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Structural results for von Neumann algebras of poly-hyperbolic groups

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STRUCTURAL RESULTS FOR VON NEUMANN ALGEBRAS OF POLY-HYPERBOLIC GROUPS

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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ABSTRACT

This work is a compilation of structural results for the von Neumann algebras of poly-hyperbolic groups established in a series of works done jointly with I. Chifan and T. Sinclair; and S. Pant. These works provide a wide range of circumstances where the product structure, a discrete structural property, can be recovered from the von Neumann algebra (a continuous object).

The primary result of Chifan, Sinclair and myself is as follows: if $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is a product of non-elementary hyperbolic icc groups and Λ is a group such that $L(\Gamma) \cong L(\Lambda)$, then Λ decomposes as an n-fold product of infinite groups. This provides a group-level strengthening of the unique prime decomposition of Ozawa and Popa by eliminating any assumption assumption on the target group Λ. The methods necessary to establish this result provide a malleable procedure which allows one to rebuild the product of a group from the algebra itself.

Modifying the techniques found in the previous work, Pant and I are able to demonstrate that the class of poly-groups exhibit a similar phenomenon. Specifically, if Γ is a poly-hyperbolic group whose corresponding algebra is non-prime, then the group must necessarily decompose as a product of infinite groups.

PUBLIC ABSTRACT

The von Neumann algebra was originally introduced by John von Neumann, motivated by is formulations of quantum mechanics and representation theorem. His works provide constructions which transform well-understood mathematical objects (in our case, we take the mathematical structures groups) into von Neumann algebras. Since their inception, there have been various attempts to characterize features of these objects by exploiting the structure of the generating objects.

If equivalent von Neumann algebras arise from different groups, the first called the source and the latter the target, we aim to understand which properties are shared by both the source and target. The groups we consider are products; i.e. those composed from smaller groups comparable to how one builds numbers from primes. By enriching the data, we demonstrate if the source and target yield equivalent von Neumann algebras and the source is a product of groups, we provide a procedure to decompose the target into a product. This vastly generalizes previous investigations into this type of rigidity by S. Popa and N. Ozawa by completely removing their balanced assumption on the target data. This demonstrates an exciting new phenomena: the complete recovery of the product structure.

The analogy of decomposing numbers into primes carries over to von Neumann algebras. As part of a subsequent investigation, we assume the von Neumann algebra is decomposable into smaller components and successfully adapt our methods to demonstrate the initial data must a product of groups.

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CHAPTER 1
INTRODUCTION AND PRELIMINARIES

The formulation of a von Neumann algebra appeared in the late 30's with the seminal works of Murray and von Neumann. A von Neumann algebra unital, self-adjoint subalgebra of $B(\mathcal{H})$, the collection bounded operators of a bounded operators on a complex Hilbert space $\mathcal{H}$, closed under the strong operator topology. Clearly, $B(\mathcal{H})$ is a von Neumann algebra regardless of the underlying space $\mathcal{H}$. When $\mathcal{H}$ is a an n-dimensional space, $B(\mathcal{H})$ can be identified with the $n \times n$ dimensional matrices $M_n(\mathbb{C})$. An alternate example is the algebra of essentially bounded functions $L^\infty(X, \mu)$ acting on $L^2(X, \mu)$ as multiplication operators. In fact, all abelian von Neumann algebras may be identified with the essentially bounded functions $L^\infty(X, \mu)$ of some measure space (X, μ).

The problem Murray and von Neumann faced was to provided other examples of von Neumann algebras other than those previously described. To address this concern, their work provided a method to assign to every (discrete) group Γ its group von Neumann algebra $L(\Gamma)$; this program would examine which group-level properties would lift to invariants of the algebra. This forms one of the most intractable questions in the field: determine what algebraic data of the group is detectable in the algebra. For instance, a central question in this field still unresolved to this day is to determine if the group von Neumann algebras of the non-amenable free groups $\mathbb{F}_n$ can be distinguished by the number of generators.

The results presented here will focus on an elementary construction of groups:

the direct product of groups. We demonstrate there is a large class of groups whose group von Neumann algebras "remember" the product structure of the underlying group; that is, any group von Neumann algebra isomorphic that arising from a product of groups in this class admits a unique prime decomposition at the level of the group in the sense of Ozawa and Popa [OP03]. Furthering these techniques, we provide a class of groups which exhibit decompose as a product whenever the group von Neumann algebra can be decomposed as a tensor product.

1.1 Preliminaries and Statement of the Problem

The notion of a von Neumann algebra was introduced in a series of works by F. Murray and J. von Neumann as an axiomitization of the phenomena observed in quantum mechanics. The significance of this formulation is highlighted as it offers a unifying perspective of of both classical and quantum mechanical physics. In short, the states of a system are (unit) vectors in the Hilbert space, the observables are the Hermitian operators, and the symmetries are the unitary operators. It would be the identification of symmetries with unitaries would motivate the further development of unitary group representations. In fact, an elementary example of a von Neumann algebra is given via unitary group representations: if $\rho : \Gamma \to \mathcal{U}(\mathcal{H}) \subset B(\mathcal{H})$ is a unitary representation of a group Γ into the bounded operators of some Hilbert space $\mathcal{H}$, the collection

$$M = \{x \in B(\mathcal{H}) : x\rho(\gamma) = \rho(\gamma)x \ \forall \gamma \in \Gamma\}$$

form a von Neumann.

The von Neumann algebras of our interest arise from countably infinite groups through the *left regular representation* $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$, where $\ell^2(\Gamma)$ denotes all square summable complex-valued sequences indexed by Γ. The von Neumann algebra generated by the collection of unitaries $\rho(\Gamma) \subset B(\ell^2(\Gamma))$ is the *group von Neumann algebra* and is denoted by $L(\Gamma)$. The classification problem of von Neumann algebras is a quest to discover so-called rigidity phenomena of groups: group-theoretic properties with ability to distinguish the resulting algebra. Murray and von Neumann were able to distinguish $L(\mathbb{S}_\infty \times \mathbb{F}_2)$ from $L(\mathbb{F}_2)$, where $\mathbb{S}_\infty$ is the groups of finitely supported permutations on a countably infinite set, by translating the group theoretic information into asymptotic behavior (see Definition 2.7 in §2.3.2). However, insight into the general question of rigidity remained elusive.

The landmark result of Alain Connes highlights the difficulty of this program: all icc amenable groups give rise to isomorphic group von Neumann algebras[Co76]. Thus elementary group invariants such as torsion, rank, and relations are alone insufficient to distinguish von Neumann algebras and require a deeper analysis of representation-theoretic aspects of the group. Connes' famous conjecture form the 70's states all property (T) groups, a representation theoretic property in diametric opposition to amenability, should give von Neumann algebras which retain all information of the underlying group. This conjecture remains an open problem at this time, and in fact not a single example of an icc Property (T) group is known to support this conjecture.

In the early 2000's Popa introduced a new conceptual framework to study von Neumann algebras now termed *Popa's deformation/rigidity theory.* This novel approach spurred spectacular progress in the field, settling many many longstanding problems in the classification of von Neumann algebras [Po01, Po03, Po04, IPP05, Po06, Po06b]. By working in this scheme, Ioana, Popa, and Vaes provided the first examples of groups which can be completely recovered from the von Neumann algebra [IPV10].

To understand the internal structure of type II_1 von Neumann algebras, Popa introduced the notion of prime von Neumann aglebras in the early 80's [Po83]: von Neumann algebras which are indecomposible as a tensor product of type II_1 subalgebras. Popa demonstrated how the von Neumann non-separable algebra $L(\mathbb{F}_\infty)$ exhibits this indecomposability. Ge extended this result to all von Neumann algebras of non-amenable free groups using Voiculescu's free probability theory[Ge98, Vo90, Vo96]. Furthering this progress, Ozawa, using C^* algebraic techniques, showed the group von Neumann algebra of all non-amenable hyperbolic groups are also prime, [Oz02], which enabled Ozawa and Popa to demonstrate a rigidity phenomenon for direct products of hyperbolic groups termed unique prime decomposition, [OP03]. We elaborate on this result in Sections 1.3 and 5.1.

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examples of groups which can be completely recovered from the von Neumann algebra [IPV10]. For our purposes, Popa's deformation/theory provide the technology in which Chifan and Sinclair would recover and extend the prime and solidity results for the von Neumann algebras associated to hyperbolic groups.

These results became the inspirations and focus of the work of myself along with I. Chifan and T. Sinclair, and subsequently become the motivation of the combined efforts of S. Pant and myself.

1.2 Notations

Let us denote a discrete group by Γ. For any subset $G \subset \Gamma$, $\langle G \rangle$ is the smallest subgroup of Γ containing G. Fixing $\gamma, \lambda \in \Gamma$ and subgroup $\Lambda < \Gamma$, we will denote $\gamma^\lambda = \lambda^{-1}\gamma\lambda$ and $\gamma^\Lambda = \left\{\gamma^\lambda : \lambda \in \Lambda\right\}$. The *centralizer* of Γ inside Λ is the subgroup $C_\Gamma(\Lambda) := \{\gamma \in \Gamma : \gamma\lambda\gamma^{-1} = \lambda \,\forall \lambda \in \Lambda\}$. The *virtual centralizer* of Λ inside Γ is the subgroup $\mathcal{V}_\Gamma(\Lambda) := \left\{\gamma \in \Gamma : |\gamma^\Gamma| < \infty\right\}$.

If $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is an n-fold product of groups and $F \subset \{1, \ldots, n\}$, then we will denote the subgroups $\Gamma_F, \Gamma_{F^c} < \Gamma$ to be the product groups $\times_{i\in F}\Gamma_i$ and $\times_{i\notin F}\Gamma$, respectively. We will also denote $\widehat{\Gamma}_j = \times_{i\neq j}\Gamma_i$, the product of all factors except generating Γ except the j-th entry; note $\Gamma = \Gamma_i \times \widehat{\Gamma}_i = \Gamma_F \times \Gamma_{F^c}$.

Fix a von Neumann algebra $M \subset B(\mathcal{H})$. We denote the

- *projections of M* by $\mathcal{P}(M)$
- *the unitaries of M* by $\mathcal{U}(M)$
- *the partial isometries of M* by $\mathcal{J}(M)$

- *unit ball of* M by $(M)_1$.

Let $A, B \subset M$ be von Neumann subalgebras of M. Then $A \vee B \subset M$ is the smallest von Neumann subalgebra of M generated by A and B. If we have a finite collection of subalgebras $A_1, \ldots, A_n, \subset M$, and $F \subset \{1, \ldots, n\}$ denote by

$$A_F = \bigvee_{i_F} A_i$$

the algebra generated by the subcollection $\{A_i : i \in F\}$. For brevity, we denote the algebra generated by all except the j -th algebra by $\widehat{A}_j = \bigvee_{i \neq j} A_i$.

1.3 Results Obtained

When $\Gamma = \Gamma_1 \times \Gamma_2$ is a product of icc groups, the resulting algebra $L(\Gamma) \cong L(\Gamma_1)\bar{\otimes}L(\Gamma_2)$ is non-prime. If $L(\Gamma)$ is non-prime, in what situations does it follow that Γ decomposes as a direct product of groups, i.e. when does a decomposition of the algebra into commuting subalgebras necessarily imply a direct-product decomposition of Γ?

Before stating the result, let us proceed with a preliminary analysis of what details a *unique prime decomposition* needs to take into account. If $M = P_1\bar{\otimes}\cdots\bar{\otimes}P_m \cong Q_1\bar{\otimes}\cdots\bar{\otimes}Q_n$ are different decomposition of M into prime von Neumann algebras, then we would like to conclude $n = m$ and $P_1 \cong Q_i$ after permuting indices. In general, the latter conclusion need not follow as one can conjugate each algebra by element of the unitary group of M; furthermore, for every $t > 0$ we can identify $P_1\bar{\otimes}P_2$ with $P_1^t\bar{\otimes}P_2^{1/t}$. Taking these details into consideration, we say M has a unique prime decomposition if whenever $M = P_1\bar{\otimes}\cdots\bar{\otimes}P_m \cong Q_1\bar{\otimes}\cdots\bar{\otimes}Q_n$ are different decomposition of M into prime von Neumann algebras, then $n = m$ and we may find a unitary $\mathcal{U}(M)$ along with $t_1 \cdots t_m > 0$ so that $uP_i^{t_i}u^* = Q_i$ and $t_1 \dots t_m = 1$.

Our research into this problem stems from Ozawa and Popa's *unique prime decomposition*: if we have $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ and $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$ are each products of non-elementary hyperbolic groups so that $L(\Gamma) \cong L(\Lambda)$, then $n = m$ and we may identify $L(\Gamma_i)$ with an amplification of $L(\Lambda_i)$. This shows how the group von Neumann algebra of products hyperbolic groups necessarily have a maximal decomposition dictated by the number of factors in the product, with the caveat that one presupposes a symmetric structure of the algebra $L(\Lambda)$. With this in mind, we eliminate all assumptions on the target group Λ and determine what, if any, algebraic data can be recovered.

Theorem 1.1. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of non-elementary hyperbolic groups and suppose there exists some group $L(\Gamma) \cong L(\Lambda)$. Then there exists a product decomposition $\Lambda_1 \times \cdots \times \Lambda_n = \Lambda$ and scalars $t_1, \cdots t_n > 0$ so that $1 = t_1 \times \cdots t_n$, and $L(\Gamma_i) \cong L(\Lambda)_i^{t_i}$.*

This provides a significant group-level strengthening of the unique prime decomposition theorem of Ozawa and Popa by removing all assumptions on the group Λ. In fact, we verify the theorem holds verbatim even when Γ is a product of weakly amenable icc groups in Ozawa's class $\mathcal{S}$(See Definition 3.3). We note the proof of Theorem 1.1 can be modified to forgo the an additional assumption that groups $\Gamma_1, \dots, \Gamma_n$ be weakly amenable to obtain the following theorem.

Theorem 1.2. *Let* $\Gamma_1, \Gamma_2 \in \mathcal{S}_{nf}$*, and denote* $\Gamma = \Gamma_1 \times \Gamma_2$*. Let* Λ *be an arbitrary group such that* $M = L(\Gamma) = L(\Lambda)$*. Then one can find subgroups* $\Lambda_1, \Lambda_2 < \Lambda$ *with* $\Lambda_1 \times \Lambda_2 = \Lambda$*, a scalar* $s > 0$*, and a unitary* $v \in M$ *such that* $vL(\Lambda_1)v^* = L(\Gamma_1)^s$ *and* $vL(\Lambda_2)v^* = L(\Gamma_2)^{1/s}$*.*

These results maybe seen as the group von Neumann algebraic analog of Monod and Shalom's [MS06, Theorem 1.10] orbit equivalence rigidity theorem for products of groups in the class $\mathcal{C}_{reg}$, though the relation between the two results is imperfect and the proofs do not seem in any precise way to depend on a common framework. One important point of contrast is the need in [MS06] for a mild mixingness assumption on the target action, while in the case of the above theorem the icc condition on the target group (corresponding to "plain ergodicity") suffices; however, this is likely accounted for by the fact that in orbit equivalence one is working over a parameter space on the group algebra. Secondly, Monod and Shalom are able to deduce honest isomorphism of the groups while in the above theorem the identification of the product factors up to stable isomorphism is sharp. We elaborate the intricacies of the necessity of amplification in Chapter 5.

As a consequence of these results we may apply Margulis' Normal Subgroup Theorem [Ma79, Z84] to deduce indecomposability of group factors of higher-rank irreducible lattices over a product of group factors of groups in the class $\mathcal{S}_{nf}$.

Corollary 1.3. *If* Λ *is an irreducible lattice in a higher rank semisimple Lie group, then* $L(\Lambda)$ *is neither isomorphic to a factor* $L(\Gamma_1 \times \Gamma_2)$ *where* Γ_1, Γ_2 *are groups in the class* $\mathcal{S}_{nf}$*, nor is it isomorphic to a factor of the form* $L(\Gamma_1 \times \cdots \times \Gamma_n)$ *where each*

$\Gamma_i \in \mathcal{S}_{nf}$ *and is weakly amenable.*

In particular if $\Lambda = PSL_2(\mathbb{Z}[\sqrt{2}])$, then $L(\Lambda)$ is not isomorphic to $L(\mathbb{F}_2 \times \mathbb{F}_2)$, even though these groups are measure equivalent in the sense of [Fu99a]. These add new natural examples to the ones found earlier [CI11]. We make note of the improvement of this result by Drimbe, Hoff and Ioana who prove the von Nuemann algebra $L(\Lambda)$ is in fact prime [DHI16].

In work done jointly with Pant, we investigate a class of groups whose group von Neumann algebras would exhibit a similar analytic structure to those of hyperbolic groups which would form the base class of the *poly hyperbolic groups*(more generally the poly-$\mathcal{C}_{\text{rss}}$). A previous investigation by Chifan, Kida and Pant would conclude the group von Neumann algebras are in fact prime, provided the groups admit a restrictive cohomological structure[CKP14]. Removing this additional structure, we adapt the techniques from the first result determine that group must (virtually) decompose as a product when the group von Neumann algebra is not prime.

Theorem 1.4. *Let* $\Gamma \in \text{Quot}_n(\mathcal{C}_{rss})$ *be an icc group and suppose* $A_1\bar{\otimes}\cdots\bar{\otimes}A_k \subset L(\Gamma)$ *is a finite index inclusion of* II_1 *factors. Then* $k \leq n$ *and there exist groups* $\Gamma_i \in \text{Quot}_{n_i}(\mathcal{C}_{rss})$ *so that* $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$*. Furthermore, we may find projections* $p_i \in A_i$*,* $q_i \in L(\Gamma_i)$ *so that* $p_iAp_i \subset q_iL(\Gamma_i)q_i$ *are finite index inclusions of* II_1 *factors.*

In the previous results, the decomposition of the algebra as a collection of commuting algebras is in one-to-one correspondence with a decomposition of the group as a product. However, Theorem 1.1 and Corollary 1.2 does not guarantee the target group Λ decomposes as a product of hyperbolic (or even bi-exact groups).

Focusing on Theorem 1.3, we obtain a sharper statement for the class of icc groups in Quot($\mathcal{C}_{\text{rss}}$). as the groups which decompose Γ as a product remain in this class.

The final chapter evinces structural results for von Neumann algebras which are amalgamated free products (AFP) of II_1 factors. The techniques employed expected to be adaptable to a wide range of APF groups in order to establish a product rigidity variant and possibly tree-graph product rigidity statement for this collection. At the moment, this analysis remains in its infancy but we postulate it to be fruitful.

CHAPTER 2
BACKGROUND

The von Neumann algebra as a subalgebra of the bounded operators on a Hilbert space $M \subset B(\mathcal{H})$ has been the subject of interest for nearly a century. Since its inception, the primary subject of interest is the classification of von Neumann subalgebras. Murray and von Neumann's introduction of the subject included a prescription to assign to every (countable) group a von Neumann algebra, an approach which would attempt to introduce group-theoretic invariants as possible sources of distinguishing criteria.

We introduce elementary Hilbert space terminology which will be necessary to define von Neumann algebras and present classical constructions such: the group von Neumann algebra, the group measure space construction, and crossed product algebras. We next introduce the fundamental notions of subalgebras such a index, amenability and solidity. The subsequent gives an elementary overview of Popa's intertwining techniques and provide a new intertwining condition for multiple algebras.

The chapter closes with a description of the groups we shall examine. We first introduce a class of groups which exhibit a sort of "negative curvature" akin to that of hyperbolic groups, Ozawa's class of bi-exact groups. Expanding this class of groups to all groups whose algebras exhibit the strong solidty property for hyperbolic groups, we then take all groups which are finite step extensions in a sense akin to the construction of solvable groups. We then provide structural results for these groups, which give evidence of a parallel structural result for their resulting algebras.

2.1 Operator Algebras

To a Hilbert space $\mathcal{H}$ with sesquilinear form $\langle \cdot, \cdot \rangle$, one induces a norm on the collection of linear maps $x : \mathcal{H} \to \mathcal{H}$ via $\|x\| = \sup_{\eta \in \mathcal{H}} \|A\eta\|_{\mathcal{H}} / \|\eta\|_{\mathcal{H}}$. The collection of linear operators which are bounded in this norm are denoted by $B(\mathcal{H})$. $B(\mathcal{H})$ forms a linear spaces over $\mathbb{C}$. This is in fact a Banach algebra with multiplication given by composition of linear maps. When $\mathcal{H}$ is an n-dimensional Hilbert space, $B(\mathcal{H})$ is the $n \times n$ matrices $M_n(\mathbb{C})$. The bounded operators also admit an anti-linear anti-isomorphic involution $* : B(\mathcal{H}) \to B(\mathcal{H})$: For any element $x \in B(\mathcal{H})$, x^* is the unique bounded linear operator satisfying

$$\langle x\eta, \xi \rangle = \langle \eta, x\xi \rangle$$

for every $\eta, \xi \in \mathcal{H}$. This operation satisfies

$$\|xx^*\| = \|x\|^2,$$

the so called C^* identity. This analysis motivates the following classical definition of functional analysis.

Definition 2.1. Let $\mathcal{A}$ be a normed algebra over $\mathbb{C}$. $\mathcal{A}$ is a C^* *(pronounced C-star) algebra* if $\mathcal{A}$ is complete in the norm (Banach algebra) and there is a map $* : \mathcal{A} \to \mathcal{A}$ with the following properties:

- $(x^*)^* = x^{**} = x$ for every $x \in \mathcal{A}$ (involution),
- $(\lambda x + y)^* = \bar{\lambda} x^* + y^*$ for each $\lambda \in \mathbb{C}$ and every $x, y \in \mathcal{A}$ (anti-linear)
- $(xy)^* = y^* x^*$ (anti-automorphic)

- $\|x^*x\| = \|x\|^2 = \|x\|\|x^*\|$.

The algebra of $n \times n$ matrices (more generally $B(\mathcal{H})$) is the elementary example of non-commutative C^* algebra. To create a commutative C^* algebra one begins with a (locally) compact Hausdorff space X. The $f : X \to \mathbb{C}$ which are continuous and compactly supported are denoted by $C_0(X)$. When $C_0(X)$ is endowed with the norm $\|f\| = \sup_{x \in X} |f(x)|$ and map $f \mapsto \bar{f}$ where $\bar{f}(x) := f(\bar{x})$, one can easily verify $C_0(X)$ is a C^* algebra and is unital precisely when X is compact. In fact, the classical theorem of Gelfand states every commutative unital C^* algebra $\mathcal{A}$ arises in this way.

Theorem 2.1 (Gelfand). *Let $\mathcal{A}$ is a unital abelian C^* algebra and endows the collection of continuous $*$-homomorphisms $\sigma(\mathcal{A})$ with the weak-$*$ topology from $\mathcal{A}^*$. $\mathcal{A}$ is isometrically isomorphic to $C(\sigma(\mathcal{A}))$.*

We note that if X and Y are compact Hausdorff spaces then one can verify $C(X) \cong C(Y)$ if and only if X and Y are homeomorphic, thereby classifying all abelian C^* algebras through topological considerations. We make note that similar result holds by removing the hypothesis that $\mathcal{A}$ be unital even though this document will focus exclusively on unital algebras.

2.2 The GNS Construction

Given a unital C^* algebra, the *state space* is the collection of all linear functionals $\varphi : \mathcal{A} \to \mathbb{C}$ such that $\|\varphi\| = 1$. When $\mathcal{A} \subset B(\mathcal{H})$ is a C^* subalgebra acting on some Hilbert space $\mathcal{H}$, every vector $\xi \in \mathcal{H}$ with $\|\xi\|_{\mathcal{H}} = 1$ induces a *vector state*

$\varphi_\xi : \mathcal{A} \to \mathbb{C}$ by the mapping

$$x \mapsto \langle x\xi, \xi \rangle.$$

In quantum mechanics, a vector state provides a mathematical formalism for the probabilistic interpretations of elementary properties of quantum particles (position). When x is self-adjoint (x is an observable using terminology found in physics literature) $x \mapsto \langle x\xi, \xi\rangle$ measure the probability an observable can be found in a particular location. We momentarily pause to introduce terminology necessary to induce a vector state from an arbitrary one acting on an abstract C^* algebra.

A *representation* of a unital C^* algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ is a ring homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$ such that $\pi(x^*) = \pi(x)^*$ and $\pi(1_\mathcal{A}) = 1_\mathcal{H}$. A vector $\xi \in \mathcal{H}$ is a *cyclic* if $\{\pi(\mathcal{A})\xi\} \subset \mathcal{H}$ is dense and $\pi : \mathcal{A} \to B(\mathcal{H})$ is a *cyclic $*$-representation* if there exists a cyclic vector. Thus a cyclic representation $\pi : \mathcal{A} \to B(\mathcal{H})$ allows us to define a vector state by taking the inner product against the cyclic vector. The theorem of Gelfand, Naimark, and Siegal demonstrates every state $\varphi : \mathcal{A} \to \mathbb{C}$ arises in this fashion.

Theorem 2.2 (The GNS Construction [GN43)**.** *] Let $\pi : \mathcal{A} \to B(\mathcal{H})$ be a cyclic representation of a unital C^* algebra $\mathcal{A}$ with cyclic vector $\xi \in \mathcal{H}$. Then the linear functional $\varphi_\pi : \mathcal{A} \to \mathbb{C}$ defined by $\varphi_\pi(a) = \langle \pi(a)\xi, \xi\rangle$ is a state. Conversely if $\varphi : \mathcal{A} \to \mathbb{C}$ is a state, then there exists a Hilbert space $\mathcal{H}_\pi$ with inner product $\langle \cdot, \cdot \rangle_\pi$ and cyclic vector ξ_π such that*

$$\varphi(x) = \langle x\xi_\pi, \xi\rangle_\pi$$

for every $x \in \mathcal{A}$.

Proof. Since the forward implication follows from the previous discussion, we only need to verify the converse.

We may define a sesqilinear form $\langle \cdot, \cdot \rangle : \mathcal{A} \times A \to \mathbb{C}$ by

$$\langle a, b\rangle = \varphi(b^* a),\ \forall a, b \in \mathcal{A}.$$

By the Cauchy-Schwartz inequality, the set $\mathcal{I} = \{x : \varphi(x^* x)\}$ form a vector subspace of $\mathcal{A}$ and is in fact a left ideal of $\mathcal{A}$. Thus the quotient space $\mathcal{J} = \mathcal{A}/\mathcal{I}$ is an inner product space with well-defined inner product given by $\langle \bar{a}, \bar{b}\rangle_\varphi := \langle a, b\rangle$. Taking the completion of $\mathcal{J}$ with respect to the norm induced by $\langle \cdot, \cdot \rangle_\pi$ produces a Hilbert space $\mathcal{H}_\varphi$ upon where $\mathcal{A}$ will be represented.

To construct a representation $\pi_\varphi : \mathcal{A} \to B(\mathcal{H}_\pi)$, we first define $\pi(a)$ as the operator $\pi_\varphi(a)(\bar{x}) = \bar{a}\bar{x}$ for every $a \in \mathcal{A}, \bar{x} \in \mathcal{J}$. Since $\mathcal{J}$ is by construction a dense subset of $\mathcal{H}$, we then extend $\pi_(a)$ to an bounded operator $\mathcal{H}_\varphi$ by taking approximates. It can be checked that π_φ is in fact a representation of $\mathcal{A}$ acting on $\mathcal{H}_\pi$ with cyclic vector $\xi_\pi = [1_\mathcal{A}]$. □

The GNS Construction will play a central role in representing the C^* algebras of interest in this document. These algebras will have a distinguished state with an additional property $\varphi(xy) = \varphi(yx)$, the trace.

2.3 von Neumann Algebras

While $B(\mathcal{H})$ is a topological spaces, one can see the uniform topology, the topology induced by the norm on $B(\mathcal{H})$, is insufficient for analytic purposes. For instance, multiplication and the star operation need not be continuous in topology induced by the norm. This motivates the construction of several weaker locally convex topologies on $B(\mathcal{H})$.

Definition 2.2. Let $\mathcal{H}$ be a Hilbert space. The *uniform topology* on $B(\mathcal{H})$ is the topology induced by the norm $\|\cdot\|_\infty$ on $B(\mathcal{H})$.

The *strong operator topology* (SOT) on $B(\mathcal{H})$ is defined by the family of semi-norms $x \mapsto \|x\xi\|_{\mathcal{H}}$ for every $\xi \in \mathcal{H}$, i.e. the SOT topology is the weakest topology such that the map $x \mapsto \|x\xi\|_{\mathcal{H}}$ is continuous for every $x \in B(\mathcal{H})$ and every $\xi \in \mathcal{H}$.

The *weak operator topology* (WOT) is defined by the family of semi-norms $x \mapsto |\langle x\xi, \eta\rangle|$ for every ξ, η, i.e. the WOT topology is the weakest topology such that the map $x \mapsto |\langle x\xi, \eta\rangle|$ is continuous for every $x \in B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$

From coarsest to finest, we have WOT $\prec$ SOT $\prec$ Uniform. To illuminate the nature of these topologies, let us assume $B(\mathcal{H})$ is separable and interpret these topologies in terms of convergence of sequences: x_n in $B(\mathcal{H})$ converges $x \in B(\mathcal{H})$

- uniformly if $\|x_n - x\| \to 0$,
- in the *strong operator topology* if for every $\xi \in \mathcal{H}$, $\|(x_n - x)\xi\|_{\mathcal{H}} \to 0$
- in the *weak operator topology* if for every $\xi, \eta \in \mathcal{H}$, $|\langle (x_n - x)\xi, \eta\rangle| \to 0$

These topologies are equivalent in when $\mathcal{H}$ is any finite dimensional Hilbert space and thus SOT and WOT become interesting when $\mathcal{H}$ is infinite dimensional. While multiplication of operators is continuous in SOT, continuity of the adjoint operation cannot be guaranteed unless one imposes the weak operator topology on $B(\mathcal{H})$.

Definition 2.3. Let $M \subset B(\mathcal{H})$ be a unital, self-adjoint subalgebra of $B(\mathcal{H})$. If M is closed with respect to the WOT topology, then M is a *von Neumann algebra*.

Thus a von Neumann algebra is a concrete C^* algebra (in the sense that it is naturally represented on a Hilbert space) closed in a weaker topology. Before introducing examples of von Neumann algebras, we explore additional properties of interest of von Neumann algebras.

The *commutant* of M, denoted M', is the set of operators

$$M' = \{x \in B(\mathcal{H}) : xm = mx \ \forall m \in M\}.$$

We use the notation $M'' = (M')'$ to denote the *bicommutant* of M, and notice that for any unital, self-adjoint set M we have the inclusion $M \subset M''$; the question is how much larger is the resulting set. A remarkable result von Neumann is the relationship between the the weak operator topology and the bicommutant.

Theorem 2.3 (The Bicommutant Theorem)**.** *If $M \subset B(\mathcal{H})$ is a unital, self-adjoint collection of operators, then M' is a von Neumann algebra. A self-adjoint maximal abelian collection of operators $M \subset B(\mathcal{H})$ is a von Neumann algebra. Furthermore, if M is a unital, self-adjoint collection of operators, then M is a von Neumann algebra if and only if $M = M''$ (M is its own bicommutant).*

The bicommutant theorem is analogous to the elementary property of Hilbert subspaces: $\mathcal{H}_0 \subset \mathcal{H}$ is a closed subspace of $\mathcal{H}$ if and only if $\mathcal{H}_0 = (\mathcal{H}_0^{\perp})^{\perp}$. Thus the weak operator topology on $B(\mathcal{H})$ is precisely the topology which is "balanced" under the double commutation. Furthermore, Theorem 2.3 allows one to generate a pair of von Neumann algebra M' and M'' for a given collection self-adjoint, unital collection of operators M; this will form the primary method of generating the algebras of interest in the following section.

If $M \subset B(\mathcal{H})$ is a von Neumann algebra, *center* of M is the von Neumann algebra $\mathcal{Z}(M) = M \cap M'$. A von Neumann algebra $M \subset B(\mathcal{H})$ is a called a *factor* if its center $\mathcal{Z}(M)$ is isomorphic to the complex numbers. A deep result of von Neumann decomposes an arbitrary von Neumann algebra M into a *direct integral* of factors over $\mathcal{Z}(M)$ and thus the classification of these algebras can be reduced to the classification of factors [vN49].

The algebras of matrices $M_n(\mathbb{C})$, and more generally $B(\mathcal{H})$ for an arbitrary Hilbert space $\mathcal{H}$, are the elementary examples of von Neumann algebra since these algebras are trivially WOT closed and in fact are factors. When (X, μ) is a σ-finite measure space, we view $L^\infty(X, \mu) \subset B(L^2(X, \mu))$ as a maximal abelian von Nuemann algebra by defining $M_f(g(x)) = f(x)g(x)$ for all functions $f \in L^\infty(X, \mu)$, $g \in L^2(X, \mu)$. A generalization of Gelfand's theorem(Theorem 2.1) allows one to identify every abelian von Neumann algebra as $L^\infty(X, \mu)$ for some measure space (X, μ).

Theorem 2.4. *Let $\mathcal{A} \subset B(\mathcal{H})$ be an abelian seperable von Neumann algebra. Then there exists a compact Hausdorff space K and Radon measure μ on K and a unitary $u : L^2(K, \mu) \to \mathcal{H}$ such that $u\mathcal{A}u^* = L^\infty(K, \mu) \subset B(L^2(K, \mu))$.*

Hence, classification of abelian von Neumann algebras is in correspondence with the classification of measure spaces. Due to this analogy of Theorem 2.1, the study of von Neumann algebras is often referred to as noncommutative measure theory with states playing the role of the measure.

2.3.1 Projections, Types and Traces

A well-established fact states the linear span of the projections $\mathcal{P}(\mathcal{M})$ are WOT dense in M. When M is an abelian von Neumann algebra, this is precisely the fact that the simple functions are dense in $L^\infty(X, \mu)$. The following analysis of projections of a von Neumann algebra $\mathcal{P}(M)$ leads to the type decomposition and the construction of the trace of type II_1 factors, a distinguished state on M which acts as a continuous version of the (normalized) traces on $M_n(\mathbb{C})$.

The comparison theory of projections was established in the works of Murray and von Neumann through geometric considerations of projections [MvN36]. To introduce the comparison theory of projections, fix a von Neumann algebra $M \subset B(\mathcal{H})$.

Definition 2.4. Given two projections $p, q \in \mathcal{P}(M)$, we say p is subequivalent to q, and denote this by $p \preceq q$, if and only if there exists a partial isometry $v \in \mathcal{J}(M)$ such that $p = vv^*$ and $v^*v \leq q$ as operators and write $p \prec q$ is the inequality is strict. Projections p and q are equivalent, written $p \sim q$ if and only if partial isometry $v \in \mathcal{J}(M)$ such that $p = vv^*$ and $v^*v = q$.

Since every projection $p \in \mathcal{P}(M)$ corresponds to subspace of $\mathcal{H}$, subequivalence of projections $p \preceq q$ corresponds to an isometric embedding of of the Hilbert subspace associated to p into the subspace associated to q. Moreover, the comparison theorem of projections states subequivalence forms a total ordering of the equilvance classes of projections of a factor M.

Theorem 2.5 (Cantor-Bernstein Thrm)**.** *Let $p, q \in \mathcal{P}(M)$ be projections in a factor M. Then either $p \prec q$, $q \prec q$, or $p \sim q$.*

Projections decompose an arbitrary von Neumann algebra as Type I, II_1, II_∞, III. The properties of each type can be distinguished by the properties and existence of a dimension function:

Theorem 2.6. *Let $M \subset B(\mathcal{H})$ be a factor. Then there exists a function $D : \mathcal{P}(M) \to [0, \infty]$, unique up to a multiplicative constant, so that*

- *$p \preceq q$ if and only if $D(p) \leq D(q)$ (monotonicity)*
- *p is finite if and only if $D(p) < \infty$*
- *If $\{p_n\}$ is any collection of orthogonal projections $D(\sum p_n) = \sum D(p_n)$ (normality)*

Furthermore D cans be normalized such that $D(\mathcal{P}(M))$ falls into exactly one of the following cases:

1. $\{0, 1, \ldots, n\}$
2. $\{0, 1, \ldots, \infty\}$
3. $[0, 1]$
4. $[0, \infty]$
5. $\{0, \infty\}$

The function D, called the *dimension function*, plays the role of a non commutative integral when extended linearly to M. When the range of D is precisely $[0,1]$, M is a type II_1 factor. We note in this case the dimension function satisfies $D(p) = D(q)$ if and only if $p \sim q$ which in turn implies $D(vv^*) = D(v^*v)$ for every $v \in \mathcal{J}(M)$. By linearly extending D to the span of the $\mathcal{P}(M)$, we state without proof that monotonicity implies that for every pair of elements $x, y \in \mathcal{P}(M)$ we have $D(xy) = D(yx)$. Normality subsequently allows us to extend D to a state $\tau : M \to \mathbb{C}$ by approximating elements of M with linear combinations of projections with the additional property that $\tau(xy) = \tau(yx)$ for every $x, y \in M$. Thus when M is a type II_1 factor we may extend D to a normal, faithful, tracial state $\tau : M \to \mathbb{C}$ such that $\tau(\mathcal{P}(M)) = [0,1]$.

Theorem 2.7. *Let M be type II_1 factor. There exists a normal, faithful, state called the* trace $\tau : M \to \mathbb{C}$ *such that $\tau(xy) = \tau(yx)$ for all $x, y \in M$. Moreover, the trace τ is unique.*

Let us suppose M is a type II_1 factor with trace τ. The GNS construction (Theorem 2.2) with respect to τ gives a faithful representation of $M \subset B(L^2(M,\tau))$. Furthermore, since τ is a faithful state we have a faithful dense embedding of $M \hookrightarrow L^2(M,\tau)$; hence we define the 2-norm of an operator $x \in M$ is the Hilbert space norm of the vector $x \in L^2(M,\tau)$ given by $\|x\|_2 := \tau(x^*x)^{1/2}$. This is the *standard form* for M and will be important for describing inclusions of algebras and the construction of the conditional expectation: an operator valued projection defined for an inclusion of subalgebras. For instance, if we have a unital inclusion of II_1 factors $N \subset M$, this induces an inclusion of their corresponding Hilbert spaces $L^(N,\tau) \subset L^2(M,\tau)$ which gives a projection e_N from the larger space to the smaller one which will restrict to a projection of the operator algebras. Properties of this map will be explored further in Section 2.4.

The classic construction of a II_1 factor arises via an inductive limit of matrix algebras: for each $n \in \mathbb{N}$ we construct a unital embedding of matrix algebra $M_{2^n}(\mathbb{C}) \hookrightarrow M_{2^{n+1}}(\mathbb{C})$ by mapping $x \mapsto x \otimes I_2$. This embedding will in fact be a trace preserving embedding, provided we take the normalized trace on the matrix algebras. The hyperfinite II_1 factor $\mathcal{R}$ is defined as

$$\mathcal{R} := \bigcup_{n \geq 1} M_{2^n}(\mathbb{C})$$

with trace built as the inductive limit of the finite dimensional traces.

Any II_1 factor M can be "enlarged" to generate a new II_1 factor by taking the tensor product with with the algebra of matrices $M_n(\mathbb{C})\bar{\otimes}M \cong M_n(M)$ which has trace given by $[x_{i,j}] \mapsto \frac{1}{n}\sum_{i=1}^{n} \tau(x_{i,i})$.

Definition 2.5. Let $M \subset B(\mathcal{H})$ be a type II_1 factor with tracial state τ and define $\tau_n : M_n(M) \to \mathbb{C}$ by $\tau_n([x_{i,j}]) = \frac{1}{n}\sum_{i=1}^{n} \tau(x_{i,i})$. The *amplification* of M with parameter t is the von Neumann algebra $M^t := pMp$ where $p \in \mathcal{P}(M_n(M))$ is a projection with trace $\tau_n(p) = t/n$

We make note that for a fixed value of $n \geq t > 0$ the isomorphism class of M^t is independent of the choice of projection p since if $p \sim q$, there exists a unitary $u \in \mathcal{U}(M_n(M))$ so that $upMpu^* = qMq$. With these properties of von Neumann algebras we continue by introducing the class of algebras of interest: the group von

Neumann algebras.

2.3.2 The Group von Neumann Algebra

The bicommutant theorem of Murray and von Neumann allows one to associate a von Neumann algebra to a unitary representation of a group $\rho : \Gamma \to \mathcal{U}(\mathcal{H})$. Naturally $\rho(\Gamma)'$ and $\rho(\Gamma)'' \subset B(\mathcal{H})$ form von Neumann algebras. We draw our attention to the *left regular representation*. Let $\ell^2(\Gamma)$ denote the Hilbert space of all square summable sequences indexed by Γ, i.e. $\eta \in \ell^2(\Gamma)$ if and only if $\sum_{\gamma\in\Gamma} |\eta(\gamma)|^2 < \infty$. Then Γ can be viewed as a subgroup of $\mathcal{U}(\ell^2(\Gamma))$ by the the covariant action

$$\gamma \cdot \eta(\lambda) := \eta(\gamma^{-1}\lambda) \quad \forall \gamma, \lambda \in \Gamma,\, \eta \in \ell^2\Gamma$$

Definition 2.6. Let $\Gamma \subset \mathcal{U}(\ell^2(\Gamma))$. The group von Neumann algebra is the algebra generated by these unitaries

$$L(\Gamma) := \{\gamma\}''_{\gamma\in\Gamma} = \overline{\mathbb{C}[\Gamma]}^{WOT} \subset B(\ell^2(\Gamma)).$$

We may also construct $R(\Gamma)$ as the bicommutant of the collection of unitaries $\{\gamma^0\}_{\gamma\in\Gamma}$ where

$$\gamma^0 \cdot \eta(\lambda) = \eta(\lambda\gamma) \quad \forall \gamma, \lambda \in \Gamma,\, \eta \in \ell^2\Gamma$$

and note by construction $L(\Gamma) \subset R(\Gamma)'$.

This approach gives little insight into the nature of the elements which live in the algebra. An alternative construction of the group von Neumann algebra arises by viewing elements of $\ell^2(\Gamma)$ acting on (a dense subset) ℓ^2 by convolution operators.

For every $\eta \in \ell^2(\Gamma)$ define a linear map $L_\eta : \ell^2(\Gamma) \to \ell^\infty(\Gamma)$ by

$$L_\eta(\xi)(\gamma) := \sum_{\lambda \in \Gamma} \xi(\gamma)\eta(\gamma^{-1}x) = (\xi * \eta)(\gamma) \quad \forall \gamma \in \Gamma$$

whence, by the Cauchy-Schwartz inequality, $L_\eta(\xi) \in \ell^\infty(\Gamma)$ since for every $\gamma \in \Gamma$ we have the estimate $|L_\eta(\xi)(\gamma)| \leq \|\xi\|_2\|\eta\|_2$. We define the *domain of* L_η to be the set $D_\eta := \{\xi \in \xi^2(\Gamma) : L_\eta(\xi) \in \ell^2(\Gamma)\}$ and note this is a dense subset of $\ell^2(\Gamma)$ containing the standard basis vectors of $\ell^2\Gamma$. Thus $L_\eta : D_\eta \to \ell^2(\Gamma)$ is a closed unbounded operator on $\ell^2(\Gamma)$ with dense domain and adjoint $L_\eta^* = L_{\bar{\eta}}$. If $D_\eta = \ell^2(\Gamma)$, then η is a *left convolver* and note the space of of left convolvers forms a linear subspace of $B(\rceil\updownarrow\updownarrow^{\in}(-))$. An elementary calculation shows $L_{\delta_\gamma}(\eta) = \gamma \cdot \eta$ for every standard basis element $\delta_\gamma \in \ell^2(\gamma)$ and $\eta \in \ell^2(\Gamma)$; therefore every finite linear combination of basis elements are left convolvers. Moreover, the left convolvers form an algebra under composition by the identity

$$L_(\eta)(L_\xi) = L_{\eta * \xi}.$$

With foresight, we define $L(\Gamma) := \{L_\eta : D_\eta = \ell^2(\Gamma)\} \subset B(\ell^2(\Gamma))$ as the *algebra of left convolvers*. We analogously define right convolution operators $R_\eta(\xi)(\gamma) := \xi * \eta(\gamma)$ and let denote $R(\Gamma)$ to be the algebra of right convolvers.

We now show defining $L(\Gamma) \subset B(\ell^2(\Gamma))$ as the algebra of left convolvers is not an abuse of notation.

Theorem 2.8. *Let* Γ *be a discrete group. Then* $L(\Gamma) = R(\Gamma)' = \{\gamma^o\}'_{\gamma \in \Gamma}$ *and* $R(\Gamma) = \{\gamma\}'_{\gamma \in \Gamma} = L(\Gamma)'$. *In particular, the algebra of left convolvers* $L(\Gamma)$ *is a von Neumann algebra such that* $L(\Gamma) = \{\gamma\}''_{\gamma \in \Gamma}$.

Proof. Note we trivially have the inclusions $L(\Gamma) \subset R(\Gamma)' \subset \{\gamma^0\}'_{\gamma\in\Gamma}$ and thus by von Neumann's bicommutant theorem (Theorem 2.3) we only need to show $\{\gamma^0\}'_{\gamma\in\Gamma} \subset L(\Gamma)$. To this end, suppose $r \in \{\gamma^0\}'_{\gamma\in\Gamma}$ and let $\xi = r\delta_e$ where e is the neutral element of Γ. Then for all $\gamma \in \Gamma$, we have

$$\eta * \delta_{\gamma^{-1}} = \gamma^o \cdot (r\delta_e) = r(\gamma^o \cdot \delta_e) = r\delta_{\gamma^{-1}}$$

and hence by linearity $r = L_\eta \in L(\Gamma)$. □

This presentation of the group von Neumann algebra allows us to clearly define a trace and determine precisely when $L(\Gamma)$ is a II_1 factor. When Γ is a discrete group, $\tau(x) = \langle x\delta_e, \delta_e\rangle$ is a normal, faithful tracial sate since by direct calculation once can verify $\tau(x^*x) = 0$ if and only if $x = 0$ and $\tau(\gamma\lambda) = \tau(\lambda\gamma)$ for every $\gamma, \lambda \in \Gamma$. We expand every element $x \in L(\Gamma)$ via its Fouier expansion $x = \sum_{\gamma\in\Gamma} x_\gamma \gamma$, where $x_\gamma = \tau(\gamma^{-1}x) \in \mathbb{C}$ and the convergence of the right-hand-side is taken in ℓ^2. Noting the Fourier coefficients are constant along conjugacy classes ($x_\gamma = x_{\lambda^{-1}\gamma\lambda}$), von Neumann demonstrated $L(\Gamma)$ is a factor precisely when Γ is an *icc* group, i.e. $|\gamma^\Gamma| = \infty$ for all $\gamma \in \Gamma \setminus \{e\}$. Thus $L(\Gamma)$ is II_1 factor with trace $\tau : L(\Gamma) \to \mathbb{C}$ given by

$$\tau(x) = \langle x\delta_e, \delta_e\rangle$$

where δ_e is the canonical basis vector indexed by the identity element $e \in \Gamma$. Elementary candidates of icc groups include the non-abelian free groups $\mathbb{F}_n, n \geq 2$; the group of finitely supported permutations on infinite symbols $\mathbb{S}_\infty$; fundamental groups of genus $g \geq 2$ surfaces, $\pi_1(\Sigma_g)$; free products $\Gamma_1 * \Gamma_2$ where $|\Gamma_1|, |\Gamma_2| \geq 2$; central quotients of pure braid groups; and $PSL_2(\mathbb{Z})$.

Murray and von Neumann introduced group von Neumann algebras in an attempt provide examples of non-isomorpic II_1 factors by distinguish the algebras from algebraic information. While this particular effort was largely unsuccessful, they were able to provide an invariant to discern $L(\mathbb{S}_\infty \times \mathbb{F}_2)$ from $L(\mathbb{F}_2)$ by demontrating the existence of a sequence of unitaries which asymptotically commute with every element in the former algebra, a property that does not exist in the latter algebra.

Definition 2.7. A II_1 factor M with trace τ has *Property Gamma* if and only if there exists a sequence of unitaries $\{u_n\} \in \mathcal{U}(M)$ such that $\tau(u_n) = 0$ and

$$\|xu_n - u_n x\|_2 \to 0$$

as $n \to \infty$ for every $x \in M$.

Using an approach motivated by the original proof of Murray and von Neumann, we show $L(\mathbb{F}_2)$ does not have Property Gamma and show $L(\mathbb{S}_\infty \times \mathbb{F}_2)$ indeed does. Letting $\mathbb{F}_2 = \langle a, b \rangle$, a combintatorial argument shows

$$\|x - \tau(x)1\|_2 \leq 10\|ax - xa\|_2 + 10\|bx - xb\|_2$$

which clearly implies the absence of trace 0 asymptotically commuting unitaries u_n in $\mathcal{U}(L(\mathbb{F}_2))$. One can easily verify that element $\gamma \in \mathbb{S}_\infty$ commutes with the transposition $(n,\, n+1) \in \mathbb{S}_\infty, n \in \mathbb{N}$ when n is sufficiently large. Thus, these transposition asymptotically commute with the generators and hence with every element of $L(\mathbb{S}_\infty \times \mathbb{F}_2)$; these form the sequence of unitaries required in Definition 2.7. It should be noted Murray and von Neumann were unable to make substantial progress in finding invariants of these algebras, but they did provide the fundamental framework for this field.

We bring to attention the the fundamental question of rigidity: if $L(\Gamma) \cong L(\Lambda)$, what group-theoretic properties are shared by the groups? How does one attempt to recover discrete algebraic data from analytic objects? The difficulty in answering these question will be discussed in Section 2.4.2.

2.3.3 The Crossed Product

One if the far-reaching applications of von Neumann algebras is in the field of ergodic theory. In fact, one of the earliest constructions the various types of von Neumann algebras arose from a group action on a measure space.

Fix an action $\Gamma \curvearrowright (X, \mu)$ of a discrete group on a σ-finite measure space. The action of Γ is *quasi-invariant* if for every $\gamma \in \Gamma$, for every measurable set $E \in X$ we have that γE is also measurable and the push-forward measure $\gamma \cdot \mu(E) := \mu(g^{-1} \cdot E)$ is equivalent to μ. In this situation Γ acts on X by measurable automorphisms. A quasi-invariant action action of a discrete group $\Gamma \curvearrowright (X, \mu)$ on a σ-finite measure space induces an action action on $L^\infty(X, \mu)$ by

$$\gamma \cdot (f(x)) = f(\gamma^{-1}x) \text{ for all } f \in L^\infty(X, \mu), \gamma \in \Gamma, x \in X.$$

Note that if $f \in L^\infty(X, \mu)$, $g \in L^2(X, \mu)$ and $\gamma \in \Gamma$, then $\|f\|_\infty = \|\gamma \cdot f\|_\infty$ and $\|g\|_2 = \|\gamma \cdot g\|_2$. Specifically, we may view $\Gamma \subset \mathcal{U}(L^2(X, \mu))$. We now construct a von Neumann algebra which encodes both the measure theoretic and dynamical aspects of the group action.

When $\Gamma \curvearrowright (X,\mu)$ is a measure preserving action ($\gamma\mu = \mu$), we view $L^\infty(X,\mu)$ as a subalgebra of $B(L^2(X,\mu)\bar{\otimes}\ell^2(\Gamma))$ by extending

$$f(g\otimes\xi) = (fg)\otimes\xi \quad \forall f\in L^\infty(x,\mu),\, g\in L^2(X,\mu),\, \xi\in\ell^2(\Gamma) \tag{2.1}$$

We also consider the diagonal action of Γ on $L^2(X,\mu)\bar{\otimes}\ell^2(\Gamma)$

$$\gamma\cdot(g\otimes\xi) = (\gamma\cdot g)\otimes(\gamma\cdot\xi) \quad \forall \gamma\in\Gamma,\, g\in L^2(X,\mu),\, \xi\in ell^2(\Gamma) \tag{2.2}$$

where the action of $\Gamma \curvearrowright \ell^2(\Gamma)$ is given by the left regular representation. The diagonal action of Γ allows us to embed the group $\Gamma \subset \mathcal{U}(L^2(X,\mu)\bar{\otimes}\ell^2(\Gamma))$.

Definition 2.8. The *group measure space construction*, $L^\infty(X)\rtimes\Gamma$, is the von Neumann subalgebra of $B(L^2(X)\bar{\otimes}\ell^2(\Gamma))$ generated by the operators

$$\{f\}_{f\in L^\infty(X)} \cup \{\gamma\}_{\gamma\in\Gamma} \subset B(L^2(X,\mu)\bar{\otimes}\ell^2(\Gamma))$$

whose actions are described in Equations (2.1) and (2.2).

Note the operators satisfy the covariance condition $\gamma f\gamma^{-1} = \gamma\cdot f$ under the identification above. By construction both von Neumann algebras $L^\infty(X,\mu), L(\Gamma)$ are subalgebras of $L^\infty(X,\mu)\rtimes\Gamma$. Note that if X is singleton set endowed with the counting measure, $L^\infty(X)\rtimes\Gamma$ is naturally isomorphic to $L(\Gamma)$.

The group measure space construction $L^\infty(X)\rtimes\Gamma$ associated to an action $\Gamma \curvearrowright (X,\mu)$ is a type II_1 factor with trace trace $\tau(x) = \int_X \langle x\delta_e, \delta_e\rangle\, d\mu$ if this is a free ergodic measure-preserving action on a non-atomic probability space.

More generally, we take $\Gamma \curvearrowright (N,\tau)$ a *trace preserving* action of a countable group on a II_1 factor ($\tau(\gamma x) = \tau(x)\,\forall x\in N, \gamma\in\Gamma$). Viewing $N\subset B(L^2(N,\tau))$ in standard form, we extend the group action $\Gamma \curvearrowright L^2(N,\tau)$ which induces a unitary representation of $\Gamma \subset \mathcal{U}(L^2(N),\tau)$. The diagonal action of Γ on $L^2(N,\tau)\bar{\otimes}\ell^2(\Gamma)$ now allows us to embed $\Gamma \subset \mathcal{U}(L^2(N,\tau)\bar{\otimes}\ell^2(\Gamma))$. N acts on $L^2(N,\tau)\bar{\otimes}\ell^2(\Gamma)$ by extending the cannonical action on elementary tensors of the form $n \otimes \xi \in L^2(N,\tau)\bar{\otimes}\ell^2(\Gamma)$. A procedure analogous to the group measure space construction yields *crossed product von Neumann algebra* $N \rtimes \Gamma \subset B(L^2(N,\tau)\bar{\otimes}\ell^2(\Gamma))$ which gives a Fourier expansion $N \rtimes \Gamma \ni m = \sum m_\gamma \gamma$ where $m_\gamma \in N$. If Γ is an icc group, $B \rtimes \Gamma$ is a II_1 factor with trace $x \mapsto \tau(x_e)$ When Γ acts trivially, $N \rtimes \Gamma \cong N\bar{\otimes}L(\Gamma)$.

2.4 Inclusions of Algebras

2.4.1 Index and the Basic Construction

Let us suppose $N \subset M$ is a unital inclusion of von Neumann algebras. A map $E_N : M \to N$ is a *conditional expectation* of M onto N if $E_N(n) = n$ for all $n \in N$ and $E_N(n_1 m n_2) = n_1 E_N(m) n_2$ for every $n_1, n_2 \in N$ and $m \in M$.

Theorem 2.9 (Umegaki)**.** *Let M be a II_1 factor with trace τ and let $N \subset M$ be a von Neumann subalgebra. Then there exists a unique normal conditional expectation $E_N : M \to N$ such that $\tau \circ E_N = \tau$.*

The proof roughly proceeds as follows:

Given a tracial von Neumann algebra (M,τ), we define $L^2(M)$ Let $N \subset M$ be an inclusion tracial of von Neumann algebras with trace τ. The GNS construction of N and M with respect to τ induces an embedding of Hilbert spaces $L^2(N,\tau) \subset L^2(M,\tau)$ which gives the existence of an orthogonal projection $e_N : L^2(M,\tau) \to L^2(N,\tau)$. This restricts to a projection $E_N : M \to N$ which can be shown to satisfy the axioms of a conditional expectation.

The *basic construction* [Ch79] $\langle M, e_N \rangle \subset \mathbb{B}(L^2(M))$ is the smallest von Neumann algebra generated by M and the orthogonal projection $e_N : L^2(M) \to L^2(N)$. $\langle M, e_N \rangle$ is endowed with a faithful semi-finite trace[1] Tr given by $Tr(xe_Ny) = \tau(xy)$, for all $x, y \in M$.

If $P \subset M$ is an inclusion of II_1 factors, then the *Jones index* of the inclusion, denoted by $[M : P]$ is the dimension of $L^2(M)$ as a left P-module. In the basic context of the basic construction, a finte index inclusion of II_1 factors will imply $\langle M, e_P \rangle$ a finite von Neumann algebra with trace Tr. Various properties for finite index inclusions of II_1 factors, most notably Jones' index rigidity theorem, can be found in the revolutionary work of V.F.R. Jones, [Jo81]. This definition was reformulated by Pimnser and Popa into a "probabilistic" notion of index to generalize index for arbitrary inclusions $P \subset Q$ of type II_1 von Neumann algebras. Pimnser and Popa's definition in fact recovers the classical Jones' index when $P \subset M$ is an inclusion of II_1 factors.

Definition 2.9 (Pimsner & Popa, [PP86])**.** Let (M, τ) be a tracial von Neumann algebra with a von Neumann subalgebra P. Let

$$\lambda = \inf \{ \|E_P(x)\|_2^2 / \|x\|_2^2 \; : \; x \in M_+ \}.$$

[1] Tr is a semi-finite trace if it is a normal, faithful, tracial linear functional. If the trace corresponds to a measure on a finte measure space, a semi-finite traces is analogous to a σ-finite measure space.

The *Pimnser-Popa index of the inclusion* $P \subseteq M$ is defined as $[M : P]_{\mathrm{PP}} = \lambda^{-1}$, under the convention that $\frac{1}{0} = \infty$.

Theorem 2.10 ([Jo81, PP86])**.** *Suppose $P \subset M$ is an inclusion of tracial von Neumann algebras. Then the following hold:*

1. *If $P \subset M$ is an inclusion of II_1 factors, then $[M : P]_{PP} = [M : P]$*
2. *If $[M : P]_{PP} < \infty$ and $p \in P$ is a projection, $[pMp : pPp] < \infty$;*
3. *If P is a II_1 factor and $[M : P]_{PP} < \infty$ then $P' \cap M$ is finite dimensional;*
4. *If $P \subset M$ is an inclusion of II_1 factors with $[M : P]_{PP} < \infty$, then $\dim_{\mathbb{C}}(P' \cap M) < \infty$.*

Let Γ a discrete group and Λ a finite index subgroup, it follows $[L(\Gamma) : L(\Lambda)]_{PP}$ is finite as well. The followring result, [CdSS15, Proposition 2.6], is the converse for certain inclusions of group von Neumann algebras. We include the essential technical lemmas and proofs here for completeness.

Lemma 2.11. *Let (M, τ) be a finite von Neumann algebra together with a projection $e \in M$ and a subset $S \subseteq \mathcal{U}(M)$. Given $\varepsilon > 0$, there exists $\eta > 0$ so that if there exists a function $\phi : S \to \mathbb{R}_+$ satisfying the following properties:*

1. *$\tau(exex^*) \leq \eta + \phi(x)$, for all $x \in S$;*
2. *for every $\delta > 0$ and every finite set $F \subset S$ there exists $u \in S$ such that $\phi(u^*y) \leq \delta$ for all $y \in F$.*

then $\tau(e) \leq \varepsilon$.

Corollary 2.12. *Let (M, τ) be a finite von Neumann algebra together with a projection $e \in M$ and a subset $S \subseteq \mathcal{U}(M)$. Assume for every $\varepsilon > 0$ there exists a function $\phi_\varepsilon : S \to \mathbb{R}_+$ satisfying the following properties:*

1. *$\tau(exex^*) \le \varepsilon + \phi_\varepsilon(x)$, for all $x \in S$;*
2. *for every $\delta > 0$ and every finite set $F \subset S$ there exists $u \in S$ such that $\phi_\varepsilon(u^*y) \le \delta$ for all $y \in F$.*

Then $e = 0$.

Proof. Applying the previous lemma, for every $\varepsilon > 0$ we have $\tau(e) \le \varepsilon$; thus $e = 0$. □

Proposition 2.13 (Proposition 2.6, [CdSS15])**.** *Let $\Omega \leqslant \Lambda \leqslant \Theta$ be groups. If there are $p \in \mathcal{P}(L(\Omega))$, $z \in \mathcal{P}(L(\Lambda)' \cap L(\Theta))$ so that $pz \neq 0$ and $pL(\Omega)pz \subseteq pL(\Lambda)pz$ admits a finite Pimsner-Popa basis then $[\Lambda : \Omega] < \infty$.*

Proof. Let $e \in \mathcal{Z}(L(\Omega))$ be the support projection of $E_{L(\Omega)}(z)$ and notice $z \le e$. For $t > 0$ denote by $e_t = 1_{(t,\infty)}(E_{L(\Omega)}(z)) \in \mathcal{Z}(L(\Omega))$ and notice e_t SOT-converges to e, as $t \to 0$. By assumption there exist elements $m'_1, m'_2, \ldots, m'_s \in pL(\Lambda)pz$ such that for every $x \in pL(\Lambda)pz$ we have $x = \sum_i E_{pL(\Omega)pz}(xm'^*_i)m'_i$. If we denote by $m_i = e_t m'_i e_t$, this further implies that for every $x \in pe_tL(\Lambda)pe_tz$ we have

$$x = \sum_i E_{pe_tL(\Omega)pe_tz}(xm_i^*)m_i. \tag{2.3}$$

Consider the map

$$L(\Lambda) \ni y \mapsto pE_{L(\Omega)}(yz)pE_{L(\Omega)}(e_tz)^{-1}e_tz \in L(\Omega)pe_tz,$$

where $E_{L(\Omega)}(e_tz)^{-1}$ is the inverse of $E_{L(\Omega)}(e_tz)$ under e_t. This map can be checked to be normal, $L(\Omega)pe_tz$-bimodular, and trace preserving, whence $E_{L(\Omega)pe_tz}(pe_tzype_tz) =$

$pE_{L(\Omega)}(yz)pE_{L(\Omega)}(e_tz)^{-1}e_tz$ for all $y \in L(\Lambda)$ by uniqueness of the conditional expectation. Therefore

$$\|E_{L(\Omega)pe_tz}(pe_tzype_tz)\|_2 \leq t^{-1}\|E_{L(\Omega)}(yz)\|_2$$

for all $y \in L(\Lambda)$. This together with (2.3) and basic approximations of m_i's further imply that for every $\varepsilon > 0$ one can find a constant $c_\varepsilon > 0$ and a finite subset $L_\varepsilon \subset \Lambda$ such that for every $x \in L(\Lambda)$ we have

$$\tau((pe_tz)x(pe_tz)x^*) \leq \varepsilon + c_\varepsilon \sum_{s\in L_\varepsilon} \|E_{L(\Omega)}(xs)\|_2^2. \tag{2.4}$$

Setting $S = \Omega$ and $\phi_\varepsilon(x) = c_\varepsilon \sum_{s\in L_\varepsilon} \|E_{L(\Omega)}(xs)\|_2^2$ we see (2.4) shows that property (1) in Corollary 2.12 is satisfied.

To finish, assume by contradiction $[\Lambda : \Omega] = \infty$. Since we have infinitely many representatives of left cosets of Ω in Λ then for every finite subset $F \subset \Omega$ there exists $\lambda \in \Lambda$ such that $E_{L(\Omega)}(u\lambda^{-1}\sigma) = 0$, for all $\sigma \in F$. This further shows ϕ_ε also satisfies (2) in Corollary 2.12 and hence $pe_tz = 0$. Since this holds for every $t > 0$ and e_t *SOT*-converges to $e \geq z$ we get $pz = 0$, which is a contradiction. Hence $[\Lambda : \Omega] < \infty$. □

Corollary 2.14. *For every infinite group Λ and every $p \in \mathcal{P}(L(\Lambda))$ the von Neumann algebra $pL(\Lambda)p$ is diffuse.*

Proof. Assuming otherwise, there exists a projection $0 \neq q \in \mathcal{Z}(L(\Lambda))$ so that $L(\Lambda)q = \mathbb{C}q$. By Proposition 2.13 this further implies Λ is a finite, contradicting the hypothesis. □

2.4.2 Relative Amenability

Let $P \subset M$ be an inclusion of von Neumann algebras. A state $\phi : M \to \mathbb{C}$ is said to be P-central if $\phi(mx) = \phi(xm)$ for every $x \in P$ and every $m \in M$. A tracial von Neuman algebra (M, τ) is *amenable* if there exists an M-central state $\phi : B(L^2(M)) \to \mathbb{C}$ so that $\phi|_M = \tau$. By the celebrated result of A. Connes, a von Neumann algebra is amenable if and only if it is approximately finite dimensional, i.e. $M = \varinjlim M_n$ for an increasing sequence of finite dimensional algebras M_n[Co76]. This reveals the difficulty in providing group-level invariants for group von Neumann algebras since every amenable icc group Γ will give $L(\Gamma) \cong \mathcal{R}$; strikingly different groups such as the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ and S_∞ will be indistinguishable at the level of the algebra.

Popa introduced relative notion for von Neumann algebras for inclusions of algebras $Q \subset M$ in the framework of bi-modules, [Po86]. Since amenable algebras are in a sense "small," relative amenability is a notion which states M is small relative to Q. This naturally occurs when the inclusion is of finite (Pimnser-Popa) index, or $M \subset M \rtimes \Gamma$ is the crossed product construction with an amenable group Γ. More recently, this construction can be extended to compare pairs of subalgebras $P, Q \subset M$ inside a common algebra, which we present in its fullest generality.

Definition 2.10. [OP07, Definition 2.2] Let (M, τ) be a tracial von Neumann algebra, $p \in M$ a projection, and $P \subset pMp, Q \subset M$ von Neumann subalgebras. We say that P is *amenable relative to Q inside M* if there exists a P-central state $\phi : p\langle M, e_Q\rangle p \to \mathbb{C}$ such that $\phi(x) = \tau(x)$, for all $x \in pMp$.

The classical notion of amenability is recovered when $Q = \mathbb{C}$. Furthermore, if Q is amenable, P is necessarily amenable. As an amenable group Neumann algebra corresponds to an amenable group, relative amenability is in the group von Neumann algebras is also reflected in the underlying group. Namely, if we have subgroups $\Lambda_1, \Lambda_2 < \Gamma$, Γ_1 is amenable relative to Λ_2 inside Γ if and only if $L(\Lambda_1)$ is amenable relative to $L(\Lambda_2)$ inside $L(\Gamma)$.

2.4.3 Popa's Intertwining Techniques

To describe the structure of von Neumann algebras, Popa introduced a powerful new conceptual framework: *deformation/rigidity theory*. This methodology includes a powerful criteria for identifying intertwiners between subalgebras of type II_1 factors, now called *Popa's intertwining-by-bimodules techniques*. Much of the recent progress in classifying von Neumann algebras can be largely attributed to this philosophy.

To motivate the definition of Popa's intertwining techniques, if we suppose $M = P_1 \bar{\otimes} P_1 = Q_1 \bar{\otimes} Q_2$ admits two different decomposition as II_1 factors, then what relationship exists between P_i and Q_i? A naive solution would be an identification $P_i = Q_i$, can be seen to be foolish by noticing $Q_i = u^* P_i u$ would be an alternate valid decomposition for M. Furthermore, every $t \in \mathbb{R}_+$ allows an identification of the algebras $P_1^t \bar{\otimes} P_2^{1/t} \cong P_1 \bar{\otimes} P_2$. Accounting for these obstacles, unique prime decomposition theorem would be of a statement of the form: if $M = P_1 \bar{\otimes} \cdots \bar{\otimes} P_m = Q_1 \bar{\otimes} \cdots \bar{\otimes} Q_n$ for some prime II_1 factors, then $n = m$ and $u^* P_i^{t_i} u = Q_i$ for some unitary $u \in \mathcal{U}(M)$

and $t_i \in R_+$ with $t_1 \cdots t_m = 1$. To establish such as result, Popa's approach was to first weaken the condition that u must be a unitary and instead create a condition which guarantees the existence of an embedding $\psi : P_1^t \to qQ_1q$ and a *partial isometry* $v \in \mathcal{J}(M)$such that the map $x \mapsto y$ satisfies $vx = yv$ for all $x \in P_1^t$

Theorem 2.15 (Popa, [Po03]). *Let (M, τ) be a separable tracial von Neumann algebra and P, Q be two (not necessarily unital) von Neumann subalgebras of M. The following are equivalent:*

1. *There exist non-zero projections $p \in P, q \in Q$, a $*$-homomorphism $\theta : pPp \to qQq$ and a non-zero partial isometry $v \in qMp$ such that $\theta(x)v = vx$, for all $x \in pPp$.*
2. *Let $\mathcal{G} \subset P$ be a group of unitaries generating P as a von Neumann algebra. There is no sequence $u_n \in \mathcal{G}$ satisfying $\|E_Q(xu_ny)\|_2 \to 0$, for all $x, y \in M$.*

If either of the equivalent conditions above hold, we say P *intertwines into Q over M*, denoted $P \preceq_M Q$. While the intertwining criteria is sufficient for many purposes, we require a stronger version of intertwining. Should $Pz \preceq_M Q$ for every $z \in P' \cap M$, then we say P *strongly intertwines into* Q and denote this property by $P \preceq^s_M Q$. The benefit of working with strong intertwining $\preceq^s$ is transitive where as intertwining in general need not be.

Fixing a trace preserving action $\Gamma \curvearrowright (B, \tau)$, S a collection of subgroups $M = B \rtimes \Gamma$, a set $\mathcal{F} \subset \Gamma$ is *small relative to* $\mathcal{S}$ if $\mathcal{F}$ is contained in a finite union of left/right translates of groups in $\mathcal{S}$, i.e. $g\Sigma h$ where $g, h \in \Gamma$ and $\Sigma \in \mathcal{S}$[BO08]. Given a set $\mathcal{F} \subset \Gamma$, $P_{\mathcal{F}}$ is the orthogonal projection from $L^2(M)$ to the closed linear span of

$\{b_g g : b_g \in B, g \in \mathcal{F}\}$. When $\mathcal{S}$ is a collection of normal groups, we need only consider right translates.

Definition 2.11. Whenever $\mathcal{V} \subset M$ is a norm-bouned subset of a von Neumann algebra M, we write $\mathcal{V} \subset_{approx} N \rtimes \mathcal{S}$ if for all $\varepsilon > 0$ there exists $\mathcal{F} \subset \Gamma$ that is small relative to $\mathcal{S}$ such that

$$\|b - \mathcal{P}_{\mathcal{F}}(b)\|_2 = \|\mathcal{P}_{\Gamma \setminus \mathcal{F}}(b)\|_2 \leq \varepsilon \quad \text{for all } b \in \mathcal{V}.$$

While various analyses examining the interplay between small sets and intertwining exist, we direct those with interest to [Va10]. We now demonstrate that if commuting algebras strongly intertwine into Q in the sense of Popa, $P_1, P_2 \preceq^s_M Q$ then the algebras they generate intertwine into Q, $(P_1 \vee P_2) \preceq_M Q$.

Proposition 2.16. *Let $\Gamma \curvearrowright (B, \tau)$ be a trace preserving action of a group on a II_1 factor B and denote by $M = B \rtimes \Gamma$. Suppose there exists normal subgroups $\Sigma_1, \ldots, \Sigma_k \lhd \Gamma$ and pairwise commuting subalgebras $A_1, \ldots, A_k \subset M$ so that $A_i \preceq^s_M B \rtimes \Sigma_i$. Then*

$$\bigvee_{i=1}^{k} A_i \preceq_M B \rtimes \Sigma_1 \cdots \Sigma_k.$$

Proof. Fixing $1 > \varepsilon > 0$, $A_1 \preceq^s_M B \rtimes \Sigma_1$ implies there exists a set $\mathcal{F}_1 \subset \bigcup_{i=1}^{j} \Sigma_1 \gamma_i$ small relative to Σ_1 so that

$$\|a_1 - \mathcal{P}_{\mathcal{F}_1}(a_1)\| < \varepsilon/2.$$

We may recursively choose $\mathcal{F}_i \subset \cup_{j=1}^{j_i} \Sigma_i \gamma_{j,i}$ small relative to Σ_i so that

$$\|a_i - \mathcal{P}_{\mathcal{F}_i}(a_i)\| \leq \varepsilon/(k j_1 \cdots j_{i-1})$$

whenever $a_i \in A_i$ with $\|a_i\| \leq 1$.

$$\|a_1 \cdots a_k - \mathcal{P}_{\mathcal{F}}(a_1 \cdots a_k)\|_2 \leq \|a_1 \cdots a_k - \mathcal{P}_{\mathcal{F}_1}(a_1) \cdots \mathcal{P}_{\mathcal{F}_k}(a_k)\|_2 \tag{2.5}$$

$$\leq \|(a_1 - \mathcal{P}_{\mathcal{F}_1}(a_1))(a_2 \cdots a_k)\|_2 \tag{2.6}$$

$$+ \|\mathcal{P}_{\mathcal{F}_1}(a_1)(a_2 \cdots a_k - \mathcal{P}_{\mathcal{F}_2}(a_1) \cdots \mathcal{P}_{\mathcal{F}_k}(a_k))\|_2$$

$$\leq \varepsilon/k + \|\mathcal{P}_{\mathcal{F}_1}(a_1)(a_2 \cdots a_k - \mathcal{P}_{\mathcal{F}_2}(a_1) \cdots \mathcal{P}_{\mathcal{F}_k}(a_k))\|_2. \tag{2.7}$$

Since $\mathcal{F}_1$ is contained j_1 left translates of Σ_1, $\|\mathcal{P}_{\mathcal{F}_1}(a_1)\|_2 \leq j_1\|a_1\|_2$. Thus previous inequality becomes

$$\|a_1 \cdots a_k - \mathcal{P}_{\mathcal{F}}(a_1 \cdots a_k)\|_2 \leq \varepsilon/k + j_1\|a_1\|\|a_2 \cdots a_k - \mathcal{P}_{\mathcal{F}_2}(a_1) \cdots \mathcal{P}_{\mathcal{F}_k}(a_k)\|_2$$

$$\leq \varepsilon/k + j_1\|(a_2 - \mathcal{P}_{\mathcal{F}_2}(a_2))(a_3 \cdots a_k)\|_2$$

$$+ j_1\|\mathcal{P}_{\mathcal{F}_2}(a_2)(a_3 \cdots a_k - \mathcal{P}_{\mathcal{F}_3}(a_3) \cdots \mathcal{P}_{\mathcal{F}_k}(a_k))\|_2$$

$$\leq \varepsilon/k + j_1\|a_2 - \mathcal{P}_{\mathcal{F}_2}(a_2)\|_2$$

$$j_1\|\mathcal{P}_{\mathcal{F}_2}(a_2)(a_3 \cdots a_k - \mathcal{P}_{\mathcal{F}_3}(a_3) \cdots \mathcal{P}_{\mathcal{F}_k}(a_k))\|_2$$

$$\leq 2\varepsilon/k + j_1 j_2\|a_3 \cdots a_k - \mathcal{P}_{\mathcal{F}_3}(a_3) \cdots \mathcal{P}_{\mathcal{F}_k}(a_k)\|_2$$

Repeated analysis will yield

$$\|a_1 \cdots a_k - \mathcal{P}_{\mathcal{F}}(a_1 \cdots a_k)\|_2 < \varepsilon. \tag{2.8}$$

Note $\mathcal{F} = \mathcal{F}_1 \cdots \mathcal{F}_k$ is contained in a finite union of right translates of $\Sigma_1, \ldots, \Sigma_k \lhd \Gamma$. Thus there exist $\lambda_1, \ldots, \lambda_n \in \Gamma$ so that

$$\mathcal{F} \subset \bigcup_{i=1}^{n} (\Sigma_1 \cdots \Sigma_k)\lambda_i.$$

Thus $\| \sum_{i=1}^n P_{\Sigma_1 \cdots \Sigma_k}(a_1 \cdots a_k \lambda_i)\|_2^2 \geq \|\mathcal{P}_{\mathcal{F}}(a_1 \cdots a_k)\|_2^2 > 1 - \varepsilon$ for every $a_i \in A_i$ with $\|a_i\| \leq 1$, and hence

$$\sum_{i=1}^n \|E_{B \rtimes \Sigma_1 \cdots \Sigma_k}(w\lambda_i)\|_2^2 \geq \| \sum_{i=1}^n E_{B \rtimes \Sigma_1 \cdots \Sigma_k}(w\lambda_i)\|_2^2 > 1 - \varepsilon$$

where w is any unitary of the form $w = u_1 \cdots u_k$ with $u_i \in \mathcal{U}(A_i)$. As unitaries of this form generate $A_1 \vee \cdots \vee A_k$, Theorem 2.15 establishes the result. □

2.5 Ultrapowers of von Neumann Algebras

An ultrafiler on $\mathbb{N}$ is a subset $\omega \subset \mathcal{P}(\mathbb{N})$ such that

1. $\emptyset \notin \omega$
2. If $A \subset B$ and $A \in \omega$, then $B \in \omega$
3. If $A, B \in \omega$, then $A \cap B \in \omega$
4. For all $A \in \mathcal{P}(\mathbb{N})$, then either $A \in \omega$ or $A^c \in \omega$.

Given a point $n \in \mathbb{N}$, the collection $\omega = \{A \in \mathcal{P}(\mathbb{N}) : n \in A\}$ forms a *principal ultrafilter*. An ultrafilter ω containing all co-finite set is called *co-final* or *non-principal*. The collection of all ultrafilters on $\mathbb{N}$ is in one-to-one correspondence with the Stone-Cech compactification of $\mathbb{N}$, denoted $\beta(\mathbb{N})$ and under this identification the non-principal ultrafilters correspond to $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. More generally, we may modify the definition of an ultrafilters to an ultrafilter on any partially ordered set $(S, \leqslant_S)$ by replacing containment $\subset$ with the partial ordering ordering $\leqslant_S$.

When $\omega \in \beta(\mathbb{N})$ is an ultrafilter on $\mathbb{N}$ and $\{(M_n, \tau_n)\}_{n=1}^\infty$ is a sequence of tracial von Neumann algebras, the *tracial ultrapoduct* is defined as the

$$(M_n, \tau_n)^\omega := \ell^\infty(\mathbb{N}, M_n)/\mathcal{I}_\omega(\mathbb{N}, M_n)$$

where $\ell^\infty(\mathbb{N}, M_n)$ is the C* algebra of bounded sequences of $\prod_{\mathbb{N}} M_n$ and $\mathcal{I}_\omega(\mathbb{N}, M_n)$ is the ideal consisting of all sequences such that $\lim_{n\to\omega} \tau_n(x_n^* x_n) = 0$, the so-called trace ideal. When we have a constant sequence of tracial von Neumann algebras($(M_n, \tau_n) = (M, \tau)$ for every $n \in \mathbb{N}$) we denote $M^\omega := (M_n, \tau_n)^\omega$ and is called the *ultrapower of* M. If (M, τ) is type II$_1$, then M^ω is a non-separable type II$_1$ von Neumann algebra with trace $\tau^\omega(x_n) = \lim_{n\to\omega} \tau(x_n)$ for any non-principal ultrafilter ω.

Ultrapower analysis of von Neumann algebras has been shown to be a fundamental tool to interpret asymptotic information into exact statements. For example, if M is a tracial von Neumann algebra then M embeds into M^ω by identifying each element with the constant sequence and notice $M' \cap M^\omega$ can be seen as the elements which asymptotically commute with M. Thus we can interpret Definition 2.7 as the following: M has Property Gamma if and only if $M' \cap M^\omega \neq \mathbb{C}1$ for every non-principal ultrafilter ω on $\mathbb{N}$. For our purposes, we modify an ultrapower argument of of Ioana for group von Neumann algebras [Io11, Theorem 3.1] to transfer the existence of commuting algebras commuting subgroups by taking an ultrapower along a partially ordered set. We elaborate the details of this analysis in Section 4.3.

CHAPTER 3
GEOMETRIC AND ALGEBRAIC ASPECTS OF GROUPS

In geometric group theory, Γ is a Gromov hyperbolic group if it is a finitely generated discrete group satisfying abstract characterizations of the notion of negative curvature of classical hyperbolic space. Building on the works of Minyev, Monod, and Shalom, this geometric structure of the groups is reinterpreted by Chifan and Sinclair into cohomological properties which encapsulate aspects of negative curvature at the level of representation theory. This perspective would allow the latter two authors to build s-malleable deformations in the sense of Popa.

We begin by presenting classical the classical and cohomological description of hyperbolic groups of Gromov and Minyev-Monod-Shalom, respectively.

3.1 Hyperbolic and Bi-Exact Groups

Definition 3.1. Suppose Γ be a finitely generated group with a set of generators S, and let X be the Cayley graph of Γ with respect to S. Γ is δ-*hyperbolic* if there exists $\delta > 0$ such that for any triple $x, y, z \in \Gamma$ $[x, z] \subset B_\delta([x, y]) \cup B_\delta([y, z])$, where $[a, b]$ denotes the geodesic from a to b relative to S.

The above property, δ-thin triangle condition, is a discrete analog of the negative curvature. While this property appears to rely upon a careful choice of the generating set, a well-known fact is that this property is independent of the generating set.

Hyperbolic groups, for our purposes, are partitioned into those commensurable to the integers $\mathbb{Z}$ (elementary hyperbolic groups) and those which contain a copy of the free group $\mathbb{F}_2$ (non-elementary). Examples of non-elementary hyperbolic icc groups include the non-abelian free groups $\mathbb{F}_n$, fundamental of genus $g \geq 2$ surfaces $\pi_1(\Sigma_g)$, and $PSL_2(\mathbb{Z})[\sqrt{2}]$.

A representation-theoretic approach to hyperbolic groups requires a brief detour towards cohomological properties of groups. The following is a brief description of the treatise given in [CS11]. Give orthogonal representation of a discrete group $\pi : \gamma \to \mathcal{O}(\mathcal{H}_\pi)$, a map $q : \gamma \to \mathcal{H}_\pi$ is a *quasi-1-cocycle* for π if there exists a constant D so that for every $\gamma, \lambda \in \Gamma$

$$\|q(\gamma\lambda) - q(\gamma) - \pi(\gamma)q(\lambda)\| < D.$$

Notice we recapture the classical definition of a 1-cocycle when $D = 0$. Given the context above, a map $q : \Gamma \to \mathcal{H}_\pi$ is an *array* if for every finite set $F \subset \Gamma$, there exists $K \geq 0$ such that for all $\gamma \in F$ and $\lambda \in \Gamma$

$$\|\pi(\gamma)(q(\lambda)) - q(\gamma\lambda)\| \leq K.$$

An array $q : \Gamma \to \mathcal{H}_\pi$ is *proper* if the set $\{\lambda : \|q(\lambda)\| < L\}$ is finite for every $L > 0$ and is *symmetric* if $\pi(\gamma)(q(\gamma^{-1})) = q(\gamma)$ for every $\gamma \in \Gamma$.

Definition 3.2 (Definition 1.6, CS11)**.** A group Γ is said to be in the class $\mathcal{QH}_{\text{reg}}$ if it admits a proper, symmetric array $q : \Gamma \to \mathcal{H}_\pi$ for some weakly-ℓ^2 representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$.

Proposition 3.1. *The following are true for the class* $\mathcal{QH}_{reg}$*:*

1. *All non-elementary hyperbolic groups belong to* $\mathcal{QH}_{reg}$*, [MMS04, Th09];*

2. *Lattices in a simple connected Lie group with real rank one belong to* $\mathcal{QH}_{reg}$*, [Sh00];*

3. $\mathcal{QH}_{reg}$ *is closed under free products, [Th09].*

This class is intimately related to Ozawa's class $\mathcal{S}$, the collection of all of bi-exact groups.

Definition 3.3 (Oz 02)**.** Γ is said to be bi-exact if there exist a sequence of maps $\mu_n : \Gamma \to \ell^1(\Gamma)$ such that $\|\mu_n(\gamma)\| = 1$ and $\lim \|\mu_n(s\gamma t) - \lambda_s \mu_n(\gamma)\| = 0$ for all $s, \gamma, t \in \Gamma$.

By [CS11, PV12], $\Gamma \in \mathcal{S}$ if and only if Γ is *exact* and $\Gamma \in \mathcal{QH}_{reg}$. We denote $\mathcal{S}_{nf}$ as the class of all icc groups in $\mathcal{S}$.

3.2 Finite Step Extensions of Groups

A fixed collection of discrete groups $\mathcal{C}$ may be enlarged by considering groups which can be formed by elementary extensions of groups in $\mathcal{C}$.

Definition 3.4. Let $\mathcal{C}$ be a fixed class of discrete groups. Γ is a *finite step extension by* $\mathcal{C}$ if there exists a chain

$$\Gamma_n \to \Gamma_{n-1} \to \cdots \to \Gamma_1 \to 1 \tag{3.1}$$

such that

- $\Gamma = \Gamma_n$

- $\Gamma_j \to \Gamma_{j-1}$ is an endomorphism with kernel in $\mathcal{C}$.

We denote the collection of all finite step extensions by $\mathcal{C}$ as $\mathrm{Quot}(\mathcal{C})$. When the value of n is minimal then $\Gamma \in \mathrm{Quot}_n(\mathcal{C})$.

If $\mathcal{C}$ is the collection of all finite cyclic groups, then $\Gamma \in \mathrm{Quot}(\mathcal{C})$ precisely defines the collection of all solvable groups.

Definition 3.5. Discrete groups Γ and Λ are *commensurable (or virtually isomorphic)* if there exist finite index subgroups $\Gamma_1 < \Gamma$, $\Lambda_1 < \Lambda$ such that $\Gamma_1 \cong \Lambda_1$. Γ and Λ are *commensurable up to finite kernel* f there exist finite index subgroups $\Gamma_1 < \Gamma$, $\Lambda_1 < \Lambda$, group Δ, and surjections $\varphi_\Lambda : \Lambda_1 \to \Delta$, $\varphi_\Gamma : \Gamma_1 \to \Delta$ such that $\ker(\varphi_\Gamma)$ and $\ker(\varphi_\Lambda)$ are finite.

It can be checked commensurability up to finite kernel is the smallest equivalence relation of discrete groups that is closed under both finite index groups and quotients under finite index normal subgroups. When working with a class $\mathcal{C}$ which is closed under commensurability (up to finite kernel), we need to modify slightly Definition 3.4.

Definition 3.6. If Γ is a discrete group such that there exists a chain

$$\Gamma_n \to \Gamma_{n-1} \to \cdots \to \Gamma_1 \to 1 \tag{3.2}$$

such that

1. Γ is commensurable to Γ_n,
2. $\Gamma_j \to \Gamma_{j-1}$ is an endomorphism with kernel in $\mathcal{C}_{\mathrm{rss}}$

then Γ an extension by C and say $\Gamma \in \mathrm{Quot}_n(\mathcal{C})$.

We alternatively characterize finite step extensions by $\mathcal{C}$ as those which are *poly*-$\mathcal{C}$: there exists a resolution

$$\Lambda_n \triangleright \Lambda_{n-1} \triangleright \cdots \triangleright \Lambda_1 \triangleright 1$$

where Γ is commensurable to Λ_n and $\Lambda_j/\Lambda_{j-1} \in \mathcal{C}$.

Proposition 3.2 ([CIK13, CKP14]). *Let $\mathcal{C}$ be a class of groups. The following hold:*

1. *If $\rho : \Lambda \to \Gamma$ with $\Gamma \in \mathrm{Quot}_n(\mathcal{C})$ and $\ker \rho \in \mathcal{C}$, then $\Lambda \in \mathrm{Quot}_{n+1}(\mathcal{C})$.*
2. *If $\Gamma_i \in \mathrm{Quot}_{n_i}(\mathcal{C})$, $\Gamma_1 \times \cdots \times \Gamma_k \in \mathrm{Quot}_{n_1+\cdots+n_k}(\mathcal{C})$.*
3. *If $\mathcal{C}$ is closed under commensurability (up to finite kernel), then so is $\mathrm{Quot}_n(\mathcal{C})$.*
4. *If $\Gamma \in \mathrm{Quot}_n(\mathcal{C})$ with $\pi_n : \Gamma_k \to \Gamma_{k-1}$ a family as in the definition of $\mathrm{Quot}_n(\mathcal{C})$ and $\rho_k := \pi_2 \circ \cdots \circ \pi_n$, then $\ker \rho_n \in \mathrm{Quot}_{n-1}(C)$.*

A generalization of part (2) of Proposition 3.2 may be verified using the following construction. Given any group $\Gamma \in \mathrm{Quot}_n(\mathcal{C})$ with $n > 2$ we may recursively define the following family of groups: First let $\Gamma_n^{(0)} = \Gamma_n$. For $0 < j \leq n-1$, suppose we have

$$\Gamma_n^{(j-1)} \xrightarrow{\pi_n} \Gamma_{n-1}^{(j-1)} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_{j+1}} \Gamma_j^{(j-1)} \to 1$$

with be a sequence of surjections with $\ker(\pi_i) \in \mathcal{C}$. Defining $\rho_k^{(j-1)} = \pi_{j+1} \circ \cdots \circ \pi_k$ and $\Gamma_k^{(j)} = \ker \rho_k^{(j-1)}$, by appropriately restricting π_k we now have

$$\Gamma_n^{(j)} \xrightarrow{\pi_n} \Gamma_{n-1}^{(j)} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_{j+1}} \Gamma_j^{(j)} = 1$$

is a chain satisfying the conditions implying $\Gamma_n^{(j)} \in \mathrm{Quot}_{n-j}(\mathcal{C})$. More generally we have for $0 \leq j \leq k \leq n$:

1. $\Gamma_k^{(j)} \in \mathrm{Quot}_{k-j}(\mathcal{C})$,

2. $\Gamma_k^{(j)} \triangleright \Gamma_k^{(j-1)}$,

3. $\Gamma_k^{(j)}/\Gamma_k^{(j-1)} \in \mathcal{C}$,

4. $\Gamma_n \triangleright \Gamma_n^{(1)} \triangleright \cdots \triangleright \Gamma_n^{(n-1)} \triangleright 1$ with $\Gamma_{n-1}^{(j)}/\Gamma_n^{(j+1)} \in \mathcal{C}$

Hence an equivalent characterization of $\Gamma \in \mathrm{Quot}_n(\mathcal{C})$ is Γ is *poly-*$\mathcal{C}$ with Hirsch length n. When $\mathcal{C}$ is the set of all abelian groups, $\mathrm{Quot}(\mathcal{C})$ is precisely collection of all solvable groups.

If $\mathcal{C}$ is a class of groups closed under commensurability (up to finite kernel), then we must modify the definition of $\mathrm{Quot}_n(\mathcal{C})$ as commensurabilty may introduce unexpected variability. For instance, if we take the family of all non-amenable free groups $\mathcal{F}$, naturally $\mathbb{F}_4 \in \mathrm{Quot}_1(\mathcal{F})$. The cannonical surjection $\mathbb{F}_4 \to \mathbb{F}_2$ demonstrates the fact $\mathbb{F}_4 \in \mathrm{Quot}_2(\mathcal{F})$. In general, $\mathbb{F}_{2n} \in \mathrm{Quot}_n(\mathcal{F})$. As all non-amenable free groups are commensurable, $\mathbb{F}_2 \in \mathrm{Quot}_n(\mathcal{F})$ for every n. Thus we impose the following minimality condition in the definition of $\mathrm{Quot}_n(\mathcal{C})$:

Definition 3.7. Let $\mathcal{C}$ be a class of groups closed under commensurabilty (up to finite kernel). $\Gamma \in \mathrm{Quot}(\mathcal{C})$ if there exists a chain of surjections

$$\Gamma_n \xrightarrow{\pi_n} \Gamma_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} \Gamma_1 \xrightarrow{\pi_1} 1$$

so that Γ is commensurable to Γ_n, $\ker(\pi_k) \in \mathcal{C}$. $\Gamma \in \mathrm{Quot}_n(\mathcal{C})$ if n is the smallest number such that Γ is a k-step extension by $\mathcal{C}$.

The following important classes of groups will for the elementary examples of a larger collection of groups

Definition 3.8. Denote by

- $\mathcal{F}$ is the collection of non-amenable free groups.
- $\mathcal{H}$ is the collection of non-elementary hyperbolic groups and non-trivial free products.
- $\mathcal{S}$ is the collection of bi-exact groups (exact groups in $\mathcal{QH}_{reg}$)
- $\mathcal{S}_{nf}$ is the collection of icc groups in Ozawa's class $\mathcal{S}$

We have the elementary conclusions of groups $\mathcal{F} \subset \mathcal{H} \subset \mathcal{S}_{nf} \subset \mathcal{S}$. The following groups are known to be poly-hyperbolic and/or poly-free:

Proposition 3.3. *The following groups are known to belong in these classes of groups.*

1. $\mathcal{H}$*, the class of all non-elementary hyperbolic groups*
2. *If* Γ *hyperbolic relative to the family of residually finite, exact, infinite, proper subgroups, then* $\Gamma \in \mathrm{Quot}_2(\mathcal{H})$*,[Os06, DGO11].*
3. *If* $k \geq 4$ *then* $\mathrm{Mod}(S_{0,k}) \in \mathrm{Quot}_{k-3}(\mathcal{F})$
4. *If* $k \geq 1$ *then* $\mathrm{Mod}(S_{1,k}) \in \mathrm{Quot}_{k}(\mathcal{F})$
5. *If* $k \geq 0$ *then* $\mathrm{Mod}(S_{2,k}) \in \mathrm{Quot}_{k+3}(\mathcal{F})$
6. $\tilde{B}_k, \tilde{P}_k \in \mathrm{Quot}_{k-2}(\mathcal{F})$*, [CKP14, Corollary 3.6].*

3.3 The Class $\mathrm{Quot}(\mathcal{C}_{\mathbf{rss}})$

The von Neumann algebras of hyperbolic icc groups, and more generally groups in generally groups in $\mathcal{QH}_{reg}$, are known to be *strongly solid*, [CS11], i.e. the normalizers of an amenable subalebra of $L(\Gamma)$ generate an amenable subalgebra. Using deformation/rigidity techniques, Popa and Vaes demonstrated the von Neumann algebras of an icc hyperbolic group admit the following dichotomy for amenable subalgebras:

Theorem 3.4 (PV09). *Let $\Gamma \curvearrowright (N,\tau)$ act by trace preserving action of a hyperbolic icc group and suppose $A \subset pN \rtimes \Gamma p$ is amenable relative to N inside $pN \rtimes \Gamma p$. Then either*

1. *$A \preceq_{N\rtimes\Gamma} N$ or*
2. *$\mathcal{N}_{pN\rtimes\Gamma p}(A)''$ is amenable relative to N inside $N \rtimes \Gamma$.*

We collect all non-amenable groups Γ whose von Neumann algebras display the dichotomy of Popa and Vaes into the following class of algebras

Definition 3.9. A group Γ belongs to the class $\mathcal{C}_{\text{rss}}$ if it is exact the following dichotomy holds: Assume $\Gamma \curvearrowright (N,\tau)$ is any trace preserving action on a tracial von Neumann algebra and denote by $M = N \rtimes \Gamma$. Let $p \in \mathcal{P}(M)$ be a projection and $A \subset pMp$ a von Neumann algebra that is amenable relative to B inside M. Then either

1. $A \preceq_M B$, or
2. $\mathcal{N}_{pMp}(A)''$ is amenable relative to N inside M.

We summarize the known properties of the class $\mathcal{C}_{rss}$:

1. Any weakly amenable group with positive first ℓ^2-Betti number such as $\mathbb{F}_n$, $(n \geq 2)$ belongs to $\mathcal{C}_{\text{rss}}$,[PV11, Theorem 3.1, Lemma 4.1, and Theorem 7.1].
2. Any weakly amenable, non-amenable, bi-exact group (the weakly amenable groups in the class S_{nf}) belongs to $\mathcal{C}_{\text{rss}}$; [PV12, Theorem 3.1].

3. $\mathcal{C}_{rss}$ is closed under commensurability up to finite kernel, [VV14].

Hence both the poly-hyperbolic and poly-free groups form a sub-collection of the poly-$\mathcal{C}_{\rm rss}$. Before we establish structural results of the von Neumann algebras associated to the class Quot($\mathcal{C}_{\rm rss}$), we require additional algebraic properties of the groups in $\mathcal{C}_{rss}$.

CHAPTER 4
TECHNOLOGY AND PREREQUISITE RESULTS

4.1 Prime, Solid, and Strongly Solid von Neumann Algebras

To examine the internal structure of a von Neumann algebra, Popa proposed an investigation into demonstrability properties of von Neumann algebras. Popa was specifically interested in finding examples of *prime* II_1 factors, von Neumann algebras which do not admit a decomposition tensor product of II_1 subalgebras. The first von Neumann algebra known to exhibit this criterion was $L(\mathbb{F}_\infty)$, the non-separable group von Neumann algebra of the free group with uncountably many generators. Ge subsequently extended Popa's result to the algebras of all non-amenable free groups via Voiculescu's free entropy theory. Popa's deformation/rigidity theory is useful in deducing sources of prime von Neumann algebras. A prerequisite for establishing the unique prime decomposition for hyperbolic groups is a stronger restriction of the internal structure of the von Neumann algebra.

A von Neumann algebra M is *solid* if any diffuse subalgebra $D \subset M$ has amenable relative commutant $D' \cap M$. Ozawa established the solidity for the von Neumann algebras of hyperbolic groups via an argument which relies on a surprising interplay C*-algebraic and von Neumann algebraic techniques[Oz02]. By recasting this problem in the context of deformation/rigidity, Popa provided an argument for the solidity of the of the free group factors [Po06], an act which would spur further progress proving primness and other structural results for various von Neumann algebras[CH08, CI08, Pe09, CSU11, BHR12, Is12, Is14, CIK13, BC14].

Chifan and Sinclair's cohomological characterization of bi-exact groups allowed for a remarkable improvement of the respective solidity results of Ozawa, Peterson and Popa. Developing a methodology which unifies the previous works, Chifan and Sinclair implemented cohomological data of bi-exact groups in the deformation/rigidity framework to prove these algebras are *strongly solid*: the normalizer $\mathcal{N}_M(A)$ of an amenable subalgebra $A \subset M$ will remain amenable. Their result further implies the absence of Cartan subalgebras for all bi-exact icc groups; a result previously know only for the free group factors [CS11].

When the von Neumann algebra is non-amenable, we naturally have strong solidity implies solidity, which in turn implies primeness. We exploit this internal rigidity of II_1 factors arising from hyperbolic to recover the

4.2 Co-Multiplication of von Neumann Algebras

The works in [PV09, Io10, IPV10] developed the deformation/rigidity analysis of group von Neumann algebras through usage of diagonal embeddings of the form $\Delta : M \hookrightarrow M\bar{\otimes}M$. In [Io10], Ioana provided the framework for analysis of this co-multiplication style embeddings for crossed product von Neumann algebras of Bernoulli actions by Property (T) groups. This analysis was subsequently applied to generalized Bernoulli crossed product von Neumann algebras to construct the first examples of W*-superrigid groups[IPV10].

In its simplest form, the co-multiplication of a group von Neumann algebra

$L(\Gamma)$ is the algebra $*$-homomorphism $\Delta : L(\Gamma) \to L(\Gamma)\bar{\otimes}L(\Gamma)$ given by

$$\gamma \mapsto \gamma \otimes \gamma,$$

where $\gamma \in L(\Gamma)$ are the canonical group unitaries generating $L(\Gamma)$ as a von Neumann algebra. In [CIK13, CKP14], the authors make use of a generalized co-multiplication map induced by a group homomorphism. A group homomorphism $\rho : \Gamma \to \Lambda$ lifts to a $*-$homormorphism of von Neumann algebras $\Delta : L(\Gamma) \to L(\Gamma)\bar{\otimes}L(\Lambda)$ by extending the map

$$\gamma \mapsto \gamma \otimes \rho(\gamma)$$

where $\{\gamma\}_{\gamma\in\Gamma}, \{\lambda\}_{\lambda\in\Lambda}$ are the canonical unitaries of $L(\Gamma)$ and $L(\Lambda)$, respectively. When $\rho : \Gamma \to \Gamma$ is the identity, this is precisely the compultiplication map along Γ. For a group $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$, we will consider the group homomorphism $\rho_n : \Gamma_n \to \Gamma_1$ defined by the concatenation of the homomorphisms witnessing Γ as an extension by $\mathcal{C}_{\mathrm{rss}}$.

When $L(\Gamma) \cong L(\Lambda)$, section 4.3 makes use of the comultiplication map $\Delta : M \to M\bar{\otimes}M$ along Λ and compare $\Delta(\Gamma)$ against this description. In the situation where $\Gamma = \Gamma_1 \times \Gamma_2$ is a product of hyperbolic groups, we analyze all possible embeddings in the sense of Popa of the subalgebras of large commuting subalgebras $L(\Gamma_1) \subset L(\Gamma)$ into $L(\Sigma)$, where Σ belongs to an arbitrary family of subgroups of Λ, and determine criteria deduce if $\Delta(\Gamma_i) \preceq_{M\bar{\otimes}M} L(\Sigma)\bar{\otimes}L(\Sigma)$. Reversing the comultiplication then will allow us to conclude if $L(\Gamma_1) \preceq L(\Sigma)$ for some proper subgroup $\Sigma < \Gamma$.

4.3 Discretization of the Commutation Relation

This portion provides the crucial step in recovering of the product structure for any group which is W* equivalent to a product of hyperbolic groups. In particular, we demonstrate if $\Gamma = \Gamma_1 \times \Gamma_2$ is a product of icc hyperbolic groups such that $L(\Gamma) \cong L(\Lambda)$, then Λ contains two non-amenable commuting subgroups Λ_1, Λ_2 such that $L(\Gamma_i)$ embeds into $L(\Lambda_i)$ in the sense of Popa. To do so, we make use of a powerful ultrapower technique pioneered by Ioana which transforms analytic data (the existence of large commuting algebras) to discrete structures (commuting subgroups)[Io11].

Theorem 4.1. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be an n-fold product of icc bi-exact groups and denote by $M = L(\Gamma)$. If $t > 0$ and Λ is an arbitrary group such that $M^t \cong L(\Lambda)$, then for every non-empty family $\mathcal{G}$ of subgroups of Λ there exists $1 \leq j \leq n$ so that either:*

1. *$L(\widehat{\Gamma}_k)^t \preceq_{M^t} L(\Sigma)$ for some $\Sigma \in \mathcal{G}$, or*
2. *$L(\Gamma_j)^t \preceq_{M^t} L(\Omega)$ where $\Omega = \cup_{m\geq 1} C_\Lambda(\Sigma_m)$ for some descending sequence of subgroups $\{\Sigma_m\}_{m=1}^{\infty}$ such that $\Sigma_m \notin \mathcal{G}$ for all m.*

Proof. Let $\{\gamma\}_{\gamma\in\Gamma}$ and $\{\lambda\}_{\lambda\in\Lambda}$ denote the canonical unitaries Γ and Λ unitaries generating M and M^t, respectively. Define a comultiplication map $\Delta : M^t \to M^t \bar{\otimes} M^t$ by linearly extending the map $\lambda \mapsto \lambda \otimes \lambda$. Note we may view $M^t \bar{\otimes} M^t = L(\Gamma)^t \bar{\otimes} L(\Gamma)^t$. Since we may write $\Gamma \cong \widehat{\Gamma}_j \times \Gamma_j$ for every $1 \leq j \leq n$, then for every integer $k \geq t$ and projection $p \in M^t$ with trace $s = t/k$ we may identify

$$L(\Lambda) = M^t = (M^k)^s = L(\widehat{\Gamma}_j)^k \bar{\otimes} L(\Gamma_j)^s.$$

With these identifications in mind, we have

$$\Delta(L(\Gamma_j)^s), \Delta(L(\widehat{\Gamma}_j)^k) \subset \Delta(L(\widehat{\Gamma}_j)^k \bar{\otimes} L(\Gamma_j)^s) \subset L(\widehat{\Gamma}_j)^k \bar{\otimes} L(\Gamma_j)^s.$$

By [BO08, Lemma 15.3.3] Γ is bi-exact relative to the family

$$\left\{ \Gamma \times \widehat{\Gamma}_i, \widehat{\Gamma}_j \times \Gamma : 1 \leq i, j, \leq n \right\},$$

by [BO08, Theorem 15.1.5] there exists i have either

1. $\Delta(L(\widehat{\Gamma}_1)^t) \preceq_{M^k \bar{\otimes} M^k} M^k \bar{\otimes} L(\widehat{\Gamma}_i)^k$, or
2. $\Delta(L(\widehat{\Gamma}_1)^t) \preceq_{M^k \bar{\otimes} M^k} L(\widehat{\Gamma}_i)^k \bar{\otimes} M^k$.

Symmetry allows us to treat the case $\Delta(L(\widehat{\Gamma}_1)^t) \preceq_{M^k \bar{\otimes} M^k} M^k \bar{\otimes} L(\widehat{\Gamma}_i)^k$ where $1 \leq i \leq n$ is fixed for the remainder of the proof.

By Popa's intertwining techniques, there exists $F \subset M^k$ finite and $c' \geq 0$ so that

$$\sum_{x \in F} \| E_{M^k \bar{\otimes} L(\hat{\Gamma}_i)^k} (\triangle(y) \cdot (1 \otimes x)) \|_2^2 \geq c', \text{ for all } y \in \mathcal{U}(L(\hat{\Gamma}_1)^t).$$

Writing $y \in L(\widehat{\Gamma}_1)^t$ as $y = \sum y_\lambda \lambda$, the previous equation implies

$$\sum_{x \in F} \sum_{\lambda \in \Lambda} |y_\lambda|^2 \| E_{L(\widehat{\Gamma}_1)} (\lambda x) \|_2^2 \geq c, \text{ for all } y \in \mathcal{U}(L(\widehat{\Gamma}_1)^t). \tag{4.1}$$

Now let $\mathcal{G}$ be an arbitrary family of subgroups of Λ and recall that a set $S \subset \Lambda$ is small relative to $\mathcal{G}$ if it is contained in finitely many left-right translates of elements of $\mathcal{G}$. If it were the case that $L(\widehat{\Gamma}_1)^t \nprec_{M^t} L(\Sigma)$ for every $\Sigma \in \mathcal{G}$, it follows from Popa's intertwining techniques that for each $\varepsilon > 0$ and $S \subset \Gamma$ small relative to $\mathcal{G}$ there exists $y \in \mathcal{U}(L(\widehat{\Gamma}_1)^t)$ so that $\sum_{\lambda \in S} |y_\lambda|^2 < \varepsilon$. Indexing over all sets S small relative to $\mathcal{G}$, it follows that for any $y \in \mathcal{U}(L(\widehat{\Gamma}_1)^t)$

$$\sup_{\Lambda \setminus S} \|E_{L(\hat{\Gamma}_i)^k}(\lambda x)\|_2^2 \geq \left[\sum_{\lambda \in \Lambda \setminus S} |y_\lambda|^2 \|E_{L(\hat{\Gamma}_i)^k}(\lambda x)\|_2^2\right] \left[\sum_{\lambda \in \Lambda \setminus S} |y_\lambda|^2\right]^{-1}, \tag{4.2}$$

whence

$$\begin{aligned} \sum_{x \in F} \sum_{\lambda \in \Lambda \setminus S} |y_\lambda|^2 \|E_{L(\hat{\Gamma}_i)^k}(\lambda x)\|_2^2 \geq & \sum_{x \in F} \sum_{\lambda \in \Lambda} |y_\lambda|^2 \|E_{L(\hat{\Gamma}_i)^k}(\lambda x)\|_2^2 \\ & - \sum_{x \in F} \sum_{\lambda \in S} |y_\lambda|^2 \|E_{L(\hat{\Gamma}_i)^k}(\lambda x)\|_2^2 \\ \geq & c - |F| \max_{x \in F} \|x\|_2^2 \sum_{x \in S} |y_\lambda|^2. \end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small and choosing $y \in \mathcal{U}(L(\widehat{\Gamma}_1)^t)$ so that $\sum_{\lambda \in S} |y_\lambda|^2 < \varepsilon$ we see

$$\sum_{x \in F} \sum_{\lambda \in \Lambda \setminus S} |y_\lambda|^2 \|E_{L(\hat{\Gamma}_i)^k}(\lambda x)\|_2^2 \geq c - |F| \max_{x \in F} \|x\|_2^2 \varepsilon > 2^{-1} c > 0.$$

Using this together with (4.2) we see that for every S small with respect to $\mathcal{G}$ we have

$$\sup_{\lambda \in \Lambda \setminus S} \sum_{x \in F} \|E_{L(\hat{\Gamma}_i)^k}(\lambda x)\|_2^2 \geq 2^{-1}(1-\varepsilon)^{-1} c > 2^{-1} c. \tag{4.3}$$

In the spirit of [Io11, Theorem 3.3], we let I be the partially ordered set of all $S \subset \Lambda$ so that S is small relative to $\mathcal{G}$ and fix a co-final ultrafilter on I. Letting $\Omega = \Lambda \cap \theta \Lambda \theta$ for some $\theta \in \prod_{S \in \omega} \Lambda \setminus S$, we assume by way of contradiction

$L(\Gamma_i)^t \not\preceq_{M^t} L(\Theta)$. This implies $L(\Gamma_i)^s \not\preceq_{M^t} L(\Omega)$ which, by Popa's intertwining techniques, yields the existence of a sequence $\{y_n\} \subset \mathcal{U}(L(\widehat{\Gamma}_1)^s)$ such that

$$\|E_{L(\Omega)}(xy_ny)\|_2^2 \to 0 \tag{4.4}$$

as $n \to \infty$ for all $x, y \in M^t$. Let $\mathcal{K} \subset L^2((M^k)^\omega)$ be the closed span of $\left\{M^k p\theta p M^k\right\}$ and $P_{\mathcal{K}} : L^2((M^k)^\omega) \to \mathcal{K}$ be the orthogonal projection onto $\mathcal{K}$. Proceeding as in the proof of [Io11, Theorem 3.1], equation (4.4) implies $\langle y_n \xi y_n^*, \eta\rangle \to 0$ for all $\xi, \eta \in \mathcal{K}$. Defining $v_\theta = (v_S)_S$ where $v_S \in \Lambda \setminus S$ satisfies inequality (4.3), we have that $\sum_{x\in F} \|E_{(L(\hat{\Gamma}_i)^k)^\omega}(v_\theta x)\|_2 \geq c/2$. Take $x \in F$ so that $x' = E_{(L(\hat{\Gamma}_i)^k)^\omega}(v_\theta x) \neq 0$ and $\xi_0 = P_{\mathcal{K}}(x')$. Then $\|v_\theta x - x'\|_2 < \|v_\theta x\|_2 = \|x\|_2$ which in turn gives $\|v_\theta x - \xi_0\|_2 < \|v_\theta x\|_2$, showing $\xi_0 \neq 0$. Since $[L(\Gamma_i)^s, L(\hat{\Gamma}_i)^k] = 1$, we have $\|\xi_0\|_2 = \langle y_n y_n^* \xi_0, \xi_0\rangle = \langle y_n \xi_0 y_n^*, \xi_0\rangle \to 0$, contradicting $\xi_0 \neq 0$. Thus $L(\Gamma_i)^t \preceq_{M^t} L(\Theta)$. Proceeding as in the proof of [Io11, Theorem 3.1] there exists a descending sequence of subgroups $\Sigma_s \leqslant \Lambda$ such that $\Sigma_s \notin \mathcal{G}$ and $\Theta = \cup_s \Omega_s$, where $\Omega_s = C_\Lambda(\Sigma_s)$ and the conclusion and part (2) then follows. □

We close this chapter with the fundamental result: the location of commuting non-amenable groups which will deconstruct the target group Λ as a direct product.

Corollary 4.2. *Let Γ, Λ be as in the hypothesis of Theorem 4.1. There exists a non-amenable group $\Sigma \subset \Lambda$ with non-amenable centralizer $C_\Lambda(\Sigma)$ and $1 \leq k \leq n$ so that $L(\widehat{\Gamma}_k)^t \preceq_{M^t} L(\Sigma)$.*

Proof. Let $\mathcal{G}$ be the collection of all subgroups Σ with non-amenable centralizer

$C_\Lambda(\Sigma)$; this is a non-empty family since $C_\Lambda(\{e\}) = \Lambda$ is non-amenable by construction. Applying Theorem 4.1, there exists $1 \leq k \leq n$ such that either

1. $L(\widehat{\Gamma}_k)^t \preceq_{M^t} L(\Sigma)$ for some $\Sigma \in \mathcal{G}$, or

2. there exists a descending sequence of subgroups of Λ, $\Sigma_m \notin \mathcal{G}$ such that $L(\Gamma_k)^t \preceq_{M^t} L(\Omega)$ where $\Omega = \cup_{m\geq 0} C_\Lambda(\Sigma_m)$.

If we assume the second case were to hold, this would imply $C(\Sigma_m)$ is an ascending tower of amenable groups which implies Ω is amenable. But this would imply $L(\Gamma_k)^s \not\preceq_{M^t} L(\Omega)$ since $L(\Gamma_k)$ is a non-amenable factor, contradicting our assumption.

Thus we must have the first case. Moreover, since $L(\widehat{\Gamma}_k)$ is a factor with no amenable direct summand, Σ is a non-amenable group with non amenable centralizer.

$\square$

We will revisit the previous result in Chapter 5 to show the groups Σ and $C_\Lambda(\Sigma)$ will almost form a direct product decomposition of Λ.

4.4 Structural Properties for Groups in $\mathrm{Quot}(\mathcal{C}_{\mathbf{rss}})$

We devote this section to explore the consequences of the interplay between the analytic and group-theoretic properties of the class $\mathrm{Quot}(\mathcal{C}_{rss})$. We demonstrate that if a group $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$ is an icc group is virtually generated by k commuting infinite subgroups, then those groups can be perturbed to commensurable groups which (virtually) decompose Γ as a direct product. Moreover, we necessarily have that $k \leq n$ and the groups $\Sigma_1, \ldots, \Sigma_k$ are also in the class $\mathrm{Quot}(\mathcal{C}_{\mathrm{rss}})$. This provides a partial converse to part (2) of Proposition 3.2. Notably, this provides group-level evidence of Theorem 1.3.

The following lemma is essentially contained in the base case for the inductive argument; we include the proof for instructive purposes and convenience of the reader.

Lemma 4.3. *Let $p \in L(\Gamma)$ be a projection and $A, B \subset pL(\Gamma)p$ be two diffuse commuting subalgebras of $pL(\Gamma)p$ with $\Gamma \in \mathcal{C}_{rss}$. Then $[pL(\Gamma)p : A \vee B]_{PP} = \infty$.*

Proof. Since $\Gamma \in \mathcal{C}_{\text{rss}}$ non-amenable, no diffuse corner of $L(\Gamma)$ is amenable. Thus if we assume by way of contradiction that $[pL(\Gamma)p : A \vee B]_{PP} < \infty$. Thus, either A or B must be non-amenable. By symmetry, we shall assume A is a non-amenable von Neumann algebra.

Note that if we take an amenable von Neumann subalgebra $A_0 \subset A$, A_0 is amenable relative to $\mathbb{C}$. The dichotomy for normalizers of the class $\mathcal{C}_{\text{rss}}$ then implies either

1. $A_0 \preceq_{L(\Gamma)} \mathbb{C}$, or
2. $\mathcal{N}_{pL(\Gamma)p}(A_0)''$ is amenable (relative to $\mathbb{C}$).

Note (1) cannot hold since A_0, and hence every corner of A_0, is diffuse. Thus $\mathcal{N}_{pL(\Gamma)p}(A_0)''$, which contains B, is amenable. Since amenability is sub-hereditary, B is also amenable and hence amenable relative to $\mathbb{C}$. Again applying the dichotomy for normalizers of the class $\mathcal{C}_{\text{rss}}$ we obtain

3. $B \preceq_{L(\Gamma)} \mathbb{C}$, or
4. $\mathcal{N}_{pL(\Gamma)p}(B)''$ is amenable (relative to $\mathbb{C}$).

If (4) holds, this would imply $A \subset \mathcal{N}_{pL(\Gamma)p}(B)''$ is an amenable subalgebra, a contradiction. However, (3) would imply B has an atomic corner, once again contradicting our assumption. Thus we cannot have two diffuse commuting subalgebras generating $pL(\Gamma)p$. □

As a corollary, we obtain a group-level structural property.

Corollary 4.4. *Suppose $\Gamma \in \mathcal{C}_{rss}$. If $\Sigma_1, \Sigma_2 < \Gamma$ are commuting subgroups such that $[\Gamma : \Sigma_1\Sigma_2] < \infty$, then either Σ_1 or Σ_2 is finite. More generally, Γ is not virtually decomposible as a product of two infinite groups.*

Proof. If we assume to the contrary $[\Gamma : \Sigma_1\Sigma_2] < \infty$, then $L(\Sigma_1)$ and $L(\Sigma_2)$ would be diffuse commuting subalgebras of $L(\Gamma)$ which generate a finite index subalgebra of $L(\Gamma)$, contradicting Proposition 4.3. □

We pause to note this result follows strictly from the analytic properties of the class $\mathcal{C}_{\mathrm{rss}}$. In the proof of Lemma 4.3, we proceeded by repeatedly exploiting the dichotomy property to obtain the following *trichotomy*

1. $A_0 \preceq_{L(\Gamma)} \mathbb{C}$, or
2. $B \preceq_{L(\Gamma)} \mathbb{C}$, or
3. $\mathcal{N}_{pL(\Gamma)p}(B)$ is amenable (relative to $\mathbb{C}$).

The recovery of the product structure for the groups in $\mathrm{Quot}(\mathcal{C}_{\mathrm{rss}})$ (Theorem 1.3) will follow from analysis of a relative version of the above trichotomy. One can readily verify case (3) leads to a contradiction, a result which occur even in the general setting of relative amenabilty (see Section 5.2).

While the result as stated implies the group vo Neumann algebras of groups in $\mathcal{C}_{\text{rss}}$ are prime, we actually obtain a much stronger property for the algebras of this class of groups: if $\Gamma \in \mathcal{C}_{\text{rss}}$ then the normalizer of any diffuse amenable subalgebra $D \subset L(\Gamma)$ generates an amenable subalgebra of $L(\Gamma)$. Thus groups in $\mathcal{C}_{\text{rss}}$ generate *strongly solid* von Neumann algebras (see Chapter 4).

Proposition 4.5. *Let* $\Gamma \in \text{Quot}_n(\mathcal{C}_{rss})$ *and suppose there exist commuting subgroups* $\Sigma_1, \Sigma_2 < \Gamma$ *generating a finite index subgroup of* Γ *with* Σ_2 *is amenable. Then* Σ_2 *is finite.*

Proof. When $n = 1$, the result follows immediately from 4.4. Thus we assume the result holds for all groups Γ in $\text{Quot}(\mathcal{C}_{\text{rss}})$ which are at most $n-1 \geq 0$ step extensions by $\mathcal{C}_{\text{rss}}$. Suppose $\Gamma \in \text{Quot}_n(\mathcal{C}_{\text{rss}})$ and

$$\Gamma_n \overset{\pi_n}{\twoheadrightarrow} \Gamma_{n-1} \overset{\pi_{n-1}}{\twoheadrightarrow} \cdots \overset{\pi_2}{\twoheadrightarrow} \Gamma_1 \overset{\pi_1}{\twoheadrightarrow} 1 \tag{4.5}$$

is a chain witnessing $\Gamma \in \text{Quot}_n(\mathcal{C}_{\text{rss}})$. Since Γ is commensurable to Γ_n, we may assume $\Sigma_1, \Sigma_2 < \Gamma_n$ are commuting subgroups which generate Γ_n. Letting $\rho_n = \pi_2 \circ \cdots \circ \pi_n$ be the concatenation of the homomorphisms in (4.5), we see $\rho_n(\Sigma_1), \rho(\Sigma_2) < \Gamma_1 \in \mathcal{C}_{\text{rss}}$ are commuting subgroups. By Corollary 4.4, either $\rho_n(\Sigma_1)$ or $\rho_n(\Sigma_2)$ is finite. We claim $\rho_n(\Sigma_2)$ must be finite. Indeed, since Σ_2 is amenable, $\rho_n(\Sigma_2)$ which means $\rho(\Sigma_2)$ is non-amenable and therefore infinite. Hence $\rho(\Sigma_1)$ is finite.

Thus passing to a finite index subgroup of $\Lambda_2 = \ker(\rho_n|_{\Sigma_2}) < \Sigma_2$ and restricting $\rho_n : \Sigma_1 \times \Lambda_2 \to \Gamma_1$, we have ρ_n surjects onto Γ_1 with $\ker(\rho_n) = \Lambda_1\Lambda_2 = \Gamma_n^{(1)}$ where $\Lambda_1 = \ker(\rho_n|_{\Sigma_1})$. Since Λ_2 is commensurable Σ_2, Λ_2 is amenable. By the induction hypothesis, Λ_2 is finite gives Σ_2 is also finite. $\square$

This *strong solidity* propagates through the class $\mathrm{Quot}(\mathcal{C}_{\mathrm{rss}})$ in a relative form which yields an upper bound on the number of large commuting infinite subgroups which exist simultaneously in a given group. Moreover, the class $\mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$ exhibits a surprising closure property. Namely, if $\Gamma \in \mathrm{Quot}(\mathcal{C}_{\mathrm{rss}})$ with commuting infinite subgroups $\Sigma_1, \ldots, \Sigma_k < \Gamma$ such that $[G : \Sigma_1 \cdots \Sigma_k] < \infty$, then the groups $\Sigma_i \in \mathrm{Quot}(\mathcal{C}_{\mathrm{rss}})$.

Proposition 4.6. $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$. *Suppose there exist infinite groups such that* Γ *is commensurable to* $\Sigma_1 \times \Sigma_2$*. Then we may find* $n_1, n_2 > 0$ *such that* $n_1 + n_2 = n$ *with* $\Sigma_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{rss})$.

Proof. As $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$, there exists a chain

$$\Gamma_n \overset{\pi_n}{\twoheadrightarrow} \Gamma_{n-1} \overset{\pi_{n-1}}{\twoheadrightarrow} \cdots \overset{\pi_2}{\twoheadrightarrow} \Gamma_1 \overset{\pi_1}{\twoheadrightarrow} 1 \tag{4.6}$$

with Γ commensurable to Γ_n. As $\mathrm{Quot}_j(\mathcal{C}_{rss})$ is closed under commensurability, we may assume $\Sigma_1 \times \Sigma_2 = \Gamma_n$ by passing to a finite index subgroup.. Denote by $\rho_n : \Gamma_n \to \Gamma_1$ the concatenation of the homomorphisms $\rho_n = \pi_2 \circ \cdots \circ p_n$ given in (4.6).

Suppose $n = 2$. Since $\pi_2 = \rho_2$ is surjective, $\rho_2(\Sigma_1), \rho_2(\Sigma_2)$ are commuting subgroups which generate $\Gamma_1 \in \mathcal{C}_{\mathrm{rss}}$, which by Corollary 4.4 implies either $\rho_2(\Sigma_1)$ or $\rho_2(\Sigma_2)$ is finite. By symmetry, we assume $\rho_2(\Sigma_1)$ is finite. Thus $\ker(\rho_2|_{\Sigma_1}) = \Lambda_1 < \Sigma_1$ is a finite index normal subgroup of Σ_1 such that $\Lambda_1 \times \Sigma_2 < \Gamma_n$ is a finite index inclusion of groups. Moreover, restricting $\rho_2 : \Lambda_1 \times \Sigma_2 \to \Gamma_1$ yields $\ker(\rho_2) = \Lambda_1 \times \ker(\rho_2|_{\Sigma_2}) \in \mathcal{C}_{\mathrm{rss}}$. Noticing Λ_1 is infinite gives $\ker(\rho_2|_{\Sigma_2})$ is finite. Thus we pass to a finite index subgroup $\Lambda_2 < \Sigma_2$ and once again restrict $\rho_2 : \Lambda_1 \times \Lambda_2$ to obtain $\ker(\rho_2) = \Lambda_1 \in \mathcal{C}_{\text{rss}}$ and $\Lambda_2 \cong \Gamma_1 \in \mathcal{C}_{\text{rss}}$. As $\mathcal{C}_{\text{rss}}$ is closed under commensurability, we have both $\Sigma_1, \Sigma_2 \in \mathcal{C}_{\text{rss}} = \text{Quot}_1(\mathcal{C}_{\text{rss}})$.

Now assume the conclusion holds up to some integer $n - 1 \in \mathbb{N}$ for all groups $\Lambda \in \text{Quot}_{n-1}(\mathcal{C}_{\text{rss}})$. In the setting of (4.6) with $\rho_n = \pi_2 \circ \cdots \circ p_n$, we have that $\rho_n(\Sigma_1), \rho_n(\Sigma_2)$ are commuting groups generating Γ_1. Lemma 4.3 implies either $\rho_n(\Sigma_1)$ or $\rho_n(\Sigma_2)$ is finite; by symmetry we only need to consider the case where $\rho_n(\Sigma_1)$ is finite. Thus $\ker(\rho_n|_{\Sigma 1}) = \Lambda_1 < \Sigma_1$ and $\Lambda_1 \times \Sigma_2 < \Gamma_1$ are finite index inclusions of groups. We restrict $\rho_n : \Lambda_1 \times \Sigma_2$ and note $\ker(\rho_n) = \Lambda_1 \times \ker(\rho_n|_{\Sigma_2}) = \Gamma_n^{(1)} \in \text{Quot}_{n-1}(\mathcal{C}_{\text{rss}})$. If $\ker(\rho_n|_{\Sigma_2})$ were finite, then Λ_1 is commensurable to $\Gamma_n^{(1)} \in \text{Quot}_{n-1}(\mathcal{C}_{\text{rss}})$. Hence we conclude $\Lambda_1 \in \mathcal{C}_{\text{rss}}$ and $\Sigma_2 \in \mathcal{C}_{\text{rss}}$. Since Λ_1 is commensurable to Σ_1, we reach the desired conclusion. Suppose now $\Lambda_1, \ker(\rho_n|_{\Sigma_2})$ are both infinite groups. By the induction hypothesis there must exists integers $j, k > 0$ such that $\Lambda_1 \in \text{Quot}_j(\mathcal{C}_{\text{rss}})$ and $\ker(\rho_n|_{\Sigma_2}) \in \text{Quot}_k(\mathcal{C}_{\text{rss}})$. Thus letting $n_1 = j$ and $n_2 = k + 1$ we see $\Lambda_i \in \text{Quot}_{n_i}(\mathcal{C}_{\text{rss}})$, whence by commensurabilty implies $\Lambda_i \in \text{Quot}_{n_i}(\mathcal{C}_{\text{rss}})$.

□

Corollary 4.7. *Let* $\Gamma \in \text{Quot}_n(\mathcal{C}_{rss})$ *and suppose* Γ *is commensurable to* $\Sigma_1 \times \Sigma_2$ *with* $\Sigma_1 \in \text{Quot}_j(\mathcal{C})$ *for some* $j \in \mathbb{N}$. *Then either* $n = j$ *and* Σ_2 *is finite, or* $j < n$ *and* $\Sigma_2 \in \text{Quot}_{n-j}(\mathcal{C}_{rss})$.

Proof. By the minimality constraint in Definition 3.7, we naturally have $j \leq n$. If $j = n$ and Σ_2 were infinite, Proposition 4.6 yields $\Sigma_i \in \text{Quot}_{n_i}(\mathcal{C})$ for some $n > n_i \geq 1$, once again contradicting minimality.

Now suppose $1 \leq j < n$. If $n = 2$, by Proposition 4.6, we have the result. We momentarily define $\mathrm{Quot}_0(\mathcal{C}_{\mathrm{rss}})$ as the collection of all finite groups. Proceeding as in the proof of the previous proposition, we have either

1. $\Gamma_n^{(1)}$ is commensurable to $\Sigma_1 \times \Lambda_2$
2. $\Gamma_n^{(1)}$ is commensurable to $\Lambda_1 \times \Sigma_2$,

where $\Lambda_i = \ker(\rho_n|_{\Sigma_i})$. In case (1), $\Sigma_1 \in \mathrm{Quot}_a(\mathcal{C}_{\mathrm{rss}})$ and $\Lambda_2 \in \mathrm{Quot}_b(\mathcal{C}_{\mathrm{rss}})$, for $a, b \geq 0$ with $a + b = n - 1$. Hence $\Sigma_2 \in \mathrm{Quot}_{b+1}(\mathcal{C}_{\mathrm{rss}})$ By minimality $a \geq j$. If $a > j$, this would imply $\Gamma \in \mathrm{Quot}_{j+b+1}(\mathcal{C}_{\mathrm{rss}})$ where $j + b + 1 < n$ once again contradicting minimality. In case (2), a similar argument will guarantee $\Lambda \in \mathrm{Quot}_{j-1}(\mathcal{C}_{\mathrm{rss}})$ and thus $\Sigma \in \mathrm{Quot}_{n-j}(\mathcal{C}_{\mathrm{rss}})$. □

As a corollary, the following holds true for groups in $\mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$.

Corollary 4.8. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$. If there exist infinite subgroups $\Sigma_1, \dots, \Sigma_k < \Gamma$ such that $\Sigma_1 \times \cdots \times \Sigma_k < \Gamma$ is a finite index inclusion of groups, then we may find integers $n_1, \dots, n_k > 0$ so that $\Sigma_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{rss})$ with $\sum_{i=1}^{k} n_i = n$.*

Proof. If $k = 2$, then this is exactly the statement in Corollary 4.6. Now suppose the result holds up to some integer $k = 1$.

Let $\Lambda_1 = \Sigma_1 \times \cdots \times \Sigma_{k-1}$ and $\Lambda_2 = \Sigma_2$. Applying Corollary 4.6 to $\Gamma = \Lambda_1 \times \Lambda_2$ gives integers $m_1, m_2 > 0$ with $n = m_1 + m_2$ so that $\Lambda_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{\mathrm{rss}})$. Now since $\Lambda_1 = \Sigma_1 \times \Sigma_{k-1} \in \mathrm{Quot}_{m_1}(\mathcal{C}_{\mathrm{rss}})$ is a $k - 1$-fold product of infinite groups, we again apply Corollary 4.6 to obtain integers $n_1, \dots, n_{k-1}$ with $n_1 + \cdots + n_{k-1} = m_1$. Letting

$n_k = m_2$ will give the conclusion. □

We may in fact remove the product structure and obtain the following generalization of Corollary 4.7. The proof of the this theorem is omitted as it follows almost by a sequence of arguments almost identical to those which establish Proposition 4.6 and Corollary 4.7.

Theorem 4.9. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$. If there exist infinite pairwise commuting subgroups $\Sigma_1, \ldots, \Sigma_k < \Gamma$ such that $\Sigma_1 \cdots \Sigma_k < \Gamma$ is a finite index inclusion of groups, then we may find integers $n_1, \ldots, n_k > 0$ so that $\Sigma_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{rss})$ with $\sum_{i=1}^k n_i = n$.*

The collection of groups $\mathrm{Quot}(\mathcal{C}_{\mathrm{rss}})$ exhibit a remarkable closure property for large subgroups which give credence to the product rigidity statement of Theorem 1.4. We close this chapter with a result demonstrating a weak intersection property for large commuting subgroups of $\Gamma \in \mathrm{Quot}(\mathcal{C}_{\mathrm{rss}})$.

Theorem 4.10. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$. If there exist infinite pairwise commuting subgroups $\Sigma_1, \ldots, \Sigma_k < \Gamma$ such that $\Sigma_1 \cdots \Sigma_k < \Gamma$ is a finite index inclusion of groups, then $\Sigma_i \cap \Sigma_j$ is finite for every $i \neq j$, $1, \leq i, j, \leq k$. Moreover, if Γ is icc then $\Sigma_1 \cdots \Sigma_k = \Sigma_1 \times \cdots \times \Sigma_k < \Gamma$*

Proof. Let $\Lambda = \langle \Sigma_i \cap \Sigma_j : i \neq j, 1, \leq i, j, \leq k \rangle$ be the group generated by the intersections and note this is an abelian, and hence amenable, subgroup of Γ. Thus Λ is finite by Proposition 4.5.

Now if Γ is an icc group, Λ will be finite; otherwise elements $\lambda \in \Lambda$ will have orbits under conjugation of size at most $|\lambda^\Gamma| \leq [\Gamma : \Sigma_1 \cdots \Sigma_k]$ contradicting the icc criterion. □

CHAPTER 5
MAIN RESULTS

We devote this chapter to establishing the main results cataloged in Chapter 1. The first section is divided into the following parts, establishing the main theorem, a proof which removes the weak amenability assumption for only two groups, and finally an application of product rigidity for lattices in Lie groups.

When $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is a product of icc hyperbolic groups, or weakly amenable icc groups in $\Gamma_i \in \mathcal{S}_{nf}$ with $L(\Gamma) \cong L(\Lambda)$, then the previous chapter gave a non-trivial verification of the existence of commuting non-amenable subgroups in Λ. Section 5.1 exploit solidity and strong solidity of the generating groups Γ_i to upgrade these groups to an honest direct product decomposition of Λ.

In Section 5.2 we adapt the techniques used to reconstruct the direct product to develop a product rigidity to the class $\text{Quot}_n(\mathcal{C}_{\text{rss}})$ and generalize the main theorem of [CKP14]. To this end, we establish a set of claims which parallel the group theoretic properties for groups in this class which appear as virtual products of infinite groups.

5.1 Product Rigidity for Hyperbolic Groups

We return our attention of the primary question of rigidity for group von Neumann algebras: if two groups Γ and Λ give rise to isomorphic von Neumann algebras, then what properties are common to both Γ and Λ. An elementary rigidity result easily verified is if Γ is an icc group, then so is Λ. Indeed since $L(\Gamma)$ is a factor, so is $L(\Lambda)$ and hence Λ is an icc group. Moreover, if Γ is an icc amenable group, the

the same must hold for Λ. However, these rigidity results do not recover fundamental properties about the group.

The following theorem of Ozawa and Popa became the impetus for the first product rigidity theorem of Chifan, Sincair and myself.

Theorem 5.1. *([OP03, Corollary 3]) Suppose* $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ *and* $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_m$ *are each products of hyperbolic icc groups. If* $L(\Gamma) \cong L(\Lambda)$*, then* $n = m$ *and there exit* $t_1, \cdots, t_n > 0$ *with* $t_1 \cdots t_k = 1$ *such that after permutation of indices and unitary conjugacy* $L(\Gamma_i)^{t_i} \cong L(\Lambda_i)$*.*

This unique prime decomposition theorem, so called due to the structural parallel with the fundamental theorem of arithmetic, along with the insights derived from Ioana's ultrapower "discretization" technique for group von Neumann algebras, [Io11], form the principles which allow us to recover the product structure of the group Λ after any removing all assumptions on this group.

Before we establish Theorem 1.1, which forms the basis for the remainder of the document, let us first examine the technicalities present in the statement of the theorem. One may ask if we can in fact deduce a group-level isomorphism theorem. Leading by example, let $\Gamma = \mathbb{F}_2 \times \mathbb{F}_9$, $\Lambda = \mathbb{F}_5 \times \mathbb{F}_3$ and recall that for any pair of II_1 factors M, N and $t > 0$ we have the canonical identification $M \bar{\otimes} N \cong M^s \bar{\otimes} N^{1/s}$. Applying Voiculescu's compression formula for the free group factors [Vo90] gives the following identification

$$L(\Gamma) = L(\mathbb{F}_2) \bar{\otimes} L(\mathbb{F}_9) \cong L(\mathbb{F}_2)^{1/2} \bar{\otimes} L(\mathbb{F}_9)^2 \cong L(\mathbb{F}_3) \bar{\otimes} L(\mathbb{F}_5) = L(\Lambda).$$

Thus one cannot obtain a sharper group-theoretic identification for a class containing the hyperbolic groups (more generally, the same holds the class $\mathcal{S}_{nf}$) and thus our results provides the most generals statement one can obtain for the class of hyperbolic groups.

If we assume the hypothesis that Γ is an n-fold product of hyperbolic groups such that $L(\Gamma)^t \cong L(\Lambda)$, the proof of Theorem 1.1 is divided into the following main steps:

1. Show the $L(\Lambda)$ cannot be generated by two commuting diffuse algebras with one amenable ($L(\Gamma)$ is virtually McDuff)
2. Locate Σ_1, Σ_2 commuting subgroups which generate finite index subgroup of Λ
3. Pass to groups commensurable with Σ_1 and Σ_2 to induce a splitting of Λ as a direct product.
4. Partition $\{1, \ldots, n\}$ into $S \cup T$ such that $L(\Gamma_S) \cong L(\Lambda_1)^s$ and $L(\Gamma_T) \cong L(\Gamma_S) \cong L(\Lambda)^{t/s}$ where the identification is via a unitary.
5. Induct this process to obtain a splitting of the identification $L(\Gamma_S) \cong L(\Lambda_1)^s$ and $L(\Gamma_T) \cong L(\Lambda)^{t/s}$.

We mimic this procedure to deduce a similar result for the class $\text{Quot}(\mathcal{C}_{\text{rss}})$.

Lemma 5.2. *For $n \geq 2$ let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n \in \mathcal{S}_{nf}$ with $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ and denote by $M = L(\Gamma)$. Let $t > 0$ be a scalar and let $Q, A \subset M^t$ be commuting subalgebras such that A is diffuse abelian and $Q = T_1 \vee T_2 \vee \cdots \vee T_{n-1}$, where $T_i \subset M^t$ are commuting, non-amenable II_1 subfactors. Then $Q \vee A \not\prec_{M^k} L(\hat{\Gamma}_s)^k$ for all integers $1 \leq s \leq n$ and $k \geq t$.*

Proof. Assume by contradiction $Q \vee A \preceq_{M^k} L(\hat{\Gamma}_s)^k$, for some $1 \leq s \leq n$ and $k \geq t$. Then there must exist a projection $p \in \mathcal{P}(Q \vee A)$, a scalar $k \geq t_1 > 0$, and a unital injective $*$-homomorphism $\phi : p(Q \vee A)p \to L(\hat{\Gamma}_s)^{t_1}$. Since Q is a II_1 factor we have $\mathcal{Z}(Q \vee A) = A$ and denote by $E_A : Q \vee A \to A$ the central trace. Since $0 \neq E_A(p)$ there exist $\mu > 0$ and a projection $0 \neq e \in A$ so that $E_A(pe) \geq \mu e$. Moreover, since T_1 is a II_1 factor there exists a projection $r \in T_1 \subseteq Q$ so that $\tau_Q(r) = \mu$ and $E_A(re) = \mu e$. Thus $E_A(pe) \geq E_A(re)$ and since E_A is a central trace there exists $w \in \mathcal{I}(Q \vee A)$ so that $pe \geq w^*w$ and $ww^* = re$. Letting $u \in \mathcal{U}(Q \vee A)$ with $w = reu$ one can check that $\phi' = \phi \circ ad(u^*) : re(Q \vee A)re \to L(\hat{\Gamma}_s)^{t_1}$ is an injective $*$-homomorphism; moreover, cutting $L(\hat{\Gamma}_s)^{t_1}$ by a projection we can assume ϕ' is unital. By construction $A_1 = \phi'(Are)$ is diffuse abelian, and $T_i^1 = \phi'(reT_ire) \subset L(\hat{\Gamma}_s)^{t_1}$ are commuting, non-amenable II_1 subfactors. Moreover $A_1, Q_0 = T_1^1 \vee \cdots \vee T_{n-1}^1 \subseteq L(\hat{\Gamma}_s)^{t_1}$ are commuting subalgebras. By [CSU11, Theorem 6.1], one can find $s_1 \in \{1, ..., n\} \setminus \{s\}$ so that $Q_1 \vee A_1 \preceq_{L(\hat{\Gamma}_s)^k} L(\hat{\Gamma}_{\{s,s_1\}})^k$, where $Q_1 = T_1^1 \vee \cdots \vee T_{n-2}^1$. Applying the previous argument $n-2$ times one can find $1 \leq s_{n-1} \leq n$, $k \geq t_{n-1} > 0$, and commuting subalgebras $A_{n-1}, Q_{n-2} \subset L(\Gamma_{s_{n-1}})^{t_{n-1}}$ with A_{n-1} diffuse abelian and Q_{n-2} non-amenable II_1 factor. This however contradicts solidity of $L(\Gamma_{s_{n-1}})$. □

Lemma 5.3. *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n \in \mathcal{S}_{nf}$ with $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ and denote by $M = L(\Gamma)$. If $t > 0$ is a scalar and $Q, A \subset M^t$ are commuting subalgebras such that A is diffuse amenable then $[M^t : A \vee Q]_{PP} = \infty$.*

Proof. Assume by contradiction $[M^t : A \vee Q]_{PP} < \infty$. Thus Q is non-amenable and by [CSU11, Theorem 6.1] there exist a subset $F_1 \subset \{1, ..., n\}$ with $|F_1| = n-1$ and an integer $t \leq k$ such that $A \preceq_{M^k} L(\Gamma_{F_1})^k$. Using [CKP14, Proposition 2.4] one can find $k \geq t_1 > 0$ and diffuse amenable subalgebra $A_1 \subset L(\Gamma_{F_1})^{t_1}$ so that $[L(\Gamma_{F_1})^{t_1} : A_1 \vee Q_1]_{PP} < \infty$ where $Q_1 = A_1' \cap L(\Gamma_{F_1})^{t_1}$. Also note that Q_1 is non-amenable. Applying the same argument recursively, after $n-1$ steps one can find $F_{n-1} \subset \{1, ..., n\}$ with $|F_{n-1}| = 1$, $t_{n-1} > 0$, and diffuse amenable subalgebra $A_{n-1} \subset L(\Gamma_{F_{n-1}})^{t_{n-1}}$ so that $[L(\Gamma_{F_{n-1}})^{t_{n-1}} : A_{n-1} \vee Q_{n-1}]_{PP} < \infty$, where $Q_{n-1} = A_{n-1}' \cap L(\Gamma_{F_{n-1}})^{t_{n-1}}$. Since $L(\Gamma_{F_{n-1}})$ is solid and A_{n-1} is diffuse it follows Q_{n-1} is amenable. Hence $A_{n-1} \vee Q_{n-1}$ is amenable and by [PP86, Lemma 2.3] and [OP07, Proposition 2.4] we conclude that $L(\Gamma_{F_{n-1}})^{t_{n-1}}$ is amenable. By factoriality this further implies that $\Gamma_{F_{n-1}}$ is amenable which is a contradiction. □

Theorem 5.4. *For $n \geq 2$, let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n \in \mathcal{S}_{nf}$ be weakly amenable with $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ and denote by $M = L(\Gamma)$. Let $t > 0$ be a scalar and let Λ be an arbitrary group such that $M^t = L(\Lambda)$. Then there exist commuting, non-amenable, icc subgroups $\Sigma_1, \Sigma_2 < \Lambda$ such that $[\Lambda : \Sigma_1\Sigma_2] < \infty$.*

Proof. The proof is quite technically involved, so it will be divided into a series of claims. Throughout proof we will be denoting by $\{u_\lambda\}_{\lambda\in\Lambda}$ the canonical group unitaries implementing $L(\Lambda) = M^t$. By Corollary 4.2 there exists non-amenable subgroup $\Sigma < \Lambda$ with non-amenable centralizer $C_\Lambda(\Sigma)$ such that $L(\hat{\Gamma}_\ell)^t \preceq_{M^t} L(\Sigma)$ for some $1 \leq \ell \leq n$. For simplicity, we assume $\ell = n$ and denote by $P = L(\hat{\Gamma}_n)^t$, $N = L(\Gamma_n)$, noticing that $M^t = P\bar{\otimes}N$. Thus, there exist $p \in \mathcal{P}(P)$, $q \in \mathcal{P}(L(\Sigma))$, $v \in \mathcal{J}(M^t)$, and an injective $*$-homomorphism $\phi : pPp \to qL(\Sigma)q$ so that

$$\phi(x)v = vx \text{ for all } x \in pPp. \tag{5.1}$$

Moreover, we can assume that $q = vv^*$. For ease of notation let $Q = \phi(pPp)$.

Claim 5.5. *Without loss of generality we may assume that $Q \subset qL(\Sigma)q$ is a finite index inclusion of II_1 factors.*

Proof of Claim 5.5. By [CKP14, Proposition 2.4] we can assume that $[qL(\Sigma)q : Q \vee (Q' \cap qL(\Sigma)q)]_{PP} < \infty$.

We first show that $Q' \cap qL(\Sigma)q$ is purely atomic. Assume by contradiction there exists a diffuse corner $r(Q' \cap qL(\Sigma)q)r$. Hence there exists a diffuse, abelian subalgebra $A \subseteq r(Q' \cap qL(\Sigma)q)r$. However, since $C_\Lambda(\Sigma)$ is non-amenable and $L(C_\Lambda(\Sigma))$ commutes with $Q \vee A$, then [CSU11, Theorem 6.1] implies $Q \vee A \preceq_{M^k} L(\hat{\Gamma}_s)^k$, for some integers $1 \leq s \leq n$ and $k \geq t$, contradicting Lemma 5.2.

As a consequence we note that by choosing a nonzero minimal, central projection q' of $Q' \cap qL(\Sigma)q$ and replacing ϕ with $\phi'(x) := \phi(x)q'$ in (5.1) above, we can assume w.l.o.g. that $[qL(\Sigma)q : Q]_{PP} < \infty$. Moreover, letting $v' := q'v \neq 0$ instead of v above, one can check that all previous equations still hold, including (5.1). Since Q is a II_1 factor and $[qL(\Sigma)q : Q]_{PP} < \infty$, Theorem 2.10 (3) implies that $Q' \cap qL(\Sigma)q$ is finite dimensional. Since $\mathcal{Z}(qL(\Sigma)q) \subseteq Q' \cap qL(\Sigma)q$, $\mathcal{Z}(qL(\Sigma)q)$ is finite dimensional as well. Thus, multiplying v above by a minimal, central projection of $\mathcal{Z}(qL(\Sigma)q)$ and using Theorem 2.10 (1) the claim obtains. ∎

Claim 5.6. *There is a nonzero projection $f \in \mathcal{P}(L(\Sigma)' \cap M^t)$ such that*

$$f(L(\Sigma) \vee (L(\Sigma)' \cap M^t))f \subseteq fM^t f$$

is a finite Jones index inclusion of II_1 factors.

Proof of Claim 5.6 Performing the downward basic construction [Jo81, Lemma 3.1.8], there exists a projection $e \in \mathcal{P}(qL(\Sigma)q)$ and a II_1 subfactor $R \subseteq Q \subseteq qL(\Sigma)q = \langle Q, e\rangle$ such that $[Q : R] = [qL(\Sigma)q : Q]$ and $Re = eL(\Sigma)e$. Keeping with the same notation, by relation (5.1) the restriction $\phi^{-1} : R \to pPp$ is an injective $*$-homomorphism such that $T = \phi^{-1}(R) \subseteq pPp$ is a finite Jones index subfactor and

$$\phi^{-1}(y)v^* = v^*y, \text{ for all } y \in R. \tag{5.2}$$

Let $\theta' : Re \to R$ be the $*$-isomorphism given by $\theta(xe) = x$. Letting $v' = ev$, we first check that $v' \neq 0$. Indeed, suppose by contradiction that this is not the case. Since $vv^* \in Q' \cap qM^tq$ we would have $vv^*xe = xvv^*e = 0$ for all $x \in Q$, whence as $\langle Q, e\rangle e = Qe$, the same holds true for all $y \in \langle Q, e\rangle$. However, since 1 is the central support of e in $\langle Q, e\rangle$, this would imply that $vv^* = 0$, a contradiction. It follows that $Re = eL(\Sigma)e$ together with (5.2) shows that $\theta = \phi^{-1} \circ \theta' : eL(\Sigma)e \to pPp$ is an injective $*$-homomorphism satisfying $\theta(eL(\Sigma)e) = T$ and

$$\theta(y)(v')^* = (v')^*y, \text{ for all } y \in eL(\Sigma)e. \tag{5.3}$$

Let w be the partial isometry in the polar decomposition of v', so that equation (5.3) holds with v' replaced by w. Notice that $w^*w \in (T' \cap pPp)\bar{\otimes}N$. Since every projection in $(T' \cap pPp)\bar{\otimes}N$ is equivalent to a projection in $\mathcal{Z}(T' \cap pPp)\bar{\otimes}N$, we can assume that $w^*w \in \mathcal{Z}(T' \cap pPp)\bar{\otimes}N$. Since $[pPp : T] < \infty$ then $T' \cap pPp$ is finite dimensional and so is $\mathcal{Z}(T' \cap pPp)$. Thus, replacing the partial isometry w by $wr_0 \neq 0$, for some minimal projection $r_0 \in \mathcal{Z}(T' \cap pPp)$, we see that all relations above still hold including relation (5.3). Moreover, we can assume that $w^*w = z_1 \otimes z_2$, for some nonzero projections $z_1 \in \mathcal{Z}(T' \cap pPp)$ and $z_2 \in N$. Using relation (5.3) we get

$$w^* L(\Sigma) w = \theta(eL(\Sigma)e) w^* w = T z_1 \otimes z_2. \tag{5.4}$$

Since $T \subseteq pPp$ is finite index inclusion of II_1 factors then by the local index formula [Jo81] it follows $Tz_1 \subseteq z_1 P z_1$ is a finite index inclusion of II_1 factors as well. Also, we have

$$(w^* L(\Sigma) w)' \cap (z_1 \otimes z_2) M^t (z_1 \otimes z_2) = ((Tz_1)' \cap z_1 P z_1) \bar{\otimes} z_2 N z_2. \tag{5.5}$$

Altogether, the previous relations imply that

$$\begin{aligned} T z_1 \bar{\otimes} z_2 N z_2 &\subseteq T z_1 \vee (T z_1' \cap z_1 P z_1) \bar{\otimes} z_2 N z_2 \\ &= w^* L(\Sigma) w \vee w^* (L(\Sigma)' \cap M^t) w \\ &= w^* L(\Sigma) w \vee \big((w^* L(\Sigma) w)' \cap (z_1 \otimes z_2) M^t (z_1 \otimes z_2) \big) \\ &\subseteq z_1 P z_1 \bar{\otimes} z_2 N z_2. \end{aligned} \tag{5.6}$$

Since $Tz_1 \subseteq z_1 P z_1$ if a finite index inclusion of II_1 factors then by Theorem 2.10 (4) so is $T z_1 \bar{\otimes} z_2 N z_2 \subseteq z_1 P z_1 \bar{\otimes} z_2 N z_2$. Letting $f = ww^*$ and $u \in \mathcal{U}(M^t)$ be a unitary such that $w^* = uww^* = uf$, relation (5.6) further implies that $f(L(\Sigma) \vee (L(\Sigma)' \cap M^t))f = L(\Sigma)f \vee f(L(\Sigma)' \cap M^t)f \subseteq fM^t f$ is an inclusion of finite Pimsner-Popa index. Moreover (5.6) and Theorem 2.10 (6) further imply that $\dim_{\mathbb{C}}(\mathcal{Z}(f(L(\Sigma) \vee (L(\Sigma)' \cap M^t))f)) \leq [z_1 P z_1 \bar{\otimes} z_2 N z_2 : T z_1 \bar{\otimes} z_2 N z_2] < \infty$. Thus shrinking f if necessary, we can assume that

$$f(L(\Sigma) \vee (L(\Sigma)' \cap M^t))f \subseteq fM^t f \tag{5.7}$$

is a finite index inclusion of II_1 factors. ■

For the remainder of the proof it will be convenient to introduce the following notation.

Notation 5.7. *For $\lambda \in \Lambda$, recall λ^Σ is the orbit of λ under the conjugation by Σ and let $\Omega := \mathcal{V}_\Lambda(\Sigma) = \{\lambda \in \Lambda : |\lambda^\Sigma| < \infty\}$ and notice it is a subgroup of Λ normalized by Σ. Let $\{\mathcal{O}_i : i \in \mathbb{N}\}$ be a (countable) enumeration of all the finite orbits of action by conjugation of Σ on Λ and notice $\Omega = \cup_i \mathcal{O}_i$. Note that $\Omega_k := \langle \cup_{i \leq k} \mathcal{O}_i \rangle$ is an ascending sequence of subgroups of Ω normalized by Σ such that $\cup_k \Omega_k = \Omega$.*

Next, we show the following:

Claim 5.8. $[\Lambda : \Omega\Sigma] < \infty$.

Proof of Claim 5.8. By construction we have $L(\Sigma)' \cap M^t \subseteq L(\Omega)$ and hence $f(L(\Sigma) \vee (L(\Sigma)' \cap M^t))f \subseteq fL(\Omega\Sigma)f$. Proceeding as in the proof of Equation (5.7), shrinking f more if necessary, we obtain that $fL(\Omega\Sigma)f \subseteq fM^t f = fL(\Lambda)f$ is a finite index inclusion of II_1 factors. By Theorem 2.10 (5) it follows that the inclusion $fL(\Omega\Sigma)f \subseteq fM^t f = fL(\Lambda)f$ has a finite Pimsner-Popa basis and hence the claim follows from Proposition 2.13. ■

Claim 5.9. *$\Sigma \cap \Omega$ is finite.*

Proof of Claim 5.9. Let $\mathcal{O}_i' = \mathcal{O}_i \cap \Sigma$ and notice $\Sigma \cap \Omega = \cup_i \mathcal{O}_i'$. For each k let $R_k = \langle \cup_{i \leq k} \mathcal{O}_i' \rangle$ and notice it forms an ascending sequence of normal subgroups of Σ such that $\cup_k R_k = \Sigma \cap \Omega$ and $[\Sigma : S_k] < \infty$, where $S_k = C_\Sigma(R_k)$. Since $R_k \cap S_k$ is abelian and $[\Sigma : S_k] < \infty$ it follows that R_k is virtually abelian; thus, $\Sigma \cap \Omega$ is a normal amenable subgroup of Σ.

From **Claim 5.5** we have obtained that $Q \subseteq qL(\Sigma)q$ is a finite index inclusion of non-amenable II_1 factors. Denoting by $z = z(q)$, the central support of q in $L(\Sigma)$, we see that $L(\Sigma)z$ is a non-amenable II_1 factor. Moreover there exists a scalar $s > 0$ such that $(qL(\Sigma)q)^s = L(\Sigma)z$. By above $Q^s \subseteq (qL(\Sigma)q)^s = L(\Sigma)z$ is a finite index inclusion of non-amenable II_1 factors. Perform the basic construction $Q^s \subseteq L(\Sigma)z \subseteq \langle L(\Sigma)z, e_{Q^s}\rangle$ and notice that $\langle L(\Sigma)z, e_{Q^s}\rangle \cong Q^\mu$ where $\mu = s[qL(\Sigma)q : Q]^2$.

Finally we show $\Sigma \cap \Omega$ is finite. Notice the normalizer $\mathcal{N}_{L(\Sigma)z}(L(\Sigma\cap\Omega)z))'' = L(\Sigma)z$ has finite index in Q^μ. From Equation (5.1) we have $Q^\mu = N^{t_1/k}$ where $N = L(\hat{\Gamma}_n)\bar{\otimes}M_k(\mathbb{C})$ with $t_1 = \tau(p)\mu$ and $t_1 \leq k \in N$. Thus we can write $Q^\mu = N^r$ where $N = L(\hat{\Gamma}_n)\bar{\otimes}M_k(\mathbb{C})$, for some $k \in \mathbb{N}$ and $0 < r < 1$. $L(\Sigma)z$ is a factor and $\Gamma_s \in \mathcal{S}_{nf}$, for all $1 \leq s \leq n-1$ then [PV12, Theorem 1.6] implies that $L(\Sigma \cap \Omega)z \preceq^s_N L(\hat{\Gamma}_{\{n,s\}})\bar{\otimes}M_k(\mathbb{C})$, for every $1 \leq s \leq n-1$. Thus using [Va10, Lemma 2.5] in the terminology therein (also see Definition 2.11) we have $(L(\Sigma\cap\Omega)z)_1 \subset_{approx} L(\hat{\Gamma}_{\{n,s\}})\bar{\otimes}M_k(\mathbb{C})$ for each $1 \leq s \leq n-1$. Then by [Va10, Lemma 2.7] we further have that $(L(\Sigma\cap\Omega)z)_1 \subset_{approx} M_k(\mathbb{C})$ and hence a corner of $L(\Sigma\cap\Omega)z$ is purely atomic. This together with Proposition 2.13 implies that $\Sigma\cap\Omega$ is finite. ■

Claim 5.10. *There exists ℓ such that $\Omega \subseteq \{\lambda \in \Lambda : |\lambda^\Sigma| \leq \ell\}$.*

Proof of the Claim 5.10. By (5.5) one can find $m_1, m_2, \ldots, m_s \in fM^tf$ a finite Pimsner-Popa basis such that for every $x \in fM^tf$ we have

$$x = \sum_i E_{f(L(\Sigma)\vee(L(\Sigma)'\cap M^t))f}(xm_i^*)m_i.$$

For $x \in fM^tf$, let $\|x\|_{2,f}$ be the 2-norm coming from the normalized trace on fM^tf.

Thus for all $x \in fM^tf$ we have $\|x\|_{2,f}^2 = \sum_i \|E_{f(L(\Sigma)\vee(L(\Sigma)'\cap M^t))f}(xm_i^*)\|_{2,f}^2$ and hence for all $x \in fL(\Omega\Sigma)f$ we have $\|x\|_{2,f}^2 = \sum_i \|E_{f(L(\Sigma)\vee(L(\Sigma)'\cap M^t))f}(xn_i^*)\|_{2,f}^2$, where $n_i = E_{fL(\Omega\Sigma)f}(m_i)$.

Together with basic approximations of n_i's, this shows that for every $\varepsilon > 0$ there exist $c_\varepsilon > 0$ and a finite subset $L_\varepsilon \subset \Omega$ such that for each $x \in fL(\Omega\Sigma)f$ we have

$$\tau(f)^{-1}\tau(fxfx^*) = \|x\|_{2,f}^2 \leq \varepsilon + c_\varepsilon \tau(f)^{-1} \sum_{s\in L_\varepsilon} \|E_{L(\Sigma)\vee(L(\Sigma)'\cap M^t)}(xu_s)\|_2^2. \tag{5.8}$$

Observe that for every $x \in L(\Omega)$ we have $E_{L(\Sigma)\vee(L(\Sigma)'\cap M^t)}(x) = E_{L(\Sigma\cap\Omega)\vee(L(\Sigma)'\cap M^t)}(x)$, noting that for every $x \in L(\Omega), a \in L(\Sigma), b \in L(\Sigma)' \cap M^t$ we have

$$\begin{aligned}
&\tau(E_{L(\Sigma)\vee(L(\Sigma)'\cap M^t)}(x)ab) = \tau(xab) = \tau(aE_{L(\Sigma)}(bx)) \\
&= \tau(aE_{L(\Sigma\cap\Omega)}(bx)) = \tau(bxE_{L(\Sigma\cap\Omega)}(a)) = \tau(bxE_{L(\Sigma\cap\Omega)\vee(L(\Sigma)'\cap M^t)} \circ E_{L(\Sigma\cap\Omega)}(a)) \\
&= \tau(bxE_{L(\Sigma\cap\Omega)\vee(L(\Sigma)'\cap M^t)} \circ E_{L(\Omega)}(a)) = \tau(bxE_{L(\Sigma\cap\Omega)\vee(L(\Sigma)'\cap M^t)}(a)) \\
&= \tau(xE_{L(\Sigma\cap\Omega)\vee(L\Sigma'\cap M)}(ab)) = \tau(E_{L(\Sigma\cap\Omega)\vee(L(\Sigma)'\cap M^t)}(x)ab)
\end{aligned} \tag{5.9}$$

This formula together with (5.8) gives that for every $\varepsilon > 0$ there exist $c_\varepsilon > 0$ and a finite subset $L_\varepsilon \subset \Omega$ such that for all $x \in L(\Omega)$ we have

$$\tau(fxfx^*) \leq \varepsilon + c_\varepsilon \sum_{s\in L_\varepsilon} \|E_{L(\Sigma\cap\Omega)\vee(L(\Sigma)'\cap M^t)}(xu_s)\|_2^2. \tag{5.10}$$

By **Claim 5.9**, the group $\Sigma \cap \Omega$ is finite and hence $L(\Sigma\cap\Omega) \vee (L(\Sigma)' \cap M^t)$ admits a finite Pimsner-Popa basis over $L(\Sigma)' \cap M^t$. Approximating the elements in this basis together with (5.10) show that for every $\varepsilon > 0$ there exist $d_\varepsilon > 0$ and a finite subset $R_\varepsilon \subset \Omega$ such that for every $x \in L(\Omega)$ we have

$$\tau(fxfx^*) \leq \varepsilon + d_\varepsilon \sum_{s \in R_\varepsilon} \|E_{L(\Sigma)' \cap M^t}(xu_s)\|_2^2. \tag{5.11}$$

Setting up $S = \Omega$ and $\phi_\varepsilon(x) = d_\varepsilon \sum_{s \in L_\varepsilon} \|E_{L(\Sigma)' \cap M^t}(xu_s)\|_2^2$, we see (5.11) shows that part (1) in Corollary 2.12 is satisfied.

We will now show that if there are elements in Ω whose orbits under conjugation by Σ have arbitrarily large size, this would imply that ϕ_ε satisfies part (2) in Corollary Corollary 2.12 as well; hence, we would have that $f = 0$, a contradiction.

To this end fix $\delta > 0$ and a finite subset $F \subset \Omega$. For every $\sigma \in \Omega$ we have $E_{L(\Sigma)' \cap M^t}(\sigma) = |\sigma^\Sigma|^{-1} \sum_{\lambda \in \sigma^\Sigma} \lambda$. Since for all $t, v \in \Omega$ we have $|(vt)^\Sigma| \leq |v^\Sigma||t^\Sigma|$, the set FR_ε is finite, and there exist elements in Ω whose orbits under conjugation by Σ have arbitrarily large size, then one can find $\sigma \in \Omega$ such that $|(\sigma^{-1}\mu s)^\Sigma| \geq \delta^{-1} d_\varepsilon |R_\varepsilon|$, for all $\mu \in F$, $s \in R_\varepsilon$. Thus for every $\mu \in F$ we have

$$\phi_\varepsilon(u_\sigma^* u_\mu) = d_\varepsilon \sum_{s \in R_\varepsilon} \|E_{L(\Sigma)' \cap M^t}(u_{\sigma^{-1}\mu s})\|_2^2 = d_\varepsilon \sum_{s \in R_\varepsilon} |(\sigma^{-1}\mu s)^\Sigma|^{-1} \leq \delta. \tag{5.12}$$

This finishes the proof of Claim 5.10. ■

Claim 5.11. *For $k \in \mathbb{N}$, let $\Sigma_k := C_\Sigma(\Omega_k)$. There exist $k \in \mathbb{N}$, a finite set $F \subset \Omega_k$, and $C > 0$ such that for every $\sigma \in \Sigma_k$ there exists $s \in F$ such that $|(s\sigma)^\Omega| \leq C$.*

Proof of the Claim 5.11. **Claim 5.10** implies

$$\|E_{L(\Sigma)' \cap M^t}(\lambda)\|_2^2 = \||\lambda^\Sigma|^{-1} \sum_{\mu \in \lambda^\Sigma} u_\mu\|_2^2 = |\lambda^\Sigma|^{-1} \geq \ell^{-1},$$

for all $\lambda \in \Omega$. Since Ω generates $L(\Omega)$, Popa's intertwining techniques further imply $L(\Omega) \preceq_{L(\Omega)} L(\Sigma)' \cap M^t$. Thus, one can find $c \in \mathcal{P}(L(\Omega))$, $d \in \mathcal{P}(L(\Sigma)' \cap M^t)$, $w_1 \in \mathcal{I}(L(\Omega))$, and a $*$-homomorphism $\alpha : cL(\Omega)c \to d(L(\Sigma)' \cap M^t)d$ so that

$$\alpha(x)w_1 = w_1 x, \text{ for all } x \in cL(\Omega)c. \tag{5.13}$$

Let $u' \in \mathcal{U}(L(\Omega))$ and $r_0 \in \mathcal{P}(cL(\Omega)c)$ so that $w_1 = u'r_0$. Denoting by $B = \alpha(cL(\Omega)c) \subset d(L(\Sigma)' \cap M^t)d$ and $r_1 = u'r_0(u')^* \in B' \cap cL(\Omega)c$, relation (5.13) implies $r_1 L(\Omega) r_1 = Br_1$.

Fix $0 < \varepsilon < 1$. Since $\Omega = \cup_k \Omega_k$ there exist $k_\varepsilon \in \mathbb{N}$ and a projection $p_\varepsilon \in L(\Omega_{k_\varepsilon})$ so that

$$\|p_\varepsilon - r_1\|_2 \leq \varepsilon. \tag{5.14}$$

Note that $[\Sigma : \Sigma_{k_\varepsilon}] < \infty$. Then (5.14) together with $[B, L(\Sigma_{k_\varepsilon})] = 0$ shows that for all $x \in L(\Omega)$ and $y \in L(\Sigma_{k_\varepsilon})$ we have $\|r_1 x r_1 y - y r_1 x r_1\|_2 \leq 2\varepsilon$. Altogether, these properties show that for every $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that for all $x \in \mathcal{U}(L(\Omega))$ and $y \in \mathcal{U}(L(\Sigma_{k_\varepsilon}))$ we have $\tau(r_1 x r_1 x^*) = \|r_1 x r_1\|_2^2 \leq 6\varepsilon + Re\tau((r_1 y)^* x r_1 y x^*)$. This further implies that

$$\tau((ar_1a^*)(br_1b^*)) \leq 6\varepsilon + Re\tau((a(r_1y)^*a^*)(br_1yb^*)) \text{ for all } a, b \in \mathcal{U}(L(\Omega)). \tag{5.15}$$

Consider a sequence of convex combinations $\xi_n = \sum_{i=1}^{k_n} \lambda_i b_i r_1 b_i^*$ with $b_i \in \mathcal{U}(L(\Omega))$ which WOT-converges to $E_{L(\Omega)' \cap M^t}(r_1)$. Denote by $\eta_n = \sum_{i=1}^{k_n} \lambda_i b_i r_1 y b_i^*$, and notice that after passing to a subsequence we can assume that η_n WOT-converges to an element $y_1 \in (M^t)_1$. Inequality (5.15) implies that $\tau(ar_1a^*\xi_n) \leq 6\varepsilon + \tau(a(r_1y)^*a^*\eta_n)$

for all $n \in \mathbb{N}$ and $a \in \mathcal{U}(L(\Omega))$. Taking limit over n we get $\tau(ar_1a^*E_{L(\Omega)'\cap M^t}(r_1)) \leq 6\varepsilon + \tau(a(r_1y)^*a^*y_1)$ for all $a \in \mathcal{U}(L(\Omega))$. Since $a^*E_{L(\Omega)'\cap M^t}(r_1) = E_{L(\Omega)'\cap M^t}(r_1)a^*$ the previous inequality gives

$$\|E_{L(\Omega)'\cap M^t}(r_1)\|_2^2 = \tau(ar_1a^*E_{L(\Omega)'\cap M^t}(r_1)) \tag{5.16}$$

$$\leq 6\varepsilon + Re\tau(a(r_1y)^*a^*y_1), \text{ for all } a \in \mathcal{U}(L(\Omega)). \tag{5.17}$$

Consider a sequence of convex combinations $\zeta_n = \sum_{i=1}^{l_n} \mu_i a_i r_1 y a_i^*$ with $a_i \in \mathcal{U}(L(\Omega))$ which WOT-converges to $E_{L(\Omega)'\cap M^t}(r_1y)$. Using (5.16) we get

$$\|E_{L(\Omega)'\cap M^t}(r_1)\|_2^2 \leq 6\varepsilon + Re\tau(\zeta_n^* y_1)$$

for all n, whence passing to the limit over n we have $\|E_{L(\Omega)'\cap M^t}(r_1)\|_2^2 \leq 6\varepsilon + Re\tau(E_{L(\Omega)'\cap M^t}((r_1y)^*)y_1)$. Using the Cauchy-Schwarz inequality and $\|y_1\|_\infty \leq 1$ this further implies that

$$\|E_{L(\Omega)'\cap M^t}(r_1)\|_2^2 \leq 6\varepsilon + \|E_{L(\Omega)'\cap M^t}(r_1y)\|_2, \text{ for all } y \in \mathcal{U}(L(\Sigma_{k_\varepsilon})). \tag{5.18}$$

Using (5.14) and (5.18), for every $\varepsilon > 0$ there exist $k_\varepsilon \in \mathbb{N}$ and a finite subset $K_\varepsilon \subset \Omega_{k_\varepsilon}$ such that

$$\|E_{L(\Omega)'\cap M^t}(r_1)\|_2^2 \leq 8\varepsilon + \sum_{s\in K_\varepsilon} \|E_{L(\Omega)'\cap M^t}(sy)\|_2, \text{ for all } y \in \mathcal{U}(L(\Sigma_{k_\varepsilon})). \tag{5.19}$$

We are now ready to prove the claim. Fix $\varepsilon > 0$. Using (5.19) there exist $k_\varepsilon \in \mathbb{N}$ and a finite subset $K_\varepsilon \subset \Omega_{k_\varepsilon}$ such that for all $\sigma \in \Sigma_{k_\varepsilon}$ we have

$$\|E_{L(\Omega)'\cap M^t}(r_1)\|_2^2 \leq 8\varepsilon + \sum_{s\in K_\varepsilon} \|E_{L(\Omega)'\cap M^t}(s\sigma)\|_2 = 8\varepsilon + \sum_{s\in K_\varepsilon} |(s\sigma)^\Omega|^{-1/2}. \tag{5.20}$$

Assume by contradiction the claim does not hold; hence, for every $k \in \mathbb{N}$, $F \in \Omega_k$ and $C > 0$ there exists $\sigma \in \Sigma_k$ such that for all $s \in F$ we have $|(s\sigma)^{\Sigma}| \geq C$. Since K_ε is finite one can find $\sigma \in \Sigma_{k_\varepsilon}$ such that $\sum_{s \in K_\varepsilon} |(s\sigma)^{\Omega}|^{-1/2} < \varepsilon$, by (5.20) we have $\|E_{L(\Omega)' \cap M^t}(r_1)\|_2^2 \leq 9\varepsilon$. As $\varepsilon > 0$ is arbitrary we get $r_1 = 0$, a contradiction. ■

Claim 5.12. *There exist $R > 0$ and $k \in \mathbb{N}$ so that there is a finite index subgroup $\Theta \leqslant \Sigma_k$, such that $\Theta \subseteq \{\lambda \in \Lambda : |\lambda^{\Omega}| \leq R\}$.*

Proof of the Claim 5.12. Using **Claim 5.11** there exist $D > 0$ and a finite cover $\cup_{i=1}^{r} K_i = \Sigma_k$ such that $\cup_i f_i K_i \subseteq \{\lambda \in \Lambda : |\lambda^{\Omega}| \leq D\}$ where $\{f_1, f_2, ..., f_r\} = F$. Using the inequality $|(vt)^{\Omega}| \leq |v^{\Omega}||t^{\Omega}|$ for all $t, v \in \Omega$ one can find $s_i \in \Sigma_k$ for $1 \leq i \leq r$ such that $\cup_i s_i K_i \subseteq \{\lambda \in \Lambda : |\lambda^{\Omega}| \leq D^2\}$. Considering the subgroups $\Theta_i = \langle s_i K_i \rangle \leqslant \Sigma_k$, the previous relations imply that $\cup_{i=1}^{r} \Theta_i \subseteq \{\lambda \in \Lambda : |\lambda^{\Omega}| < \infty\}$ and $\cup_{i=1}^{r} s_i^{-1} \Theta_i = \Sigma_k$. Thus at least one the subgroups $\Theta := \Theta_i \leqslant \Sigma_k$ has finite index. Denoting by $\tilde{K}_i = K_i \cap \Theta$ we still have $\cup_i f_i \tilde{K}_i \subseteq \{\lambda \in \Lambda : |\lambda^{\Omega}| \leq D\}$ and $\cup_{i=1}^{r} \tilde{K}_i = \Theta$ and as before there exist $t_i \in \Theta$ so that $\cup_i t_i \tilde{K}_i \subseteq \{\lambda \in \Lambda : |\lambda^{\Omega}| \leq D^2\}$. Since $|(t_i^{-1})^{\Omega}| < \infty$, the previous containment shows that $\Theta \subseteq \{\lambda \in \Lambda : |\lambda^{\Omega}| \leq R\}$ where $R = D^2 \max_{i=1,r} |(t_i^{-1})^{\Omega}|$. ■

Claim 5.13. *If we denote by $\Omega' = \mathcal{V}_{\Lambda}(\Omega) = \{\lambda \in \Lambda : |\lambda^{\Omega}| < \infty\}$ then $\Theta \leqslant \Omega'$ has finite index.*

Proof of the Claim 5.13. Assume by contradiction that $\{h_i\}$ is a infinite sequence of representatives of distinct right cosets of Ω' in Θ. Since $\Theta \leqslant \Sigma$ and $\Omega\Sigma \leqslant \Lambda$ are finite index and Σ normalizes Ω if follows that $\Omega\Theta \leqslant \Lambda$ is also finite index. Thus passing to an infinite subsequence of $\{h_i\}$ one can find $\lambda \in \Lambda$ and $x_i \in \Omega\Theta$ so that $h_i = x_i\lambda$ for all $i \geq 1$. Thus $h_i h_1^{-1} \in \Omega\Theta$ for all $i \geq 2$. Hence one can find $\sigma_i \in \Theta$ and $\omega_i \in \Omega \cap \Omega'$ so that $h_i h_1^{-1} = \sigma_i \omega_i$. From construction we have $\Theta\omega_i \neq \Theta\omega_j$ for all $i \neq j$. Since $\Theta \leqslant \Sigma$ one can check that $|\omega_i^{\Omega\Theta}| < \infty$ for all i. Indeed $S_i := \omega_i^{\Theta} \subset \Omega'$ is a finite set for each i and $\omega_i^{\Omega\Theta} = \bigcup_{\zeta \in S_i} \zeta^{\Omega}$. Since $\Omega\Theta \leqslant \Lambda$ has finite index we further have $|\omega_i^{\Lambda}| < \infty$ for all $i \geq 2$. This however contradicts the fact that Λ is icc. ■

We are now ready to complete the proof of the Theorem 1.1. Combining **Claims 5.9** and **5.13** it follows that $\Omega' \cap \Omega$ is a finite group. Since Ω normalizes Ω' and Θ normalizes Ω, for all $\sigma \in \Theta$, $\omega \in \Omega$ the words $\omega\sigma\omega^{-1}\sigma^{-1}, \sigma\omega\sigma^{-1}\omega^{-1} \in \Omega' \cap \Omega$. Hence the commutator $\Omega' \cap \Omega \cap \Theta\Omega$ is a finite normal subgroup of $\Omega\Theta$ containing all commutators. Since $\Omega\Theta$ has finite index in Λ and the latter is icc it follows that $[\Theta, \Omega] = 1$. Letting $\Sigma_1 := \Theta$ and $\Sigma_2 := \Omega$ we get our conclusion. □

If in the previous theorem we assume Λ is finitely generated, since $[\Lambda : \Omega\Sigma] < \infty$, it follows that $\Omega\Sigma$ is finitely generated as well. Since from construction we have that $\Omega\Sigma_k$ is an increasing tower of subgroups such that $\cup_k \Omega\Sigma_k = \Omega\Sigma$, by finite generation, there exists $\ell \in \mathbb{N}$ such that $\Omega\Sigma_\ell = \Omega\Sigma$. Hence $\Sigma_\ell = C_\Sigma(\Omega_\ell)$ is a finite index subgroup of Σ which commutes with Ω_ℓ. Thus $\Sigma_\ell\Omega_\ell$ has finite index in Λ and the conclusion of the theorem follows. Therefore in this situation one needs neither the weak amenability assumption nor the **Claims 5.9-5.12**. This would be the case if we assume for instance that the groups Γ_i's belong to class S_{nf} and have property (T).

We require one more intermediate result before the proof of Theorem 1.1.

Theorem 5.14. *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n \in \mathcal{S}_{nf}$ which are all weakly amenable. Let $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$, and denote by $M = L(\Gamma)$. Let $t > 0$ be a scalar and let Λ be an arbitrary discrete group such that $M^t = L(\Lambda)$. Then one can find icc subgroups $\Phi_1, \Phi_2 < \Lambda$ with $\Phi_1 \times \Phi_2 = \Lambda$, a scalar $s > 0$, a proper subset $F \subset \{1, \ldots, n\}$ and a unitary $v \in \mathcal{U}(M)$ such that $vL(\Phi_1)v^* = L(\Gamma_F)^s$ and $vL(\Phi_2)v^* = (L(\Gamma_{F^c}))^{t/s}$.*

Proof. From Theorem 5.4 there exist commuting, non-amenable, icc subgroups Σ_1, Σ_2 of Λ such that $[\Lambda : \Sigma_1\Sigma_2] < \infty$. Fix an integer $r > t$. Applying [CSU11, Theorem 6.1] there exist $1 \leq k \leq 2$ and a smallest $1 \leq i \leq n-1$ such that

1. $L(\Sigma_k) \preceq_{M^r} L(\Gamma_F)^r$ for some $F \subset \{1, \ldots, n\}$ with $|F| = i$; and
2. $L(\Sigma_k') \not\preceq_{M^r} L(\Gamma_K)^{r'}$, for all integers $r' \geq r$, all $K \subset \{1, \ldots, n\}$ with $|K| = i-1$, and all subgroups $\Sigma_k' \leqslant \Sigma_k$ with $[\Sigma_k : \Sigma_k'] < \infty$.

For simplicity we assume throughout the proof that $k = 1$. Denote by $M_1 = L(\Gamma_F)^r$ and $M_2 = L(\Gamma_{F^c})^{t/r}$ and notice that $M^t = M_1 \bar{\otimes} M_2$. Proceeding as in the proof of [OP03, Proposition 12] there exist a scalar $\mu > 0$ and a partial isometry $u \in \mathcal{I}(M^t)$ satisfying $p = uu^* \in M_2^{1/\mu}$, $q = u^*u \in L(\Sigma_1)' \cap M^t$ and

$$uL(\Sigma_1)u^* \subseteq M_1^{\mu}p. \tag{5.21}$$

Consider the group $\Omega_2 = \{\lambda \in \Lambda : |\lambda^{\Sigma_1}| < \infty\}$. Since $\Sigma_2 \leqslant \Omega_2$ then from above it follows that $[\Lambda : \Omega_2\Sigma_1] < \infty$. Letting $\Omega_1 = C_{\Sigma_1}(\Omega_2)$, it is a straightforward exercise to see that $\Omega_1, \Omega_2 < \Lambda$ are commuting, non-amenable, icc subgroups. Since $[\Omega_2 : \Sigma_2] \leq [\Lambda : \Sigma_1\Sigma_2]$, we have that $[\Sigma_1 : \Omega_1] < \infty$ so $[\Lambda : \Omega_1\Omega_2] < \infty$. By construction we have $L(\Sigma_1)' \cap M^t \subseteq L(\Omega_2)$ and the latter is a II_1 factor. Relation

(5.21) gives $uL(\Omega_1)u^* \subseteq M_1^\mu p$. As in the proof of [OP03, Proposition 12], since $L(\Omega_2)$ and $M_2^{1/\mu}$ are factors, then for a large integer m there exist $w_1, \ldots, w_m \in L(\Omega_2)$ and $u_1, \ldots, u_m \in M_2^{1/\mu}$ partial isometries satisfying $w_i w_i{}^* = q' \leq q$, $u_i{}^* u_i = p' = uq'u^* \leq p$ for each $1 \leq i \leq m$ and $\sum_j w_j{}^* w_j = 1_{L(\Omega_2)}$, $\sum_j u_j u_j{}^* = 1_{M_2^{1/\mu}}$. Combining with the above, one can check $v = \sum_j u_j u w_j \in M^t$ is a unitary satisfying $vL(\Omega_1)v^* \subseteq M_1^\mu$. This further implies that

$$v(L(\Omega_1)' \cap M^t)v^* \supseteq M_2^{1/\mu}. \tag{5.22}$$

Consider the groups $\Theta_2 = \{\lambda \in \Lambda : |\lambda^{\Omega_1}| < \infty\}$ and $\Theta_1 = C_{\Omega_1}(\Theta_2)$ and proceeding as before we have that $\Theta_1, \Theta_2 < \Lambda$ are commuting, non-amenable, icc subgroups such that $[\Lambda : \Theta_1\Theta_2] < \infty$ and $[\Sigma_1 : \Theta_1] < \infty$. Moreover, we have that $L(\Theta_2) \supseteq L(\Omega_1)' \cap M^t$ whence by (5.22) we have $vL(\Theta_2)v^* \supseteq M_2^{1/\mu}$. Since $M^t = M_1^\mu \bar{\otimes} M_2^{1/\mu}$ by [Ge96, Theorem A] we have that $vL(\Theta_2)v^* = B \bar{\otimes} M_2^{1/\mu}$, where $B \subseteq M_1^\mu$ is a factor. Thus B and $vL(\Theta_1)v^*$ are commuting subfactors generating a finite index subfactor of M_1^μ. Since $L(\Theta_1) \nprec_M L(\Gamma_K)^{r'}$, for $r' \geq r$ and all $K \subset \{1, \ldots, n\}$ with $|K| = i - 1$ and since $vL(\Theta_1)v^*$ is non-amenable, it follows from [CSU11, Theorem 6.1] that B is amenable. Also if we assume that B is diffuse then by Lemma 5.3 one contradicts the solidity of $L(\Gamma_i)$; thus, B is a completely atomic factor whence $B = M_k(\mathbb{C})$, for some integer k. Altogether, this shows that $vL(\Theta_2)v^* = M_2^s$ where $s = k/\mu$. Since $M = M_1^{1/s} \bar{\otimes} M_2^s$ we also have $v(L(\Theta_2)' \cap M)v^* = M_1^{1/s}$. Let $\Phi_1 = \{\lambda \in \Lambda : |\lambda^{\Theta_2}| < \infty\}$ and since Θ_2 is icc it follows that $\Phi_1 \cap \Theta_2 = 1$. By construction we have that $vL(\Phi_1)v^* \supseteq v(L(\Theta_2)' \cap M)v^* = M_1^{1/s}$; hence, using [Ge96, Theorem A] again we get $vL(\Phi_1)v^* = A \bar{\otimes} v(L(\Theta_2)' \cap M)v^*$ for some $A \subseteq vL(\Theta_2)v^*$.

In particular, we have that $A = vL(\Phi_1)v^* \cap vL(\Theta_2)v^* = vL(\Phi_1 \cap \Theta_2)v^* = \mathbb{C}1$. This further implies that $vL(\Phi_1)v^* = v(L(\Theta_2)' \cap M)v^*$ whence Φ_1 and Θ_2 are commuting, non-amenable subgroups of Λ such that $\Phi_1 \cap \Theta_2 = 1$, $\Phi_1\Theta_2 = \Lambda$, $vL(\Phi_1)v^* = M_1^{1/s}$, and $vL(\Theta_2)v^* = M_2^s$. Letting $\Phi_2 = \Theta_2$ we get the desired conclusion. □

Proceeding inductively using Theorem 5.14, we establish Theorem 1.3.

Theorem 5.15 (Theorem 1.3)**.** *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n \in \mathcal{S}_{nf}$ be weakly amenable groups with $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ and denote by $M = L(\Gamma)$. Let $t > 0$ be a scalar and let Λ be an arbitrary group such that $M^t = L(\Lambda)$. Then one can find subgroups $\Lambda_1, \Lambda_2, \ldots, \Lambda_n < \Lambda$ with $\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n = \Lambda$, scalars $t_i > 0$ with $t_1 t_2 \cdots t_n = t$, and a unitary $u \in \mathcal{U}(M)$ such that $uL(\Lambda_i)u^* = L(\Gamma_i)^{t_i}$ for all $1 \leq i \leq n$.*

Proof. Assume the statement is satisfied for any collection of $2 \leq k \leq n-1$ groups belonging to $\mathcal{S}_{nf}$. By Theorem 5.14 one can find subgroups $\Lambda_1', \Lambda_2' < \Lambda$ with $\Lambda_1' \times \Lambda_2' = \Lambda$, a scalar $s > 0$, a proper subset $F \subset \{1, \ldots, n\}$, and a unitary $v \in \mathcal{U}(M^t)$ such that $vL(\Lambda_1')v^* = L(\Gamma_F)^s$ and $vL(\Lambda_2')v^* = L(\Gamma_{F^c})^{t/s}$. Since $vL(\Lambda_1')v^* = L(\Gamma_F)^s$ then by the inductive hypothesis there exist subgroups $\Lambda_i < \Lambda_1'$, $i \in F$ with $\times_{i\in F}\Lambda_i = \Lambda_1'$, scalars $t_i > 0$ with $\Pi_{i\in F} t_i = s$, and $u_1 \in \mathcal{U}(L(\Gamma_F)^s)$ such that $u_1 L(\Lambda_i)u_1^* = L(\Gamma_i)^{t_i}$, for all $i \in F$. Similarly, one can find subgroups $\Lambda_i < \Lambda_2'$, $i \in F^c$ with $\times_{i\in F^c}\Lambda_i = \Lambda_2'$, scalars $t_i > 0$ with $\Pi_{i\in F^c} t_i = t/s$, and $u_2 \in \mathcal{U}(L(\Gamma_{F^c})^{t/s})$ such that $u_2 L(\Lambda_i)u_2^* = L(\Gamma_i)^{t_i}$, for all $i \in F^c$. Altogether, the previous relations imply $\times_{j=1}^n \Lambda_j = \Lambda_1' \times \Lambda_2' = \Lambda$ and $\Pi_{j=1}^n t_j = (\Pi_{i\in F} t_i)(\Pi_{i\in F^c} t_i) = s \cdot t/s = t$. Moreover, letting $u = (u_1 \otimes u_2)v$ we get $uL(\Lambda_i)u^* = L(\Gamma_i)^{t_i}$ for all $1 \leq i \leq n$, as desired. □

5.1.1 Remarks on the Use of Weak Amenability

Notice that in the proof of Theorem 1.3 the assumption that the Γ_i's are weakly amenable is used only in the proof of **Claim 5.9**, class $\mathcal{S}_{nf}$ being sufficient for all the other steps of the proof. In fact when we are dealing with only two groups the weak amenability assumption can be dropped.

Theorem 5.16. *Let $\Gamma_1, \Gamma_2 \in \mathcal{S}_{nf}$ with $\Gamma = \Gamma_1 \times \Gamma_2$ and denote by $M = L(\Gamma)$. Let $t > 0$ be a scalar and let Λ be an arbitrary group such that $M^t = L(\Lambda)$. Then there exist commuting, non-amenable, icc subgroups $\Sigma_1, \Sigma_2 < \Lambda$ such that $[\Lambda : \Sigma_1\Sigma_2] < \infty$.*

Proof. The proof is identical with the proof of Theorem 5.4. Thus we borrow the same notations and set-up as there. We only include a proof for Claim 5.9 which bypasses the usage of weak amenability, the rest of the proof following the same arguments. Under the same notations as in the proof of Theorem 5.4 we show that:

Claim 5.17. *$\Sigma \cap \Omega$ is finite.*

Let $\mathcal{O}'_i = \mathcal{O}_i \cap \Sigma$ and notice $\Sigma \cap \Omega = \cup_i \mathcal{O}'_i$. For each k let $R_k = \langle \cup_{i \in I_k} \mathcal{O}'_i \rangle$ and notice it forms an ascending sequence of normal subgroups of Σ such that $\cup_k R_k = \Sigma \cap \Omega$ and $[\Sigma : \Sigma_k] < \infty$, where $\Sigma_k = C_\Sigma(R_k)$. Since $R_k \cap \Sigma_k$ is abelian and $[\Sigma : \Sigma_k] < \infty$ it follows that R_k is virtually abelian; thus, $\Sigma \cap \Omega$ is a normal amenable subgroup of Σ.

In the first part of the proof of Theorem 5.3 (see Claim 5.5) we have obtained that $Q \subseteq qL(\Sigma)q$ is a finite index inclusion of non-amenable II_1 factors. Denoting by $z = z(q)$, the central support of q in $L(\Sigma)$, we see that $L(\Sigma)z$ is a non-amenable II_1

factor. Moreover there exists a scalar $s > 0$ such that $(qL(\Sigma)q)^s = L(\Sigma)z$, so $Q^s \subseteq (qL(\Sigma)q)^s = L(\Sigma)z$ is a finite index inclusion of non-amenable II_1 factors. Perform the basic construction $Q^s \subseteq L(\Sigma)z \subseteq \langle L(\Sigma)z, e_{Q^s}\rangle$, and notice that $\langle L(\Sigma)z, e_{Q^s}\rangle = Q^\mu$ where $\mu = s[qL(\Sigma)q : Q]^2$.

To begin we argue that each R_k is finite. Since $C_k := R_k \cap \Sigma_k \leqslant R_k$ has finite index it suffices to show that C_k is finite. From construction note that

$$L(C_k) \subseteq \mathcal{Z}(L(\Sigma_k)) \subseteq L(\Sigma_k)' \cap L(\Sigma). \tag{5.23}$$

By passing to a finite index subgroup we can further assume w.l.o.g. that $\Sigma_k \lhd \Sigma$ is normal and $[\Sigma : \Sigma_k] = r$. Fix $\gamma_1, \gamma_2, ..., \gamma_r \in \Sigma$ a complete set of representatives for the cosets of Σ_k in Σ. One can check the map $E_{\mathcal{Z}(L(\Sigma))} : L(\Sigma_k)' \cap L(\Sigma) \to \mathcal{Z}(L(\Sigma))$ given by $E_{\mathcal{Z}(L(\Sigma))}(x) = r^{-1}\sum_i u_{\gamma_i} x u_{\gamma_i}^*$ for all $x \in L(\Sigma_k)' \cap L(\Sigma)$ is a trace-preserving conditional expectation satisfying $\|E_{\mathcal{Z}(L(\Sigma))}(x)\|_2^2 \geq r^{-1}\|x\|_2^2$; hence $[L(\Sigma_k)' \cap L(\Sigma) : \mathcal{Z}(L(\Sigma))]_{PP} \leq r$. By Theorem 2.10 (2) we have $[L(\Sigma_k)' \cap L(\Sigma)z : \mathcal{Z}(L(\Sigma))z]_{PP} \leq r$ and since $L(\Sigma)z$ is a factor it follows that $L(\Sigma_k)' \cap L(\Sigma)z$ is finite dimensional. Hence there exists $z_0 \in \mathcal{P}(L(\Sigma_k)' \cap L(\Sigma))$ so that $L(\Sigma_k)' \cap L(\Sigma)z_0 = \mathbb{C}z_0$. Combining with (5.23) we have $z_0 \in \mathcal{P}(L(C_k)' \cap L(\Sigma))$ and $L(C_k)z_0 = \mathbb{C}z_0$. By functional calculus one can find $z_1 \in \mathcal{P}(\mathcal{Z}(L(C_k)))$ so that $L(C_k)z_1 = \mathbb{C}z_1$ and Corollary 2.14 entails C_k is finite.

We now show $\Sigma \cap \Omega$ is finite. Proceeding by contradiction assume that $\Sigma \cap \Omega$ is infinite whence $L(\Sigma \cap \Omega)z \subset L(\Sigma)z$ is a diffuse subalgebra. Also notice that $z \in \mathcal{Z}(L(\Sigma)) \subseteq L(\Sigma \cap \Omega)$ and $\mathcal{Z}(L(\Sigma)) = z$. Let $\mathcal{G} = \{\sigma z : \sigma \in \Sigma\} \subset \mathcal{N}_{L(\Sigma)z}(L(\Sigma \cap \Omega)z)$. Since $L(R_k)z$ is an increasing family of $\mathcal{G}$-invariant finite-dimensional subalgebras whose union is dense in $L(\Sigma\cap\Omega)z$, it follows that the natural action by conjugation $\mathcal{G}\curvearrowright L(\Sigma\cap\Omega)z$ is compact, whence weakly compact.

From (5.1) we have $Q^{\mu}=L(\Gamma_1)^{t_1}$ with $t_1=\tau(p)\mu$, and from above we get that $L(\Sigma)z\subseteq L(\Gamma_1)^{t_1}$ is a finite index inclusion of II_1 factors. Since $\Gamma_1\in\mathcal{S}_{nf}$ then same proof as in [CSU11, Theorem 6.1] show that $\mathcal{G}''=L(\Sigma)z$ is amenable. However, this contradicts the non-amenability of $L(\Gamma_1)$. Hence $\Sigma\cap\Omega$ is finite. □

Theorem 5.18 (Corollary 1.2)**.** *Let $\Gamma_1,\Gamma_2\in\mathcal{S}_{nf}$ with $\Gamma=\Gamma_1\times\Gamma_2$ and denote by $M=L(\Gamma)$. Let $t>0$ be a scalar and let Λ be an arbitrary group such that $M^t=L(\Lambda)$. Then one can find subgroups $\Lambda_1,\Lambda_2<\Lambda$ with $\Lambda_1\times\Lambda_2=\Lambda$, a scalar $s>0$, a proper subset $F\subset\{1,\ldots,n\}$ and a unitary $v\in\mathcal{U}(M)$ such that $vL(\Lambda)_1v^*=L(\Gamma_1)^s$ and $vL(\Lambda)_2v^*=L(\Gamma_2)^{t/s}$.*

Proof. If follows from Theorem 5.16, proceeding as in the proof of Theorem 5.14. □

5.1.2 von Neumann Algebas Generated by Lattices

In this section we show how our main result above can be used to give new families of examples of icc discrete groups which are measure equivalent, but whose group von Neumann algebras are not stably isomorphic. The first such examples were constructed by the first Chifan and Ioana [CI11]. We recall that discrete groups Γ and Λ are said to be *measure equivalent* if there exists an essentially free action $\Gamma\times\Lambda\curvearrowright(\Omega,m)$ on a standard σ-finite measure space such that the restriction of the action to each of Γ and Λ admits a Borel fundamental domain of finite measure. The prototypical examples of measure equivalent discrete groups are pairs of lattices in semisimple Lie groups: see [Fu99a] for a thorough treatment of measure equivalence.

In order to establish the result, we will need to make use of a celebrated theorem of Margulis in order to distinguish the algebraic structure of certain lattices in the same ambient Lie group. We briefly recall that if G is a semisimple Lie group without compact factors, then a lattice $\Gamma < G$ is said to be *reducible* if G admits semisimple factors G_1 and G_2 so that setting $\Gamma_1 := \Gamma \cap G_1$ and $\Gamma_2 := \Gamma \cap G_2$ we have that $\Gamma_1\Gamma_2$ is a finite index subgroup of Γ. For example, a finite product of lattices is a reducible lattice in the product of the respective ambient Lie groups. A lattice is *irreducible* if it is not reducible. For instance, the natural inclusion of $\mathbb{Z}[\sqrt{2}]$ as a lattice in $\mathbb{R} \times \mathbb{R}$ induces an irreducible lattice inclusion of $PSL(2, \mathbb{Z}[\sqrt{2}])$ into $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$. Note that if a group Γ virtually decomposes as a direct product of infinite groups, e.g., Γ is a reducible lattice, then Γ admits an infinite normal subgroup with infinite index. However it is a consequence of the following famous theorem of Margulis [Ma79, Z84] that irreducible lattices can admit no direct product decomposition:

Theorem 5.19 (Normal Subgroup Theorem)**.** *If $\Gamma < G$ is an irreducible lattice in a higher-rank semisimple Lie group G with no compact factors then Γ is virtually simple, i.e., any normal subgroup of Γ is either finite or has finite index in Γ.*

Note that by the Borel density theorem, if Γ is a lattice in a semi-simple Lie group with no compact factors and trivial center, then Γ is icc, whence if irreducible cannot be decomposed as *any* direct product of groups.

The following corollary is a direct application of the Normal Subgroup Theorem to Theorem 1.3 and Corollary 1.1.

Corollary 5.20 (Corollary 1.2)**.** *If Λ is an irreducible lattice in a higher rank semi-simple Lie group, then $L(\Lambda)$ is neither isomorphic to a factor $L(\Gamma_1 \times \Gamma_2)$ where Γ_1, Γ_2 are groups in the class $\mathcal{S}_{nf}$, nor is it isomorphic to a factor of the form $L(\Gamma_1 \times \cdots \times \Gamma_n)$ where each $\Gamma_i \in \mathcal{S}_{nf}$ and is weakly amenable.*

We apply this corollary in the following more specific situation. Let $G_1, \ldots, G_n$ be non-compact, real simple Lie groups of rank one with trivial center, and let $\Gamma_1, \ldots, \Gamma_n$ be respective lattices. It follows from [CH89] that each Γ_i is weakly amenable and from [Oz06] that each Γ_i belongs to the class $\mathcal{S}_{nf}$. Thus if $\Lambda < G_1 \times \cdots \times G_2$ is an irreducible lattice, then Λ and $\Gamma_1 \times \cdots \times \Gamma_n$ are measure equivalent groups whose associated group factors are not stably isomorphic.

5.2 Product Rigidity for $\mathrm{Quot}_n(\mathcal{C}_{\mathbf{rss}})$

In [CKP14], the authors proved the collection of groups $\mathcal{P} = \mathrm{Quot}(\mathcal{C}_{\mathrm{rss}}) \cap \mathcal{NC}_1$ will have prime von Neumann algebras. The authors considered groups which lie at the intersection of the class $\mathcal{C}_{rss}$ and the class NC_1: the collection of all non-amenable groups which admit an unbounded quasi-cocycle valued in a mixing, weakly-ℓ^2, orthogonal representation. The authors demonstrated how existence of the quasi-cocycle with the properties above lift to *s-malleable deformations* which in conjunction with the dichotomy for normalizers of the class $\mathcal{C}_{\mathrm{rss}}$ "control" the deformations to deduce primeness of the corresponding von Neumann algebras. This theorem applies to a fairly large collection of groups includes Johnson kernel and Torelli groups for surfaces of low genus with large number of boundary components; central quotients of pure braid groups; and a large collection of groups hyperbolic relative to a family of exact, residually finite, infinite proper subgroups.

Theorem 5.21. *([CKP14, Theorem A]) Let* Γ *be any group commensurable to a group in* $\mathcal{NC}_1 \cap \text{Quot}(\mathcal{C}_{rss})$*. Then* $L(\Gamma)$ *is prime, and in particular* $L(\Gamma) \ncong L(\Lambda \times \Sigma)$ *for any infinite groups* Λ, Σ*.*

The above hypotheses can be altered to remove the cohomological criteria to deduce Lemma 4.3 and Corollary 4.4 which shows $L(\Gamma)$ is prime whenever $\Gamma \in \mathcal{C}_{\text{rss}}$. Motivated by the results contained in Section 5.1, Pant and I removed the cohomological condition $\mathcal{NC}_1$ and investigated what structural results can be proven beyond primeness. Precisely, we wanted to investigate the possible tensor decomposition of these groups algebras. Thus requires a more sophisticated analysis that we develop by building on the techniques

The first intertwining result is a substitute for the discretization of the commutation relation of Theorem 4.1. In this way, we identify a large algebra with a subgroup. The remainder of the statements closely mimic the proof of Theorem 5.3 to recover a virtual product in the group Λ, which in conjunction with the icc condition allows us to pass to commensurable groups and deduce an honest direct product. We make note the bulk of this section of this document can be found in [dSP17].

Lemma 5.22. *Let* M *be a type* II_1 *factor with* $A_1, \ldots, A_k \subset M$ *diffuse commuting* II_1 *factors such that* $A_1 \vee \cdots \vee A_k \subset M$ *is a finite index inclusion of algebras. Then there exists a projection* $z \in M$ *so that* $\mathcal{Z}(Q_i z) \cong \mathbb{C}$*, where* $\mathcal{N}_M(A_i)'' = Q_i$*.*

Proof. Letting $A = A_1 \vee \cdots \vee A_k$, Theorem 2.10 part (4) implies $A' \cap M$ is finite dimensional. Notice $A \subset Q_i$ for every $i = 1, \ldots, k$ since A_i and $A_i' \cap M$ are both subalgebras of Q_i. Thus $\mathcal{Z}(Q_i)$ is finite dimensional since $\mathcal{Z}(Q_i) \subset Q_i' \cap M \subset A' \cap M$.

Now for any projection $z \in \mathcal{Z}(Q_i)$, we claim $Q_i z = \mathcal{N}_{zMz}(A_i z)''$[1]. It suffices to show $\mathcal{N}_M(A_i)z = \mathcal{N}_{zMz}(A_i z)$. This follows clearly form the following facts: given any unitary $u \in \mathcal{N}_M(A_i)$, $(uz)^* uz = z = uz(uz)^*$; if $v \in \mathcal{U}(zMz)$ is a normalizing unitary of $A_i z$, then $v + (1 - z) \in \mathcal{U}(M)$ is a normalizing unitary of A_i.

Since the algebras $\mathcal{Z}(Q_i)$ pairwise commute, we may take any minimal projections $z_i \in \mathcal{Z}(Q_i)$ so that $z_i z_j = z_j z_1 \neq 0$. Then

$$Az \subset zMz = N$$

is a finite index inclusion of algebras with $\mathcal{Z}(\mathcal{N}_N(A_i z)'') = \mathcal{Z}(Q_i z) = \mathbb{C}z$. □

Proposition 5.23. *Let* $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$ *with* $p \in L(\Gamma)$ *a non-zero projection. Suppose there exist* $A, B \subset pL(\Gamma)p$ *diffuse commuting subalgebras so that* $[pL(\Gamma)p : A \vee B]_{PP} < \infty$. *Then either*

1. $A \preceq_{L(\Gamma)} L(\Gamma_n^{(1)})$, *or*
2. $B \preceq_{L(\Gamma)} L(\Gamma_n^{(1)})$.

Proof. First note $n \geq 2$ by Lemma 4.3. Let $A_0 \subset A$ be an amenable subalgebra. Letting $M = L(\Gamma)$, we see $\Delta(A_0)$ and $\Delta(B)$ are diffuse commuting subalgebras of $M \bar{\otimes} L(\Gamma_1)$ with $\Delta(A_0)$ amenable. Since $\Gamma_1 \in \mathrm{Quot}_1(\mathcal{C}_{\mathrm{rss}}) = \mathcal{C}_{\mathrm{rss}}$, we have either

[1] This holds in much greater generality: the projection z can be taken to be any projection in $Q_i' \cap M$

1. $\Delta(A_0) \preceq M\bar{\otimes}1$, or

2. $\mathcal{N}_{M\bar{\otimes}L(\Gamma_1)}(\Delta(A_0))''$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$.

Assume (2) holds. Noting $\Delta(B) \subset \mathcal{N}_{M\bar{\otimes}L(\Gamma_1)}(\Delta(A_0))''$ yields $\Delta(B)$ is amenable relative to $M\bar{\otimes}1$ inside $M\bar{\otimes}L(\Gamma_1)$. Applying the dichotomy property of $\mathcal{C}_{\text{rss}}$ either

3. $\Delta(B) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$, or

4. $\mathcal{N}_{M\bar{\otimes}L(\Gamma_1)}(\Delta(B))''$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$.

To summarize we have either

5. $\Delta(A_0) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$,

6. $\Delta(B) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$, or

7. $\Delta(A \vee B) \subset \mathcal{N}_{M\bar{\otimes}L(\Gamma_1)}(\Delta(B))''$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$.

We first show case (7) is impossible. Since $A \vee B$ is a finite index subalgebra of M, $M \preceq^s A \vee B$ and hence M is amenable relative to $A \vee B$. By [OP07, Proposition 2.4], $\Delta(M)$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$. However, [CIK13, Proposition 3.5] would imply $\rho(\Gamma) = \Gamma_1$ is amenable, a contradiction. Thus we only have case (5) and (6). Since A_0 was an arbitrary amenable subalgebra of A, by [BO08] we have either

8. $\Delta(A) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$, or

9. $\Delta(B) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$.

Noting $\ker(\rho_n) = L(\Gamma_n^{(1)})$, the result follows from application of [CIK13, Proposition 3.4]. $\square$

Lemma 5.24. *Let* $\Gamma \in \text{Quot}_n(\mathcal{C}_{rss})$*. If* $A, B \subset pL(\Gamma)p$ *are diffuse commuting subalgebras with* A *amenable, then* $[pL(\Gamma)p : A \vee B]_{PP} = \infty$*.*

Proof. We proceed by induction on n. When $n = 1$, this follows from Lemma 4.3. Now assume the statement holds for all groups in $\text{Quot}_k(\mathcal{C}_{rss})$ where $k \leq n-1$ and take $\Gamma \in \text{Quot}_n(\mathcal{C}_{\text{rss}})$ with $A, B \subset pL(\Gamma)p$ as stated. Then $\Delta(A)$ is amenable and therefore amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$. Since $\Gamma_1 \in \mathcal{C}_{\text{rss}}$, either

1. $\Delta(A) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$, or
2. $\mathcal{N}_{M\otimes L(\Gamma_1)}(\Delta(A))''$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$.

We first show (1) is impossible. If (1) were to hold, [CIK13, Proposition 3.4] implies $A \preceq L(\Gamma_n^{(1)})$. By [CKP14, Proposition 2.4], there exists a $*$-isomorphism $\psi : p_1Ap_1 \to A_1 \subset qL(\Gamma_n^{(1)})q$ such that $A_1 \vee (A_1{}' \cap qL(\Gamma_n^{(1)})q) \subset qL(\Gamma_n^{(1)})q$ is a finite index inclusion of algebras. Since A is a diffuse amenable algebra, A_1 is also diffuse amenable. Since $\Gamma_n^{(1)}$ is non-amenable, $A_1' \cap qL(\Gamma_n^{(1)})q$ is non-amenable. Supposing $A_1' \cap qL(\Gamma_n^{(1)})q$ has an atomic corner, cutting by a minimal central projection z, $A_1z \subset qzL(\Gamma_n^{(1)})qz$ is a finite index inclusion of algebra. Since A_1z is amenable is an amenable corner of $L(\Gamma_n^{(1)})$, this is would imply $\Gamma_n^{(1)}$ is an amenable group, a contradiction. If instead $A_1' \cap qL(\Gamma_n^{(1)})q$ were diffuse, this would contradict the induction hypothesis.

Now if (2) holds, the assumption $[M : A \vee B]_{PP} < \infty$ implies $\Delta(A \vee B) \subset M\bar{\otimes}L(\Gamma_1)$ is also a finite index inclusion of algebras. Since $\Delta(A \vee B)$ is a subalgebra of $\mathcal{N}_{M\bar{\otimes}L(\Gamma_1)}(\Delta(A))''$, $\Delta(A \vee B)$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$. By [OP07, Proposition 2.4], $M\bar{\otimes}L(\Gamma_1)$ is amenable relative to $M\bar{\otimes}1$. However, this is impossible as [CIK13, Proposition 3.5] would imply $\Gamma_1 \in \mathcal{C}_{rss}$ is amenable. □

To establish the main result, we show that the maximal number of commuting diffuse subalgebras is controlled by the Hirsh length. Note that we have an upper bound rather than equality. Central quotients of braid groups are poly-free groups which give rise to prime von Neumann alrebras [CKP14, Theorem A].

Lemma 5.25. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$ and suppose $A_1, \ldots, A_k \subset qL(\Gamma)q$ are diffuse commuting II_1 factors. If $A_1 \vee \cdots \vee A_k \subset qL(\Gamma)q$ generate a finite index subalgebra, then $k \leq n$.*

Proof. When $n = 1$, Lemma 4.3 proves the assertion. Now suppose the conclusion holds for all groups in $\mathrm{Quot}_m(\mathcal{C}_{rss})$ up to $m = n - 1$. Now let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$ and assume to the contrary there are $k > n$ diffuse subalgebras $A_1, \ldots, A_k \subset qL(\Gamma)q$ generating a finite index subalgebra of $qL(\Gamma)q$. Without loss of generality, we may assume $k = n+1$. Then by Proposition 5.23, for every $j \in \{1, \ldots, k\}$, either $\hat{A}_j \preceq_{L(\Gamma)} L(\Gamma_n^{(1)})$ or $A_j \preceq_{L(\Gamma)} L(\Gamma_n^{(1)})$.

Now if $\hat{A}_j \preceq L(\Gamma_n^{(1)})$, by [CKP14, Proposition 2.4] there exists a $*$-isomorphism $\psi : p\hat{A}_j p \to A \subset rL(\Gamma_n^{(1)})r$ so that $A \vee A' \cap rL(\Gamma_n^{(1)})r \subset rL(\Gamma_n^{(1)})r$ is a finite index inclusion of algebras. We may assume $p = p_1 \cdots p_k$, $p_i \in A_i$ for $i \neq j$. Hence $\psi(p\hat{A}_j p) = \psi(\bigvee_{i \neq j} p_i A_i p) = \bigvee_{i \neq j} \psi(p_i A_i p)$. Thus

$$\bigvee_{i \neq j} \psi(p_i A_i p) \vee (\psi(p\hat{A}_j p)' \cap rL(\Gamma_n^{(1)})r) \subset rL(\Gamma_n^{(1)})r$$

is a finite index inclusion of algebras. By Lemma 5.24, the center $\mathcal{Z}(\bigvee_{i \neq j} \psi(p_i A_i p) \vee (\psi(p\hat{A}_j p)' \cap rL(\Gamma_n^{(1)})r))$ cannot be diffuse. Thus, cutting by a minimal central projection we may assume

$$\bigvee_{i \neq j} \psi(p_i A_i p) \vee (\psi(p\hat{A}_j p)' \cap rL(\Gamma_n^{(1)})r) \subset rL(\Gamma_n^{(1)})r$$

is a finite index inclusion of factors. However, this would contradict the induction hypothesis as it would allow for at least n commuting diffuse non-amenable subalgebras of $rL(\Gamma_n^{(1)})r$.

If instead $A_j \preceq_{L(\Gamma)} L(\Gamma_n^{(1)})$ for all j, [Va10, Lemma 2.5, Proposition 2.6], in conjunction with the factoriality of each A_j, imply $A_j \preceq^s_{L(\Gamma)} L(\Gamma_n^{(1)})$. Proposition 2.16 would then give $L(\Gamma) \preceq_{L(\Gamma)}$, which implies $\Gamma_n^{(1)}$ is finite index in Γ once again leading to a contradiction. □

The following proposition is the key ingredient in decomposing a group as a product: if we may find an subgroup of $\Sigma < \Gamma \in \mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$ then we may also find another subgroup commuting with Σ so that Γ is commensurable to the direct product $\Sigma \times \Omega$. The proof of this proposition closely follows the proof of Theorem 5.3 in the previous section.

Proposition 5.26. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$ be an icc group and denote by $L(\Gamma) = M$. Suppose we have subgroup $\Sigma < \Gamma$, and a projection $p \in L(\Sigma)' \cap M$ so that $\Sigma \in \mathrm{Quot}_j(\mathcal{C}_{rss})$ and*

$$p[L(\Sigma) \vee (L(\Sigma)' \cap M)]p \subset pMp$$

is a finite index inclusion of II_1 factors. Then we may find commuting subgroups $\Sigma_1, \Sigma_2 < \Gamma$ such that $[\Sigma : \Sigma_1] < \infty$ and $[\Gamma : \Sigma_1 \times \Sigma_2] < \infty$. Furthermore, if $\Sigma \in \mathrm{Quot}_j(\mathcal{C}_{rss})$, then $\Sigma_2 \in \mathrm{Quot}_{n-j}(\mathcal{C}_{rss})$.

Proof. Letting $\Sigma_2 = \left\{\gamma \in \Gamma : |\gamma^{\Sigma}| < \infty\right\}$ and proceeding as in **Claim 5.8**, we see $[\Gamma : \Sigma_2\Sigma] < \infty$. Now, the first half of **Claim 5.9** demonstrates $\Sigma \cap \Sigma_2$ is amenable since it can be written as an increasing tower of amenable groups. Let Γ_1 act trivially on $\mathbb{C}$, $\Gamma \cong \Gamma_n \to \cdots \to \Gamma_1$ is a chain witnessing $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$. Since $L(\Sigma \cap \Sigma_2)$ is amenable and Σ normalizes $\Sigma_2 \cap \Sigma$, the dichotomy of $\mathcal{C}_{rss}$ will imply $L(\Sigma_2 \cap \Sigma) \preceq \mathbb{C}1$. Thus, 2.13 implies $\Sigma \cap \Sigma_2$ is finite.

Claims 5.10–5.13 in the proof of Theorem 5.3 provides the existence of a subgroup $\Sigma_1 \leqslant \Sigma$ satisfying $[\Sigma : \Sigma_1] < \infty$, $[\Gamma : \Sigma_1\Sigma_2] < \infty$, and $[\Sigma_2, \Sigma_1] = \{e\}$. Since $\Sigma \cap \Sigma_2 \geqslant \Sigma_1 \cap \Sigma_2$, it follows $\Sigma_1 \cap \Sigma_2$ is finite as well. Since Γ is icc and $[\Gamma : \Sigma_1\Sigma_2] < \infty$ then $\Sigma_1 \cap \Sigma_2 = \{e\}$. Thus $[\Gamma : \Sigma_1 \times \Sigma_2] < \infty$.

Now if we also had assumed $\Sigma \in \mathrm{Quot}_j(\mathcal{C}_{\mathrm{rss}})$, Corollary 4.9 yields $\Sigma_2 \in \mathrm{Quot}_{n-j}(\mathcal{C}_{\mathrm{rss}})$. □

Theorem 5.27. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$ be an icc group and $q \in L(\Gamma)$ a projection. Suppose $A_1, \ldots, A_k \subset qL(\Gamma)q = M$ are diffuse commuting II_1 factors such that $[M : A_1 \vee \cdots \vee A_k] < \infty$. Then there exist icc groups $\Sigma_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{rss})$, non-zero projections $p_i \in A_i$, finite index subfactors $D_i \subset p_i A_i p_i$, and a unitary $u \in M$ such that*

- *Γ is commensurable to $\Sigma_1 \times \cdots \times \Sigma_k$,*
- *$\sum_{i=1}^k n_i = n$,*
- *$D_i \subset p_i u^* L(\Sigma_i) u p_i$ is a finite index inclusion of II_1 factors.*

Proof. As our theorem is taken up to commensurability, we will treat the case when $\Gamma = \Gamma_n$ where

$$\Gamma_n \to \Gamma_{n-1} \to \cdots \to \Gamma_1 \to 1$$

witnesses $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$. Furthermore, Lemma 5.22 implies we may assume the normalizers $\mathcal{N}_{pMp}(A_i)''$ are factors. By Proposition 5.23, for every i we have either $\hat{A}_i \preceq_M L(\Gamma_n^{(1)})$ or $A_i \preceq_M L(\Gamma_n^{(1)})$. If we assume $A_i \preceq_M L(\Gamma_n^{(1)})$, then $A_i \preceq_M^s L(\Gamma_n^{(1)})$ and hence $A_1 \vee \cdots \vee A_k \preceq L(\Gamma_n^{(1)})$. Since $A_1 \vee \cdots \vee A_k$ is a finite index subalgebra of $L(\Gamma)$, then $[\Gamma : \Gamma_n^{(1)}] < \infty$ contradicting that $\Gamma/\Gamma_n^{(1)} \in \mathcal{C}_{\text{rss}}$. Thus, there exists i such that $\hat{A}_i \preceq_M L(\Gamma_n^{(1)})$ but $A_i \not\preceq_M L(\Gamma_n^{(1)})$. For simplicity, we consider the case $i = k$. [CKP14, Proposition 2.4] give the existence of projections $p \in \hat{A}_k, q_1 \in L(\Gamma_n^{(1)})$, a partial isometry $v \in M$, and a $*$-isomorphism $\psi : p\hat{A}_kp \to B \subset q_1L(\Gamma_n^{(1)})q_1$ such that

(a) $B \vee (B' \cap q_1L(\Gamma_n^{(1)})q_1) \subset q_1L(\Gamma_n^{(1)})q_1$ is a finite index inclusion of algebras,

(b) $\psi(x)v = vx$ for all $x \in p\hat{A}_kp$.

As in the proof of Lemma 5.25, we may assume $p = p_1 \cdots p_{k-1}$ where $p_i \in A_i$ are projections such that $B = \psi(p\hat{A}_kp) = \psi(p_1A_ip) \vee \cdots \vee \psi(p_{k-1}A_{k-1}p) = B_1 \vee \cdots \vee B_{k-1}$. Thus we have

(c) $\psi(p_iA_ip) = B_i$,

(d) $\psi(x)v = vx$ for all $x \in p\hat{A}_kp$,

(e) $B_1 \vee \cdots \vee B_{k-1} \vee (B' \cap q_1L(\Gamma_n^{(1)})q_1) \subset q_1L(\Gamma_n^{(1)})q_1$ is a finite index inclusion of algebras.

We first assume $n = 2$. In this case, Lemma 5.25 implies $k = 2$. Since $\Gamma_n^{(1)} \in \text{Quot}_1(\mathcal{C}_{rss})$, Lemma 4.3 implies $\mathcal{Z}(B' \cap q_1L(\Gamma_n^{(1)})q_1)$ cannot have any diffuse part and therefore is completely atomic. Thus multiplying v by some minimal central projection $q' \in B' \cap q_1L(\Gamma_n^{(1)})q_1$ so that $vq' \neq 0$, we may assume $\psi(pA_1p) = B \subset q_1L(\Gamma_n^{(1)})q_1$ is a finite index inclusion of factors. Moreover, $\dim_{\mathbb{C}}(\mathcal{Z}(qL(\Gamma_n^{(1)}))q) \leq [qL(\Gamma_n^{(1)})q : B]_{PP} < \infty$ since B is a II_1 factor. Thus, after multiplying again by a minimal central projection, we may assume $B \subset q_1 L(\Gamma_n^{(1)}) q_1$ is a finite index inclusion of II_1 factors. We claim there exists a projection $r \in L(\Gamma_n^{(1)})' \cap M$ such that.

$$r[L((\Gamma_n^{(1)}) \vee L(\Gamma_n^{(1)})' \cap M)]r \subset rMr \tag{5.24}$$

is a finite index inclusion of II_1 factors.

To this end, the downward basic construction [Jo81, Lemma 3.1.8] gives a projection $e \in q_1 L(\Gamma_n^1) q_1$ and a subfactor $C \subset B \subset q_1 L(\Gamma_n^{(1)}) q_1 = \langle B, e \rangle$ such that $[B : C] = [q_1 L(\Gamma_n^{(1)}) q_1 : B]$, $Ce = eL(\Gamma_n^{(1)})e$ and $Ce \cong C$. Then the restriction $\psi^{-1} : C \to D \subset pA_1p$ is a $*$-isomorphism such that $[pA_1p : D] < \infty$ with $\psi^{-1}(y)v^* = v^*y$ for all $y \in C$. Let $\theta : Ce \to C$ be the $*$-isomorphism given by $xe \mapsto x$ and denote by $v' = ev$. If we suppose $v' = 0$, we would have $vv^*xe = xvv^*e = 0$ for all $x \in B$. As $\langle B, e \rangle e = Be$, $vv^*t = 0$ for all $t \in \langle B, e \rangle$. However, since q is the central support of e in $\langle B, e \rangle$, this would yield $vv^* = 0$. Thus it follows that $\varphi = \psi^{-1} \circ \theta : eL(\Gamma_n^{(1)})e \to D$ is a $*$-isomorphism satisfying

$$\varphi(y)w^* = w^*x \text{ for all } y \in eL(\Gamma_n^{(1)})e \tag{5.25}$$

where w^* is the partial isometry from the polar decomposition of $v^*e = |v^*e|w^*$. Note that $s = w^*w \in D' \cap pMp$ and $ww^* \leq e$. Thus equation (5.25), we obtain

$$w^* L(\Gamma_n^{(1)}) w = \varphi(eL(\Gamma_n^{(1)})e) w^*w = Ds \tag{5.26}$$

$$(w^* L(\Gamma_n^{(1)}) w)' \cap sMs = (Ds)' \cap sMs. \tag{5.27}$$

First note $A_2p \subset D' \cap pMp$. Since $D \subset pA_1p$ is a finite index inclusion, so are the inclusions $D \vee A_2p \subset p(A_1 \vee A_2)p \subset pMp$ and hence $D \vee A_2p \subset pMp$ is a finite index inclusion of algebras. By the local index formula, we also have $Ds \vee s(D' \cap M)s \subset sMs$ is also a finite index inclusion of II_1 factors.

Let $r = ww^*$ and $u \in M$ a unitary with $w^* = ur$. Conjugating (5.26) and (5.27) by u implies $r[L(\Gamma_n^{(1)}) \vee L(\Gamma_n^{(1)})' \cap M]r = L(\Gamma_n^{(1)})r \vee (L(\Gamma_n^{(1)})' \cap rMr) \subset rMr$ is a finite index inclusion of II_1 factors (after shrinking r is necessary). By Proposition 5.26, there exists a finite index subgroup $\Sigma_1 < \Gamma_n^{(1)}$ such that $\Sigma_i \in \mathrm{Quot}_1(\mathcal{C}_{\mathrm{rss}})$ with $[\Gamma : \Sigma_1 \times \Sigma_2] < \infty$ and $rA_2r \subset ru^*L(\Sigma_2)ur$, where $\Sigma_2 = V_\Gamma(\Gamma_n^{(1)})$. Since $ru^*L(\Gamma_n^{(1)})ur \subset rA_1r$ is a finite index inclusion of II_1 factors, so is the inclusion $ru^*L(\Sigma_1)r \subset rA_1r$. Performing the downward basic construction gives a subfactor $A_1f \subset ru^*L(\Sigma_1)r$ where $f \in A_1' \cap rMr$.

Since $rA_2r \subset r(A_1' \cap M)r$ is a finite Pimnser-Popa index inclusion of algebras, so is the inclusion $rA_2r \subset ru^*L(\Omega)ur$. Thus cutting once again by a minimal projection we have $r_2A_2r_r \subset r_2u^*L(\Sigma_2)ur_2$ is a finite index inclusion of II_1 factors.

Now suppose the result holds for all icc groups $\Lambda \in \mathrm{Quot}_{n-1}(\mathcal{C}_{\mathrm{rss}})$ for some $n \in \mathbb{N}$. Take $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$ an icc group. Proceeding as in the case when $n = 2$, we may assume $\hat{A}_k \preceq_M L(\Gamma_n^{(1)})$. More precisely, since the center of $A_1 \vee \cdots \vee A_k$ is trivial, by [Va10, Lamma 2.5, Proposition 2.6] $\hat{A}_k \preceq^s_M L(\Gamma_n^{(1)})$. [CKP14, Proposition 2.4] give the existence of projections $p \in A_1, q \in L(\Gamma_n^{(1)})$, a partial isometry $v \in M$ and a $*$-isomorphism $\psi : pA_1p \to B \subset q_1L(\Gamma_n^{(1)})q_1$ such that

(f) $B \vee (B' \cap q_1L(\Gamma_n^{(1)})q_1) \subset q_1L(\Gamma_n^{(1)})q_1$ is a finite index inclusion of algebras,

(g) $\psi(x)v = vx$ for all $x \in pA_1p$.

If $B_k = B' \cap q_1L(\Gamma_n^{(1)})q_1$ was not diffuse, we cutting by a minimal central projection to obtain $B \subset q_1 L(\Gamma_n^{(1)})q_1$ is finite Pimnser-Popa index inclusion of algebras. As before, $\mathcal{Z}(qL(\Gamma_n^{(1)}))$ is finite dimensional. Thus, we cut by an appropriate minimal central projection to obtain a finite index inclusion of II_1 factors and proceed exactly as in the case when $n = 2$.

Now suppose B and $B_k = B' \cap q_1 L(\Gamma_n^{(1)})q_1$ are both diffuse. Then, by cutting by a minimal central projection if necessary, we have $B \vee B_k \subset q_1 L(\Gamma_n^{(1)})q_1$ is a finite index inclusion of II_1 factors. By the induction hypothesis, there exists a unitary $w \in q_1 L(\Gamma_n^{(1)})q_1$, $\Lambda 1, \cdots, \Lambda_k$ subgroups of $\Gamma_n^{(1)}$ and projections $p_i \in B_i$, $q_i \in L(\Lambda_i)$ so that

- $w(p_1 B_i p_1 w^* \subset q_1 L(\Lambda_i) q_1$ is a finite index inclusion of II_1 factors
- $\Gamma_n^{(1)}$ is commensurable to $\Lambda_1 \times \cdots \times \Lambda_k$
- $\Lambda \in \mathrm{Quot}_{m_1}(\mathcal{C}_{\mathrm{rss}})$ with $1 \leq m_1 < n - 1$.
- $\sum m_i = n - 1$.

Letting $\Lambda = \Lambda_1 \times \cdots \times \Lambda_k$ and proceeding as in the case when $n = 2$, there exists a projection s such that

$$s[L(\Lambda) \vee L(\Lambda)' \cap M]s \subset sMs$$

is a finite index inclusion of II_1 factors. Once we apply Lemma 5.26 and follow the same procedure as in the case when $n = 2$, we may find a finite index subgroup $\Lambda_1 < \Lambda$ so that $r\hat{A}_j r \subset ru^* L(\Lambda_1)ur$ and $rA_k r \subset L(\Lambda_2)$ are finite index inclusions of II_1 factors with $[\Gamma : \Lambda_1 \times \Lambda_2] < \infty$. Furthermore, we may assume

$$r\hat{A}_j r = rA_1 r \vee \cdots \vee rA_{k-1} r.$$

Letting $\Gamma_k = \Sigma_2$ and once again applying the induction hypothesis, we may appropriately identify corners of A_i with groups Γ_i so that $\Gamma_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{\mathrm{rss}})$ with $n_1 + \cdots + n_k = n$.

□

CHAPTER 6
FURTHER AND FINAL REMARKS

6.1 Broader Impacts

Previous investigations into the structural properties of von Neumann algebras culminated establishing primeness results for a large class of von Neumann algebras, [IPP05, Oz05, CH08, Pe06]. In a work in progress, we obtained structural results in the converse for non-prime von Neumann algebras arising from a large class of groups. Specifically, we provide instances where one is able infer the product group structure of the group is in one to one correspondence with a non-prime von Neumann algebra.

We demonstrate another application of product rigidity for the von Neumann algebras of *amalgamated free products* of groups, hereby initialized as AFP groups. While it is well known that non-trivial free products of von Neumann algebas product prime von Neumann algebras, very few structural results exist for amalgamated free products. By applying an intertwining results of Ioana and Vaes [Io12, Va13] for amalgamated free products, akin to the dichotomy for normalizers in crossed products by hyperbolic groups, preliminary analysis conducted by Chifan, Sucpikarnon, and myself the following result for AFP von Neumann algebras.

Theorem 6.1. *Consider $M = M_1 *_P M_2$ be an amalgamated free product von Neumann algebra and let $A_1, A_2 \subset M$ be two commuting diffuse subalgebras that generate a finite index subalgebra $A_1 \vee A_2 \subset M$. Then there exists $i = 1, 2$, $p \in \mathcal{P}(P)$, $r \in \mathcal{P}(A_i)$, $u \in \mathcal{U}(M)$, $B \subset pPp$ and a partial isometry $v \in M$ with $vv^* \in B' \cap pMp$*

such that the following hold:

1. $urA_iru^* = Bvv^*$;

2. $[pPp : B \vee B' \cap pPp]_{PP} < \infty$;

3. $[pM_kp : B \vee B' \cap pM_kp]_{PP} < \infty$ *for all* $k = 1, 2$, *and*

4. $[pMp : B \vee B' \cap pMp]_{PP} < \infty$.

Moreover, we expect to establish a variant of product rigidity in the case when the von Neumann algebra M arises from a group. However, this result immediately implies primeness results for the von Neumann algebras groups the following classes of groups.

Corollary 6.2. *Let* Γ *be a group in one of the following classes:*

1. *the integral two-dimensional Cremona group* $Aut_k(k[X, Y])$;

2. *the Higman group and its variations introduced by Monod [Mo16];*

3. *the Burger-Mozes groups [BM01];*

Then $L(\Gamma)$ *is prime.*

6.2 Future Research Directions

The direct and amalgamated free products are but a small subclass of larger elementary constructions of groups known as tree and graph products. The tree product of Serre is a powerful organizational tool for the description groups which arise as generalized amalgamated free products, such as HNN extensions, and their subgroups. In this description, Serre associates a group Γ to a graph $\mathcal{G}$ built from the following data:

1. A connected graph $\mathcal{G}$.

2. An assignment of a group Γ_v to every vertex $v \in \mathcal{V}(\mathcal{G})$.

3. An assignment of a group A_e for every $e \in \mathcal{E}(\mathcal{G})$ such that if $v_1, v_2 \in \mathcal{V}(\mathcal{G})$ are connected by e, then $A_e \leqslant \Gamma_{v_i}$.

In the case where the graph has exactly two vertices v_1, v_2 corresponding to groups Γ_1, Γ_2 and a single edge e associated to the common subgroup $\Lambda \leqslant \Gamma_i$, this construction gives the AFP group $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$.

A parallel tool is the graph product of groups originally introduced and studied in the dissertation of E. Green [Gr90]. Let $\mathcal{G}$ be a graph without loops and multiple edges and denote is vertex set by $\mathcal{V}(\mathcal{G}) = \mathcal{V}$. Let $\{\Gamma_v\}_{v\in\mathcal{V}}$ be a family of groups, indexed over the vertices of $\mathcal{G}$ and called vertex groups. The graph product of $\{\Gamma_v\}_{v\in\mathcal{V}}$ is defined as the quotient of the free product $*_{v\in\mathcal{V}}\Gamma_v$ by the relations $[\Gamma_u, \Gamma_v] = 1$ whenever u and v are adjacent vertices in $\mathcal{G}$. These groups are natural generalizations of the right angled Artin and Coxeter groups.

Merging the constructions, we define colored graph product of groups where the coloring of an edge determines either amalgamation over a common subgroup or commutation of the groups at the corresponding vertices. While there are procedures where one can iterate the construction of the graphy product by "removing" vertices and edges for graph and tree product, complete analysis of the combined constructions remain unknown. We expect to find instances where we can recover combinatorial data of the graph from the resulting von Neumann algebra such as the existence of a bi-partition.

6.3 Final Remarks

We close by noting the results in Chapter 5 are the first deduce structural results for von Neumann algebras which are not *virtually prime*, i.e. there exists a splitting of the von Neumann algebra as a tensor product of II_1 factors up to finite index. This encourages us to consider virtual statements about von Nuemann algebras. In essence we propose an investigation into structural results for II_1 factors M and N have a common finite index subalgebra P. [CKP14, Theorem 2.4] is an intertwining result of

However, this is not the entire story as we have yet to determine a construction of von Neumann algebras which corresponds to commensurability up to finite kernel for groups. Our current investigations suggest this corresponds to the notion of *stable equivalence*, i.e. isomorphism of von Neumann algebras up to compression. for von Neumann algebras. If this should be the case, one might investigate a weaker form of W^* rigidity for groups, which we term *commensurable rigidity.* Namely, does there exists an icc group Γ such that whenever $L(\Gamma) \cong L(\Lambda)^t$ it follows Γ and Λ are commensurable up to finite kernel?

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