Masthead Logo

University of Iowa
Iowa Research Online

Theses and Dissertations

Summer 2017

Structural results in group von Neumann algebra

Sujan Pant
University of Iowa

Copyright © 2017 Sujan Pant

This dissertation is available at Iowa Research Online: https://ir.uiowa.edu/etd/5822

Recommended Citation
Pant, Sujan. "Structural results in group von Neumann algebra." PhD (Doctor of Philosophy) thesis, University of Iowa, 2017.
https://doi.org/10.17077/etd.dysxly3f

Follow this and additional works at: https://ir.uiowa.edu/etd

Part of the Mathematics Commons

STRUCTURAL RESULTS IN GROUP VON NEUMANN ALGEBRA

by

Sujan Pant

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

August 2017

Thesis Supervisor: Ionut Chifan, Associate Professor

Copyright by
SUJAN PANT
2017
All Rights Reserved

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Sujan Pant

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the August 2017 graduation.

Thesis Committee: ______________________________
Ionut Chifan, Thesis Supervisor

____
Richard Baker

____
Victor Camillo

____
Palle Jorgensen

____
Surjit Khurana

ACKNOWLEDGEMENTS

Firstly, I want to thank mom, dad, aunt and my sister, in particular. To my sister, from the moment I came to the USA for my undergraduates 10 years ago all the way to my Phd now, your unconditional support was paramount in my success academically and life in general. I want to thank my spouse Eboni for being the source of love and energy throughout this process.

I want to thank my adviser Ionut Chifan for his patience and guidance. You showed me how to be a mathematician and endure in spite of setbacks along the way to succeed in all facet of life. You helped me navigate and overcome various difficulties that I faced in the past 4 years.

I want to thank my thesis committee members Richard Baker, Victor Camillo, Palle Jorgensen, and Surjit Khurana for their time, patience and availability. I also want to acknowledge Paul Muhly, and Fred Goodman for their encouragement and advice throughout my time at University of Iowa.

Finally, I want to thank many mentors I met during my undergraduate degree. In particular, I will be thankful forever to my undergraduate research advisor Dennis Merino for his encouragement and guidance. Lastly, I want to thank Katherine Pedersen, you believed in me more than I believed in myself.

ABSTRACT

Chifan, Kida, and myself introduced a new class of non-amenable groups denoted by $\mathbf{NC} \cap \mathbf{Quot}(\mathcal{C}_{rss})$ which gives rise to *prime* von Neumann algebras. This means that for every $\Gamma \in \mathbf{NC} \cap \mathbf{Quot}(\mathcal{C}_{rss})$ its group von Neumann algebra $L(\Gamma)$ cannot be decomposed as a tensor product of diffuse von Neumann algebras. The class $\mathbf{NC} \cap \mathbf{Quot}(\mathcal{C}_{rss})$ is fairly large as it contains many natural examples of groups, some intensively studied in various areas of mathematics: all infinite central quotients of pure surface braid groups; all mapping class groups of (punctured) surfaces of genus $0, 1, 2$; most Torelli groups and Johnson kernels of (punctured) surfaces of genus $0, 1, 2$; and, all groups hyperbolic relative to finite families of residually finite, exact, infinite, proper subgroups.

In a separate investigation, de Santiago and myself were able to extend the previous techniques that allowed us to eliminate the usage of the $\mathbf{NC}$ condition and ultimately classify all the possible tensor factorization of the von Neumann algebras of groups that belong solely to $\mathbf{Quot}(\mathcal{C}_{rss})$. This provides a far-reaching generalization of the aforementioned primeness results; for instance, we were able to show that if Γ is a poly-hyperbolic group, then whenever we have a tensor decomposition $L(\Gamma) \cong P_1 \bar{\otimes} P_2 \bar{\otimes} \cdots \bar{\otimes} P_n$ then there exists a product decomposition $\Gamma \cong \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ with $\Gamma_i \in \mathbf{Quot}(\mathcal{C}_{rss})$ and, up to amplifications, we have $L(\Gamma_i) \cong P_i$ for all $i = 1, n$.

PUBLIC ABSTRACT

John von Neumann started the study of von Neumann algebra to study group representations and quantum mechanics. Analogous to prime numbers, prime von Neumann algebra cannot be decomposed into tensor product of diffuse subalgebras.

By analogy, Popa introduced the notion of prime von Neumann algebras. He showed that the von Neumann algebra arising from free group on uncountably generated free group is prime. Since one way to construct Von Neumann algebras is through groups, understanding the shared properties between the initial data (the group) and derived group von Neumann algebra is one of the main areas of study in this field. We show that poly-hyperbolic groups are prime and provide vast class of of groups that gives rise to von Neumann algebras.

By analogy with decomposing numbers into products of primes there is a natural notion of decomposing von Neumann algebras as tensor product of von Neumann subalgebras. In this document we are able to classify, in terms of the initial data, all possible prime decompositions for the von Neumann algebras arising from extensions of hyperbolic groups.

TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

In pioneering work [Po81] Sorin Popa discovered that the (non-separable) factors $L(\mathbb{F})$ arising from uncountably generated free groups $\mathbb{F}$ are *prime*, i.e., $L(\mathbb{F})$ cannot be decomposed as a tensor product of diffuse factors. Much later, using Voiculescu's influential free probability theory [VDN92, V94, V96], Liming Ge was able to show primeness for all factors associated with countably generated, non-abelian free groups as well, [G98]. In the context of free probability other examples of prime factors were subsequently unveiled [Shl00, Shl04, Ju07].

By developing a different perspective, largely based on C^*-algebraic methods, Narutaka Ozawa obtained a far-reaching generalization of these results by showing that all factors $L(\Gamma)$ associated with non-elementary hyperbolic groups Γ are in fact *solid*, i.e., for every diffuse, amenable subalgebra $A \subset L(\Gamma)$ its relative commutant $A' \cap L(\Gamma)$ is again amenable; in particular, it follows that $L(\Gamma)$ is prime. Notice that Ozawa's solidity result holds for all factors associated with bi-exact groups [Oz03, Oz05, BO08].

In the early 2000's Popa introduced a completely new conceptual framework to study von Neumann algebras, now termed *Popa's deformation/rigidity theory*. This novel approach has generated spectacular progress over the last decade , leading to complete solutions to many longstanding open problems in the classification of von Neumann algebras and equivalence relations arising from group actions, [Po01, Po03, Po04, IPP05, Po06, Po06b]. The theory develops a powerful technical apparatus designed to incorporate meaningful cohomological, geometric, and algebraic information of a group and its actions in the analytic context of von Neumann algebras. Overtime, these methods became more and more precise and sophisticated and reveled unprecedented connections with cohomological, geometric, and dynamical aspects in group theory.

These techniques are very suitable to study the primeness phenomenon as well. Indeed, using his free malleable deformations in combination with a novel spectral gap argument, Popa was able to find a new, elementary proof for Ozawa's solidity result for the non-amenable free group factors, [Po06]. His approach laid out the foundations for many important subsequent developments regarding the algebraic structure of factors. For instance, it provided the correct insight which later allowed Ozawa and Popa to show in a remarkable work [OP07] that all non-amenable free group factors $L(\mathbb{F})$ are in fact *strongly solid*, i.e., for every diffuse amenable subalgebra $A \subset L(\mathbb{F})$, its normalizing group $\mathcal{N}_{L(\mathbb{F})}(A) = \{u \in \mathcal{U}(L(\mathbb{F})) : uAu^* = A\}$ generates an amenable von Neumann subalgebra in $L(\mathbb{F})$—a result of great influence for the entire subsequent development on the classification of normalizers of algebras in many classes of factors.

Exploiting a new viewpoint which originates in his recent ingenious study of unbounded derivations on von Neumann algebras, Jesse Peterson further showed that every non-amenable, icc group with positive first ℓ^2-Betti number gives rise to a prime factor, [Pe06].

These results along with Ozawa's earlier solidity results have spawned a rich activity in the classification of von Neumann algebras. Numerous technical outgrowth of these methods by several authors have led overtime to the discovery to many striking structural results including primeness, (strong) solidity, uniqueness of Cartan subalgebra, and beyond for large classes of von Neumann algebras, [Po08, OP07, OP08, CH08, CI08, Pe09, PV09, FV10, Io10, IPV10, HPV10, CP10, Si10, Va10, Fi11, CS11, CSU11, Io11, PV11, HV12, PV12, Io12, Bo12, BHR12, Is12, BV13, Va13, Is14, CIK13, VV14, BC14, Ho15, DHI16].

1.1 Statements of main results

In this thesis we introduce new families of groups which give rise to prime von Neumann algebras. Many of these groups are intensively studied in various areas of Mathematics, especially topology and geometric group theory. Over time, via deep topological and geometric methods, many strong classification results emerged regarding the structure of these groups in both discrete and measurable setting. However, momentarily, little is known about the structure of the von Neumann algebras associated with these groups and this thesis initiates a study in this direction.

In order to properly introduce our result we recall several definitions. For more details the reader may consult next chapter of this document.

Definition 1.1.1. A group Γ belongs to class $\mathbf{NC} \cap \mathbf{Quot}(\mathcal{C}_{rss})$ if the following two conditions are satisfied:

• **NC**(stands for "negatively curved" groups) Γ is non-amenable and admits an unbounded quasi-cocycle valued into one of its mixing, weakly-ℓ^2, orthogonal representations;

- $\mathbf{Quot}(\mathcal{C}_{rss})$: Γ is a finite-step extension of groups belonging to $\mathcal{C}_{rss}$—the collection of all non-elementary hyperbolic groups and non-amenable, non-trivial free products of exact groups.

While this definition is somehow technical and may not be entirely illuminating, the class $\mathbf{NC}\cap\mathbf{Quot}(\mathcal{C}_{rss})$ contains in fact many natural examples of groups. For instance, it includes all groups that are commensurable with the following concrete families of groups:

1. Any infinite, central quotient of the pure braid group $PB_n(S_{g,k})$ of n strands on a surface $S_{g,k}$—in particular, all surface pure braid groups $PB_n(S_{g,k})$, for $n \geq 1$ and either $g = 1$ and $k \geq 1$ or $g \geq 2$ and $k \geq 0$;
2. Any mapping class group $\mathrm{Mod}(S_{g,k})$, for $0 \leq g \leq 2$ and $2g + k \geq 4$;
3. Any Torelli group $\mathcal{I}(S_{g,k})$ and Johnson kernel $\mathcal{K}(S_{g,k})$, for $g = 1, 2$ and $2g+k \geq 4$;
4. Any group that is hyperbolic relative to a finite family of exact, residually finite, infinite, proper subgroups.

The central result of this document shows that all groups in $\mathbf{NC}\cap\mathbf{Quot}(\mathcal{C}_{rss})$ give rise to prime von Neumann algebras; in particular, for any $\Gamma \in \mathbf{NC}\cap\mathbf{Quot}(\mathcal{C}_{rss})$ its von Neumann algebra *cannot* be decomposed as $L(\Gamma) = L(\Omega \times \Sigma)$, for any infinite groups Ω and Σ.

Theorem A. *Let Γ be a group that can be realized as a finite-by-($\mathbf{NC}\cap\mathbf{Quot}(\mathcal{C}_{rss})$) group. Denote by $L(\Gamma)$ its corresponding von Neumann algebra. If $p \in L(\Gamma)$ is* *a nonzero projection, then any two diffuse, commuting subalgebras* $B, C \subseteq pL(\Gamma)p$ *generate together a von Neumann subalgebra* $B \vee C$ *which has infinite Pimsner-Popa index in* $pL(\Gamma)p$*. In particular,* $L(\Gamma)$ *is prime and hence* $L(\Gamma) \ncong L(\Omega \times \Sigma)$*, for any infinite groups* Ω *and* Σ*.*

In this form the result is sharp, as in general there are groups Γ in $\mathbf{NC} \cap \mathbf{Quot}(\mathcal{C}_{rss})$ whose algebras $L(\Gamma)$ *do* contain commuting, non-amenable, diffuse subalgebras which, together generate subalgebras in $L(\Gamma)$ of infinite index. To see some basic examples, consider B_n to be the braid group on $n \geq 6$ strands and denote by Z its center. By [FM11, Section 9.2], the quotient $\Gamma = B_n/Z$ can be realized as a subgroup of index n inside the mapping class group of a $(n+1)$-punctured surface of genus zero and hence Theorem 2.1.4 and Examples 2.1.6 c) further imply that $\Gamma \in \mathbf{NC}$. Moreover, using Birman short exact sequence, Corollary 2.2.6 shows that Γ is a $(n-2)$-step extension of non-abelian free groups and hence $\Gamma \in \mathcal{P}$. On the other hand, notice that B_n contains mixed braid subgroups of the form $B_p \times B_q < B_n$, where $p+q = n$ with $p, q \geq 3$. Then one can check that the quotients $\Gamma_1 = B_p/Z$ and $\Gamma_2 = B_q/Z$ are commuting, non-amenable subgroups of Γ which together generate a subgroup $\langle \Gamma_1, \Gamma_2 \rangle < \Gamma$ of infinite index. This canonically implies that $L(\Gamma_1)$ and $L(\Gamma_2)$ are commuting, non-amenable, diffuse subalgebras of $L(\Gamma)$ which together generate a von Neumann subalgebra $L(\langle \Gamma_1, \Gamma_2 \rangle) \subset L(\Gamma)$ of infinite index. Notice that, all such groups in $\mathbf{NC} \cap \mathbf{Quot}(\mathcal{C}_{rss})$ will give rise to prime von Neumann algebras which are not solid in the sense of Ozawa, [Oz03].

We believe that Theorem A can be further improved by showing that the algebra $B \vee C$ is actually never co-amenable inside $pL(\Gamma)p$, [Po86]. Notice that this will follow verbatim from our current proofs if one will be able to show an analogue of Proposition 3.1.12 in the context of co-amenable inclusions rather that finite index inclusions.

The proof of our result is based on Popa's deformation/rigidity theory and is obtained by induction on n where $\Gamma \in \mathbf{NC} \cap \mathbf{Quot}_n(\mathcal{C}_{rss})$. For the induction step we use in an essential way recent, powerful results due to Popa and Vaes [PV11, PV12] and to Ioana [Io12] regarding the classification of normalizers of subalgebras in von Neumann algebras arising from actions by non-elementary hyperbolic groups and by free product groups, respectively. Assuming by contradiction that $B \vee C \subseteq pL(\Gamma)p$ has finite index then condition $\Gamma \in \mathbf{Quot}_n(\mathcal{C}_{rss})$ enables us to employ these structural results, via the methods developed in [CIK13], to intertwine B (or C) onto a subalgebra $B_0 \subseteq qL(\Sigma)q \subset qL(\Gamma)q$, where $\Sigma \lhd \Gamma$ is a normal subgroup satisfying $\Sigma \in \mathbf{Quot}_{n-1}(\mathcal{C}_{rss})$ and $q \in B_0$ is a nonzero sub-projection of p. Moreover there exists a subalgebra $C_0 \subset qL(\Sigma)q$ commuting to B_0 such that $B_0 \vee C_0 \subseteq qL(\Sigma)q$ has finite index. Since $\Gamma \in \mathbf{NC}$ then by [CSU13, Theorem 2.1] we have $\Sigma \in \mathbf{NC} \cap \mathbf{Quot}_{n-1}(\mathcal{C}_{rss})$ and by the induction hypothesis one can find a nonzero corner rB_0r of finite index in $rL(\Sigma)r$. On the other hand, since $\Gamma \in \mathbf{NC}$, then a spectral gap argument shows that the corresponding weak deformations V_t on $L(\Gamma)$ arising from an unbounded quasi-cocycle on Γ [CS11] will converge uniformly to the identity on the unit ball $(B)_1$. From this, developing new aspects in the infinitesimal analysis of V_t (Section 4.1) we further show that V_t converges uniformly to the identity on the unit ball $(rB_0r)_1$ and by the finite index assumption it follows that a V_t has a uniform decay on the unit ball $(r(L\Sigma)r)_1$. Hence the quasi-cocycle is bounded on Σ and by [CSU13, Theorem 2.1] it is bounded on Γ, which is a contradiction; thus $B \vee C$ must have infinite index in $pL(\Gamma)p$.

Notice that, a spectral gap argument [CS11, Proposition 1.7 (3)] shows that for any group $\Gamma \in \mathbf{NC}$, any two commuting, infinite subgroups $\Gamma_1, \Gamma_2 < \Gamma$ generate an infinite index subgroup $\langle \Gamma_1, \Gamma_2 \rangle$ of Γ (in other words, Γ is not presentable by products). This should be seen as evidence supporting the far-reaching conjecture that Theorem A actually holds for all groups Γ satisfying only condition **NC** (even, without the mixing assumption on the representation). However, from a technical point of view, a successful implementation of this argument in the von Neumann algebra setting seems out of reach momentarily and depending heavily on investigating new aspects of the infinitesimal analysis of the weak deformations arising from quasi-cocycles. Unlike in the case of 1-cocycles, the weak deformations arising from quasi-cocycles of groups seem to lack good averaging and uniform bimodularity properties which makes their analysis quite difficult. In our situation some of these difficulties could be by-passed through the knowledge that Γ admits a "finite resolution" by groups in $\mathcal{C}_{rss}$. Hence our result can be viewed as a first instance when knowing a little bit more information about the group (in addition of being in **NC**) could decisively enhance the analysis on the weak deformations to conclude primeness results for many such group von Neumann algebras. It is then conceivable that there is actually an entire spectrum of such properties and a thorough investigation may reveal interesting results in this direction.

As a byproduct of these methods, we obtain new applications of deformation/rigidity techniques to the algebraic structure of groups. Indeed, outgrowths of our methods in combination with the techniques developed in [CSU13] allows us to deduce a result which complements [CSU13, Theorem 3.5] in the case of groups satisfying condition **NC** above.

Theorem B. *For every* $\Gamma \in \mathbf{NC}$ *there exists a short exact sequence of groups* $1 \to F \to \Gamma \to \Gamma_0 \to 1$*, where* F *is a finite and* Γ_o *is infinite conjugacy class. In particular, if* Γ *is assumed torsion free then* Γ *is infinite conjugacy class and non-inner amenable.*

In particular, this provides a more quasi-cohomological explanation for some recent results on non-inner amenability of acylindrically hyperbolic groups by Dahmani, Guirardel, and Osin [DGO11], Osin [Os13], and Minasyan and Osin [MO13]. This result also implies that every groups in $\mathbf{NC} \cap \mathbf{Quot}(\mathcal{C}_{rss})$ gives von Neumann algebra with finite dimensional center, result that is implicitly used in the proof of Theorem A.

Recently together with De Santiago I was able to find new technology that allowed us to classify all possible tensor decompositions of $L(\Gamma)$ whenever Γ belongs solely in $\mathbf{Quot}(C_{rss})$. In particular this provides a far-reaching generalization of Theorem A. The precise statement is the following:

Theorem C. *Let* $\Gamma \in \mathbf{Quot}_n(\mathcal{C}_{rss})$ *be an icc group and suppose there exists* $A_1, \ldots, A_k \subset L(\Gamma)$ *commuting diffuse subalgebras of* $L(\Gamma)$ *generating a finite index subalgebra.*

Then there exists a projection $p_i \in A_i$, finite index subfactors $D_i \subset p_i A_i p_i$, groups $\Gamma_1, \ldots, \Gamma_k$, and a unitary $u \in pL(\Gamma)p$ such that

1. $D_i \subset p_1 u^* L(\Gamma_i) u p_i$ *is a finite index inclusion of algebras,*
2. $\Gamma_i \in \mathbf{Quot}_{n_i}(\mathcal{C}_{rss})$
3. $\sum_{i=1}^{k} n_i = n$,
4. Γ *is commensurable to* $\Gamma_1 \times \cdots \times \Gamma_k$.

In particular $L(\Gamma)$ is prime if and only if Γ is virtually indecomposible as a product of groups in $\mathbf{Quot}(\mathcal{C}_{rss})$.

In addition this theorem can be used to obtain an enhanced version of the unique prime factorization phenomenon à la Ozawa-Popa [OP03] for groups in class $\mathbf{Quot}(C_{rss})$. Namely, we showed the following:

Theorem D. *Suppose $\Gamma_i \in \mathbf{Quot}t_{n_i}(\mathcal{C}_{rss})$ and $\Lambda_i \in \mathbf{Quot}_{m_j}(\mathcal{C}_{rss})$ are poly-hyperbolic icc groups with $L(\Gamma_1 \times \cdots \times \Gamma_n) \cong L(\Lambda_1 \times \cdots \times \Gamma_m)$. Then $n = m$ and we have $L(\Gamma_i) \cong L(\Lambda_i)$, up to permutation and amplification.*

CHAPTER 2
GENERALITIES ON GROUPS

We introduce new families of groups which give rise to prime von Neumann algebras. Many of these groups are intensively studied in various areas of Mathematics, especially topology and geometric group theory. Over time, via deep topological and geometric methods, many strong classification results emerged regarding the structure of these groups in both discrete and measurable setting. However, momentarily, little is known about the structure of the von Neumann algebras associated with these groups. Formally, we define our class of groups as follows:

Definition 2.0.1. An orthogonal representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ is called *mixing* if for every $\xi, \eta \in \mathcal{H}$ we have $\lim_{\gamma\to\infty}\langle \pi_\gamma(\xi), \eta\rangle = 0$.

Basic examples are any multiple of the (real) left regular representation, $\oplus\ell^2_{\mathbb{R}}(\Gamma)$. The mixing property is preserved under many basic operations on representations including: taking sub-representations, restrictions to infinite subgroups, direct sums, tensor products, and inductions to finite index supra-groups.

Definition 2.0.2. A representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ is called *weakly-ℓ^2* if it is weakly contained in the left regular representation $\ell^2_{\mathbb{R}}(\Gamma)$.

It follows from the definitions that any restriction of a weakly-ℓ^2 representation to any of its subgroups is again weakly-ℓ^2. Moreover, the weakly-ℓ^2 property is preserved under direct sum and under induction to supragroups (this follows from a similar proof with [BHV05, Theorem F.3.5]). Thus, if Γ is a group and $\{\Sigma_i\}$ is a countable family of amenable subgroups then by above it follows that the multiple $\oplus_i \ell^2_{\mathbb{R}}(\Gamma/\Sigma_i)$ is weakly-ℓ^2.

Definition 2.0.3 (Notation 0.1 in [CSU13])**.** We say that a group Γ belongs to class NC if it is non-amenable and there exists a weakly-ℓ^2, mixing, orthogonal representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ such that $\mathcal{QH}^1_{as}(\Gamma, \pi) \neq \emptyset$.

Definition 2.0.4. A group Γ belongs to class $\mathbf{NC} \cap \mathbf{Quot}(\mathcal{C}_{rss})$ if the following two conditions are satisfied:

- **NC**(stands for "negatively curved" groups) Γ is non-amenable and admits an unbounded quasi-cocycle valued into one of its mixing, weakly-ℓ^2, orthogonal representations;

- $\mathbf{Quot}(\mathcal{C}_{rss})$: Γ is a finite-step extension of groups belonging to $\mathcal{C}_{rss}$—the collection of all non-elementary hyperbolic groups and non-amenable, non-trivial free products of exact groups.

2.1 Class NC

Let Γ be a countable discrete group and let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation. A map $q : \Gamma \to \mathcal{H}$ is called a *quasi-cocycle* for π if there exists a constant $C \geq 0$ such that

$$\sup_{\gamma,\lambda\in\Gamma} \|q(\gamma\lambda) - q(\gamma) - \pi_\gamma(q(\lambda))\| \leq C. \tag{2.1.1}$$

The infimum of all constants C satisfying equation (2.1.1) is called the *defect* of quasi-cocycle q and is denoted by $D(q)$. When the defect vanishes the quasi-cocycle q

is actually a 1-cocycle with coefficients in π, [BHV05]. Throughout this document we will assume, without any loss of generality, that any quasi-cocycle is *anti-symmetric*, i.e., $q(\gamma) = -\pi_\gamma(q(\gamma^{-1}))$, for all $\gamma \in \Gamma$. We can make this assumption because in fact every quasi-cocycle is within a bounded distance from an anti-symmetric quasi-cocycle, [Tho09]. The set of all unbounded, anti-symmetric quasi-cocycles for π will be denoted by $\mathcal{QH}^1_{as}(\Gamma, \pi)$. If $\mathcal{B}_{as}(\Gamma, \pi)$ denotes the set of all bounded, anti-symmetric maps $b : \Gamma \to \mathcal{H}$ then we obviously have that $\mathcal{QH}^1_{as}(\Gamma, \pi) + \mathcal{B}_{as}(\Gamma, \pi) = \mathcal{QH}^1_{as}(\Gamma, \pi)$ and $(\mathbb{R} \setminus \{0\}) \cdot \mathcal{QH}^1_{as}(\Gamma, \pi) = \mathcal{QH}^1_{as}(\Gamma, \pi)$.

The following result will be used in the sequel.

Lemma 2.1.1. *Let Γ be a group and let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be a weakly-ℓ^2, mixing orthogonal representation such that $\mathcal{QH}^1(\Gamma, \pi) \neq \emptyset$. If we fix $q \in \mathcal{QH}^1(\Gamma, \pi)$ then for every infinite subgroup $\Lambda \subset \Gamma$ we have that the centralizer $C_\Gamma(\Lambda)$ is amenable or there exists $C \geq 0$ such that $\langle \Lambda, C_\Gamma(\Lambda) \rangle \subseteq B^q_C = \{\lambda \in \Gamma : \|q(\lambda)\| \leq C\}$. Here, for every subset $H \subseteq \Gamma$, we have denoted by $C_\Gamma(H) = \{\gamma \in \Gamma : \gamma h = h\gamma, \text{ for all } h \in H\}$.*

Proof. Assume that the centralizer $C_\Gamma(\Lambda)$ is non-amenable. Then proceeding as in the first part of the proof of [CSU11, Proposition 2.6] one can find $C_1 \geq 0$ such that $\Lambda \subseteq B^q_{C_1}$. Finally since every element of $C_\Gamma(\Lambda)$ normalizes the subgroup Λ, π is mixing, and Λ is infinite, then by the first part in [CSU13, Theorem 2.1] one can find $C \geq 0$ such that $\langle \Lambda, C_\Gamma(\Lambda) \rangle \subseteq B^q_C$. □

Notice that this class includes the class $\mathcal{D}_{reg}$ introduced by Thom in [Tho09]. In the remaining part of this subsection we underline some basic properties of the class NC. Some of these have been already discussed in [CSU13, Section 1] but we will include them here to make the text more self contained. As we will see this class is quite rich, containing large families of groups which are intensively studied in various areas of mathematics, especially topology, geometric group theory, or logic. In addition, we study permanence properties of NC under various canonical constructions in group theory. Many of these properties are either folklore or already appeared in the literature so many of their proofs will be skipped.

Proposition 2.1.2. *The following properties hold:*

a) If $\Sigma < \Gamma_1, \Gamma_2$ *are groups with* Σ *finite and* $[\Gamma_1 : \Sigma] \geq 2, [\Gamma_2 : \Sigma] \geq 3$ *then* $\Gamma_1 *_\Sigma \Gamma_2 \in NC$*,[PV09, CP10];*

b) Given $\Sigma < \Gamma$ *groups with* Σ *nontrivial finite,* Γ *infinite and* $\Theta : \Sigma \to \Gamma$ *is a monomorphism denote by* $HNN(\Gamma, \Sigma, \Theta)$ *the corresponding HNN-extension; then* $HNN(\Gamma, \Sigma, \Theta) \in NC$*, [FV10, CP10];*

c) If a non-amenable group Γ *acts on a tree with finite stabilizers on edges then* $\Gamma \in NC$*;*

d) The class NC *is closed under taking non-amenable normal subgroups;*

e) If Γ *is either a direct product of infinite groups or admits a infinite normal amenable subgroup then* $\Gamma \notin NC$*, [MS04, Po08, Pe06, CS11] .*

Proof. We will only very briefly justify d). Let $\Gamma \in NC$ and let $\Sigma \lhd \Gamma$ be any non-amenable, normal subgroup. Then there exists a weakly-ℓ^2, mixing, orthogonal representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ and $q \in \mathcal{QH}^1_{as}(\Gamma, \pi)$. From the previous observations it follows that the restriction $\pi_{|\Sigma} : \Sigma \to \mathcal{O}(\mathcal{H})$ is again weakly-ℓ^2 and mixing. Moreover, since Σ is normal in Γ then [CSU13, Corollary 2.3] implies that the restriction $q_{|\Sigma} \in \mathcal{QH}^1_{as}(\Sigma, \pi_{|\Sigma})$, and hence $\Sigma \in NC$. $\square$

Proposition 2.1.3. *The class NC is closed under taking supragroups of finite index.*

Proof. Let $\Omega \in NC$ and let $\Omega < \Gamma$ be finite index supragroup. By passing to a finite index subgroup of Ω we can assume without any loss of generality that Ω is normal in Γ. Since Γ satisfies NC there exists a mixing, weakly-ℓ^2, orthogonal representation $\pi : \Omega \to \mathcal{O}(\mathcal{H})$ and an unbounded quasi-cocycle $q : \Omega \to \mathcal{H}$. To get our conclusion we use a construction as in the proof of Kaloujnine-Krasner embedding theorem [KK50], or more commonly known as the induced representation of π from Ω to Γ.

Denote by $\hat{\Gamma} = \Gamma/\Omega = \{\hat{\lambda} = \Omega\lambda : \lambda \in \Gamma\}$ the (finite) quotient group. Fix $\{t_{\hat{\lambda}} : \lambda \in \Gamma\} \subset \Gamma$ a complete set of representatives for the cosets Ω in Γ and notice that $\widehat{t_{\hat{\lambda}}} = \hat{\lambda}$, for every $\lambda \in \Gamma$; hence for every $\gamma, \lambda \in \Gamma$, we have $f_\gamma(\hat{\lambda}) := t_{\hat{\lambda}} \gamma t^{-1}_{\widehat{\lambda\gamma}} \in \Omega$.

Now we define $\tilde{\pi} : \Gamma \to \mathcal{O}(\oplus_{\hat{\Gamma}}\mathcal{H})$ by letting $\tilde{\pi}_\gamma(\oplus_{\hat{\lambda}}\xi_{\hat{\lambda}}) = \oplus_{\hat{\lambda}}(\pi_{f_\gamma(\hat{\lambda})}(\xi_{\widehat{\lambda\gamma}}))$, for every $\gamma \in \Gamma$ and $\oplus_{\hat{\lambda}}\xi_{\hat{\lambda}} \in \oplus_{\hat{\Gamma}}\mathcal{H}$. It is straightforward exercise to check that $\tilde{\pi}$ is an orthogonal mixing representation.

Also since π is weakly-ℓ^2 a similar argument with [BHV05, Theorem F.3.5]) shows $\tilde{\pi}$ is also weakly-ℓ^2.

In the remaining part we show that the map $\tilde{q} : \Gamma \to \oplus_{\hat{\Gamma}}\mathcal{H}$ defined by $\tilde{q}(\gamma) = \oplus_{\hat{\lambda}} q(f_\gamma(\hat{\lambda}))$, for every $\gamma \in \Gamma$, is an unbounded quasi-cocycle, which will conclude the proof. To see this, fix $\gamma, \delta \in \Gamma$ and using the definitions together with the quasi-cocycle inequality for q we get

$$\begin{aligned}\|\tilde{q}(\gamma\delta)-\tilde{\pi}_\gamma(\tilde{q}(\delta))-\tilde{q}(\gamma)\|^2 &= \sum_{\hat{\lambda}}\|q(f_{\gamma\delta}(\hat{\lambda}))-\pi_{f_\gamma(\hat{\lambda})}(q(f_\delta(\widehat{\lambda\gamma})))-q(f_\gamma(\hat{\lambda}))\|^2\\ &= \sum_{\hat{\lambda}}\|q(f_\gamma(\hat{\lambda})f_\delta(\widehat{\lambda\gamma}))-\pi_{f_\gamma(\hat{\lambda})}(q(f_\delta(\widehat{\lambda\gamma})))-q(f_\gamma(\hat{\lambda}))\|^2\\ &\le \sum_{\hat{\lambda}} D^2(q)=|\hat{\Gamma}|D^2(q).\end{aligned}$$

This computation shows that $\tilde{q}$ is a quasi-cocycle satisfying $D(\tilde{q})\le|\hat{\Gamma}|^{1/2}D(q)$.

Finally, picking the section map such that $t_{\hat{e}}=e$ one can see that $\|\tilde{q}(\gamma)\|\ge\|q(\gamma)\|$ and hence $\tilde{q}$ is unbounded. □

The following result parallels similar results for the class of acylindrically hyperbolic groups, [MO13, Lemma 3.8].

Theorem 2.1.4. *The class NC is closed under commensurability. Moreover, every group of the form finite-by-NC belongs to NC.*

Proof. Assuming Γ_i are groups such that $\Gamma_1\in NC$ and Γ_1 is commensurable with Γ_2 we need to show that $\Gamma_2\in NC$. From assumptions there exists a group H such that $H<\Gamma_1$, $H<\Gamma_2$ and $[\Gamma_1:H]<\infty$, $[\Gamma_2:H]<\infty$. Thus one can find a subgroup $H_0<H$ which is normal in Γ_1 and satisfies $[H:H_0]\le[\Gamma_1:H_0]<\infty$. Since $\Gamma_1\in NC$, then by part d) in Proposition 2.1.2 we have that $H_0\in NC$. Since $[\Gamma_2:H_0]=[\Gamma_2:H][H:H_0]<\infty$ then Lemma 2.1.3 further implies that $\Gamma_2\in NC$, which finishes the proof of the first part of the statement.

For the second part, if Γ is a finite-by-NC group there exists $\Omega\lhd\Gamma$ a finite normal subgroup such that $\Lambda=\Gamma/\Omega\in NC$. Denote by $p:\Gamma\to\Lambda$ the canonical projection. Thus there exist $\pi : \Lambda \to \mathcal{O}(\mathcal{H})$ a weakly-ℓ^2, mixing, orthogonal representation and $q \in \mathcal{QH}^1(\Lambda, \pi)$ an unbounded quasi-cocycle. Notice that for every $\gamma \in \Gamma$ and $\xi \in \mathcal{H}$ the formula $\tilde{\pi}_\gamma(\xi) = \pi_{p(\gamma)}(\xi)$ defines an orthogonal representation $\tilde{\pi} : \Gamma \to \mathcal{O}(\mathcal{H})$. Since π is mixing and weakly-ℓ^2 and Ω is finite it follows that $\tilde{\pi}$ is also mixing and weakly-ℓ^2. Also it is a straightforward exercise to check that the map $\tilde{q} : \Gamma \to \mathcal{H}$ given by $\tilde{q}(\gamma) = q(p(\gamma))$ for every $\gamma \in \Gamma$ defines an unbounded quasi-cocycle for $\tilde{\pi}$ whose defect satisfies $D(\tilde{q}) \leq D(q)$; thus $\Gamma \in NC$. □

The following result from [CSU13, Theorem 2.3]

Corollary 2.1.5. *Let Γ be a countable, discrete group together with a family of subgroups G, let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation that is mixing with respect to Γ. Assume $q : \Gamma \to \mathcal{H}$ is a quasi-cocycle and for every $C \geq 0$ we denote by*

$$B_C := \gamma \in \Gamma : \|q(\gamma)\| \leq C.$$

Let $C \geq 0$ and let $k_1, k_2, .., k_l \in \Gamma$ be elements such that, for each $1 \leq i \leq l$ there exists am infinite sequence $(\gamma_{n,i})_n \in B_C$ (without repetitions) such that we have $k_1\gamma_{n,1}k_2\gamma_{n,2}k_3\gamma_{n,3}\cdots k_l\gamma_{n,l} = e$, for all n. Then there exists $1 \leq j \leq n$ such that $k_j \in G_{l_{C+D}}$.

In the end of this subsection we describe some recent important progress in building quasi-cocycles through innovative methods in geometric group theory. Some of the first results emerged from the seminal work of Mineyev [Mi01] and Mineyev, Monod, and Shalom [MMS03], who showed that every Gromov hyperbolic group Γ

admits an unbounded (even proper) quasi-cocycle into a finite multiple of its left-regular representation and hence it is in $\mathcal{D}_{reg}$. This was generalized by Mineyev and Yaman to relatively hyperbolic groups, [MY09]. Hamenstädt [Ha] showed that all weakly acylindrical groups, in particular, non-elementary mapping class groups and $Out(\mathbb{F}_n)$, $n \geq 2$, belong to the class $\mathcal{D}_{reg}$. More recently, Hull and Osin [HO11] and independently Bestvina, Bromberg and Fujiwara [BBF13] where able to find a unified approach to these results by showing that for every group which admits a non-degenerate, hyperbolically embedded subgroup belongs to the class $\mathcal{D}_{reg}$. Their key results are some beautiful extension theorems on quasi-cohomology. In fact, by very recent work of Osin [Os13] the weak curvature conditions used in both papers, as well as Hamenstädt's weak acylindricity condition, are equivalent to the notion of acylindrical hyperbolicity formulated by Bowditch, cf. [op. cit.]. Collecting these results together, the following families of groups are known to be acylindrically hyperbolic. In particular, they belong to the class $\mathcal{D}_{reg}$ and thus will be contained in class NC.

Examples 2.1.6. The following classes of groups belong to NC:

a. Gromov hyperbolic groups [Mi01, MMS03];

b. Groups which are hyperbolic relative to a family of subgroups as in [MY09, HO11];

c. The mapping class group $\mathrm{Mod}(S_{g,k})$ of a surface $S_{g,k}$ with $3g + k - 4 \geq 0$, [Ha];

d. $Out(\mathbb{F}_n)$, $n \geq 2$, [Ha];

e. Non-virtually cyclic graph products $G\{\Gamma_v\}_{v\in V}$ of non-trivial groups with respect to some finite irreducible graph G with at least two vertices; in particular, non-virtually cyclic right angled Artin groups which do not split as a products, [MO13].

We end this section by noting some more examples of groups in NC

Proposition 2.1.7. *Let $M = S_{g,b}$ be a surface and let k be a positive integer. Then, the following assertions hold:*

(i) If M is large, then $PB_k(M) \in NC$.

(ii) If either $g = 0$, $b \leq 2$ and $b + k \geq 4$ or $g = 1$, $b = 0$ and $k \geq 2$, then $\widetilde{PB}_k(M) \in NC$.

Proof. Let $S = S_{g,b+k}$ be the surface in the exact sequence (2.2.3). Suppose that M is large. As mentioned right before Remark 2.2.7, $PB_k(M)$ is isomorphic to a normal subgroup of $\mathrm{PMod}(S)$. By Proposition 2.1.2 d) and Examples 2.1.6 c., we have $PB_k(M) \in NC$ which gives assertion (i).

Next, we suppose the conditions in assertion (ii) are satisfied. By Remark 2.2.7, we have that $\widetilde{PB}_k(M)$ is isomorphic to a normal subgroup of $\mathrm{PMod}(S)$. Then Proposition 2.1.2 d) and Examples 2.1.6 c. again imply that $\widetilde{PB}_k(M) \in NC$, which gives assertion (ii). □

2.2 Class $Quot(\mathcal{C})$

Let $\mathcal{C}$ be a class of groups. We define $Quot_1(\mathcal{C}) = \mathcal{C}$. Given an integer $n \geq 2$, we say that a group Γ belongs to the class $Quot_n(\mathcal{C})$ if the following are satisfied:

1. there exist a collection of groups Γ_k, $1 \leq k \leq n$ and a collection of surjective homomorphisms $\pi_k : \Gamma_k \to \Gamma_{k-1}$ such that $\Gamma_1 \in \mathcal{C}$ and $ker(\pi_k) \in \mathcal{C}$, for all $2 \leq k \leq n$;
2. Γ and Γ_n are commensurable.

Definition 2.2.1. We denote by $Quot(\mathcal{C}) := \cup_{n \in \mathbb{N}} Quot_n(\mathcal{C})$ and any group $\Gamma \in Quot(\mathcal{C})$ is called a *finite-step extension of groups in* $\mathcal{C}$.

Below we list some useful basic algebraic properties of this family of groups. We will omit most of the proofs as they are either straightforward or already contained in [CIK13, Lemmas 2.9-2.10].

Proposition 2.2.2. *The following properties hold:*

1. *If* $\Gamma \in Quot_n(\mathcal{C})$ *and* $p : \Lambda \to \Gamma$ *is a surjective homomorphism such that* $ker(p) \in \mathcal{C}$ *then* $\Lambda \in Quot_{n+1}(\mathcal{C})$.
2. *If* $\Gamma_i \in Quot_{n_i}(\mathcal{C})$ *for all* $1 \leq i \leq k$ *then* $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k \in Quot_{n_1+n_2+\cdots+n_k}(\mathcal{C})$.
3. *If* $\mathcal{C}$ *is closed under commensurability then* $Quot_n(\mathcal{C})$ *is also closed under commensurability.*
4. *Let* $\Gamma \in Quot_n(\mathcal{C})$ *and let* $\pi_k : \Gamma_k \to \Gamma_{k-1}$ *be the surjective homomorphisms satisfying the previous definition and let* $p_n = \pi_2 \circ \pi_3 \circ \cdots \circ \pi_n : \Gamma_n \to \Gamma_1$. *Then the following hold:*
   (a) *If* $\Lambda < \Gamma_1$ *is a subgroup such that* $\Lambda \in \mathcal{C}$ *then* $p_n^{-1}(\Lambda) \in Quot_n(\mathcal{C})$;
   (b) $\ker(p_n) \in Quot_{n-1}(\mathcal{C})$; *moreover, if* $\mathcal{C}$ *is closed under commensurability and* $\Lambda < \Gamma$ *is a subgroup such that* $p_n(\Lambda)$ *is finite then* $p_n^{-1}(p_n(\Lambda)) \in$

$Quot_{n-1}(\mathcal{C})$.

5. *If $\mathcal{C}$ is closed under commensurability up to finite kernel then $Quot_n(\mathcal{C})$ is also closed under commensurability up to finite kernel.*

6. *If all the groups in $\mathcal{C}$ are exact then so are all the groups in $Quot_n(\mathcal{C})$, [DL14].*

Before we present the proof we explain the terminology used in (5) above; two groups Γ_1 and Γ_2 are called *commensurable up to finite kernel* if there exist finite index subgroups $\Lambda_i \leqslant \Gamma_i$, a group H, and surjections $\psi_i : \Lambda_i \to H$ such that $ker(\psi_i)$ are finite. One can easily check that this is the smallest equivalence relation whose classes are closed under taking both finite index subgroups and quotients under normal finite subgroups.

Proof of Proposition 2.2.2 We show only (5). Assume that Λ_1 is commensurable up to finite kernel with Λ_2 and $\Lambda_1 \in Quot_n(\mathcal{C})$. Thus one can find finite index subgroups $\Sigma_i < \Lambda_i$, a group H, and surjective homomorphisms $\phi_i : \Sigma_i \to H$ with finite kernels $\Omega_i := ker(\phi_i)$. Denote by $\tilde{\phi}_i : \Sigma_i/\Omega_i \to H$ the induced isomorphisms. Since $\Lambda_1 \in Quot_n(\mathcal{C})$, $\Sigma_1 < \Lambda_1$ is finite index, and $\mathcal{C}$ is closed under taking finite index subgroups, it follows that $\Sigma_1 \in Quot_n(\mathcal{C})$. Thus there exist surjections $\pi_k : \Gamma_k \to \Gamma_{k-1}$ such that $ker(\pi_k) \in \mathcal{C}$ for every $2 \leq k \leq n$, $\Gamma_1 \in \mathcal{C}$, and $\Sigma_1 = \Gamma_n$. Let $\Theta_n := \Omega_1$ and then for every $2 \leq k \leq n$ define recursively $\Theta_{k-1} := \pi_k(\Theta_k) < \Gamma_{k-1}$; since π_k's are surjections then Θ_{k-1} is a finite normal subgroup of Γ_{k-1}. Moreover, for each $2 \leq k \leq n$ the map $\tilde{\pi}_k : \Gamma_k/\Theta_k \to \Gamma_{k-1}/\Theta_{k-1}$ given by $\tilde{\pi}_k(x\Theta_k) = \pi_k(x)\Theta_{k-1}$ is a surjective homomorphisms and $ker(\tilde{\pi}_k) = ker(\pi_k)\Theta_k$. By the isomorphism theorem we have $ker(\tilde{\pi}_k) \cong ker(\pi_k)/(ker(\pi_k) \cap \Theta_k)$ and since Θ_k is finite and $\mathcal{C}$ is closed under commensurability by finite kernel it follows that $ker(\tilde{\pi}_k) \in \mathcal{C}$, for all $2 \leq k \leq n$ and $\Gamma_1/\Theta_1 \in \mathcal{C}$. If $p : \Sigma_2 \to \Sigma_2/\Omega_2$ denotes the canonical projection then the formulas $\tilde{\pi}'_n := \tilde{\pi}_n \circ \tilde{\phi}_2^{-1} \circ \tilde{\phi}_1 \circ p : \Sigma_2 \to \Gamma_{n-1}/\Theta_{n-1}$ and $\tilde{\pi}'_k := \tilde{\pi}_k : \Gamma_k/\Theta_k \to \Gamma_{k-1}/\Theta_{k-1}$ for every $2 \leq k \leq n-1$ define surjective homomorphisms. Moreover, since the kernel $ker(\tilde{\pi}'_n) = p^{-1}(\tilde{\phi}_1^{-1}(\tilde{\phi}_2(ker(\pi_n))))$ satisfies $ker(\tilde{\pi}'_n)/\Omega_2 \cong ker(\pi_k)$ and Ω_2 is finite we have $ker(\tilde{\pi}'_n) \in \mathcal{C}$. By above we have $\ker(\tilde{\pi}'_k) \in \mathcal{C}$, for all $2 \leq k \leq n-1$, and $\Gamma_1/\Theta_1 \in \mathcal{C}$ which further imply that $\Sigma_2 \in Quot_n(\mathcal{C})$. Finally, since from construction Σ_2 is a finite index subgroup of Λ_2 we conclude that $\Lambda_2 \in Quot_n(\mathcal{C})$. $\square$

As a common generalization for poly-cyclic, poly-free [Me84], and poly-hyperbolic groups [DY08] we introduce the notion of poly-$\mathcal{C}$ groups, where $\mathcal{C}$ is a fixed class of groups. A group Γ is called *poly-$\mathcal{C}$* if there exists a positive integer n and a chain of groups $\langle e \rangle = \Lambda_0 \lhd \Lambda_1 \lhd \cdots \lhd \Lambda_{n-1} \lhd \Lambda_n = \Gamma$ such that the quotients $\Lambda_i/\Lambda_{i-1} \in \mathcal{C}$, for all $0 \leq i \leq n-1$. Below we show the poly-$\mathcal{C}$ groups are closely related with the family $Quot(\mathcal{C})$ introduced at the beginning of the section.

Proposition 2.2.3. *Given a group Γ, the following conditions are equivalent:*

a. *There exist a collection of groups Γ_k, $1 \leq k \leq n$ satisfying $\Gamma = \Gamma_n$ and $\Gamma_1 \in \mathcal{C}$ and a collection of surjective homomorphisms $\pi_k : \Gamma_k \to \Gamma_{k-1}$ satisfying $ker(\pi_k) \in \mathcal{C}$, for all $2 \leq k \leq n$;*

b. *Γ is a poly-$\mathcal{C}$ group.*

Proof. First assume condition a. above is satisfied. For every integer $2 \leq k \leq n$ consider the surjective homomorphism $p_k : \Gamma_n \to \Gamma_{n-k+1}$ defined by $p_k = \pi_{n-k+2} \circ \pi_{n-k+3} \circ \cdots \circ \pi_{n-1} \circ \pi_n$. This family of homomorphisms naturally gives rise to following chain of normal subgroups: $\langle e\rangle \lhd \ker(p_2) \lhd \ker(p_3) \lhd \cdots \lhd \ker(p_{n-1}) \lhd \ker(p_n) \lhd \Gamma_n = \Gamma$. Using the surjectivity of π_k's and the isomorphism theorem we see that $\Gamma_n/\ker(p_n) \cong \Gamma_1 \in \mathcal{C}$, $\ker(p_n)/\ker(p_{n-1}) \cong \ker(\pi_2) \in \mathcal{C}$, $\ker(p_{n-1})/\ker(p_{n-2}) \cong \ker(\pi_3) \in \mathcal{C}$, ..., $\ker(p_3)/\ker(p_2) \cong \ker(\pi_{n-1}) \in \mathcal{C}$, $\ker(p_2) = \ker(\pi_n) \in \mathcal{C}$. This shows that Γ is poly-$\mathcal{C}$.

To see the converse assume Γ is poly-$\mathcal{C}$. Hence one can find a chain of normal subgroups: $\langle e\rangle = \Lambda_0 \lhd \Lambda_1 \lhd \cdots \lhd \Lambda_{n-1} \lhd \Lambda_n = \Gamma$ such that the quotients $\Lambda_i/\Lambda_{i-1} \in \mathcal{C}$, for all $0 \leq i \leq n-1$. For every $2 \leq k \leq n$ consider the surjective groups homomorphism $\pi_k : \Lambda_n/\Lambda_{n-k} \to \Lambda_n/\Lambda_{n-k+1}$ defined by $\pi_k(x\Lambda_{n-k}) = x\Lambda_{n-k+1}$, for all $x \in \Lambda_n$. From assumptions we have that $\Lambda_n/\Lambda_{n-1} \in \mathcal{C}$ and furthermore, using the isomorphism theorem, we see that $\ker(\pi_k) \cong (\Lambda_n/\Lambda_{n-k})/(\Lambda_n/\Lambda_{n-k+1}) \cong \Lambda_{n-k+1}/\Lambda_{n-k} \in \mathcal{C}$, for all $2 \leq k \leq n$. Finally, denoting by $\Gamma_1 := \Lambda_n/\Lambda_{n-1}$ and $\Gamma_k := \Lambda_n/\Lambda_{n-k}$, for all $2 \leq k \leq n$, we see the conditions enumerated in part a. are satisfied. $\square$

2.2.1 Relative hyperbolic groups

Let Γ be a group that is hyperbolic relative to $\mathcal{P} = \{P_i : i \in I\}$ a finite family of infinite, proper, residually finite subgroups. Using very deep methods in geometric group theory Denis Osin was able to prove a powerful algebraic Dehn filling analog for this class of groups. As a consequence, it was shown in [Os06, Theorem 1.1] that there exist a non-elementary hyperbolic group H, a surjective homomorphism $\psi : \Gamma \to H$, and finite index, normal subgroups $N_i \lhd P_i$ such that $ker(\psi)$ is the normal closure of $\cup_i N_i$ in Γ, i.e., $ker(\psi) = \langle\langle \cup_i N_i \rangle\rangle^{\Gamma}$. More recently, using the concept of very rotating family of subgroups, Dahmani, Guirardel, and Osin [DGO11, Theorem

7.9] were able to describe more concretely the structure of $ker(\psi)$, as a nontrivial free product. Precisely, they showed that there exist a family of nonempty subsets $T_i \subset \Gamma$ such that $ker(\psi) = *_{i\in I}(*_{\gamma\in T_i} N_i^{\gamma})$, where $N_i^{\gamma} = \gamma N_i \gamma^{-1}$, (see also Osin's argument in [CIK13, Corollary 5.1]). Summarizing we have the following:

Theorem 2.2.4 ([Os06, DGO11])**.** *Let $\mathcal{H}$ be the class consisting of all non-elementary hyperbolic groups and all non-amenable, non-trivial free product of exact groups. Then for any group Γ that is hyperbolic relative to a finite family of residually finite, exact, infinite, proper subgroups we have $\Gamma \in Quot_2(\mathcal{H})$.*

2.2.2 Mapping class groups

Let $S_{g,k}$ be a connected, compact and orientable surface of genus g with k boundary components. Throughout this thesis, we assume that a surface satisfies these conditions unless otherwise mentioned. Denote by $\mathrm{Mod}(S_{g,k})$ the *mapping class group* of $S_{g,k}$, i.e., the group of isotopy classes of orientation-preserving homeomorphisms from $S_{g,k}$ onto itself, where isotopy may move points of the boundary of $S_{g,k}$. Using the Birman short exact sequence in combination with the earlier results of Birman and Hilden [BH73] it was shown in [CIK13, Section 4.3] the following result.

Theorem 2.2.5. *If $\mathcal{F}$ denotes the class of all groups commensurable with non-abelian free groups then we have the following:*

(i) If $g = 0$, $k \geq 4$ then $\mathrm{Mod}(S_{g,k}) \in Quot_{k-3}(\mathcal{F})$;

(ii) If $g = 1$, $k \geq 1$ then $\mathrm{Mod}(S_{g,k}) \in Quot_{k}(\mathcal{F})$;

(iii) If $g = 2$, $k \geq 0$ then $\mathrm{Mod}(S_{g,k}) \in Quot_{k+3}(\mathcal{F})$.

It follows from [FM11] that, for every positive integer k, the central quotient of the braid group with k strands $\tilde{B}_k$ can be canonically identified with a finite index subgroup of the mapping class group $\mathrm{Mod}(S_{0,k+1})$. Moreover, notice that the central quotient of the pure braid group with k strands $\tilde{P}_k$ is a normal subgroup of index $k!$ in $\tilde{B}_k$. Combining with the above theorem we have

Corollary 2.2.6. *For every $k \geq 3$, we have that $\tilde{B}_k, \tilde{P}_k \in Quot_{k-2}(\mathcal{F})$.*

2.2.3 Surface braid groups

Throughout this subsection, we let $M = S_{g,b}$ be a surface. Consider $\mathrm{PMod}(M)$ the *pure mapping class group* of M, i.e., the subgroup of $\mathrm{Mod}(M)$ that consists of isotopy classes of orientation-preserving homeomorphisms from M onto itself which also preserve each component of ∂M as a set. Then $\mathrm{PMod}(M)$ is a subgroup in $\mathrm{Mod}(M)$ of index $b!$.

Fix k a positive integer. Denote by $F_k(M)$ the space of ordered k distinct points of M and let $PB_k(M)$ be the fundamental group of $F_k(M)$. The group $PB_k(M)$ is called the *pure braid group of k strands* on M. From definitions we have the equality $PB_1(M) = \pi_1(M)$. For the basic properties of the group $PB_k(M)$ we refer to [B69a, B69b, B74, PR99].

We fix $x_1, \ldots, x_k$, mutually distinct points of M. By [FaNe62, Theorem 3], the map from $F_{k+1}(M)$ into $F_k(M)$ sending a point $(t_1, \ldots, t_{k+1})$ of $F_{k+1}(M)$ to $(t_1, \ldots, t_k)$ is a locally trivial fibration which has the fiber $F_1(M \setminus \{x_1, \ldots, x_k\})$. Thus, we have the following exact sequence

$$\cdots \to \pi_2(F_{k+1}(M)) \to \pi_2(F_k(M))$$
$$\to \pi_1(M \setminus \{x_1, \ldots, x_k\}) \to PB_{k+1}(M) \to PB_k(M) \to 1. \quad (2.2.1)$$

If $(g, b) \neq (0, 0)$, then by [FaNe62, Corollary 2.2] we have $\pi_2(F_l(M)) = 0$, for any positive integer l, and hence (2.2.1) gives the following short exact sequence

$$1 \to \pi_1(M \setminus \{x_1, \ldots, x_k\}) \to PB_{k+1}(M) \to PB_k(M) \to 1. \qquad (2.2.2)$$

Following the terminology from [PR99], we say that M is *large* if the group $\pi_1(M)$ is non-elementarily hyperbolic. This is the case if and only if

$$(g, b) \neq (0, 0), (0, 1), (0, 2), (1, 0)$$

. Let $S = S_{g,b+k}$ be a surface and choose k components of ∂S. We suppose that M is obtained by filling a disk to each of these k components of ∂S. Thus we get the following Birman exact sequence

$$PB_k(M) \xrightarrow{j} \mathrm{PMod}(S) \to \mathrm{PMod}(M) \to 1 \qquad (2.2.3)$$

corresponding to the canonical embedding of S into M ([B69b, Theorem 1], [FM11, Theorem 9.1] and [Iv02, Theorem 2.8.C]). Denote by $Z_k(M)$ the center of $PB_k(M)$ and notice that by [B69b, Corollary 1.2] we have $\ker j < Z_k(M)$. Denote by $\widetilde{PB}_k(M) := PB_k(M)/Z_k(M)$. When M large then [PR99, Proposition 1.6] implies that the center $Z_k(M)$ is trivial and hence j is an injective homomorphism and also $\widetilde{PB}_k(M) = PB_k(M)$.

Remark 2.2.7. For any M non-large, we will argue below that $\ker j = Z_k(M)$; in particular, the group $\tilde{PB}_k(M)$ can be identified with a normal subgroup of $\mathrm{PMod}(S)$ through j.

To justify our claim notice that when $g = 0$ and $b \leq 2$ the group $\mathrm{PMod}(M)$ is trivial. Then using the exact sequence (2.2.3) we have that $PB_k(M)/\ker j \simeq \mathrm{PMod}(S)$. Since by [FM11, Section 3.4] the center of $\mathrm{PMod}(S)$ is trivial, this further implies that $\ker j = Z_k(M)$; in this case we also get that $\widetilde{PB}_k(M) \simeq \mathrm{PMod}(S)$.

Now assume that $g = 1$ and $b = 0$. In [B69b, Corollary 1.3], two generators of $\ker j$ were described through the presentation of $PB_k(M)$ in [B69a, Theorem 5], and it was shown that $\ker j$ is isomorphic to $\mathbb{Z}^2$. Combining this with [PR99, Proposition 4.2], we obtain the equality $\ker j = Z_k(M)$.

Theorem 2.2.8. *For every positive integer k, the following assertions hold:*

(i) If M is large and $b \geq 1$, then $PB_k(M) \in Quot_k(\mathcal{F})$;

(ii) If M is large and $b = 0$, then $PB_k(M) \in Quot_k(\mathcal{H})$;

(iii) If $g = 0$, $b \leq 2$ and $b + k \geq 4$, then $\widetilde{PB}_k(M) \in Quot_{b+k-3}(\mathcal{F})$;

(iv) If $g = 1$, $b = 0$ and $k \geq 2$, then $\widetilde{PB}_k(M) \in Quot_{k-1}(\mathcal{F})$.

Proof. Assertions (i) and (ii) follow inductively from the short exact sequence (2.2.2) and the equality $PB_1(M) = \pi_1(M)$.

If $g = 0$, $b \leq 2$, and $b + k \geq 4$ then using the Remark 2.2.7 above we have that $\tilde{PB}_k(M) \simeq \mathrm{PMod}(S)$. Thus, assertion (iii) follows from Theorem 2.2.5 (i).

Suppose that $g = 1$, $b = 0$ and $k \geq 2$. Let $\rho\colon PB_{k+1}(M) \to PB_k(M)$ be the surjection from the exact sequence (2.2.2). As mentioned in Remark 2.2.7, two generators of $Z_k(M)$ are described in [B69b, Corollary 1.3] and [PR99, Proposition 4.2]. The homomorphism ρ is induced by the map from $F_{k+1}(M)$ into $F_k(M)$ which sends a point $(t_1, \ldots, t_{k+1})$ of $F_{k+1}(M)$ to $(t_1, \ldots, t_k)$. It follows from the description of generators of $Z_k(M)$ that $\rho(Z_{k+1}(M)) = Z_k(M)$ which further gives rise to a surjection $\tilde{\rho}\colon \widetilde{PB}_{k+1}(M) \to \widetilde{PB}_k(M)$.

Since $\pi_1(M \setminus \{x_1, \ldots, x_k\})$ is isomorphic to the free group of rank $k+1$ and its center is trivial, then (2.2.2) induces in a canonical way a short exact sequence

$$1 \to \pi_1(M \setminus \{x_1, \ldots, x_k\}) \to \widetilde{PB}_{k+1}(M) \to \widetilde{PB}_k(M) \to 1. \tag{2.2.4}$$

Since $PB_1(M) = \pi_1(M) \simeq \mathbb{Z}^2$ then the group $\widetilde{PB}_1(M)$ is trivial. Hence, assertion (iv) follows from (2.2.4), by induction on k. □

Remark 2.2.9. Next we briefly discuss the exceptional cases for $PB_k(M)$ not covered by Theorem 2.2.8. If $(g, b) = (0, 0)$ and $k \leq 3$, then the first theorem in [FV62, Section VI.2] implies that $PB_k(M)$ is finite. If $(g, b) = (0, 1)$, then $PB_1(M)$ is trivial, and $PB_2(M)$ is the Artin pure braid group of two strands and thus is isomorphic to $\mathbb{Z}$ ([FM11, Section 9.3]). If $(g, b) = (0, 2)$, then $PB_1(M) \simeq \mathbb{Z}$. If $(g, b) = (1, 0)$, then $PB_1(M) \simeq \mathbb{Z}^2$.

2.2.4 The Torelli group and the Johnson kernel

Let $S = S_{g,k}$ be a surface. A simple closed curve in S is called *essential* in S if it is neither homotopic to a single point of S nor isotopic to a component of ∂S. When there is risk no confusion, by a curve in S we mean either an essential simple closed curve in S or its isotopy class. A curve α in S is called *separating* in S if $S \setminus \alpha$ is not connected. Otherwise α is called *non-separating* in S. Whether α is separating in S or not depends only on the isotopy class of α. A pair of non-separating curves in S, $\{\beta, \gamma\}$, is called a *bounding pair (BP)* in S if β and γ are disjoint and non-isotopic and $S \setminus (\beta \cup \gamma)$ is not connected. This condition depends only on the isotopy classes of β and γ. Given a curve α in S, we denote by $t_\alpha \in \mathrm{PMod}(S)$ the *Dehn twist* about α.

We define the *Torelli group* $\mathcal{I}(S)$ to be the group generated by all elements of the form t_α and $t_\beta t_\gamma^{-1}$ with α a separating curve in S and $\{\beta, \gamma\}$ a BP in S. We define the *Johnson kernel* $\mathcal{K}(S)$ as the group generated by all t_α with α a separating curve in S. We refer the reader to [FM11, Chapter 6] for more background of these groups.

When $g \geq 2$ and $k \leq 1$, the Torelli group of S is originally defined as the group of elements of $\mathrm{Mod}(S)$ acting on $H_1(S, \mathbb{Z})$ trivially. Using the results in [Jo79] and [Pow78], this original group turns out to be equal to the group $\mathcal{I}(S)$ defined above. When $g = 0$, any curve in S is separating in S, and therefore we have $\mathcal{I}(S) = \mathcal{K}(S) = \mathrm{PMod}(S)$.

Theorem 2.2.10. *The following assertions hold:*

(i) If $g = 1$ and $k \geq 2$, then $\mathcal{I}(S), \mathcal{K}(S) \in Quot_{k-1}(\mathcal{F})$;

(ii) If $g = 2$ and $k \geq 0$, then $\mathcal{I}(S), \mathcal{K}(S) \in Quot_{k+1}(\mathcal{F})$.

Proof. If $g = 2$ and $k = 0$, then the equality $\mathcal{I}(S) = \mathcal{K}(S)$ holds, and it is further isomorphic to the free group of infinite rank by results in [Me92, BBM10]. The theorem for this case follows.

Suppose that either $g = 1$ and $k \geq 2$ or $g = 2$ and $k \geq 1$. Choose a component of ∂S, and denote by R the surface obtained by filling up a disk in this chosen component of ∂S. Then by [FM11, Theorem 4.6] we have the Birman exact sequence

$$1 \to \pi_1(R) \xrightarrow{j} \mathrm{PMod}(S) \xrightarrow{q} \mathrm{PMod}(R) \to 1 \tag{2.2.5}$$

corresponding to the canonical embedding of S into R. By [Iv02, Lemma 4.1.I], through the injection j, each of standard generators of $\pi_1(R)$ induces either the element t_α with α a separating curve in S, its inverse, or the element $t_\beta t_\gamma^{-1}$ with $\{\beta, \gamma\}$ a BP in S. It follows that $j(\pi_1(R)) < \mathcal{I}(S)$. Also, by the definition of $\mathcal{I}(S)$ we have $q(\mathcal{I}(S)) = \mathcal{I}(R)$. Combining these with (2.2.5) we obtain the following short exact sequence

$$1 \to \pi_1(R) \to \mathcal{I}(S) \to \mathcal{I}(R) \to 1. \tag{2.2.6}$$

If $g = 1$ and $k = 2$, then $\mathcal{I}(R)$ is trivial because there exist neither a separating curve in R nor a BP in R. Hence, when $g = 1$ and $k \geq 2$ we get $\rangle(\mathcal{S}) \in \mathcal{Q}\sqcap\wr\sqcup_{\|-\infty}(\mathcal{F})$, by using (2.2.6) and induction on k. Similarly, when $g = 2$ and $k \geq 0$ we have $\mathcal{I}(S) \in Quot_{k+1}(\mathcal{F})$, by induction on k, starting from the result in the first paragraph of the proof.

Suppose again that either $g = 1$ and $k \geq 2$ or $g \geq 2$ and $k \geq 1$. Restricting the the short exact sequence (2.2.5), we obtain the following short exact sequence

$$1 \to j(\pi_1(R)) \cap \mathcal{K}(S) \to \mathcal{K}(S) \to \mathcal{K}(R) \to 1. \tag{2.2.7}$$

Denote by $N = j(\pi_1(R)) \cap \mathcal{K}(S)$. Through the injection j, any simple loop in R

surrounding exactly one component of ∂R and cutting a cylinder from R induces either the element t_α with α a separating curve in S or its inverse; in particular, it follows that N is infinite. Also, relying on [Jo80] it was proved in [Ki09, Proposition 2.2] that for any BP $\{\beta, \gamma\}$ in S, no non-zero power of $t_\beta t_\gamma^{-1}$ lies in $\mathcal{K}(S)$. This further implies that N has infinite index in $j(\pi_1(R))$ and hence, by [DD89, Theorem V.12.5], N is a free group of infinite rank. Finally, we obtain the conclusion of the theorem about $\mathcal{K}(S)$, by using (2.2.7) and induction on k. □

CHAPTER 3
VON NEUMANN ALGEBRA

3.1 Some Preliminaries on Intertwining Results

3.1.1 Introduction

Let $\mathcal{H}$ be a Hilbert space. The uniform topology, the topology induced by the norm on $B(\mathcal{H})$, is insufficient for analytic purposes. Multiplication and the star operation need not be continuous in topology induced by the norm.

Definition 3.1.1. Let $\mathcal{H}$ be a Hilbert space. The *uniform topology* on $B(\mathcal{H})$ is the topology induced by the norm $\|\cdot\|$ on $B(\mathcal{H})$.
The *strong operator topology* (SOT) on $B(\mathcal{H})$ is defined by the family of semi-norms $x \mapsto \|x\xi\|_{\mathcal{H}}$ for every $\xi \in H$, i.e. the SOT topology is the weakest topology such that the map $x \mapsto \|x\xi\|_{\mathcal{H}}$ is continuous for every $x \in B(\mathcal{H})$ and every $\xi \in \mathcal{H}$.
The *weak operator topology* (WOT) is defined by the family of semi-norms $x \mapsto |\langle x\xi, \eta\rangle|_{\mathcal{H}}$ for every ξ; η, i.e. the WOT topology is the weakest topology such that the $x \mapsto |\langle x\xi, \eta\rangle|_{\mathcal{H}}$ is continuous for every $x \in B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$.

These topologies are equivalent in when $\mathcal{H}$ is finite dimensional. In general, from coarsest to finest, we have

$$WOT \prec SOT \prec Uniform$$

Definition 3.1.2. Let $M \subset B(\mathcal{H})$ be a unital, self-adjoint subalgebra of $B(\mathcal{H})$. If M is closed with respect to the WOT topology, then M is a *von Neumann algebra.*

The *commutant* of M, denoted by M', is the set of operators

$$M' = x \in B(\mathcal{H}) : xm = mx \, \forall m \in M$$

. The *bicommutant* of M is denoted by $M'' = (M')'$. The following result by Von Neumann shows that in a von Neumann algebra, the weak operator topology and the bicommutant are the same. This remarkable result connects analytic and algebraic properties of von Neumann algebra.

Theorem 3.1.3. *If $M \subset B(\mathcal{H})$ is a unital, self-adjoint collection of operators, then M' is a von Neumann algebra. A self-adjoint maximal abelian collection of operators $M \subset B(\mathcal{H})$ is a von Neumann algebra. Furthermore, if M is a unital, self-adjoint collection of operators, then M is a von Neumann algebra if and only if $M = M''$ (M is its own bicommutant)*

If $M \subset B(\mathcal{H})$ is a von Neumann algebra, *center* of M is the von Neumann algebra $\mathcal{Z}(M) = M \cap M'$. A von Neumann algebra $M \subset B(\mathcal{H})$ is a called a *factor* if its center $\mathcal{Z}$ is isomorphic to the complex numbers.

3.1.2 The Group von Neumann Algebra

The bicommutant theorem of Murray and von Neumann allows one to associate a von Neumann algebra to a unitary representation of a group $\rho : \Gamma \to \mathcal{U}(\mathcal{H})$. Notice $\rho(\Gamma)'$ and $\rho(\Gamma)'' \in B(\mathcal{H}$ form von Neumann algebras. Let $l^2(\Gamma)$ denote the Hilbert space of all square summable sequences indexed by Γ, i.e. $\eta \in \ell^2(\Gamma)$ if and only if $\sum_{\gamma \in \Gamma} |\eta(\gamma)|^2 < \inf$. Then Γ can be viewed as a subgroup of $\mathcal{U}(\ell^2(\Gamma))$ by the the covariant action

$$\gamma \cdot \eta(\lambda) := \eta(\gamma^{-1}\lambda) \forall \gamma, \lambda \in \Gamma, \eta \in \ell^2(\Gamma)$$

Definition 3.1.4. Let $\Gamma \in \mathcal{U}(\ell^2(\Gamma))$. The group von Neumann algebra is the algebra generated by these unitaries

$$L(\Gamma) := \{\gamma\}''_{\gamma \in \Gamma} = \overline{\mathbb{C}[\Gamma]}^{WOT} \subset B(\ell^2(\Gamma)$$

. An alternative construction of the group von Neumann algebra arises by viewing elements of $\ell^2(\Gamma)$ acting on (a dense subset) ℓ^2 by convolution operators.

Theorem 3.1.5 (Murray-von Neumann '36)**.** *$L(\Gamma)$ is a II_1 factor iff Γ is infinite conjugacy class (icc) group.*

3.1.3 Popa's intertwining techniques

Over a decade ago Popa developed a powerful technology for conjugating subalgebras of tracial von Neumann algebras, now termed the *intertwining-by-bimodules techniques*, [Po03, Theorem 2.1 and Corollary 2.3]. For further reference we recall the following theorem.

Theorem 3.1.6 (Popa, [Po03])**.** *Let (M, τ) be a separable tracial von Neumann algebra and P, Q be two (not necessarily unital) von Neumann subalgebras of M.*

Then the following are equivalent:

1. *There exist non-zero projections $p \in P, q \in Q$, a $*$-homomorphism $\theta : pPp \to qQq$ and a non-zero partial isometry $v \in qMp$ such that $\theta(x)v = vx$, for all $x \in pPp$.*

2. *There is no sequence* $u_n \in \mathcal{U}(P)$ *satisfying* $\|E_Q(xu_ny)\|_2 \to 0$, *for all* $x, y \in M$.

If one of the two equivalent conditions in the theorem above holds then we say that *a corner of* P *embeds into* Q *inside* M, and write $P \preceq_M Q$. If in addition we have that $Pp' \preceq_M Q$, for any non-zero projection $p' \in P' \cap 1_P M 1_P$, then we write $P \preceq_M^s Q$.

Proposition 3.1.7. *Let* $\Gamma \curvearrowright (B, \tau)$ *be a trace preserving action of a group on a* II_1 *factor* B *and denote by* $M = B \rtimes \Gamma$. *Suppose there exists normal subgroups* $\Sigma_1, \ldots, \Sigma_k \lhd \Gamma$ *and pairwise commuting subalgebras* $A_1, \ldots, A_k \subset M$ *so that* $A_i \preceq_M^s B \rtimes \Sigma_i$. *Then*

$$\bigvee_{i=1}^{k} A_i \preceq_M B \rtimes \Sigma_1 \cdots \Sigma_k.$$

Proof. Fixing $1 > \varepsilon > 0$, $A_1 \preceq_M^s B \rtimes \Sigma_1$ implies there exists a set $\mathcal{F}_1 \subset \bigcup_{i=1}^{j} \Sigma_1 \gamma_i$ small relative to Σ_1 so that

$$\|a_1 - \mathcal{P}_{\mathcal{F}_1}(a_1)\| < \varepsilon/2.$$

We may recursively choose $\mathcal{F}_i \subset \cup_{j=1}^{j_i} \Sigma_i \gamma_{j,i}$ small relative to Σ_i so that

$$\|a_i - \mathcal{P}_{\mathcal{F}_i}(a_i)\| \leq \varepsilon/(kj_1 \cdots j_{i-1})$$

whenever $a_i \in A_i$ with $\|a_i\| \leq 1$.

$$\|a_1 \cdots a_k - \mathcal{P}_{\mathcal{F}}(a_1 \cdots a_k)\|_2 \leq \|a_1 \cdots a_k - \mathcal{P}_{\mathcal{F}_1}(a_1) \cdots \mathcal{P}_{\mathcal{F}_k}(a_k)\|_2 \tag{3.1.1}$$

$$\leq \|(a_1 - \mathcal{P}_{\mathcal{F}_1}(a_1))(a_2 \cdots a_k)\|_2 \tag{3.1.2}$$

$$+ \|\mathcal{P}_{\mathcal{F}_1}(a_1)(a_2 \cdots a_k - \mathcal{P}_{\mathcal{F}_2}(a_1) \cdots \mathcal{P}_{\mathcal{F}_k}(a_k))\|_2$$

$$\leq \varepsilon/k + \|\mathcal{P}_{\mathcal{F}_1}(a_1)(a_2\cdots a_k - \mathcal{P}_{\mathcal{F}_2}(a_1)\cdots\mathcal{P}_{\mathcal{F}_k}(a_k))\|_2. \tag{3.1.3}$$

Since $\mathcal{F}_1$ is contained j_1 left translates of Σ_1, $\|\mathcal{P}_{\mathcal{F}_1}(a_1)\|_2 \leq j_1\|a_1\|_2$. Thus previous inequality becomes

$$\begin{aligned}\|a_1\cdots a_k - \mathcal{P}_{\mathcal{F}}(a_1\cdots a_k)\|_2 \leq& \varepsilon/k + j_1\|a_1\|\|a_2\cdots a_k - \mathcal{P}_{\mathcal{F}_2}(a_1)\cdots\mathcal{P}_{\mathcal{F}_k}(a_k)\|_2\\ \leq& \varepsilon/k + j_1\|(a_2 - \mathcal{P}_{\mathcal{F}_2}(a_2))(a_3\cdots a_k)\|_2\\ &+ j_1\|\mathcal{P}_{\mathcal{F}_2}(a_2)(a_3\cdots a_k - \mathcal{P}_{\mathcal{F}_3}(a_3)\cdots\mathcal{P}_{\mathcal{F}_k}(a_k))\|_2\\ \leq& \varepsilon/k + j_1\|a_2 - \mathcal{P}_{\mathcal{F}_2}(a_2)\|_2\\ & j_1\|\mathcal{P}_{\mathcal{F}_2}(a_2)(a_3\cdots a_k - \mathcal{P}_{\mathcal{F}_3}(a_3)\cdots\mathcal{P}_{\mathcal{F}_k}(a_k))\|_2\\ \leq& 2\varepsilon/k + j_1 j_2\|a_3\cdots a_k - \mathcal{P}_{\mathcal{F}_3}(a_3)\cdots\mathcal{P}_{\mathcal{F}_k}(a_k)\|_2\end{aligned}$$

Repeated analysis will yield

$$\|a_1\cdots a_k - \mathcal{P}_{\mathcal{F}}(a_1\cdots a_k)\|_2 < \varepsilon. \tag{3.1.4}$$

Note $\mathcal{F} = \mathcal{F}_1\cdots\mathcal{F}_k$ is contained in a finite union of right translates of $\Sigma_1\cdots\Sigma_k \triangleleft \Gamma$. Thus there exist $\lambda_1, \cdots, \lambda_n \in \Gamma$ so that

$$\mathcal{F} \subset \bigcup_{i=1}^{n}(\Sigma_1\cdots\Sigma_k)\lambda_i.$$

Thus $\|\sum_{i=1}^{n} P_{\Sigma_1\cdots\Sigma_k}(a_1\cdots a_k u_{\lambda_i})\|_2^2 \geq \|\mathcal{P}_{\mathcal{F}}(a_1\cdots a_k)\|_2^2 > 1-\varepsilon$ for every $a_- \in A_i$ with $\|a_i\| \leq 1$.

$$\sum_{i=1}^{n}\|E_{B\rtimes\Sigma_1\cdots\Sigma_k}(wu_{\lambda_i})\|_2^2 \geq \|\sum_{i=1}^{n}E_{B\rtimes\Sigma_1\cdots\Sigma_k}(wu_{\lambda_i})\|_2^2 > 1-\varepsilon$$

where w is any unitary of the form $w = u_1 \cdots u_k$ with $u_i \in \mathcal{U}(A_i)$. As unitaries of this form generate $A_1 \vee \cdots \vee A_k$, Theorem 3.1.6 establishes the result. □

3.1.4 Finite index inclusions of tracial von Neumann algebras

If $P \subseteq M$ are II_1 factors, then the *Jones index* of the inclusion $P \subseteq M$, denoted $[M : P]$, is the dimension of $L^2(M)$ as a left P-module. M. Pimsner and S. Popa showed that the number $[M : P]$ can be interpreted as the best constant appearing in several inequalities involving the conditional expectation E_P [PP86, Theorem 2.2]. It also follows from their work that these constants can be used to define a "probabilistic" index of any inclusion of tracial von Neumann algebras [PP86, Remark 2.4].

Definition 3.1.8 (Pimsner & Popa, [PP86])**.** Let (M, τ) be a tracial von Neumann algebra with a von Neumann subalgebra P. Let

$$\lambda = \inf \{\|E_P(x)\|_2^2 / \|x\|_2^2 \ : \ x \in M_+\}.$$

The *index of the inclusion* $P \subseteq M$ is defined as $[M : P] = \lambda^{-1}$, under the convention that $\frac{1}{0} = \infty$.

Theorem 3.1.9 ([Jo81, PP86])**.** *Suppose $P \subset M$ is an inclusion of tracial von Neumann algebras. Then the following hold:*

1. *If $P \subset M$ is an inclusion of II_1 factors, then $[M : P]_{PP} = [M : P]$*
2. *If $[M : P]_{PP} < \infty$ and $p \in P$ is a projection, $[pMp : pPp] < \infty$;*
3. *If P is a II_1 factor and $[M : P]_{PP} < \infty$ then $P' \cap M$ is finite dimensional;*

4. *If $P \subset M$ is an inclusion of II_1 factors with $[M : P]_{PP} < \infty$, then $\dim_C(P' \cap M) < \infty$.*

Let Γ a discrete group and Λ a finite index subgroup., it follows $[L(\Gamma) : L(\Lambda)]_{PP}$ is finite as well. We have the following generalized converse to this fact:

Proposition 3.1.10 ([CdSS15])**.** *Let $\Omega \leqslant \Lambda \leqslant \Theta$ be groups such that there exists projections $p \in L(\Omega)$ and $z \in L(\Lambda)' \cap L(\Theta)$ so that $pz \neq 0$ and $[pL(\Lambda)pz : pL(\Omega)pz]_{PP} < \infty$. Then $[\Lambda : \Omega] < \infty$.*

For further use we note the following basic facts:

Lemma 3.1.11. *[PP86, Lemma 2.3] Let (M, τ) be a tracial von Neumann algebra and $P \subseteq M$ be a von Neumann subalgebra such that $[M : P] < \infty$. Then the following hold:*

1. *for every projection $p \in P$ we have $[pMp : pPp] < \infty$;*
2. *$M \preceq^s_M P$.*

As explained in [CIK13], it turns out that this precise notion of index is a well suited technical tool to study global decomposition properties for von Neumann algebras, up to intertwining. For instance, it enables one to show the following version of [CIK13, Proposition 3.6] involving commuting subalgebras rather than masa's. Its proof is similar with the one presented in [CIK13] but we include all details for reader's convenience.

Proposition 3.1.12. *Let (M, τ) be a tracial von Neumann algebra and let $z \in M$*

be a non-zero projection. Assume that $P \subseteq zMz$ and $N \subseteq M$ are von Neumann subalgebras such that $P \vee (P' \cap zMz) \subseteq zMz$ has finite index and that $P \preceq_M N$.

Then one can find a scalar $s > 0$, non-zero projections $r \in N, p \in P$, a subalgebra $P_0 \subseteq rNr$, and a $*$-isomorphism $\theta : pPp \to P_0$ such that the following properties are satisfied:

1. *$P_0 \vee (P_0' \cap rNr) \subseteq rNr$ has finite index;*
2. *there exist a non-zero partial isometry $v \in M$ such that $rE_N(vv^*) = E_N(vv^*)r \geq sr$ and $\theta(pPp)v = P_0 v = rvpPp$;*
3. *$E_N(v(pP'p \cap pMp)v^*)'' \subseteq P_0' \cap rNr$.*

Proof. Since $P \preceq_M N$, one can find nonzero projections $p \in P$ and $q \in N$, a nonzero partial isometry $v \in pMq$, and a $*$-homomorphism $\theta : pPp \to qNq$ such that $\theta(x)v = vx$, for all $x \in pPp$. Notice that $v^*v \in pPp' \cap pMp$ and $q' := vv^* \in \theta(pPp)' \cap qMq$. Moreover, without any loss of generality, we can assume that the support projection of $E_N(q')$ equals q. Observe that for every $x \in pPp$ and $y \in pPp' \cap pMp$ and we have

$$\theta(x)vyv^* = vxyv^* = vyxv^* = vyv^*\theta(x).$$

Thus we have $v(pPp' \cap pMp)v^* \subseteq \theta(pPp)' \cap qMq$ and hence

$$E_N(v(pPp' \cap pMp)v^*) \subseteq \theta(pPp)' \cap qNq. \tag{3.1.5}$$

Since the inclusion $P \vee (P' \cap zMz) \subseteq zMz$ has finite index then also $pPp \vee (pPp' \cap pMp) = p(P \vee (P' \cap M))p \subseteq pMp$ has finite index and hence $v(pPp \vee (pPp' \cap pMp))v^* \subseteq vpMpv^* = q'Mq'$ has finite index too.

For every $s > 0$ we denote by $q_s = 1_{[s,\infty)}(E_N(q'))$ and notice that $\|q_s - q\| \to 0$, as $s \to 0$ and $q_s E_N(q') = E_N(q') q_s \geq s q_s$. This further implies that $\|q_s v - v\| \to 0$, as $s \to 0$; in particular, we can pick $s > 0$ such that $q_s v \neq 0$. Applying [CIK13, Lemma 2.3] (see also [Io11, Lemma 1.6(1)]) it follows that the inclusion

$$q_s E_N(v(pPp \vee (pPp' \cap pMp))v^*)'' q_s \subseteq q_s N q_s \tag{3.1.6}$$

has finite index. Since $v^*v \in pPp' \cap pMp$ then $v(pPp \vee (pPp' \cap pMp))v^* \subseteq v(pPp)v^* \vee v(pPp' \cap pMp)v^* = \theta(pPp)vv^* \vee v(pPp' \cap pMp)v^*$. This further implies

$$\begin{aligned} E_N(v(pPp \vee (pPp' \cap pMp))v^*)'' &\subseteq E_N(\theta(pPp)vv^* \vee v(pPp' \cap pMp)v^*)'' \\ &\subseteq \theta(pPp) \vee E_N(v(pPp' \cap pMp)v^*)''. \end{aligned}$$

This last containment together with relation (3.1.6) give that the inclusion $\theta(pPp)q_s \vee q_s E_N(v(pPp' \cap pMp)v^*)'' q_s \subseteq q_s N q_s$ has finite index. Then the statement follows from this and (3.1.5) by letting $r := q_s$, and $P_0 := \theta(pPp)q_s \subset q_s N q_s$. □

3.1.5 Dichotomy for normalizers inside crossed products

Recently, Sorin Popa and Stefaan Vaes obtained a series of ground breaking results regarding the classification of normalizers of algebras inside crossed products arising from large families of groups including free groups [PV11, Theorem 1.6] or hyperbolic groups [PV12, Theorem 1.4]. Motivated by these results and by the remarkable subsequent developments in the realm of amalgamated free products due to Adrian Ioana [Io12] (see also [Va13]) we will introduce a new class of groups. However to be able to state it properly we need one more definition.

Definition 3.1.13. [OP07, Section 2.2] Let (M, τ) be a tracial von Neumann algebra,

$p \in M$ a projection, and $P \subset pMp, Q \subset M$ von Neumann subalgebras. We say that P is *amenable relative to* Q *inside* M if there exists a P-central state $\phi : p\langle M, e_Q\rangle p \to \mathbb{C}$ such that $\phi(x) = \tau(x)$, for all $x \in pMp$.

Definition 3.1.14. A group Γ belongs to class $\mathcal{C}_{rss}$ if it is exact [KW99] and the following dichotomy property holds: Assume $\Gamma \curvearrowright B$ is any trace preserving action on a tracial von Neumann algebra (B, τ) and denote by $M = B \rtimes \Gamma$. Let $p \in M$ be a projection and $A \subset pMp$ a von Neumann subalgebra that is amenable relative to B inside M. Then either $A \preceq_M B$ or $P := \mathcal{N}_{pMp}(A)''$ is amenable relative to B inside M.

Summarizing the results described above, the following classes of groups belong to $\mathcal{C}_{rss}$:

1. [PV11, Theorem 3.1, Lemma 4.1, and Theorem 7.1] Any weakly amenable group with positive first ℓ^2-Betti number;
2. [PV12, Theorem 3.1] Any weakly amenable, non-amenable, bi-exact group;
3. [Io12, Theorem 1.6],[Va13, Theorem A] Any non-amenable, nontrivial free product $\Gamma = \Gamma_1 * \Gamma_2$, where Γ_i are exact.

In the same spirit as in [VV14] one can establish that the class $\mathcal{C}_{rss}$ is closed under commensurability.

3.2 The class $NC \cap Quot(\mathcal{C}_{rss})$

Since the first two theorem of our main structural results are applicable to von Neumann algebras arising from groups belonging to $NC \cap Quot(\mathcal{C}_{rss})$, it would be interesting to thoroughly investigate this class of groups. While a complete understanding of this class of groups remains an open problem for future study, following the previous two subsections, we know it includes all groups that are commensurable with the following concrete groups:

1. Any infinite, central quotient of the pure braid group $PB_n(S_{g,k})$ of n strands on a surface $S_{g,k}$—in particular, all surface pure braid groups $PB_n(S_{g,k})$, for $n \geq 1$ and either $g = 1$ and $k \geq 1$ or $g \geq 2$ and $k \geq 0$;
2. Any mapping class group $\mathrm{Mod}(S_{g,k})$, for $0 \leq g \leq 2$ and $2g + k \geq 4$;
3. Any Torelli group $\mathcal{I}(S_{g,k})$ and Johnson kernel $\mathcal{K}(S_{g,k})$, for $g = 1, 2$ and $2g+k \geq 4$;
4. Any group that is hyperbolic relative to a finite family of exact, residually finite, infinite, proper subgroups.

CHAPTER 4
DEFORMATION ON VON NEUMANN ALGEBRA

4.1 Gaussian Deformations Arising From Quasi-cocycles on Groups

Throughout this section we will assume that Γ is a countable group and $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ is an orthogonal representation such that $\mathcal{QH}^1_{as}(\Gamma,\pi) \neq \emptyset$. Following [PS09, Si10] to the orthogonal representation π one can associate, via the Gaussian construction, a probability measure space (Y_π, μ_π) and a family $\{\omega(\xi) : \xi \in \mathcal{H}\}$ of unitaries in $L^\infty(Y_\pi,\mu_\pi)$ such that $L^\infty(Y_\pi,\mu_\pi)$ is generated as a von Neumann algebra by the $\omega(\xi)$'s and the following relations hold:

1. $\omega(0) = 1$, $\omega(\xi_1 + \xi_2) = \omega(\xi_1)\omega(\xi_2)$, $\omega(\xi)^* = \omega(-\xi)$, for all $\xi, \xi_1, \xi_2 \in \mathcal{H}$;
2. $\tau(\omega(\xi)) = \exp(-\|\xi\|^2)$, where τ is the trace on $L^\infty(Y_\pi)$ given by integration.

Furthermore, there is a p.m.p. action $\Gamma \curvearrowright^{\hat{\pi}} (Y_\pi, \mu_\pi)$ called the *Gaussian action associated to* π which in turn induces a trace preserving action $\Gamma \curvearrowright^{\hat{\pi}} L^\infty(Y_\pi,\mu_\pi)$ that satisfies $\hat{\pi}_\gamma(\omega(\xi)) = \omega(\pi_\gamma(\xi))$, for all $\gamma \in \Gamma$ and $\xi \in \mathcal{H}$.

Assume that (N,τ) is a finite von Neumann algebra endowed with a trace τ, $\Gamma \curvearrowright^\sigma (N,\tau)$ is a trace preserving action and denote by $M = N \rtimes_\sigma \Gamma$ the corresponding crossed product von Neumann algebra. Then the *Gaussian dilation* of M is defined as the crossed product algebra $\tilde{M} = (N \bar{\otimes} L^\infty(Y_\pi,\mu_\pi)) \rtimes_{\sigma\otimes\hat{\pi}} \Gamma$.

Fix $q \in \mathcal{QH}^1_{as}(\Gamma,\pi)$ an unbounded quasi-cocycle. Following [Si10] (see also [CS11]) we construct a deformation arising from q through a canonical exponentiation procedure—throughout the text this will be referred to as the *Gaussian deformation* *associated with* q. For every $t \in \mathbb{R}$ consider the unitary $V_t \in \mathcal{U}(L^2(N)\bar{\otimes}L^2(Y_\pi, \mu_\pi)\bar{\otimes}\ell^2(\Gamma))$ defined by the formula

$$V_t(x \otimes y \otimes \delta_\gamma) = x \otimes \omega(tq(\gamma))y \otimes \delta_\gamma,$$

for every $x \in L^2(N)$, $y \in L^2(Y_\pi, \mu_\pi)$, and $\gamma \in \Gamma$. In [CS11] it was proved that V_t is a strongly continuous one parameter group of unitaries also satisfying the following transversality property, [Po08]:

Proposition 4.1.1. *[CS11, Lemma 2.8] Under the previous assumptions, for each t and any* $\xi \in L^2(M)$*, we have*

$$2||e_M^\perp \cdot V_t(\xi)||_2^2 \geq ||\xi - V_t(\xi)||_2^2 \geq ||e_M^\perp \cdot V_t(\xi)||_2^2, \tag{4.1.1}$$

where e_M *denotes the orthogonal projection of* $L^2(\tilde{M})$ *onto* $L^2(M)$ *and* $e_M^\perp = 1 - e_M$.

Notice also that the deformation V_t satisfies an "asymptotic bimodularity" property, a key notion to incorporate the "bounded equivariance" of quasi-cocycles into von Neumann algebra context.

Theorem 4.1.2. *[CS11, Lemma 2.6] For every* $x, y \in N \rtimes_{\sigma,r} \Gamma$ *(the reduced crossed product) we have that*

$$\lim_{t\to 0}\left(\sup_{\xi\in(L^2(M))_1} \|xV_t(\xi)y - V_t(x\xi y)\|_2\right) = 0. \tag{4.1.2}$$

In the remaining part of the section we prove a few other basic convergence properties for V_t that will be of essential use in the sequel. Before we proceed to the concrete statements we notice that when q is a 1-cocycle (i.e. when $D(q) = 0$) these properties follow easily and they are already used in one form or another throughout the literature, [Va10].

Lemma 4.1.3. *There exists a function* $f:\mathbb{R}\to[0,2^{1/2}]$ *satisfying* $\lim_{t\to 0} f(t)=0$ *and such that for every* $x,y\in M$ *and* $z\in N\rtimes_{alg}\Gamma$ *we have the following inequality:*

$$\begin{aligned}&\max\{\|V_t(xy)-xV_t(y)\|_2,\|V_t(yx)-V_t(y)x\|_2\}\\&\leq 2\|x\|_\infty\|y-z\|_2+\|z\|_\infty\,|sup(z)|^{1/2}\left(\|V_{2^{1/2}t}(x)-x\|_2+f(t)\|x\|_2\right).\end{aligned}\tag{4.1.3}$$

Here we denoted by $|sup(z)|$ *the cardinality of the support of* z *in* Γ.

Proof. Using the triangle inequality and the fact that V_t is a unitary we have that

$$\begin{aligned}\|V_t(xy)-xV_t(y)\|_2&\leq\|V_t(x(y-z))\|_2+\|xV_t(y-z)\|_2+\|V_t(xz)-xV_t(z)\|_2\\&\leq 2\|x\|_\infty\|y-z\|_2+\|V_t(xz)-xV_t(z)\|_2.\end{aligned}\tag{4.1.4}$$

Next, let $z=\sum_\mu z_\mu u_\mu\in N\rtimes_{alg}\Gamma$, with $z_\mu\in N$, and let $x=\sum_\lambda x_\lambda u_\lambda$, with $x_\lambda\in N$, be the Fourier decompositions of z and x respectively. Thus, using the Cauchy-Schwarz inequality together with the formula for V_t, we see that

$$\begin{aligned}\|V_t(xz)-xV_t(z)\|_2&=\|\sum_\mu V_t(xz_\mu u_\mu)-xV_t(z_\mu u_\mu)\|_2\\&\leq|\operatorname{sup}(z)|^{1/2}\left(\sum_\mu\|V_t(xz_\mu u_\mu)-xV_t(z_\mu u_\mu)\|_2^2\right)^{1/2}\\&=|\operatorname{sup}(z)|^{1/2}\left(\sum_{\mu,\lambda}\|x_\lambda\sigma_\lambda(z_\mu)\otimes(\omega(tq(\lambda\mu))-\omega(t\pi_\lambda(q(\mu))))\|_2^2\right)^{1/2}\\&=|\operatorname{sup}(z)|^{1/2}\left(\sum_{\mu,\lambda}(2-2e^{-t^2\|q(\lambda\mu)-\pi_\lambda(q(\mu))\|^2})\|x_\lambda\sigma_\lambda(z_\mu)\|_2^2\right)^{1/2}\end{aligned}\tag{4.1.5}$$

Furthermore, using successively the basic inequalities $\|x_\lambda \sigma_\lambda(z_\mu)\|_2 \leq \|z_\mu\|_\infty \|x_\lambda\|_2 \leq \|z\|_\infty \|x_\lambda\|_2$, $\|q(\lambda\mu) - \pi_\lambda(q(\mu))\|^2 \leq 2\|q(\lambda)\|^2 + 2D(q)^2$, and $e^{-2t^2\|q(\lambda)\|^2} \leq 1$ we see that the last term above is smaller than

$$
\begin{aligned}
&\leq |\sup(z)|\,\|z\|_\infty \left(\sum_\lambda (2 - 2e^{-2t^2\|q(\lambda)\|^2 - 2t^2 D(q)^2})\|x_\lambda\|_2^2\right)^{1/2}\\
&\leq |\sup(z)|\,\|z\|_\infty \left(\sum_\lambda (2 - 2e^{-2t^2\|q(\lambda)\|^2})\|x_\lambda\|_2^2 + \sum_\lambda (2 - 2e^{-2t^2 D(q)^2})\|x_\lambda\|_2^2\right)^{1/2}\\
&= |\sup(z)|\,\|z\|_\infty \left(\|V_{2^{1/2}t}(x) - x\|_2^2 + \|x\|_2^2(2 - 2e^{-2t^2 D(q)^2})\right)^{1/2}\\
&\leq |\sup(z)|\,\|z\|_\infty \left(\|V_{2^{1/2}t}(x) - x\|_2 + \|x\|_2(2 - 2e^{-2t^2 D(q)^2})^{1/2}\right)
\end{aligned}
\tag{4.1.6}
$$

So, letting $f(t) = (2 - 2e^{-2t^2 D(q)^2})^{1/2}$, the inequalities (4.1.4), (4.1.5), and (4.1.6) give the first inequality in (4.1.3). The second inequality in (4.1.3) follows similarly and we leave the details to the reader. □

Proposition 4.1.4. *Let $X \subseteq (M)_1$ be a set and let $F \subseteq M$ be a finite subset and denote by $cu(F) = \{\sum \lambda_i x_i \,:\, x_i \in F, \lambda_i \in \mathbb{C}, |\lambda_i| \leq 1\}$. If $V_t \to Id$ (or equivalently $e_M^\perp \cdot V_t \to 0$) uniformly on X then $V_t \to Id$ (or equivalently $e_M^\perp \cdot V_t \to 0$) uniformly on $cu(F) \cdot X \cdot cu(F)$.*

Proof. To show our statement, it is sufficient to prove that $V_t \to \mathrm{Id}$ uniformly on $X \cdot cu(F)$ and on $cu(F) \cdot X$. We will only show the former convergence as the later will follow in a similar manner. Moreover, using the triangle inequality it suffices to show that $V_t \to \mathrm{Id}$ uniformly on $X \cdot cu(F)$ only when F is a singleton, thus assume that $F = \{y\}$.

Fix $\varepsilon > 0$ and by Kaplansky's density theorem let $y_\varepsilon \in N \rtimes_{alg} \Gamma$ with $\|y_\varepsilon\|_\infty \leq \|y\|_\infty$ such that

$$\|y - y_\varepsilon\|_2 \leq \varepsilon/6. \tag{4.1.7}$$

Also, since $V_t \to \mathrm{Id}$ uniformly on $X \cup \{y\}$ and since $f(t) \to 0$ as $t \to 0$, one can find $t_\varepsilon > 0$ such that for all $|t| \leq t_\varepsilon$ and all $x \in X$ we simultaneously have

$$\begin{aligned} &\|V_t(x) - x\|_2 \leq \varepsilon/\left(6\|y_\varepsilon\|_\infty |sup(y_\varepsilon)|^{1/2}\right); \\ &\|V_t(y) - y\|_2 \leq \varepsilon/3; and \\ &f(t) \leq \varepsilon/\left(6\|y_\varepsilon\|_\infty |sup(y_e)|^{1/2}\right). \end{aligned} \tag{4.1.8}$$

Thus, applying the triangle inequality and then using the first inequality in (4.1.3) in the previous lemma together with (4.1.8), (4.1.7), and $\|x\|_2 \leq \|x\|_\infty \leq 1$ we see that for every $|t| \leq t_\varepsilon/2^{1/2}$ we have

$$\begin{aligned} \|V_t(xy) - xy\|_2 &\leq \|V_t(xy) - xV_t(y)\|_2 + \|V_t(y) - y\|_2 \\ &\leq 2\|y - y_\varepsilon\|_2 + \|y_\varepsilon\|_\infty |sup(y_\varepsilon)|^{1/2} \left(\|V_{2^{1/2}t}(x) - x\|_2 + f(t)\right) \\ &+ \|V_t(y) - y\|_2 \\ &\leq \varepsilon/3 + \varepsilon/6 + \varepsilon/6 + \varepsilon/3 = \varepsilon, \end{aligned}$$

which finishes the proof. □

Corollary 4.1.5. *Let $X \subseteq (M)_1$ be a subset and for each $i = 1, 2$ let $a_i \in M_+$ be positive elements, $f_i \in M$ be projections, and $\lambda_i > 0$ be scalars satisfying $f_i a_i = a_i f_i \geq \lambda_i f_i$. If $e_M^\perp \cdot V_t \to 0$ uniformly on $a_1 X a_2$ then $e_M^\perp \cdot V_t \to 0$ uniformly on $f_1 X f_2$.*

Proof. Since $f_i a_i = a_i f_i \geq \lambda_i f_i$ then one can find $x_i \in M$ such that $a_i x_i = f_i$. The statement follows then from the transversality property (Proposition 4.1.1) and Proposition 4.1.4. $\square$

Remark 4.1.6. More generally, instead of being a quasi-cocycle assume that q is an *s-array* on Γ, i.e., there exists a linear function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|q(\lambda\gamma) - \pi_\lambda(q(\gamma))\| \leq \psi(\|q(\lambda)\|)$, for all $\lambda, \gamma \in \Gamma$. Then, one can easily check that a version of Lemma 4.1.3 still holds, with some constants in the inequality (4.1.3) and the formula for f there slightly modified. Consequently, Proposition 4.1.4 and Corollary 4.1.5 will also hold in this case.

Lemma 4.1.7. *Let $\Sigma < \Gamma$ be a subgroup and let $a \in N \rtimes \Gamma$ be an element satisfying $0 \leq a \leq 1$ and put $P = N \rtimes \Gamma$. Then for every $\epsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $|t| \leq t_\varepsilon$ and all $x \in (P)_1$ we have*

$$\|e_M^\perp \cdot V_t(a'x)\|_2^2 \leq \|e_M^\perp \cdot V_t(ax)\|_2^2 + \varepsilon, \tag{4.1.9}$$

where $a' = E_P(a)$.

Proof. As before consider the Gaussian dilations $\tilde{M} = (N \bar{\otimes} L^\infty(X_\pi, \mu_\pi)) \rtimes \Gamma$ and by $\tilde{P} = (N \bar{\otimes} L^\infty(X_\pi, \mu_\pi)) \rtimes \Sigma$ and we notice the following commuting square condition $E_{\tilde{P}} \circ E_M = E_M \circ E_{\tilde{P}} = E_P$.

Fix $\varepsilon > 0$. By Kaplansky density theorem there exists $a_\varepsilon \in N \rtimes_{alg} \Gamma$ with $\|a_\varepsilon\|_\infty \leq 1$ such that $\|a - a_\varepsilon\|_2 \leq \varepsilon/4$. Also, using Theorem 4.1.2, there exists $t_\varepsilon > 0$

such that for all $x \in (P)_1$ and all $0 \leq |t| \leq t_\varepsilon$ we have

$$\begin{aligned} &\|e_M^\perp \cdot V_t(a_\varepsilon x) - a_\varepsilon e_M^\perp \cdot V_t(x)\|_2 \leq \varepsilon/4, \text{ and} \\ &\|e_M^\perp \cdot V_t(a'_\varepsilon x) - a'_\varepsilon e_M^\perp \cdot V_t(x)\|_2 \leq \varepsilon/4, \end{aligned} \tag{4.1.10}$$

where we have denoted by $a'_\varepsilon = E_P(a_\varepsilon)$. Thus, inequalities (4.1.10) in combination with relations $e_{\tilde{P}} \cdot e_M^\perp \cdot V_t(x) = e_M^\perp \cdot V_t(x)$, for all $x \in P$, $e_{\tilde{P}} a_\varepsilon e_{\tilde{P}} = E_P(a_\varepsilon) e_{\tilde{P}}$, and the basic inequality $\|e_{\tilde{P}}(\xi)\|_2 \leq \|\xi\|_2$, for all $\xi \in L^2(\tilde{M})$, show that for every $|t| \leq t_\varepsilon$ we have

$$\begin{aligned} \|e_M^\perp \cdot V_t(ax)\|_2 &\geq \|e_M^\perp \cdot V_t(a_\varepsilon x)\|_2 - \varepsilon/4 \\ &\geq \|a_\varepsilon e_M^\perp \cdot V_t(x)\|_2 - \varepsilon/2 \\ &= \|a_\varepsilon \cdot e_{\tilde{P}} \cdot e_M^\perp \cdot V_t(x)\|_2 - \varepsilon/2 \\ &\geq \|e_{\tilde{P}} \cdot a_\varepsilon \cdot e_{\tilde{P}} e_M^\perp \cdot V_t(x)\|_2 - \varepsilon/2 \\ &= \|E_P(a_\varepsilon) e_M^\perp \cdot V_t(x)\|_2 - \varepsilon/2 \\ &\geq \|e_M^\perp \cdot V_t(E_P(a_\varepsilon)x)\|_2 - 3\varepsilon/4 \\ &\geq \|e_M^\perp \cdot V_t(E_P(a)x)\|_2 - \varepsilon. \end{aligned} \tag{4.1.11}$$

Since $\varepsilon > 0$ was arbitrary then inequality (4.1.9) follows by squaring (4.1.11). □

Lemma 4.1.8. *Let $\Sigma < \Gamma$ be a subgroup and let $p \in N \rtimes \Sigma =: P$ be a nonzero projection. Assume for every $\varepsilon > 0$ there exists $t_\varepsilon^1 > 0$ such that for all $|t| \leq t_\varepsilon^1$ and all $x \in (pPp)_1$ we have*

$$\|e_M^\perp \cdot V_t(x)\|_2^2 \leq \varepsilon. \tag{4.1.12}$$

Then there exists a nonzero element $r \in \mathcal{Z}(P)$ with $0 < r \leq 1$ such that for every

$\varepsilon > 0$ *one can find* $t_\varepsilon > 0$ *such that for all* $|t| \leq t_\varepsilon$ *and all* $y \in (P)_1$ *we have*

$$\|e_M^\perp \cdot V_t(yr)\|_2^2 \leq \varepsilon. \tag{4.1.13}$$

Proof. First we claim that there exists $r' \in \mathcal{Z}(P)$ a projection such that $r'p \neq 0$ and for every $\varepsilon_1 > 0$ there exists $t_{\varepsilon_1} > 0$ such that for all $|t| \leq t_{\varepsilon_1}$, $y \in (P)_1$ we have

$$\|e_M^\perp \cdot V_t(yr'p)\|_2^2 \leq \varepsilon_1. \tag{4.1.14}$$

To see this we use a standard convexity argument [CP10], and [Va10]. Notice that the set $\mathcal{G} := \{nu_\gamma : n \in \mathcal{U}(N), \gamma \in \Sigma\}$ forms a dense subgroup of P. Consider the closed convex hull $\mathcal{K}(p) = \overline{conv}^{\|\cdot\|_2}\{zpz^* : z \in \mathcal{G}\}$ and denote by $q \in K(p)$ the unique element of minimal $\|\cdot\|_2$. Notice that since $\|zqz^*\|_2 = \|q\|_2$ and $zqz^* \in \mathcal{K}(p)$ for every $z \in \mathcal{G}$ then by uniqueness we have that $zqz^* = q$ for every $z \in \mathcal{G}$. Thus $q \in P \cap \mathcal{G}' = \mathcal{Z}(P)$ and since $ctr_P(\mathcal{K}(p)) = ctr_P(p)$ we conclude that $q = ctr_P(p) \in \mathcal{K}(p)$.

Fix $1 \geq \varepsilon > 0$. Thus from the definition of $\mathcal{K}(p)$ one can find a finite subset $\mathcal{F}_\varepsilon \subset \mathcal{G}$ and $0 < c_s, s \in \mathcal{F}_\varepsilon$ with $\sum_{s\in\mathcal{F}_\varepsilon} c_s = 1$ such that

$$\|q - \sum_{s\in\mathcal{F}_\varepsilon} c_s sps^*\|_2 \leq \varepsilon/8. \tag{4.1.15}$$

Moreover, by Proposition 4.1.1 and Theorem 4.1.2 above there exists $t_\varepsilon^1 > 0$ such that for all $|t| \leq t_\varepsilon^1$, $s \in \mathcal{F}_\varepsilon$, and $x \in (P)_1$ we have

$$\|se_M^\perp \cdot V_t(ps^*xp) - e_M^\perp \cdot V_t(sps^*xp)\|_2 \leq \varepsilon/8. \tag{4.1.16}$$

Also from (4.1.12) there exists $t_\varepsilon^2 > 0$ such that for all $|t| \leq t_\varepsilon^2$, $s \in \mathcal{F}_\varepsilon$, and $x \in (P)_1$ we have

$$\|e_M^\perp \cdot V_t(ps^*xp)\|_2^2 \leq \varepsilon/8. \tag{4.1.17}$$

Using the triangle inequality together with $\|V_t(\xi)\|_2 \leq \|\xi\|_2$, for $\xi \in L^2(M)$, (4.1.15), (4.1.16), (4.1.17) and Cauchy-Schwarz inequality for every $x \in (P)_1$ and $|t| \leq \min\{t^1_\varepsilon, t^2_\varepsilon\}$, we have

$$\begin{aligned}
\|e_M^\perp \cdot V_t(qxp)\|_2^2 &\leq \left(\|e_M^\perp \cdot V_t((q - \sum_{s\in\mathcal{F}_\varepsilon} c_s sps^*)xp)\|_2 + \sum_{s\in\mathcal{F}_\varepsilon} c_s \|e_M^\perp \cdot V_t(sps^*xp)\|_2 \right)^2 \\
&\leq (\|e_M^\perp \cdot V_t((q - \sum_{s\in\mathcal{F}_\varepsilon} c_s sps^*)xp)\|_2 + \\
&\quad + \sum_{s\in\mathcal{F}_\varepsilon} c_s \|e_M^\perp \cdot V_t(sps^*xp) - se_M^\perp \cdot V_t(ps^*xp)\|_2 + \\
&\quad + \sum_{s\in\mathcal{F}_\varepsilon} c_s \|e_M^\perp \cdot V_t(ps^*xp)\|_2)^2 \\
&\leq \left(\varepsilon/8 + \varepsilon/8 + \sum_{s\in\mathcal{F}_\varepsilon} c_s \|e_M^\perp \cdot V_t(ps^*xp)\|_2 \right)^2 \\
&\leq \varepsilon^2/16 + \varepsilon/2 + \sum_{s\in\mathcal{F}_\varepsilon} c_s \|e_M^\perp \cdot V_t(ps^*xp)\|_2^2 \\
&\leq \varepsilon^2/16 + \varepsilon/2 + \varepsilon/8 \leq \varepsilon
\end{aligned}$$

Altogether, we have obtained that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $|t| \leq t_\varepsilon$ and all $x \in (P)_1$ we have

$$\|e_M^\perp \cdot V_t(qxp)\|_2^2 \leq \varepsilon. \tag{4.1.18}$$

For every $\mu > 0$ we denote by q_μ the spectral projection of q corresponding to the interval (μ, ∞) and notice that $q_\mu \nearrow r := supp(q)$ increasingly in SO- topology, as $\mu \searrow 0$. Thus there exists $\delta > 0$ such that $q_\delta q \neq 0$ and since $q = ctr_P(p)$ it follows that $q_\delta p \neq 0$. Moreover, since $q_\delta q \geq \delta q_\delta$ there exists an element $x_\delta \in \mathcal{Z}(P)$ such that $qq_\delta x_\delta = q_\delta$ and $\|x_\delta\|_\infty \leq \delta^{-1}$.

Fix $\varepsilon_1 > 0$. From (4.1.18) there exists t_{ε_1} such that for all $|t| \leq t_{\varepsilon_1}$ and all

$x \in (P)_1$ we have

$$\|e_M^{\perp} \cdot V_t(qxp)\|_2^2 \leq \varepsilon_1 \delta^2.$$

If in this inequality we let $x = \delta q_\delta x_\delta y$ for arbitrary $y \in (P)_1$ then we get our claim for $r' = q_\delta$.

Finally, we notice that our claim together with same averaging argument as used in its proof further implies (4.1.13), where $r = ctr_P(r'p)$. In fact, the arguments presented above apply verbatim and we leave the details to the reader. □

Lemma 4.1.9. *Assume the previous notations and let* $N \rtimes \Sigma =: P$. *Assume that there exists a finite subgroup* $\Omega < \Sigma$ *such that* $\mathcal{Z}(P) \subseteq N \rtimes \Omega$. *Also suppose there is a nonzero element* $0 \leq r \leq 1$ *with* $r \in \mathcal{Z}(P)$ *such that for every* $\varepsilon > 0$ *there exists* $t_\varepsilon > 0$ *such that for all* $|t| \leq t_\varepsilon$ *and all* $x \in (P)_1$ *we have*

$$\|e_M^{\perp} \cdot V_t(xr)\|_2^2 \leq \varepsilon. \tag{4.1.19}$$

Then the quasi-cocycle q *is bounded on* Σ.

Proof. Fix $0 < \varepsilon < 1$ and applying the assumption for $\varepsilon\|r\|_2^2$ it follows that there exists $t > 0$ such that for all $\gamma \in \Sigma$ we have

$$\|e_M^{\perp} \cdot V_t(u_\gamma r)\|_2^2 \leq \varepsilon\|r\|_2^2. \tag{4.1.20}$$

Consider $r = \sum_{\omega \in \Omega} r_\omega u_\omega$ be the its Fourier decomposition. Thus using the definition of V_t we see that (4.1.20) is equivalent to

$$\sum_{\omega \in \Omega} (1 - e^{-t^2\|q(\gamma\omega)\|^2})\|r_\omega\|_2^2 \leq \varepsilon \sum_{\omega \in \Omega} \|r_\omega\|_2^2.$$

In particular, this inequality implies that for every $\gamma \in \Sigma$ there exists $\omega \in \Omega$ such that $1 - e^{-t^2\|q(\gamma\omega)\|^2} \leq \varepsilon$ or equivalently $\|q(\gamma\omega)\| \leq (\ln(2/(1-\varepsilon))^{1/2})/t$. Using the quasi-cocycle relation this, further entails that $\|q(\gamma)\| \leq D(q) + \|q(\omega)\| + (\ln(2/(1-\varepsilon))^{1/2})/t$. Altogether, we have $\|q(\gamma)\| \leq D(q) + \sup_{\omega\in\Omega} \|q(\omega)\| + (\ln(2/(1-\varepsilon))^{1/2})/t$, for all $\gamma \in \Sigma$, and since Ω is finite it follows that q is bounded on Σ. $\square$

Remark 4.1.10. Finally, we leave to the reader to check that all the previous Lemmas 4.1.7- 4.1.9 still hold if instead of being a quasi-cocycle one assumes that q is just an (anti)symmetric [CS11] s-array on Γ, as defined in Remark 4.1.6. Essentially, all the proofs will follow in the same way.

CHAPTER 5
MAIN RESULTS

5.1 Primeness Results for von Neumann Algebras of Groups in $NC \cap Quot(\mathcal{C}_{rss})$

In this section we will use the technical results from the previous sections to derive the proof of Theorem A. Our arguments are similar in essence with the ones used in [CIK13] but they have slightly different technical forms. For the sake of completeness we include all details. To simplify the writing in the main proof we first introduce a notation:

Notation 5.1.1. Fixing a group $\Gamma_n \in Quot_n(\mathcal{C}_{rss})$, there exist groups $\Gamma_1, \Gamma_2, \ldots, \Gamma_{n-1}$ and a collection of surjective homomorphisms $\pi_k : \Gamma_k \to \Gamma_{k-1}$ such that $\Gamma_1 \in \mathcal{C}_{rss}$, and $ker(\pi_k) \in \mathcal{C}_{rss}$, for all $2 \leq k \leq n$. Then we define $\theta_n = \pi_2 \circ \pi_3 \circ \cdots \circ \pi_n : \Gamma_n \to \Gamma_1$ and notice that by Proposition 2.2.2 we have that $ker(\theta_n) \in Quot_{n-1}(\mathcal{C}_{rss})$.

Theorem 5.1.2. *Let Γ be a group that can be realized as a finite-by-($\mathbf{NC}_1 \cap \mathbf{Quot}(\mathcal{C}_{rss})$) group. Denote by $L(\Gamma)$ its corresponding von Neumann algebra. If $p \in L(\Gamma)$ is a nonzero projection, then any two diffuse, commuting subalgebras $B, C \subseteq pL(\Gamma)p$ generate together a von Neumann subalgebra $B \vee C$ which has infinite Pimsner-Popa index in $pL(\Gamma)p$. In particular, $L(\Gamma)$ is prime and hence $L(\Gamma) \ncong L(\Omega \times \Sigma)$, for any infinite groups Ω and Σ.*

Proof of theorem A. Since both NC and $Quot_n(C_{rss})$ are closed under commensurability then it will be sufficient to treat the case $\Lambda/\Omega = \Gamma_n \in NC \cap Quot_n(C_{rss})$, where $\Omega \lhd \Lambda$ is a finite normal subgroup.

Throughout the proof we will denote by $\{u_\lambda\}_{\lambda\in\Lambda} \subset L(\Lambda) =: M$ and $\{v_\gamma\}_{\gamma\in\Gamma_1} \subset L(\Gamma_1)$ the canonical unitaries. We will prove our statement by induction on n.

First we argue for $n = 1$. Since $\Gamma_1 \in \mathcal{C}_{rss}$ denote by $\theta : \Lambda \to \Gamma_1$ the canonical projection. Consider the $*$-homomorphism $\tilde{\theta} : M \to M\bar{\otimes}L(\Gamma_1)$ given by $\tilde{\theta}(u_\lambda) = u_\lambda \otimes v_{\theta(\lambda)}$, for all $\lambda \in \Lambda$. Assume by contradiction that $B, C \subseteq pMp$ are two commuting, diffuse subalgebras such that the inclusion $B \vee C \subseteq pMp$ has finite index. Hence $\tilde{\theta}(B)$, $\tilde{\theta}(C)$ are commuting, diffuse subalgebras of $\tilde{\theta}(p)\,(M\bar{\otimes}L(\Gamma_1))\,\tilde{\theta}(p)$. Let $P \subseteq B$ be an arbitrary diffuse, amenable subalgebra. Then $\tilde{\theta}(P)$ is amenable and hence it is amenable relatively to $M \otimes 1$ inside $M\bar{\otimes}L(\Gamma_1)$. Thus by the dichotomy property we either have that

1. $\tilde{\theta}(P) \preceq_{M\bar{\otimes}L(\Gamma_1)} M \otimes 1$, or
2. $\tilde{\theta}(C)$ is amenable relative to $M \otimes 1$ inside $M\bar{\otimes}L(\Gamma_1)$.

Moreover, the case (2) above further implies, by the same dichotomy theorem, that either

(3) $\tilde{\theta}(C) \preceq_{M\bar{\otimes}L(\Gamma_1)} M \otimes 1$, or

(4) $\tilde{\theta}(B \vee C)$ is amenable relative to $M \otimes 1$ inside $M\bar{\otimes}L(\Gamma_1)$.

As in the proof of [CIK13, Theorem 3.1] we show that case (4) above will lead to a contradiction. Indeed since $B\vee C \subseteq pMp$ has finite index then $pMp \preceq^s_{pMp} B \vee C$ and hence pMp is amenable relative to $B \vee C$ inside pMp. This implies that $\tilde{\theta}(pMp)$ is amenable relative to $\tilde{\theta}(B \vee C)$ inside $M\bar{\otimes}L(\Gamma_1)$ and by [OP07, Proposition 2.4] it follows that $\tilde{\theta}(pMp)$ is amenable relative to $M \otimes 1$ inside $M\bar{\otimes}L(\Gamma_1)$. Finally, [CIK13, Proposition 3.5] further implies that $\theta(\Lambda) = \Gamma_1$ is amenable which is a contradiction.

In conclusion, for every $P \subseteq B$ be an arbitrary diffuse, amenable subalgebra we have either (1) or (3) above. Due to symmetry, we can assume without any loss of generality that $\tilde{\theta}(B) \preceq_{M\bar{\otimes}L(\Gamma_1)} M \otimes 1$. By [CIK13, Proposition 3.4] this further implies that $B \preceq_M L(\Omega)$ and since Ω is finite it follows that B is not diffuse which contradicts the assumptions; this completely settles case $n = 1$.

Next we show the inductive step. Since $\Lambda/\Omega = \Gamma_n \in \mathcal{C}_{rss}$ there exists a surjection $\theta' = \theta_n \circ \theta : \Lambda \to \Gamma_1$, where θ_n is the homomorphism from Notation 5.1.1. This allows us to define a $*$-homomorphism $\tilde{\theta}' : M \to M\bar{\otimes}L(\Gamma_1)$ by letting $\tilde{\theta}'(u_\lambda) = u_\lambda \otimes v_{\theta'(\lambda)}$, for all $\lambda \in \Lambda$. Assume by contradiction that $B, C \subseteq pMp$ are two commuting, diffuse subalgebras such that the inclusion $B \vee C \subseteq pMp$ has finite index. Thus $\tilde{\theta}'(B)$, $\tilde{\theta}'(C)$ are two commuting diffuse subalgebras of $\tilde{\theta}'(p)\,(M\bar{\otimes}L(\Gamma_1))\,\tilde{\theta}'(p)$. Proceeding as in $n = 1$ case one of the following must hold:

(5) $\tilde{\theta}'(B) \preceq_{M\bar{\otimes}L(\Gamma_1)} M \otimes 1$;

(6) $\tilde{\theta}'(C) \preceq_{M\bar{\otimes}L(\Gamma_1)} M \otimes 1$;

(7) $\tilde{\theta}'(B \vee C)$ is amenable relative to $M \otimes 1$ inside $M\bar{\otimes}L(\Gamma_1)$.

As in that proof, case (7) implies that $\theta'(\Lambda) = \Gamma_1$ is amenable which is a contradiction and cases (5) and (6) implies that $B \preceq_M L(ker(\theta')))$ and $C \preceq_M L(ker(\theta'))$, respectively. Also, notice that if B is amenable (and hence C non-amenable!) we automatically have that $B \preceq_M L(ker(\theta')))$. Thus, by the previous discussion, it suffices to treat only the case $B \preceq_M L(\Sigma)$, where $\Sigma = ker(\theta')$. By Proposition 3.1.12 one can find $s > 0$, non-zero projections $r \in L(\Sigma), q \in B$, a subalgebra $B_o \subseteq rL(\Sigma)r$, and a $*$-isomorphism $\theta : qBq \to B_o$ such that the following properties are satisfied:

$$B_o \vee (B_o' \cap rL(\Sigma)r) \subset rL(\Sigma)r \text{ has finite index;} \tag{5.1.1}$$

$$\text{there exist a non-zero partial isometry } v \in M \text{ such that} \tag{5.1.2}$$

$$rE_N(vv^*) = E_N(vv^*)r \geq sr \text{ and } \theta(qBq)v = B_o v = rvqBq.$$

By Theorem 2.1.4 and Proposition 2.2.2, we have $\Sigma/(\Sigma\cap\Omega) = ker(\theta')/(\ker(\theta')\cap\Omega) = \ker(\theta_n) \in NC \cap Quot_{n-1}(\mathcal{C}_{rss})$. Thus, by the induction hypothesis, it follows a corner of B_o or $B_o' \cap rL(\Sigma)r =: C_o$ is completely atomic. However, since B is diffuse then from (5.1.2) and (5.1.1) it follows that B_o is diffuse too, and hence a corner of C_o is completely atomic. Thus, there exists $p_o \in C_o$ nonzero projection such that $p_o C_0 p_o = \mathbb{C}p_o$. From (5.1.1) the inclusion $B_o \vee C_o \subseteq rL(\Sigma)r$ has finite index and from Lemma 3.1.11 it follows that the inclusion $B_o p_o = B_o \vee (\mathbb{C}p_o) = p_o(B_o \vee C_o)p_o \subseteq p_o L(\Sigma)p_o$ has finite index, too.

Let $\wp \in \mathcal{QH}^1_{as}(\Lambda, \pi)$ be an unbounded quasi-cocycle and let $V_t : L^2(M) \to L^2(\tilde{M})$ be corresponding Gaussian deformation as defined in the Section 4.1, where $M \subseteq \tilde{M}$ is the Gaussian dilation of M. Denote by e_M the orthogonal projection on $L^2(\tilde{M})$ onto $L^2(M)$. Since C can always be assumed non-amenable then using the same à la Popa spectral gap argument (see for instance [CS11, Theorem 3.2]) we have that $e_M^{\perp} \cdot V_t \to 0$ uniformly on $(B)_1$, as $t \to 0$. Using Proposition 4.1.4 this further implies that $e_M^{\perp} \cdot V_t \to 0$ uniformly on $rv(qBq)_1$, as $t \to 0$. Using (5.1.2) and

Proposition 4.1.4 again we get that $e_M^{\perp} \cdot V_t \to 0$ uniformly on $(B_o)_1 rvv^*$, as $t \to 0$. Moreover, Lemma 4.1.7 further gives that $e_M^{\perp} \cdot V_t \to 0$ uniformly on $(B_o)_1 rE_N(vv^*)$, as $t \to 0$. Hence, by (5.1.2) and Corollary 4.1.5 we obtain that $e_M^{\perp} \cdot V_t \to 0$ uniformly on $(B_o)_1 r = (B_o)_1$, as $t \to 0$ and by Proposition 4.1.4 again we conclude that $e_M^{\perp} \cdot V_t \to 0$ uniformly on $(B_o)_1 p_o$, as $t \to 0$.

Since $B_o p_o \subseteq p_o L(\Sigma) p_o$ has finite Pimsner-Popa index then using part (2) in Lemma 3.1.11 we have that $p_o L(\Sigma) p_o \preceq_{p_o L(\Sigma) p_o} B_o p_o$. Hence one can find nonzero projections $p_o' \in p_o L(\Sigma) p_o$, $r_o \in B_o p_o$, nonzero partial isometry $v_o \in p_o L(\Sigma) p_o$, and an injective unital $\star$-homomorphism $\Xi : p_o' L(\Sigma) p_o' \to r_o B r_o$ such that $\Xi(x) v_o = v_o x$, for all $x \in p_o' L(\Sigma) p_o'$. Since $v_o^* v_o \in \mathcal{Z}(p_o' L(\Sigma) p_o')$ and $v_o v_o^* \in (\Xi(p_o' L(\Sigma) p_o'))' \cap r_o L(\Sigma) r_o$ it follows that $v_o v_o^* L(\Sigma) v_o v_o^* = v_o L(\Sigma) v_o^* = \Xi(p_o' L(\Sigma) p_o') v_o v_o^*$. Since $\Xi(p_o' L(\Sigma) p_o') \subseteq p_o B_o p_0$ and $e_M^{\perp} \cdot V_t \to 0$ uniformly on $(B_o)_1 p_o$, as $t \to 0$ then from Proposition 4.1.4 it follows that $e_M^{\perp} \cdot V_t \to 0$ uniformly on $(\Xi(p_o' L(\Sigma) p_o'))_1$, as $t \to 0$. Thus using the above relations together with Proposition 4.1.4 we further get that $e_M^{\perp} \cdot V_t \to 0$ uniformly on $(\Xi(p_o' L(\Sigma) p_o') v_o v_o^*)_1 = (v_o v_o^* L(\Sigma) v_o v_o^*)_1$, as $t \to 0$. Then Lemmas 4.1.8-4.1.9 and Corollary 6.1.3 imply the quasi-cocycle $\wp$ is bounded on Σ. Moreover, since Σ is normal in Λ then [CSU13, Theorem 2.1] further implies that $\wp$ is bounded on Λ and we have reached a contradiction; this settles the inductive step and hence the proof. $\square$

We strongly believe Theorem A actually holds for all groups satisfying only condition NC. A successful strategy to show such a statement seems to depend heavily on investigating new aspects of the infinitesimal analysis of the weak deformations arising from quasicocycles (the Gaussian dilation from Section 4.1). However, there are some serious technical obstacles in this direction, the most significant being the lack of uniform bimidularity as well as good averaging properties of the Gaussian dilations associated with non-proper unbounded quasicocycles.

Conjecture 5.1.3. *Let $\Gamma \in NC$ and let $L(\Gamma)$ be its the corresponding von Neumann algebra. If $p \in L(\Gamma)$ is a nonzero projection, then any two diffuse, commuting subalgebras $B, C \subseteq pL(\Gamma)p$ generate together a von Neumann subalgebra $B \vee C$ which has infinite Pimsner-Popa index in $pL(\Gamma)p$; in particular, $L(\Gamma)$ is prime.*

Our techniques can also be used to show primeness of II_1 factors arising from all free ergodic probability measure preserving actions of groups in the class $NC \cap Quot(\mathcal{C}_{rss})$. In fact one can show the following counterpart of Theorem A for actions.

5.2 Product Rigidity of Group Von Neumann Algebra

The main goal of this section is to prove the following result,

Theorem 5.2.1. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$ be an icc group and suppose there exists $A_1, \ldots, A_k \subset L(\Gamma)$ commuting diffuse subalgebras of $L(\Gamma)$ generating a finite index subalgebra. Then there exists a projection $p_i \in A_i$, finite index subfactors $D_i \subset p_i A_i p_i$, groups $\Gamma_1, \ldots, \Gamma_k$, and a unitary $u \in pL(\Gamma)p$ such that*

1. *$D_i \subset p_1 u^* L(\Gamma_i) u p_i$ is a finite index inclusion of algebras,*
2. *$\Gamma_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{rss})$*
3. *$\sum_{i=1}^{k} n_i = n$,*
4. *Γ is commensurable to $\Gamma_1 \times \cdots \times \Gamma_k$.*

In particular $L(\Gamma)$ is prime if and only if Γ is virtually indecomposible as a product of groups in $\mathrm{Quot}(\mathcal{C}_{rss})$.

Before proceeding to the proof we need to introduce more notations. Given a base class of groups $\mathcal{C}$, a discrete countable group Γ is a *finite step extension by $\mathcal{C}$* if there exists a exists a chain of groups

$$\Gamma = \Gamma_n \overset{\pi_n}{\to} \Gamma_{n-1} \overset{\pi_{n-1}}{\to} \cdots \overset{\pi_2}{\to} \Gamma_1 \overset{\pi_1}{\to} 1 \tag{5.2.1}$$

such that for every $1 \leq k \leq n$, $\pi_k : \Gamma_k \to \Gamma_{k-1}$ is a surjective homomorphism with $\ker \pi_k \in \mathcal{C}$. For each $n \in$, $\mathrm{Quot}_n(\mathcal{C})$ denotes the collection of all groups which are n-step extensions by $\mathcal{C}$. We denote the collection of all finite step extension by $\mathcal{C}$ by $\mathrm{Quot}(\mathcal{C}) = \cup \mathrm{Quot}_n(\mathcal{C})$. Given any group $\Gamma \in \mathrm{Quot}_n(\mathcal{C})$ with $n > 2$ we may recursively define the following family of groups: First let $\Gamma_n^{(0)} = \Gamma_n$. For $0 < j \leq n-1$, suppose we have

$$\Gamma_n^{(j-1)} \overset{\pi_n}{\to} \Gamma_{n-1}^{(j-1)} \overset{\pi_{n-1}}{\to} \cdots \overset{\pi_{j+1}}{\to} \Gamma_j^{(j-1)} \to 1$$

with be a sequence of surjections with $\ker(\pi_i) \in \mathcal{C}$. Defining $\rho_k^{(j-1)} = \pi_{j+1} \circ \cdots \circ \pi_k$ and $\Gamma_k^{(j)} = \ker \rho_k^{(j-1)}$, by appropriately restricting π_k we now have

$$\Gamma_n^{(j)} \overset{\pi_n}{\to} \Gamma_{n-1}^{(j)} \overset{\pi_{n-1}}{\to} \cdots \overset{\pi_{j+1}}{\to} \Gamma_j^{(j)} = 1$$

is a chain satisfying the conditions implying $\Gamma_n^{(j)} \in \mathrm{Quot}_{n-j}(\mathcal{C})$. More generally we have for $0 \leq j \leq k \leq n$:

- $\Gamma_k^{(j)} \in \mathrm{Quot}_{k-j}(\mathcal{C})$,
- $\Gamma_k^{(j)} \triangleright \Gamma_k^{(j-1)}$,

- $\Gamma_k^{(j)}/\Gamma_k^{(j-1)} \in \mathcal{C}$,
- $\Gamma_n \rhd \Gamma_n^{(1)} \rhd \cdots \rhd \Gamma_n^{(n-1)} \rhd 1$ with $\Gamma_{n-1}^{(j)}/\Gamma_n^{(j+1)} \in \mathcal{C}$

Hence an equivalent characterization of $\Gamma \in \mathrm{Quot}_n(\mathcal{C})$ is Γ is *poly*-$\mathcal{C}$ with Hirsch length n. When $\mathcal{C}$ is the set of all abelian groups, $\mathrm{Quot}(\mathcal{C})$ is precisely collection of all solvable groups.

If $\mathcal{C}$ is a class of groups closed under commensurability (up to finite kernel), then we generalize the definition of $\mathrm{Quot}_n(\mathcal{C})$ there exists a chain as in (5.2.1)

$$\Gamma_n \to \Gamma_{n-1} \to \cdots \to \Gamma_1 \to 1 \tag{5.2.2}$$

with Γ commensurable (up to finite kernel) to Γ_n. In this situation, we must take care as commensurabilty may introduce unexpected variability. For instance, if we take the family of all non-amenable free groups $\mathcal{F}$, naturally $\mathbb{F}_4 \in \mathrm{Quot}_1(\mathcal{F})$. The cannonical surjection $\mathbb{F}_4 \to \mathbb{F}_2$ demonstrates the fact $\mathbb{F}_4 \in \mathrm{Quot}_2(\mathcal{F})$. In general, $\mathbb{F}_{2n} \in \mathrm{Quot}_n(\mathcal{F})$. As all non-amenable free groups are commensurable, $\mathbb{F}_2 \in \mathrm{Quot}_n(\mathcal{F})$ for every n. Thus we impose the following minimality condition in the definition of $\mathrm{Quot}_n(\mathcal{C})$:

Definition 5.2.2. Let $\mathcal{C}$ be a class of groups closed under commensurabilty (up to finite kernel). $\Gamma \in \mathrm{Quot}(\mathcal{C})$ if there exists a chain of surjections

$$\Gamma_n \stackrel{\pi_n}{\to} \Gamma_{n-1} \stackrel{\pi_{n-1}}{\to} \cdots \stackrel{\pi_2}{\to} \Gamma_1 \stackrel{\pi_1}{\to} 1$$

so that Γ is commensurable to Γ_n, $\ker(\pi_k) \in \mathcal{C}$. $\Gamma \in \mathrm{Quot}_n(\mathcal{C})$ if n is the smallest number such that Γ is a k-step extension by $\mathcal{C}$.

Lemma 5.2.3. *Let $p \in L(\Gamma)$ be a projection and $A, B \subset pL(\Gamma)p$ be two diffuse commuting subalgebras of $pL(\Gamma)p$ with $\Gamma \in \mathcal{C}_{rss}$. Then $[pL(\Gamma)p : A \vee B]_{PP} = \infty$. In particular if $\Sigma_1, \Sigma_2 < \Gamma$ are commuting subgroups such that $[\Gamma : \Sigma_1\Sigma_2] < \infty$, then either Σ_1 or Σ_2 is finite.*

We omit the proof, as it is the base step of the induction argument of the main theorem of [CKP15].

Proposition 5.2.4. $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$*. Suppose there exist infinite groups such that Γ is commensurable to $\Sigma_1 \times \Sigma_2$. Then we may find $n_1, n_2 > 0$ such that $n_1 + n_2 = n$ with $\Sigma_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{rss})$.*

Proof. As $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$, there exists a chain

$$\Gamma_n \overset{\pi_n}{\twoheadrightarrow} \Gamma_{n-1} \overset{\pi_{n-1}}{\twoheadrightarrow} \cdots \overset{\pi_2}{\twoheadrightarrow} \Gamma_1 \overset{\pi_1}{\twoheadrightarrow} 1 \tag{5.2.3}$$

with Γ commensurable to Γ_n. As $\mathrm{Quot}_j(\mathcal{C}_{\mathrm{rss}})$ is closed under commensurability for all j, after passing to a finite index subgroup, we may assume $\Sigma_1 \times \Sigma_2 = \Gamma_n$.

Let $\rho_n : \Gamma_n \to \Gamma_1$ be given by the concatenation of the homomorphisms given in (5.2.3). Note $\rho_n(\Sigma_1 \times \{e\}), \rho_n(\{e\} \times \Sigma_2)$ are commuting groups generating Γ_1. Lemma 5.2.3 implies either $\rho_n(\Sigma_1 \times \{e\})$ or $\rho_n(\{e\} \times \Sigma_2)$ is finite. Symmetry allows us to assume $\ker(\pi_n|_{\Sigma_1}) = \Sigma < \Sigma_1$ is a finite index normal subgroup of Σ_1 with $\Sigma < \ker(\rho_n) = \Gamma_n^{(1)}$. $\Sigma \times \Sigma_2$ is a finite index subgroup of Γ_n. If we restrict ρ_n to $\Sigma \times \Sigma_2$, a simple calculation yields $\Gamma_n^{(1)} = \Sigma \times \Lambda$ where $\Lambda = \ker(\rho_n|_{\Sigma_2})$. Thus we have

$$\frac{\Sigma_2}{\Lambda} \cong \frac{\Sigma \times \Sigma_2}{\Sigma \times \Lambda} < \frac{\Gamma_n}{\Gamma_n^{(1)}} \in \mathcal{C}_{\mathrm{rss}} \tag{5.2.4}$$

Since Γ is commensurable to $\Sigma \times \Sigma_2$, the inclusion in (5.2.4) is a finite index inclusion of groups. Thus $\Sigma_2/\Lambda \in \mathcal{C}_{rss}$.

When $n = 2$, we have $\Gamma_2^{(1)} \in \mathcal{C}_{rss}$ and $\Gamma_2^{(1)} = \Sigma \times \Lambda$ Lemma 5.2.3 implies Λ is finite. Thus Σ is commensurable to $\Gamma_n^{(1)}$ yielding $\Sigma \in \mathrm{Quot}_1(\mathcal{C}_{\mathrm{rss}})$. Since $\Lambda \triangleleft \Sigma_2$ and Λ is finite, passing to a finite index subgroup of both Λ and Σ_2, (5.2.4) shows $\Sigma_2 \in \mathcal{C}_{\mathrm{rss}}$.

Now if the result holds for all groups in $\mathrm{Quot}_{n-1}(\mathcal{C}_{\mathrm{rss}})$ up to some integer $n-1$, we have $\Gamma_n^{(1)} \in \mathrm{Quot}_{n-1}(\mathcal{C}_{\mathrm{rss}})$ with $\Gamma_n^{(1)} = \Sigma \times \Lambda$. If Λ is a finite group, then repeating the argument when $n = 2$ proves the result for n. If instead Λ is an infinite group, then by the induction hypothesis, we may find $n_1, n_2 > 0$ so that $n_1 + n_2 = n - 1$, $\Sigma \in \mathrm{Quot}_{n_1}(\mathcal{C}_{\mathrm{rss}})$, and $\Lambda \in \mathrm{Quot}_{n_2}(\mathcal{C}_{\mathrm{rss}})$. By (5.2.4), we then have $\Sigma_2 \in \mathrm{Quot}_{n_2+1}(\mathcal{C}_{rss})$. Furthermore, Σ_1 is commensurable to Σ which Quot_n properties listed earlier ensures $\Sigma_1 \in \mathrm{Quot}_{n_1}(\mathcal{C}_{\mathrm{rss}})$. $\square$

Corollary 5.2.5. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$ and suppose Γ is commensurable to $\Sigma_1 \times \Sigma_2$ with $\Sigma_1 \in \mathrm{Quot}_j(\mathcal{C})$ for some $j \in N$. Then either $n = j$ and Σ_2 is finite, or $j < n$ and $\Sigma_2 \in \mathrm{Quot}_{n-j}(\mathcal{C}_{rss})$.*

Proof. By the minimality constraint in Definition $\mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$, we naturally have $j \leq n$. If $j = n$ and Σ_2 were infinite, Proposition 5.2.4 yields $\Sigma_i \in \mathrm{Quot}_{n_i}(\mathcal{C})$ for some $n > n_i \geq 1$, once again contradicting minimality.

Now suppose $1 \leq j < n$. If $n = 2$, by Proposition 5.2.4, we have the result. We momentarily define $\mathrm{Quot}_0(\mathcal{C}_{\mathrm{rss}})$ as the collection of all finite groups. Proceeding as in the proof of the previous proposition, we have either

1. $\Gamma_n^{(1)}$ is commensurable to $\Sigma_1 \times \Lambda_2$

2. $\Gamma_n^{(1)}$ is commensurable to $\Lambda_1 \times \Sigma_2$,

where $\Lambda_i = \ker(\rho_n|_{\Sigma_i})$. In case (1), $\Sigma_1 \in \mathrm{Quot}_a(\mathcal{C}_{\mathrm{rss}})$ and $\Lambda_2 \in \mathrm{Quot}_b(\mathcal{C}_{\mathrm{rss}})$, for $a, b \geq 0$ with $a + b = n - 1$. Hence $\Sigma_2 \in \mathrm{Quot}_{b+1}(\mathcal{C}_{\mathrm{rss}})$ By minimality $a \geq j$. If $a > j$, this would imply $\Gamma \in \mathrm{Quot}_{j+b+1}(\mathcal{C}_{\mathrm{rss}})$ where $j + b + 1 < n$ once again contradicting minimality. In case (2), a similar argument will guarantee $\Lambda \in \mathrm{Quot}_{j-1}(\mathcal{C}_{\mathrm{rss}})$ and thus $\Sigma \in \mathrm{Quot}_{n-j}(\mathcal{C}_{\mathrm{rss}})$. □

Further analysis of the claims above yields the following stronger statement. We will omit the proof as it follows almost identically

Corollary 5.2.6. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$. Suppose we have a chain witnessing $\Gamma \in \mathrm{Quot}_n(\mathcal{C})_{rss}$*

$$\Gamma_n \overset{\pi_n}{\twoheadrightarrow} \Gamma_{n-1} \overset{\pi_{n-1}}{\twoheadrightarrow} \cdots \overset{\pi_2}{\twoheadrightarrow} \Gamma_1 \overset{\pi_1}{\twoheadrightarrow} 1$$

with $[\Gamma_n : \Sigma_1\Sigma_2] < \infty$ for some infinite commuting groups $\Sigma_1, \Sigma_2 < \Gamma_n$. Then there exists $n_1, n_2 > 0$ so that $n_1 + n_2 = n$ so that $\Sigma_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{rss})$. Furthermore, if a priori we have $\Sigma_1 \in \mathrm{Quot}_j$, then $\Sigma_2 \in \mathrm{Quot}_{n-j}(\mathcal{C}_{rss})$.

To establish results in C, we first set out the following notation. Proceeding as in [CIK13], group homomorphism $\rho : \Gamma \to \Lambda$ lifts ρ to a $*-$homormorphism of von Neumann algebras $\Delta : L(\Gamma) \to L(\Gamma)\bar{\otimes}L(\Lambda)$ by extending the map $u_\gamma \mapsto u_\gamma \otimes v_{\rho(\gamma)}$ where $\{u_\gamma\}_{\gamma\in\Gamma}, \{v_\lambda\}_{\lambda\in\Lambda}$ are the canonical unitaries of $L(\Gamma)$ and $L(\Lambda)$, respectively. When $\rho : \Gamma \to \Gamma$ is the identity, this is precisely the compultiplication map along Γ. For a group $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$, we will consider the group homomorphism $\rho_n : \Gamma_n \to \Gamma_1$ as defined in the previous section. If $A_1, \ldots, A_k \subset M$ are subalgebras of M, then

$A_1 \vee \cdots \vee A_k \subset M$ is defined to be the smallest von Neumann algebra in M containing $A_1 \cup \cdots \cup A_k$. For every $j \in \{1, \ldots, k\}$,, we denote $\hat{A}_j := A_1 \vee \cdots \vee A_{j-1} \vee A_{j+1} \vee \cdots \vee A_k$.

Lemma 5.2.7. *Let M be a type II_1 factor with $A_1, \ldots, A_k \subset M$ diffuse commuting II_1 factors such that $A_1 \vee \cdots A_k \subset M$ is a finite index inclusion of algebras. Then there exists a projection $z \in M$ so that $\mathcal{Z}(Q_i z) \cong$, where $\mathcal{N}_M(A_i)'' = Q_i$.*

Proof. Letting $A = A_1 \vee \cdots \vee A_k$, Theorem 3.1.9 part (d) implies $A' \cap M$ is finite dimensional. Notice $A \subset Q_i$ for every $i = 1, \ldots, k$ since A_i and $A_i' \cap M$ are both subalgebras of Q_i. Thus $\mathcal{Z}(Q_i)$ is finite dimensional since $\mathcal{Z}(Q_i) \subset Q_i' \cap M \subset A' \cap M$.

Now for any projection $z \in \mathcal{Z}(Q_i)$, we claim $Q_i z = \mathcal{N}_{zMz}(A_i z)''$[1]. It suffices to show $\mathcal{N}_M(A_i)z = \mathcal{N}_{zMz}(A_i z)$. This follows clearly form the following facts: given any unitary $u \in \mathcal{N}_M(A_i)$, $(uz)^* uz = z = uz(uz)^*$; if $v \in \mathcal{U}(zMz)$ is a normalizing unitary of $A_i z$, then $v + (1 - z) \in \mathcal{U}(M)$ is a normalizing unitary of A_i.

Since the algebras $\mathcal{Z}(Q_i)$ pairwise commute, we may take any minimal projections $z_i \in \mathcal{Z}(Q_i)$ so that $z_i z_j = z_j z_1 \neq 0$. Then

$$Az \subset zMz = N$$

is a finite index inclusion of algebras with $\mathcal{Z}(\mathcal{N}_N(A_i z)'') = \mathcal{Z}(Q_i z) = z$. □

Proposition 5.2.8. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$ with $p \in L(\Gamma)$ a non-zero projection. Suppose there exist $A, B \subset pL(\Gamma)p$ diffuse commuting subalgebras so that $[pL(\Gamma)p : A \vee B]_{PP} < \infty$. Then either*

---

[1] This holds in much greater generality: the projection z can be taken to be any projection in $Q_i' \cap M$

1. $A \preceq_{L(\Gamma)} L(\Gamma_n^{(1)})$*, or*

2. $B \preceq_{L(\Gamma)} L(\Gamma_n^{(1)})$.

Proof. First note $n \geq 2$ by Lemma 5.2.3. Let $A_0 \subset A$ be an amenable subalgebra. Letting $M = L(\Gamma)$, we see $\Delta(A_0)$ and $\Delta(B)$ are diffuse commuting subalgebras of $M\bar{\otimes}L(\Gamma_1)$ with $\Delta(A_0)$ amenable. Since $\Gamma_1 \in \mathrm{Quot}_1(\mathcal{C}_{\mathrm{rss}}) = \mathcal{C}_{\mathrm{rss}}$, we have either

1. $\Delta(A_0) \preceq M\bar{\otimes}1$, or
2. $\mathcal{N}_{M\bar{\otimes}L(\Gamma_1)}(\Delta(A_0))''$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$.

Assume (2) holds. Noting $\Delta(B) \subset \mathcal{N}_{M\bar{\otimes}L(\Gamma_1)}(\Delta(A_0))''$ yields $\Delta(B)$ is amenable relative to $M\bar{\otimes}1$ inside $M\bar{\otimes}L(\Gamma_1)$. Applying the dichotomy property of $\mathcal{C}_{\mathrm{rss}}$ either

3. $\Delta(B) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$, or
4. $\mathcal{N}_{M\bar{\otimes}L(\Gamma_1)}(\Delta(B))''$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$.

To summarize we have either

5. $\Delta(A_0) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$,
6. $\Delta(B) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$, or
7. $\Delta(A \vee B) \subset \mathcal{N}_{M\bar{\otimes}L(\Gamma_1)}(\Delta(B))''$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$.

We first show case (7) is impossible. Since $A \vee B$ is a finite index subalgebra of M, $M \preceq^s A \vee B$ and hence M is amenable relative to $A \vee B$. By [OP07, Proposition 2.4], $\Delta(M)$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$. However, [CIK13, Proposition 3.5] would imply $\rho(\Gamma) = \Gamma_1$ is amenable, a contradiction. Thus we only have case (5) and (6). Since A_0 was an arbitrary amenable subalgebra of A, by [BO08] we have either

8. $\Delta(A) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$, or

9. $\Delta(B) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$.

Noting $\ker(\rho_n) = L(\Gamma_n^{(1)})$, the result follows from application of [CIK13, Proposition 3.4]. $\square$

Lemma 5.2.9. *Let $\Gamma \in \text{Quot}_n(\mathcal{C}_{rss})$. If $A, B \subset pL(\Gamma)p$ are diffuse commuting subalgebras with A amenable, then $[pL(\Gamma)p : A \vee B]_{PP} = \infty$.*

Proof. We proceed by induction on n. When $n = 1$, this follows from Lemma 5.2.3. Now assume the statement holds for all groups in $\text{Quot}_k(\mathcal{C}_{\text{rss}})$ where $k \leq n-1$ and take $\Gamma \in \text{Quot}_n(\mathcal{C}_{rss})$ with $A, B \subset pL(\Gamma)p$ as stated. Then $\Delta(A)$ is amenable and therefore amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$. Since $\Gamma_1 \in \mathcal{C}_{\text{rss}}$, either

1. $\Delta(A) \preceq_{M\bar{\otimes}L(\Gamma_1)} M\bar{\otimes}1$, or
2. $\mathcal{N}_{M\otimes L(\Gamma_1)}(\Delta(A))''$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$.

We first show (1) is impossible. If (1) were to hold, [CIK13, Proposition 3.4] implies $A \preceq L(\Gamma_n^{(1)})$. By [CKP15, Proposition 2.4], there exists a $*$-isomorphism $\psi : p_1Ap_1 \to A_1 \subset qL(\Gamma_n^{(1)})q$ such that $A_1 \vee ({A_1}' \cap qL(\Gamma_n^{(1)})q) \subset qL(\Gamma_n^{(1)})q$ is a finite index inclusion of algebras. Since A is a diffuse amenable algebra, A_1 is also diffuse amenable. Since $\Gamma_n^{(1)}$ is non-amenable, $A_1' \cap qL(\Gamma_n^{(1)})q$ is non-amenable. Supposing $A_1' \cap qL(\Gamma_n^{(1)})q$ has an atomic corner, cutting by a minimal central projection z, $A_1z \subset qzL(\Gamma_n^{(1)})qz$ is a finite index inclusion of algebra. Since A_1z is amenable is an amenable corner of $L(\Gamma_n^{(1)})$, this is would imply $\Gamma_n^{(1)}$ is an amenable group, a contradiction. If instead $A_1' \cap qL(\Gamma_n^{(1)})q$ were diffuse, this would contradict the induction hypothesis.

Now if (2) holds, the assumption $[M : A \vee B]_{PP} < \infty$ implies $\Delta(A \vee B) \subset M\bar{\otimes}L(\Gamma_1)$ is also a finite index inclusion of algebras. Since $\Delta(A \vee B) \subset \mathcal{N}_{M\bar{\otimes}L(\Gamma_1)}(\Delta(A))''$,

$\Delta(A \vee B)$ is amenable relative to $M\bar{\otimes}1$ in $M\bar{\otimes}L(\Gamma_1)$. By [OP07, Proposition 2.4], $M\bar{\otimes}L(\Gamma_1)$ is amenable relative to $M\bar{\otimes}1$. However, this is impossible as [CIK13, Proposition 3.5] would imply $\Gamma_1 \in \mathcal{C}_{\mathrm{r}ss}$ is amenable. $\square$

To establish the main result, we show that the maximal number of commuting diffuse subalgebras is controlled by the Hirsh length. Note that we have an upper bound rather than equality. Central quotients of braid groups are poly-free groups which give rise to prime von Neumann alrebras [CKP15, Theorem A].

Lemma 5.2.10. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$ and suppose $A_1, \ldots, A_k \subset qL(\Gamma)q$ are diffuse commuting II_1 factors. If $A_1 \vee \cdots \vee A_k \subset qL(\Gamma)q$ generate a finite index subalgebra, then $k \leq n$.*

Proof. When $n = 1$, Lemma 5.2.3 proves the assertion. Now suppose the conclusion holds for all groups in $\mathrm{Quot}_m(\mathcal{C}_{\mathrm{r}ss})$ up to $m = n - 1$. Now let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$ and assume to the contrary there are $k > n$ diffuse subalgebras $A_1, \ldots, A_k \subset qL(\Gamma)q$ generating a finite index subalgebra of $qL(\Gamma)q$. Without loss of generality, we may assume $k = n + 1$. Then by Proposition 5.2.8, for every $j \in \{1, \ldots, k\}$, either $\hat{A}_j \preceq_{L(\Gamma)} L(\Gamma_n^{(1)})$ or $A_j \preceq_{L(\Gamma)} L(\Gamma_n^{(1)})$.

Now if $\hat{A}_j \preceq L(\Gamma_n^{(1)})$, by [CKP15, Proposition 2.4] there exists a $*$-isomorphism $\psi : p\hat{A}_j p \to A \subset rL(\Gamma_n^{(1)})r$ so that $A \vee A' \cap rL(\Gamma_n^{(1)})r \subset rL(\Gamma_n^{(1)})r$ is a finite index inclusion of algebras. We may assume $p = p_1 \cdots p_k$, $p_i \in A_i$ for $i \neq j$. Hence $\psi(p\hat{A}_j p) = \psi(\bigvee_{i \neq j} p_i A_i p) = \bigvee_{i \neq j} \psi(p_i A_i p)$. Thus

$$\bigvee_{i \neq j} \psi(p_i A_i p) \vee (\psi(p\hat{A}_j p)' \cap rL(\Gamma_n^{(1)})r) \subset rL(\Gamma_n^{(1)})r$$

is a finite index inclusion of algebras. By Lemma 5.2.9, the center $\mathcal{Z}(\bigvee_{i\neq j}\psi(p_iA_ip)\vee(\psi(p\hat{A}_jp)'\cap rL(\Gamma_n^{(1)})r))$ cannot be diffuse. Thus, cutting by a minimal central projection we may assume

$$\bigvee_{i\neq j}\psi(p_iA_ip)\vee(\psi(p\hat{A}_jp)'\cap rL(\Gamma_n^{(1)})r)\subset rL(\Gamma_n^{(1)})r$$

is a finite index inclusion of factors. However, this would contradict the induction hypothesis as it would allow for at least n commuting diffuse non-amenable subalgebras of $rL(\Gamma_n^{(1)})r$.

If instead $A_j \preceq_{L(\Gamma)} L(\Gamma_n^{(1)})$ for all j, [Va10, Lemma 2.5, Proposition 2.6], in conjunction with the factoriality of each A_j, imply $A_j \preceq^s_{L(\Gamma)} L(\Gamma_n^{(1)})$. Proposition 3.1.7 would then give $L(\Gamma)\preceq_{L(\Gamma)}$, which implies $\Gamma_n^{(1)}$ is finite index in Γ once again leading to a contradiction. □

The following proposition is the key ingredient in decomposing a group as a product: if we may find an subgroup of $\Sigma<\Gamma\in \mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$ then we may also find another subgroup commuting with Σ so that Γ is commensurable to the direct product $\Sigma\times\Omega$. The proof of this proposition closely follows the proof of [CdSS15, Theorem 4.3].

Proposition 5.2.11. *Let $\Gamma\in\mathrm{Quot}_n(\mathcal{C}_{rss})$ be an icc group and denote by $L(\Gamma)=M$. Suppose we have subgroup $\Sigma<\Gamma$, and a projection $p\in L(\Sigma)'\cap M$ so that $\Sigma\in\mathrm{Quot}_j(\mathcal{C}_{rss})$ and*

$$p[L(\Sigma)\vee(L(\Sigma)'\cap M)]p\subset pMp$$

is a finite index inclusion of II_1 factors. Then we may find commuting subgroups $\Sigma_1, \Sigma_2 < \Gamma$ such that $[\Sigma : \Sigma_1] < \infty$ and $[\Gamma : \Sigma_1 \times \Sigma_2] < \infty$. Furthermore, if $\Sigma \in \mathrm{Quot}_j(\mathcal{C}_{rss})$, then $\Sigma_2 \in \mathrm{Quot}_{n-j}(\mathcal{C}_{rss})$.

Proof. Letting $\Sigma_2 = \{\gamma \in \Gamma : |\gamma|^{\Sigma} < \infty\}$ and proceeding as in [CdSS15, claim 4.7], we see $[\Gamma : \Sigma_2\Sigma] < \infty$. Now, the first half of [CdSS15, Claim 4.8] demonstrates $\Sigma \cap \Sigma_2$ is amenable since it can be written as an increasing tower of amenable groups. Let Γ_1 act trivially on , $\Gamma \cong \Gamma_n \to \cdots \to \Gamma_1$ is a chain witnessing $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$. Since $L(\Sigma \cap \Sigma_2)$ is amenable and Σ normalizes $\Sigma_2 \cap \Sigma$, the dichotomy of $\mathcal{C}_{\mathrm{rss}}$ will imply $L(\Sigma_2 \cap \Sigma) \preceq 1$. Thus, [CdSS15, Proposition 2.6] implies $\Sigma \cap \Sigma_2$ is finite.

Claims 4.9–4.12 in the proof of [CdSS15, Theorem 4.3] provides the existence of a subgroup $\Sigma_1 \leqslant \Sigma$ satisfying $[\Sigma : \Sigma_1] < \infty$, $[\Gamma : \Sigma_1\Sigma_2] < \infty$, and $[\Sigma_2, \Sigma_1] = \{e\}$. Since $\Sigma \cap \Sigma_2 \geqslant \Sigma_1 \cap \Sigma_2$, it follows $\Sigma_1 \cap \Sigma_2$ is finite as well. Since Γ is icc and $[\Gamma : \Sigma_1\Sigma_2] < \infty$ then $\Sigma_1 \cap \Sigma_2 = \{e\}$. Thus $[\Gamma : \Sigma_1 \times \Sigma_2] < \infty$.

Now if we also had assumed $\Sigma \in \mathrm{Quot}_j(\mathcal{C}_{\mathrm{rss}})$, Corollary 5.2.6 yields $\Sigma_2 \in \mathrm{Quot}_{n-j}(\mathcal{C}_{\mathrm{rss}})$. □

Theorem 5.2.12. *Let $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$ be an icc group and $q \in L(\Gamma)$ a projection. Suppose $A_1, \ldots, A_k \subset qL(\Gamma)q = M$ are diffuse commuting II_1 factors such that $[M : A_1 \vee \cdots \vee A_k] < \infty$. Then there exist icc groups $\Sigma_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{rss})$, non-zero projections $p_i \in A_i$, finite index subfactors $D_i \subset p_iA_ip_i$, and a unitary $u \in M$ such that*

- *Γ is commensurable to $\Sigma_1 \times \cdots \times \Sigma_k$,*
- *$\sum_{i=1}^k n_i = n$,*
- *$D_i \subset p_iu^*L(\Sigma_i)up_i$ is a finite index inclusion of II_1 factors.*

Proof. As our theorem is taken up to commensurability, we will treat the case when $\Gamma = \Gamma_n$ where

$$\Gamma_n \to \Gamma_{n-1} \to \cdots \to \Gamma_1 \to 1$$

witnesses $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{\mathrm{rss}})$. Furthermore, Lemma 5.2.7 implies we may assume $\mathcal{N}_{pMp}(A_i)''$ is a factor. By Proposition 5.2.8, for every i we have either $\hat{A}_i \preceq_M L(\Gamma_n^{(1)})$ or $A_i \preceq_M L(\Gamma_n^{(1)})$. If we assume $A_i \preceq_M L(\Gamma_n^{(1)})$, then $A_i \preceq_M^s L(\Gamma_n^{(1)})$ and hence $A_1 \vee \cdots \vee A_k \preceq L(\Gamma_n^{(1)})$. Since $A_1 \vee \cdots \vee A_k$ is a finite index subalgebra of $L(\Gamma)$, then $[\Gamma : \Gamma_n^{(1)}] < \infty$ contradicting that $\Gamma/\Gamma_n^{(1)} \in \mathcal{C}_{\mathrm{rss}}$. Thus, there exists i such that $\hat{A}_i \preceq_M L(\Gamma_n^{(1)})$ but $A_i \not\preceq_M L(\Gamma_n^{(1)})$. For simplicity, we consider the case $i = k$. [CKP15, Proposition 2.4] give the existence of projections $p \in \hat{A}_k, q_1 \in L(\Gamma_n^{(1)})$, a partial isometry $v \in M$, and a $*$-isomorphism $\psi : p\hat{A}_k p \to B \subset q_1 L(\Gamma_n^{(1)})q_1$ such that

(a) $B \vee (B' \cap q_1 L(\Gamma_n^{(1)})q_1) \subset q_1 L(\Gamma_n^{(1)})q_1$ is a finite index inclusion of algebras,

(b) $\psi(x)v = vx$ for all $x \in p\hat{A}_k p$.

As in the proof of Lemma 5.2.10, we may assume $p = p_1 \cdots p_{k-1}$ where $p_i \in A_i$ are projections such that $B = \psi(p\hat{A}_k p) = \psi(p_1 A_i p) \vee \cdots \vee \psi(p_{k-1} A_{k-1} p) = B_1 \vee \cdots \vee B_{k-1}$. Thus we have

(c) $\psi(p_i A_i p) = B_i$,

(d) $\psi(x)v = vx$ for all $x \in p\hat{A}_k p$,

(e) $B_1 \vee \cdots \vee B_{k-1} \vee (B' \cap q_1 L(\Gamma_n^{(1)})q_1) \subset q_1 L(\Gamma_n^{(1)})q_1$ is a finite index inclusion of algebras.

We first assume $n = 2$. In this case, Lemma 5.2.10 implies $k = 2$. Since

$\Gamma_n^{(1)} \in \mathrm{Quot}_1(\mathcal{C}_{\mathrm{rss}})$, Lemma 5.2.3 implies $\mathcal{Z}(B' \cap q_1 L(\Gamma_n^{(1)}) q_1)$ cannot have any diffuse part and therefore is completely atomic. Thus multiplying v by some minimal central projection $q' \in B' \cap q_1 L(\Gamma_n^{(1)}) q_1$ so that $vq' \neq 0$, we may assume $\psi(pA_1p) = B \subset q_1 L(\Gamma_n^{(1)}) q_1$ is a finite index inclusion of factors. Moreover, $\dim_(\mathcal{Z}(qL(\Gamma_n^{(1)}))q) \leq [qL(\Gamma_n^{(1)})q : B]_{PP} < \infty$ since B is a II_1 factor. Thus, after multiplying again by a minimal central projection, we may assume $B \subset q_1 L(\Gamma_n^{(1)}) q_1$ is a finite index inclusion of II_1 factors. We claim there exists a projection $r \in L(\Gamma_n^{(1)})' \cap M$ such that.

$$r[L((\Gamma_n^{(1)}) \vee L(\Gamma_n^{(1)})' \cap M)]r \subset rMr \tag{5.2.5}$$

is a finite index inclusion of II_1 factors.

To this end, the downward basic construction [Jo81, Lemma 3.1.8] gives a projection $e \in q_1 L(\Gamma_n^1) q_1$ and a subfactor $C \subset B \subset q_1 L(\Gamma_n^{(1)}) q_1 = \langle B, e \rangle$ such that $[B : C] = [q_1 L(\Gamma_n^{(1)}) q_1 : B]$, $Ce = eL(\Gamma_n^{(1)})e$ and $Ce \cong C$. Then the restriction $\psi^{-1} : C \to D \subset pA_1p$ is a $*$-isomorphism such that $[pA_1p : D] < \infty$ with $\psi^{-1}(y)v^* = v^*y$ for all $y \in C$. Let $\theta : Ce \to C$ be the $*$-isomorphism given by $xe \mapsto x$ and denote by $v' = ev$. If we suppose $v' = 0$, we would have $vv^*xe = xvv^*e = 0$ for all $x \in B$. As $\langle B, e \rangle e = Be$, $vv^*t = 0$ for all $t \in \langle B, e \rangle$. However, since q is the central support of e in $\langle B, e \rangle$, this would yield $vv^* = 0$. Thus it follows that $\varphi = \psi^{-1} \circ \theta : eL(\Gamma_n^{(1)})e \to D$ is a $*$-isomorphism satisfying

$$\varphi(y)w^* = w^*x \text{ for all } y \in eL(\Gamma_n^{(1)})e \tag{5.2.6}$$

where w^* is the partial isometry from the polar decomposition of $v^*e = |v^*e|w^*$. Note that $s = w^*w \in D' \cap pMp$ and $ww^* \leq e$. Thus equation (5.2.6), we obtain

$$w^*L(\Gamma_n^{(1)})w = \varphi(eL(\Gamma_n^{(1)})e)w^*w = Ds \tag{5.2.7}$$

$$(w^*L(\Gamma_n^{(1)})w)' \cap sMs = (Ds)' \cap sMs. \tag{5.2.8}$$

First note $A_2p \subset D' \cap pMp$. Since $D \subset pA_1p$ is a finite index inclusion, so are the inclusions $D \vee A_2p \subset p(A_1 \vee A_2)p \subset pMp$ and hence $D \vee A_2p \subset pMp$ is a finite index inclusion of algebras. By the local index formula, we also have $Ds \vee s(D' \cap M)s \subset sMs$ is also a finite index inclusion of II_1 factors.

Let $r = ww^*$ and $u \in M$ a unitary with $w^* = ur$. Conjugating (5.2.7) and (5.2.8) by u implies $r[L(\Gamma_n^{(1)}) \vee L(\Gamma_n^{(1)})' \cap M]r = L(\Gamma_n^{(1)})r \vee (L(\Gamma_n^{(1)})' \cap rMr) \subset rMr$ is a finite index inclusion of II_1 factors (after shrinking r is necessary). By Proposition 5.2.11, there exists a finite index subgroup $\Sigma_1 < \Gamma_n^{(1)}$ such that $\Sigma_i \in \mathrm{Quot}_1(\mathcal{C}_{\mathrm{rss}})$ with $[\Gamma : \Sigma_1 \times \Sigma_2] < \infty$ and $rA_2r \subset ru^*L(\Sigma_2)ur$, where $\Sigma_2 = V_\Gamma(\Gamma_n^{(1)})$. Since $ru^*L(\Gamma_n^{(1)})ur \subset rA_1r$ is a finite index inclusion of II_1 factors, so is the inclusion $ru^*L(\Sigma_1)r \subset rA_1r$. Performing the downward basic construction gives a subfactor $A_1f \subset ru^*L(\Sigma_1)r$ where $f \in A_1' \cap rMr$.

Since $rA_2r \subset r(A_1' \cap M)r$ is a finite Pimnser-Popa index inclusion of algebras, so is the inclusion $rA_2r \subset ru^*L(\Omega)ur$. Thus cutting once again by a minimal projection we have $r_2A_2r_r \subset r_2u^*L(\Sigma_2)ur_2$ is a finite index inclusion of II_1 factors.

Now suppose the result holds for all icc groups $\Lambda \in \mathrm{Quot}_{n-1}(\mathcal{C}_{\mathrm{rss}})$ for some $n \in$. Take $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$ an icc group. Proceeding as in the case when $n = 2$, we may assume $\hat{A}_k \preceq_M L(\Gamma_n^{(1)})$. More precisely, since the center of $A_1 \vee \cdots \vee A_k$ is trivial, by [Va10, Lamma 2.5, Proposition 2.6] $\hat{A}_k \preceq_M^s L(\Gamma_n^{(1)})$. [CKP15, Proposition 2.4] give the existence of projections $p \in A_1, q \in L(\Gamma_n^{(1)})$, a partial isometry $v \in M$ and a $*$-isomorphism $\psi : pA_1p \to B \subset q_1L(\Gamma_n^{(1)})q_1$ such that

(f) $B \vee (B' \cap q_1L(\Gamma_n^{(1)})q_1) \subset q_1L(\Gamma_n^{(1)})q_1$ is a finite index inclusion of algebras,

(g) $\psi(x)v = vx$ for all $x \in pA_1p$.

If $B_k = B' \cap q_1L(\Gamma_n^{(1)})q_1$ was not diffuse, we cutting by a minimal central projection to obtain $B \subset q_1L(\Gamma_n^{(1)})q_1$ is finite Pimnser-Popa index inclusion of algebras. As before, $\mathcal{Z}(qL(\Gamma_n^{(1)}))$ is finite dimensional. Thus, we cut by an appropriate minimal central projection to obtain a finite index inclusion of II_1 factors and proceed exactly as in the case when $n = 2$.

Now suppose B and $B_k = B' \cap q_1L(\Gamma_n^{(1)})q_1$ are both diffuse. Then, by cutting by a minimal central projection if necessary, we have $B \vee B_k \subset q_1L(\Gamma_n^{(1)})q_1$ is a finite index inclusion of II_1 factors. By the induction hypothesis, there exists a unitary $w \in q_1L(\Gamma_n^{(1)})q_1$, $\Lambda 1, \cdots, \Lambda_k$ subgroups of $\Gamma_n^{(1)}$ and projections $p_i \in B_i$, $q_i \in L(\Lambda_i)$ so that

- $w(p_1B_ip_1w^* \subset q_1L(\Lambda_i)q_1$ is a finite index inclusion of II_1 factors
- $\Gamma_n^{(1)}$ is commensurable to $\Lambda_1 \times \cdots \times \Lambda_k$
- $\Lambda \in \mathrm{Quot}_{m_1}(\mathcal{C}_{\mathrm{rss}})$ with $1 \leq m_1 < n - 1$.
- $\sum m_i = n - 1$.

Letting $\Lambda = \Lambda_1 \times \cdots \times \Lambda_k$ and proceeding as in the case when $n = 2$, there exists a projection s such that

$$s[L(\Lambda) \vee L(\Lambda)' \cap M]s \subset sMs$$

is a finite index inclusion of II_1 factors. Once we apply Lemma 5.2.11 and follow the same procedure as in the case when $n = 2$, we may find a finite index subgroup $\Lambda_1 < \Lambda$ so that $r\hat{A}_j r \subset ru^* L(\Lambda_1) ur$ and $rA_k r \subset L(\Lambda_2)$ are finite index inclusions of II_1 factors with $[\Gamma : \Lambda_1 \times \Lambda_2] < \infty$. Furthermore, we may assume

$$r\hat{A}_j r = rA_1 r \vee \cdots \vee rA_{k-1} r.$$

Letting $\Gamma_k = \Sigma_2$ and once again applying the induction hypothesis, we may appropriately identify corners of A_i with groups Γ_i so that $\Gamma_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{\mathrm{rss}})$ with $n_1 + \cdots + n_k = n$.

$\square$

Note the analysis made of the theorems in Section 5 hold if instead we assume $A_1 \vee A_k \subset pL\Gamma p$ has finite dimensional center. By carrying out a similar procedure we have the following generalization of the main theorem:

Theorem 5.2.13. *Suppose $A_1 \vee \cdots \vee A_k \subset L(\Gamma)$ where $\Gamma \in \mathrm{Quot}_n(\mathcal{C}_{rss})$. Then there exist commuting groups $\Sigma_i \in \mathrm{Quot}_{n_i}(\mathcal{C}_{rss})$ so that Γ is commensurable to the product $\Sigma_1 \cdots \Sigma_k$.*

CHAPTER 6
APPLICATIONS

6.1 Applications to Group Theory

The presence of a non-trivial quasi-cocycle on a group taking values in its left regular representation restricts significantly the internal structure of the group; for instance, it excludes most relations of "order one" like (asymptotic) commutation, etc. In the same spirit, we will show that any such group has at most finitely many finite conjugacy classes. More precisely, appealing to the representation theory techniques which steam mainly from [CSU13] we show the following more general statement.

Theorem 6.1.1. *Let Γ be a non-amenable group together with $\Sigma \lhd \Gamma$, a normal non-amenable subgroup. If $\Gamma \in NC$ then there are only finitely many finite orbits for the action of Σ on Γ by conjugation.*

Proof. Since $\Gamma \in NC$ there exists a weakly-ℓ^2, mixing orthogonal representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ and $q \in \mathcal{QH}^1_{as}(\Gamma, \pi)$.

Let $\{\mathcal{O}_n : n \in \mathbb{N}\}$ be an enumeration of all the (disjoint) finite orbits for the action of Σ on Γ by conjugation. Notice that $\cup_{n\in\mathbb{N}}\mathcal{O}_n = \{\gamma \in \Gamma : [\Sigma : C_\Sigma(\gamma)] < \infty\} =: \Lambda$, where $C_\Gamma(\gamma)$ is the centralizer of γ in Γ. Since Σ is normal in Γ, it is a straightforward exercise to show that Λ is a normal subgroup of Γ. For every $n \in \mathbb{N}$ denote by $\Lambda_n := \cup_{i=1}^n \mathcal{O}_i$.

The proof relies heavily on the techniques used in [CSU13, Theorems 3.1 and 3.5] so we will only include a brief sketch on how to fit together these results. Denote by $M = L(\Gamma)$ the corresponding group von Neumann algebra and let u_γ, with $\gamma \in \Gamma$ be the canonical group unitaries. For every $n \in \mathbb{N}$ denote by $\xi_n = |\Lambda_n|^{-1/2} \sum_{a\in\Lambda_n} u_a \in M \subset L^2(M)$. Then a basic calculation shows that for every $\gamma \in \Sigma$ and $n \in \mathbb{N}$ we have

$$u_\gamma \xi_n = \xi_n u_\gamma.$$

Denote by $\tilde{M} = L^\infty(Y^\pi) \rtimes \Gamma$ the Gaussian dillation associated with π. Let $V_t : L^2(M) \to L^2(\tilde{M})$, with $t \in \mathbb{R}$, be the Gaussian deformation corresponding to q as defined in Section 4.1. Since Σ is non-amenable and π is weakly-ℓ^2 one can find a finite subset $E \subset \Sigma$ and $K \geq 0$ such that for every $\xi \in L^2(\tilde{M}) \ominus L^2(M)$ we have that

$$\sum_{\gamma\in E} \|u_\gamma \xi - \xi u_\gamma\|_2 \geq K\|\xi\|_2.$$

Then Proposition 4.1.2 above combined with the same spectral gap argument from the beginning of the proof of theorem [CSU13, Theorem 3.1] show that

$$\lim_{t\to 0}\left(\sup_n \|e_M^\perp \cdot V_t(\xi_n)\|_2\right) = 0.$$

Thus the transversality property (Proposition 4.1.1) will further imply that

$$\lim_{t\to 0}\left(\sup_n \|\xi_n - V_t(\xi_n)\|_2\right) = 0.$$

Then a simple calculation shows that for every $\varepsilon > 0$ there exists $C \geq 0$ such that

$$\sup_n \|\xi_n - P_{B'_C}(\xi_n)\|_2 \leq \varepsilon. \tag{6.1.1}$$

As before, we have denoted by $P_{B'_C}$ the orthogonal projection from $\ell^2(\Gamma)$ onto the Hilbert subspace $\ell^2(B'_C)$ with $B'_C = \{\lambda \in \Gamma : \|q(\lambda)\| \leq C, \lambda \neq e\}$ being the ball of radius C centered and pierced at the identity element e.

Then the same argument as in the proof of [CSU13, Theorem 3.5, pages 15-16] shows that for every $\varepsilon > 0$ there exists $C \geq 0$ such that for every $\gamma \in \Sigma$ we have

$$\limsup_n \|P_{A_\gamma}(\xi_n)\|_2^2 \geq 1 - 6\varepsilon^2, \tag{6.1.2}$$

where $A_\gamma = \gamma B'_C \gamma^{-1} \cap B'_C$. Since Σ is normal in Γ then by [CSU13, Theorem 2.1] the quasi-cocycle is unbounded on Σ. Thus, since γ does not depend on ε or C, one can pick $\gamma \in \Sigma \setminus B_{2C+2D(q)}$ and by [CSU13, Theorem 2.1] again it follows A_γ is finite. Moreover, using the definition of ξ_n we see that $\|P_{A_\gamma}(\xi_n)\|_2^2 = |A_\gamma \cap \Lambda_n||\Lambda_n|^{-1}$, for every n. This together with (6.1.2) imply that there exists an integer n_0 such that $\Lambda_k = \Lambda_{n_0}$, for every $k \geq n_0$; hence $\Lambda = \Lambda_{n_0}$ is finite. $\square$

If we let $\Sigma = \Gamma$ in the previous theorem we notice the following immediate corollary.

Corollary 6.1.2. *Let $\Gamma \in NC$. Then Γ has only finitely many finite conjugacy classes. Hence there exists a short exact sequence of groups $1 \to F \to \Gamma \to \Gamma_0 \to 1$, where F is a finite and Γ_o is infinite conjugacy class. In particular, if Γ is assumed torsion free then Γ is infinite conjugacy class.*

The following corollary is a straightforward consequence of the previous results.

Corollary 6.1.3. *Let $\Gamma \in NC$ be a non-amenable group together with $\Sigma \lhd \Gamma$ a non-amenable normal subgroup. Assume that $\Gamma \curvearrowright N$ is a trace preserving action on a finite von Neumann algebra N. If $N \rtimes \Gamma$ denotes the corresponding crossed product von Neumann algebra then there exists $\Lambda \lhd \Gamma$ a finite normal subgroup such that $\mathscr{Z}(N \rtimes \Sigma) \subseteq (N \rtimes \Sigma)' \cap (N \rtimes \Gamma) \subseteq N \rtimes \Lambda$.*

REFERENCES

[BHV05] B. Bekka P. de la Harpe, and A. Valette, *Kazhdan property (T)*, New Mathematical Monographs, vol. 11, CUP, Cambridge 2008. xiv+472 pages.

[BV13] M. Berbec and S. Vaes, *W^*-superrigidity for group von Neumann algebras of left-right wreath products*, Proc. Lond. Math. Soc. **108** (2014), 1116–1152.

[BBF13] M. Bestvina, K. Bromberg, and K. Fujiwara, *Bounded cohomology via quasi-trees*, Preprint June 2013, ArXiv:1306.1542(v1).

[BBM10] M. Bestvina, K.-U. Bux, and D. Margalit, *The dimension of the Torelli group*, J. Amer. Math. Soc. **23** (2010), 61–105.

[B69a] J. S. Birman, *On braid groups*, Comm. Pure App. Math. **22** (1969), 41–72.

[B69b] J. S. Birman, *Mapping class groups and their relationship with braid groups*, Comm. Pure App. Math. **22** (1969), 213–238.

[B74] J. S. Birman, *Braids, links, and mapping class groups*, Ann. of Math. Stud., 82, Princeton Univ. Press, Princeton, N.J., 1974.

[BH73] J. S. Birman and H. M. Hilden, *On Isotopies of Homeomorphisms of Riemann Surfaces*, Ann. of Math. (2) **97** (1973), 424–439.

[Bo12] R, Boutonnet, *W^*-superrigidity of mixing Gaussian actions of rigid groups*, Adv. Math., **244** (2013), 69-9-0.

[BC14] R, Boutonnet and A. Carderi, *Maximal amenable von Neumann subalgebras arising from maximal amenable subgroups*, Geom. Funct. Anal., **25** (2015),1688-1705.

[BHR12] R. Boutonnet, C. Houdayer, and S. Raum, *Amalgamated free product type III factors with at most one Cartan subalgebra*. Compos. Math. **150** (2014), 143–174.

[BO08] N. P. Brown and N. Ozawa, C^**-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, AMS, Providence, RI.

[CI08] I. Chifan and A. Ioana, *Ergodic subequivalence relations induced by a Bernoulli action*, Geom. Funct. Anal. **20** (2010), 53–67.

[CH08] I. Chifan and C. Houdayer, *Bass-Serre rigidity results in von Neumann algebras*, Duke Math. J. **153** (2010), 23–54.

[CK14] I. Chifan and Y. Kida, *OE and W^* superrigidity results for actions by surface braid groups*, Proc. Lond. Math. Soc. **111** (2015), 1431-1470.

[CS11] I. Chifan and T. Sinclair, *On the structural theory of II_1 factors of negatively curved groups*, Ann. Sci. École Norm. Sup. **46** (2013), no. 1, 1–34.

[CP10] I. Chifan and J. Peterson, *Some unique group measure space decomposition results*, Duke Math. J. **162** (2013), 1923–1966.

[CIK13] I. Chifan, A. Ioana, and Y. Kida, *W^*-superrigidity for arbitrary actions of central quotients of braid groups*, Math. Ann. **361** (2015), 563–582.

[CKP15] I. Chifan, Y. Kida, S. Pant, *Primeness results for von Neumann algebras associated with surface braid groups,* Int. Math. Res. Not. Vol. 2016 (2016), no. 16, 4807–4848.

[CdSS15] I. Chifan, R. de Santiago, T. Sinclair, *W^* rigidity for the von Neumann algebras of hyperbolic groups* Geom. and Func. Analysis **26** (2016),139–159.

[CSU13] I. Chifan, T. Sinclair, and B. Udrea, *Inner amenability for groups and central sequences in factors*, Ergodic Theory Dynam. Systems, **36** (2016), 1106–1129.

[CSU11] I. Chifan, T. Sinclair, and B. Udrea, *On the structural theory of II_1 factors of negatively curved groups, II: Actions by product groups*, Adv. Math. **245** (2013), 208–236.

[Co75] A. Connes, *Classification of injective factors*, Ann. of Math. (2) **104** (1976), 73–115.

[dSP17] R. de Santiago and S. Pant, *Product rigidity for poly-hyperbolic groups*, in preparation.

[DHI16] D. Drimbe, D. Hoff, and A. Ioana, *Prime II_1 factors arising from irreducible lattices in products of rank one simple Lie groups*, preprint, arXiv:1611.02209.

[DY08] F. Dahmani and A. Yaman, *Symbolic dynamics and relatively hyperbolic groups*, Groups Geom. Dyn. **2** (2008), no.2, 165–184.

[DGO11] F. Dahmani, V. Guirardel, D. Osin, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, Memoirs of the A.M.S., **245** (2017).

[DL14] S. Deprez and K. Li, *Permanence properties of property A and coarse embeddability for locally compact groups*,(2014).

[DD89] W. Dicks and M. J. Dunwoody, *Groups acting on graphs*, Cambridge Stud. Adv. Math. **17**, Cambridge University Press, Cambridge, 1989.

[FV62] E. Fadell and J. Van Boskirk, *The braid groups of E^2 and S^2*, Duke Math. J. **29** (1962), 243–258.

[FaNe62] E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand. **10** (1962), 111–118.

[FM11] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Math. Ser., October 2011.

[Fi11] P. Fima, *A note on the von Neumann algebra of a Baumslag-Solitar group*, C. R. Acad. Sci. Paris, Ser. I **349** (2011), 25–27.

[FV10] P. Fima and S. Vaes, *HNN extensions and unique group measure space decomposition of II_1 factors*, Trans. Amer. Math. Soc. **364** (2012), 2601–2617.

[FoNe62] R. H. Fox and L. Neuwirth, *The braid groups*, Math. Scand. **10** (1962), 119–126.

[GaJu07] M. Gao and M. Junge, *Examples of prime von Neumann algebras*, Int. Math. Res. Not., Vol. 2007, Article ID rnm 042, 34 pages.

[G98] L. Ge, *Applications of free entropy to finite von Neumann algebras. II*, Ann. of Math. (2) **147** (1998), no. 1, 143–157.

[GoGu04] D.L. Gonçalves and J. Guaschi, *The roots of the full twist for surface braid groups*, Math. Proc. Camb. Phil. Soc. **137** (2004), 307–320.

[Ha] U. Hamenstädt, *Bounded cohomology and isometry groups of hyperbolic spaces*, J. Eur. Math. Soc. **10** (2008), 315–349.

[Ho15] D. Hoff, *Von Neumann Algebra of Equivalence Relation with Nontrivial One-Cohomology*, J. Funct. Anal., **270** (2016), no. 4, 1501-1536.

[HPV10] C. Houdayer, S. Popa, and S. Vaes, *A class of groups for which every action is W^*-superrigid*, Groups Geom. Dyn. **7** (2013), 577–590.

[HV12] C. Houdayer and S. Vaes, *Type III factors with unique Cartan decomposition*, J. Math. Pures Appl. **100** (2013), 564–590.

[HO11] M. Hull, D. Osin, *Induced quasi-cocycles on groups with hyperbolically embedded subgroups*, Alg. Geom. Topol. **13** (2013), no. 5, 2635–2665.

[Io10] A. Ioana, *W^*-superrigidity for Bernoulli actions of property (T) groups*, J. Amer. Math. Soc. **24** (2011), 1175–1226.

[Io11] A. Ioana, *Uniqueness of the group measure space decomposition for Popa's $\mathcal{HT}$ factors*, Geom. Funct. Anal. **22** (2012), 699–732.

[Io12] A. Ioana, *Cartan subalgebras of amalgamated free product II_1 factors*, Ann. Sci. Éc. Norm. Sup., **48** (2015), 71–130.

[IPP05] A. Ioana, J. Peterson, and S. Popa, *Amalgamated free products of weakly rigid factors and calculation of their symmetry groups*, Acta Math. **200** (2008), 85–153.

[IPV10] A. Ioana, S. Popa, and S. Vaes, *A class of superrigid group von Neumann algebras*, Ann. of Math. **178** (2013), 231–286..

[Is12] Y. Isono, *Examples of factors which have no Cartan subalgebras*, Trans. Amer. Math. Soc., **367** (2015), 7917–7937.

[Is14] Y. Isono, *Some prime factorization results for free quantum group factors*, J. Reine Angew. Math., **722** (2015), 215–250.

[Iv02] N. V. Ivanov, *Mapping class groups*, Handbook of geometric topology, 523–633, North-Holland, Amsterdam, 2002.

[Jo79] D. Johnson, *Homeomorphisms of a surface which act trivially on homology*, Proc. Amer. Math. Soc. **75** (1979), 119–125.

[Jo80] D. Johnson, *Abelian quotients of the mapping class group $\mathcal{I}_g$*, Math. Ann. **249** (1980), 225–242.

[Jo81] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.

[Ju07] K. Jung, *Strongly 1-bounded von Neumann algebras*, Geom. Funct. Anal. **17** (2007), no. 4, 1180–1200.

[Ki08] Y. Kida, *Orbit equivalence rigidity for ergodic actions of the mapping class group*, Geom. Dedicata **131** (2008), 99–109.

[Ki09] Y. Kida, *Automorphisms of the Torelli complex and the complex of separating curves*, J. Math. Soc. Japan **63** (2011), 363–417.

[KW99] E. Kirchberg and S. Wassermann, *Exact groups and continuous bundles of C^*-algebras*, Math. Ann. **315** (2) (1999), 169–203.

[KK50] Krasner M. and Kaloujnine L. *Produit complet de groupes de permutations et probleme d'extension de groupes, I, II, III*, Acta Univ. Szeged, vol. 13 (1950), 208–230; 14 (1951), 39–66, 69–82.

[Me84] D. Meier, *On polyfree groups*, Illinois J. Math. **28** (1984), no.3, 437–443.

[Me92] G. Mess, *The Torelli groups for genus* 2 *and* 3 *surfaces*, Topology **31** (1992), no.4, 775–790.

[MO13] A. Minasyan and D. Osin, *Acylindrical hyperbolicity of groups acting on trees*, Math. Ann., **362** (2001), 1055–1105.

[Mi01] I. Mineyev, *Straightening and bounded cohomology of hyperbolic groups*, Geom. Funct. Anal. **11** (2001), 807–839.

[MY09] I. Mineyev and A. Yaman, *Relative hyperbolicity and bounded cohomology*, Preprint, University of Urbana-Champaign, 2006.

[MMS03] I. Mineyev, N. Monad, and Y. Shalom, *Ideal bicombings for hyperbolic groups and applications*, Topology **43** (2004), No. 6, 1319–1344.

[MS04] N. Monod and Y. Shalom, *Orbit equivalence rigidity and bounded cohomology*, Ann. of Math. **164** (2006), No. 3, 825–878.

[Neu54] B. H. Neumann, *Groups covered by permutable sets*, J. Lond. Math. Soc. (2) **29** (1954), No. 2, 236–248.

[Os13] D. Osin, *Acylindrically hyperbolic groups*, to appear in Trans. Amer. Math. Soc., ArXiv:1304.1246.

[Os06] D. Osin, *Peripheral fillings of relatively hyperbolic groups*, Invent. Math. **167** (2007), no. 2, 295–326.

[Oz03] N. Ozawa, *Solid von Neumann algebras*, Acta Math. **192** (2004), 111–117.

[Oz05] N. Ozawa, *A Kurosh type theorem for type II_1 factors*, Int. Math. Res. Not. Volume 2006, Article ID97560

[OP03] N. Ozawa and S. Popa, *Some prime factorization results for type II_1 factors*, Invent. Math. **156** (2004), 223–234.

[OP08] N. Ozawa and S. Popa, *On a class of II_1 factors with at most one Cartan subalgebra. II*, Amer. J. Math. **132** (2010), 841–866.

[OP07] N. Ozawa and S. Popa, *On a class of II_1 factors with at most one Cartan subalgebra*, Ann. of Math. **172** (2010), 713–749.

[PR99] L. Paris and D. Rolfsen, *Geometric subgroups of surface braid groups*, Ann. Inst. Fourier (Grenoble) **49** (1999), 417–472.

[Pe06] J. Peterson, *L^2-rigidity in von Neumann algebras*, Invent. Math. **175** (2009), no. 2, 417–433.

[Pe09] J. Peterson, *Examples of group actions which are virtually $W*$-superrigid*, Preprint February 2009, ArXiv:1002.1745.

[PS09] J. Peterson and T. Sinclair, *On cocycle superrigidity for Gaussian actions*, Ergodic Theory Dynam. Systems **32** (2012), no. 1, 249–272.

[PP86] M. Pimsner and S. Popa, *Entropy and index for subfactors*, Ann. Sci. Éc. Norm. Sup. **19** (1986), 57–106.

[Po81] S. Popa, *Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras*, J. Oper. Theory **9** (1983), 253–268.

[Po86] S. Popa, *Correspondences*, Preprint INCREST, 1986.

[Po01] S. Popa, *On a class of type II_1 factors with Betti numbers invariants*, Ann. of Math. **163** (2006), 809–899.

[Po03] S. Popa, *Strong Rigidity of II_1 Factors Arising from Malleable Actions of w-Rigid Groups I*, Invent. Math. **165** (2006), 369–408.

[Po04] S. Popa, *Strong Rigidity of II_1 Factors Arising from Malleable Actions of w-Rigid Groups II*, Invent. Math. **165** (2006), 409–453.

[Po08] S. Popa, *On the superrigidity of malleable action with spectal gap*, J. Amer. Math. Soc. **21** (2008), 981–1000.

[Po06] S. Popa, *On Ozawa's property for free group factors*, Int. Math. Res. Not. Vol. 2007 : article ID rnm036, 10 pages.

[Po06b] S. Popa, *Deformation and rigidity for group actions and von Neumann algebras*, Proceedings of the International Congress of Mathematicians, (Madrid 2006), Volume I, EMS Publishing House, Zürich 2006/2007, pp. 445–479.

[PV09] S. Popa and S. Vaes, *Group measure space decomposition of II_1 factors and W^*-superrigidity*, Invent. Math. **182** (2010), 371–417.

[PV11] S. Popa and S. Vaes, *Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups*, Acta Math. **212** (2014), no. 1, 141–198.

[PV12] S. Popa and S. Vaes, *Unique Cartan decomposition for II_1 factors arising from arbitrary actions of hyperbolic groups*, J. Reine Angew. Math. **694** (2014), 215–239.

[Pow78] J. Powell, *Two theorems on the mapping class group of surfaces*, Proc. Amer. Math. Soc. **68** (1978), 347–350.

[Shl04] D. Shlyakhtenko, *Some estimates for non-microstates free entropy dimension, with applications to q-semicircular families*, Int. Math. Res. Not. **51** (2004), 2757–2772.

[Shl00] D. Shlyakhtenko, *Prime type III factors*, Proc. Nat. Acad. Sci., **97** (2000), 12439–12441.

[Si10] T. Sinclair, *Strong solidity of group factors from lattices in $SO(n,1)$ and $SU(n,1)$*, J. Funct. Anal. **260** (2011), no. 11, 3209–3221.

[SW12] O. Sizemore and A. Winchester, *A unique prime decomposition result for wreath product factors*, Pacific J. Math. **265** (2013), no. 1, 221–232.

[Tho09] A. Thom, *Low degree bounded cohomology and L^2-invariants for negatively curved groups*, Groups Geom. Dyn. **3** (2009), no. 2, 343–358.

[Va07] S. Vaes, *Explicit computations of all finite index bimodules for a family of II_1 factors*, Ann. Sci. Éc. Norm. Sup. **41** (2008), 743–788.

[Va10] S Vaes, *One-cohomology and the uniqueness of the group measure space decomposition of a II_1 factor*, Math. Ann. **355** (2013), 661–696.

[Va13] S. Vaes, *Normalizers inside amalgamated free products von Neumann algebras*, Publ. Res. Inst. Math. Sci. **50** (2014), 695–721.

[VV14] S. Vaes and P. Verraedt, *Classification of type III Bernoulli crossed products*, Adv. Math. **281** (2015), 296–332.

[V94] D-V. Voiculescu, *The analogues of entropy and of Fisher's information measure in free probability theory. II*, Invent. Math. **118** (1994), no. 3, 411–440.

[V96] D-V. Voiculescu, *The analogues of entropy and of Fisher's information measure in free probability theory: the absence of Cartan subalgebras*, Geom. Funct. Anal. **6** (1996), no. 1,172–199.

[VDN92] D-V. Voiculescu, K.J. Dykema, and A. Nica, *Free random variables*, CRM Monograph Series 1. American Mathematical Society, Providence, RI, 1992.