^d)$, then $v \perp \mathcal{L}_a W$ in $L^2(\mathbb{R}^d)$. Similarly, we have

$$\langle u, 1\rangle_{L^2(S^d)} = c_1 \int v(y)\mathcal{L}_a W = 0.$$

This also implies that

$$\int_{S^d} |\nabla_{S^d} u|^2 dx \ge \lambda_1 \int_{S^d} |u|^2 dx.$$

Together with (3.6), this yields (3.5). $\square$

From the variational characterization of W and the implicit function theorem, one can obtain the following lemma and the proof is similar to [17].

Lemma 3.3. *There exists $\delta_0 > 0$ such that for all $f \in \dot{H}^1_a$ with $E(f) = E(W)$, $\mathbf{d}(f) < \delta_0$, there exists a couple (θ, μ) in $\mathbb{R} \times (0, \infty)$ with $f_{[\theta,\mu]} \perp iW, W_1$. The parameters θ and μ are unique in $\mathbb{R} / 2\pi\mathbb{Z} \times \mathbb{R}$, and mapping $f \mapsto (\theta, \mu)$ is C^1.*

Proof. Define functional J_0, J_1 on $\mathbb{R} \times (0, \infty) \times \dot{H}^1_a$:

$$J_0(\theta, \mu, f) = \left\langle f_{[\theta,\mu]}, iW \right\rangle_{\dot{H}^1_a}, \qquad J_1(\theta, \mu, f) = \left\langle f_{[\theta,\mu]}, W_1 \right\rangle_{\dot{H}^1_a}.$$

From definition of iW and W_1, we have $J_0(0, 1, W) = J_1(0, 1, W) = 0$, and

$$\begin{aligned} \frac{\partial J_0}{\partial \theta}(0, 1, W) &= \|W\|^2_{\dot{H}^1_a}, \qquad & \frac{\partial J_0}{\partial \mu}(0, 1, W) = 0; \\ \frac{\partial J_1}{\partial \theta}(0, 1, W) &= 0, & \frac{\partial J_1}{\partial \mu}(0, 1, W) = -\|W_1\|^2_{\dot{H}^1_a}. \end{aligned}$$

Implicit Function Theorem implies that there exists ε_0,$\eta_0 > 0$. For any $f \in \dot{H}^1_a$ with $\|f - W\|_{\dot{H}^1_a} < \varepsilon_0$, $\exists!(\theta, \mu)$ satisfying $|\theta| + |\mu - 1| \le \eta_0$ and $\left\langle f_{[\theta,\mu]}, iW \right\rangle_{\dot{H}^1_a} = \left\langle f_{[\theta,\mu]}, W_1 \right\rangle_{\dot{H}^1_a} = 0$.

When $\|f - W\|_{\dot{H}^1_a} \ge \varepsilon_0$, from the variational characterization of W, if $\mathbf{d}(f) < \delta_0$ for some small constant $\delta_0 > 0$ to be defined later, then $\exists(\theta_1, \mu_1)$ such that $\left\| f_{[\theta_1,\mu_1]} - W \right\|_{\dot{H}^1_a} \le \varepsilon(\mathbf{d}(f))$. By taking δ_0 small, $\left\| f_{[\theta_1,\mu_1]} - W \right\|_{\dot{H}^1_a} < \varepsilon_0$ and similarly $\exists!(\theta, \mu)$ such that $\left\langle f_{[\theta,\mu]}, iW \right\rangle_{\dot{H}^1_a} = \left\langle f_{[\theta,\mu]}, W_1 \right\rangle_{\dot{H}^1_a} = 0$.

Note that $f \mapsto (\theta, \mu)$ is C^1 also follows from the Implicit Function Theorem.

□

Let u be a solution of NLS_a on a time interval I such that $E(u_0) = E(W)$, $\mathbf{d}(u(t)) < \delta_0$ for all $t \in I$. From Lemma 3.3, for any $t \in I$, $\exists!(\theta(t), \mu(t))$ such that

$$u_{[\theta(t),\mu(t)]}(t) = (1 + \alpha(t))W + \tilde{u}(t), \tag{3.8}$$

where $\tilde{u}(t) \perp H$ and $1+\alpha(t) = \left\langle u_{[\theta(t),\mu(t)]}(t), W \right\rangle_{\dot{H}^1_a} \big/ \|W\|^2_{\dot{H}^1_a}$. We also denote

$$v(t) = \alpha(t)W + \tilde{u}(t) = u_{[\theta(t),\mu(t)]}(t) - W. \tag{3.9}$$

Lemma 3.4. *Let u be a solution of NLS_a on a time interval I such that $E(u_0) = E(W)$, $\mathbf{d}(u(t)) < \delta_0$ for all $t \in I$. Taking a smaller δ_0 if necessary, for any $t \in I$,*

$$|\alpha(t)| \sim \|v(t)\|_{\dot{H}^1_a} \sim \|\tilde{u}(t)\|_{\dot{H}^1_a} \sim \mathbf{d}(u(t)), \tag{3.10}$$

$$|\alpha'(t)| + |\theta'(t)| + \left|\frac{\mu'(t)}{\mu(t)}\right| \leq C\mu^2(t)\mathbf{d}(u(t)). \tag{3.11}$$

Proof. For simplicity, we omit the parameter t in this proof.

Proof of (3.10)**:** From above construction,

$$\|v\|^2_{\dot{H}^1_a} = \alpha^2 \|W\|^2_{\dot{H}^1_a} + \|\tilde{u}\|^2_{\dot{H}^1_a}. \tag{3.12}$$

Since $\tilde{u} \perp H$, $\int \tilde{u}(t)\mathcal{L}_a W = \int \tilde{u}(t) W^{p_c} = 0$. Direction computation shows $Q(v) = Q(W)\alpha^2 + Q(\tilde{u})$. By (3.3) and $E(W+v) = E(W)$,

$$|Q(W)\alpha^2 + Q(\tilde{u})| = |Q(v)| \lesssim \|v\|^3_{\dot{H}^1_a}.$$

By Lemma 3.2, $Q(\tilde{u}) \approx \|\tilde{u}\|^2_{\dot{H}^1_a}$. Thus,

$$\|\tilde{u}\|^2_{\dot{H}^1_a} \lesssim \|v\|^3_{\dot{H}^1_a} + \alpha^2,\ \alpha^2 \lesssim \|\tilde{u}\|^2_{\dot{H}^1_a} + \|v\|^3_{\dot{H}^1_a}. \tag{3.13}$$

Note that $\|v\|_{\dot{H}^1_a} \to 0$ when $\mathbf{d}(u) \to 0$ follows from the variational characterization of W. Together with (3.12) and (3.13), this yields that for small $\mathbf{d}(u)$,

$$|\alpha| \approx \|v\|_{\dot{H}^1_a} \approx \|\tilde{u}\|_{\dot{H}^1_a}.$$

By definition,

$$\mathbf{d}(u) = |\,\|W+v\|_{\dot{H}_a^1}^2 - \|W\|_{\dot{H}_a^1}^2\,| = |\,\|v\|_{\dot{H}_a^1}^2 + 2\alpha\,\|W\|_{\dot{H}_a^1}^2\,|.$$

Thus, (3.10) holds.

Proof of (3.11)**:** Introducing variables y and s such that

$$\mu(t)y = x, \quad ds = \mu^2(t)dt,$$

then we can rewrite (NLS_a) as

$$(i\partial_s - \mathcal{L}_a)u_{[\theta,\mu]} + |u_{[\theta,\mu]}|^{p_c-1}u_{[\theta,\mu]} + \theta_s u_{[\theta,\mu]} + i\frac{\mu_s}{\mu}\left(\frac{d-2}{2}u_{[\theta,\mu]} + y\cdot\nabla u_{[\theta,\mu]}\right) = 0. \quad (3.14)$$

From $u_{[\theta,\mu]} = W + v$, $\mathcal{L}_a W = W^{p_c}$, and $\frac{d-2}{2}W + y\cdot\nabla W = W_1$, we get

$$\partial_s v + \mathcal{L}(v) + R(v) - i\theta_s(W+v) + \frac{\mu_s}{\mu}W_1 + \frac{\mu_s}{\mu}(\frac{d-2}{2}v + y\cdot\nabla v) = 0.$$

Together with $v = \tilde{u} + \alpha(s)W$, this yields

$$\begin{aligned} & \partial_s\tilde{u} + \alpha_s W + (-\mathcal{L}_a + W^{p_c-1})\tilde{u} + i(\mathcal{L}_a - p_c W^{p_c-1})\tilde{u} \\ & -i\alpha(p_c-1)W^{p_c} - i\theta_s W + \frac{\mu_s}{\mu}W_1 \\ = \; & -R(v) + i\theta_s v - \frac{\mu_s}{\mu}(\frac{d-2}{2}v + y\cdot\nabla v), \end{aligned} \quad (3.15)$$

where $R(v) = -i|W+v|^{p_c-1}(W+v) + iW^{p_c} + ip_c W^{p_c-1}v_1 - W^{p_c-1}v_2$. By (3.10) and $\|R(v)\|_{L^{\frac{2d}{d+2}}} \lesssim \|v\|_{\dot{H}_a^1}^2 + \|v\|_{\dot{H}_a^1}^{p_c}$, we obtain

$$\begin{aligned} & |\,\langle(3.15), W\rangle_{\dot{H}_a^1}\,| + |\,\langle(3.15), iW\rangle_{\dot{H}_a^1}\,| + |\,\langle(3.15), W_1\rangle_{\dot{H}_a^1}\,| \\ \lesssim \; & \mathbf{d}(u(s))\left(\mathbf{d}(u(s)) + |\theta_s(s)| + \frac{\mu_s}{\mu}(s)\right). \end{aligned}$$

Denote $\mathcal{E}(s) = \mathbf{d}(u(s))\left(\mathbf{d}(u(s)) + |\theta_s(s)| + \frac{\mu_s}{\mu}(s)\right)$ and project the equation involving (3.15) in $\dot{H}^1_a$ on W, iW and W_1 separately, we get

$$\begin{cases} \alpha_s \|W\|^2_{\dot{H}^1_a} = \langle(\mathcal{L}_a - W^{p_c-1})\tilde{u}_2, W\rangle_{\dot{H}^1_a} + O(\mathcal{E}(s)) \\ \theta_s \|W\|^2_{\dot{H}^1_a} = \langle(\mathcal{L}_a - W^{p_c-1})\tilde{u}_1, W\rangle_{\dot{H}^1_a} - \alpha(p_c-1)\langle W^{p_c}, W\rangle_{\dot{H}^1_a} + O(\mathcal{E}(s)) \\ \frac{\mu_s}{\mu} \|W_1\|^2_{\dot{H}^1_a} = \langle(\mathcal{L}_a - W^{p_c-1})\tilde{u}_2, W_1\rangle_{\dot{H}^1_a} + O(\mathcal{E}(s)) \end{cases} .$$

In addition,

$$\begin{aligned} |\langle\mathcal{L}_a\tilde{u}_2, W\rangle_{\dot{H}^1_a}| &= |\langle\mathcal{L}_a\tilde{u}_2, \mathcal{L}_a W\rangle_{L^2}| = |\langle\tilde{u}_2, W^{p_c}\rangle_{\dot{H}^1_a}| \lesssim \|\tilde{u}\|_{\dot{H}^1_a}, \\ |\langle\mathcal{L}_a\tilde{u}_2, W_1\rangle_{\dot{H}^1_a}| &= |\langle\mathcal{L}_a\tilde{u}_2, \mathcal{L}_a W_1\rangle_{L^2}| = |\left\langle\tilde{u}_2, W^{p_c-1}W_1\right\rangle_{\dot{H}^1_a}| \lesssim \|\tilde{u}\|_{\dot{H}^1_a}, \\ |\left\langle W^{p_c-1}\tilde{u}_k, W\right\rangle_{\dot{H}^1_a}| &= |\left\langle W^{p_c-1}\tilde{u}_2, \mathcal{L}_a W\right\rangle_{L^2}| \lesssim \|\tilde{u}\|_{L^{\frac{2d}{d-2}}} \lesssim \|\tilde{u}\|_{\dot{H}^1_a}. \end{aligned}$$

Thus, $|\alpha_s(s)| + |\theta_s(s)| + |\frac{\mu_s}{\mu}(s)| \lesssim \|\tilde{u}\|_{\dot{H}^1_a} + \mathcal{E}(s)$. (3.10) then implies that

$$|\alpha_s(s)| + |\theta_s(s)| + |\frac{\mu_s}{\mu}(s)| \lesssim \mathbf{d}(u(s)),$$

which proves (3.11) by changing of variables. The proof of this lemma is complete. □

3.1.3 Convergence in the sub-critical case

In this sub-section, we classify radial solutions u to (NLS$_a$) satisfying

$$E(u) = E(W), \ \|u_0\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}; \tag{3.16}$$

$$\|u\|_{S(0,\infty)} = \infty. \tag{3.17}$$

Proposition 3.5. *Let u be a radial solution to (NLS$_a$) such that* (3.16) *and* (3.17) *hold. Then there exist $\theta \in \mathbb{R}$, $\mu > 0$, and $c, C > 0$, such that $\forall t \geq 0$,*

$$\|u(t)_{[\theta,\mu]} - W\|_{\dot{H}^1_a} \leq Ce^{-ct}. \tag{3.18}$$

Corollary 3.6. *Any radial solution u to (NLS$_a$) satisfying* (3.16) *does not fulfill*

$$\|u(t)\|_{S(-\infty,0)} = \|u(t)\|_{S(0,\infty)} = \infty. \tag{3.19}$$

3.1.3.1 Global existence and compactness of sub-critical threshold solutions

For the focusing energy critical NLS, Duyckaerts and Merle [17] first discovered that sub-critical threshold solutions(in the sense of (3.20) with $a = 0$) are all global in time solutions. In addition, if the solution does not scatter, then it only blows up at one time direction and has precompactness on the blow-up direction after scaling.

Proposition 3.7. *Fix $0 > a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$. Let u be a solution of NLS$_a$ with radial initial data u_0 such that*

$$E(u_0) = E(W) \text{ and } \|u_0\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}. \tag{3.20}$$

Then $u(t)$ exists globally in time. If we further assume that $\|u\|_{S(0,\infty)} = \infty$, then there exists a map λ defined on $[0, \infty)$ such that the set $\{u(t)_{[\lambda(t)]} : t \in [0,\infty)\}$ is precompact in $\dot{H}^1_a$. An analogous result holds on $(-\infty, 0]$.

The case $a = 0$ has been proved(or sketched) in [17], and the proof relies on the rigidity and compactness arguments in [26]. When $0 > a > -(\frac{d-2}{2})^2 + (\frac{d-2}{d+2})^2$, the precompactness part follows essential the same proof as Theorem 2.19 in Chapter 2; the global existence part can be proved by way of contradiction, see Claim 2.21 in Chapter 2. One consequence of Proposition 3.7 is the existence of a mixed behavior solution, for a proof, see [17].

Corollary 3.8. *There exists a global solution u to NLS_a such that* (3.20) *holds and*

$$\|u\|_{S(0,\infty)} = \infty, \ \|u\|_{S(-\infty,0)} < \infty.$$

3.1.3.2 Proof of Proposition 3.5 and Corollary 3.6

By Proposition 3.7, (3.16) and (3.17) imply that there exists a function $\lambda(t)$ such that

$$K = \{u(t)_{[\lambda(t)]} : t \geq 0\} \text{ is precompact in } \dot{H}_a^1. \tag{3.21}$$

From the ArzeIà–Ascoli theorem, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\int_{|x|\geq C(\varepsilon)} |\nabla u(t)_{[\lambda(t)]}|^2 + |u(t)_{[\lambda(t)]}|^{2^*} + \frac{|u(t)_{[\lambda(t)]}|^2}{|x|^2} dx \leq \varepsilon. \tag{3.22}$$

From variational characterization of W(Proposition 3.1), $\lim_{t\to\infty} \mathbf{d}(u(t)) = 0$ implies that $\lim_{t\to\infty} \inf_{\theta\in\mathbb{R},\mu>0} \|u(t)_{[\theta,\mu]} - W\|_{\dot{H}_a^1} = 0$. If we omit the exponential decay in (3.18) at this moment, we should at least proceed with showing

$$\lim_{t\to\infty} \mathbf{d}(u(t)) = 0. \tag{3.23}$$

We first slow down our step and prove an even weaker result.

Lemma 3.9. *Under the assumption in Proposition 3.5. There exists a sequence of $t_n \to \infty$ such that $\mathbf{d}(u(t_n)) \to 0$.*

Proof. Define a weighted mass quantity as in [17],

$$V_R(t) = \int_{\mathbb{R}^d} \phi_R(x)|u(t,x)|^2\, dx, \tag{3.24}$$

where $\phi_R(x) = R^2\phi(\frac{x}{R})$ and ϕ is a smooth radial cutoff function such that $\phi(x) = 0$ when $|x| \geq 2$ and $\phi(x) = |x|^2$ when $|x| \leq 1$. Taking partial derivatives on time, we have

$$\begin{aligned}
\partial_t V_R(t) &= 2\mathbf{Im}\int_{\mathbb{R}^d} \overline{u(t)}\,\nabla u(t)\cdot\nabla\phi_R dx;\\
\partial_{tt} V_R(t) &= 4\mathbf{Re}\int_{\mathbb{R}^d} (\phi_R)_{jk}(x)u_j(t)\bar{u}_k(t)\,dx - \tfrac{4}{d}\int_{\mathbb{R}^d}(\Delta\phi_R)|u(t)|^{2^*}\,dx\\
&\qquad - \int_{\mathbb{R}^d}(\Delta\Delta\phi_R)|u(t)|^2\,dx + 4a\int_{\mathbb{R}^d}\tfrac{x}{|x|^4}\nabla\phi_R|u(t)|^2\,dx.
\end{aligned}$$

From Hardy and Cauchy-Schwartz inequality, equivalence of Sobolev norms, and coercivity of energy, for any $t \in [0,\infty)$,

$$|\partial_t V_R(t)| \lesssim R^2\int_{\mathbb{R}^d}\frac{|u(t)|}{|x|}|\nabla u(t)|dx \lesssim R^2\|\nabla u(t)\|_2^2 \lesssim_{E(W)} R^2. \tag{3.25}$$

For any given $\varepsilon > 0$, by Hardy inequality, Sobolev embedding, and (3.22) with $R\lambda(t) > C(\varepsilon)$, we obtain

$$\begin{aligned}
\partial_{tt} V_R(t) &= 8\int_{\mathbb{R}^d}|\nabla u(t)|^2 + \tfrac{a}{|x|^2}|u(t)|^2 - |u(t)|^{2^*}\,dx\\
&\qquad + O(\int_{|x|\geq R}|\nabla u(t)|^2 + |u(t)|^{2^*}\,dx)\\
&= \frac{16}{d-2}\mathbf{d}(u(t)) + O(\int_{|x|\geq R\lambda(t)}|\nabla u(t)_{[\lambda(t)]}|^2 + |u(t)_{[\lambda(t)]}|^{2^*}\,dx)\\
&\geq \frac{16}{d-2}\mathbf{d}(u(t)) - \varepsilon.
\end{aligned}$$

Together with the fundamental theorem of calculus and (3.25), this yields that

$$\int_{t_0}^{T}\mathbf{d}(u(t))dt \lesssim \varepsilon(T-t_0) + R^2 \lesssim \varepsilon T, \tag{3.26}$$

by picking $R = (\varepsilon T)^{\frac{1}{2}}$ and provided $\exists t_0 > 0$ with $R\lambda(t) > C(\varepsilon)$ for all $t \geq t_0$. This process is possible as

$$\lim_{t\to\infty} \lambda(t)\sqrt{t} = \infty. \tag{3.27}$$

We apply the argument in [17] to prove (3.27). Suppose otherwise that $\lambda(t)\sqrt{t} \to \tau_0 < \infty$ as $t \to \infty$. Construct a sequence of solutions v_n to (NLS$_a$) with $v_n(\tau) = u(t_n + \frac{\tau}{\lambda(t_n)^2})_{[\lambda(t_n)]}$. From the precompactness, passing to a sub-sequence if necessary, $v_n(0) \to v_0$ in $\dot{H}^1_a$. Let $v(t)$ be the solution to (NLS$_a$) with initial data v_0. Together with conservation of energy, this implies

$$E(v(t)) = E(W), \quad \|v_0\|_{\dot{H}^1_a} \leq \|W\|_{\dot{H}^1_a}.$$

Proposition 3.7 shows that v exists globally in time. From Theorem 2.15, we have

$$\lim_{n\to\infty} u(0)_{[\lambda(t_n)]} = \lim_{n\to\infty} v_n(-\lambda(t_n)^2 t_n) = v(-\tau_0) \text{ in } \dot{H}^1_a.$$

As $\lambda(t_n) \to 0$, $u(0)_{[\lambda(t_n)]} \rightharpoonup 0$ weakly in $\dot{H}^1_a$ which implies that $v(-\tau_0) = 0$. This contradicts $E(v(t)) = E(W)$. Hence, (3.27) holds.

Next, we conclude the proof. From (3.26), we have $\frac{1}{T}\int_{t_0}^T \mathbf{d}(u(t))dt \lesssim \varepsilon$. Thus, $\overline{\lim}_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{d}(u(t))dt \lesssim \varepsilon$. By taking $\varepsilon \to 0$, we indeed have

$$\overline{\lim}_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{d}(u(t))dt = 0$$

which completes the proof of this lemma. □

To fill the gap from Lemma 3.9 to (3.23), we need the next lemma. Note that by picking $u_n = u$, $\lambda_n = \lambda$, $t_{0n} = t_n$, $t_{1n} = t_{n+1}$, Lemma 3.10 then yields (3.23).

Lemma 3.10. *Let $\{t_{0n}\}$ and $\{t_{1n}\}$ be two sequence of real numbers with $t_{0n} < t_{1n}$, $\{u_n\}$ a sequence of radial solutions to (NLS$_a$) such that $u_n(t_{0n})$ satisfying* (3.16) *and*

(3.17)*, and* $\{\lambda_n\}$ *a sequence of positive functions such that the set:* $\tilde{K} = \{u_n(t)_{[\lambda_n(t)]} : n \in \mathbb{N}, t \in (t_{0n}, t_{1n})\}$ *is precompact* $\dot{H}^1_a$*. Assume that*

$$\lim_{n\to\infty} \mathbf{d}(u_n(t_{0n})) + \mathbf{d}(u_n(t_{1n})) = 0. \tag{3.28}$$

Then

$$\lim_{n\to\infty} \sup_{t\in(t_{0n},t_{1n})} \mathbf{d}(u_n(t)) = 0. \tag{3.29}$$

Proof. From the scaling invariance and continuity of $\lambda_n(t)$, it suffices to prove the lemma under additional assumption:

$$\inf_{t\in[t_{0n},t_{1n}]} \lambda_n(t) = 1 \text{ for all } n \geq 1. \tag{3.30}$$

We will split the proof into the following steps.

Step 1: Under the assumptions in Lemma 3.10 and (3.30), we have that for any $n \geq 1$,

$$\int_{t_{0n}}^{t_{1n}} \mathbf{d}(u_n(t))dt \lesssim \mathbf{d}(u_n(t_{0n})) + \mathbf{d}(u_n(t_{1n})). \tag{3.31}$$

We define $V_{R,n}(t) = \int_{\mathbb{R}^d} \phi_R(x)|u_n(t,x)|^2\,dx$ as in (3.24). Similar to (3.24), we have

$$\partial_t V_{R,n}(t) = 2\mathbf{Im}\int_{\mathbb{R}^d} \overline{u_n(t)}\,\nabla u_n(t)\cdot\nabla\phi_R dx.$$

Similar to (3.25), $|\partial_t V_{R,n}(t)| \leq R^2\|u_n(t)\|^2_{\dot{H}^1_a}$. We fix $0 < \delta_1 < \delta_0$(δ_0 is defined in Lemma 3.4). If $\mathbf{d}(u_n(t)) \geq \delta_1$, there exists $C > 1$(C depends on δ_1) such that

$$|\partial_t V_{R,n}(t)| \leq R^2[\mathbf{d}(u_n(t)) + \|W\|^2_{\dot{H}^1_a}] \leq CR^2\mathbf{d}(u_n(t)),$$

provided $(C-1)\|W\|_{\dot{H}_a^1}^2 \leq \delta_1$. Next, we prove that $|\partial_t V_{R,n}(t)| \leq CR^2\mathbf{d}(u_n(t))$ still holds if $\mathbf{d}(u_n(t)) < \delta_1$. Indeed, we can write

$$u_n(t)_{[\theta_n(t),\mu_n(t)]} = v_n(t) + W \tag{3.32}$$

in the sense of (3.9). By Lemma 3.4, $\|v_n(t)\|_{\dot{H}_a^1} \lesssim \mathbf{d}(u_n(t))$. From change of variable $x = \frac{y}{\mu_n(t)}$, we have

$$\begin{aligned}
\partial_t V_{R,n}(t) &= 2R^2\mathbf{Im}\int_{\mathbb{R}^d} \frac{\overline{v_n+W}}{R\mu_n(t)} \nabla(v_n+W)\cdot(\nabla\phi)(\frac{y}{R\mu_n(t)})dy \\
&= 2R^2\mathbf{Im}\int_{\mathbb{R}^d} \frac{(W\nabla v_n + \nabla\bar{v}_n W + \bar{v}_n\nabla v_n)}{R\mu_n(t)}\cdot(\nabla\phi)(\frac{y}{R\mu_n(t)})dy.
\end{aligned}$$

From equivalence of Sobolev norms, Hardy and Cauchy-Schwartz inequality, we get

$$|\partial_t V_{R,n}(t)| \lesssim R^2(\|v_n\|_{\dot{H}_a^1}^2 + \|v_n\|_{\dot{H}_a^1}) \lesssim R^2\mathbf{d}(u_n(t)).$$

In all cases, we have proved that

$$|\partial_t V_{R,n}(t)| \lesssim R^2\mathbf{d}(u_n(t)). \tag{3.33}$$

Next, we aim to prove some lower bound of $\partial_{tt}V_{R,n}(t)$, then apply the fundamental theorem of calculus to conclude the proof.

From the radial assumptions on u_n and ϕ_R, direct computations as in Lemma 3.9 show that

$$\partial_{tt}V_{R,n}(t) = \frac{16}{d-2}\mathbf{d}(u_n(t)) + A_R(u_n(t)), \tag{3.34}$$

where $r=|x|$ and $A_R(u_n(t)) = \int_{|x|\geq R}(4\frac{d^2\phi_R}{dr^2}-8)|\nabla u_n|^2dx + \int_{|x|\geq R}(-\frac{4}{d}\Delta\phi_R+8)|u_n|^{2^*}dx - \int_{\mathbb{R}^d}(\Delta\Delta\phi_R)|u_n(t)|^2\,dx$. Similar to the treatment in Lemma 3.9, we can roughly write

$A_R(u_n(t)) = O(\int_{|x|\geq R} |\nabla u_n(t)|^2 + |u_n(t)|^{2^*}\,dx)$. Same proof shows that for any given $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$R\lambda_n(t) \geq C(\varepsilon) \Longrightarrow |A_R(u_n(t))| \leq \varepsilon$$

for any n and any $t \in (t_{0n}, t_{1n})$. Due to the assumption (3.30), we write our result as follows. For any given $\varepsilon > 0$, there exists $R_1 > 0$ such that

$$R \geq R_1 \Longrightarrow |A_R(u_n(t))| \leq \varepsilon \tag{3.35}$$

for any n and any $t \in (t_{0n}, t_{1n})$.

Indeed, (3.35) provides a good lower bound of $\partial_{tt} V_{R,n}(t)$ if $\mathbf{d}(u_n(t)) \gtrsim \varepsilon$. It remains to search the lower bound of $\partial_{tt} V_{R,n}(t)$ when $\mathbf{d}(u_n(t)) \lesssim \varepsilon$. Here, we play the modulation trick again. Rewriting u_n as in (3.32) with $\|v_n(t)\|_{\dot{H}^1_a} \lesssim \mathbf{d}(u_n(t)) < \delta_2$ for small $\delta_2 < \delta_0$(defined in Lemma 3.4). From equivalence of Sobolev norms, change of variables, $A_R(W) = 0$ and provided $R\mu_n(t) \geq 1$, we get

$$\begin{aligned}
|A_R(u_n(t))| &= |A_R((v_n(t) + W)_{[\mu_n(t)^{-1}]})| = |A_{R\mu_n(t)}(v_n(t) + W)| \\
&= |A_{R\mu_n(t)}(v_n(t) + W) - A_{R\mu_n(t)}(W)| \\
&\lesssim \int_{|x|\geq R\mu_n(t)} |\nabla v_n|^2 + |\nabla W \cdot \nabla v_n| + W^{2^*-1}|v_n| + |v_n|^{2^*} dx \\
&\quad + \int_{R\mu_n(t)\leq|x|\leq 2R\mu_n(t)} \frac{W|v_n| + |v_n|^2}{(R\mu_n(t))^2} dx \\
&\lesssim \|v_n\|^2_{\dot{H}^1_a} + \|v_n\|^{2^*}_{\dot{H}^1_a} + (R\mu_n(t))^{\frac{2-d}{2}\beta} \|v_n\|_{\dot{H}^1_a},
\end{aligned}$$

where in the last inequality we use the fact

$$\|\nabla W\|_{L^2(|x|\geq R)} \sim \|W\|_{L^{2^*}(|x|\geq R)} \sim R^{\frac{2-d}{2}\beta} \text{ if } R \geq 1.$$

From (3.30) and (3.40) in Step 3, $\mu_n(t) \gtrsim \lambda_n(t) \geq 1$. Thus, if $\mathbf{d}(u_n(t)) < \delta_2 < \delta_0$, there exists $R_2 > 0$ such that

$$R \geq R_2 \Rightarrow |A_R(u_n(t))| \leq \frac{8}{d-2}\mathbf{d}(u_n(t)). \tag{3.36}$$

By picking $\varepsilon = \frac{8}{d-2}\delta_2$ and $R_0 = \max\{R_1, R_2\}$, then

$$R \geq R_0 \Longrightarrow |A_R(u_n(t))| \leq \frac{8}{d-2}\mathbf{d}(u_n(t))$$

for any n and any $t \in (t_{0n}, t_{1n})$. Consequently, for $R \geq R_0$,

$$\partial_{tt} V_{R,n}(t) \geq \frac{8}{d-2}\mathbf{d}(u_n(t)).$$

From fundamental theorem of calculus and (3.33) and picking $R = 2R_0$, we get

$$\int_{t_{0n}}^{t_{1n}} \mathbf{d}(u_n(t))dt \lesssim R_0^2\mathbf{d}(u_n(t_{0n})) + R_0^2\mathbf{d}(u_n(t_{1n})) \lesssim \mathbf{d}(u_n(t_{0n})) + \mathbf{d}(u_n(t_{1n})).$$

This finishes the proof of step 1.

Step 2: If $t_n \in (t_{0n}, t_{1n})$ such that $\lambda_n(t_n) \leq C$ for some $C > 0$, then $\lim_{n\to\infty} \mathbf{d}(u_n(t_n)) = 0$.

From boundness of $\lambda_n(t_n)$ and precompactness of $\tilde{K}$ in $\dot{H}^1_a$, up to a sub-sequence, $u_n(t_n) \to V_0$ in $\dot{H}^1_a$. Thus, $E(V_0) = E(W)$ and $\|V_0\|_{\dot{H}^1_a} \leq \|W\|_{\dot{H}^1_a}$.

If $\|V_0\|_{\dot{H}^1_a} = \|W\|_{\dot{H}^1_a}$, this clearly implies $\mathbf{d}(u_n(t_n)) \to 0$. It remains to show that $\|V_0\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}$ leads to a contradiction. Indeed, if $\|V_0\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}$, we define v as the solution to (NLS$_a$) with initial data V_0. Proposition 3.7 shows that V is a global solution. In addition, for large n, $t_{1n} \geq 1 + t_n$. If otherwise, $t_{1n} - t_n \to \tau \in [0, 1]$ up to a sub-sequence. From Theorem 2.15, $u_n(t_{1n}) \to V(\tau)$ in $\dot{H}^1_a$ and

$E(V(\tau)) = E(W)$. Since $\mathbf{d}(u_n(t_{1n})) \to 0$, $\mathbf{d}(V(\tau)) = 0$. Thus, up to symmetry $V = W$ contradicting $\|V_0\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}$. Hence, for large n, $t_{1n} \geq 1 + t_n$. Note that $\mathbf{d}(V(t))$ is a continuous function of t. From $\|V_0\|_{\dot{H}^1_a} < \|W\|_{\dot{H}^1_a}$ and $t_{1n} \geq 1 + t_n$, we thus have

$$\lim_{n\to\infty} \int_{t_{0n}}^{t_{1n}} \mathbf{d}(u_n(t))dt \geq \lim_{n\to\infty} \int_{t_n}^{t_n+1} \mathbf{d}(u_n(t))dt = \int_0^1 \mathbf{d}(V(t)) > 0,$$

which contradicts

$$\lim_{n\to\infty} \int_{t_{0n}}^{t_{1n}} \mathbf{d}(u_n(t))dt \lesssim \lim_{n\to\infty} \mathbf{d}(u_n(t_{0n})) + \mathbf{d}(u_n(t_{1n})) = 0.$$

Step 3: Verify (3.29) by way of contradiction.

From assumption (3.30), there exists a sequence of $b_n \in (t_{0n}, t_{1n})$ such that $\lambda_n(b_n) \to 1$. From Step 2, $\mathbf{d}(u_n(b_n)) \to 0$. We argue by contradiction that (3.29) fails. Then for some $\delta_1 > 0$, $\sup_{t\in(t_{0n},b_n)} \mathbf{d}(u_n(t)) \geq \delta_1$(The proof is the same if $\sup_{t\in(b_n,t_{1n})} \mathbf{d}(u_n(t)) \geq \delta_1$). From the continuity of $\mathbf{d}(u_n(t))$, there exist $a_n \in (t_{0n}, b_n)$ and $0 < \delta_2 \ll \min\{\delta_1, \delta_0, 1\}$($\delta_0$ is given by Lemma 3.4) such that

$$\mathbf{d}(u_n(a_n)) = \delta_2 \text{ and } \mathbf{d}(u_n(t)) < \delta_2 \text{ for } \forall t \in (a_n, b_n). \tag{3.37}$$

Thus, from the modulation results in Section 3.1.2, we have the following decomposition on $t \in [a_n, b_n]$ in the sense of (3.8),

$$u_n(t)_{[\theta_n(t),\mu_n(t)]} = (1 + \alpha_n(t))W + \tilde{u}_n(t), \tag{3.38}$$

where all the parameters satisfy (3.10) and (3.11) as in Lemma 3.4. Thus,

$$u_n(t)_{[\lambda_n(t)]} = e^{-i\theta_n(t)}(1 + \alpha_n(t))W_{[\lambda_n(t)/\mu_n(t)]} + e^{-i\theta_n(t)}\tilde{u}_n(t)_{[\lambda_n(t)/\mu_n(t)]}.$$

From the precompactness of $\tilde{K}$, $u_n(t)_{[\lambda_n(t)]} \to V(t)$ in $\dot{H}^1_a$ up to a sub-sequence. Thus, from the orthogonality between W and $\tilde{u}_n(t)$,

$$(1+\alpha_n(t))^2\|W\|^2_{\dot{H}^1_a} + \|\tilde{u}_n(t)\|^2_{\dot{H}^1_a} \to \|V(t)\|^2_{\dot{H}^1_a} \tag{3.39}$$

We will prove that

$$\lambda_n(t) \sim \mu_n(t) \text{ for } \forall t \in [a_n, b_n]. \tag{3.40}$$

Indeed, either $\lambda_n(t)/\mu_n(t) \to \infty$ or 0, $(1+\alpha_n(t))W_{[\lambda_n(t)/\mu_n(t)]} \rightharpoonup 0$ weakly in $\dot{H}^1_a$, which leaves $e^{-i\theta_n(t)}\tilde{u}_n(t)_{[\lambda_n(t)/\mu_n(t)]} \rightharpoonup V(t)$ weakly in $\dot{H}^1_a$. From the weak lower semi-continuity of the $\dot{H}^1_a$ norm,

$$\|V\|_{\dot{H}^1_a} \le \liminf_{n\to\infty} \|\tilde{u}_n(t)_{[\lambda_n(t)/\mu_n(t)]}\|_{\dot{H}^1_a} = \liminf_{n\to\infty} \|\tilde{u}_n(t)\|_{\dot{H}^1_a}.$$

Together with (3.39), this shows that $(1+\alpha_n(t))^2\|W\|^2_{\dot{H}^1_a} \to 0$ which contradicts $|\alpha_n(t)| \sim \mathbf{d}(u_n(t)) \le \delta_2 \ll 1$. Thus, (3.40) holds, which implies that the sequence $\mu_n(t)$ is bounded from below in view of $\lambda_n(t) \ge 1$ in (3.30).

A further exploration of $\{\mu_n(t)\}$ shows that $\sup_{n,t\in[a_n,b_n]} \mu_n(t) < \infty$. From $\lambda_n(b_n) \to 1$ and (3.40), we may assume $\mu_n(b_n) \to \mu_0 \in (0,\infty)$. Suppose otherwise $\sup_{n,t\in[a_n,b_n]} \mu_n(t) = \infty$. From the continuity of $\mu_n(t)$, for large n, there exists $c_n \in (a_n, b_n)$ such that $\mu_n(t) < 2\mu_0$ for all $t \in (c_n, b_n]$ and $\mu_n(c_n) = 2\mu_0$. From fundamental theorem of calculus, Lemma 3.4, and Step 1, we have

$$\left|\frac{1}{\mu_n(b_n)^2} - \frac{1}{\mu_n(c_n)^2}\right| \le \int_{c_n}^{b_n} \left|\frac{\mu_n'(t)}{\mu_n(t)^3}\right| dt \lesssim \int_{c_n}^{b_n} \mathbf{d}(u_n(t))dt \to 0. \tag{3.41}$$

This contradicts that $\mu_n(b_n) \to \mu_0$ and $\mu_n(c_n) = 2\mu_0$.

Hence, $\sup_{n,t\in[a_n,b_n]} \mu_n(t) < \infty$ which implies that $\sup_{n,t\in[a_n,b_n]} \lambda_n(t) < \infty$ in view of (3.40). In particular, $\lambda_n(a_n)$ is bounded. From Step 2, we should have

$\mathbf{d}(u_n(a_n)) \to 0$ which contradicts our construction $\mathbf{d}(u_n(a_n)) = \delta_2$ in (3.37). The proof of this lemma is complete. $\square$

Arguing similar to the proof of (3.41), one can get the non-oscillatory behavior of μ_n on (t_{0n}, t_{1n}).

Corollary 3.11. *Under the assumption in Lemma 3.10, for all n large such that $\mathbf{d}(u_n(t)) < \delta_0$ for $t \in (t_{0n}, t_{1n})$. We have the decomposition $u_n(t)_{[\theta_n(t),\mu_n(t)]} = (1 + \alpha_n(t))W + \tilde{u}_n(t)$ in the sense of* (3.8) *with all parameters satisfying the result in Lemma 3.4. Then,*

$$\lim_{n\to\infty} \frac{\sup_{t\in(t_{0n},t_{1n})} \mu_n(t)}{\inf_{t\in(t_{0n},t_{1n})} \mu_n(t)} = 1.$$

With all above preparation, we are ready to prove Proposition 3.5 and Corollary 3.6.

Proof of Proposition 3.5. From the discussion before the statement of Lemma 3.10, we have proved that $\lim_{t\to\infty} \mathbf{d}(u(t)) = 0$. For large t, we have the decomposition in the sense of (3.8),

$$u(t)_{[\theta(t),\mu(t)]} = (1 + \alpha(t))W + \tilde{u}(t),$$

where all the parameters satisfy (3.10) and (3.11) as in Lemma 3.4.

Convergence of $\mu(t)$: $\lim_{t\to\infty} \mu(t) = \mu_\infty \in (0, \infty)$.

If this fails, then there exists two sequence of times t_{0n} and t_{1n}($t_{0n} < t_{1n}$ and $t_{0n} \to \infty$) such that $\mu(t_{0n})/\mu(t_{1n}) \to \mu_0 \neq 1$. Apply Corollary 3.11 with $u_n(t) = u(t)$ and $\lambda_n(t) = \lambda(t)$, we get $\mu(t_{0n})/\mu(t_{1n}) \to 1$ which is an expected contradiction.

Exponential decay: $\alpha(t) \sim \|\tilde{u}(t)\|_{\dot{H}^1_a} \sim \mathbf{d}(u(t)) \le Ce^{-ct}$ for t large.

We first derive the decay of $\mathbf{d}(u(t))$ via the estimate

$$\exists C > 0, \forall t \geq 0, \quad \int_t^\infty \mathbf{d}(u(s))ds \leq C\ \mathbf{d}(u(t)). \tag{3.42}$$

If (3.42) fails, there exists a sequence $t_{0n} \to \infty$ such that $\int_{t_{0n}}^\infty \mathbf{d}(u(s))ds \geq n\ \mathbf{d}(u(t_{0n}))$. For any $t_{1n} > t_{0n}$, from Step 1 in Lemma 3.10, we have

$$\int_{t_{0n}}^{t_{1n}} \mathbf{d}(u(s))ds \lesssim \mathbf{d}(u(t_{0n})) + \mathbf{d}(u(t_{1n}))$$

where the assumption in Step 1 remains valid as $\mu(t) \to \mu_\infty$ implies that $\lambda(t)$ is bounded from below in view of (3.40). Taking $t_{1n} \to \infty$, we get $\int_{t_{0n}}^\infty \mathbf{d}(u(s))ds \lesssim \mathbf{d}(u(t_{0n}))$ which contradicts $\int_{t_{0n}}^\infty \mathbf{d}(u(s))ds \geq n\ \mathbf{d}(u(t_{0n}))$. Grönwall's inequality implies that for some constants $c, C > 0$, $\int_t^\infty \mathbf{d}(u(s))ds \leq Ce^{-ct}$. As $\mu(t)$ is bounded by $\mu(t) \to \mu_\infty$, from Lemma 3.4 and fundamental theorem of Calculus, we have

$$\mathbf{d}(u(t)) \sim |\alpha(t)| = |\int_t^\infty \alpha'(s)ds| \lesssim |\int_t^\infty \mathbf{d}(u(s))ds| \leq Ce^{-ct}. \tag{3.43}$$

Convergence of $\theta(t)$: $\lim_{t\to\infty} \theta(t) = \theta_\infty \in [0, \infty)$.

Picking any sequence $t_n \to \infty$, similar to (3.43), we have $|\theta(t_n) - \theta(t_m)| = |\int_{t_n}^{t_m} \theta'(s)ds| \lesssim |\int_{t_n}^{t_m} \mathbf{d}(u(s))ds| \leq Ce^{-c\min\{t_n,t_m\}}$, which shows that $\{\theta(t_n)\}$ is a Cauchy sequence in $\mathbb{R}$, thus converges. As the sequence is arbitrarily taken, we have $\theta(t) \to \theta_\infty$.

Therefore, in view of the decomposition $u(t)_{[\theta(t),\mu(t)]} = (1+\alpha(t))W + \tilde{u}(t)$, from $\mu(t) \to \mu_\infty$, $\theta(t) \to \theta_\infty$, and $\alpha(t) \sim \|\tilde{u}(t)\|_{\dot{H}^1_a} \leq Ce^{-ct}$, (3.18) holds. The proof of Proposition 3.5 is complete. $\square$

Proof of Corollary 3.6. We argue by contradiction as in [17]. Suppose u is a solution to (NLS$_a$) satisfying (3.16) and (3.19). Applying Proposition 3.5 both forward and backward in time, we see that $\{u(t) : t \in \mathbb{R}\}$ is precompact in $\dot{H}^1_a$. Indeed, for any sequence $\{t_n\}$, passing to a sub-sequence, if $t_n \to t_*(t_*$ is finite), from continuity of the flow, we have $u(t_n) \to u(t_*)$ in $\dot{H}^1_a$; if $t_n \to \pm\infty$, $u(t_n) \to W_{[-\theta,\mu^{-1}]}$ in $\dot{H}^1_a$ for some $\theta \in \mathbb{R}$, $\mu > 0$ by Proposition 3.5. In addition, $\lim_{t\to\pm\infty} \mathbf{d}(u(t)) = 0$ follows from $\mathbf{d}(u(t)) \leq Ce^{-ct}$ as proven in (3.43). Set $u_n = u, t_{0n} = -n, t_{1n} = n, \lambda_n = 1$. All assumptions are fulfilled for (3.31), and we get

$$\int_{-\infty}^{\infty} \mathbf{d}(u(t))dt = \lim_{n\to\infty} \int_{-n}^{n} \mathbf{d}(u(t))dt \lesssim \lim_{n\to\infty} [\mathbf{d}(u(-n)) + \mathbf{d}(u(n))] = 0.$$

This shows that $\mathbf{d}(u(t)) = 0$ for all $t \in \mathbb{R}$ which contradicts $\mathbf{d}(u_0) > 0$. The proof is complete. $\square$

3.1.4 Convergence in the super-critical case

In this sub-section, we classify radial solutions u to (NLS$_a$) defined on $[0, \infty)$ or $(-\infty, 0]$ satisfying

$$E(u) = E(W),\ \|u_0\|_{\dot{H}^1_a} > \|W\|_{\dot{H}^1_a}. \tag{3.44}$$

In the sub-critical case, the precompactness of the set K in (3.21) plays an essential role in the proof of Proposition 3.5. In the super-critical case, current method proposed by [17] requires additional regularity of the initial data, i.e. $u_0 \in L^2$.

Proposition 3.12. *Let u be a radial solution to (NLS$_a$) defined on $[0, \infty)$ and satisfying (3.44). In addition, we assume that the initial data $u_0 \in L^2$. Then there exist $\theta \in \mathbb{R}$, $\mu > 0$, and $c, C > 0$, such that $\forall t \geq 0$,*

$$\|u(t)_{[\theta,\mu]} - W\|_{\dot{H}^1_a} \leq Ce^{-ct}. \tag{3.45}$$

A similar result holds if u is defined on $(-\infty, 0]$.

Corollary 3.13. *Under the assumption of Proposition 3.12, the solution u is not defined on $\mathbb{R}$.*

Proof of Corollary 3.13. Suppose otherwise that u is defined on $\mathbb{R}$. Proposition 3.12 then yields that $\lim_{t\to\pm\infty} \mathbf{d}(u(t)) = 0$. In addition, (3.46) holds for all $t \in \mathbb{R}$. Thus, $\partial_t V_R(t) \to 0$ as $t \to \pm\infty$ and $\partial_t V_R(t)$ is a non-increasing function. Then $\partial_t V_R(t) = 0$ for all $t \in \mathbb{R}$, and $\partial_{tt} V_R(t) = 0$ for all $t \in \mathbb{R}$. However, $\partial_{tt} V_R(0) \le -\frac{8}{d-2}\mathbf{d}(u_0) < 0$. We get a contradiction and Corollary 3.13 is proved. □

Proof of Proposition 3.12. **Step 1.** There exist $C, R_0 > 0$(depending only on $M(u_0)$), such that for $R \ge R_0$, and all $t \ge 0$,

$$\partial_t V_R(t) \le CR^2 \mathbf{d}(u(t)), \quad \partial_{tt} V_R(t) \le -\frac{8}{d-2}\mathbf{d}(u(t)), \tag{3.46}$$

where $V_R(t) = \int_{\mathbb{R}^d} |u(t)|^2 \phi_R(x) dx$, $\phi_R(x) = R^2 \phi(\frac{x}{R})$, and ϕ is a non-negative radial C_0^∞ function such that $\phi(r) = r^2$ when $r \le 1$ and $\frac{d^2\phi}{dr^2} \le 2$ when $r \ge 0$.

The same proof of (3.33) shows that $\partial_t V_R(t) \le CR^2 \mathbf{d}(u(t))$. Similar to (3.34), (3.44) yields that

$$\partial_{tt} V_R(t) = -\frac{16}{d-2}\mathbf{d}(u(t)) + A_R(u(t)), \tag{3.47}$$

where $r = |x|$ and $A_R(u(t)) = \int_{|x|\ge R} (4\frac{d^2\phi_R}{dr^2} - 8)|\nabla u|^2 dx + \int_{|x|\ge R} (-\frac{4}{d}\Delta\phi_R + 8)|u|^{2^*} dx - \int_{\mathbb{R}^d} (\Delta\Delta\phi_R)|u(t)|^2 \, dx$.

As $\frac{d^2\phi_R}{dr^2} \leq 2$ and $\Delta\Delta\phi_R = 0$ when $|x| < R$, we have

$$\begin{aligned}
A_R(u(t)) &\lesssim \int_{|x|\geq R} |u|^{2^*} dx + \frac{\|u(t)\|_2^2}{R^2} \\
&\lesssim \|u(t)\|_{L^\infty(|x|\geq R)}^{\frac{4}{d-2}} \|u(t)\|_2^2 + \frac{\|u(t)\|_2^2}{R^2} \\
&\lesssim \frac{1}{R^{\frac{2d+2}{d-2}}} \|u(t)\|_{\dot{H}^1}^{\frac{2}{d-2}} \|u(t)\|_2^{\frac{2d-2}{d-2}} + \frac{1}{R^2} \\
&\lesssim \frac{1}{R^{\frac{2d+2}{d-2}}} \|u(t)\|_{\dot{H}^1_a}^{\frac{2}{d-2}} + \frac{1}{R^2},
\end{aligned}$$

where we use the conservation of mass($\|u(t)\|_2 = \|u_0\|_2$), equivalence of Sobolev norms, and the decay property of radial functions by Strauss [45]: for any radial function $f \in H^1(\mathbb{R}^d)$, $\forall x$, $|x| \geq 1$, $|f(x)| \lesssim |x|^{\frac{1-d}{2}} \|f\|_{\dot{H}^1}^{\frac{1}{2}} \|f\|_2^{\frac{1}{2}}$. Recall that $\|u(t)\|_{\dot{H}^1_a}^2 = \mathbf{d}(u(t)) + \|W\|_{\dot{H}^1_a}^2$ in the super-critical case, thus

$$A_R(u(t)) \lesssim \frac{1}{R^{\frac{2d+2}{d-2}}} (\mathbf{d}(u(t)) + \|W\|_{\dot{H}^1_a}^2)^{\frac{1}{d-2}} + \frac{1}{R^2}. \tag{3.48}$$

When $\mathbf{d}(u(t)) \geq \delta_1 > 0$($\delta_1$ to be chosen later), (3.48) yields that, $\exists R_1(\delta_1) > 0$ such that $R > R_1$,

$$A_R(u(t)) \leq \frac{8}{d-2} \mathbf{d}(u(t)), \tag{3.49}$$

which implies that $\partial_{tt} V_R(t) \leq -\frac{8}{d-2}\mathbf{d}(u(t))$ in view of (3.47). It remains to prove (3.49) when $\mathbf{d}(u(t)) \leq \delta_1$. In view of (3.36), it suffices to prove there exists $0 < \delta_1 < \delta_0$(δ_0 is given in Lemma 3.4) such that

$$\mu^- = \inf\{\mu(t) : t \geq 0, \mathbf{d}(u(t)) \leq \delta_1\} > 0. \tag{3.50}$$

Indeed, from $u_0 \in L^2$ and the conservation of mass, we have

$$\begin{aligned}
M(u(t)) &\geq \int_{|x|\leq \frac{1}{\mu(t)}} |u(t)|^2 dx = \frac{1}{\mu(t)^2} \int_{|x|\leq 1} |u(t)_{[\theta(t),\mu(t)]}|^2 dx \\
&\geq \frac{1}{\mu(t)^2} \left(\int_{|x|\leq 1} W^2 dx - c\delta_1^2 \right).
\end{aligned}$$

Rearranging the inequality and picking δ_1 small, we get (3.50).

The proof of step 1 is complete.

Step 2. For some constants $c, C > 0$,

$$\int_t^\infty \mathbf{d}(u(s))ds \le Ce^{-ct} \text{ and } \lim_{t\to\infty} \mathbf{d}(u(t)) = 0.$$

Given $R > R_0$(defined in step 1), we first point out that $\partial_t V_R(t) > 0$ for $t \ge 0$. Suppose otherwise that at some point $t_* \ge 0$, $\partial_t V_R(t_*) \le 0$. As $\partial_t V_R(t)$ is strictly decreasing by (3.46), $\partial_t V_R(t) \le \partial_t V_R(t_*+1) < 0$ for $t \ge t_*+1$. This implies that $V_R(t)$ will be negative when t is sufficiently large and contradicts the definition of $V_R(t)$.

From fundamental theorem of Calculus and (3.46), for any $T > t > 0$, we get

$$\begin{aligned} \partial_t V_R(T) - \partial_t V_R(t) &= \int_t^T \partial_{tt} V_R(s)ds \le -\frac{8}{d-2}\int_t^T \mathbf{d}(u(s))ds, \\ \partial_t V_R(T) - \partial_t V_R(t) &> -\partial_t V_R(t) \ge -CR^2 \mathbf{d}(u(t)). \end{aligned}$$

Thus, $\int_t^T \mathbf{d}(u(s))ds \lesssim R^2\mathbf{d}(u(t))$. Fix $R > R_0$ and let $T \to \infty$, for some positive constant $C > 0$, we get

$$\int_t^\infty \mathbf{d}(u(s))ds \le C\mathbf{d}(u(t)).$$

Grönwall's inequality then yields that for some constants $c, C > 0$,

$$\int_t^\infty \mathbf{d}(u(s))ds \le Ce^{-ct}. \tag{3.51}$$

This clearly implies that there exists $\{t_n\}$ such that

$$\mathbf{d}(u(t_n)) \to 0 \text{ as } t_n \to \infty. \tag{3.52}$$

Next, we show that

$$\lim_{t\to\infty} \mathbf{d}(u(t)) = 0. \tag{3.53}$$

Suppose (3.53) fails, the continuity of $\mathbf{d}(u(t))$ yields that there exists another sequence $t_n' \to \infty (t_n' > t_n)$ such that

$$\mathbf{d}(u(t_n')) = \delta', \ 0 < \mathbf{d}(u(t)) < \delta' \text{ for } \forall t \in [t_n, t_n'), \tag{3.54}$$

where $\delta' < \delta_0(\delta_0$ is defined in Lemma 3.4). Thus, Lemma 3.4, on $t \in [t_n, t_n']$, in the sense of (3.8), we have

$$u(t)_{[\theta(t),\mu(t)]} = (1+\alpha(t))W + \tilde{u}(t), \tag{3.55}$$

where all the parameters satisfy (3.10) and (3.11). From Lemma 3.4, (3.52) and (3.54), $|\alpha(t_n)| \sim \mathbf{d}(u(t_n)) \to 0$ and $|\alpha(t_n')| \sim \mathbf{d}(u(t_n')) = \delta'$. Hence, if $\mu(t)$ is uniformly bounded on $[t_n, t_n']$, we have

$$\begin{cases} |\alpha(t_n') - \alpha(t_n)| = \left| \int_{t_n}^{t_n'} \alpha'(t) dt \right| \lesssim \int_{t_n}^{\infty} \mathbf{d}(u(t)) dt \to 0, \\ \lim_{n\to\infty} |\alpha(t_n') - \alpha(t_n)| = \lim_{n\to\infty} |\alpha(t_n')| = O(\delta') \end{cases}$$

which yields a desired contradiction. It remains to prove

$$\sup_{t\in[t_n,t_n'],n\in\mathbb{N}} \mu(t) < \infty. \tag{3.56}$$

We first show that $\sup_{n\in\mathbb{N}} \mu(t_n) < \infty$. Suppose otherwise that $\mu(t_n) \to \infty$ after passing to a sub-sequence if necessary. As $\|\tilde{u}(t_n)\|_{\dot{H}_a^1} \sim |\alpha(t_n)| \sim \mathbf{d}(u(t_n)) \to 0$, $u(t_n)_{[\theta(t_n),\mu(t_n)]} \to W$ in $\dot{H}_a^1$ which implies that for any $\varepsilon > 0$,

$$\int_{|x|\ge\varepsilon} |u(t_n)|^{2^*} = \int_{|x|\ge\mu(t_n)\varepsilon} |u(t_n)_{[\theta(t_n),\mu(t_n)]}|^{2^*} \to 0 \text{ as } n \to \infty.$$

Together with Hölder inequality, this yields

$$\begin{aligned} V_R(t_n) &= \int_{|x|\ge\varepsilon} |u(t_n)|^2 \varphi_R \, dx + \int_{|x|<\varepsilon} |u(t_n)|^2 \varphi_R \, dx \\ &\lesssim R^2 \Big(\int_{|x|\ge\varepsilon} |u(t_n)|^{2^*} \Big)^{\frac{d-2}{d}} + o_\varepsilon(1) \to 0, \end{aligned}$$

by first taking $n \to \infty$ then $\varepsilon \to 0$. While from $V_R'(t) > 0$, we get $V_R(t) < 0$ for $t \geq 0$, which is absurd. Next, we use $\sup_{n\in\mathbb{N}} \mu(t_n) < \infty$ and $\int_t^\infty \mathbf{d}(u(s))ds \leq Ce^{-ct}$ to finish the proof of (3.56). Let

$$\mu(b_n) = \sup\{\mu(t) : t \in [t_n, t_n']\}.$$

Clearly, $\mu(t_n) \leq \mu(b_n)$. In addition, from $\int_t^\infty \mathbf{d}(u(s))ds \leq Ce^{-ct}$, Lemma 3.4 and fundamental theorem of Calculus, as $n \to \infty$, we have

$$\left|\frac{1}{\mu(t_n)^2} - \frac{1}{\mu(b_n)^2}\right| \leq \int_{t_n}^{t_n'} \left|\frac{\mu'(t)}{\mu(t)^3}\right| dt \lesssim \int_{t_n}^{\infty} \mathbf{d}(u(t))dt \to 0.$$

Together with $\sup_{n\in\mathbb{N}} \mu(t_n) < \infty$, this yields (3.56).

Step 3. Conclude the proof of (3.45).

As $\lim_{t\to\infty} \mathbf{d}(u(t)) = 0$, for large t, we have the decomposition in the sense of (3.8),

$$u(t)_{[\theta(t),\mu(t)]} = (1 + \alpha(t))W + \tilde{u}(t),$$

where all the parameters satisfy (3.10) and (3.11) as in Lemma 3.4.

Convergence of $\mu(t)$: $\lim_{t\to\infty} \mu(t) = \mu_\infty \in (0, \infty)$.

Picking any sequence $t_n \to \infty$, we have

$$\left|\frac{1}{\mu(t_n)^2} - \frac{1}{\mu(t_m)^2}\right| = \left|\int_{t_n}^{t_m} \frac{\mu'(s)}{\mu(s)^3} ds\right| \lesssim \left|\int_{t_n}^{t_m} \mathbf{d}(u(s))ds\right| \leq Ce^{-c\min\{t_n,t_m\}},$$

which shows that $\{\frac{1}{\mu(t_n)^2}\}$ is a Cauchy sequence in $\mathbb{R}$, thus converges. As the sequence is arbitrarily taken, we have $\mu(t) \to \mu_\infty$. In addition, $\mu_\infty < \infty$ follows exactly the same proof as (3.56) above. $\mu_\infty > 0$, however, follows from the argument (3.50).

Convergence of $\theta(t)$: $\lim_{t\to\infty} \theta(t) = \theta_\infty \in [0, \infty)$. The proof is the same as the sub-critical case.

Exponential decay: $\alpha(t) \sim \|\tilde{u}(t)\|_{\dot{H}^1_a} \sim \mathbf{d}(u(t)) \leq Ce^{-ct}$ for t large.

As $\mu(t) \to \mu_\infty \in (0,\infty)$, $\mu(t)$ is uniformly bounded. From Lemma 3.4, fundamental theorem of Calculus and (3.51), we have

$$\begin{aligned} \mathbf{d}(u(t)) &\sim \|\tilde{u}(t)\|_{\dot{H}^1_a} \sim |\alpha(t)| = \left|\int_t^\infty \alpha'(s)ds\right| \\ &\lesssim \left|\int_t^\infty \mathbf{d}(u(s))ds\right| \leq Ce^{-ct}. \end{aligned}$$

Therefore, in view of the decomposition $u(t)_{[\theta(t),\mu(t)]} = (1+\alpha(t))W + \tilde{u}(t)$, from $\mu(t) \to \mu_\infty$, $\theta(t) \to \theta_\infty$, and $\alpha(t) \sim \|\tilde{u}(t)\|_{\dot{H}^1_a} \leq Ce^{-ct}$, (3.45) holds. The proof of Proposition 3.12 is complete. □

3.2 Spectral theory and preliminary estimates

In this section, we will explore the spectral theory of some operators generated from the equation (3.57) and related Strichartz type estimates. The corresponding problem for NLS was studied by Duyckaerts and Merle [17]. Throughout this section, we will write a complex function $f = f_1 + if_2$ or $\binom{f_1}{f_2}$ with $f_1 = \mathbf{Re}f$ and $f_2 = \mathbf{Im}f$.

To estimate the difference between u and W, we consider the quantity $u - W$. Let $v = u - W$ and u solves NLS$_a$, we then have

$$\partial_t v + \mathcal{L}(v) + R(v) = 0, \tag{3.57}$$

where $R(v) = -i|v+W|^{p_c-1}(v+W) + iW^{p_c} + ip_cW^{p_c-1}v_1 - W^{p_c-1}v_2$ and

$$\mathcal{L} = \begin{pmatrix} 0 & -\mathcal{L}_a + W^{p_c-1} \\ \mathcal{L}_a - p_cW^{p_c-1} & 0 \end{pmatrix}.$$

Note that $\mathcal{L}(v) = (-\mathcal{L}_a + W^{p_c-1})v_2 + i(\mathcal{L}_a - p_cW^{p_c-1})v_1$.

3.2.1 Spectral theory for the linearized operator $\mathcal{L}$

As $\mathcal{L}_a W = W^{p_c}$ and $(\mathcal{L}_a - p_c W^{p_c-1})W_1 = 0$, we get

$$\mathcal{L}(W) = \mathcal{L}(W_1) = 0. \tag{3.58}$$

If either $W \in L^2$ or $W_1 \in L^2$, we see that 0 is the eigenvalue of $\mathcal{L}$ with W or W_1 being the corresponding eigenfunction. However, this fails when $d \leq 5$ regarding the constrain on a. It turns out that $\mathcal{L}$ has two eigenfunctions for two non-zero real eigenvalues.

Lemma 3.14. *Let $d \geq 3$. The operator $\mathcal{L}$ admits two eigenfunctions $\mathcal{Y}_+$, $\mathcal{Y}_- \in \mathcal{S}(\mathbb{R}^d)$ with real eigenvalues*

$$\mathcal{L}(\mathcal{Y}_+) = e_0\mathcal{Y}_+,\ \mathcal{L}(\mathcal{Y}_-) = -e_0\mathcal{Y}_-,\ \mathcal{Y}_- = \overline{\mathcal{Y}_+},\ e_0 \in (0, \infty). \tag{3.59}$$

Proof. To simplify the notation, we write $V := W^{p_c-1}$. From $\overline{\mathcal{L}(v)} = -\mathcal{L}(\bar{v})$, $\mathcal{L}\mathcal{Y}_+ = e_0\mathcal{Y}_+$ implies that $\mathcal{L}\overline{\mathcal{Y}_+} = -\overline{\mathcal{L}(\mathcal{Y}_+)} = -e_0\overline{\mathcal{Y}_+}$. It suffices to show the existence of an eigenfunction $\mathcal{Y}_+$. Denote $\mathcal{Y}_1 = \mathbf{Re}\mathcal{Y}_+$ and $\mathcal{Y}_2 = \mathbf{Im}\mathcal{Y}_+$. $\mathcal{L}\mathcal{Y}_+ = e_0\mathcal{Y}_+$ is equivalent to

$$\begin{cases} (\mathcal{L}_a - V)\mathcal{Y}_2 = -e_0\mathcal{Y}_1 \\ (\mathcal{L}_a - p_cV)\mathcal{Y}_1 = e_0\mathcal{Y}_2 \end{cases}. \tag{3.60}$$

Note that $\mathcal{L}_a - V$ on L^2 with domain H^2 is self-adjoint and non-negative(by the Sharp Gagliardo-Nirenberg inequality 2.9), thus it has a unique square root $(\mathcal{L}_a - V)^{\frac{1}{2}}$ with domain H^1(see [14]).

Denote $P = (\mathcal{L}_a - V)^{\frac{1}{2}}(\mathcal{L}_a - p_cV)(\mathcal{L}_a - V)^{\frac{1}{2}}$. It suffices to show that there exists $f_1 \in H^4$ such that

$$Pf_1 = -e_0^2 f_1. \tag{3.61}$$

Indeed, $\mathcal{Y}_1 = (\mathcal{L}_a - V)^{\frac{1}{2}} f_1$ and $\mathcal{Y}_2 = \frac{1}{e_0}(\mathcal{L}_a - p_c V)(\mathcal{L}_a - V)^{\frac{1}{2}} f_1$ is a solution to the system (3.60) if (3.61) holds. Note that $P = (\mathcal{L}_a - V)^2 + (1-p_c)(\mathcal{L}_a - V)^{\frac{1}{2}} V (\mathcal{L}_a - V)^{\frac{1}{2}}$ is a relatively compact, self-adjoint perturbation of $(\mathcal{L}_a - V)^2$ with essential spectrum lies in $[0, \infty)$.

From $\langle Pf, f \rangle_2 = \left\langle (\mathcal{L}_a - p_c V)(\mathcal{L}_a - V)^{\frac{1}{2}} f, (\mathcal{L}_a - V)^{\frac{1}{2}} f \right\rangle_2$, it suffices to find $F = (\mathcal{L}_a - V) g$ with $g \in H^4$ such that

$$\langle (\mathcal{L}_a - p_c V) F, F \rangle_2 < 0. \tag{3.62}$$

When $3 \leq d \leq 2 + \frac{2}{\beta}$, then $W \notin L^2(\mathbb{R}^d)$. Denote $W_b = \chi(\frac{x}{b}) W$, then we first show that $\exists b > 0$ such that

$$\langle (\mathcal{L}_a - p_c V) W_b, W_b \rangle_2 < 0. \tag{3.63}$$

Indeed, by definition of $\mathcal{L}_a = -\Delta + \frac{a}{|x|^2}$, one can rewrite $\langle (\mathcal{L}_a - p_c V) W_b, W_b \rangle_2$ as $\int (1 - p_c) \chi(\frac{x}{b}) W^{p_c + 1} - \frac{2}{b} W (\nabla \chi)(\frac{x}{b}) \cdot \nabla W - \frac{1}{b^2} (\Delta \chi)(\frac{x}{b}) W^2$. Taking $b \gg 1$, on the support of $(\nabla \chi)(\frac{x}{b})$ and $(\Delta \chi)(\frac{x}{b})$, we have $W \leq |x|^{(\beta+1)\frac{2-d}{2}}$ and $|\nabla W| \leq |x|^{(\beta+1)\frac{2-d}{2} - 1}$. Thus,

$$\left| \int \frac{2}{b} W (\nabla \chi)(\frac{x}{b}) \cdot \nabla W| + |\int \frac{1}{b^2} (\Delta \chi)(\frac{x}{b}) W^2 \right| \lesssim b^{\beta(2-d)}.$$

Note that $0 < \beta \leq 1$, hence (3.63) holds if $b \gg 1$.

From Lemma 3.2 and the fact $W \notin L^2$, $Ker(\mathcal{L}_a - V) = 0$. As $\mathcal{L}_a - V$ is self-adjoint, Range$(\mathcal{L}_a - V)$ is dense in L^2. Given $\varepsilon > 0$, there exists $G_\varepsilon \in H^2$ such that

$$\|(\mathcal{L}_a - V) G_\varepsilon - (\mathcal{L}_a - V + 1) W_b\|_2 < \varepsilon,$$

where b is picked such that (3.63) holds. Let $F_\varepsilon = (\mathcal{L}_a - V + 1)^{-1}(\mathcal{L}_a - V)G_\varepsilon = (\mathcal{L}_a - V)(\mathcal{L}_a - V + 1)^{-1}G_\varepsilon$, we get $\|(\mathcal{L}_a - V + 1)(F_\varepsilon - W_b)\|_2 < \varepsilon$ which implies that $\|F_\varepsilon - W_b\|_{H^2} \leq \varepsilon \|(\mathcal{L}_a - V + 1)^{-1}\|_{L^2 \to H^2}$. Thus,

$$|\langle(\mathcal{L}_a - p_c V)W_b, W_b\rangle_2 - \langle(\mathcal{L}_a - p_c V)F_\varepsilon, F_\varepsilon\rangle_2| \lesssim \varepsilon.$$

Then, for small ε, (3.63) yields (3.62) for $F = F_\varepsilon$.

When $d \geq 2+\frac{2}{\beta}$, then $W \in L^2(\mathbb{R}^d)$. From $\mathcal{L}_a W = W^{p_c}$ and $(\mathcal{L}_a - p_c W^{p_c-1})W_1 = 0$, for any $\alpha \in \mathbb{R}$, we have

$$\langle(\mathcal{L}_a - p_c V)(W + \alpha W_1), W + \alpha W_1\rangle_2 = \langle(\mathcal{L}_a - p_c V)W, W\rangle_2 < 0.$$

In view of Lemma 3.2, $Ker(\mathcal{L}_a - V) = \text{Span}\{W\}$ and

$$\overline{\text{Range}(\mathcal{L}_a - V)} = \text{Span}\{W\}^\perp. \tag{3.64}$$

Choose α such that $\langle W + \alpha W_1, W\rangle_2 = 0$, then $(\mathcal{L}_a - V)W = 0$ implies that

$$\langle(\mathcal{L}_a - V + 1)(W + \alpha W_1), W\rangle_2 = \langle W + \alpha W_1, W\rangle_2 = 0.$$

Given $\varepsilon > 0$, (3.64) shows that there exists $G_\varepsilon \in H^2$ such that

$$\|(\mathcal{L}_a - V)G_\varepsilon - (\mathcal{L}_a - V + 1)(W + \alpha W_1)\|_2 < \varepsilon.$$

Similarly as previous case, for small ε, (3.62) holds for $F = (\mathcal{L}_a - V + 1)^{-1}(\mathcal{L}_a - V)G_\varepsilon$.

Therefore, (3.62) holds for all $d \geq 3$, thus also proofs (3.59). It remains to show that $\mathcal{Y}_+$, $\mathcal{Y}_- \in \mathcal{S}(\mathbb{R}^d)$ which follows almost the same treatment as in [17]. We denote $(\mathcal{P}_{k,s})$ as the property such that

$$\forall \varphi \in C_0^\infty(\mathbb{R}^d \setminus \{0\}),\ \exists C,\ \forall R \geq 1,\ \|\varphi(\frac{x}{R})\mathcal{Y}_1\|_{H^s} \leq \frac{C}{R^k}.$$

We aim to show $(\mathcal{P}_{k,s})$ holds for all $k, s \geq 0$.

Note that $\mathcal{Y}_1 = (\mathcal{L}_a - V)^{\frac{1}{2}} f_1 \in H^3$ as $f_1 \in H^4$, thus $(\mathcal{P}_{0,3})$ holds. By induction, it suffices to prove that $(\mathcal{P}_{k,s})$ holds implies $(\mathcal{P}_{k+1,s+1})$ holds. Let $\varphi, \tilde{\varphi} \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ with $\tilde{\varphi} = 1$ on the support of φ. From (3.60), we get

$$(\mathcal{L}_a^2 + e_0^2)\mathcal{Y}_1 = V\mathcal{L}_a\mathcal{Y}_1 + p_c\mathcal{L}_a(V\mathcal{Y}_1) - p_c V^2 \mathcal{Y}_1.$$

Together with the decay that $V(x) \sim \frac{1}{|x|^{2\beta+2}}$ when $|x|$ is large, this yields

$$\|\varphi(\frac{x}{R})[(\mathcal{L}_a^2 + e_0^2)\mathcal{Y}_1]\|_{H^{s-3}} \lesssim \frac{\|\tilde{\varphi}(\frac{x}{R})\mathcal{Y}_1\|_{H^s}}{R^{2\beta+2}}.$$

Note that e_0^2 lies in the resolvent of $\mathcal{L}_a^2$, thus

$$\begin{aligned}
\|\varphi(\frac{x}{R})\mathcal{Y}_1\|_{H^{s+1}} &= \|(\mathcal{L}_a^2 + e_0^2)^{-1}(\mathcal{L}_a^2 + e_0^2)[\varphi(\frac{x}{R})\mathcal{Y}_1]\|_{H^{s+1}} \\
&\leq \|(\mathcal{L}_a^2 + e_0^2)[\varphi(\frac{x}{R})\mathcal{Y}_1]\|_{H^{s-3}} \\
&\lesssim \|\varphi(\frac{x}{R})[(\mathcal{L}_a^2 + e_0^2)\mathcal{Y}_1]\|_{H^{s-3}} + \frac{\|\tilde{\varphi}(\frac{x}{R})\mathcal{Y}_1\|_{H^s}}{R} \\
&\lesssim \frac{1}{R^{k+1}}.
\end{aligned}$$

As $(\mathcal{P}_{k,s})$ holds implies that $(\mathcal{P}_{k+1,s+1})$ holds, the proof is complete. □

We also remark that if $e_1 \in \mathbb{R} \setminus \{-e_0, 0, e_0\}$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$, then by a similar argument above, one can obtain

$$(\mathcal{L} - e_1)^{-1}\phi \in \mathcal{S}(\mathbb{R}^d). \tag{3.65}$$

We define a bilinear form B on $\dot{H}^1$ associated with the quadratic form Q,

$$B(f, g) = \frac{1}{2}\mathbf{Im}\int \mathcal{L}(f)\bar{g}. \tag{3.66}$$

Consequently, we have

$$\begin{cases} B(f,f) = Q(f) \\ B(f,g) = B(g,f), \quad B(iW,f) = B(W_1,f) = 0, \ \forall f,g \in \dot{H}^1_a \\ B(\mathcal{L}f,g) = -B(f,\mathcal{L}g), \quad \forall f,g \in \dot{H}^1_a, \quad \forall \mathcal{L}f, \mathcal{L}g \in \dot{H}^1_a \\ Q(\mathcal{Y}_+) = Q(\mathcal{Y}_-) = 0, \quad B(\mathcal{Y}_+,\mathcal{Y}_-) \neq 0. \end{cases} \tag{3.67}$$

Indeed, the only nontrivial relation is $B(\mathcal{Y}_-,\mathcal{Y}_+) \neq 0$. Suppose otherwise $B(\mathcal{Y}_-,\mathcal{Y}_+) = 0$. Then $\frac{1}{2}\mathbf{Im}\int -e_0\mathcal{Y}_-\mathcal{Y}_- = 0$, which implies that either $\mathbf{Re}\mathcal{Y}_- = 0$ or $\mathbf{Im}\mathcal{Y}_- = 0$. If $\mathbf{Im}\mathcal{Y}_- = 0$, then $\mathcal{Y}_+ = \mathcal{Y}_-$ which is a contradiction; when $\mathbf{Re}\mathcal{Y}_- = 0$, then $\mathcal{Y}_+ = -\mathcal{Y}_-$. Thus, $\mathcal{L}\mathcal{Y}_+ = \mathcal{L}(-\mathcal{Y}_-) = e_0\mathcal{Y}_- = -e_0\mathcal{Y}_+$ which contradicts $\mathcal{L}\mathcal{Y}_+ = e_0\mathcal{Y}_+$. A different approach can be found in [17].

Lemma 3.15. *Let*

$$G_\perp = \{v \in \dot{H}^1_a, \ \langle iW, v\rangle_{\dot{H}^1_a} = \langle W_1, v\rangle_{\dot{H}^1_a} = B(\mathcal{Y}_+, v) = B(\mathcal{Y}_-, v) = 0\}.$$

There exists $c > 0$ such that

$$Q(f) \geq c\,\|f\|^2_{\dot{H}^1_a}, \qquad \forall f \in G_\perp. \tag{3.68}$$

Proof. Recall that $\mathcal{Y}_- = \overline{\mathcal{Y}_+}$ and $\dot{H}^1_a = H \oplus H^\perp$. Thus, one can decompose f, $\mathcal{Y}_+$ and $\mathcal{Y}_-$ into

$$f = \alpha W + \tilde{h}, \ \mathcal{Y}_+ = \theta W + \eta iW + \xi W_1 + h_+, \ \mathcal{Y}_- = \theta W - \eta iW + \xi W_1 + h_-, \tag{3.69}$$

where $\tilde{h}, h_+, h_- \in H^\perp$, $h_- = \overline{h_+}$. (3.67) then yields

$$\begin{aligned} Q(f) &= \alpha^2 Q(W) + Q(\tilde{h}), \\ Q(\mathcal{Y}_+) &= \theta^2 Q(W) + Q(h_+) + 2\theta B(W, h_+) = 0, \\ Q(\mathcal{Y}_-) &= \theta^2 Q(W) + Q(h_-) + 2\theta B(W, h_-) = 0. \end{aligned} \tag{3.70}$$

While for any $h \in H^{\perp}$, $B(W,h) = \frac{1}{2}\int(\mathcal{L}_a W - p_c W^{p_c})h_1 = \frac{1-p_c}{2}\langle W, h_1\rangle_{\dot{H}^1_a} = 0$. Thus,

$$-\theta^2 Q(W) = Q(h_+) = Q(h_-). \tag{3.71}$$

In addition, from $B(\mathcal{Y}_+, f) = B(\mathcal{Y}_-, f) = 0$, we get $\alpha\theta Q(W) + B(h_+, \tilde{h}) = \alpha\theta Q(W) + B(h_-, \tilde{h}) = 0$. Together with (3.70) and (3.71), this yields

$$Q(f) = -\frac{B(h_+, \tilde{h})B(h_-, \tilde{h})}{\sqrt{Q(h_+)}\sqrt{Q(h_-)}} + Q(\tilde{h}). \tag{3.72}$$

Next, we show that h_+ and h_- are independent in the real Hilbert space $\dot{H}^1_a$. As $h_- = \overline{h_+}$, it suffices to prove

$$\mathbf{Re}h_+ \neq 0 \text{ and } \mathbf{Im}h_+ \neq 0. \tag{3.73}$$

Suppose otherwise that $\mathbf{Im}h_+ \neq 0$, (3.69) then implies that $\mathcal{Y}_2 \in \text{Span}\{W\}$. From $\mathcal{L}_a W = W^{p_c}$ and (3.60), we see that $\mathcal{Y}_2 = \mathcal{Y}_1 = 0$, which contradicts the fact that $\mathcal{Y}_+$ is an eigenfunction. Assume that $\mathbf{Re}h_+ \neq 0$, from $(\mathcal{L}_a - p_c W^{p_c-1})W_1 = 0$, (3.69) and (3.60),

$$\begin{aligned} e_0\mathcal{Y}_2 &= (\mathcal{L}_a - p_c W^{p_c-1})(\theta W + \xi W_1) = \theta(1-p_c)W^{p_c}, \\ -e_0\mathcal{Y}_1 &= (\mathcal{L}_a - W^{p_c-1})\mathcal{Y}_2 = \frac{\theta(1-p_c)}{e_0}(\mathcal{L}_a - W^{p_c-1})W^{p_c}. \end{aligned}$$

By (3.69), $\mathcal{Y}_1 = \frac{\theta(p_c-1)}{e_0^2}(\mathcal{L}_a - W^{p_c-1})W^{p_c} \in \text{Span}\{W, W_1\}$. While direct computation shows

$$(\mathcal{L}_a - W^{p_c-1})W^{p_c} \notin \text{Span}\{W, W_1\}.$$

We get another contradiction. Hence, (3.73) holds and h_+ and h_- are independent in $\dot{H}^1_a$.

To conclude the proof of the lemma, we claim that there exists $0 < b < 1$ such that for any nontrivial $h \in H^{\perp}$,

$$\left| \frac{B(h_+, h)B(h_-, h)}{\sqrt{Q(h_+)}\sqrt{Q(h_-)}} \right| \leq bQ(h). \tag{3.74}$$

By (3.72) and Lemma 3.2, we then get

$$Q(f) \geq (1-b)Q(\tilde{h}) \gtrsim \|\tilde{h}\|^2_{\dot{H}^1_a}. \tag{3.75}$$

(3.70) yields that $bQ(\tilde{h}) \geq -\alpha^2 Q(W) = \frac{p_c - 1}{2}\alpha^2 \|W\|^2_{\dot{H}^1_a}$, thus $Q(f) \gtrsim \alpha^2 \|W\|^2_{\dot{H}^1_a}$. By (3.75), we get

$$Q(f) \gtrsim \alpha^2 \|W\|^2_{\dot{H}^1_a} + \|\tilde{h}\|^2_{\dot{H}^1_a} = \|f\|^2_{\dot{H}^1_a}.$$

Therefore, (3.68) holds and it remains to verify (3.74). Indeed, from Cauchy-Schwartz, we see that $0 < b \leq 1$. Next, we exclude the case $b = 1$. Assume that there exists $h \in H^{\perp}$ such that the equality in (3.68) holds when $b = 1$. Thus, two inequalities of Cauchy Schwartz become equality and $h \in \mathrm{Span}\{h_+\} \cap \mathrm{Span}\{h_-\} = \{0\}$. This contradicts that h is nontrivial. Hence, (3.74) holds and the proof of this lemma is complete. $\square$

Denote $\sigma(\mathcal{L})$ as the spectrum of $\mathcal{L}$ on L^2 with domain H^2. Together with (3.59), we see that $\{-e_0, e_0\} \subset \sigma(\mathcal{L}) \cap \mathbb{R}$. And we claim that

$$\sigma(\mathcal{L}) \cap \mathbb{R} \subset \{-e_0, 0, e_0\}. \tag{3.76}$$

Indeed, if the essential spectrum of $\mathcal{L}_a$ equals $[0, \infty)$, we then have $\sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\}$.

Proof of (3.76). Since $\mathcal{L}$ is a relative compact perturbation of $\begin{pmatrix} 0 & -\mathcal{L}_a \\ \mathcal{L}_a & 0 \end{pmatrix}$, the essential spectrum lies in $i\mathbb{R}$. Thus, $\sigma(\mathcal{L}) \cap \mathbb{R}$ only contains eigenvalues. Assume that

(3.76) fails, then there exists nontrivial $f_1 \in H^2$ such that $\mathcal{L}f_1 = e_1 f_1$ for some $e_1 \in \mathbb{R}\backslash\{-e_0, 0, e_0\}$. By (3.67), $B(f_1, \mathcal{Y}_+) = e_1 \frac{1}{2}\mathbf{Im}\int f\overline{\mathcal{Y}_+}$ and $B(\mathcal{Y}_+, f_1) = e_0 \frac{1}{2}\mathbf{Im}\int \mathcal{Y}_+\overline{f_1}$, thus $(e_1 + e_0)B(\mathcal{Y}_+, f_1) = 0$ which implies that $B(\mathcal{Y}_+, f_1) = 0$. Similarly, we have $B(\mathcal{Y}_-, f_1) = 0$. Hence, we can write

$$f_1 = \frac{\langle f_1, iW\rangle_{\dot{H}^1_a}}{\|iW\|^2_{\dot{H}^1_a}} iW + \frac{\langle f_1, W_1\rangle_{\dot{H}^1_a}}{\|W_1\|^2_{\dot{H}^1_a}} W_1 + g,$$

where $g \in G_\perp$. By definition of B, $B(f_1, f_1) = e_1 \frac{1}{2}\mathbf{Im}\int f_1\overline{f_1} = 0$. Together with (3.67), this yields that $B(g, g) = Q(g) = 0$. By Lemma 3.15, $g = 0$. Thus, in view of $\mathcal{L}(iW) = \mathcal{L}(W_1) = 0$, we have $\mathcal{L}f_1 = e_1 f_1 = 0$, which contradicts f_1 being nontrivial. Therefore, $\sigma(\mathcal{L}) \cap \mathbb{R} \subset \{-e_0, 0, e_0\}$. □

3.2.2 Preliminary estimates of the linearized operator and equation

Most of the estimates below serve the same purpose as in [17]. Even though we stick to the notations in [17], most of them differs from those in [17].

We can rewrite (3.57) as a nonlinear Schrödinger equation with inverse square potential

$$i\partial_t v - \mathcal{L}_a v = -\mathcal{V}(v) - iR(v), \tag{3.77}$$

where $\mathcal{V}(v) = p_c W^{p_c - 1} v_1 + iW^{p_c-1} v_2$. Next, we develop several Strichartz type estimates associated with the equation (3.77).

Lemma 3.16 (Nonlinear estimate I)**.** *Let $f \in L^{2^*}$, then*

$$\|\mathcal{V}(f)\|_{\frac{2d}{d+2}} \lesssim \|f\|_{2^*}.$$

Let I be a finite time interval and $f \in S(I)$ such that $\nabla f \in Z(I)$. Then, there exists

C independent of I, f, and g such that

$$\begin{cases} \|f\|_{S(I)} \leq C\|\nabla f\|_{Z(I)} \\ \|\sqrt{\mathcal{L}_a}\mathcal{V}(f)\|_{N(I)} \sim \|\nabla \mathcal{V}(f)\|_{N(I)}, \|\nabla f\|_{Z(I)} \sim \|\sqrt{\mathcal{L}_a} f\|_{Z(I)} \\ \|\sqrt{\mathcal{L}_a}\mathcal{V}(f)\|_{N(I)} \lesssim |I|^{\theta_1}\|\sqrt{\mathcal{L}_a} f\|_{Z(I)} \end{cases},$$

where $\theta_1 = \frac{4\sigma - 12d + 2d\sigma + 5d^2 + 4}{(d+2)^2(d-2)}$.

Proof. From Hölder inequality, $\|\mathcal{V}(f)\|_{\frac{2d}{d+2}} \lesssim \|W^{p_c-1}\|_{\frac{d}{2}}\|f\|_{2^*} \lesssim \|f\|_{2^*}$. And $\|f\|_{S(I)} \lesssim \|\nabla f\|_{Z(I)}$ follows from Sobolev embedding. Note that $\left|\frac{\nabla W(x)}{W(x)}\right| \sim \frac{1}{|x|}$ no matter x is large or small, from Hölder and Hardy inequality,

$$\begin{aligned} \|\nabla \mathcal{V}(f)\|_{N(I)} &\lesssim \|W^{p_c-1}\nabla f\|_{N(I)} + \|W^{p_c-1}(\frac{\nabla W}{W})f\|_{N(I)} \\ &\lesssim \|W\|_{S(I)}^{p_c-1}\|\nabla f\|_{Z(I)} \lesssim |I|^{\theta_1}\|\nabla f\|_{Z(I)} \end{aligned}$$

An application of the equivalence of Sobolev norms completes the proof. □

Lemma 3.17 (Nonlinear estimate II)**.** *Let f, g be functions in* L^{2^*}*. Then*

$$\|R(f) - R(g)\|_{\frac{2d}{d+2}} \lesssim \|f - g\|_{2^*}(\|f\|_{2^*} + \|g\|_{2^*} + \|f\|_{2^*}^{p_c-1} + \|g\|_{2^*}^{p_c-1}).$$

Let I be a finite time interval and f, g be functions in S(I), such that ∇f*,* $\nabla g \in Z(I)$*. Then*

$$\begin{aligned} \|\nabla[R(f) - R(g)]\|_{N(I)} &\lesssim \|\nabla(f - g)\|_{Z(I)} \\ &\quad \cdot \left[|I|^{\theta_2}(\|\nabla f\|_{Z(I)} + \|\nabla g\|_{Z(I)}) + \|\nabla f\|_{Z(I)}^{p_c-1} + \|\nabla g\|_{Z(I)}^{p_c-1}\right], \end{aligned}$$

where $\theta_2 = \frac{(6-d)(4\sigma - 12d + 2d\sigma + 5d^2 + 4)}{4(d-2)(d+2)^2}$*. Equivalence of Sobolev norms then yields that above estimate holds after replacing* ∇ *by* $\sqrt{\mathcal{L}_a}$*.*

Proof. From Lemma 5.6 in [17], we have

$$
\begin{aligned}
|R(f)-R(g)| &\lesssim |f-g|(W^{p_c-2}(|f|+|g|)+|f|^{p_c-1}+|g|^{p_c-1}) \\
|\nabla(R(f)-R(g))| &\lesssim |(A)|+|(B)|+|(C)|,
\end{aligned}
$$

where

$$
\begin{aligned}
(A) &= W^{-1}\nabla W(f-g)(W^{p_c-2}|f|+W^{p_c-2}|g|+|f|^{p_c-1}+|g|^{p_c-1}), \\
(B) &= W\nabla(W^{-1}f-W^{-1}g)(W^{p_c-2}|f|+|f|^{p_c-1}), \\
(C) &= W\nabla(W^{-1}g)|f-g|(W^{p_c-2}+|f|^{p_c-2}+|g|^{p_c-2}).
\end{aligned}
$$

Note that $\left|\frac{\nabla W(x)}{W(x)}\right| \sim \frac{1}{|x|}$ no matter x is large or small, from Hölder and Hardy inequality,

$$
\begin{aligned}
\|(A)\|_{N(I)} &\lesssim \|\nabla(f-g)\|_{Z(I)}(\|W\|_{S(I)}^{p_c-2}(\|f\|_{S(I)}+\|g\|_{S(I)})+\|f\|_{S(I)}^{p_c-1}+\|g\|_{S(I)}^{p_c-1}) \\
&\lesssim \|\nabla(f-g)\|_{Z(I)}(|I|^{\theta_2}(\|f\|_{S(I)}+\|g\|_{S(I)})+\|f\|_{S(I)}^{p_c-1}+\|g\|_{S(I)}^{p_c-1}).
\end{aligned}
$$

Terms $\|(B)\|_{N(I)}$ and $\|(C)\|_{N(I)}$ can be estimated analogously. Equivalence of Sololev norms yields the conclusion. $\square$

From above non-linear estimates, a continuity argument in [17] shows the following. As the proof of this lemma is identical to [17], for simplicity, we skip it.

Lemma 3.18 (Strichartz estimate, [17])**.** *Let v be a solution of* (3.57). *Assume for some $c_0>0$,*

$$\|v(t)\|_{\dot{H}^1_a} \lesssim e^{-c_0 t} \quad \text{for all } t \geq 0.$$

Then for any admissible pair (q,r),

$$\|v\|_{S(t,\infty)}+\|\sqrt{\mathcal{L}_a}v\|_{q,r(t,\infty)} \lesssim e^{-c_0 t}.$$

As $N(I) \neq L_t^2 L_x^{\frac{2d}{d+2}}$, we get above control for all admissible pair (q,r). We also recall one neat result which simplifies the argument on solutions with exponential decay or increase on time.

Claim 3.19 (Sums of exponential, [17]). *Let $t_0 > 0$, $p \in (1,\infty)$, $a_0 \neq 0$, E a normed vector space, and $f \in L^p([t_0,\infty);E)$ such that*

$$\exists \tau_0, C_0 > 0,\ \forall t \geq t_0,\ \|f\|_{L^p([t,t+\tau_0);E)} \leq C_0 e^{a_0 t}. \tag{3.78}$$

Then for $t \geq t_0$,

$$\begin{cases} \|f\|_{L^p([t,\infty);E)} \leq \frac{C_0 e^{a_0 t}}{1-e^{a_0\tau_0}} & \text{if } a_0 < 0 \\ \|f\|_{L^p([t_0,t);E)} \leq \frac{C_0 e^{a_0 t}}{1-e^{-a_0\tau_0}} & \text{if } a_0 > 0 \end{cases}. \tag{3.79}$$

Proof. If $a_0 < 0$, we sum up (3.78) at time t, $t+\tau_0$, $t+2\tau_0$, $\cdots$. If $a_0 > 0$, we sum up (3.78) at time t_0, $t_0+\tau_0$, $\cdots$, t. Triangle inequality on time then yields (3.79). $\square$

Lemma 3.20. *Consider a linearized equation of* (3.57)*: $\partial_t h + \mathcal{L}h = \varepsilon$ such that for any $t \geq 0$,*

$$\|\sqrt{\mathcal{L}_a}\varepsilon\|_{N(t,\infty)} + \|\varepsilon(t)\|_{\frac{2d}{d+2}} \lesssim e^{-c_1 t}, \tag{3.80}$$

$$\|h(t)\|_{\dot{H}^1_a} \lesssim e^{-c_0 t}, \tag{3.81}$$

where $0 < c_0 < c_1$. Then for any admissible pair (q,r) with $2 \leq r < 2^*$

- *if $e_0 \geq c_1$ or $e_0 < c_0$, for $\forall \eta > 0$,*

$$\|h(t)\|_{\dot{H}^1_a} + \|\sqrt{\mathcal{L}_a}h\|_{q,r(t,\infty)} \lesssim_\eta e^{-(c_1-\eta)t}; \tag{3.82}$$

- *if $c_0 \leq e_0 < c_1$, $\exists A_+ \in \mathbb{R}$ such that for $\forall \eta > 0$,*

$$\|h - A_+ e^{-e_0 t}\mathcal{Y}_+\|_{\dot{H}^1_a} + \|\sqrt{\mathcal{L}_a}(h - A_+ e^{-e_0 t}\mathcal{Y}_+)\|_{q,r(t,\infty)} \lesssim_\eta e^{-(c_1-\eta)t}. \tag{3.83}$$

Remark. Suppose h and ε satisfy the assumptions in Lemma 3.20, then similar arguments in Lemma 3.18 shows that for any admissible pair (q,r)

$$\|\sqrt{\mathcal{L}_a}h\|_{q,r(t,\infty)} \lesssim e^{-c_0 t}.$$

Proof. We first decompose $h(t)$ into

$$h(t) = \alpha_+(t)\mathcal{Y}_+ + \alpha_-(t)\mathcal{Y}_- + \beta(t)iW + \gamma(t)W_1 + g(t), \quad g(t) \in G_\perp, \tag{3.84}$$

where $\alpha_+ = \frac{B(h,\mathcal{Y}_-)}{B(\mathcal{Y}_+,\mathcal{Y}_-)}$, $\alpha_- = \frac{B(h,\mathcal{Y}_+)}{B(\mathcal{Y}_+,\mathcal{Y}_-)}$, $\beta = \frac{1}{\|W\|^2_{\dot H^1_a}}\langle h - \alpha_+\mathcal{Y}_+ - \alpha_-\mathcal{Y}_-, iW\rangle_{\dot H^1_a}$, and $\gamma = \frac{1}{\|W_1\|^2_{\dot H^1_a}}\langle h - \alpha_+\mathcal{Y}_+ - \alpha_-\mathcal{Y}_-, W_1\rangle_{\dot H^1_a}$. In addition, (3.84) and $\partial_t h + \mathcal{L}h = \varepsilon$ imply the following:

$$\frac{d}{dt}(e^{e_0 t}\alpha_+) = e^{e_0 t}\frac{B(\mathcal{Y}_-,\varepsilon)}{B(\mathcal{Y}_+,\mathcal{Y}_-)},\ \frac{d}{dt}(e^{-e_0 t}\alpha_-) = e^{-e_0 t}\frac{B(\mathcal{Y}_+,\varepsilon)}{B(\mathcal{Y}_+,\mathcal{Y}_-)}, \tag{3.85}$$

$$\frac{dQ(h)}{dt} = 2B(h,\varepsilon),\ \frac{d\beta}{dt} = \frac{\langle iW,\tilde\varepsilon\rangle_{\dot H^1_a}}{\|W\|^2_{\dot H^1_a}},\ \frac{d\gamma}{dt} = \frac{\langle W_1,\tilde\varepsilon\rangle_{\dot H^1_a}}{\|W_1\|^2_{\dot H^1_a}}, \tag{3.86}$$

where $\tilde\varepsilon = \varepsilon - \frac{B(\mathcal{Y}_-,\varepsilon)}{B(\mathcal{Y}_+,\mathcal{Y}_-)}\mathcal{Y}_+ - \frac{B(\mathcal{Y}_+,\varepsilon)}{B(\mathcal{Y}_+,\mathcal{Y}_-)}\mathcal{Y}_- - \mathcal{L}g$.

Decay of α_+, α_-:

$$|\alpha_-(t)| \lesssim e^{-c_1 t},\ |\alpha_+(t)| \lesssim \begin{cases} e^{-c_1 t} & \text{if } e_0 < c_0 \\ e^{-c_1 t} + e^{-e_0 t} & \text{if } e_0 \geq c_0 \end{cases}.$$

By definition, we can write

$$B(f,g) = \frac{1}{2}\int \mathcal{L}_a f_1 \cdot g_1 + \mathcal{L}_a f_2 \cdot g_2 - p_c W^{p_c-1} f_1 g_1 - W^{p_c-1} f_2 g_2.$$

Hölder inequality, Sobolev embedding, and equivalence of Sobolev norms yield that on some finite time interval I,

$$\begin{aligned}
&\int_I |B(f,g)| \qquad (3.87)\\
\lesssim\ & \|\sqrt{\mathcal{L}_a} f\|_{N'(I)} \|\sqrt{\mathcal{L}_a} g\|_{N(I)} + |I|^{1-\frac{\theta}{m}} \|W^{p_c-1}\|_X \|f\|_{\infty,2^*} \|g\|_{\frac{m}{\theta},b}\\
\lesssim\ & \|\sqrt{\mathcal{L}_a} f\|_{N'(I)} \|\sqrt{\mathcal{L}_a} g\|_{N(I)} + |I|^{1-\frac{\theta}{m}} \|\sqrt{\mathcal{L}_a} f\|_{\infty,2} \|\sqrt{\mathcal{L}_a} g\|_{N(I)}^{\theta} \|g\|_{\infty,\frac{2d}{d+2}}^{1-\theta},
\end{aligned}$$

where $\theta = \frac{8\sigma-8d+4d\sigma+2d^2+8}{4\sigma-12d+2d\sigma+3d^2+12}$, $m = \frac{4(d^2-4)}{4\sigma-12d+2d\sigma+5d^2+4}$, $X = \frac{d(d^2-4)}{4\sigma-4d+2d\sigma+d^2+4}$, and $b = \frac{2d(d^2-4)}{d^3-8\sigma+4d-4d\sigma-16}$. Note that

$$\|g\|_{\frac{m}{\theta},b} \lesssim \|g\|^{\theta}_{m,\frac{-2d(d^2-4)}{4\sigma-8d+2d\sigma+3d^2-d^3+12}} \|g\|^{1-\theta}_{\infty,\frac{2d}{d+2}} \lesssim \|\sqrt{\mathcal{L}_a} g\|^{\theta}_{N(I)} \|g\|^{1-\theta}_{\infty,\frac{2d}{d+2}}.$$

We remind that this is different from the estimate in [17], as W is not bounded in our case. Also, if $f = \mathcal{Y}_\pm$ or is a Schwartz function, we have

$$\int_I |B(f,\varepsilon)| \lesssim |I|^{1-\frac{1}{m}} \|\sqrt{\mathcal{L}_a}\varepsilon\|_{N(I)} + |I|^{1-\frac{\theta}{m}} \|\sqrt{\mathcal{L}_a}\varepsilon\|^{\theta}_{N(I)} \|\varepsilon\|^{1-\theta}_{\infty,\frac{2d}{d+2}}. \qquad (3.88)$$

Together with (3.80), this yields

$$\int_t^{t+1} |e^{-e_0 s} B(\mathcal{Y}_+, \varepsilon(s))| ds \lesssim e^{-(e_0+c_1)t}. \qquad (3.89)$$

Claim 3.19 then implies that $\int_t^\infty |e^{-e_0 s} B(\mathcal{Y}_+, \varepsilon(s))| ds \lesssim e^{-(e_0+c_1)t}$. By integrating the equation of $\alpha_-(t)$ in (3.85) from t to ∞, we get $|\alpha_-(t)| \lesssim e^{-c_1 t}$.

Next, we prove the upper bound for $|\alpha_+|$.

When $e_0 < c_0$, by (3.88), $|e^{e_0 t}\alpha_+| \lesssim e^{(e_0-c_0)t}$, thus $\lim_{t\to\infty} e^{e_0 t}\alpha_+ = 0$. Similar to (3.89), $\int_t^{t+1} |e^{e_0 t} B(\mathcal{Y}_-, \varepsilon(s))| ds \lesssim e^{(e_0-c_1)t}$. Claim 3.19 then yields

$$\int_t^{\infty} |e^{e_0 t} B(\mathcal{Y}_-, \varepsilon(s))| ds \lesssim e^{(e_0-c_1)t}.$$

By integrating the equation of $\alpha_+(t)$ in (3.85) from t to ∞, $|\alpha_+(t)| \lesssim e^{-c_1 t}$.

When $e_0 \geq c_0$, above tricks fail as the limit of $e^{e_0 t}\alpha_+$ as $t \to \infty$ is not clear. By integrating the equation of $\alpha_+(t)$ in (3.85) from 0 to t, we have $e^{e_0 t}\alpha_+(t) = \alpha_+(0) + \int_0^t e^{e_0 s}\frac{B(\mathcal{Y}_-,\varepsilon(s))}{B(\mathcal{Y}_+,\mathcal{Y}_-)}ds$. Thus, by (3.88) and (3.80), we obtain

$$\begin{aligned}
|\alpha_+(t)| &\leq |\alpha_+(0)|e^{-e_0 t} + e^{-e_0 t}\int_0^t e^{e_0 s}|B(\mathcal{Y}_-,\varepsilon(s))|ds \\
&\leq |\alpha_+(0)|e^{-e_0 t} + Ce^{-e_0 t}\int_0^t e^{(e_0-c_1)s}ds \\
&\lesssim e^{-e_0 t} + e^{-c_1 t}.
\end{aligned}$$

Decay of $|\beta(t)|, |\gamma(t)|$, **and** $\|g(t)\|_{\dot{H}^1_a}$:

$$|\beta(t)| + |\gamma(t)| + \|g(t)\|_{\dot{H}^1_a} \lesssim e^{-\frac{(c_0+c_1)}{2}t}. \tag{3.90}$$

By (3.67) and $g \in G_\perp$, $Q(h) = 2\alpha_+\alpha_- B(\mathcal{Y}_+,\mathcal{Y}_-) + Q(g)$. Lemma 3.15 then yields

$$\|g(t)\|^2_{\dot{H}^1_a} \lesssim Q(g) \lesssim |\alpha_+ \cdot \alpha_-| + |Q(h)| \lesssim e^{-(c_0+c_1)t}.$$

Indeed, by (3.87), $\int_t^{t+1}|2B(h,\varepsilon(s))|ds \lesssim e^{-(c_0+c_1)t}$.

From (3.2.2) and (3.81), $\lim_{t\to\infty}\beta(t) = 0$. Thus, from $\frac{d\beta}{dt} = \frac{\langle iW,\tilde{\varepsilon}\rangle_{\dot{H}^1_a}}{\|W\|^2_{\dot{H}^1_a}}$, (3.88), and (3.80), we obtain

$$\begin{aligned}
&\int_t^{t+1} |\langle iW,\tilde{\varepsilon}(s)\rangle_{\dot{H}^1_a}|ds \\
\lesssim\ & e^{-c_1 t} + \int_t^{t+1}|\langle iW,\mathcal{L}g(s)\rangle_{\dot{H}^1_a}|ds = e^{-c_1 t} + \int_t^{t+1}|\langle iW^{p_c},\mathcal{L}g(s)\rangle_2|ds \\
\lesssim\ & e^{-c_1 t} + \int_t^{t+1}(\|W^{2p_c-1}\|_{\frac{2d}{d+2}} + \|\mathcal{L}_a W^{p_c}\|_{\frac{2d}{d+2}})\|g(s)\|_{2^*}ds \\
\lesssim\ & e^{-c_1 t} + \int_t^{t+1}\|g(s)\|_{\dot{H}^1_a}ds \\
\lesssim\ & e^{-\frac{c_0+c_1}{2}t},
\end{aligned}$$

where $W^{2p_c-1}, \mathcal{L}_a W^{p_c} \in L^{\frac{2d}{d+2}}$ is due to the fact that $\frac{4}{d+2} < \beta < 1$. Thus, from Claim 3.19,

$$|\beta(t)| \lesssim \int_t^\infty |\langle iW, \tilde{\varepsilon}(s)\rangle_{\dot{H}^1_a}| ds \lesssim e^{-\frac{c_0+c_1}{2}t}.$$

The proof of $|\gamma(t)| \lesssim e^{-\frac{c_0+c_1}{2}t}$ is analogous.

Conclude the proof. From (3.84), (3.2.2), and (3.90), if $e_0 \geq c_1$ or $e_0 < c_0$, we have

$$\|h(t)\|_{\dot{H}^1_a} \lesssim e^{-c_1 t} + e^{-\frac{c_0+c_1}{2}t} \lesssim e^{-\frac{c_0+c_1}{2}t}.$$

In other words, we get better decay of $\|h(t)\|_{\dot{H}^1_a}$. By iterating above process, $\forall \eta > 0$, $\|h(t)\|_{\dot{H}^1_a} \lesssim_\eta e^{-(c_1-\eta)t}$. And by Lemma 3.18, $\|\sqrt{\mathcal{L}_a} h\|_{q,r(t,\infty)} \lesssim_\eta e^{-(c_1-\eta)t}$. Thus, (3.82) holds. It remains to prove (3.83).

Note that $|\alpha_+(t)| \lesssim e^{-c_1 t} + e^{-e_0 t}$ when $e_0 \geq c_0$, thus $\lim_{t\to\infty} \alpha_+(t) e^{e_0 t} = A_+$ for some $A_+ \in \mathbb{R}$. Integrating $\frac{d}{dt}(e^{e_0 t}\alpha_+) = e^{e_0 t}\frac{B(\mathcal{Y}_-,\varepsilon)}{B(\mathcal{Y}_+,\mathcal{Y}_-)}$ from t to ∞, from (3.88), we have

$$A_+ - e^{e_0 t}\alpha_+ = \int_t^\infty e^{e_0 s}\frac{B(\mathcal{Y}_-,\varepsilon)}{B(\mathcal{Y}_+,\mathcal{Y}_-)} ds = O(e^{(e_0-c_1)t}).$$

In view of the decompositon of h in (3.84), (3.2.2) and (3.90) yield that

$$\|h - A_+ e^{-e_0 t}\mathcal{Y}_+\|_{\dot{H}^1_a} \lesssim e^{-(\frac{c_0+c_1}{2})t}.$$

Note that $\partial_t(h - A_+ e^{-e_0 t}\mathcal{Y}_+) + \mathcal{L}(h - A_+ e^{-e_0 t}\mathcal{Y}_+) = \varepsilon$. Then (3.82) also holds if we replace h by $h - A_+ e^{-e_0 t}\mathcal{Y}_+$. We thus get (3.83) and complete the proof of this Lemma. $\square$

3.3 Proof of the dynamics of NLS$_a$: Theorem 1.3 and 1.4

We conclude the proofs of our main result of the dynamics of NLS$_a$ in this section. We first construct a sequence of approximate solutions to NLS$_a$. The contraction argument in Proposition 3.22 demonstrates the existence and uniqueness of a solution to NLS$_a$ which are exponentially close to W in the sense of (3.96). Recall that any solution u under assumptions in Proposition 3.5 or 3.12 is exponetially close to W in the sense that $\|u(t)_{[\theta,\mu]} - W\|_{\dot{H}^1_a} \leq Ce^{-ct}$, but no further information of the solution is given in Section 3.1. Here, we apply Lemma 3.23 and prove that above solutions are unique up to symmetries.

3.3.1 Contraction argument near an approximate solution

Lemma 3.21 ([17]). *Let $\alpha \in \mathbb{R}$. There exists functions $\{\Phi_j^\alpha\}$ in $\mathcal{S}(\mathbb{R}^d)$, such that $\Phi_1^\alpha = \alpha\mathcal{Y}_+$ and if*

$$W_k^\alpha = W + \sum_{j=1}^{k} e^{-je_0t}\Phi_j^\alpha, \tag{3.91}$$

then as $t \to \infty$,

$$(i\partial_t - \mathcal{L}_a)W_k^\alpha + |W_k^\alpha|^{p_c-1}W_k^\alpha = O(e^{-(k+1)e_0t}) \text{ in } \mathcal{S}(\mathbb{R}^d). \tag{3.92}$$

Proof. Recall that in [17], $R(z) = W^{p_c}J(W^{-1}z)$, where $J(z) = -i[|1+z|^{p_c-1}(1+z) - 1 - \frac{p_c+1}{2}z - \frac{p_c-1}{2}\bar{z}]$ is real analytic on $\{|z| < 1\}$ and $J(0) = \partial_z J(0) = \partial_{\bar{z}}J(0) = 0$. Thus, when $|z| < 1$, one can write

$$J(z) = \sum_{j_1+j_2\geq 2} a_{j_1j_2}z^{j_1}\bar{z}^{j_2} \tag{3.93}$$

with convergence of the series and all its derivatives.

Denote $v_k = W_k^\alpha - W = \sum_{j=1}^k e^{-je_0t}\Phi_j^\alpha$. Then (3.92) can be equivalently formulated as

$$\varepsilon_k = \partial_t v_k + \mathcal{L}v_k + R(v_k) = O(e^{-(k+1)e_0t}) \text{ in } \mathcal{S}(\mathbb{R}^d). \tag{3.94}$$

We prove this lemma by induction. Note that $\mathcal{L}W = 0$, $\mathcal{L}\mathcal{Y}_+ = e_0\mathcal{Y}_+$ and when $k = 1$, $\Phi_1^\alpha = \alpha\mathcal{Y}_+$, thus $\partial_t v_1 + \mathcal{L}v_1 = 0$. Then, $\varepsilon_1 = R(v_1)$ which satisfies that $|R(v_1)| \lesssim W^{p_c}|W^{-1}v_1|^2$ for large t by (3.93). As $v_1 = \alpha e^{-e_0t}\mathcal{Y}_+$, (3.94) holds for $k = 1$.

Next, suppose that there exists $\Phi_1^\alpha, \cdots, \Phi_k^\alpha$ such that (3.94) holds. From $v_k = \sum_{j=1}^k e^{-je_0t}\Phi_j^\alpha$ and (3.93), there exists $F_j \in \mathcal{S}(\mathbb{R}^d)$ $(1 \le j \le k+1)$ such that

$$\varepsilon_k = \sum_{j=1}^k e^{-je_0t}(-je_0\Phi_j^\alpha + \mathcal{L}\Phi_j^\alpha) + R(v_k) = \sum_{j=1}^{k+1} e^{-je_0t}F_j + O(e^{-(k+2)e_0t}).$$

By our induction hypothesis, $F_j = 0$ for all $1 \le j \le k$. Hence,

$$\varepsilon_k = e^{-(k+1)e_0t}F_{k+1} + O(e^{-(k+2)e_0t}) \text{ in } \mathcal{S}(\mathbb{R}^d). \tag{3.95}$$

Define $\Phi_{k+1}^\alpha = -(\mathcal{L} - (k+1)e_0)^{-1}F_{k+1}$ which is well defined as $(k+1)e_0 \notin \sigma(\mathcal{L})$. In addition, $\Phi_{k+1}^\alpha \in \mathcal{S}(\mathbb{R}^d)$ by (3.65). Write $v_{k+1} = v_k + e^{-(k+1)e_0t}\Phi_{k+1}^\alpha$, we obtain

$$\begin{aligned}
\varepsilon_{k+1} &= \partial_t(v_k + e^{-(k+1)e_0t}\Phi_{k+1}^\alpha) + \mathcal{L}(v_k + e^{-(k+1)e_0t}\Phi_{k+1}^\alpha) + R(v_{k+1}) \\
&= \varepsilon_k + [\mathcal{L} - (k+1)e_0](e^{-(k+1)e_0t}\Phi_{k+1}^\alpha) - R(v_k) + R(v_{k+1}) \\
&= \varepsilon_k - e^{-(k+1)e_0t}F_{k+1} - R(v_k) + R(v_{k+1}).
\end{aligned}$$

From (3.93), $R(v_{k+1}) - R(v_k) = O(e^{-(k+2)e_0t})$ in $\mathcal{S}(\mathbb{R}^d)$. (3.95) then yields that $\varepsilon_{k+1} = O(e^{-(k+2)e_0t})$ in $\mathcal{S}(\mathbb{R}^d)$ which completes the induction argument. □

Proposition 3.22. *Let $\alpha \in \mathbb{R}$. There exists $k_0 > 0$ such that for any $k \geq k_0$, there exists $t_k \geq 0$ and a solution W^α of NLS_a such that for $t \geq t_k$,*

$$\|\sqrt{\mathcal{L}_a}(W^\alpha - W_k^\alpha)\|_{Z(t,\infty)} \leq e^{-(k+\frac{1}{2})e_0 t}. \tag{3.96}$$

In addition, W^α is the unique solution to NLS_a satisfying (3.96) *for large t, and W^α is independent of k such that*

$$\|W^\alpha - W - \alpha e^{-e_0 t}\mathcal{Y}_+\|_{\dot{H}^1_a} \leq e^{-\frac{3}{2}e_0 t}. \tag{3.97}$$

Proof. Let $h = W^\alpha - W_k^\alpha$, $\omega^\alpha = W^\alpha - W$, and $v_k = W_k^\alpha - W$. Then $h = \omega^\alpha - v_k$ and $\partial_t \omega^\alpha + \mathcal{L}\omega^\alpha + R(\omega^\alpha) = 0$. In view of (3.94), this implies that

$$i\partial_t h - \mathcal{L}_a h = -i\varepsilon_k - \mathcal{V}(h) - iR(v_k + h) + iR(v_k).$$

From the Duhamel formula, it suffices to show that

$$\mathcal{M}_k(h) = -\int_t^\infty e^{-i(t-s)\mathcal{L}_a}[\varepsilon_k + i\mathcal{V}(h) + R(v_k+h) - R(v_k)]ds$$

is a contraction from $B^k = \left\{h \in E^k, \|h\|_{E^k} \leq 1\right\}$ to B^k in the Banach space

$$E^k = \left\{h \in S(t_k,\infty), \sqrt{\mathcal{L}_a}h \in Z(t_k,\infty);\ \|h\|_{E^k} < \infty\right\},$$

where $\|h\|_{E^k} = \sup_{t\geq t_k} e^{(k+\frac{1}{2})e_0 t}\|\sqrt{\mathcal{L}_a}h\|_{Z(t,\infty)}$. We will need the following estimates which can be derived by equivalence of Sobolev norms and Claim 6.4 in [17]: There exists $k_0 > 0$ such that for $k \geq k_0$ the following hold: for all $f \in E^k$

$$\|\sqrt{\mathcal{L}_a}(\mathcal{V}(f))\|_{N(t,\infty)} \leq \frac{1}{4C^*}e^{-(k+\frac{1}{2})e_0 t};$$

there exists a constant C_k depending only on k such that for all f, $g \in B^k$ and $t \geq t_k$

$$\begin{cases} \|\sqrt{\mathcal{L}_a}(R(v_k+f)-R(v_k+g))\|_{N(t,\infty)} \leq C_k e^{-(k+\frac{3}{2})e_0 t}\|f-g\|_{E^k} \\ \|\sqrt{\mathcal{L}_a}\varepsilon_k\|_{N(t,\infty)} \leq C_k e^{-(k+1)e_0 t} \end{cases}.$$

Indeed, Strichartz estimate yields that for some constant $C^* > 0$,

$$\begin{aligned} & \|\mathcal{M}_k(h)\|_{E^k} \\ \leq\ & C^* \sup_{t\geq t_k} e^{(k+\frac{1}{2})e_0 t}\|\sqrt{\mathcal{L}_a}[\varepsilon_k + i\mathcal{V}(h) + R(v_k+h) - R(v_k)]\|_{N(t,\infty)} \\ \leq\ & \frac{1}{4} + C^* C_k e^{-e_0 t_k} + C^* C_k e^{-\frac{1}{2}e_0 t_k}, \end{aligned}$$

$$\begin{aligned} & \|\mathcal{M}_k(f) - \mathcal{M}_k(g)\|_{E^k} \\ \leq\ & C^* \sup_{t\geq t_k} e^{(k+\frac{1}{2})e_0 t}\|\sqrt{\mathcal{L}_a}[i\mathcal{V}(f-g) + R(v_k+f) - R(v_k+g)]\|_{N(t,\infty)} \\ \leq\ & \|f-g\|_{E^k}(\frac{1}{2} + C^* C_k e^{-e_0 t_k}). \end{aligned}$$

By taking t_k sufficiently large, $\mathcal{M}_k$ is a contraction from B^k to B^k.

Next, we show that W^α is independent of k. Assume that W^α solves NLS such that (3.96) holds for $t \geq t_k$. Clearly, the uniqueness of W^α still holds for any $t \geq t_{\tilde{k}}$ with $t_{\tilde{k}} > t_k$($\mathcal{M}_k$ again is a contraction from B^k to B^k). Let $k < \tilde{k}$ and W^α and $\tilde{W}^\alpha$ be the corresponding solutions constructed above for k and $\tilde{k}$. Thus, on $t \geq t_{\tilde{k}}$, both W^α and $\tilde{W}^\alpha$ satisfy (3.96). From the uniqueness of the fixed point argument, $W^\alpha(t) = \tilde{W}^\alpha(t)$ for $t \geq t_{\tilde{k}}$. The uniqueness of NLS$_a$ then yields that $W^\alpha = \tilde{W}^\alpha$. Hence, W^α is independent of k.

Finally, let $h = W^\alpha - W^\alpha_k$ as defined above with $k \geq k_0 > 0$. From Strichartz estimate,

$$\begin{aligned}\|W^\alpha - W_k^\alpha\|_{\dot{H}_a^1} &\le C^*\|\sqrt{\mathcal{L}_a}[\varepsilon_k + i\mathcal{V}(h) + R(v_k+h) - R(v_k)]\|_{N(t,\infty)} \\ &\lesssim e^{-(k+\frac{1}{2})e_0 t}.\end{aligned}$$

Together with $W_k^\alpha - (W + \alpha e^{-e_0 t}\mathcal{Y}_+) = O(e^{-2e_0 t})$, this yields (3.97). The proof of this proposition is complete. □

3.3.2 Proof of the main results

Proof of Theorem 1.3. Let $\mathcal{Y}_1 = \mathbf{Re}\mathcal{Y}_+ = \mathbf{Re}\mathcal{Y}_-$. We first point out that $\langle W, \mathcal{Y}_1\rangle_{\dot{H}_a^1} \neq 0$. From (3.66) and $\mathcal{L}W = i(1-p_c)\mathcal{L}_a W$, we get

$$B(W,\mathcal{Y}_+) = B(W,\mathcal{Y}_-) = \frac{1-p_c}{2}\langle W, \mathcal{Y}_1\rangle_{\dot{H}_a^1}.$$

If $\langle W, \mathcal{Y}_1\rangle_{\dot{H}_a^1} = 0$, then $W \in G_\perp$. By Lemma 3.15, $Q(W) \gtrsim \|W\|_{\dot{H}_a^1}^2$ which contradicts $Q(W) < 0$. Replacing $\mathcal{Y}_\pm$ by $-\mathcal{Y}_\pm$ when necessary, we assume that

$$\langle W, \mathcal{Y}_1\rangle_{\dot{H}_a^1} > 0.$$

By (3.97) and the conservation of energy, $\lim_{t\to\infty} W^\alpha = W$ in $\dot{H}_a^1$ and $E(W^\alpha) = E(W)$. In addition,

$$\|W^\alpha\|_{\dot{H}_a^1}^2 = \|W\|_{\dot{H}_a^1}^2 + 2ae^{-e_0 t}\langle W, \mathcal{Y}_1\rangle_{\dot{H}_a^1} + O(e^{-\frac{3}{2}e_0 t}).$$

Denote $W^\pm = W^{\pm 1}$, and recall that $\langle W, \mathcal{Y}_1\rangle_{\dot{H}_a^1} > 0$, for large $t > 0$, we have

$$\|W^+(t)\|_{\dot{H}_a^1} > \|W\|_{\dot{H}_a^1},\ \|W^-(t)\|_{\dot{H}_a^1} < \|W\|_{\dot{H}_a^1}. \tag{3.98}$$

From Remark 1.2, (3.98) holds for all t on the time existence of $W^\pm$. Proposition 3.7 and Corollary 3.6 yield that W^- exists globally in time and $\|W^-(t)\|_{S(-\infty,0)} < \infty$.

Note that $\frac{4}{d+2} < \beta < 1$ yields $W \in L^2(\mathbb{R}^6)$ and $W \notin L^2(\mathbb{R}^d)$ when $d \leq 5$. This is different from the case $a = 0$ for NLS: the ground solition is in $L^2(\mathbb{R}^d)$ when $d \geq 5$. It remains to show $W^\alpha \in L^2(\mathbb{R}^6)$.

Define $F_R(t) = \int_{\mathbb{R}^6} |W^\alpha|^2 \psi(\frac{x}{R}) dx$, where ψ is a smooth radial cut-off function with $\psi(x) = 1$ when $|x| \leq 1$ and $\psi(x) = 0$ when $|x| \geq 2$. As W^α solves NLS$_a$, we get

$$\begin{aligned}
F_R'(t) &= \frac{2}{R}\mathbf{Im}\int \overline{W}^\alpha \nabla W^\alpha \cdot (\nabla\psi)(\frac{x}{R})dx \\
&= \frac{2}{R}\mathbf{Im}\int [W\nabla(W^\alpha - W)\cdot(\nabla\psi)(\frac{x}{R}) + (\overline{W}^\alpha - W)\nabla W \cdot (\nabla\psi)(\frac{x}{R}) \\
&\quad + (\overline{W}^\alpha - W)\nabla(W^\alpha - W)\cdot(\nabla\psi)(\frac{x}{R})]dx,
\end{aligned}$$

By (3.97), Hardy inequality, and equivalence of Sobolev norms, we obtain

$$\begin{aligned}
|F_R'(t)| &\lesssim \|W^\alpha - W\|_{\dot{H}^1}(\|W^\alpha\|_{\dot{H}^1} + \|W\|_{\dot{H}^1}) \\
&\lesssim \|W^\alpha - W\|_{\dot{H}^1_a} \lesssim e^{-e_0 t}.
\end{aligned}$$

Also, $\lim_{t\to\infty} F_R(t) = \int_{\mathbb{R}^6} |W|^2\psi(\frac{x}{R})dx$. Indeed,

$$\begin{aligned}
& |\int_{\mathbb{R}^6} (|W^\alpha(t)|^2 - |W|^2)\psi(\frac{x}{R})dx| \\
\leq & \int_{\mathbb{R}^6} |W^\alpha - W|(|W^\alpha| + |W|)\psi(\frac{x}{R})dx \\
\lesssim & \|W^\alpha - W\|_3 (\|W^\alpha\|_3 + \|W\|_3)R^2 \\
\lesssim & R^2\|W^\alpha - W\|_{\dot{H}^1_a}(\|W^\alpha\|_{\dot{H}^1_a} + \|W\|_{\dot{H}^1_a}) \to 0, \text{ as } t \to \infty.
\end{aligned}$$

Thus, for t large enough,

$$|F_R(t) - \int_{\mathbb{R}^6} |W|^2\psi(\frac{x}{R})dx| \lesssim e^{-e_0 t}.$$

Taking $R \to \infty$, we see that $W^\alpha \in L^2(\mathbb{R}^6)$ and $\|W^\alpha\|_2 = \|W\|_2$. Hence, $W^+ \in L^2(\mathbb{R}^6)$ and Corollary 3.13 shows that W^+ blows up backward at finite time. $\square$

Lemma 3.23. *If u is a solution of NLS_a such that $\exists \gamma_0 > 0$ and for some positive constant C,*

$$\|u(t) - W\|_{\dot{H}^1_a} \leq Ce^{-\gamma_0 t},\ E(u) = E(W). \tag{3.99}$$

then $\exists! \alpha \in \mathbb{R}$, $u = W^\alpha$. For any $\alpha \neq 0$, there exists $T_\alpha \in \mathbb{R}$ such that $W^\alpha(t) = W^+(t - T_\alpha)$ when $\alpha > 0$ and $W^\alpha(t) = W^-(t - T_\alpha)$ when $\alpha < 0$.

Proof. Let $u = W + v$. Then $\partial_t v + \mathcal{L}(v) + R(v) = 0$ by (3.57). The conclusion $\exists! \alpha \in \mathbb{R}$, $u = W^\alpha$ follows from the following two steps.

Step 1. $\exists!\ \alpha \in \mathbb{R}$ such that $\forall \eta > 0$,

$$\|v(T) - \alpha e^{-e_0 T}\mathcal{Y}_+\|_{\dot{H}^1_a} + \|\sqrt{\mathcal{L}_a}[v(t) - \alpha e^{-e_0 t}\mathcal{Y}_+]\|_{Z(T,\infty)} \leq C_\eta e^{-(2-\eta)e_0 T}. \tag{3.100}$$

Step 2. $\forall m > 0$, $\exists t_0 > 0$, for all $t \geq t_0$,

$$\|u(t) - W^\alpha(t)\|_{\dot{H}^1_a} + \|\sqrt{\mathcal{L}_a}(u - W^\alpha)\|_{Z(t,\infty)} \leq Ce^{-mt}. \tag{3.101}$$

Indeed, pick $m = (k_0 + 1)e_0$, where k_0 is defined in Proposition 3.22, by (3.96) and (3.101), we get

$$\|\sqrt{\mathcal{L}_a}(u - W^\alpha_{k_0})\|_{Z(t,\infty)} \leq Ce^{-(k_0 + \frac{1}{2})e_0 t}.$$

Uniqueness of Proposition 3.22 implies that $u = W^\alpha$.

Proof of step1. In view of Lemma 3.20($h = v$, $\varepsilon = -R(v)$, $c_0 = e_0$ and $c_1 = 2e_0$), (3.100) indeed follows from

$$\|v(t)\|_{\dot{H}^1_a} \leq Ce^{-e_0 t},\ \|R(v(t))\|_{L^{\frac{2d}{d+2}}} + \|\sqrt{\mathcal{L}_a}R(v)\|_{N(t,\infty)} \leq Ce^{-2e_0 t}.$$

While the bound on $R(v)$ follows from the bound on $v(t)$ by Lemma 3.17 and Claim 3.19. It suffices to prove

$$\|v(t)\|_{\dot{H}^1_a} \leq Ce^{-e_0 t}. \tag{3.102}$$

By Lemma 3.18, $\|v(t)\|_{\dot{H}^1_a} + \|\sqrt{\mathcal{L}_a} v\|_{Z(t,\infty)} \leq Ce^{-\gamma_0 t}$. Then from Lemma 3.17 and Claim 3.19, we get

$$\|R(v(t))\|_{L^{\frac{2d}{d+2}}} + \|\sqrt{\mathcal{L}_a} R(v(t))\|_{N(t,\infty)} \leq Ce^{-2\gamma_0 t}.$$

From Lemma 3.20, we obtain(take $\eta = \frac{1}{2}$)

$$\|v(t)\|_{\dot{H}^1_a} \leq C(e^{-e_0 t} + e^{-\frac{3}{2}\gamma_0 t}).$$

Thus, if $\frac{3}{2}\gamma_0 \geq e_0$, (3.102) holds; when $\frac{3}{2}\gamma_0 < e_0$, we indeed get another good upper bound for $\|v(t)\|_{\dot{H}^1_a}$, i.e., $\|v(t)\|_{\dot{H}^1_a} \leq Ce^{-\frac{3}{2}\gamma_0 t}$. By iterating above process, we get (3.100).

Proof of step 2. By picking $\eta = \frac{1}{2}$ in (3.100) and (3.91), (3.101) holds for $m = \frac{3}{2}e_0$. Next, we show that if (3.101) holds for m, then it also holds $m + \frac{e_0}{2}$. By iterating this process finite many times, we get (3.101). Denote $\omega^\alpha = W^\alpha - W$ and $h = u - W^\alpha = v - \omega^\alpha$. By (3.57),

$$\partial_t h + \mathcal{L} h = R(\omega^\alpha) - R(v).$$

From the induction assumption, $\|v(t) - \omega^\alpha(t)\|_{\dot{H}^1_a} + \|\sqrt{\mathcal{L}_a}(v - \omega^\alpha)\|_{Z(t,\infty)} \leq e^{-mt}$. By Lemma 3.17 and Claim 3.19, we get

$$\|R(\omega^\alpha(t)) - R(v(t))\|_{L^{\frac{2d}{d+2}}} + \|\sqrt{\mathcal{L}_a}[R(\omega^\alpha) - R(v)]\|_{N(t,\infty)} \leq Ce^{-(m+e_0)t}.$$

In view of Lemma 3.20, we get

$$\|v(t)-\omega^\alpha(t)\|_{\dot H^1_a}+\|\sqrt{\mathcal{L}_a}(v-\omega^\alpha)\|_{Z(t,\infty)}\le Ce^{-(m+\frac{e_0}{2})t},$$

which shows that (3.101) holds for $m+\frac{e_0}{2}$. The proof of step 2 is complete.

Step 3. Classify the solution W^α when $\alpha\neq 0$.

Let T_α be such that $e^{-e_0T_\alpha}|\alpha|=1$. From Proposition 3.22,

$$\|W^\alpha(t+T_\alpha)-W\mp e^{-e_0t}\mathcal{Y}_+\|_{\dot H^1_a}\lesssim e^{-\frac{3}{2}e_0t}. \tag{3.103}$$

As $W^\alpha(t+T_\alpha)$ satisfies (3.99), $\exists!\tilde\alpha$ such that $W^\alpha(t+T_\alpha)=W^{\tilde\alpha}(t)$. (3.103) then shows that $\tilde\alpha=1$ if $\alpha>0$ and $\tilde\alpha=-1$ when $\alpha<0$. A time substitution yields the conclusion. □

Proof of Theorem 1.4. Assume $E(u_0)=E(W)$ and u_0 is radial in $\dot H^1_a$.

(a) $\|u_0\|_{\dot H^1_a}<\|W\|_{\dot H^1_a}$: Proposition 3.7 shows that u exists globally in time. From Proposition 3.5, if $\|u\|_{S(0,\infty)}=\infty$, then there exist $\theta\in\mathbb{R}$, $\mu>0$, and $c,C>0$, such that $\forall t\ge 0$, $\|u(t)_{[\theta,\mu]}-W\|_{\dot H^1_a}\le Ce^{-ct}$. In addition, $u(t)_{[\theta,\mu]}$ clearly satisfies the assumption (3.99), from $\|u(t)_{[\theta,\mu]}\|_{\dot H^1_a}=\|u(t)\|_{\dot H^1_a}<\|W\|_{\dot H^1_a}$, Lemma 3.23 implies that $u(t)_{[\theta,\mu]}=W^-(t-T_\alpha)$, or equivalently, $u(t)=W^-(t-T_\alpha)_{[-\theta,\mu^{-1}]}$.

(b) $\|u_0\|_{\dot H^1_a}=\|W\|_{\dot H^1_a}$: The conclusion follows from the Sharp Gagliardo-Nirenberg inequality.

(c) $\|u_0\|_{\dot H^1_a}>\|W\|_{\dot H^1_a}$: Suppose $u_0\in L^2$ and u is defined on $[0,\infty)$. From Proposition 3.12, there exist $\theta\in\mathbb{R}$, $\mu>0$, and $c,C>0$, such that $\forall t\ge 0$, $\|u(t)_{[\theta,\mu]}-W\|_{\dot H^1_a}\le Ce^{-ct}$. In addition, $u(t)_{[\theta,\mu]}$ clearly satisfies the assumption (3.99), from

$\|u(t)_{[\theta,\mu]}\|_{\dot{H}^1_a} = \|u(t)\|_{\dot{H}^1_a} > \|W\|_{\dot{H}^1_a}$, Lemma 3.23 implies that $u(t)_{[\theta,\mu]} = W^+(t-T_\alpha)$, or equivalently, $u(t) = W^+(t-T_\alpha)_{[-\theta,\mu^{-1}]}$.

The proof of Theorem 1.4 is complete. $\square$

CHAPTER 4
FUTURE DISCUSSIONS

4.1 Dynamics of energy critical NLW with inverse square potential

For focusing energy critical nonlinear wave equation(NLW), the ground state solution is still W_0. The energy of this solution has been proved as the threshold for the dynamical behavior of NLW solutions: blow-up, global well-posedness, scattering, ground state solution convergence. In dimension $d = 3, 4, 5$ for NLW, the result is proved by Duyckaerts-Merle [18]; when $d \geq 6$, it is obtained by Li-Zhang [38]. Similarly, when $a < 0$, W_a is the ground state solution of NLW with inverse square potential; W_a is also the threshold between blowup and scattering of solutions to NLW. Several preliminary estimates derived in my thesis can be directly applied. I will work with my advisor on this project soon.

4.2 Symplectic non-squeezing of Hartree equations

4.2.1 Mass subcritical Hartree equations

This subsection reviews my previous work [52] on symplectic non-squeezing of mass subcritical Hartree equations.

Symplectic non-squeezing of finite dimensional Hamiltonian PDEs is a classical result due to Gromov [21]. He observed that if the complex dimension $n \geq 2$, then the symplectomorphism can not map a ball into a thinner cylinder. The counterpart in infinite dimensions was not available long time after Gromov's finite dimensional result. In 1994, Bourgain [5] proved the non-squeezing for 1d cubic NLS on torus which provides one infinite dimensional PDE example of the symplectic non-squeezing. Afterwards, there are couple of results verifying the non-squeezing for various PDEs on torus.

Over the last twenty years, there are quite a few attempts solving this problem by establishing the connection between finite dimensional PDE and infinite dimensional PDE, or between PDEs on torus and PDEs on the whole space case. The first successful try was given by Killip-Visan-Zhang [35, 36], who are able to prove the result for cubic NLS in 1d and 2d. The significance of their results is not only they are the first to show the symplectic non-squeezing for problems with infinite dimension and infinite volume, but also they provide effective approach to approximate the infinite dimensional problem on $\mathbb{R}^d$ by the finite dimensional problem on large torus. Stimulated by their work, I prove the non-squeezing properties of all mass subcritical Hartree equations on $\mathbb{R}^d (d \geq 2)$:

$$\begin{cases} (i\partial_t + \Delta)u = F(u) \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^d) \end{cases}, \tag{4.1}$$

where $F(u) = \pm(|x|^{-\lambda} * |u|^2)u$ and $0 < \lambda < 2$.

Theorem 4.1 (Symplectic Non-squeezing, [52])**.** *Fix z_*, $l \in L^2(\mathbb{R}^d)$ with $\|l\|_2 = 1$, $\alpha \in \mathbb{C}$, $T > 0$ and $0 < r < R < \infty$. Then there exists $u_{\infty,0} \in B(z_*, R)$ such that the solution u_∞ to (4.1) with initial data $u(0) = u_{\infty,0}$ satisfies*

$$|\langle l, u_\infty(T)\rangle - \alpha| > r.$$

The result is achieved by elaborating and generalizing some of the techniques in [35], and two major ingredients are weak well-posedness and approximation to infinite dimensional problems by finite dimensional problems.

4.2.2 Mass critical Hartree equations

To show the non-squeezing properties of mass critical Hartree equations, one needs to obtain the weak well-posedness and approximation results similar to previous work. The approximation argument in [35] requires several kernel and communicator estimates on $\mathbb{R}^2$, which has been extended to higher dimensions in my paper [52]. Obtaining the corresponding weak well-posedness result is challenging. Although non-squeezing result is local in time, showing that the non-squeezing holds for any fixed time requires global well-posedness of the equation. Due to the criticality, the well-posedness of NLS with Littlewood-Paley projections is not available. Thus, a modified version of frequency projection was introduced in [35]. Similar to Bourgain's induction on energy argument, solutions to these modified NLS are global well-posed. Similar problem of the mass critical Hartree equation is the major obstacle I want to overcome later.

4.3 Obstacle problem of nonlinear Schrödinger equation

4.3.1 Energy subcritical obstacle problem

This subsection reviews the main result in [51].

Let Ω be the exterior domain of a smooth compact strictly convex obstacle in $\mathbb{R}^3$(a classical example is the region outside a sphere), we consider the initial value problem NLS_Ω with Dirichlet boundary condition:

$$\begin{cases} iu_t + \Delta u = -\left|u\right|^p u \\ \quad u|_{x\in\partial\Omega} = 0 \\ u|_{t=0} = u_0 \in H_0^1(\Omega) \end{cases}, \tag{NLS_Ω}$$

where $p = \frac{4}{3-2s}$ and $0 < s < 1$.

For NLS, the critical space is $\dot{H}^s$ since the natural scaling for the equation $u_\lambda(t,x) = \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x)$ solves NLS and keeps the $\dot{H}^s$ norm of the initial data $\|u(0,x)\|_{\dot{H}^s(\mathbb{R}^3)} = \|u_\lambda(0,x)\|_{\dot{H}^s(\mathbb{R}^3)}$. For NLS_Ω, $\dot{H}^s$ is still considered as the critical space, since adding an obstacle does not change the dimensionality of the problem. In this paper, we will discuss the global well-posedness and scattering of the solution.

On the time of existence, the solution preserves mass and energy

$$\begin{aligned} M(u(t)) &= \int_{\mathbb{R}^3} |u(t,x)|^2 dx = M(u_0), \\ E(u(t)) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t,x)|^2 dx - \frac{1}{p+2}\int_{\mathbb{R}^3} |u(t,x)|^{p+2} dx = E(u_0). \end{aligned}$$

Here, $u(t,x)$ can be considered as a function in $H^1(\mathbb{R}^3)$ by applying the 0-extension outside Ω.

From the local theory, we know that if the initial data is sufficiently small in H_0^1, $u(t)$ is global and scatters in both time directions, see also [4, 23, 34] and the reference therein. Scattering for large solutions was first obtained by Killip-Visan-Zhang [34] for the focusing cubic NLS_Ω. Similar result in the whole space case was first achieved by the earlier work of Holmer-Roudenko [22] for cubic radial NLS, Duyckaerts-Holmer-Roudenko [16] for cubic non-radial NLS, and Fang-Xie-Cazenave

[19] for general focusing energy subcritical NLS. In [51], I extend the result in [19] to NLS_Ω by elaborating the techniques in [32, 34].

Indeed, both in the whole space case and obstacle case, the scattering result of large solutions is highly related to the ground state solution (see [22, 51] for more discussions on this topic). The ground state solution $Q(x)$ is the positive, radial, exponentially decaying solution with minimal mass for the nonlinear elliptic equation $-Q + \triangle Q + |Q|^p Q = 0$ in $\mathbb{R}^3$.

Theorem 4.2 (Scattering, [51])**.** *Let $u_0 \in H_0^1(\Omega)$ satisfy*

$$\begin{cases} \|u_0\|_{L_x^2(\Omega)}^{1-s} \|\nabla u_0\|_{L_x^2(\Omega)}^{s} < \|Q\|_{L_x^2(\mathbb{R}^3)}^{1-s} \|\nabla Q\|_{L_x^2(\mathbb{R}^3)}^{s} \\ M(u_0)^{1-s} E(u_0)^s < M(Q)^{1-s} E(Q)^s \end{cases},$$

where $Q(x)$ is the ground state solution. Then, there exists a unique global solution $u(t,x)$ such that the scattering norm

$$S_{\mathbb{R}}(u) = \|u(t,x)\|_{L_{t,x}^{c^*}(\mathbb{R}\times\Omega)} < \infty.$$

In particular, the bounded space-time norm implies the scattering: there exists $\phi^{\pm} \in H_0^1(\Omega)$ such that

$$\lim_{t\to\pm\infty} \left\| u(t) - e^{it\Delta_\Omega}\phi^{\pm} \right\|_{H_0^1(\Omega)} = 0.$$

As observed in [34], the difficulty of inducting on the scale invariant quantity $M^{1-s}(u)E^s(u)$ is that no nontrivial weak limit can be extracted even if the product $M^{1-s}(u)E^s(u)$ is bounded. For example, the mass of the sequence u_n diverge and the energy converge to 0 while the product are a fixed constant. In $\mathbb{R}^3$, one can simply fix this issue by scaling. In the obstacle case, applying scaling transformation change the geometry of the problem, hence significantly increase the complexity of the problem. Instead, we apply a simple solution proposed in [34] by doing the induction on the algebraic combination of $E(u)$ and $M(u)$.

The region $M(u_0)^{1-s}E(u_0)^s < M(Q)^{1-s}E(Q)^s$ on the $E - M$ plane can be seen as the union of all the triangles below the curve $M^{1-s}E^s = M(Q)^{1-s}E(Q)^s$.

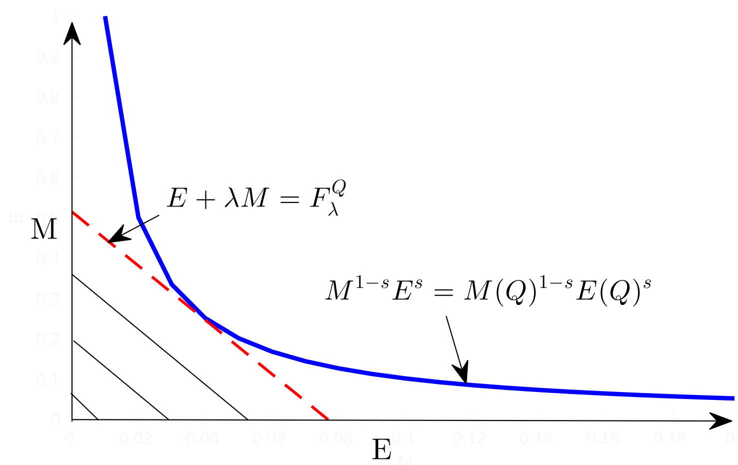

Figure 4.1: Threshold of scattering for obstacle NLS

Based on this geometry observation, the condition

$$M(u_0)^{1-s}E(u_0)^s < M(Q)^{1-s}E(Q)^s$$

is equivalent to that, for any $0 < \lambda < \infty$,

$$F^\lambda(u_0) = E(u_0) + \lambda M(u_0) < F^\lambda(Q),$$

where $E + \lambda M = F^\lambda(Q)$ is the tangent line at the intersection point of the curve

$M^{1-s}E^s = M(Q)^{1-s}E(Q)^s$. From this reformulation, we transfer the condition into linear combination of mass and energy which simplifies the induction argument.

4.3.2 $\dot{H}^{\frac{1}{2}}$ critical obstacle problem

In [26], Kenig and Merle showed that if a solution of the defocusing non-radial cubic NLS in $\mathbb{R}^3$ remains bounded in $\dot{H}^{\frac{1}{2}}$ in its maximal interval of existence, then the interval is infinite(solution is global in time) and the solution scatters. We expect that the corresponding result holds on exterior domains for NLS_Ω.

4.3.3 Energy critical obstacle problem

The global well-posedness and scattering of the energy critical NLS on exterior domains in dimension three was proved in [33]. The corresponding result in higher dimensions is still open. Similar to the whole space case, one crucial tool is the stability result. When the dimension is greater than 6, as lack of the exotic Strichartz and dispersive estimates, the stability result is open. However, it is promising to extend the result in [33] to $\mathbb{R}^d (4 \leq d \leq 6)$. Indeed, the main difficulty is generalizing the convergence of linear Schrödinger flows from $d = 3$ to higher dimensions. This extension requires fine analysis which involves spectral theory, differential geometry, Harmonic analysis, matrix theory. This is one of my future research topic.

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