Fall 2017

An application of the theory of moments to Euclidean relativistic quantum mechanical scattering

Gordon J. Aiello
University of Iowa

AN APPLICATION OF THE THEORY OF MOMENTS TO EUCLIDEAN RELATIVISTIC QUANTUM MECHANICAL SCATTERING

by

Gordon J. Aiello

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Applied Mathematics and Computational Sciences
in the Graduate College of
The University of Iowa

December 2017

Thesis Supervisors: Professor Wayne N. Polyzou
Professor Palle E.T. Jorgensen

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Gordon J. Aiello

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Applied Mathematics and Computational Sciences at the December 2017 graduation.

Thesis Committee: ______________________________
Wayne Polyzou, Thesis Supervisor

____
Palle Jorgensen, Thesis Supervisor

____
Paul Muhly

____
Vincent Rodgers

____
Richard Baker

To my brother, Alan. I admire you for becoming the man you are today and I truly look up to you for having made the climb. Your thoughtfulness, generosity, and devotion impact my life on a daily basis for which I am immeasurably grateful. I am proud to be your brother.

To my father, James. Through your example I have learned the importance of values, integrity, dedication, and measuring twice to cut once. That's what makes you that guy. I am proud to be your son. Go Lions!

To my love and inspiration, Brittany. Thank you for continually believing in me and being there every step of the way. This road would have been entirely impassable if not for the light you've given to my life.

Finally, to my mother, Mary. There are no words. Thank you for always believing in me. Thank you for teaching me to value the beauty of dreams. Thank you for showing me how to sustain grace, conviction, and courage under fire. Thank you for teaching me the difference one person can make. Thank you for showing me what real heroes look like. I am proud to be your son. This one's definitely for you, Ma!

ACKNOWLEDGEMENTS

Thank you to my advisors, Wayne Polyzou and Palle Jorgensen, and the rest of my thesis committee. I am especially grateful to Professor Polyzou for his time and endless patience. I hope this work serves as a testament to all the encouragement you've given me during my time at Iowa and the seemingly limitless extent of your wisdom and expertise. To Professor Jorgensen, thank you for always finding time to talk and for being so willing to share your passion and vast knowledge of all things math.

I would like to thank Professor Vincent Rodgers for always opening his door to me. The depth of your knowledge is incomprehensible. I hope to someday impact aspiring scholars and teachers to the degree you have me.

Thank you to Iowa City, the University of Iowa, the Applied Mathematics and Computational Sciences program, and to Professor Weimin Han for encouraging and supporting students whose passions cross many interdisciplinary lines. Thank you also to Professors Laurent Jay, Vic Camillo, Colleen Mitchell, Richard Baker, and Gerhard Strohmer for your time, encouragement, and letters of recommendation. A very special thank you also goes out to Margaret Driscol and Cindy Van Ark, the pillars of the mathematics department.

Thank you to all of my remarkably talented colleagues, each of whom has been generous with their time and encouragement, I hope you all know who you are. Special thanks go to K. Czuprynski, R. de Santiago, T. Kindred, K. Betancourt, R. Ligo, A.

Appuhamy, A. Mahat, K. McCormick, T. Noffsinger, C. Stiegler, T. Takhtaganov, M. Barela, C. Caples, C. Patterson, N. Richmond, A. Perrine, A. Sistko, Z. Zhao, Y. Yang, G. Tanner, J. Verniero, J. Park, and S. Yaros. A very special thanks goes to Matt McCurry. Your humor, wit, and intellect are unsurpassed and I'm lucky to have you as a friend, chief.

Thank you to all of my friends and family. Special recognition goes out to the Anderson, Aiello, and Griffin families, A. Simon-Brissette, C. and S. Hanna, J. Masten, the RF 8's, E. Anderson and A. Gutiérrez, M. Pooley, P. Fortuna, C. Noyalis, and M. Geltz. A very special thanks goes to my Aunt Chris for all the love and cards she has sent over the years. An enormous thanks goes to my Uncle John. You have supported me in every way possible and are a constant source of wisdom and encouragement. Of all the John and Chris Andersons in the world, I'm sure glad I got matched up with the likes of you two.

In memoriam of my greatest mathematical influence, Sam B. Nadler, Jr. I am forever indebted to Professor Nadler for showing me the beauty of mathematical thought and expression. Though I know he would abhor the lack of pictures, I hope this is, nevertheless, a document of which he would be proud.

ABSTRACT

One recipe for mathematically formulating a relativistic quantum mechanical scattering theory utilizes a two-Hilbert space approach, denoted by $\mathscr{H}$ and $\mathscr{H}_0$, upon each of which a unitary representation of the Poincaré Lie group is given. Physically speaking, $\mathscr{H}$ models a complicated interacting system of particles one wishes to understand, and $\mathscr{H}_0$ an associated simpler (i.e., free/noninteracting) structure one uses to construct "asymptotic boundary conditions" on so-called scattering states in $\mathscr{H}$. Simply put, $\mathscr{H}_0$ is an attempted idealization of $\mathscr{H}$ one hopes to realize in the large time limits $t \to \pm\infty$.

The above considerations lead to the study of the existence of strong limits of operators of the form $e^{iHt}Je^{-iH_0t}$, where H and H_0 are the self-adjoint generators of the time translation subgroup of the unitary representations of the Poincaré group on $\mathscr{H}$ and $\mathscr{H}_0$, and J is a contrived mapping from $\mathscr{H}_0$ into $\mathscr{H}$ that provides the internal structure of the scattering asymptotes.

The existence of said limits in the context of Euclidean quantum theories (satisfying precepts known as the Osterwalder-Schrader axioms) depends on the choice of J and leads to a marvelous connection between this formalism and a beautiful area of classical mathematical analysis known as the Stieltjes moment problem, which concerns the relationship between numerical sequences $\{\mu_n\}_{n=0}^{\infty}$ and the existence/uniqueness of measures $\alpha(x)$ on the half-line satisfying

$$\mu_n = \int_0^{\infty} x^n d\alpha(x).$$

PUBLIC ABSTRACT

It's an exciting time for particle physics! As technology improves, our ability to investigate the subatomic world grows stronger. For example, The Large Hadron Collider in Switzerland has recently confirmed the existence of a particle known as the Higgs boson. Underlying such groundbreaking discoveries is scattering theory – the physics governing particle collisions and interactions.

Essential to physics is the need for both theoretical tools and experimental data. The job of the theorist is to identify the underlying structure of a physical system, and then find methods for performing calculations within the framework of this system to provide experimenters with predictions.

The reason relativistic quantum scattering is a complicated subject, however, is that on the atomic and subatomic scale, one cannot possibly follow the detailed paths of the particles as they interact with one another. Instead, one considers statistical data obtained through the scattering of many repeated collision experiments.

The current state of scattering theory requires the computation of unwieldy quantities known as Minkowski Green's functions. The purpose of this thesis is to show that it's possible to develop the theory of calculating physically relevant quantities in scattering experiments using a simpler framework than what's available at present. In so doing, the troublesome Minkowski Green's functions are replaced with a simpler collection of objects called Schwinger functions, which help simplify the necessary mathematics into which the underlying physics typically dragoons theorists.

TABLE OF CONTENTS

CHAPTER

CHAPTER 1
THE MOMENT PROBLEM

The purpose of this first chapter is to provide an introduction to a beautiful area of classical mathematical analysis known as the moment problem. The material found herein has a direct, though seemingly underutilized, application to relativistic quantum mechanical scattering theory; we will have much to say about the mathematical formalism of the scattering problem when the time comes. For the moment (get it?!), it suffices to know that the scattering framework with which we will be concerned is modeled by a Hilbert space whose inner product structure is governed by a collection of distributions – known as "Euclidean Green's functions" – that satisfy certain physically reasonable assumptions. While working with this construct, one is interested in establishing the existence of a strong limit – known to physicists as a "scattering state" – the proof of which reduces to a question of polynomial density on the half-line $[0, \infty)$ with respect to a measure induced by the aforementioned Green's functions. Simply put, from a mathematical point of view we are faced with the following question: Given a non-decreasing measure function $\alpha(x)$ on the half-line, do there exist conditions such that if $\alpha(x)$ were to satisfy said provisos, then the linear subspace of all polynomials will be dense in $L^2_\alpha(0, \infty)$?

We will find the answer to be in the affirmative, the confirmation of which rests on establishing the following two complementary theorems:

Carleman's Condition on the Half Line ([36], Page 20)**.** *A sufficient condition* *that the Stieltjes moment problem be determined is that*

$$\sum_{n=1}^{\infty} \mu_n^{-1/2n} = \infty,$$

where

$$\mu_n = \int_0^{\infty} x^n d\alpha(x).$$

Theorem 1.1 ([3], Page 45)**.** *If $\alpha(x)$ is the solution of a determinate moment problem, the set of all polynomials is dense in L^2_α.*

Our intent for this chapter is to provide the reader with a certain level of intuition for Carleman's condition and **Theorem 1.1**, saving the more technical components of their proofs – of which there are a great deal – for subsequent chapters and appendices.

1.1 Preliminaries

Definition 1.1 ([41], Page 10)**.** Let $p \geq 1$, and let $\alpha(x)$ be a non-decreasing function in $[a, b]$ which is not constant. Let the functions

$$f_0(x), f_1(x), f_2(x), \cdots, f_n(x), \cdots \tag{1.1}$$

be of the class $L^p_\alpha(a, b)$. The system (1.1) is called closed in $L^p_\alpha(a, b)$ if for every $f(x)$ in $L^p_\alpha(a, b)$ and for every $\epsilon > 0$, a function of the form

$$k(x) = c_0 f_0(x) + c_1 f_1(x) + \cdots + c_n f_n(x)$$

exists such that

$$\int_a^b |f(x) - k(x)|^p \, d\alpha(x) < \epsilon.$$

Given a non-decreasing measure function $\alpha(x)$, it is known from the theory of L^p-spaces (see [33], Page 69) that the collection of continuous functions of compact support $C_0(a,b)$ is dense in $L^p_\alpha(a,b)$. That is, given $f \in L^p_\alpha(a,b)$ and $\epsilon > 0$, there exists a continuous function F of compact support such that

$$\int_a^b |f(x) - F(x)|^p \, d\alpha(x) < \epsilon.$$

Furthermore, the Stone-Weierstrass theorem (see [32], Page 159) tells us that if f is a continuous complex-valued function on a *compact* interval $[a,b]$, there exits a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a,b]$, and if f is real-valued, the P_n may be taken real-valued also. Combining these two theorems, we obtain a result tantalizingly close to what we're after.

Theorem 1.2 ([41], Page 11)**.** *Let $p \geq 1$, and a and b finite. Given $f \in L^p_\alpha(a,b)$ and $\epsilon > 0$, there exists a polynomial P such that*

$$\int_a^b |f(x) - P(x)|^p \, d\alpha(x) < \epsilon.$$

That is, the system of monomials $\{x^n\}_{n=0}^\infty$ is closed in $L^p_\alpha(a,b)$ in the sense of **Definition 1.1** *for compact intervals $[a,b]$.*

The question that now bears asking is: Is **Theorem 1.2** robust enough to permit us setting $a = 0$ and letting $b \to \infty$? This is decidedly not the case, as the following example illustrates.

Example 1.1 ([48], Pages 126 & [41], Pages 40-41)**.** Define

$$\alpha(x) = \int_0^x e^{-t^{1/4}} dt \qquad \Longrightarrow \qquad d\alpha(x) = e^{-x^{1/4}} dx \qquad \text{(see [48], Page 12)}$$

and note $\sin(x^\mu)$ is a non-zero element of $L^2_\alpha(0,\infty)$ for all $\mu \geq 0$. Taking $\mu = 1/4$, direct calculation (via the change of variables $x \mapsto u^4$ and integration by parts) yields

$$\begin{aligned}\int_0^\infty x^n \sin\left(x^{1/4}\right) d\alpha(x) &= -2i \int_0^\infty \left[e^{(i-1)u} - e^{-(i+1)u}\right] u^{4n+3} du \\ &= -2i(4n+3)! \left[\frac{1}{(1-i)^{4n+4}} - \frac{1}{(1+i)^{4n+4}}\right]\end{aligned}$$

Writing $1 \pm i = \sqrt{2}e^{\pm i\pi/4}$, the above reduces to

$$\begin{aligned}-2i(4n+3)! \left[\frac{1}{(1-i)^{4n+4}} - \frac{1}{(1+i)^{4n+4}}\right] &= -2i4^{-(n+1)}(4n+3)! \left[e^{i\pi(n+1)} - e^{-i\pi(n+1)}\right] \\ &= 4^{-n}(4n+3)! \sin\left[(n+1)\pi\right] \\ &= 0, \qquad n = 0, 1, 2, \cdots,\end{aligned}$$

so that

$$\int_0^\infty x^n \sin\left(x^{1/4}\right) d\alpha(x) = 0 \tag{1.2}$$

holds for all non-negative integers n. Through linearity of the integral, equation (1.2) immediately implies

$$\int_0^\infty P(x) \sin\left(x^{1/4}\right) d\alpha(x) = 0, \tag{1.3}$$

for all polynomials $P(x)$. Applying equation (1.3) in conjunction with Hölder's inequality, we deduce for all polynomials $P(x)$

$$\begin{aligned}
0 &< \int_0^\infty \sin^2(x^{1/4})d\alpha(x) \\
&= \int_0^\infty \sin\bigl(x^{1/4}\bigr)\left[\sin\bigl(x^{1/4}\bigr) - P(x)\right] d\alpha(x) \\
&\leq \int_0^\infty \bigl|\sin\bigl(x^{1/4}\bigr) - P(x)\bigr| \, d\alpha(x) \\
&\leq \left[\int_0^\infty \bigl|\sin\bigl(x^{1/4}\bigr) - P(x)\bigr|^2 \, d\alpha(x)\right]^{1/2} \left[\int_0^\infty d\alpha(x)\right]^{1/2} \\
&= 2\sqrt{6}\,\bigl\|\sin\bigl(x^{1/4}\bigr) - P(x)\bigr\|_{L^2_\alpha(0,\infty)}.
\end{aligned}$$

Hence, we have

$$0 < \frac{1}{2\sqrt{6}} \int_0^\infty \sin^2(x^{1/4})d\alpha(x) \leq \bigl\|\sin\bigl(x^{1/4}\bigr) - P(x)\bigr\|_{L^2_\alpha(0,\infty)}$$

for all polynomials $P(x)$, from which it follows that the system $\{x^n\}_{n=0}^\infty$ is *not* closed in $L^2_\alpha(0,\infty)$.

Remark. Constructing alternative examples that yield equivalent results is perhaps more delicate than the previous calculation seems to indicate. For example, the above computation fails to yield analogous results in the case of $d\alpha(x) = e^{-x}dx$ and $\sin(x)$ on the half-line.

How does one press on from this point, given that unbounded intervals allow for cases in which the system $\{x^n\}_{n=0}^\infty$ fails to be closed? It would appear that if we are to remove the compactness condition on our interval, some other stipulation must be put in place to save the day, as it were.

Perhaps there are many ways in which to proceed. One such path, however, comes to us courtesy of a brief remark given by Szegö (see [41], Page 40). Modestly paraphrasing, we learn that the condition

$$\int_0^\infty x^n \sin\bigl(x^{1/4}\bigr) d\alpha(x) = 0, \qquad n = 0, 1, 2, \cdots, \tag{1.4}$$

encountered in **Example 1.1** is closely connected with a uniqueness concern regarding our chosen integration measure, which we discuss further in the next section.

Remark. Equation (1.4) can be interpreted as an orthogonality condition between $\sin\bigl(x^{1/4}\bigr)$ and each monomial in the system $\{x^n\}_{n=0}^\infty$ with respect to the inner product of $L^2_\alpha(0, \infty)$.

1.2 The Moment Problem and Further Considerations

In [3], Akhiezer states that the moment problem was originally studied by the Dutch mathematician Thomas Stieltjes in the mid to late 19th century. From a given numerical sequence, Stieltjes was interested in determining a distribution of positive mass on the half-line $[0, \infty)$ whose static moment of order n is precisely the nth term in the given sequence. From a mathematical point of view, the moment problem can be formulated on any given interval. We will be most concerned, though, with the full and half-line cases, which are referred to in the literature as the Hamburger and Stieltjes problems, respectfully. As we shall see later (**Lemma 1.3**, Page 10), Stieltjes moment problems can always be viewed as a special type of "symmetric" Hamburger problem. Nevertheless, we state each separately, as well as what is meant by a "unique" solution with regards to the moment problem.

The Hamburger Moment Problem ([3], Page 29)**.** *Given an infinite sequence of real numbers* $\{\mu_n\}_{n=0}^\infty$*, it is required to find a non-decreasing function* $\alpha(x)$ *on the* *real line satisfying the equations*

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x), \qquad n = 0, 1, 2, \cdots. \tag{1.5}$$

The Stieltjes Moment Problem ([3], Page 76)**.** *Given an infinite sequence of real numbers* $\{\mu_n\}_{n=0}^{\infty}$*, it is required to find a non-decreasing function* $\alpha(x)$ *on the half-line* $[0, \infty)$ *satisfying the equations*

$$\mu_n = \int_{0}^{\infty} x^n d\alpha(x), \qquad n = 0, 1, 2, \cdots. \tag{1.6}$$

Definition 1.2 ([3], Page 29)**.** Two solutions of the moment problem are not considered to be distinct if their difference is a constant at all points where that difference is continuous. The moment problem is called determinate if in this sense it has only a single solution, and indeterminate otherwise.

In either case, one need only consider the positivity of the even degree moments to see that the class of sequences for which Hamburger of Stieltjes solutions exist is not completely arbitrary. For example, a necessary and sufficient condition for a solution to the Hamburger problem to exist is that each of the (Hermitian) Hankel matrices

$$H_0 = \mu_0, \quad H_1 = \begin{bmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{bmatrix}, \quad H_2 = \begin{bmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{bmatrix}, \cdots$$

$$H_n = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & & & \vdots \\ \mu_2 & & & & \vdots \\ \vdots & & & & \mu_{2n-4} \\ \vdots & & & \mu_{2n-4} & \mu_{2n-3} \\ \mu_{n-1} & \cdots & \mu_{2n-4} & \mu_{2n-3} & \mu_{2n-2} \end{bmatrix}, \quad \cdots$$

be positive definite (see [3], Page 30); i.e., $x^T H_n x > 0$ for all non-zero $x \in \mathbb{R}^n$ and all $n \geq 0$. A wonderful result accredited to the mathematician J. J. Sylvester is that the above condition is equivalent to the fact that each of the determinants

$$\mathrm{Det}(H_n) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & & & \vdots \\ \mu_2 & & & & \vdots \\ \vdots & & & & \mu_{2n-4} \\ \vdots & & & \mu_{2n-4} & \mu_{2n-3} \\ \mu_{n-1} & \cdots & \mu_{2n-4} & \mu_{2n-3} & \mu_{2n-2} \end{vmatrix}$$

be positive (see [3], Page 1). In particular, we have that $\mu_0 > 0$, which we will later use to "normalize" our moment sequence to a convenient starting value for μ_0. It is assumed henceforth that only moment sequences $\{\mu_n\}_{n=0}^{\infty}$ satisfying this positivity condition will be considered, thereby guaranteeing the existence of a solution to the corresponding moment problem. We table these ideas for now, making further use of them in the coming chapters (e.g., see **Section 2.1**, Page 24).

We close this section by returning to **Example 1.1** (Page 4). In so doing, we look to clearly illustrate the meaning of **Definition 1.2**, as well as the connection between the orthogonality condition expressed in equation (1.4) and the indeterminacy of the moment problem, as intimated by Szegö in [41] and discussed above in this work.

Example 1.2. Consider the sequence

$$\mu_n = 4(4n+3)! \qquad n = 0, 1, 2, \cdots.$$

Calculations similar to those performed in **Example 1.1** show (also see [48], Page 126) that the function

$$\alpha_1(x) = \int_0^x e^{-t^{1/4}} dt, \qquad x \geq 0,$$

is a non-decreasing solution to the moment problem

$$\mu_n = \int_0^\infty x^n d\alpha(x), \qquad n = 0, 1, 2, \cdots.$$

Furthermore, our work in **Example 1.1** shows that

$$\alpha_2(x) = \int_0^x \left[1 - \sin\left(t^{1/4}\right)\right] e^{-t^{1/4}} dt, \qquad x \geq 0,$$

is also a solution to the moment problem

$$\mu_n = \int_0^\infty x^n d\alpha(x), \qquad n = 0, 1, 2, \cdots.$$

The difference

$$\alpha_1(x) - \alpha_2(x) = \int_0^x \sin\left(t^{1/4}\right) e^{-t^{1/4}} dt, \qquad x \geq 0,$$

is continuous but non-constant on the half-line $[0, \infty)$. Hence, as a result of equation (1.2), the Stieltjes moment problem

$$\mu_n = \int_0^\infty x^n d\alpha(x), \qquad n = 0, 1, 2, \cdots,$$

is indeterminate in the sense of **Definition 1.2**.

1.3 Stieltjes Problems Are Hamburger Problems

Our aim for this section is in establishing a lemma proving Stieltjes moment problems can always be expressed as Hamburger moment problems, which we presaged on Page 6. While complete details are provided, the proof is essentially a combination of the change of variables theorem for Stieltjes integration (see [48], Page 19) and the fact that the mapping $x \mapsto x^2$ defines a homeomorphism of the half-line $[0, \infty)$.

Lemma 1.3 ([25], Page 296)**.** *Let $\{\mu_n\}_{n=0}^{\infty}$ be a Stieltjes moment sequence and define the corresponding symmetric Hamburger moment sequence $\{\mu'_n\}_{n=0}^{\infty}$ by*

$$\mu'_{2n} = \mu_n, \qquad \mu'_{2n+1} = 0, \qquad n = 0, 1, 2, \cdots;$$

i.e.,

$$\{\mu'_n\}_{n=0}^{\infty} = \{\mu_0,\, 0,\, \mu_1,\, 0,\, \mu_2,\, 0,\, \mu_3,\, 0,\, \mu_4, \cdots\}.$$

Then the set of solutions to the Stieltjes moment problem is in bijection with the set of solutions to the symmetric Hamburger moment problem.

Proof. Suppose $\alpha(x)$ is a solution to the Stieltjes moment problem, so that

$$\mu_n = \int_0^{\infty} x^n d\alpha(x), \qquad n = 0, 1, 2, \cdots,$$

and define (see [36], Page 19)

$$\beta(x) = \begin{cases} \dfrac{1}{2}\alpha(x^2) & x \geq 0 \\ -\dfrac{1}{2}\alpha(x^2) & x < 0. \end{cases}$$

We claim $\beta(x)$ is a solution to the corresponding symmetric Hamburger moment problem $\{\mu'_n\}_{n=0}^{\infty}$. Indeed,

$$\begin{aligned}
\int_{-\infty}^{\infty} x^n d\beta(x) &= \frac{1}{2}\int_0^{\infty} x^n d\alpha(x^2) - \frac{1}{2}\int_{-\infty}^{0} x^n d\alpha(x^2) \\
&= \frac{1}{2}\int_0^{\infty} x^n d\alpha(x^2) - \frac{1}{2}(-1)^n \int_{\infty}^{0} x^n d\alpha(x^2) \\
&= \frac{1}{2}\int_0^{\infty} x^n d\alpha(x^2) + \frac{1}{2}(-1)^n \int_{0}^{\infty} x^n d\alpha(x^2) \\
&= \frac{1}{2}\int_0^{\infty} x^{n/2} d\alpha(x)\left[1 + (-1)^n\right],
\end{aligned}$$

where we have used the change of variables formula for Stieltjes integrals to justify the last equality (see [48], Page 19). Examining the above we are led directly to

$$\int_{-\infty}^{\infty} x^{2n} d\beta(x) = \int_0^{\infty} x^n d\alpha(x) = \mu_n = \mu'_{2n}, \qquad \int_{-\infty}^{\infty} x^{2n+1} d\beta(x) = 0, \qquad n = 0, 1, 2, \cdots,$$

as desired.

Conversely, suppose $\beta(x)$ is a solution to the (symmetric) Hamburger moment problem

$$\mu'_{2n} = \int_{-\infty}^{\infty} x^{2n} d\beta(x), \qquad 0 = \int_{-\infty}^{\infty} x^{2n+1} d\beta(x), \qquad n = 0, 1, 2, \cdots,$$

and define (see [48], Page 137)

$$\sigma(x) = \frac{\beta(x) - \beta(-x)}{2}, \qquad x \in \mathbb{R}.$$

Since $\beta(x)$ is non-decreasing – by definition of a solution to the Hamburger moment problem – $\sigma(x)$ is non-decreasing as well. Moreover, calculations similar to those performed above show that $\sigma(x)$ satisfies

$$\mu'_{2n} = \int_{-\infty}^{\infty} x^{2n} d\sigma(x), \qquad 0 = \int_{-\infty}^{\infty} x^{2n+1} d\sigma(x), \qquad n = 0, 1, 2, \cdots.$$

Writing

$$\mu'_{2n} = \int_{-\infty}^{\infty} x^{2n} d\sigma(x) = \int_{0}^{\infty} x^{2n} d\sigma(x) + \int_{-\infty}^{0} x^{2n} d\sigma(x)$$

and using the change of variables $x \mapsto -x$ in the second integration, we obtain

$$\mu'_{2n} = \int_{0}^{\infty} x^{2n} d\sigma(x) - \int_{0}^{\infty} x^{2n} d\sigma(-x).$$

Noting that $\sigma(x)$ is odd, the above becomes

$$\mu'_{2n} = 2 \int_{0}^{\infty} x^{2n} d\sigma(x).$$

Using the change of variable $x \mapsto x^{1/2}$ (see [48], Page 19) and setting $\alpha(x) = 2\sigma(x^{1/2})$ for $x \geq 0$, we obtain

$$\mu'_{2n} = \int_{0}^{\infty} x^{n} d\alpha(x) = \mu_n, \qquad n = 0, 1, 2, \cdots,$$

where $\alpha(x)$ is non-decreasing on the half-line because $x \mapsto x^{1/2}$ and $\sigma(x)$ are non-decreasing on $[0, \infty)$. Hence, $\alpha(x)$ is a solution to the Stieltjes moment problem corresponding to a symmetric Hamburger moment problem.

To conclude that the two solution sets are in bijection, we note that the above mappings are inverses of one another. $\square$

In light of the previous result, we proceed with our analysis of the Stieltjes and Hamburger moment problems in a parallel fashion to that of the classical literature. Specifically, we will state and prove theorems for the general Hamburger case; that is, full-line moment problems that are not necessarily symmetric. These results will then be decanted down to the Stieltjes half-line problem for their eventual use in our scattering theory application.

1.4 The Function $I(z;\alpha)$

The next step on our journey requires us to introduce the integral transform

$$I(z;\alpha) = \int_{-\infty}^{\infty} \frac{1}{z-x}\, d\alpha(x), \qquad \text{Im } z \neq 0,$$

where $\alpha(x)$ is a solution to a given Hamburger moment problem. The utility of the function $I(z;\alpha)$ cannot be overstated, as it is the primary tool used to establish both Carleman's condition and **Theorem 1.1**. Underscoring its importance, we dedicate an appendix to the elucidation of the many technical components that surround $I(z;\alpha)$. For now, we focus on merely getting a feel for how $I(z;\alpha)$ connects the dots, so to speak.

Standard arguments from single variable complex analysis show that $I(z;\alpha)$ is analytic on the domain Im $z \neq 0$, its values being conjugate at two conjugate points (see **Lemma A.1**, Page 87). Moreover, $I(z;\alpha)$ can be used to *essentially uniquely determine* the measure $\alpha(x)$ through what is known as the Stieltjes-Perron inversion formula (equation (1.7) below). Further discussion of this result will resume after its statement is given.

Theorem 1.4 ([16], Page 591)**.** *Let $\alpha(x)$ be a bounded, non-decreasing real function defined on $(-\infty,\infty)$, and, as above, let $I(z;\alpha)$ be defined by*

$$I(z;\alpha) = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\alpha(t), \qquad \text{Im}\, z \neq 0.$$

Then for arbitrary real a and b

$$\frac{1}{2}\left[\alpha(b^+)+\alpha(b^-)\right] - \frac{1}{2}\left[\alpha(a^+)+\alpha(a^-)\right] = -\frac{1}{\pi}\lim_{y\to 0^+} \text{Im} \int_a^b I(x+iy;\alpha)dx. \qquad (1.7)$$

1.5 $I(z;\alpha)$ Essentially Determines Moment Problem Solutions

We now make good on our promise to discuss the fundamental role $I(z;\alpha)$ plays in essentially determining moment problem solutions. We will see that if $I(z;\alpha_1) = I(z;\alpha_2)$, then $\alpha_1(x)$ and $\alpha_2(x)$ are equivalent solutions to the moment problem, and it is in this sense what the word "essentially" is taken to mean here. To do so, we make use of a concept known as the *normalization* of a bounded, non-decreasing function $\alpha(x)$ (see [48], Pages 13–14), which we define by

$$\alpha^*(x) = \frac{\alpha(x^+) + \alpha(x^-)}{2} - \alpha(-\infty^+), \qquad -\infty < x < \infty,$$

and from whence it follows

$$\alpha^*(-\infty^+) = 0$$
$$\alpha^*(\infty^-) = \alpha(\infty^-) - \alpha(-\infty^+).$$

Applying results from Widder in [48] (see **Theorems 7a** & **8b** on Pages 12 and 14, respectfully), we know that if $f(x)$ is continuous, then

$$\int_{-\infty}^{\infty} f(x)d\alpha(x) = \int_{-\infty}^{\infty} f(x)d\alpha^*(x).$$

In particular, $\alpha(x)$ and $\alpha^*(x)$ generate the same Hamburger moment sequence

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x) = \int_{-\infty}^{\infty} x^n d\alpha^*(x), \qquad n = 0, 1, 2, \cdots,$$

and are equivalent in the sense of **Definition 1.2**, because the difference

$$\alpha(x) - \alpha^*(x) = \alpha(-\infty^+) \tag{1.8}$$

is a constant at any point of continuity for $\alpha(x)$.

Moreover, if $I(z;\alpha_1) = I(z;\alpha_2)$, then $\alpha_1^*(x) = \alpha_2^*(x)$. Indeed, upon application of the Stieltjes-Perron inversion formula (1.7) we have

$$\begin{aligned}
\alpha_1^*(x) &= \frac{1}{2}\left[\alpha_1(x^+) + \alpha_1(x^-)\right] - \alpha_1(-\infty^+) \\
&= -\frac{1}{\pi} \lim_{y \to 0^+} \operatorname{Im} \int_{-\infty}^{x} I(s+iy;\alpha_1)ds \\
&= -\frac{1}{\pi} \lim_{y \to 0^+} \operatorname{Im} \int_{-\infty}^{x} I(s+iy;\alpha_2)ds \\
&= \frac{1}{2}\left[\alpha_2(x^+) + \alpha_2(x^-)\right] - \alpha_2(-\infty^+) \\
&= \alpha_2^*(x).
\end{aligned}$$

Now note that if $\alpha_1(x)$ and $\alpha_2(x)$ are solutions to a Hamburger moment problem, then, as a result of their monotonic nature, $\alpha_1(x)$ and $\alpha_2(x)$ can be discontinuous on at most a countable set (see **Theorem 4.30** in [32], Page 96). Thus the set of shared points of continuity of two such functions is dense in the real line. Hence, if $I(z;\alpha_1) = I(z;\alpha_2)$, then $\alpha_1^*(x) = \alpha_2^*(x)$, and using equation (1.8) it follows that at any point x in the dense set of shared continuity

$$\alpha_1(x) - \alpha_2(x) = \alpha_1(x) - \alpha_1^*(x) + \alpha_2^*(x) - \alpha_2(x) = \alpha_1(-\infty^+) - \alpha_2(-\infty^+);$$

i.e., $\alpha_1(x)$ and $\alpha_2(x)$ differ by a constant at all shared points of continuity. We express this result in the form of a theorem.

Theorem 1.5 ([16], Page 593, & [48], Pages 12–14). *If $I(z;\alpha_1) = I(z;\alpha_2)$ for two solutions of a Hamburger moment problem, then $\alpha_1(x)$ and $\alpha_2(x)$ are equivalent in the sense of* **Definition 1.2** *(see Page 7).*

At this point we hope the fog has begun to lift. We see from **Theorem 1.5** that $I(z;\alpha)$ will alert us to the equivalence of two *particular* solutions to a moment problem. Since we would ultimately like to apply **Theorem 1.1** to our relativistic quantum mechanical model, it is (perhaps) natural to ask if there exist conditions on candidate sequences $\{\mu_n\}_{n=0}^{\infty}$ that are sufficient to guarantee $I(z;\alpha)$ deems *all* solutions to the corresponding moment problem equivalent. For if such a condition were to exist, **Theorem 1.5** tells us that we are indeed working with a determinate moment problem and we could then appeal to **Theorem 1.1** to obtain the density of polynomials in our relativistic quantum mechanical Hilbert space, per our primary intent.

Such a provision comes to us courtesy of Carleman's condition, which takes slightly different, but closely related (**Lemma 1.3**, Page 10), forms on the full and half-line.

Theorem 1.6 ([36], Page 19)**.** *A sufficient condition that the Hamburger moment problem be determined is that*

$$\sum_{n=1}^{\infty} \mu_{2n}^{-1/2n} = \infty. \tag{1.9}$$

Theorem 1.7 ([36], Page 20)**.** *A sufficient condition that the Stieltjes moment problem be determined is that*

$$\sum_{n=1}^{\infty} \mu_{n}^{-1/2n} = \infty. \tag{1.10}$$

There is still much work to do before we can formally prove Carleman's condition on the full line. However, assuming the validity of **Theorem 1.6**, we can immediately establish **Theorem 1.7** for the half-line case.

Proof. By **Lemma 1.3** (Page 10), the set of solutions to the Stieltjes moment problem

$$\mu_n = \int_0^\infty x^n d\alpha(x)$$

is in bijection with the set of solutions to the symmetric Hamburger moment problem

$$\mu'_{2n} = \mu_n, \qquad \mu'_{2n+1} = 0, \qquad n = 0, 1, 2, \cdots.$$

Hence,

$$\sum_{n=1}^{\infty} \mu_n^{-1/2n} = \infty \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} (\mu'_{2n})^{-1/2n} = \infty,$$

from which **Theorem 1.6** allows us to conclude that the Stieltjes moment problem

$$\mu_n = \int_0^\infty x^n d\alpha(x)$$

is determinate, whenever $\displaystyle\sum_{n=1}^{\infty} \mu_n^{-1/2n} = \infty$. □

1.6 Closing Examples

We illustrate the relevance of **Theorem 1.7** to several examples, starting with a continuation of our discussion from Page 8.

Example 1.3. Recall from **Example 1.2** that the sequence

$$\mu_n = 4(4n+3)! \qquad n = 0, 1, 2, \cdots,$$

foments an indeterminate Stieltjes moment problem. We chronicle here the details showing

$$\sum_{n=1}^{\infty} \mu_n^{-1/2n} < \infty,$$

as need be the case, per **Theorem 1.7**.

Using the concavity of the logarithm function together with a right endpoint approximation, we have for $n \geq 1$

$$\int_1^n \ln x \, dx \leq \sum_{k=1}^{n} \ln\, k,$$

from which it follows

$$n \ln\left(\frac{n}{e}\right) + 1 \leq \ln n!$$

Exponentiation then gives

$$e\left(\frac{n}{e}\right)^n \leq n! \tag{1.11}$$

Using (1.11) with $n \mapsto 4n$, we have

$$\mu_n = 4(4n+3)! \geq (4n)! \geq e\left(\frac{4n}{e}\right)^{4n}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{e^2}{e^{1/2n}(4n)^2} < \infty \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} \mu_n^{-1/2n} < \infty,$$

as required.

The conclusion of **Example 1.3** represents the completion of a loop we began early on in this work. We hope the reader is now convinced that the closure of the system $\{x^n\}_{n=0}^{\infty}$ is a delicate issue when dealing with unbounded intervals. Moreover, we trust that our working example has assured oneself that there is an intimate connection between said closure, the uniqueness of measures, and growth demands placed on the measure's associated moments by Carleman's condition.

We end the present section with two further examples. The first provides an illustration of a determinate moment problem and is similar to the situation we will encounter in our application to scattering theory, in that a moment-generating measure is given at the onset and we wish to establish its uniqueness - see **Section 5.10**. The second is an application of our work to quantum mechanics and uses language more familiar to the physicist.

Example 1.4 ([36], Page 22)**.** Fix $\lambda \geq 1/2$, define

$$\alpha(x) = \int_0^x e^{-\lambda t^\lambda} dt \qquad \Longrightarrow \qquad d\alpha(x) = e^{-\lambda x^\lambda} dx,$$

and note the Stieltjes moment sequence generated by $\alpha(x)$ is given by

$$\mu_n = \int_0^\infty x^n d\alpha(x) = \int_0^\infty x^n e^{-\lambda x^\lambda} dx.$$

Now, choose δ so that $0 < \delta < \lambda$, write

$$\mu_n = \int_0^\infty x^n e^{-\lambda x^\lambda} e^{\delta x^{1/2}} e^{-\delta x^{1/2}} dx, \tag{1.12}$$

and consider the sequence of functions

$$f_n(x) = x^n e^{-\delta x^{1/2}}, \qquad n \geq 1.$$

Solving $f_n'(x) = 0$, one finds

$$0 \leq f_n(x) \leq f_n\left(\left(\frac{2n}{\delta}\right)^2\right) = \left(\frac{2n}{\delta}\right)^{2n} e^{-2n}, \qquad x \geq 0.$$

Then, by (1.12),

$$\mu_n = \int_0^\infty x^n e^{-\lambda x^\lambda} dx \leq \left(\frac{2n}{\delta}\right)^{2n} e^{-2n} \int_0^\infty e^{-\lambda x^\lambda} e^{\delta x^{1/2}} dx.$$

Since $0 < \delta < \lambda$ and $\lambda \geq 1/2$, the latter integral converges to a constant, which we denote by C. Hence,

$$\mu_n \leq C \left(\frac{2n}{\delta}\right)^{2n} e^{-2n},$$

so that

$$\sum_{n=1}^{\infty} \frac{e\delta}{2n} = \infty \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} \mu_n^{-1/2n} = \infty.$$

By Carleman's condition, the Stieltjes moment problem

$$\mu_n = \int_0^{\infty} x^n d\alpha(x)$$

is determinate in the sense of **Definition 1.2**.

We begin our final example with a brief discussion of the axioms underlying quantum mechanics for the reader less familiar with these ideas. A passing sense for the spectral theorem's ability to connect expected values of self-adjoint operators to integration with respect to a particular measure over the operator's spectrum is assumed (see, e.g., [13], Page 141).

Example 1.5 ([13], Page 67)**.** In the mathematical treatment of classical physics, one considers a set known as "phase space" in which all possible states of a particular system are represented; e.g., for a particle moving in one-dimension, the classical phase space is $\mathbb{R}^2$, which we think of as pairs (x, p) with x being the particle's position and p its momentum. Within this construct, *physical* observables (i.e., measurements) made on the *real-world* system are formulated *mathematically* as functions on the phase space; e.g., in the one-dimensional motion case, the measurement of the particle's position is represented by the function $f(x, p) = x$.

One route – there are several – to mathematically dealing with the experimentally observed quantized nature of Nature is to modify the previous framework, which we indicate diagrammatically below:

$$\begin{aligned}
\text{Point in phase space} &\longleftrightarrow \text{Unit vector in an appropriate Hilbert space;}\\
\text{Functions } f \text{ on phase space} &\longleftrightarrow \text{Self-adjoint operators } \hat{f} \text{ on quantum Hilbert space;}\\
\text{Determinism of measurement} &\longleftrightarrow \text{Probabilistic interpretation of measurement outcomes.}
\end{aligned}$$

The remainder of the present discussion focuses primarily on the third correspondence above. Stated explicitly, quantum theory requires that the probability distribution for the measured outcomes of an observable f on a quantum system characterized by the unit vector ψ must satisfy

$$E(f^m) = \langle \psi, (\hat{f})^m \psi \rangle, \tag{1.13}$$

where $E(\cdot)$ denotes the expected value. This seemingly innocuous equation expresses a beautiful synthesis of physics with mathematics, and represents the crux of axiomatic quantum mechanics. On the left we have a quantity attainable only through repeated trials (i.e., *experimental* means), and the right declares that the structure of our *mathematical* model (i.e., the inner product space) must accurately capture this probabilistic datum. For those more familiar with the matters discussed heretofore, it is *exactly* this axiomatized equation that allows one to interpret the measures obtained from the spectral theorem for self-adjoint operators as probability distributions in the realm of quantum mechanics (see [13], Page 66).

One illustration between the current discussion and Carleman's uniqueness condition comes from considering unit norm eigenvectors of a self-adjoint operator $\hat{f}$. Suppose ψ satisfies $\hat{f}\psi = \lambda\psi$, where $\|\psi\| = 1$ (note: since $\hat{f}$ is self-adjoint, its spectrum is a subset the real line, so that λ is a real number - see [13], Pages 57 & 177). The spectral theorem then says that there is a measure $\alpha(x)$ on $\mathbb{R}$, such that

$$\int_{-\infty}^{\infty} x^n d\alpha(x) = \langle \psi, (\hat{f})^n \psi \rangle = \lambda^n \|\psi\|^2 = \lambda^n, \qquad n \geq 1.$$

Viewed as a Hamburger moment problem with $\mu_n = \lambda^n$, it follows from the calculation

$$\sum_{n=1}^{\infty} \mu_{2n}^{-1/2n} = \sum_{n=1}^{\infty} \frac{1}{|\lambda|} = \infty$$

and **Theorem 1.6** (see Page 16) that $\alpha(x)$ is, in fact, unique in the sense of **Definition 1.2** (see Page 7). Such a measure is given by the shifted Heaviside function

$$\alpha(x) = H(x - \lambda) = \begin{cases} 0 & x < \lambda \\ 1 & x \geq \lambda, \end{cases}$$

because

$$\int_{-\infty}^{\infty} x^n d\alpha(x) = \int_{-\infty}^{\infty} x^n \delta(x - \lambda) dx = \lambda^n.$$

As discussed above, equation (1.13) allows us to interpret the measure $\alpha(x)$ as a probability distribution for the measured outcomes of the observable f on a physical system in the state represented mathematically by ψ. Hence, the uniqueness of $\alpha(x)$ and

$$d\alpha(x) = \delta(x - \lambda) dx$$

tells us that the measurement of f on a quantum system represented by the state ψ does not yield random values, but rather always returns the quantity λ. That is, the probability of measuring the value of f to be λ on a system characterized by ψ is 100%.

CHAPTER 2
TWO SPECIAL POLYNOMIAL FAMILIES

In Chapter 1 we aimed to strike a balance between heuristic and a proof-oriented approaches to the moment problem and future applications thereof. In an effort to establish **Theorems 1.1** & **1.6** (Pages 2 & 16, respectively), we begin gently steering the form of our exposition towards the latter, acquiring the necessary tools for doing so along the way. Specifically, we will introduce two families of polynomials – $P_n(x)$ and $Q_n(x)$ – which, when combined with our old friend $I(z;\alpha)$, will be seen as terms in the Fourier expansion for

$$f_0(x) = \frac{1}{x-z}, \qquad \text{where } \operatorname{Im} z \neq 0 \text{ is fixed;}$$

i.e., (minus) the kernel of the integral transform $I(z;\alpha)$. The primary source for our discussion is [3], a masterfully written and eminently readable text.

2.1 The Polynomials $P_n(x)$

Recall from our discussion on Page 7 that a necessary and sufficient condition for the Hamburger moment problem

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x)$$

to possess a solution is that each of the Hankel matrices

$$H_n = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & & & \vdots \\ \mu_2 & & & & \vdots \\ \vdots & & & & \mu_{2n-4} \\ \vdots & & & \mu_{2n-4} & \mu_{2n-3} \\ \mu_{n-1} & \cdots & \mu_{2n-4} & \mu_{2n-3} & \mu_{2n-2} \end{bmatrix}$$

have positive determinant. Hence, $\mu_0 = \mathrm{Det}(H_1) > 0$, and we can normalize any given moment sequence via $\mu_n \mapsto \dfrac{\mu_n}{\mu_0}$ so that $\mu_0 = 1$. We assume this to be the case henceforth.

Given a moment sequence $\{\mu_n\}_{n=0}^{\infty}$, we construct the family of polynomials $P_n(x)$ via

$$P_0(x) = 1,$$
$$P_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \qquad n = 1, 2, \cdots,$$

where $D_n = \mathrm{Det}(H_{n+1})$ for $n \geq 0$, and we set $D_{-1} = 1$ for consistency with $P_0(x) = 1$ (see [3], Page 3). Defining the functional $\mathfrak{S}$ on the space of all polynomials by

$$\mathfrak{S}\{R(x)\} = r_0\mu_0 + r_1\mu_1 + \cdots + r_n\mu_n, \tag{2.1}$$

where

$$R(x) = r_0 + r_1 x + \cdots + r_n x^n,$$

we will show that the polynomials $P_n(x)$ have the following properties, which characterize them fully (see [3], Page 3):

1. $P_n(x)$ is a polynomial of degree n and its leading coefficient is positive;

2. The orthogonality relation

$$\mathfrak{S}\{P_m(x)P_n(x)\} = \delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

holds.

Indeed, the first property is a consequence of expanding the determinant in $P_n(x)$ around the x^n term and applying the positivity condition $\mathrm{Det}(H_n) > 0$, while the second follows from observing that

$$\mathfrak{S}\left\{P_n(x)x^m\right\} = \frac{1}{\sqrt{D_{n-1}D_n}}\mathfrak{S}\left\{\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ x^m & x^{m+1} & \cdots & x^{m+n} \end{vmatrix}\right\}$$

$$= \frac{1}{\sqrt{D_{n-1}D_n}}\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_m & \mu_{m+1} & \cdots & \mu_{m+n} \end{vmatrix}$$

$$= \begin{cases} 0 & 0 \le m \le n-1 \\ \sqrt{\frac{D_n}{D_{n-1}}} & m = n \end{cases} \tag{2.2}$$

and

$$P_n(x) = \sqrt{\frac{D_{n-1}}{D_n}}x^n + R_{n-1}(x), \tag{2.3}$$

where $R_{n-1}(x)$ is a polynomial of at most degree $n-1$.

2.2 A Finite Difference Equation for the $P_n(x)$'s

As a result of properties (1)-(2), any polynomial may be expanded in terms of the $P_n(x)$'s, and we use this fact to write

$$xP_n(x) = a_{n,n+1}P_{n+1}(x) + a_{n,n}P_n(x) + a_{n,n-1}P_{n-1}(x) + \cdots. \tag{2.4}$$

Comparing the leading coefficients on both sides of the above and using (2.3), we have

$$a_{n,n+1} = \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}.$$

For $0 \le k \le n$, we multiply both sides of (2.4) by $P_k(x)$ and apply the functional $\mathfrak{S}$ together with the heretofore established properties of the $P_n(x)$'s to obtain

$$a_{n,k} = 0, \qquad k = 0, 1, \cdots, n-2,$$

$$a_{n,n-1} = \mathfrak{S}\left\{xP_n(x)P_{n-1}(x)\right\},$$

$$a_{n,n} = \mathfrak{S}\left\{xP_n(x)P_n(x)\right\}.$$

Substituting

$$xP_{n-1}(x) = a_{n-1,n}P_n(x) + R_{n-1}(x)$$

into

$$a_{n,n-1} = \mathfrak{S}\left\{xP_n(x)P_{n-1}(x)\right\},$$

we deduce that

$$a_{n,n-1} = a_{n-1,n}.$$

Setting

$$a_n = a_{n,n} = \mathfrak{S}\left\{xP_n(x)P_n(x)\right\}, \qquad b_n = a_{n,n+1} = \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}, \qquad \text{and} \;\; b_{-1} = 0,$$

equation (2.4) takes the form

$$xP_n(x) = b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x), \qquad n = 0, 1, 2, \cdots.$$

Hence, the $P_n(x)$'s satisfy the second-order finite-difference equation

$$b_{n-1}y_{n-1} + a_ny_n + b_ny_{n+1} = xy_n, \qquad n = 1, 2, 3, \cdots, \tag{2.5}$$

with the initial conditions

$$P_0(x) = 1, \qquad P_1(x) = \frac{x - a_0}{b_0}.$$

2.3 The Polynomials $Q_n(x)$

A second linearly independent set of solutions to (2.5) – which we denote by $Q_n(x)$ – can be obtained by using the initial conditions

$$Q_0(x) = 0, \qquad Q_1(x) = \frac{1}{b_0}.$$

Observe that $Q_n(x)$ can be expressed in terms of $P_n(x)$ via

$$Q_n(x) = \mathfrak{S}_t \left\{ \frac{P_n(x) - P_n(t)}{x - t} \right\}. \tag{2.6}$$

Indeed, for $n = 0$ and $n = 1$, the result follows directly from $P_0(x) - P_0(t) = 0$ and $P_1(x) - P_1(t) = \dfrac{x-t}{b_0}$, respectively. For $n \geq 2$ the right side of (2.6) satisfies (2.5) because

$$\begin{aligned}
xQ_n(x) &= x\,\mathfrak{S}_t \left\{ \frac{P_n(x) - P_n(t)}{x - t} \right\} \\
&= \mathfrak{S}_t \left\{ \frac{xP_n(x) - xP_n(t)}{x - t} \right\} \\
&= \mathfrak{S}_t \left\{ \frac{xP_n(x) - xP_n(t)}{x - t} \right\} + \mathfrak{S}_t \left\{ P_n(t) \right\} \\
&= \mathfrak{S}_t \left\{ \frac{xP_n(x) - tP_n(t)}{x - t} \right\} \\
&= b_{n-1}Q_{n-1}(x) + a_nQ_n(x) + b_nQ_{n+1}(x),
\end{aligned}$$

where in the third equality we used the orthogonality relation between $P_n(t)$ and $P_0(t) = 1$, and in the final equality we utilized (2.5).

We close the current section by noting that $P_n(x)$ and $Q_n(x)$ satisfy the discrete analogue of the Liouville-Ostrogradskii formula from the theory of linear differential equations (see [3], Page 9):

$$P_{n-1}(z)Q_n(z) - P_n(z)Q_{n-1}(z) = \frac{1}{b_{n-1}}, \qquad n = 1, 2, 3, \cdots, \tag{2.7}$$

where z is the same fixed z appearing in $f_0(x) = 1/(x-z)$ (see Page 24). The proof of (2.7) can be readily obtained using mathematical induction and the initial conditions satisfied by $P_n(x)$ and $Q_n(x)$.

2.4 The Relationship Between $P_n(x)$, $Q_n(x)$, and $I(z;\alpha)$

We shall see presently that $P_n(x)$, $Q_n(x)$, and $I(z;\alpha)$ are precisely the terms needed to write down a generalized Fourier expansion of the kernel function

$$f_0(x) = \frac{1}{x-z}, \qquad \text{where } \operatorname{Im} z \neq 0 \text{ is fixed,}$$

for $I(z;\alpha)$ with respect to the family of polynomials $\{P_n(x)\}_{n=0}^{\infty}$, whenever $\alpha(x)$ is a solution to a Hamburger moment problem. To verify this, suppose a moment sequence $\{\mu_n\}_{n=0}^{\infty}$ is given and that $\alpha(x)$ satisfies

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x), \qquad n = 0, 1, 2, \cdots.$$

One then notes that the functional $\mathfrak{S}$, whose essence is to replace $x^n \mapsto \mu_n$ (see equation (2.1), Page 25), can be expressed in terms of integration with respect to $\alpha(x)$; i.e.,

$$\mathfrak{S}\{R(x)\} = \int_{-\infty}^{\infty} R(x) d\alpha(x)$$

for any polynomial

$$R(x) = r_0 + r_1 x + \cdots + r_n x^n.$$

From this observation one can use equation (2.6) (Page 28) to write

$$Q_n(z) = \int_{-\infty}^{\infty} \frac{P_n(x) - P_n(z)}{x - z} d\alpha(x), \qquad n = 0, 1, 2, \cdots,$$

where, as before, z above is the same fixed z appearing in $f_0(x) = 1/(x-z)$. Boiling these ingredients in sawdust, salting them in glue, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x-z} P_n(x) d\alpha(x) &= \int_{-\infty}^{\infty} \frac{P_n(x) - P_n(z)}{x-z} d\alpha(x) + P_n(z) \int_{-\infty}^{\infty} \frac{1}{x-z} d\alpha(x) \\ &= Q_n(z) - I(z;\alpha) P_n(z).\end{aligned}$$

Hence, $f_0(x)$ has the following formal generalized Fourier expansion

$$f_0(x) \sim \sum_{n=0}^{\infty} \left[Q_n(z) - I(z;\alpha)P_n(z)\right] P_n(x), \tag{2.8}$$

an expression which neatly ties together all of the major players in our theory thus far.

2.5 Bessel's Inequality

It is known from the theory of Fourier analysis (see [41], Page 38) that the partial sum

$$\sum_{k=0}^{n} \left[Q_k(z) - I(z;\alpha)P_k(z)\right] P_k(x)$$

is the best degree n approximation to $f_0(x)$ in the $L^2_\alpha(\mathbb{R})$ metric. Namely, one has

$$\min_{\deg\{R(x)\} \le n} \|f_0 - R\|^2_{L^2_\alpha} = \|f_0\|^2_{L^2_\alpha} - \sum_{k=0}^{n} \left|Q_k(z) - I(z;\alpha)P_k(z)\right|^2,$$

Letting $n \to \infty$ above we obtain Bessel's inequality, which takes the form

$$\sum_{n=0}^{\infty} \left|Q_n(z) - I(z;\alpha)P_n(z)\right|^2 \le \int_{-\infty}^{\infty} \frac{1}{|x-z|^2}\, d\alpha(x). \tag{2.9}$$

But

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{|x-z|^2}\, d\alpha(x) &= \int_{-\infty}^{\infty} \frac{1}{z-\bar{z}} \left[\frac{1}{x-z} - \frac{1}{x-\bar{z}}\right] d\alpha(x) \\ &= \frac{-I(z;\alpha) + I(\bar{z};\alpha)}{z - \bar{z}}.\end{aligned} \tag{2.10}$$

Upon writing $w = -I(z;\alpha)$, (2.9) now reads

$$\sum_{n=0}^{\infty} |Q_n(z) + wP_n(z)|^2 \leq \frac{w - \overline{w}}{z - \overline{z}}, \tag{2.11}$$

an inequality of which we will make explicit use in the following chapters.

2.6 Chapter 2 Summary

As promised, the present chapter has been more technical in nature than the previous one. Despite all the bells and whistles, the truly valuable content is that of **Sections 2.4** & **2.5** – specifically, equation (2.7), expression (2.8), and inequality (2.11). We kindly ask the reader to keep these three results in the front of their mind as they provide the foundation for proving **Theorems 1.1** & **1.6**, which is our goal in the forthcoming chapters.

CHAPTER 3
THE PROOF OF CARLEMAN'S CONDITION

We have come a long way from our humble beginnings in Chapter 1. To wit, we have defined moment problems and have seen through various examples how they are related to the uniqueness of measures and polynomial density in L^2-spaces. Moreover, *assuming* the validity of Carleman's condition on the full line (**Theorem 1.6**, Page 16), we were able to prove that it holds for the half-line case (**Theorem 1.7**, Page 16). We use the present chapter to plug this gap in our exposition, providing the reader with a great level of detail for substantiating **Theorem 1.6** – an alternative proof relying more directly on the analyticity of $I(z;\alpha)$ can be found in **Appendix B**. Looking on the bright side, much of the heavy lifting has already been performed at this point.

Remark. In this chapter, the terminology "moment problem" is taken to mean a Hamburger moment problem

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x),$$

unless otherwise noted. We also remind the reader that, as was the case in Chapter 2, we assume our moment sequence is normalized – i.e., $\mu_n \mapsto \dfrac{\mu_n}{\mu_0}$ – so that $\mu_0 = 1$.

3.1 Bessel's Inequality and the Circles $K_n(z)$

We saw in equation (2.11) (Page 31) that Bessel's inequality takes the form

$$\sum_{n=0}^{\infty} |Q_n(z) + wP_n(z)|^2 \leq \frac{w - \overline{w}}{z - \overline{z}},$$

for the generalized Fourier expansion of $f_0(x) = 1/(x-z)$, where $w = -I(z,\alpha)$ and Im $z \neq 0$ is fixed. As the next theorem shows, Bessel's inequality is related to a family of circular contours $K_n(z)$ in the half-plane.

Theorem 3.1 ([3], Page 11)**.** *Let z be fixed in the half-plane* Im $z > 0$ *(*Im $z < 0$*), let t vary along the whole real axis, and define*

$$w_n(z,t) = -\frac{Q_n(z) - tQ_{n-1}(z)}{P_n(z) - tP_{n-1}(z)}.$$

Then $w = w_n(z,t)$ describes a circular contour $K_n(z)$ in the half-plane Im $w > 0$ *(*Im $w < 0$*), whose equation may be written in the form*

$$\frac{w-\overline{w}}{z-\overline{z}} - \sum_{k=0}^{n-1} |wP_k(z) + Q_k(z)|^2 = 0.$$

The center of this circle is at the point

$$-\frac{Q_n(z)\overline{P_{n-1}(z)} - Q_{n-1}(z)\overline{P_n(z)}}{P_n(z)\overline{P_{n-1}(z)} - P_{n-1}(z)\overline{P_n(z)}},$$

and its radius is

$$\frac{1}{|z-\overline{z}|}\frac{1}{\sum_{k=0}^{n-1}|P_k(z)|^2}. \tag{3.1}$$

For points w lying outside the circle $K_n(z)$ the inequality

$$\frac{w-\overline{w}}{z-\overline{z}} - \sum_{k=0}^{n-1} |wP_k(z) + Q_k(z)|^2 < 0$$

holds, while for points within the circle $K_n(z)$ one has the inequality

$$\frac{w-\overline{w}}{z-\overline{z}} - \sum_{k=0}^{n-1} |wP_k(z) + Q_k(z)|^2 > 0.$$

Lastly, the circle $K_n(z)$ lies entirely within the circle $K_{n-1}(z)$ and the circumferences of the circles touch.

Proof. One uses the finite difference equations (2.5) and (2.7) together with (2.6) to concoct three further relations satisfied by the $P_n(x)$'s and $Q_n(x)$'s. With these established, proving **Theorem 3.1** checks with a modest amount of algebraic manipulation. See [3], Pages 8–13. $\square$

Remark. The above theorem shows us that Bessel's inequality distinguishes between the interiors and exteriors of a family of disks – also denoted $K_n(z)$ – in the complex w-plane. When a point w lies on the boundary of the limiting disk $K_\infty(z)$, the equality

$$\frac{w-\overline{w}}{z-\overline{z}} = \sum_{k=0}^{\infty} |wP_k(z) + Q_k(z)|^2$$

holds. In this case, the point w is related to a solution $\alpha(x)$ of the moment problem known as an *N-extremal solution at the point z* (see [3], Page 43). Determinate moment solutions are subsumed by the theory of N-extremal moment solutions, though we proceed only with the determinate case in the name of consistency. Note, however, that little, if anything, need be changed in what's to come to obtain results for the N-extremal case.

The keen-eyed reader will notice that we have defined the variable w in two seemingly different ways: (i) $w = -I(z;\alpha)$ and (ii) $w = w_n(z,t)$. As the next theorem shows, this all comes out in the wash.

Theorem 3.2 ([3], Page 40)**.** *The set of all values assumed at the point z (*$\operatorname{Im} z \neq 0$*) by all functions $-I(z;\alpha)$, coincides with the point set $K_\infty(z)$. In other words, as*

$\alpha(x)$ ranges over all possible solutions to a given moment problem, the set of values

$$-I(z;\alpha) = \int_{-\infty}^{\infty} \frac{1}{x-z} d\alpha(x), \qquad \text{Im } z \neq 0 \textit{ fixed},$$

gives precisely the limiting disk (or point) $K_\infty(z)$.

Proof. See [3], Pages 40 – 41. □

3.2 Equation (2.7) and Determinate Moment Problems

Recalling the discrete Liouville-Ostrogradskii formula (2.7)

$$P_n(z)Q_{n+1}(z) - P_{n+1}(z)Q_n(z) = \frac{1}{b_n}, \qquad n = 0, 1, 2, \cdots,$$

from **Section 2.3** (Page 28), we shall prove the following.

Theorem 3.3 ([3], Page 24)**.** *If*

$$\sum_{n=0}^{\infty} \frac{1}{b_n} = \infty,$$

then the limiting disk $K_\infty(z)$ is a single point; i.e., the moment problem

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x)$$

is determinate.

Proof. ([11], Page 264). By way of contradiction, suppose the limiting disk $K_\infty(z)$ is not a single point. From equation (3.1) (Page 33), it follows that

$$\sum_{n=0}^{\infty} |P_n(z)|^2 < \infty. \tag{3.2}$$

We know from **Theorem 3.1** (Page 33) that

$$w_n(z,0) = -\frac{Q_n(z)}{P_n(z)}$$

lies on the circle $K_n(z)$, so that (also by **Theorem 3.1**)

$$\frac{w_n(z,0)-\overline{w_n(z,0)}}{z-\bar{z}}=\sum_{k=0}^{n-1}|w_n(z,0)P_k(z)+Q_k(z)|^2.$$

Using the initial conditions $P_0(z)=1$ and $Q_0(z)=0$ (Page 27), the above equality implies

$$\left|\frac{Q_n(x)}{P_n(z)}\right|^2=|w_n(z,0)|^2=|w_n(z,0)P_0(z)+Q_0(z)|^2\leq\frac{w_n(z,0)-\overline{w_n(z,0)}}{z-\bar{z}}.$$

The triangle inequality then gives

$$|w_n(z,0)|^2\leq\frac{2|w_n(z,0)|}{|z-\bar{z}|},$$

so that for fixed Im $z\neq 0$, we have the bound

$$|w_n(z,0)|\leq\frac{2}{|z-\bar{z}|}\qquad\Longrightarrow\qquad|Q_n(z)|^2\leq\left(\frac{2}{|z-\bar{z}|}\right)^2|P_n(z)|^2.$$

It then follows immediately from equation (3.2) that

$$\sum_{n=0}^{\infty}|Q_n(z)|^2<\infty \tag{3.3}$$

as well.

However, (3.2) and (3.3) together give

$$\sum_{n=0}^{\infty}\frac{1}{b_n}<\infty.$$

Indeed, from the triangle and Cauchy-Schwarz inequalities we have (note $b_n>0$ by definition – see Page 27)

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{b_n} &= \sum_{n=0}^{\infty} |P_n(z)Q_{n+1}(z) - P_{n+1}(z)Q_n(z)| \\
&\leq \sum_{n=0}^{\infty} |P_n(z)Q_{n+1}(z)| + \sum_{n=0}^{\infty} |P_{n+1}(z)Q_n(z)| \\
&\leq \sqrt{\left(\sum_{n=0}^{\infty} |P_n(z)|^2\right)\left(\sum_{n=0}^{\infty} |Q_{n+1}(z)|^2\right)} + \sqrt{\left(\sum_{n=0}^{\infty} |P_{n+1}(z)|^2\right)\left(\sum_{n=0}^{\infty} |Q_n(z)|^2\right)} \\
&< \infty,
\end{aligned}
$$

contradicting

$$\sum_{n=0}^{\infty} \frac{1}{b_n} = \infty.$$

Therefore, $K_\infty(z)$ reduces to a single point whenever

$$\sum_{n=0}^{\infty} \frac{1}{b_n} = \infty.$$

Thus, by **Theorem 3.2** (Page 34), $I(z;\alpha_1) = I(z;\alpha_2)$ for *all* solutions $\alpha_1(x)$ and $\alpha_2(x)$ to the moment problem

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x), \qquad n = 0, 1, 2, \cdots.$$

Hence, by **Theorem 1.5** (Page 15), the Hamburger moment problem is determinate whenever

$$\sum_{n=0}^{\infty} \frac{1}{b_n} = \infty.$$

□

3.3 The Proof of Carleman's Condition for the Hamburger Problem

We are now firmly in position to prove **Theorem 1.6**, which we restate here for convenience

Theorem 1.6 ([3], Page 85 & [36], Page 19)**.** *If*

$$\sum_{n=1}^{\infty} \mu_{2n}^{-1/2n} = \infty, \tag{3.4}$$

the moment problem

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x)$$

is determinate.

Remark. The forthcoming argument will make use of Carleman's inequality for non-negative real numbers u_n (see [3], Page 86):

$$\sum_{n=1}^{\infty} \sqrt[n]{u_1 u_2 \cdots u_n} \leq e \sum_{n=1}^{\infty} u_n, \tag{3.5}$$

the proof of which can be found in **Appendix B**.

Proof. Recall the orthogonality relation

$$\mathfrak{S}\{P_m(x)P_n(x)\} = \delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

and equation (2.2)

$$\mathfrak{S}\{P_n(x)x^m\} = \begin{cases} 0 & 0 \leq m \leq n-1 \\ \sqrt{\frac{D_n}{D_{n-1}}} & m = n \end{cases}$$

for the family of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ (**Section 2.1**, Pages 25 and 26, respectively).

We have seen that the functional $\mathfrak{S}$ can be expressed as

$$\mathfrak{S}\{R(x)\} = \int_{-\infty}^{\infty} R(x) d\alpha(x)$$

for any polynomial $R(x)$ (Page 29). Hence, remembering that (Page 27)

$$b_n = \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}, \qquad n = 0, 1, 2, \cdots,$$

where

$$D_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}, \qquad n = 0, 1, 2, \cdots,$$

with $D_{-1} = 1$ (Page 25), we have

$$b_0 b_1 \cdots b_{n-1} = b_0 b_1 \cdots b_{n-1} \int_{-\infty}^{\infty} [P_n(x)]^2 \, d\alpha(x) = \int_{-\infty}^{\infty} x^n P_n(x) d\alpha(x),$$

for $n \geq 1$. Applying the Cauchy-Schwarz inequality we obtain the bound

$$b_0 b_1 \cdots b_{n-1} \leq \sqrt{\int_{-\infty}^{\infty} x^{2n} d\alpha(x)} \sqrt{\int_{-\infty}^{\infty} [P_n(x)]^2 \, d\alpha(x)} = \sqrt{\mu_{2n}},$$

for $n \geq 1$. Hence, from Carleman's inequality (3.5), we have

$$\sum_{n=1}^{\infty} \mu_{2n}^{-1/2n} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{b_0 b_1 \cdots b_{n-1}}} \leq e \sum_{n=1}^{\infty} \frac{1}{b_n}.$$

Thus,

$$\sum_{n=1}^{\infty} \mu_{2n}^{-1/2n} = \infty \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} \frac{1}{b_n} = \infty$$

and it follows from **Theorem 3.3** (Page 35) that the Hamburger moment problem

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x), \qquad n = 0, 1, 2, \cdots,$$

is determinate in the sense of **Definition 1.2** (Page 7). $\square$

CHAPTER 4
THE PROOF OF THEOREM 1.1

At long last, we set our sights on completing the justification of **Theorem 1.1** (Page 2). While relatively brief, providing a separate chapter for the coming proof is intended to signify its place of importance in our theory. In an effort to keep this work as self-contained as possible, we quote two pillars of modern mathematical analysis, of which we will make use posthaste.

The Hahn-Banach Theorem ([33], Page 107)**.** *Let M be a linear subspace of a normed linear space X, and let $x_0 \in X$. Then x_0 is in the closure of M if and only if there is no bounded linear functional L on X such that $L(x) = 0$ for all $x \in M$ but $L(x_0) \neq 0$.*

The Riesz Representation Theorem ([13], Page 540)**.** *If $L : H \to \mathbb{C}$ is a bounded linear functional defined on the Hilbert space H, then there exists a unique element $y \in H$ such that*

$$L(x) = \langle y, x \rangle$$

for all $x \in H$. Furthermore, the operator norm of L as a linear functional is equal to the norm of y as an element of H.

4.1 The Proof of Theorem 1.1

We will establish the following.

Theorem 1.1 ([3], Page 45)**.** *If $\alpha(x)$ is the solution of a determinate moment problem,* *the set of all polynomials is dense in* L^2_α.

Proof. Suppose $\alpha(x)$ is the solution of a determinate moment problem. Since $\alpha(x)$ is unique, for each fixed Im $z \neq 0$, the set $K_\infty(z)$ is a single point in the complex w-plane by way of **Theorem 3.2** (Page 34). According to **Theorem 3.1** (Page 33), it follows that

$$\sum_{k=0}^{\infty} |Q_k(z) + wP_k(z)|^2 = \frac{w - \overline{w}}{z - \bar{z}}, \tag{4.1}$$

where $w = -I(z; \alpha)$.

As we saw in equation (2.11) (Page 31), Bessel's inequality takes the form

$$\sum_{n=0}^{\infty} |Q_n(z) + wP_n(z)|^2 \leq \frac{w - \overline{w}}{z - \bar{z}},$$

for the generalized Fourier expansion of $f_0(x) = 1/(x - z)$, where $w = -I(z, \alpha)$ and Im $z \neq 0$ is fixed. Since equality holds in (4.1), it follows that

$$\inf_{\text{All Polynomials } R(x)} \|f_0 - R\|^2_{L^2_\alpha} = \|f_0\|^2_{L^2_\alpha} - \sum_{n=0}^{\infty} |Q_n(z) + wP_n(z)|^2 = 0,$$

because

$$\|f_0\|^2_{L^2_\alpha} = \frac{w - \overline{w}}{z - \bar{z}}$$

from equation (2.10) (Page 30). Thus, the function $f_0(x)$ can be approximated in L^2_α by a polynomial to any degree of accuracy (see [41], Page 38). Additionally, note that this result is also true for

$$\bar{f}_0(x) = \frac{1}{x - \bar{z}}.$$

We claim that the same holds for the functions

$$f_1(x) = \frac{1}{(x - z)^2}, \qquad \bar{f}_1(x) = \frac{1}{(x - \bar{z})^2}$$

as well. Indeed, writing $b = \text{Im } z \neq 0$ we have

$$\int_{-\infty}^{\infty} \left| \frac{1}{(x-z)^2} - \frac{A}{x-z} - \sum_{k=0}^{n} B_k x^k \right|^2 d\alpha(x)$$
$$\leq \frac{1}{b^2} \int_{-\infty}^{\infty} \left| \frac{1}{x-z} - A - (x-z) \sum_{k=0}^{n} B_k x^k \right|^2 d\alpha(x),$$

where the bounding quantity can be made arbitrarily small, as shown above. Continuing in this manner, we conclude that the functions

$$f_k(x) = \frac{1}{(x-z)^{k+1}}, \qquad \bar{f}_k(x) = \frac{1}{(x-\bar{z})^{k+1}}, \qquad k = 0, 1, 2, \cdots,$$

can all be approximated to any degree of accuracy by polynomials in L^2_α.

Now, by way of contradiction, suppose that the set of all polynomials is not dense in L^2_α. By the **Hahn-Banach Theorem** there is a bounded linear functional L which is zero for all polynomials but is not identically zero on L^2_α. By the **Riesz Representation Theorem** there is an element $g(x) \in L^2_\alpha$ such that

$$L(f) = \int_{-\infty}^{\infty} \overline{f(x)} g(x) d\alpha(x) \tag{4.2}$$

for all $f \in L^2_\alpha$. By Hölder's inequality

$$\|\bar{f} g\|_{L^1_\alpha} \leq \|f\|_{L^2_\alpha} \|g\|_{L^2_\alpha},$$

so that L not identically zero implies

$$\int_{-\infty}^{\infty} |g(x)|^2 d\alpha(x) \neq 0. \tag{4.3}$$

On the other hand, we have from our assumption on L that $L(x^n) = 0$ for all $n = 0, 1, 2, \cdots$; i.e.,

$$\int_{-\infty}^{\infty} x^n g(x) d\alpha(x) = 0, \qquad n = 0, 1, 2, \cdots,$$

so that for *any* polynomial $P(x)$ we have

$$\int_{-\infty}^{\infty} g(x)P(x)d\alpha(x) = 0.$$

Hence, for *any* polynomial $P(x)$ it follows that

$$\int_{-\infty}^{\infty} \frac{g(x)}{(x-z)^k} d\alpha(x) = \int_{-\infty}^{\infty} g(x)\left[\frac{1}{(x-z)^k} - P(x)\right] d\alpha(x), \qquad k = 0, 1, 2, \cdots.$$

Appealing to Hölder's inequality yet again we obtain

$$\left|\int_{-\infty}^{\infty} \frac{g(x)}{(x-z)^k} d\alpha(x)\right| \leq \left\|\frac{1}{(x-z)^k} - P(x)\right\|_{L^2_\alpha} \|g\|_{L^2_\alpha}, \qquad k = 0, 1, 2, \cdots.$$

As we saw above, the quantity

$$\left\|\frac{1}{(x-z)^k} - P(x)\right\|_{L^2_\alpha}, \qquad k = 1, 2, 3, \cdots,$$

can be made arbitrarily small by a correct choice of $P(x)$. Hence,

$$\int_{-\infty}^{\infty} \frac{g(x)}{(x-z)^k} d\alpha(x) = 0 \tag{4.4}$$

and similarly

$$\int_{-\infty}^{\infty} \frac{\overline{g(x)}}{(x-z)^k} d\alpha(x) = 0, \tag{4.5}$$

for $k = 1, 2, 3, \cdots$.

Arguments similar to those given proving $I(z;\alpha)$ is analytic (see **Appendix A**) also show that the functions

$$\phi(\lambda) = \int_{-\infty}^{\infty} \frac{g(x)}{x-\lambda} d\alpha(x), \qquad \psi(\lambda) = \int_{-\infty}^{\infty} \frac{\overline{g(x)}}{x-\lambda} d\alpha(x)$$

are analytic in the upper and lower λ-planes. Equations (4.4) and (4.5) demonstrate the fact that $\phi(\lambda)$ and $\psi(\lambda)$ along with their derivatives become zero together at the point $\lambda = z$. Thus, $\phi(\lambda) = 0$ and $\psi(\lambda) = 0$ identically. By a *slightly* generalized version of the Stieltjes-Perron formula (see [3], Page 125), it follows that

$$\int_{-\infty}^{x} g(t)d\alpha(t) = 0, \qquad -\infty < x < \infty.$$

But this contradicts (4.3) because

$$0 = \int_{-\infty}^{\infty} \overline{g(x)}\, d\left[\int_{-\infty}^{x} g(t)d\alpha(t)\right] = \int_{-\infty}^{\infty} |g(x)|^2 d\alpha(x) \neq 0.$$

Thus, the collection of all polynomials is dense in L^2_α, whenever $\alpha(x)$ is the solution to a determinate moment problem. $\square$

CHAPTER 5
SCATTERING ASYMPTOTIC CONDITIONS IN EUCLIDEAN RELATIVISTIC QUANTUM THEORY

Portions of this chapter appeared in the earlier work

[2] G. J. Aiello and W. N. Polyzou, Scattering Asymptotic Conditions in Euclidean Relativistic Quantum Theory, *Physical Review D,* Volume **93**, Issue 5 (2016).

In the words of J. R. Taylor ([42], Page 557), scattering experiments are the single most powerful tool for investigating the structure of atomic and subatomic objects. In such an experiment one fires a stream of projectiles, such as electrons or protons, at a target object – an atom or atomic nucleus, for example – and, by observing the distribution of scattered projectiles as they emerge from the collision, one can gain information about the target and its interactions with the projectile. Perhaps the most famous scattering experiment was the discovery by Ernest Rutherford of the structure of the atom: Rutherford and his assistants fired streams of α particles (the positively charged nuclei of helium atoms) at a thin layer of gold atoms in a sheet of gold foil; by measuring the distribution of the scattered α particles, they were able to deduce that most of the mass of an atom is concentrated in a tiny, positively charged nucleus at the center of the atom. Since that time, most discoveries in atomic and subatomic physics were made with the help of scattering experiments, in which a stream of projectiles were directed at a suitable target and the outgoing particles carefully monitored.

The reason scattering theory is a rather complicated topic is that on the atomic and subatomic scale one cannot possibly follow the detailed orbit of the projectiles as they interact with the target. Hence, very little can be learned by observing a single projectile. On the other hand, if we send in many projectiles, we can observe the number of them that get scattered in different directions and from this we can learn a great deal about the statistical distribution of the many scattered projectiles, which is the central concept of quantum scattering theory.

One approach to mathematically formulating a relativistic quantum mechanical scattering theory utilizes a two-Hilbert space formalism, denoted by $\mathscr{H}$ and $\mathscr{H}_0$, upon each of which a unitary representation of the Poincaré Lie group is given. Physically speaking, $\mathscr{H}$ models a complicated interacting system of particles one wishes to understand, and $\mathscr{H}_0$ an associated simpler (i.e., free/noninteracting) structure one uses to construct "asymptotic boundary conditions" on scattering states in $\mathscr{H}$. This separates the internal structure of the asymptotic states from their space-time properties. Simply put, $\mathscr{H}_0$ is an attempted idealization of $\mathscr{H}$ one hopes to realize in the large time limits $t \to \pm\infty$ (see [7], Page 19).

The above considerations lead to the study of the existence of strong limits of operators of the form $e^{iHt}Je^{-iH_0t} : \mathscr{H}_0 \to \mathscr{H}$. Here H and H_0 are self-adjoint generators of the time translation subgroup of unitary the representations of the Poincaré group on $\mathscr{H}$ and $\mathscr{H}_0$, respectively, and J is a contrived mapping from $\mathscr{H}_0$ into $\mathscr{H}$ – referred to by physicists as an "injection operator" – that provides the internal structure of the scattering asymptotes. When the strong time limits $t \to \pm\infty$ of $e^{iHt}Je^{-iH_0t}$ exist, the resulting mappings from $\mathscr{H}_0$ into $\mathscr{H}$ are known as "wave operators" (see [7], Page 19).

The existence of said limits in the context of Euclidean quantum theories depends on the properties of the connected portions of a collection of distributions called Euclidean Green's/Schwinger functions – which are assumed to satisfy a set of precepts known as the Osterwalder-Schrader axioms, see **Section 5.2** below – as well as on the choice of $J : \mathscr{H}_0 \to \mathscr{H}$. The primary purpose of this chapter is to show how to construct the J mappings from the asymptotic Hilbert space, $\mathscr{H}_0$, to the physical Hilbert space, $\mathscr{H}$, that are needed to establish the existence of wave operators using a result known as Cook's method (e.g., see [30], Page 20). The construction of the injection operator J forms the bridge between the Euclidean scattering formalism we will utilize and the Stieltjes moment problem of Chapters 1-4 in this work.

5.1 Scattering in Euclidean Space: An Overview

Perturbative/Path Integral methods for calculating quantum observables often rely on the use of a technique known to physicists as a Wick rotation, whereby one makes the substitution $t \mapsto \pm it$ (effectively converting the Minkowski "metric" on $\mathbb{R}^4$ to the standard Euclidean structure) allowing for the solution of a problem in Minkowski space to be evaluated in the Euclidean domain. Mathematically speaking, the Wick rotation procedure is tantamount to analytically continuing the quantity of interest to a complex domain. For example, in [12] (see Pages 461–462) Gross shows that the ground state transition matrix element $\langle 0|0\rangle^J$ in the presence of a source term J (of no relation to the injection operator J discussed above) is given by

$$\langle 0|0\rangle^J = \lim_{\substack{T_i \to i\infty \\ T_f \to -i\infty}} \frac{e^{iE_0(T_f - T_i)} \left\langle q', T_f \middle| q, T_i \right\rangle^J}{\phi_0(q')\phi_0^*(q)}, \qquad \phi_0(q) = \langle q|0\rangle \, ,$$

and that by rotating the time variables to the imaginary axis, the most rapid convergence of this limit is obtained. In [51] (see Page 12), Zee states that he counts on path integrals

$$\int Dq(t)e^{i\int dtL(\dot{q},q)} \tag{5.1}$$

to converge because the oscillatory phase factors from different paths tend to cancel out, though he does concede that performing the Wick rotation $t \mapsto -it$ is a more rigorous approach to the problem.

In [35] Julian Schwinger, motivated by his wartime work on waveguide problems for radar equipment at MIT (see [23], Page 6), argued that the boundary conditions needed to properly relate the Minkowski Green's functions to vacuum expectation values of a time-ordered product of field operators is equivalent to the imposition of the requirement that these same functions "should remain a regular function when you make the time coordinate complex in a specific way, and that you never find an exponential that becomes unlimitedly large" (see [23], Page 9). These analytically continued functions – the aforementioned Euclidean Green's functions – satisfy Schwinger-Dyson equations and are moments of Euclidean path integrals

$$S_n(x_1, \dots, x_n) = \frac{\int D[\phi]e^{-A[\phi]}\phi(x_1)\cdots\phi(x_n)}{\int D[\phi']e^{-A[\phi']}},$$

where $A[\phi]$ is the classical action functional and with the dissolution of the imaginary term from (5.1) indicating the transition to the Euclidean domain. According to [23] (see Page 10), Schwinger viewed the Euclidean formalism as empirically equivalent to, but better mathematically behaved than, the Minkowski space theory.

The purpose of this chapter is to argue that it is possible to calculate scattering observables directly in the Euclidean representation of quantum field theory. This avoids the need for analytic continuation and the demand to solve singular integral equations. The essential observation, which is a consequence of the Osterwalder-Schrader reconstruction theorem [24], is that there is a representation of the physical Hilbert space – $\mathcal{H}$ above – directly in terms of the Euclidean Green's functions without analytic continuation. There is also a representation of the Poincaré Lie algebra on this space. According to [49], this defines a relativistic quantum theory. Cluster properties of the Schwinger functions suggest that it should be possible to formulate scattering problems directly in this representation.

Below we list the steps that are needed to construct transition matrix elements. These will be discussed in more detail in the sections that follow. While some of these steps have been discussed in a previous work of P. Kopp and W. N. Polyzou (see [21]), we include all of them to make this work self contained. The discussion in this chapter is limited to spinless particles. Spin does not introduce any additional complications that impact the formulation of the scattering problem. The treatment of particles with spin can be found in [27].

1. An asymptotic Hilbert space, $\mathcal{H}_0$, is defined as the direct sum of tensor products of irreducible representation spaces of the Poincaré group. The mass and spin of the particles in the target, beam, and measured in the detectors determine the mass and spin labels of the irreducible representations.

2. Mappings from the asymptotic Hilbert space, $\mathcal{H}_0$, to the Euclidean represen-

tation of the physical Hilbert space, $\mathscr{H}$, are constructed. These mappings add the internal structure of the asymptotic particles to the space-time properties, including particle cloud effects. They also control the joint energy-momentum spectrum of the asymptotic states, which leads to strong limits.

3. The scattering asymptotic condition is formulated and sufficient conditions on the connected parts of the Schwinger functions for the existence of the channel Møller wave operators are given.

The mapping from the asymptotic Hilbert space to the Euclidean representation of the Hilbert space is the Euclidean analog of a Haag-Ruelle quasi-local field operator (see [18] and [30]). This mapping controls the four-momentum spectrum of the asymptotic states, which is needed to isolate asymptotic states with different mass and the same energy. This can be done in the Euclidean representation because we have explicit representations of the four momentum operators. The result is that the scattering asymptotic condition can be formulated as a strong limit, like it is in non-relativistic quantum mechanics.

In **Sections 5.6** and **5.7**, the results of these considerations are illustrated using the Lehmann representation of a two-point Schwinger function. Methods for isolating the discrete part of the Lehmann weight of these functions are central to the formulation of the scattering asymptotic condition. **Section 5.10** contains the main technical results of this chapter. It shows that orthogonal polynomials in the mass squared operator are complete in this representation of the Hilbert space. This is needed to ensure that the injection operators that are used to formulate the asymptotic condition have positive relative-time support, so their range is in the Euclidean representation of the Hilbert space.

5.2 The Osterwalder-Schrader Axioms

Following [28] (see Page 133), we begin the present section with essential notational considerations. Define I_+^n to be the set of all n-tuples of non-negative integers $k = (k_1, k_2, \cdots, k_n)$ and let $|k| = \sum_{i=1}^n k_i$. Denote by D^k the operator

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}},$$

and by x^k the product $x^k = x_1^{k_1} \cdots x_n^{k_n}$. The Schwartz space of test functions $\mathscr{S}(\mathbb{R}^n)$ is then defined to be the functions of rapid decrease on $\mathbb{R}^n$; i.e., $\mathscr{S}(\mathbb{R}^n)$ is the set of all infinitely differentiable complex-valued functions $f(x)$ on $\mathbb{R}^n$ for which

$$\sup_{x \in \mathbb{R}^n} |x^k D^m f(x)| < \infty, \qquad \forall\, k, m \in I_+^n.$$

Two important subspaces of $\mathscr{S}(\mathbb{R}^{4n})$ must be singled out before the Osterwalder-Schader axioms can be formally stated (see [24], Page 86). We emphasize the fact that we are now specifically considering $\mathscr{S}(\mathbb{R}^{4n})$, not $\mathscr{S}(\mathbb{R}^n)$. By ${}^0\mathscr{S}(\mathbb{R}^{4n})$ we mean the subspace of all functions in $\mathscr{S}(\mathbb{R}^{4n})$ with the property that $f(x_1, \ldots, x_n)$ together with all of its derivatives $D^k f(x_1, \ldots, x_n)$ vanish if $x_i = x_j$ for some $i \neq j$, where $x_j \in \mathbb{R}^4$ for each $1 \leq j \leq n$. We also define the subspace ${}^0\mathscr{S}_+(\mathbb{R}^{4n})$ of ${}^0\mathscr{S}(\mathbb{R}^{4n})$ to be the set of all functions $f \in {}^0\mathscr{S}(\mathbb{R}^{4n})$ such that $f(x)$ and all of its derivatives satisfy a "time-ordering" condition on the half-line; i.e., $D^k f(x) = 0$ for all $k \in I_+^{4n}$ unless $0 < x_1^0 < x_2^0 < \cdots < x_n^0 < \infty$. Equipped with the induced topology from

$\mathscr{S}(\mathbb{R}^{4n})$, ${}^0\mathscr{S}(\mathbb{R}^{4n})$ and ${}^0\mathscr{S}_+(\mathbb{R}^{4n})$ are closed subspaces of $\mathscr{S}(\mathbb{R}^{4n})$ (see [24], Page 86). Lastly, we take ${}^0\underline{\mathscr{S}_+}$ to be the set of all sequences $\underline{f} = (f_0, f_1, f_2, ...)$ where $f_0 \in \mathbb{C}$ and $f_n \in {}^0\mathscr{S}_+(\mathbb{R}^{4n})$, $n = 1, 2, ...$, with the property that all but finitely many f_n's are equal to zero.

The Osterwalder-Schrader axioms given below are taken directly from [24] (see Page 88) and define conditions on a collection of Euclidean Green's/Schwinger functions $\{S_n\}_{n=0}^{\infty}$ that allow the reconstruction of a relativistic quantum theory.

E0: **Temperedness.** $S_0 = 1$ and $S_n(x_1, ..., x_n) \in {}^0\mathscr{S}'(\mathbb{R}^{4n})$, for $n = 1, 2, ...$, where ${}^0\mathscr{S}'(\mathbb{R}^{4n})$ denotes the dual space of ${}^0\mathscr{S}(\mathbb{R}^{4n})$ (see [28], Pages 43 and 134).

E1: **Euclidean Invariance.** The collection of Schwinger functions is assumed to be Euclidean invariant; i.e.,

$$S_n(f) = S_n(f_{(a,R)})$$

for all $R \in SO(4)$, $a \in \mathbb{R}^4$, and $f \in {}^0\mathscr{S}(\mathbb{R}^{4n})$, where

$$f_{(a,R)}(x_1, ..., x_n) = f(Rx_1 + a, ..., Rx_n + a).$$

Here the Euclidean nature of the formalism manifests itself in the use of the special orthogonal group $SO(4)$ to define invariance, as opposed to its counterpart in Minkowski space theory; i.e., the proper orthochronous subgroup of the Lorentz group (see [40], Page 10).

E2: **Reflection Positivity.** For all $\underline{f} \in {}^0\underline{\mathscr{S}_+}$,

$$\sum_{n,m} S_{n+m}\left(\Theta f_n^* \times f_m\right) \geq 0,$$

where

$$(\Theta f)_n(x_1, \dots, x_n) = f_n((-x_1^0, \mathbf{x}_1), \dots, (-x_n^0, \mathbf{x}_n))$$

denotes the "time reversal" operator. This positivity stipulation is used in [24] to construct an inner product structure for the Hilbert space of the field theory, expressed directly in terms of Euclidean variables, without the use of an explicit analytic continuation.[1]

E3: **Symmetry.** For all $f \in {}^0\mathscr{S}(\mathbb{R}^{4n})$ and all permutations π in the permutation group P_n,

$$S_n(f) = S_n(f^\pi),$$

where

$$f^\pi(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)}).$$

E4: **Cluster Property.** For all $\underline{f}, \underline{g} \in {}^0\underline{\mathscr{S}_+}$ and all $a = (0, \mathbf{a})$ in $\mathbb{R}^4$, we have

$$\lim_{\lambda\to\infty} \sum_{n,m} \left[S_{n+m}\left(\Theta f_n^* \times g_{m(\lambda a,1)}\right) - S_n\left(\Theta f_n^*\right) S_n\left(g_m\right)\right] = 0.$$

5.3 The Physical Hilbert Space

As discussed in **Section 5.2**, Hilbert space vectors in the Euclidean framework are represented by sequences of Schwartz test functions of the form

$$f(x) = (f_0, f_1(x_{11}), f_2(x_{21}, x_{22}), \dots)$$

[1] It was originally concluded that the reflection positivity axiom implied that the evenly labeled Schwinger functions, S_{2n}, are distributions of positive type (see [28], Page 331; [29], Page 14). Although these two notions of positivity are analogous, they are not identical. This misstep does not affect the work of this thesis moving forward.

where the individual functions obey the time support restrictions

$$f_n(x_{n1}, x_{n2}, \dots, x_{nn}) = 0 \qquad \text{unless} \qquad 0 < x^0_{n1} < x^0_{n2} < \dots < x^0_{nn} < \infty. \tag{5.2}$$

Sequences of functions $\{f_n\}_{n=1}^{\infty}$ satisfying this condition are called positive relative time functions. This is a linear subspace of the space of sequences of Schwartz test functions in Euclidean variables.

The Euclidean time reflection operator Θ is defined on these sequences via

$$\Theta f(x) = (f_0, (\Theta f)_1(x_{11}), (\Theta f)_2(x_{21}, x_{22}), \dots)$$

where, as above,

$$(\Theta f)_n(x_1, \dots, x_n) = f_n((-x^0_1, \mathbf{x}_1), \dots, (-x^0_n, \mathbf{x}_n))$$

The Hilbert space inner product of f with g is defined by

$$\langle f|g\rangle = \sum_{n,m} S_{n+m}\left(\Theta f^*_n \times g_m\right). \tag{5.3}$$

The reflection positivity axiom (E2) is the condition that this space does not have negative norm states:

$$\langle f|f\rangle \geq 0.$$

Reflection positivity makes (5.3) into a Hilbert space inner product. There are states, $|f\rangle$, with zero norm. The physical Hilbert space, $\mathcal{H}$, is obtained by identifying sequences whose difference has zero norm. This space is made complete by identifying Cauchy sequences with vectors in this space. While the Euclidean time supports must be disjoint, the order in (5.2) does not matter because the Schwinger functions of a local field theory are symmetric (E3), so the arguments can be relabeled so (5.2) is satisfied.

Reflection positivity is not automatic; it is a property of acceptable Schwinger functions. By Euclidean invariance (E1), reflection positivity must hold for any choice of the Euclidean time axis. In addition, without the support restriction, functions that are odd with respect to Euclidean time reflection will have negative norm, so the support condition, or an alternative restriction is necessary for positivity.

In what follows we assume that the collection of Schwinger functions are reflection positive. This condition needs to be verified in models or approximations.

5.4 Spatial Versus Temporal Translations

We consider here the effect the time reversal operator Θ has in determining a unitary representation of the four-dimensional translation group on $\mathcal{H}$. The ideas discussed in this section have direct analogs for determining unitary representations of the boost and rotation operators as well. In **Section 5.5** we will combine these elements to obtain a representation of the Poincaré Lie algebra on $\mathcal{H}$.

For $a = (0, \mathbf{a})$, we define $U_s(a)$ on $\mathcal{H}$ by

$$(U_s(a)f)_n(x_1, \dots, x_n) = f_n(x_1 - a, \dots, x_n - a).$$

Writing

$$(\Theta f)^*_n(x_1, \dots, x_n) = f^*_n((-x^0_1, \mathbf{x}_1 - \mathbf{a} + \mathbf{a}), \dots, (-x^0_n, \mathbf{x}_n - \mathbf{a} + \mathbf{a})),$$

the Euclidean invariance axiom (E1) gives

$$\langle f | U_s(a) g \rangle = \langle U_s(-a) f | g \rangle \, ,$$

so that

$$\langle U_s(a)f|U_s(a)g\rangle = \langle f|g\rangle\,, \qquad \forall\, f,g \in {}^0\underline{\mathscr{S}_+}.$$

That is, $U_s(a)$ defines a unitary representation of the three-dimensional spatial translation group. According to Stone's theorem (see [13], Page 210), the generator of $U_s(a)$ is a densely defined self-adjoint operator on $\mathscr{H}$. Commutativity of the one-dimensional translation operators $U_s((0,a_1,0,0)), U_s((0,0,a_2,0)), U_s((0,0,0,a_3))$ implies that the translation operator $U_s(a)$ can be written as

$$U_s(a) = e^{-i\mathbf{P}\cdot\mathbf{a}}$$

(see [45], Page 239; [14], Page 29; [46], Page 171). Applying the multivariable form of Taylor's theorem (see [46], Page 6) to the infinitesimal translation $U_s(da)$, where $da = (0, da_1, da_2, da_3)$, we obtain

$$\mathbf{P} f_n(x_1, \dots, x_n) = -i \sum_{k=1}^{n} \frac{\partial}{\partial \mathbf{x}_k} f_n(x_1, \dots, x_n). \tag{5.4}$$

Now, for $t \geq 0$, define T^t on ${}^0\underline{\mathscr{S}_+}$ by

$$(T^t f)_n(x_1, \dots, x_n) = f_n((x_1^0 - t, \mathbf{x}_1), \dots, (x_n^0 - t, \mathbf{x}_n)). \tag{5.5}$$

We remark that the translation operator T^t preserves the positive relative time support condition characteristic of the elements of ${}^0\underline{\mathscr{S}_+}$.

Typically in non-relativistic quantum mechanics, one considers time translations $t' \mapsto t' + t$ and runs a completely analogous argument to the one given above for spatial translations to determine an expression for the self-adjoint generator for these transformations, known to physicists as the Hamiltonian. Doing so one finds that the Hamiltonian has the same form as those for the spatial translations, save for the sign change $-i \mapsto i$ (see [45], Pages 94 and 157). Given our definition of T^t in (5.5), one might naïvely expect, then, that the Hamiltonian operator and $\mathbf{P}$ will have the same form in the framework we are using here.

A moment's thought, however, shows that this is not the case because the time reversal operator Θ present in the definition of the inner product on $\mathscr{H}$ spoils the unitarity of the operator T^t. Similar to what was done above, we write

$$(\Theta f)_n^*(x_1, \ldots, x_n) = f_n^*((-x_1^0 + t - t, \mathbf{x}_1), \ldots, (-x_n^0 + t - t, \mathbf{x}_n)),$$

the Euclidean invariance axiom (E1) now gives

$$\langle T^t f | g \rangle = \langle f | T^t g \rangle \qquad \forall\, f, g \in {}^0\underline{\mathscr{S}_+}.$$

In other words, the family $\{T^t\}_{t \geq 0}$ is a one parameter semigroup of symmetric operators on ${}^0\underline{\mathscr{S}_+}$ and the loss of unitarity is owed completely to the time reversal operator Θ.

Nevertheless, we can recover a one parameter group of unitary operators whose generator is a positive self-adjoint operator. One might venture to guess that this can be accomplished by considering the transformations T^{it} instead. This is indeed the case; however, we cannot justify the unitarity of T^{it} by appealing to the Euclidean invariance of the Schwinger functions (E1) – as we did in the case of spatial translations – because $(it, 0, 0, 0,) \notin \mathbb{R}^4$.

A more rigorous approach to the problem goes as follows. According to the regularity theorem for tempered distributions (see [28], Page 139; [40], Pages 34 - 35), each of the Schwinger functions can be expressed as some number of weak derivatives of a polynomially bounded continuous function. That is to say, for each S_n, there is $k \in I_+^{4n}$, a positive integer $j \in \mathbb{N}$, and a continuous function $g(x)$ on $\mathbb{R}^{4n}$, all depending on S_n, which together satisfy

$$|g(x)| \leq C(1 + |x|^j) \tag{5.6}$$

and

$$S_n(f) = \int_{\mathbb{R}^{4n}} (-1)^{|k|} g(x) \left(D^k f\right)(x)\, d^{4n}x, \qquad \forall f \in {}^0\mathscr{S}(\mathbb{R}^{4n}).$$

Applying this result, we find that for any $f \in {}^0\underline{\mathscr{S}_+}$

$$\left|\sum_{n,m} S_{n+m}\left(\Theta f_n^* \times T^t f_m\right)\right| = \left|\sum_{n,m} \int_{\mathbb{R}^{4n+4m}} (-1)^{|k(n,m)|} g_{(n,m)}(x_1, \dots, x_n, y_1, \dots, y_m) \right.$$
$$\left. \times \left(D^{k(n,m)} \Theta f_n^* \times T^t f_m\right)(x,y)\, d^{4n}x\, d^{4m}y\right|,$$

where we have introduced notation for the explicit dependence of k, j, and $g(x)$ above on the indices n and m labeling S_{n+m}. Using the bound (5.6) and $t \geq 0$, we obtain

$$\left|\sum_{n,m} S_{n+m}\left(\Theta f_n^* \times T^t f_m\right)\right| \leq \sum_{n,m} \int_{\mathbb{R}^{4n+4m}} C_{(n,m)} \left(1 + (|x|_{\mathbb{R}^{4n}} + |y|_{\mathbb{R}^{4m}} + t)^{j(n,m)}\right)$$
$$\times \left|\left(D^{k(n,m)} \Theta f_n^* \times f_m\right)(x,y)\right| d^{4n}x\, d^{4m}y.$$

By definition of $f \in {}^0\underline{\mathscr{S}_+}$, only finitely many non-zero terms appear in the above sums. Hence, upon integrating over the x and y variables, we have

$$|\langle f | T^t f \rangle| \leq P(t) \tag{5.7}$$

for some polynomial $P(t)$, which depends on our chosen f (see [24], Page 91). Using the Cauchy-Schwarz inequality in concert with the symmetric nature of T^t, we can iterate the bound in (5.7) to obtain

$$\begin{aligned}
|\langle f|T^t f\rangle| &\leq \|f\| \, \|T^t f\| \\
&= \|f\| \, \left\langle f \middle| T^{2t} f \right\rangle^{\frac{1}{2}} \\
&\leq \|f\|^{\sum_{k=0}^{n-1} 2^{-k}} \left\langle f \middle| T^{2^n t} f \right\rangle^{2^{-n}} \\
&\leq \|f\|^{\sum_{k=0}^{n-1} 2^{-k}} \left(P(2^n t)\right)^{2^{-n}}, \qquad \forall n \in \mathbb{N}.
\end{aligned}$$

Taking the limit as $n \to \infty$ we have (see [24], Page 92)

$$|\langle f|T^t f\rangle| \leq \|f\|^2.$$

Hence, we see that $\{T^t\}_{t\geq 0}$ defines a contractive symmetric local semigroup. It now follows from arguments made in a masterfully written paper of A. Klein and L. J. Landau that there is a positive self-adjoint generator $H \geq 0$ such that $T^t = e^{-tH}$ (see [20], Pages 124 & 128), where we note that the spectral condition $H \geq 0$ characterizes the physical requirement that the energy of our system be bounded below. According to the functional calculus for self-adjoint operators, defining $T^{it} = e^{itH}$ gives rise to a strongly continuous one-parameter unitary group $\{T^{it}\}_{t\in\mathbb{R}}$ on $\mathscr{H}$ (see [13], Pages 208 - 209). Applying the multivariable form of Taylor's theorem (see [46], Page 6) to the infinitesimal translation T^{idt}, we obtain

$$Hf_n(x_1, \dots, x_n) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k^0} f_n(x_1, \dots, x_n)$$

and we note that the factor of i present in the momentum operator $\mathbf{P}$ of (5.4) does not appear in the representation of the Hamiltonian operator H.

5.5 The Poincaré Lie Algebra

As we've seen in **Section 5.4**, the time reflection operator Θ breaks the Euclidean invariance of (5.3):

$$\langle f|g\rangle = \sum_{n,m} S_{n+m}\left(\Theta f_n^* \times g_m\right).$$

As a result the group of real Euclidean transformations on sequences of Euclidean test functions

$$f_n(x_1, \ldots, x_n) \to f'_n(x_1, \ldots, x_n) = f_n(Rx_1 - a, \ldots, Rx_n - a),$$

where $R \in O(4)$ and $a = (t, \mathbf{a})$ is a Euclidean four vector, becomes a subgroup of the complex Poincaré group with respect to the inner product (5.3). This is because the covering groups of the Lorentz group and $O(4)$ have the same analytic continuation.

The only technical issue is that these complex transformations do not generally preserve the positive relative time condition; however, it is preserved by Euclidean time translations when $t \geq 0$. Similarly, rotations in Euclidean space-time planes are defined for small angles on a subspace of positive relative-time functions with support on wedge shaped regions $x_k^0 > b|\mathbf{x}_k|$, where b is a positive constant.

In the case of Euclidean time translations, these transformations form contractive Hermetian semigroups on the physical Hilbert space and can be extended to imaginary-time time translations, giving rise to a unitary representation of the four-dimensional translation group on $\mathcal{H}$.

Similar to the case of spacial translations, time reversal is a non-factor in defining a unitary representation of the pure rotational group. Methods familiar to the reader from the theory of angular momentum in quantum mechanics (e.g., see [45], Page 262) yield the following representation for the generators of the pure rotational group:

$$\mathbf{J} f_n(x_1, \ldots, x_n) = -i \sum_{k=1}^{n} \left(\mathbf{x}_k \times \frac{\partial}{\partial \mathbf{x}_k} \right) f_n(x_1, \ldots, x_n).$$

Rotationless boosts, however, involve temporal components in their transformation rules and, thus, must be handled in a similar fashion to the time translation case. Our work in this direction in **Section 5.4** carries over completely, save for the fact that the local symmetric semigroups associated to boosts directed along the various spatial axes do not admit the contractive property that the family $\{T^t\}_{t \geq 0}$ of time translation operators did. The only change produced by this observation is that the generators of these local symmetric semigroups no longer admit a positive spectral condition (see [20], Pages 124 & 128), as was the case for $H \geq 0$. The important point is that in both cases the generators of contractive and non-contractive local symmetric semigroups are self-adjoint (see [20], Page 124).

Similar to imaginary-time time translations, rotationless Lorentz boosts are associated with imaginary rapidity. For example, upon suppressing the x_k^2 and x_k^3 labels for $1 \leq k \leq n$, an infinitesimal boost $d\zeta$ in the "first" spatial direction takes the form

$$[1 + iK_1 d\zeta]\, f_n(x_1, \ldots, x_n) = f_n\Big((x_1^0 + i x_1^1 d\zeta,\, x_1^1 - i x_1^0 d\zeta, \ldots), \ldots, (x_n^0 + i x_n^1 d\zeta,\, x_n^1 - i x_n^0 d\zeta, \ldots) \Big).$$

Taylor expanding the right side about $(x_1, \ldots, x_n) \in \mathbb{R}^{4n}$, we have

$$[1 + iK_1 d\zeta]\, f_n(x_1, \ldots, x_n) = f_n(x_1, \ldots, x_n) + i d\zeta \sum_{k=1}^{n} \left(x_k^1 \frac{\partial}{\partial x_k^0} - x_k^0 \frac{\partial}{\partial x_k^1} \right) f_n(x_1, \ldots, x_n),$$

so that

$$K_1 f_n(x_1, \dots, x_n) = \sum_{k=1}^{n} \left(x_k^1 \frac{\partial}{\partial x_k^0} - x_k^0 \frac{\partial}{\partial x_k^1} \right) f_n(x_1, \dots, x_n).$$

Generalizing this calculation to infinitesimal rotationless boosts of imaginary rapidity along the remaining spatial directions we have

$$\mathbf{K} f_n(x_1, \dots, x_n) = \sum_{k=1}^{n} \left(\mathbf{x}_k \frac{\partial}{\partial x_k^0} - x_k^0 \frac{\partial}{\partial \mathbf{x}_k} \right) f_n(x_1, \dots, x_n).$$

Summarizing our results, the infinitesimal generators H and $\mathbf{K}$ of time translations and rotationless boosts, along with the generators $\mathbf{J}$ and $\mathbf{P}$ of rotations and translations $\mathbf{P}$ have the following representations as differential operators on $\mathscr{H}$:

$$\begin{aligned}
H f_n(x_1, \dots, x_n) &= \sum_{k=1}^{n} \frac{\partial}{\partial x_k^0} f_n(x_1, \dots, x_n), \\
\mathbf{P} f_n(x_1, \dots, x_n) &= -i \sum_{k=1}^{n} \frac{\partial}{\partial \mathbf{x}_k} f_n(x_1, \dots, x_n), \\
\mathbf{J} f_n(x_1, \dots, x_n) &= -i \sum_{k=1}^{n} \left(\mathbf{x}_k \times \frac{\partial}{\partial \mathbf{x}_k} \right) f_n(x_1, \dots, x_n), \\
\mathbf{K} f_n(x_1, \dots, x_n) &= \sum_{k=1}^{n} \left(\mathbf{x}_k \frac{\partial}{\partial x_k^0} - x_k^0 \frac{\partial}{\partial \mathbf{x}_k} \right) f_n(x_1, \dots, x_n).
\end{aligned}$$

Direct computation shows that these operators satisfy the commutation relations of the Poincaré Lie algebra (see [34], Page 84; [47], Page 61):

$$\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k, & [J_i, P_j] &= i\epsilon_{ijk} P_k, \\
[J_i, K_j] &= i\epsilon_{ijk} K_k, & [K_i, P_j] &= i\delta_{ij} H, \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k, & [K_i, H] &= -iP_i, \\
[J_i, H] &= 0, & [P_i, H] &= 0,
\end{aligned}$$

where ϵ_{ijk} is the three-dimensional Levi-Civita symbol and δ_{ij} the Kronecker delta function.

As generators of strongly continuous one-parameter unitary groups on $\mathcal{H}$, Hermiticity of $\mathbf{P}$ and $\mathbf{J}$ follows from Stone's theorem (see [13], Page 210). In the case of $H \geq 0$ and $\mathbf{K}$, Hermiticity is a consequence of their relationship to (contractive/non-contractive) local symmetric semigroups (see [20], Pages 124 & 128). Also, we again note that the usual factor of i is missing from both H and $\mathbf{K}$ because they are linear in x_k^0 or $\dfrac{\partial}{\partial x_k^0}$, so the usual sign change from complex conjugation is replaced by the Euclidean time reflection.

In the Euclidean framework the Poincaré generators are simple differential operators; the dynamical information is contained in the Schwinger functions. A collection of reflection positive Schwinger functions and the expressions for the Poincaré generators define the relativistic quantum theory.

5.6 Two-Point Functions

The two-point Schwinger function, $S_2(x, y)$, will be used in this section to illustrate the relation of the Euclidean representation of a relativistic quantum theory defined above to the conventional treatment of a relativistic particle. Furthermore, the structure of the two-point function is also used to formulate the scattering asymptotic condition in the Euclidean framework. Note here the change in notation: $x_1 \mapsto x$ and $x_2 \mapsto y$.

Translational invariance of the Schwinger functions (E1) allows us to write $S_2(x, y) = S_2(x - y)$ (see [24], Page 90). The structure of $S_2(x, y)$ is given by its so-called Lehmann representation, which has the form (see [29], Page 70)

$$S_2(x-y) = \frac{1}{(2\pi)^4} \int \frac{e^{ip\cdot(x-y)}}{p^2+m^2}\,\rho(m)\,dm\,d^4p, \tag{5.8}$$

where $\rho(m)$, known as the Lehmann weight, has support on the mass spectrum of the states that have non-zero matrix elements with $\phi(x)\,|0\rangle$, where $\phi(x)$ is the Minkowski Heisenberg field and $|0\rangle$ is the vacuum. The spectral condition requires that the support of $\rho(m)$ be contained in the half-line $[0,\infty)$. According to the regularity theorem for tempered distributions (see [40], Page 34), the measure $\rho(m)\,dm$ is polynomially bounded; i.e.,

$$\int_{-D}^{D} \rho(m)\,dm = \int_{0}^{D} \rho(m)\,dm \le C(1+D^n),$$

for some $C>0$, $n\in\mathbb{N}$, and all $D>0$, where we have used the support condition on $\rho(m)$ to justify the equality.

Using this general two-point Schwinger function to compute the inner product of vectors $f,g\in\mathscr{H}$ represented by a single function – i.e., $f(x)=(f_1(x_{11}),0,0,...)$ and $g(y)=(g_1(y_{11}),0,0,...)$, where we "equate" $f(x)=f_1(x_{11})$ and $g(y)=g_1(y_{11})$ for notational convenience – with positive-time support gives

$$\begin{aligned}
\langle f|g\rangle &= \sum_{n,m} S_{n+m}\left(\Theta f_n^* \times g_m\right) \\
&= S_2\left(\Theta f^* \times g\right) \\
&= \frac{1}{(2\pi)^4}\int \Theta f^*(x)\frac{e^{ip\cdot(x-y)}}{p^2+m^2}\,g(y)\rho(m)\,dm\,d^4p\,d^4x\,d^4y \\
&= \frac{1}{(2\pi)^4}\int f^*(x)\frac{e^{-ip^0(x^0+y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{(p^0)^2+\mathbf{p}^2+m^2}\,g(y)\rho(m)\,dm\,d^4p\,d^4x\,d^4y,
\end{aligned} \tag{5.9}$$

where the final equality is a result of the change of variables $x^0\mapsto -x^0$. Using a semicircular contour in the lower p^0 half-plane and the residue theorem (see [38], Page 76) to calculate the p^0 integral, we find

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-ip^0(x^0+y^0)}}{(p^0)^2+\mathbf{p}^2+m^2}\,dp^0=\frac{e^{-\sqrt{\mathbf{p}^2+m^2}(x^0+y^0)}}{\sqrt{\mathbf{p}^2+m^2}},$$

where $\omega_m(\mathbf{p})=\sqrt{\mathbf{p}^2+m^2}$ is the energy of a particle with mass m and momentum $\mathbf{p}$ (see [10], Page 59; [44], Page 91). Using this result in (5.9), we obtain

$$\begin{aligned}\langle f|g\rangle &= \frac{1}{(2\pi)^3}\int f^*(x)\frac{e^{-\omega_m(\mathbf{p})(x^0+y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{2\omega_m(\mathbf{p})}\,g(y)\rho(m)\,dm\,d^3\mathbf{p}\,d^4x\,d^4y\\ &= \int\frac{\chi^*(\mathbf{p})\psi(\mathbf{p})}{2\omega_m(\mathbf{p})}\,\rho(m)\,dm\,d^3\mathbf{p},\end{aligned}\tag{5.10}$$

where $\chi(\mathbf{p})$ and $\psi(\mathbf{p})$ are the momentum-space wave functions defined by

$$\begin{aligned}\chi(\mathbf{p}) &= \frac{1}{(2\pi)^{3/2}}\int f(x^0,\mathbf{x})e^{-\omega_m(\mathbf{p})x^0-i\mathbf{p}\cdot\mathbf{x}}\,d^4x,\\ \psi(\mathbf{p}) &= \frac{1}{(2\pi)^{3/2}}\int g(y^0,\mathbf{y})e^{-\omega_m(\mathbf{p})y^0-i\mathbf{p}\cdot\mathbf{y}}\,d^4y.\end{aligned}$$

The support condition on the Euclidean times of $f(x)$ and $g(y)$ allow for the calculation of the integrals in (5.10). Moreover, we list several other important observations about this expression.

1. If $f(x)=g(x)$, then $\langle f|f\rangle\geq 0$ in (5.10). Hence, the Lehmann representation (5.8) of two-point Schwinger functions is consistent with the reflection positivity axiom (E2).

2. Equation (5.10) has the form of an ordinary Lorentz invariant inner product with Lorentz invariant measure $d^3\mathbf{p}/2\omega_m(\mathbf{p})$ (see [29], Pages 70 & 74). This shows how the physical Hilbert space inner product emerges from this Euclidean quadrature without analytic continuation.

3. The origin of the mass dependence, which provides the dynamical relation between energy and momentum via $\omega_m(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$, emanates from the Lehmann weight $\rho(m)$ in the two-point Schwinger function (5.8).

4. For $c > 0$, the distribution $f(x) = \delta(x^0 - c)\tilde{f}(\mathbf{x})$ is square integrable with respects to the inner product (5.10) when $\tilde{f}(\mathbf{x})$ is a Schwartz test function in three variables. This is due to the non-trivial kernel in the scalar product. Indeed, using $(\Theta f)^*(x) = \delta(-x^0 - c)\tilde{f}^*(\mathbf{x})$, calculations similar to those establishing (5.10) show that for such $f(x)$:

$$\langle f|f\rangle = \int |\phi(\mathbf{p})|^2 \frac{e^{-2\omega_m(\mathbf{p})}}{2\omega_m(\mathbf{p})} \rho(m) dm\, d^3\mathbf{p} < \infty,$$

where

$$\phi(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \int \tilde{f}(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{x}} d^3\mathbf{x}$$

is the three-dimensional Fourier transform of $\tilde{f}(\mathbf{x})$ (see [40], Page 43). The finiteness of $\langle f|f\rangle$ in this case is a result of the exponential damping of the polynomially bounded weight $\rho(m)$, together with the fact that the Fourier transform (in any dimension) is an isomorphism on the space of Schwartz test functions (see [40], Page 44); i.e., $\phi(\mathbf{p})$ is a Schwartz test function, as defined in **Section 5.2**. This means that the Hilbert space of normalizable vectors includes expressions with delta functions in Euclidean time. This observation has useful computational consequences which will be used in **Section 5.8**.

5. The mass square Casimir operator is the four-dimensional Euclidean Laplacian:

$$M^2 = \nabla^2_{\text{Euclidean}} = \left(\frac{\partial}{\partial x^0}\right)^2 + \boldsymbol{\nabla}^2_{\mathbf{x}}. \tag{5.11}$$

6. $\rho(m) = \delta(m - m_0)$ is the Lehmann weight for a free particle of mass m_0 (see [26], Pages 22 - 24; [43], Page 129).

The Lehmann weight $\rho(m)$ has the general structure

$$\rho(m) = \sum_{k=1}^{N} c_k \delta(m - m_k) + \rho_{ac}(m), \tag{5.12}$$

where

$$0 < m_1 < m_2 < \cdots < m_N < \text{support}(\rho_{ac}).$$

The support of the continuous part of the Lehmann weight, $\rho_{ac}(m)$, which is associated with multiparticle states, is not bounded from above and it is this fact that will require us to make use of our work from Chapters 1-4 on the Stieltjes moment problem when constructing the injection operator $J : \mathscr{H}_0 \to \mathscr{H}$. In what's to follow, we will make explicit use of the polynomial boundedness property of $\rho(m)$. Moreover, we also assume the support of the Lehmann weight includes discrete masses.

5.7 Scattering Asymptotic Conditions

As mentioned in the introduction to this chapter, a two-Hilbert space framework will be used to formulate the scattering asymptotic condition. Superb references on the matter can be found in [7], [8], [30], and [50]. Detectors respond to a particle's mass, spin, linear momentum, and spin polarization. The internal structure of the particle is of no consequence to the detector. The two-Hilbert space formulation of scattering separates the degrees of freedom that are measured asymptotically, which are modeled on $\mathscr{H}_0$, from the internal degrees of freedom, which are modeled on $\mathscr{H}$.

An n_α-particle scattering state in scattering channel α asymptotically looks like a direct product of wave packets in each particle's momentum and magnetic quantum number. These n_α-particle states span a channel subspace defined by

$$\mathcal{H}_\alpha = \bigotimes_{i=1}^{n_\alpha} \mathcal{H}_{m_i j_i}$$

where $\mathcal{H}_{m_i j_i}$ is a mass $m_i > 0$ spin j_i irreducible representation space of the Poincaré group, associated with mass and spin of each of the n_α asymptotic particles in the scattering channel α. The functions in $\mathcal{H}_{m_i j_i}$ are square integrable functions of the three-momentum and magnetic quantum number of a particle of mass m_i and spin j_i.

A channel α injection operator, J_α,

$$J_\alpha : \mathcal{H}_\alpha \to \mathcal{H}$$

is a mapping from the asymptotic channel α Hilbert space to the full Hilbert space. This operator combines the internal structure degrees of freedom for each asymptotically separated particle in the channel α with their momentum and spin degrees of freedom. The purpose of this and subsequent sections is to discuss the construction of channel injection operators in the Euclidean representation presented above.

The asymptotic Hilbert space, $\mathcal{H}_0$, is defined as the direct sum of the channel subspaces over the set of scattering channels $\mathcal{A}$,

$$\mathcal{H}_0 = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{H}_\alpha$$

and the two-Hilbert space injection operator, J, as the sum of all of the channel injection operators

$$J = \sum_{\alpha \in \mathcal{A}} J_\alpha .$$

While the channel sum includes an infinite number of channels, initial scattering states are vectors in a single channel subspace, $\mathcal{H}_\alpha$, and if the energy of the initial state is bounded, the number of open final channels is finite, so the infinite sum does not lead to convergence problems.

There is a natural unitary representation $U_0(\Lambda, a)$ of the Poincaré group on the asymptotic Hilbert space, which is given by the direct sum of tensor products of unitary irreducible representations:

$$U_0(\Lambda, a) = \bigoplus_{\alpha \in \mathcal{A}} \left(\bigotimes_{i \in \alpha} D^{m_i j_i}(\Lambda, a) \right)$$

where the $D^{m_i j_i}(\Lambda, a)$ are unitary irreducible representations of the Poincaré group for a particle of mass m_i and spin j_i:

$$D^{m_i j_i}(\Lambda, a) = \langle (m_i, j_i)\mathbf{p}_i, \mu_i | U_i(\Lambda, a) | (m_i, j_i)\mathbf{p}'_i, \mu'_i \rangle$$

where μ_i is the magnetic quantum number; i.e. the projection of a spin (canonical, light front, or helicity) on an axis. These representations are known analytically [19]. In the asymptotic Hilbert space there is no distinction between elementary and composite particles. These distinctions appear in the injection operator.

A scattering state is a solution of the Schrödinger equation that evolves into a state that asymptotically looks like a collection of asymptotically separated free particles. In the two-Hilbert space formalism the asymptotic condition has the form

$$\lim_{t \to \pm\infty} \| U(I, t) \, |\psi_\pm\rangle - J U_0(I, t) \, |\psi_A\rangle \, \| = 0, \tag{5.13}$$

with $|\psi_\pm\rangle \in \mathcal{H}$, $|\psi_A\rangle \in \mathcal{H}_0$, and where $U(I, t)$ and $U_0(I, t)$ are the time evolution operators on $\mathcal{H}$ and $\mathcal{H}_0$, respectively. Using the unitarity of $U(I, t)$ this can be expressed as

$$\lim_{t\to\pm\infty} \| \, |\psi_\pm\rangle - U(I,-t)JU_0(I,t)\,|\psi_A\rangle \, \| = 0, \tag{5.14}$$

where $|\psi_A\rangle$ is a normalizable state in the asymptotic Hilbert space $\mathcal{H}_0$.

The existence of this limit depends on properties of time evolution subgroup of the Poincaré group, $U(I,t)$, and the choice of injection operator, J. A sufficient provision for the convergence of this limit is the Cook condition (see [9]; [30], Page 20; [50], Page 84), which in the two-Hilbert space framework is

$$\int_a^{\pm\infty} \|(HJ - JH_0)U_0(I,t)\,|\psi_A\rangle\,\|\,dt < \infty, \tag{5.15}$$

with $a \in \mathbb{R}$ finite, and where H and H_0 are the self-adjoint generators of $U(I,t)$ and $U_0(I,t)$, respectively. For this to hold, $(HJ - JH_0)$ needs to be a short-ranged operator. Since $|\psi_A\rangle$ is normalizable, this asymptotic condition can be verified one channel at a time; i.e.,

$$\int_a^{\pm\infty} \|(HJ_\alpha - J_\alpha H_\alpha)U_\alpha(I,t)\,|\psi_\alpha\rangle\,\|dt < \infty, \tag{5.16}$$

where $|\psi_\alpha\rangle \in \mathcal{H}_\alpha$ is the α-component of $|\psi_A\rangle$.

The discussion above applies to any representation of a quantum theory. In particular, Cook's method can be brought to bear in the Euclidean representation defined in **Section 5.3**, whose inner product structure was given in (5.3) by

$$\langle f|g\rangle = \sum_{n,m} S_{n+m}\left(\Theta f_n^* \times g_m\right). \tag{5.17}$$

5.8 Injection Operators and One-Body Solutions

In this section we discuss the construction of injection operators that can be used to formulate the scattering asymptotic condition (5.13). The basic strategy employed will be a Euclidean reformulation of Haag-Ruelle scattering theory (see [30], Page 317).

One difference between quantum theories of a fixed (finite) number of particles and quantum field theory is that, while the field theory Hamiltonian has one-body eigenstates, there are no states corresponding to N *free* particles. The absence of a subspace corresponding to N free particles impacts the formulation of the scattering asymptotic condition. The important observation is that the asymptotic condition defines the initial data for the Schrödinger equation at a time when the particles are asymptotically separated. In particular, the asymptotic condition requires that at this time the solution looks like another state of N asymptotically separated particles. These other states provide labels for different N-particle scattering solutions of the Schrödinger equation. What these other states look like when the particles are not asymptotically separated is irrelevant to the formulation of this initial condition, however the choice of these states impact the labels that distinguish different scattering solutions.

To formulate a suitable asymptotic condition in the field theory case Haag and Ruelle construct localized field operators that create single-particle states out of the vacuum. For scalar particles these quasi-local field operators have the form

$$\phi_h(x) = \frac{1}{(2\pi)^2} \int h(-p^2) e^{ip\cdot(x-y)} \phi(y) d^4p d^4y,$$

where $h(-p^2)$ is a smooth function that is 1 when $p^2 = -m^2$ is the mass of the asymptotic particle, and vanishes on the rest of the support of the Lehmann weight of the two-point function. They then isolate the part of $\phi_h(x)$ that asymptotically behaves like a creation operator for a particle of mass m:

$$a^\dagger(f) = i \int \phi_h(x) \overset{\leftrightarrow}{\partial}_t f(x) d^3x,$$

where $f(x)$ is a positive-energy solution of the Klein-Gordon equation for a particle of mass m. When applied to the vacuum this operator creates a time-independent single-particle state out of the vacuum. Products of these operators with different positive-energy Klein-Gordon solutions, $f_i(x_i)$, are time dependent and represent

$$U(I,-t)J_\alpha U_\alpha(I,t) \left|\psi_\alpha\right\rangle$$

in (5.14) (see [30]). Ruelle demonstrated the existence of the strong limits ($t \to \pm\infty$) using locality and the assumed existence of a mass gap. The existence of the strong limits in the Haag-Ruelle case is a result of using the quasi-local fields $\phi_h(x)$, rather than local interpolating fields $\phi(x)$ used in LSZ theory, to formulate the scattering asymptotic condition. Ruelle was able to show that the integrand in (5.15) asymptotically behaves like $t^{-3(n-1)/2}$ (see [18]), which is sufficient to satisfy the Cook condition.

In order to illustrate the corresponding construction in the Euclidean representation we focus on the Cook condition for elastic two-particle scattering. This can be formulated in terms of the four-point Schwinger function, $S_4(x_1, x_2, y_1, y_2)$.

In order to support a scattering theory the four-point Schwinger function should have a cluster expansion of the form:

$$S_4(x_1, x_2, y_1, y_2) = S_2(x_1, y_1)S_2(x_2, y_2) + S_2(x_1, y_2)S_2(x_2, y_1) + S_c(x_1, x_2, y_1, y_2),$$

where $S_c(x_1, x_2, y_1, y_2)$ is translationally invariant, but otherwise connected (see [47], Page 178). The two-point functions $S_2(x_i, y_j)$ have a standard Lehmann representations of the form (5.8). In the physically interesting case the Lehmann weight, $\rho(m)$, in $S_2(x, y)$ has both discrete and continuous parts; i.e., as in (5.12),

$$\rho(m) = \sum_{k=1}^{N} c_k \delta(m - m_k) + \rho_{ac}(m).$$

For this illustration the Lehmann weight is assumed to have at least two distinct discrete masses, m_1 and m_2.

Equation (5.13) has two distinct contributions; one from the connected part,

$$S_c(x_1, x_2, y_1, y_2),$$

of the four-point function and the other from the disconnected parts,

$$S_2(x_1, y_1)S_2(x_2, y_2) \qquad \text{and} \qquad S_2(x_1, y_2)S_2(x_2, y_1).$$

The quantities that appear in (5.13) are:

$$H_\alpha = \omega_{m_1}(\mathbf{p}_1) + \omega_{m_2}(\mathbf{p}_2), \qquad \omega_{m_i}(p_i) = \sqrt{\mathbf{p}_i^2 + m_i^2},$$
$$U_\alpha(I, t) = e^{-i(\omega_{m_1}(\mathbf{p}_1) + \omega_{m_2}(\mathbf{p}_2))t}, \qquad H = \frac{\partial}{\partial x_1^0} + \frac{\partial}{\partial x_2^0}.$$

In this example the channel α is assocated with asymptotic two-particle states with particles of mass m_1 and mass m_2.

To verify the Cook condition of (5.16), we insert into the expression

$$\|(HJ_\alpha - J_\alpha H_\alpha)U_\alpha(I, t)\,|\psi_\alpha\rangle\,\|^2 = \langle\psi_\alpha|U_\alpha(I, -t)(J_\alpha^\dagger H - H_\alpha J_\alpha^\dagger)(HJ_\alpha - J_\alpha H_\alpha)U_\alpha(I, t)|\psi_\alpha\rangle$$

complete sets of states labeled by $|\mathbf{p}_1, \mathbf{p}_2\rangle$, $|\mathbf{p}_1', \mathbf{p}_2'\rangle$, $|x_1, x_2\rangle$, and $|y_1, y_2\rangle$. Doing so, (5.17) gives

$$\|(HJ_\alpha - J_\alpha H_\alpha)U_\alpha(I,t)\,|\psi_\alpha\rangle\,\|^2 =$$

$$\int f_1^*(\mathbf{p}_1) f_2^*(\mathbf{p}_2) e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t} \left(\frac{\partial}{\partial x_1^0} + \frac{\partial}{\partial x_2^0} - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2)\right) \langle \mathbf{p}_1, \mathbf{p}_2 | J_\alpha^\dagger | x_1, x_2\rangle \times$$

$$(S_2(\theta x_1, y_1) S_2(\theta x_2, y_2) + S_2(\theta x_1, y_2) S_2(\theta x_2, y_1) + S_c(\theta x_1, \theta x_2, y_1, y_2)) \times$$

$$\left(\frac{\partial}{\partial y_1^0} + \frac{\partial}{\partial y_2^0} - \omega_{m_1}(\mathbf{p}_1') - \omega_{m_2}(\mathbf{p}_2')\right) \langle y_1, y_2 | J_\alpha | \mathbf{p}_1', \mathbf{p}_2'\rangle \times$$

$$e^{-i(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))t} f_1(\mathbf{p}_1') f_2(\mathbf{p}_2')\, d^3\mathbf{p}_1\, d^3\mathbf{p}_2\, d^3\mathbf{p}_1'\, d^3\mathbf{p}_2'\, d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2, \quad (5.18)$$

where we represent $\langle \mathbf{p}_1, \mathbf{p}_2 | \psi_\alpha\rangle = f_1(\mathbf{p}_1) f_2(\mathbf{p}_2)$ by a product of narrow wave packets, $f_i(\mathbf{p}_i)$, in the three momentum of each asymptotic particle in channel α and where the matrix elements of the injection operator J_α have the general structure

$$\langle x_1, x_2 | J_\alpha | \mathbf{p}_1, \mathbf{p}_2\rangle\,.$$

We will demonstrate that for a suitable choice of injection operator J_α, the terms in (5.18) containing products of two-point functions vanish. When this is true the entire contribution to (5.18) comes from the connected part of the four-point function. To show this we take advantage of the observation that a delta function in the Euclidean time variable multiplied by a Schwartz test function in space variables is a normalizable vector in the Euclidean representation of the Hilbert space, as discussed on Page 66 of this work.

We consider an injection operator J_α of the form

$$\langle x_1, x_2 | J_\alpha | \mathbf{p}_1, \mathbf{p}_2\rangle = h_1\left(\nabla_1^2\right) h_2\left(\nabla_2^2\right) \delta(x_1^0 - t_1)\delta(x_2^0 - t_2) \frac{1}{(2\pi)^3} e^{i\mathbf{p}_1\cdot\mathbf{x}_1 + i\mathbf{p}_2\cdot\mathbf{x}_2}, \quad (5.19)$$

where $h_i(m^2)$ is a smooth bump function that is 1 when m^2 is the square of the asymptotic particles mass and 0 on the rest the support of the Lehmann weight (see [46], Page 140).

A few remarks are in order. First, the Euclidean time support condition can be satisfied by choosing $0 < t_1 < t_2$. The non-trivial constraint on the $h_i(\nabla_i^2)$ is that these operators map functions with positive Euclidean time support to functions with positive Euclidean time support. The construction of functions with this property will be discussed in the next section. For our current purpose, we simply assume that the chosen $h_i(\nabla_i^2)$ preserves the support condition.

With this choice for $\langle x_1, x_2|J_\alpha|\mathbf{p}_1, \mathbf{p}_2\rangle$ the contribution to (5.18) from the disconnected term $S_2(\theta x_1, y_1)S_2(\theta x_2, y_2)$ becomes

$$\int f_1^*(\mathbf{p}_1)f_2^*(\mathbf{p}_2)e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t}\left(\frac{\partial}{\partial x_1^0}+\frac{\partial}{\partial x_2^0}-\omega_{m_1}(\mathbf{p}_1)-\omega_{m_2}(\mathbf{p}_2)\right)\times$$

$$\langle\mathbf{p}_1, \mathbf{p}_2|J_\alpha^\dagger|x_1, x_2\rangle\, S_2(\theta x_1, y_1)S_2(\theta x_2, y_2)\times$$

$$\left(\frac{\partial}{\partial y_1^0}+\frac{\partial}{\partial y_2^0}-\omega_{m_1}(\mathbf{p}_1')-\omega_{m_2}(\mathbf{p}_2')\right)\langle y_1, y_2|J_\alpha|\mathbf{p}_1', \mathbf{p}_2'\rangle\times$$

$$e^{-i(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))t}f_1(\mathbf{p}_1')f_2(\mathbf{p}_2')\,d^3\mathbf{p}_1\,d^3\mathbf{p}_2\,d^3\mathbf{p}_1'\,d^3\mathbf{p}_2'\,d^4x_1\,d^4x_2\,d^4y_1\,d^4y_2 =$$

$$\int f_1^*(\mathbf{p}_1)f_2^*(\mathbf{p}_2)e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t}\left(\frac{\partial}{\partial x_1^0}+\frac{\partial}{\partial x_2^0}-\omega_{m_1}(\mathbf{p}_1)-\omega_{m_2}(\mathbf{p}_2)\right)\times$$

$$h_1(\nabla_{x_1}^2)h_2(\nabla_{x_2}^2)\delta(x_1^0-t_1)\delta(x_2^0-t_2)\,\frac{1}{(2\pi)^3}\,e^{-i\mathbf{p}_1\cdot\mathbf{x}_1-i\mathbf{p}_2\cdot\mathbf{x}_2}\times$$

$$\frac{1}{(2\pi)^3}\,\frac{e^{-\omega_m(\mathbf{k}_1)(x_1^0+y_1^0)+i\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{y}_1)}}{2\omega_m(\mathbf{k}_1)}\,\rho_1(m)\frac{1}{(2\pi)^3}\,\frac{e^{-\omega_{m'}(\mathbf{k}_2)(x_2^0+y_2^0)+i\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{y}_2)}}{2\omega_{m'}(\mathbf{k}_2)}\,\rho_2(m')\times$$

$$\left(\frac{\partial}{\partial y_1^0}+\frac{\partial}{\partial y_2^0}-\omega_{m_1}(\mathbf{p}_1')-\omega_{m_2}(\mathbf{p}_2')\right)\times$$

$$h_1(\nabla_{y_1}^2)h_2(\nabla_{y_2}^2)\delta(y_1^0-t_1)\delta(y_2^0-t_2)\,\frac{1}{(2\pi)^3}\,e^{i\mathbf{p}_1'\cdot\mathbf{y}_1+i\mathbf{p}_2'\cdot\mathbf{y}_2}\times$$

$$e^{-i(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))t} f_1(\mathbf{p}_1') f_2(\mathbf{p}_2') \times$$

$$dm\, dm'\, d^3\mathbf{k}_1\, d^3\mathbf{k}_2\, d^3\mathbf{p}_1\, d^3\mathbf{p}_2\, d^3\mathbf{p}_1'\, d^3\mathbf{p}_2'\, d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2.$$

Integrating the terms containing

$$\frac{\partial}{\partial x_1^0}, \frac{\partial}{\partial x_2^0}, \frac{\partial}{\partial y_1^0}, \frac{\partial}{\partial y_2^0}, \nabla_{x_1}^2, \nabla_{x_2}^2, \nabla_{y_1}^2, \nabla_{y_2}^2$$

by parts gives

$$\int f_1^*(\mathbf{p}_1) f_2^*(\mathbf{p}_2) e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t} \left(\frac{\partial}{\partial x_1^0} + \frac{\partial}{\partial x_2^0} - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2) \right) \times$$

$$h_1(\nabla_{x_1}^2) h_2(\nabla_{x_2}^2) \delta(x_1^0 - t_1) \delta(x_2^0 - t_2) \frac{1}{(2\pi)^3}\, e^{-i\mathbf{p}_1\cdot\mathbf{x}_1 - i\mathbf{p}_2\cdot\mathbf{x}_2} \times$$

$$\frac{1}{(2\pi)^3}\, \frac{e^{-\omega_m(\mathbf{k}_1)(x_1^0+y_1^0)+i\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{y}_1)}}{2\omega_m(\mathbf{k}_1)}\, \rho_1(m) \frac{1}{(2\pi)^3}\, \frac{e^{-\omega_{m'}(\mathbf{k}_2)(x_2^0+y_2^0)+i\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{y}_2)}}{2\omega_{m'}(\mathbf{k}_2)}\, \rho_2(m') \times$$

$$\left(\frac{\partial}{\partial y_1^0} + \frac{\partial}{\partial y_2^0} - \omega_{m_1}(\mathbf{p}_1') - \omega_{m_2}(\mathbf{p}_2') \right) \times$$

$$h_1(\nabla_{y_1}^2) h_2(\nabla_{y_2}^2) \delta(y_1^0 - t_1) \delta(y_2^0 - t_2) \frac{1}{(2\pi)^3}\, e^{i\mathbf{p}_1'\cdot\mathbf{y}_1 + i\mathbf{p}_2'\cdot\mathbf{y}_2} \times$$

$$e^{-i(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))t} f_1(\mathbf{p}_1') f_2(\mathbf{p}_2') \times$$

$$dm\, dm'\, d^3\mathbf{k}_1\, d^3\mathbf{k}_2\, d^3\mathbf{p}_1\, d^3\mathbf{p}_2\, d^3\mathbf{p}_1'\, d^3\mathbf{p}_2'\, d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2 =$$

$$\int f_1^*(\mathbf{p}_1) f_2^*(\mathbf{p}_2) e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t} (\omega_m(\mathbf{k}_1) + \omega_{m'}(\mathbf{k}_2) - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2)) \times$$

$$h_1(m^2) h_2(m'^2) \delta(x_1^0 - t_1) \delta(x_2^0 - t_2) \frac{1}{(2\pi)^3}\, e^{-i\mathbf{p}_1\cdot\mathbf{x}_1 - i\mathbf{p}_2\cdot\mathbf{x}_2} \times$$

$$\frac{1}{(2\pi)^6}\, \frac{e^{-\omega_m(\mathbf{k}_1)(x_1^0+y_1^0)-\omega_{m'}(\mathbf{k}_2)(x_2^0+y_2^0)} e^{i\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{y}_1)+i\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{y}_2)}}{2\omega_m(\mathbf{k}_1)\, 2\omega_{m'}(\mathbf{k}_2)}\, \rho_1(m)\rho_2(m') \times$$

$$(\omega_m(\mathbf{k}_1) + \omega_{m'}(\mathbf{k}_2) - \omega_{m_1}(\mathbf{p}_1') - \omega_{m_2}(\mathbf{p}_2')) h_1(m^2) h_2(m'^2) \times$$

$$\delta(y_1^0 - t_1) \delta(y_2^0 - t_2) \frac{1}{(2\pi)^3}\, e^{i\mathbf{p}_1'\cdot\mathbf{y}_1 + i\mathbf{p}_2'\cdot\mathbf{y}_2} e^{-i(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))t} f_1(\mathbf{p}_1') f_2(\mathbf{p}_2') \times$$

$$dm\, dm'\, d^3\mathbf{k}_1\, d^3\mathbf{k}_2\, d^3\mathbf{p}_1\, d^3\mathbf{p}_2\, d^3\mathbf{p}_1'\, d^3\mathbf{p}_2'\, d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2.$$

Using the identity (see [40], Page 44)

$$\delta(\mathbf{k}-\mathbf{p}) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{x}\cdot(\mathbf{k}-\mathbf{p})}\, d^3\mathbf{x}$$

on the above, then performing the integrations over x_1^0, x_2^0, y_1^0, y_2^0, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_1'$, and $\mathbf{p}_2'$, we obtain

$$\begin{gathered}
\int f_1^*(\mathbf{p}_1) f_2^*(\mathbf{p}_2) e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t} (\omega_m(\mathbf{k}_1) + \omega_{m'}(\mathbf{k}_2) - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2)) \times \\
h_1(m^2) h_2(m'^2) \delta(x_1^0 - t_1) \delta(x_2^0 - t_2) \frac{1}{(2\pi)^3}\, e^{-i\mathbf{p}_1\cdot\mathbf{x}_1 - i\mathbf{p}_2\cdot\mathbf{x}_2} \times \\
\frac{1}{(2\pi)^6} \frac{e^{-\omega_m(\mathbf{k}_1)(x_1^0+y_1^0)-\omega_{m'}(\mathbf{k}_2)(x_2^0+y_2^0)} e^{i\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{y}_1)+i\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{y}_2)}}{2\omega_m(\mathbf{k}_1)\, 2\omega_{m'}(\mathbf{k}_2)}\, \rho_1(m)\rho_2(m') \times \\
(\omega_m(\mathbf{k}_1) + \omega_{m'}(\mathbf{k}_2) - \omega_{m_1}(\mathbf{p}_1') - \omega_{m_2}(\mathbf{p}_2')) h_1(m^2) h_2(m'^2) \times \\
\delta(y_1^0 - t_1)\delta(y_2^0 - t_2) \frac{1}{(2\pi)^3}\, e^{i\mathbf{p}_1'\cdot\mathbf{y}_1 + i\mathbf{p}_2'\cdot\mathbf{y}_2} e^{-i(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))t} f_1(\mathbf{p}_1') f_2(\mathbf{p}_2') \times \\
dm\, dm'\, d^3\mathbf{k}_1\, d^3\mathbf{k}_2\, d^3\mathbf{p}_1\, d^3\mathbf{p}_2\, d^3\mathbf{p}_1'\, d^3\mathbf{p}_2'\, d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2 = \\
\int f_1^*(\mathbf{k}_1) f_2^*(\mathbf{k}_2) \left(\omega_m(\mathbf{k}_1) + \omega_{m'}(\mathbf{k}_2) - \omega_{m_1}(\mathbf{k}_1) - \omega_{m_2}(\mathbf{k}_2)\right)^2 \times \\
\left[h_1(m^2)\right]^2 \left[h_2(m'^2)\right]^2 \frac{e^{-2\omega_m(\mathbf{k}_1)t_1}\, e^{-2\omega_{m'}(\mathbf{k}_2)t_2}}{2\omega_m(\mathbf{k}_1)\, 2\omega_{m'}(\mathbf{k}_2)}\, \rho_1(m)\rho_2(m') \times \\
f_1(\mathbf{k}_1) f_2(\mathbf{k}_2)\, dm\, dm'\, d^3\mathbf{k}_1\, d^3\mathbf{k}_2.
\end{gathered}$$

Hence, the contribution to (5.18) from the disconnected term $S_2(\theta x_1, y_1) S_2(\theta x_2, y_2)$ is ultimately given by

$$\begin{gathered}
\int f_1^*(\mathbf{k}_1) f_2^*(\mathbf{k}_2) \left(\omega_m(\mathbf{k}_1) + \omega_{m'}(\mathbf{k}_2) - \omega_{m_1}(\mathbf{k}_1) - \omega_{m_2}(\mathbf{k}_2)\right)^2 \left[h_1(m^2)\right]^2 \left[h_2(m'^2)\right]^2 \times \\
\frac{e^{-2\omega_m(\mathbf{k}_1)t_1}\, e^{-2\omega_{m'}(\mathbf{k}_2)t_2}}{2\omega_m(\mathbf{k}_1)\, 2\omega_{m'}(\mathbf{k}_2)}\, \rho_1(m)\rho_2(m') f_1(\mathbf{k}_1) f_2(\mathbf{k}_2)\, dm\, dm'\, d^3\mathbf{k}_1\, d^3\mathbf{k}_2,
\end{gathered} \tag{5.20}$$

with a similar expression resulting from the disconnected term $S_2(\theta x_1, y_2) S_2(\theta x_2, y_1)$ in (5.18).

The expression in (5.20) is seen to vanish identically. Indeed, by our bump function choices for $h_1(m^2)$ and $h_2(m'^2)$, the terms

$$\left[h_1(m^2)\right]^2 \rho(m) \qquad \text{and} \qquad \left[h_2(m'^2)\right]^2 \rho_2(m')$$

reduce to $\delta(m - m_1)$ and $\delta(m' - m_2)$, respectively. Integrating (5.20) over m and m' then produces the vanishing term

$$\left(\omega_{m_1}(\mathbf{k}_1) + \omega_{m_2}(\mathbf{k}_2) - \omega_{m_1}(\mathbf{k}_1) - \omega_{m_2}(\mathbf{k}_2)\right)^2 = 0.$$

A similar analysis yields the same result for the $S_2(\theta x_1, y_2)S_2(\theta x_2, y_1)$ term in (5.18).

We emphasize that the disconnected terms in (5.18) would not vanish without the $h(m^2)$ bump functions. Instead, non-zero terms independent of time (the real time dependence cancels after performing the integrals over the spatial coordinates) would remain, which would lead to a violation of the Cook condition. In that case the strong limits that provide the existence of the desired wave operators would not exist.

The remaining terms are linear in the connected Euclidean four-point Schwinger function. The precise large time behavior depends on the form of the connected four point function $S_c(x_1, x_2, y_1, y_2)$. Using a model connected four-point function and following arguments in [18], W. N. Polyzou has shown the same $t^{-3/2}$ behavior for the integrand in (5.16) that one gets non-relativistically (see [27]). This is sufficient to satisfy the Cook condition.

5.9 The h Function

The key requirement in constructing a suitable injection operator is to construct an operator $h(\nabla^2)$ with the property that

$$h(\nabla_x^2)f(x)$$

has positive Euclidean time support when $f(x)$ has support for positive Euclidean time. This demand is met when $h(\nabla^2)$ is a polynomial in ∇^2. However, for non-polynomial $h(m^2)$ - like our bump function choice in **Section 5.8** - it is not obvious that $h(\nabla^2)$ preserves the positive time support condition when $h(m^2)$ has support on an unbounded interval. The simplest counter example is the unitary translation operator, which: (1) Can be formally represented as an infinite series in derivatives and (2) Changes (translates) the support of functions. If, however, *any* smooth bump function can be approximated by polynomials in m^2 in the sense of L^2 to any degree of accuracy, we can simultaneously preserve the positive time support condition required by the elements in our physical Hilbert space $\mathscr{H}$ and construct an injection operator J_α that will allow for the satisfaction of the Cook condition (5.16) on Page 70. Note that the polynomials must be in m^2 rather than m because the *square* of the mass is the Laplacian in this representation (see (5.11), Page 66).

In the unphysical case of a Lehmann weight with support on N discrete masses, an $h(m)$ that selects the j^{th} mass can be written as:

$$h_j(m) = \prod_{i\neq j}^{N} \frac{m - m_i^2}{m_j^2 - m_i^2}, \qquad j = 1, \dots, N,$$

where, for simplicity, we choose to ignore terms of the form $e^{-2\omega_m(\mathbf{k})t}/2\omega_m(\mathbf{k})$ appearing in (5.20), as including such terms would only introduce various constant factors to the analysis, per the (summed) Dirac delta function nature of $\rho(m)$ in this case. These functions are known in the analysis community as the Lagrange interpolating polynomials (see [4], Pages 133 - 134; [5], Page 123) and the collection $\{h_j(m)\}_{j=1}^N$ forms an orthonormal family with respect to the weight

$$\rho(m) = \sum_{k=1}^{N} \delta(m - m_k^2), \qquad 0 < m_1 < m_2 < \cdots < m_N. \tag{5.21}$$

Moreover, Carleman's condition on the half-line (see **Theorem 1.7**, Page 16) ensures that the Lagrange polynomials are dense in $L^2_\rho(0, \infty)$ for $\rho(m)$ as given in (5.21). Indeed, the moments in this case are given by

$$\mu_n = \int_0^\infty m^n \left(\sum_{k=1}^{N} \delta(m - m_k^2) \right) dm = \sum_{k=1}^{N} m_k^{2n}.$$

It immediately follows, then, that

$$\mu_n \leq N m_N^{2n} \qquad \Longrightarrow \qquad N^{-1/2n} \frac{1}{m_N} \leq \mu_n^{-1/2n}.$$

Hence,

$$\sum_{n=1}^{\infty} N^{-1/2n} \frac{1}{m_N} = \infty \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} \mu_n^{-1/2n} = \infty.$$

A more interesting, but still unphysical, case is when the support of the Lehmann weight consists of a single mass (or several masses) and a continuous spectrum with compact support. In this case, the Weierstrass approximation theorem implies a bump function choice for $h(m)$ can be uniformly approximated by a sequence of polynomials on the support of the Lehmann weight (see [6], Page 143).

The physically interesting case is when the support of the Lehmann weight possesses discrete masses and an *unbounded* continuous interval extending from the lowest multiparticle threshold to infinity; i.e.,

$$\rho(m) = \sum_{k=1}^{N} c_k \delta(m - m_k) + \rho_{ac}(m),$$

where

$$0 < m_1 < m_2 < \cdots < m_N < \text{support}(\rho_{ac}),$$

as discussed on Page 67. Inspecting (5.20) on Page 77, we see that the quantity of interest in determining whether a smooth bump function in the variable m^2 can be approximated in the sense of L^2 by a sequence of polynomials in m^2 is given by

$$\int_0^\infty \frac{e^{-2\omega_m(\mathbf{p})t}}{2\omega_m(\mathbf{p})}\,\rho(m)\,dm, \qquad t > 0,$$

where $\omega_m(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$ is the energy of a particle with mass m and momentum $\mathbf{p}$ (see [10], Page 59; [44], Page 91). We note that the change $\mathbf{k} \mapsto \mathbf{p}$ has been made to transform (5.20) into the above. If the polynomials in m^2 with respect to the weight

$$w_{t,\mathbf{p}}(m) = \frac{e^{-2\omega_m(\mathbf{p})t}}{2\omega_m(\mathbf{p})}\,\rho(m), \qquad t > 0,$$

are complete, then it is possible to approximate the required bump function operator $h(\nabla^2)$ in $L^2_w(0,\infty)$ by polynomial operators in ∇^2. Here it is enough to treat $\mathbf{p}$ as a constant which is sufficient for three-momentum states with compact support.

As we know from our work in Chapters 1-4, the relevant condition for demonstrating said polynomial density in $L^2_w(0,\infty)$ is given by Carleman's condition - see

Theorem 1.7, Page 16. Hence, it remains to show that

$$\sum_{n=1}^{\infty} \mu_n^{-1/2n} = \infty,$$

for

$$\mu_n = \int_0^{\infty} m^{2n} \frac{e^{-2\omega_m(\mathbf{p})t}}{2\omega_m(\mathbf{p})} \rho(m)\, dm.$$

Once established, there remains the practical question of how to efficiently compute a suitable polynomial approximation to $h(m^2)$.

5.10 Verifying the Carleman Condition for $w_{t,\mathbf{p}}(m)$

As discussed in the previous section, we look to prove that

$$\sum_{n=1}^{\infty} \mu_n^{-1/2n} = \infty,$$

for

$$\mu_n = \int_0^{\infty} m^{2n} \frac{e^{-2\omega_m(\mathbf{p})t}}{2\omega_m(\mathbf{p})} \rho(m)\, dm, \qquad t > 0, \tag{5.22}$$

where

$$\rho(m) = \sum_{j=1}^{N} c_j \delta(m - m_j) + \rho_{ac}(m),$$

with

$$0 < m_1 < m_2 < \cdots < m_N < \mathrm{support}(\rho_{ac}).$$

Two simplifications will aid our analysis: (1) Since the reciprocal power $-1/2n$ will ultimately be applied, bounding μ_n above is useful; (2) We know from the regularity theorem for tempered distributions (see [40], Page 34) that $\rho(m)$ is polynomially bounded. Hence, we can replace $\rho(m)$ with m^k in (5.22) to obtain a new moment sequence, which we also denote by μ_n, that bounds (5.22) above:

$$\mu_n = \int_0^\infty m^{2n+k} \frac{e^{-2\omega_m(\mathbf{p})t}}{2\omega_m(\mathbf{p})}\, dm, \qquad t > 0.$$

Let $p = |\mathbf{p}| = \sqrt{\mathbf{p}^2}$, use the hyperbolic trigonometric substitution $m = p\sinh(u)$ (see [39], Page 521), and note that

$$\cosh^2(u) - \sinh^2(u) = 1 \qquad \Longrightarrow \qquad p\cosh(u) = \sqrt{\mathbf{p}^2 + m^2}.$$

Putting these pieces together we obtain

$$\mu_n = \frac{p}{2}\int_0^\infty (p\sinh(u))^{2n+k} e^{-2pt\cosh(u)} du.$$

We now integrate via the substitution method once again by taking $x = 2pt\cosh(u)$. Note that in this case,

$$\cosh^2(u) - \sinh^2(u) = 1 \qquad \Longrightarrow \qquad p\sinh(u) = \left[\left(\frac{x}{2t}\right)^2 - p^2\right]^{1/2}.$$

Turning the change of variables crank gives

$$\mu_n = \frac{p}{4t}\int_{2pt}^\infty \left[\left(\frac{x}{2t}\right)^2 - p^2\right]^{\frac{2n+k-1}{2}} e^{-x} dx. \tag{5.23}$$

We now use (5.23) to bound μ_n above:

$$\begin{aligned}
\mu_n &= \frac{p}{4t}\int_{2pt}^\infty \left[\left(\frac{x}{2t}\right)^2 - p^2\right]^{\frac{2n+k-1}{2}} e^{-x} dx \\
&\leq \frac{p}{4t}\int_{2pt}^\infty \left[\left(\frac{x}{2t}\right)^2\right]^{\frac{2n+k-1}{2}} e^{-x} dx \\
&= \frac{p}{4t}\int_{2pt}^\infty \left(\frac{x}{2t}\right)^{2n+k-1} e^{-x} dx \\
&\leq \frac{p}{4t}\int_{0}^\infty \left(\frac{x}{2t}\right)^{2n+k-1} e^{-x} dx
\end{aligned}$$

$$= \frac{p}{4t}\,(2t)^{1-2n-k}\underbrace{\int_0^\infty x^{2n+k-1}e^{-x}dx}_{\Gamma(2n+k)}$$

$$= \frac{p}{4t}\,(2t)^{1-2n-k}\,\Gamma(2n+k)$$

$$= \frac{p}{4t}\,(2t)^{1-2n-k}\,(2n+k-1)! \tag{5.24}$$

where in the penultimate line we have utilized the well-known Gamma function (see [22], Page 156; [38], Page 160):

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt, \qquad s \in \mathbb{R} \text{ and } s > 0.$$

Using a technique known as the method of steepest descent, one can obtain the following asymptotic expansion for $\Gamma(s)$, which is commonly referred to as Stirling's formula (see [22], Page 537; [1], Page 257):

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt \sim \sqrt{2\pi}s^{s-1/2}e^{-s}, \qquad s \to \infty. \tag{5.25}$$

More precisely, H. Robbins has shown in [31] that $n! = \Gamma(n+1)$ can be written as

$$n! = \sqrt{2\pi}\,n^{n+1/2}e^{-n}e^{r_n}, \qquad \forall n \in \mathbb{N},$$

where r_n satisfies the double inequality

$$\frac{1}{12n+1} < r_n < \frac{1}{12n},$$

from which it follows $n!$ satisfies the sharp bound

$$\sqrt{2\pi}\,n^{n+1/2}\,e^{-n}\,e^{1/(12n+1)} < n! < \sqrt{2\pi}\,n^{n+1/2}\,e^{-n}\,e^{1/12n}, \qquad \forall n \in \mathbb{N}. \tag{5.26}$$

In any event, by applying (5.25) or (5.26), we see from (5.24) that

$$\frac{2et}{2n+k-1} \sim \left[\frac{p}{4t}\,(2t)^{1-2n-k}\,(2n+k-1)!\right]^{-1/2n} \le \mu_n^{-1/2n}, \qquad n \to \infty.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{2n+k-1} = \infty \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} \mu_n^{-1/2n} = \infty,$$

and it follows that the Carleman condition on the half-line is satisfied by the weight function

$$w_{t,\mathbf{p}}(m) = \frac{e^{-2\omega_m(\mathbf{p})t}}{2\omega_m(\mathbf{p})}\, \rho(m), \qquad t > 0.$$

According to **Theorem 1.7** (see Page 16), $w_{t,\mathbf{p}}(m)$ is the unique solution - in the sense of **Definition 1.2** on Page 7 - to the Stieltjes moment problem

$$\mu_n = \int_0^{\infty} m^{2n}\, \frac{e^{-2\omega_m(\mathbf{p})t}}{2\omega_m(\mathbf{p})}\, \rho(m)\, dm.$$

Theorem 1.1 (see Page 2 and Chapter 4) then confirms to us that the polynomials in the variable m^2 are dense in $L^2_w(0,\infty)$. We conclude, therefore, per our discussion in **Section 5.9**, that it is indeed possible to simultaneously preserve the positive time support condition required by the elements in our physical Hilbert space $\mathscr{H}$, and construct the injection operator $J_\alpha : \mathscr{H}_0 \to \mathscr{H}$ of (5.19)

$$\langle x_1, x_2 | J_\alpha | \mathbf{p}_1, \mathbf{p}_2 \rangle = h_1\left(\nabla_1^2\right) h_2\left(\nabla_2^2\right) \delta(x_1^0 - t_1)\delta(x_2^0 - t_2) \frac{1}{(2\pi)^3} e^{i\mathbf{p}_1\cdot\mathbf{x}_1 + i\mathbf{p}_2\cdot\mathbf{x}_2}$$

that will allow for the satisfaction of the Cook condition (5.16) on Page 70.

Remark. The discerning reader will note the striking similarity between the weight function $w_{t,\mathbf{p}}(m)$ above and $w_\lambda(x) = e^{-\lambda x^\lambda}$ of **Example 1.4** (see Page 19). Indeed, under the change of variables $2t\omega_{t,\mathbf{p}}(m) \mapsto x$, the two problems are equivalent when $\lambda = 1$.

5.11 Summary

The purpose of this thesis was to excogitate the formulation of scattering asymptotic conditions using strong limits in the framework of a quantum field theory defined by a collection of reflection-positive Schwinger functions. In this formalism the Hilbert space inner product is expressed as a quadratic form, whose kernel consists of said Schwinger functions and a Euclidean time reversal operator acting on the final states. Hilbert space vectors were taken have positive Euclidean relative time support.

The main problem addressed in this work centered on constructing the Euclidean analog of Haag-Ruelle quasilocal fields. These are needed to get a scattering theory that can be formulated in terms of strong limits and to treat scattering where the initial and final particles may be composite. To achieve this result, we used the fact that there are normalizable vectors in this representation of the Hilbert spare that are proportional to delta functions in the Euclidean time variable, as discussed in **Section 5.6**.

Lastly, it was necessary to show that polynomials in the variable m^2 are complete with respect to a particular weight function. This result was needed to ensure that the application of complicated differential operators that project positive relative time support functions on the desired asymptotic states do not change the Hilbert space positive Euclidean relative time support conditions, and was fully established in **Section 5.10**.

APPENDIX A
APPENDIX TO CHAPTER 1

This appendix serves to substantiate the claim that $I(z;\alpha)$ is analytic on $\operatorname{Im} z \neq 0$ (Page 13), as well as verify the details behind the Stieltjes-Perron inversion formula of **Theorem 1.4** (Page 13).

A.1 Analyticity of $I(z;\alpha)$

Lemma A.1 ([36], Page xiv). *The function*

$$I(z;\alpha) = \int_{-\infty}^{\infty} \frac{1}{z-x}\, d\alpha(x), \qquad \operatorname{Im} z \neq 0,$$

is analytic in the upper and lower half-planes, its values being conjugate at two conjugate points.

Proof. Let K be a given non-empty compact subset of the upper (lower) half-plane $\operatorname{Im} z > 0$ ($\operatorname{Im} z < 0$) and define

$$I_n(z;\alpha) = \int_{-n}^{n} \frac{1}{z-x}\, d\alpha(x), \qquad n = 1, 2, 3, \cdots.$$

Note that for each fixed x in $[-n, n]$, $1/(z-x)$ is analytic as function of z in the upper (lower) half-plane. It follows from a standard result in the theory of a single complex variable (see [15], Page 183) that $I_n(z;\alpha)$ is analytic in the upper (lower) half-plane for each $n = 1, 2, 3, \cdots$.

Now for $n \geq \max_{z \in K} |z| + 1$ and $|x| \geq n$, the reverse triangle inequality gives

$|z-x| \geq |x| - |z| \geq n - |z| \geq 1$ for all $z \in K$. It then follows that

$$\begin{aligned}
|I(z;\alpha) - I_n(z;\alpha)| &= \left| \int_{-\infty}^{\infty} \frac{1}{z-x}\, d\alpha(x) - \int_{-n}^{n} \frac{1}{z-x}\, d\alpha(x) \right| \\
&\leq \int_{|x|\geq n} \frac{1}{|z-x|}\, d\alpha(x) \\
&\leq \int_{|x|\geq n} d\alpha(x) \\
&= [\alpha(\infty^-) - \alpha(n)] + [\alpha(-n) - \alpha(-\infty^+)].
\end{aligned}$$

Since $\alpha(x)$ is a solution to a Hamburger moment problem and the improper Stieltjes integral is defined in analogy with the Riemann case (see [48], Page 15), we have

$$\mu_0 = \int_0^{\infty} d\alpha(x) + \int_{-\infty}^{0} d\alpha(x) = [\alpha(\infty^-) - \alpha(0)] + [\alpha(0^-) - \alpha(-\infty^+)]\,,$$

so that $\alpha(\pm\infty^{\mp})$ exist and are finite; i.e., $\alpha(x)$ is a bounded non-decreasing function. Taking the limit as $n \to \infty$ in

$$|I(z;\alpha) - I_n(z;\alpha)| \leq [\alpha(\infty^-) - \alpha(n)] + [\alpha(-n) - \alpha(-\infty^+)],$$

and noting that bounding term is independent of $z \in K$, we see that $I_n(z;\alpha)$ converges uniformly on K to $I(z;\alpha)$. Since K was an arbitrary compact subset of the upper (lower) half-plane, it follows (see [15], Page 161) that $I(z;\alpha)$ is analytic for $\operatorname{Im} z > 0$ ($\operatorname{Im}\, z < 0$). $\square$

Remark. Some authors (e.g., J. A. Shohat & J. D. Tamarkin in [36]) choose to include boundedness in the definition of a moment problem solution $\alpha(x)$.

A.2 The Stieltjes-Perron Inversion Formula Proof

Establishing the Stieltjes-Perron inversion formula requires the use of a lemma. As we're wont to do, the formal calculations are performed completely. Intuitively speaking, however, one notes that pointwise

$$\lim_{y \to 0^+} \operatorname{Arctan}\left(\frac{x}{y}\right) = \begin{cases} \dfrac{\pi}{2} & x > 0 \\ 0 & x = 0. \end{cases}$$

Hence, intuitively speaking,

$$\lim_{y \to 0^+} d\operatorname{Arctan}\left(\frac{x}{y}\right) = \frac{\pi}{2}\,\delta(x)dx.$$

The proof then follows by combining an integration by parts with the above.

Lemma A.2 ([16], Page 591)**.** *Let $\alpha(x)$ be a bounded, non-decreasing real function defined on $(-\infty, \infty)$. Then*

$$\lim_{y \to 0^+} \int_0^\infty \operatorname{Arctan}\left(\frac{x}{y}\right) d\alpha(x) = \frac{\pi}{2}\left[\alpha(\infty^-) - \alpha(0^+)\right].$$

Proof. As we are concerned with the limit as $y \to 0^+$, consider $0 < y < 1$ and express the integral of interest as

$$\int_0^\infty = \int_0^{y^2} + \int_{y^2}^{\sqrt{y}} + \int_{\sqrt{y}}^\infty.$$

Since the inequality

$$0 \leq \operatorname{Arctan}\left(\frac{x}{y}\right) \leq \frac{x}{y}$$

holds for $x \geq 0$ and $0 < y < 1$, we have

$$0 \leq \int_0^{y^2} \operatorname{Arctan}\left(\frac{x}{y}\right) d\alpha(x) \leq \int_0^{y^2} \frac{x}{y}\, d\alpha(x) \leq y\left[\alpha(y^2) - \alpha(0)\right].$$

Thus, even if $\alpha(0^+) \neq \alpha(0)$, we have

$$\lim_{y\to 0^+} \int_0^{y^2} \operatorname{Arctan}\left(\frac{x}{y}\right) d\alpha(x) = 0.$$

What remains to be shown, then, is

$$\lim_{y\to 0^+} \int_{y^2}^{\infty} \operatorname{Arctan}\left(\frac{x}{y}\right) d\alpha(x) = \frac{\pi}{2}\left[\alpha(\infty^-) - \alpha(0^+)\right],$$

which is equivalent to establishing

$$\lim_{y\to 0^+} \int_{y^2}^{\infty} \left[\frac{\pi}{2} - \operatorname{Arctan}\left(\frac{x}{y}\right)\right] d\alpha(x) = 0, \tag{A.1}$$

because

$$\lim_{y\to 0^+} \int_{y^2}^{\infty} d\alpha(x) = \alpha(\infty^-) - \alpha(0^+).$$

Using

$$0 \leq \operatorname{Arctan}\left(\frac{x}{y}\right) \leq \frac{\pi}{2} \qquad \text{and} \qquad \lim_{y\to 0^+} \alpha(\sqrt{y}) = \lim_{y\to 0^+} \alpha(y^2) = \alpha(0^+)$$

together with the bound

$$0 \leq \int_{y^2}^{\sqrt{y}} \left[\frac{\pi}{2} - \operatorname{Arctan}\left(\frac{x}{y}\right)\right] d\alpha(x) \leq \frac{\pi}{2} \int_{y^2}^{\sqrt{y}} d\alpha(x) = \frac{\pi}{2}\left[\alpha(\sqrt{y}) - \alpha(y^2)\right],$$

we have

$$\lim_{y\to 0^+} \int_{y^2}^{\sqrt{y}} \left[\frac{\pi}{2} - \operatorname{Arctan}\left(\frac{x}{y}\right)\right] d\alpha(x) = 0. \tag{A.2}$$

Similarly,

$$0 \leq \int_{\sqrt{y}}^{\infty} \left[\frac{\pi}{2} - \operatorname{Arctan}\left(\frac{x}{y}\right)\right] d\alpha(x) \leq \left[\frac{\pi}{2} - \operatorname{Arctan}\left(\frac{1}{\sqrt{y}}\right)\right] \int_{\sqrt{y}}^{\infty} d\alpha(x), \tag{A.3}$$

where we have used the fact that Arctan is an increasing function to justify the second inequality. Now, since

$$\lim_{y\to 0^+} \operatorname{Arctan}\left(\frac{1}{\sqrt{y}}\right) = \frac{\pi}{2}$$

and

$$\lim_{y\to 0^+}\int_{\sqrt{y}}^{\infty} d\alpha(x) = \alpha(\infty^-) - \alpha(0^+) \leq \alpha(\infty^-) - \alpha(0) < \infty,$$

it follows from (A.3) that

$$\lim_{y\to 0^+}\int_{\sqrt{y}}^{\infty}\left[\frac{\pi}{2} - \text{Arctan}\left(\frac{x}{y}\right)\right] d\alpha(x) = 0. \tag{A.4}$$

Combining (A.2) with (A.4) we obtain (A.1), as desired. □

With **Lemma A.2** in hand, we are a hop, skip, and a jump away from confirming the Stieltjes-Perron inversion formula (Page 13).

Proof. We have

$$-\int_a^b I(x+iy;\alpha)dx = \int_a^b\left[\int_{-\infty}^{\infty}\frac{1}{t-(x+iy)}\,d\alpha(t)\right]dx. \tag{A.5}$$

As we saw in the proof of **Lemma A.1**,

$$I_n(x+iy) = \int_{-n}^{n}\frac{1}{(x+iy)-t}\,d\alpha(t)$$

converges uniformly to $I(x+iy;\alpha)$ for fixed $y > 0$. Hence, we may interchange the order of integration in (A.5) (see [48], Page 25) to obtain

$$\begin{aligned}-\int_a^b I(x+iy;\alpha)dx &= \int_{-\infty}^{\infty}\left[\int_a^b\frac{1}{t-(x+iy)}\,dx\right]d\alpha(t)\\ &= \int_{-\infty}^{\infty}\left[\log(t-(a+iy)) - \log(t-(b+iy))\right]d\alpha(t).\end{aligned}$$

Expressing the imaginary part of the logarithm by the principal value of the arctangent function we have

$$\begin{aligned}-\text{Im}\int_a^b I(x+iy;\alpha)dx = &\int_{-\infty}^{\infty}\text{Arctan}\left(\frac{t-a}{y}\right)d\alpha(t)\\ &-\int_{-\infty}^{\infty}\text{Arctan}\left(\frac{t-b}{y}\right)d\alpha(t).\end{aligned} \tag{A.6}$$

For the moment we focus on the first integral on the right hand side above. Breaking the integration interval into two pieces about $t = a$ and using the change of variables $t \mapsto t + a$, this term becomes

$$\int_0^\infty \operatorname{Arctan}\left(\frac{t}{y}\right) d\alpha(t+a) + \int_0^\infty \operatorname{Arctan}\left(\frac{t}{y}\right) d\alpha(a-t).$$

Applying **Lemma A.2** to each of the integrals above gives

$$\lim_{y\to 0^+} \int_{-\infty}^\infty \operatorname{Arctan}\left(\frac{t-a}{y}\right) d\alpha(t) = \frac{\pi}{2}\left[\alpha(\infty^-) + \alpha(-\infty^+) - \alpha(a^+) - \alpha(a^-)\right]. \quad \text{(A.7)}$$

Employing the same technique on the second integral in (A.6) yields

$$\lim_{y\to 0^+} \int_{-\infty}^\infty \operatorname{Arctan}\left(\frac{t-b}{y}\right) d\alpha(t) = \frac{\pi}{2}\left[\alpha(\infty^-) + \alpha(-\infty^+) - \alpha(b^+) - \alpha(b^-)\right]. \quad \text{(A.8)}$$

Subtracting (A.8) from (A.7) and multiplying by $\frac{1}{\pi}$, equation (A.6) produces the Stieltjes-Perron inversion formula

$$\frac{1}{2}\left[\alpha(b^+) + \alpha(b^-)\right] - \frac{1}{2}\left[\alpha(a^+) + \alpha(a^-)\right] = -\frac{1}{\pi} \lim_{y\to 0^+} \operatorname{Im} \int_a^b I(x+iy;\alpha)dx,$$

as desired. $\square$

APPENDIX B
APPENDIX TO CHAPTER 3

The purpose of this appendix is twofold: (1) We provide a proof of Carleman's inequality (3.5) (Page 38); (2) We give a second proof of Carleman's condition for the Hamburger moment problem.

B.1 Carleman's Inequality Proof

Carleman's Inequality ([3], Page 86)**.** *Let* $\{u_n\}_{n=1}^{\infty}$ *be a collection of non-negative real numbers. Then*

$$\sum_{n=1}^{\infty} \sqrt[n]{u_1 u_2 \cdots u_n} \leq e \sum_{n=1}^{\infty} u_n.$$

Proof. Define the numbers $c_1, c_2, c_3, \cdots$ by the equations

$$c_1 c_2 \cdots c_n = (n+1)^n,$$

so that

$$c_n = \frac{(n+1)^n}{n^{n-1}} \leq ne, \tag{B.1}$$

where the inequality holds as a result of the monotonically increasing limit (see [6], Page 73)

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Applying the arithmetic-geometric mean inequality ([6], Page 28)

$$\sqrt{ab} \leq \frac{1}{2}(a+b)$$

for non-negative real numbers, the rearrangement theorem for absolutely convergent series ([6], Page 255), and inequality (B.1), we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \sqrt[n]{u_1 u_2 \cdots u_n} &= \sum_{n=1}^{\infty} \frac{\sqrt[n]{u_1 c_1 u_2 c_2 \cdots u_n c_n}}{n+1} \\
&\leq \sum_{n=1}^{\infty} \frac{u_1 c_1 + u_2 c_2 + \cdots + u_n c_n}{n(n+1)} \\
&= \sum_{n=1}^{\infty} u_n c_n \left[\sum_{k=n}^{\infty} \frac{1}{k(k+1)} \right] \\
&= \sum_{n=1}^{\infty} u_n c_n \left[\sum_{k=n}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) \right] \\
&= \sum_{n=1}^{\infty} u_n c_n \frac{1}{n} \\
&\leq e \sum_{n=1}^{\infty} u_n,
\end{aligned}$$

as desired. □

B.2 A Second Proof of Carleman's Condition on the Full Line

Our second proof of **Theorem 1.6** comes to us courtesy of two technical lemmas. Before providing these essential elements, however, we briefly outline the origins for the conditions that precipitate them. We are led once again to examine our old friend $I(z;\alpha)$, where $\alpha(x)$ is a solution to the Hamburger moment problem elicited by $\{\mu_n\}_{n=0}^{\infty}$. In so doing, we will explore more deeply the analytic properties of $I(z;\alpha)$, a rewarding enterprise in its own right.

Using the difference of powers formula

$$z^n - t^n = (z-t)\left(z^{n-1} + z^{n-2}t + \cdots + zt^{n-2} + t^{n-1}\right),$$

we obtain (see [36], Page 14)

$$I(z;\alpha) = \int_{-\infty}^{\infty} \frac{1}{z-t} d\alpha(t) = \int_{-\infty}^{\infty} d\alpha(t) \left[\frac{1}{z} + \frac{t}{z^2} + \cdots + \frac{t^{n-1}}{z^n} + \frac{t^n}{z^n(z-t)} \right]$$
$$= \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \cdots \frac{\mu_{n-1}}{z^n} + R_n(z;\alpha),$$

where

$$R_n(z;\alpha) = \frac{1}{z^n} \int_{-\infty}^{\infty} \frac{t^n}{z-t} d\alpha(t).$$

Introducing the notation

$$\mu_n^*(\alpha) = \int_{-\infty}^{\infty} |t|^n d\alpha(t)$$

(noting that $\mu_{2n}^* = \mu_{2n}$) for the "absolute moments" of $\alpha(t)$, we have for $\operatorname{Im} z \geq a > 0$

$$|R_n(z;\alpha)| \leq \frac{1}{|z|^n} \int_{-\infty}^{\infty} \frac{|t|^n}{|z-t|} d\alpha(t) \leq \frac{\mu_n^*(\alpha)}{|z|^n a}.$$

Hence, if $\alpha_1(t)$ and $\alpha_2(t)$ are two solutions to the moment problem given by $\{\mu_n\}_{n=0}^{\infty}$, then for $\operatorname{Im} z \geq a > 0$

$$|I(z;\alpha_1) - I(z;\alpha_2)| = |R_n(z;\alpha_1 - \alpha_2)| \leq \frac{\mu_n^*(\alpha_1) + \mu_n^*(\alpha_2)}{|z|^n a}. \tag{B.2}$$

If we put

$$f(z) = I(z;\alpha_1) - I(z;\alpha_2), \qquad |z| = r,$$

and

$$m_n = \mu_n^*(\alpha_1) + \mu_n^*(\alpha_2),$$

we see that $f(z)$ is analytic in the upper half-plane (**Lemma A.1**, Page 87) and

$$|f(z)| \leq \frac{m_n}{r^n} \frac{1}{a}, \qquad \text{for} \qquad \operatorname{Im} z \geq a > 0, \qquad n = 0, 1, 2, \cdots. \tag{B.3}$$

Without loss of generality (by redefining the moment sequence $\mu_n \mapsto \dfrac{\mu_n}{4\mu_0}$, if need be), we can assume $m_0 = 1/2$ and $a = 1/2$, so that $f(z)$ is analytic for $\operatorname{Im} z > 0$ and (taking $n = 0$ in inequality (B.3))

$$|f(z)| \leq 1 \qquad \text{for} \qquad \operatorname{Im} z \geq \frac{1}{2}.$$

By considering isosceles triangles with bases along the line segment from $z = 0$ to $z = i$, we have that the line $\operatorname{Im} z = 1/2$ is given by the equation $|z - i| = |z|$, and the half-plane $\operatorname{Im} z \geq 1/2$ by $|z - i| \leq |z|$. Defining the variable z' by

$$z' = \frac{i}{z} \qquad \Longleftrightarrow \qquad z = \frac{i}{z'},$$

the line $|z-i| = |z|$ corresponds to the circle $|z'-1| = 1$, and the half-plane $\operatorname{Im} z > 0$ to the half-plane $\operatorname{Re} z' > 0$ with the half-plane $\operatorname{Im} z \geq 1/2$ mapping to the interior of the circle $|z' - 1| = 1$. Denoting the circle $|z' - 1| = 1$ by γ, the interior of γ by (γ), and dropping the prime, we are dragooned by inequality (B.3) into the analysis of a class of functions $f(z)$ satisfying the following provisos:

1. $f(z)$ is analytic in (γ) and on γ, except possibly at $z = 0$;
2. $|f(z)| \leq 1$ in (γ);
3. For $n = 0, 1, 2, \cdots$ and $|z| = r$,

   $$|f(z)| \leq C m_n r^n$$

   in (γ) and on γ, except possibly at $z = 0$.

Next we define the function

$$T(x) = \sup_{n \geq 0} \frac{x^n}{m_n}$$

and emphasize the relationship of $T(x)$ to $f(z)$ via the m_n's. We also note that $T(x) \geq 2$ because $m_0 = 1/2$, and that for some values of x, $T(x)$ may be infinite.

With these considerations in mind, we are ready to state the first lemma needed for proving **Theorem 1.6**.

Lemma B.1 ([36], Page 15). *If $f(z)$ satisfies conditions (1)-(3) above and if*

$$\int_1^\infty \frac{\log T(x)}{x^2}\, dx = \infty,$$

then $f(z)$ is identically zero.

To prove **Lemma B.1** we will make use of Jensen's formula, which we cite here for convenience. The key feature of this result is that it relates the zeros of an analytic function to the value of a contour integral, an eminently reasonable tool to use given the statement of the lemma and the conditions placed on f. In addition to the source provided, Page 307 of [33] is also a superb reference on the matter.

Theorem ([38], Page 135). *Let Ω be an open set that contains the closure of a disc D_R of radius R centered at the origin and suppose that f is analytic in Ω, $f(0) \neq 0$, and f vanishes nowhere on the circle C_R of radius R centered at the origin. If $z_1, \cdots, z_N$ denote the zeros of f inside the disc counted with multiplicities, then*

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right|\, d\theta.$$

Remark. The hypothesis that $f(0) \neq 0$ is perhaps more benign than appears at first glance. If $f(z)$ has a zero of order k at $z = 0$, then the formula can be applied to the function $g(z) = f(z)/z^k$ (see [33], Page 308).

Proof. Suppose $f(z) \not\equiv 0$ satisfies conditions (1)-(3). It then suffices to show

$$\int_1^\infty \frac{\log T(x)}{x^2}\,dx < \infty.$$

Since $T(x)$ is defined using the notion of supremum, it follows that for each $x > 0$ and $\epsilon > 0$, there is a natural number $n = n(x, \epsilon)$ such that

$$T\left(\frac{1}{x}\right) - \epsilon < \frac{1}{m_n x^n}.$$

Hence, for this particular value of n,

$$m_n < \frac{1}{(T(1/x) - \epsilon)\, x^n}.$$

Taking $x = r$ and applying condition (3) from the assumptions on f, we see that in (γ) and on γ, except possibly at $z = 0$,

$$|f(z)| \leq C m_n r^n < \frac{C}{T(1/r) - \epsilon}. \tag{B.4}$$

Since we can always find n such that (B.4) holds for a chosen $\epsilon > 0$, it follows that

$$|f(z)| \leq \frac{C}{T(1/r)} \qquad \Longleftrightarrow \qquad \frac{T(1/r)}{C} \leq \frac{1}{|f(z)|}$$

in (γ) and on γ, except possibly at $z = 0$. Using the increasing nature of the logarithm, the inequality

$$\frac{T(1/r)}{C} \leq \frac{1}{|f(z)|}$$

allows us to deduce that

$$\log T(1/r) \leq \log \frac{1}{|f(z)|} + \log C,$$

from which we see

$$\int_\gamma \log \frac{1}{|f(z)|} |dz| < \infty \qquad \Longrightarrow \qquad \int_\gamma \log T(1/r) |dz| < \infty,$$

where $|dz|$ denotes the arc length parameterization of the curve γ.

Applying Jensen's formula, the assumption $f(z) \not\equiv 0$ in (γ) implies

$$\int_\gamma \log \frac{1}{|f(z)|} |dz| < \infty,$$

so that, by what was just discussed,

$$2 \int_{\gamma'} \log T(1/r) |dz| = \int_\gamma \log T(1/r) |dz| < \infty, \tag{B.5}$$

where γ' is the upper half of the circle γ.

Using the Law of Cosines, we can parameterize γ' by

$$r = 2\cos\theta \qquad \Longleftrightarrow \qquad \theta = \operatorname{Arccos}\left(\frac{r}{2}\right).$$

Hence, using (B.5) and $\log T(1/r) \geq \log 2 > 0$ (see Page 96),

$$\begin{aligned}
\infty &> \int_{\gamma'} \log T(1/r) |dz| \\
&= \int_0^2 \log T(1/r) \frac{2}{\sqrt{4 - r^2}}\, dr \\
&\geq \int_0^1 \log T(1/r) \frac{2}{\sqrt{4 - r^2}}\, dr \\
&\geq \int_0^1 \log T(1/r) dr \\
&= \int_1^\infty \frac{\log T(x)}{x^2}\, dx,
\end{aligned}$$

where in the final equality we used the substitution $r \mapsto \frac{1}{x}$. Thus, we have

$$\int_1^\infty \frac{\log T(x)}{x^2}\, dx < \infty,$$

as desired. $\square$

We waste no time and proceed directly into the second lemma needed to prove **Theorem 1.6**.

Lemma B.2 ([36], Page 16). *Let* $\beta_0 = m_0 = 1/2$, $\beta_n = m_n^{1/n}$ *for* $n \geq 1$, *and define*

$$\beta_n^* = \inf_{k \geq n} \beta_k, \qquad n = 0, 1, 2, \cdots.$$

Then the integral $\int_1^\infty \log T(x) x^{-2} dx$ *and the series* $\sum_{n=0}^{\infty} \frac{1}{\beta_n^*}$ *converge or diverge simultaneously.*

We make no bones about the length and technical nature of the coming proof. While our argument follows closely the one cited in [36], we look to provide the reader with a finer, more detailed level of substantiation than the one found therein.

Proof. Note that β_n^* is a nondecreasing sequence of nonnegative numbers ($m_n \geq 0$ by definition - see Page 95), so that either:

$$\text{i. } 0 \leq \lim_{n \to \infty} \beta_n^* < \infty \qquad \text{or} \qquad \text{ii. } \lim_{n \to \infty} \beta_n^* = \infty.$$

If $\lim_{n \to \infty} \beta_n^* < \infty$, it immediately follows that

$$\sum_{n=0}^{\infty} 1/\beta_n^* = \infty$$

because $1/\beta_n^* \nrightarrow 0$, in which case we must demonstrate

$$\int_1^\infty \frac{\log T(x)}{x^2} dx = \infty.$$

Indeed, since $\lim\limits_{n\to\infty} \beta_n^* < \infty$, there is a constant L and a subsequence $\{\beta_{n_j}\}_{j=1}^{\infty}$ such that $\beta_{n_j} = (m_{n_j})^{1/n_j} < L$ for all $j = 1, 2, \cdots$. Hence,

$$\left(\frac{x}{L}\right)^{n_j} < \frac{x^{n_j}}{m_{n_j}} \qquad \text{for all } x > 0. \tag{B.6}$$

For $x > L$ we have $\left(\frac{x}{L}\right)^{n_j} \to \infty$ as $j \to \infty$, which, in combination with (B.6) and the definition of $T(x)$ (see Page 97), implies $T(x) = \infty$ for all $x > L$. Thus, the integral

$$\int_1^{\infty} \frac{\log T(x)}{x^2}\, dx$$

diverges, as claimed.

The remaining situation to consider, then, is that for which $\beta_n^* \to \infty$ as $n \to \infty$. In this case we may write

$$\beta_n^* = \min_{k \geq n} \beta_k \qquad n = 0, 1, 2, \cdots, \tag{B.7}$$

which follows as a consequence of the increasing nature of the β_n^*'s and the fact that $\beta_n^* \to \infty$. As a result,

$$\beta_0^* = \min_{k \geq 0} \beta_k > 0.$$

Indeed, if $\beta_0^* = 0$, then $m_k^{1/k} = \beta_k = 0$ for some k. Hence, from the definition of m_k,

$$0 = \int_{-\infty}^{\infty} |t|^k d\alpha_1(t) + \int_{-\infty}^{\infty} |t|^k d\alpha_2(t),$$

which implies $\alpha_1(x)$ and $\alpha_2(x)$ are constants and thus only generate the trivial moment sequence $\mu_n = 0$, contradicting $m_0 = 1/2$.

Now, for any $x > 0$, define $n(x)$ to be the number of $\beta_n^* \leq x$. We note the following properties of $n(x)$:

i. $n(x)$ is a non-decreasing step function that assumes only integer values;

ii. For $0 \leq x < \beta_0^*$, $n(x) = 0$;

iii. Since β_n^* is a non-decreasing sequence, $\beta_{n(x)}^* \leq x$ for all $x > 0$;

iv. The step function behavior of $n(x)$ implies that $dn(x)$ is a linear combination of delta functions located at the nodes $x_n = \beta_n^*$.

Appealing to (B.7), observe that for any given $x > 0$, there is an integer $n_1(x) \geq n(x)$ such that $\beta_{n(x)}^* = \beta_{n_1(x)}$. As this holds for *all* $x > 0$, we replace $x \mapsto x/e$ to obtain

$$\beta_{n(x/e)}^* = \beta_{n_1(x/e)}, \qquad n_1(x/e) = n_1 \geq n(x/e).$$

Hence,

$$\begin{aligned} T(x) = \sup_{n \geq 0} \frac{x^n}{m_n} \geq \sup_{n \geq 1} \frac{x^n}{m_n} = \sup_{n \geq 1} \frac{x^n}{\beta_n^n} \geq \frac{x^{n_1}}{\beta_{n_1}^{n_1}} \\ = \left(\frac{x}{\beta_{n(x/e)}^*} \right)^{n_1} \geq \left(\frac{x}{\beta_{n(x/e)}^*} \right)^{n(x/e)} \geq \left(\frac{x}{x/e} \right)^{n(x/e)} = e^{n(x/e)}, \end{aligned}$$

where we have used $n_1 \geq n(x/e)$ and $\beta_{n(x/e)}^* \leq x/e < x$ to justify the final two inequalities. It then immediately follows that for all $x > 0$

$$\log T(x) \geq n(x/e). \tag{B.8}$$

We will make use of this inequality momentarily.

Now consider the function

$$V(x) = \int_0^x \frac{n(t)}{t}\, dt.$$

Integrating by parts in conjunction with properties (ii) and (iv) of $n(x)$ gives

$$V(x) = n(x) \log x - \int_0^x \log t \, dn(t) = n(x) \log x - \sum_{k=0}^{n(x)} \log \beta_k^*$$
$$= \log \frac{x^{n(x)}}{\beta_0^* \beta_1^* \cdots \beta_{n(x)}^*}.$$

Since $\beta_k^* \leq x$ for $k \leq n(x)$ (see (iii) above), $\beta_k^* > x$ for $k > n(x)$, and $\beta_n^* \uparrow \infty$, for any given $x > 0$, the maximum element of the sequence whose terms are given by

$$\frac{x^n}{\beta_0^* \beta_1^* \cdots \beta_n^*} = \frac{1}{\beta_0^*} \frac{x}{\beta_1^*} \cdots \frac{x}{\beta_n^*}$$

is

$$\frac{x^{n(x)}}{\beta_0^* \beta_1^* \cdots \beta_{n(x)}^*};$$

i.e.,

$$\max \left\{ \frac{x^n}{\beta_0^* \beta_1^* \cdots \beta_n^*} \,\middle|\, n \geq 0 \right\} = \frac{x^{n(x)}}{\beta_0^* \beta_1^* \cdots \beta_{n(x)}^*}. \tag{B.9}$$

On the other hand, for each $n \geq 1$, the increasing nature of the β_n^* sequence implies

$$\beta_1^* \beta_2^* \cdots \beta_n^* \leq (\beta_n^*)^n \leq \beta_n^n = m_n, \tag{B.10}$$

where the definition of β_n^* as a minimum over the β_n's is used to justify the second inequality. We now employ (B.10) to show that for $n \geq 0$

$$\frac{x^n}{\beta_0^* \beta_1^* \cdots \beta_n^*} \geq \frac{x^n}{m_n}. \tag{B.11}$$

Indeed, for $n = 0$, $\beta_0^* \leq \beta_0$ gives

$$\frac{1}{\beta_0^*} \geq \frac{1}{\beta_0} = \frac{1}{m_0} = 2 = \frac{x^0}{m_0},$$

and for $n \geq 1$, via (B.10), we have

$$\frac{x^n}{\beta_0^* \beta_1^* \cdots \beta_n^*} \geq 2 \frac{x^n}{\beta_1^* \beta_2^* \cdots \beta_n^*} \geq \frac{x^n}{\beta_1^* \beta_2^* \cdots \beta_n^*} \geq \frac{x^n}{m_n}.$$

Combining (B.9) with (B.11) we obtain

$$\frac{x^{n(x)}}{\beta_0^* \beta_1^* \cdots \beta_{n(x)}^*} = \sup_{n \geq 0} \frac{x^n}{\beta_0^* \beta_1^* \cdots \beta_n^*} \geq \sup_{n \geq 0} \frac{x^n}{m_n} = T(x),$$

so that

$$V(x) = \log \frac{x^{n(x)}}{\beta_0^* \beta_1^* \cdots \beta_{n(x)}^*} \geq \log T(x). \tag{B.12}$$

Next we look to establish

$$\sum_{n=0}^{\infty} \frac{1}{\beta_n^*} = \int_0^{\infty} \frac{1}{t}\, dn(t) = \int_0^{\infty} \frac{n(t)}{t^2}\, dt. \tag{B.13}$$

The first equality follows from property (iv) of $n(x)$. For the second, integration by parts yields for $R > 1$ together with property (ii) of $n(x)$ gives

$$\int_0^R \frac{1}{t}\, dn(t) = \frac{n(R)}{R} + \int_0^R \frac{n(t)}{t^2}\, dt.$$

Hence, if

$$\int_0^{\infty} \frac{1}{t}\, dn(t) = \lim_{R \to \infty} \int_0^{\infty} \frac{1}{t}\, dn(t) < \infty,$$

it follows from $\dfrac{n(R)}{R} \geq 0$ that

$$\int_0^{\infty} \frac{n(t)}{t^2}\, dt < \infty$$

as well. Conversely, if

$$\int_0^{\infty} \frac{n(t)}{t^2}\, dt < \infty,$$

then

$$\lim_{R\to\infty}\int_R^\infty \frac{n(t)}{t^2}\,dt = 0,$$

from which it follows that $\dfrac{n(R)}{R} \to 0$ as $R \to \infty$, because

$$0 \le \frac{n(R)}{R} = n(R)\int_R^\infty \frac{1}{t^2}\,dt \le \int_R^\infty \frac{n(t)}{t^2}\,dt.$$

Hence,

$$\int_0^\infty \frac{1}{t}\,dn(t) < \infty \qquad \Longleftrightarrow \qquad \int_0^\infty \frac{n(t)}{t^2}\,dt < \infty,$$

and it follows by what was just shown regarding $\dfrac{n(R)}{R}$ that

$$\int_0^\infty \frac{1}{t}\,dn(t) = \int_0^\infty \frac{n(t)}{t^2}\,dt,$$

as claimed.

Using the discreteness of sequences to note that $\beta^*_{n(x)}$ is constant for incremental changes $x \mapsto x + dx$, the calculation

$$V(x) = \log\frac{x^{n(x)}}{\beta^*_0\beta^*_1\cdots\beta^*_{n(x)}} \qquad \Longrightarrow \qquad dV(x) = dn(x)\log\frac{x}{\beta^*_0\beta^*_1\cdots\beta^*_{n(x)}} + \frac{n(x)}{x}dx,$$

together with the same reasoning used to establish (B.13), gives

$$\int_0^\infty \frac{n(t)}{t^2}\,dt = \int_0^\infty \frac{1}{t}\,dV(t) = \int_0^\infty \frac{V(t)}{t^2}\,dt, \tag{B.14}$$

and we are now in a position to use the fruits of our labor to conclude the present proof.

Suppose

$$\int_1^\infty \frac{\log\,T(x)}{x^2}\,dx < \infty.$$

By (B.8), we then have

$$\int_1^\infty \frac{n(x/e)}{x^2}\, dx < \infty. \tag{B.15}$$

Using the substitution $x \mapsto ex$, the above and (B.13) together imply

$$\sum_{n=0}^\infty \frac{1}{\beta_n^*} = \int_0^\infty \frac{n(t)}{t^2}\, dt < \infty. \tag{B.16}$$

Conversely, if we assume

$$\sum_{n=0}^\infty \frac{1}{\beta_n^*} < \infty,$$

then, by (B.13) and (B.14),

$$\int_0^\infty \frac{V(t)}{t^2}\, dt = \int_0^\infty \frac{n(t)}{t^2}\, dt = \sum_{n=0}^\infty \frac{1}{\beta_n^*} < \infty.$$

Therefore, via (B.12),

$$\int_1^\infty \frac{\log T(x)}{x^2}\, dx < \infty,$$

as desired. □

Remark. There is a *minor* consideration to be made when concluding the validity of (B.16) from (B.15) that concerns whether $\beta_0^* \le 1/e$ or $1/e \le \beta_0^*$. In either case, the integral is convergent.

At long last, we now proceed with our second proof of **Theorem 1.6**.

Proof. Suppose $\sum_{n=1}^\infty \mu_{2n}^{-1/2n} = \infty$, let $\alpha_1(x)$ and $\alpha_2(x)$ be *any* two solutions to the Hamburger moment problem

$$\mu_n = \int_{-\infty}^\infty x^n d\alpha(x),$$

and define

$$\beta_n^* = \inf_{k \geq n} \left[\mu_k^*(\alpha_1) + \mu_k^*(\alpha_2)\right]^{1/k}, \qquad n = 1, 2, \cdots,$$

where $\mu_k^*(\alpha_1)$ and $\mu_k^*(\alpha_2)$ are the absolute moments of $\alpha_1(x)$ and $\alpha_2(x)$, respectfully (see Page 95). Since $\mu_{2n}^*(\alpha_1) = \mu_{2n}^*(\alpha_2) = \mu_{2n}$ for each $n = 1, 2, \cdots$, we have

$$\beta_n^* \leq 2^{1/2n} \mu_{2n}^{1/2n} \leq 2\mu_{2n}^{1/2n}, \qquad \text{for } n = 1, 2, \cdots.$$

Hence,

$$\sum_{n=1}^{\infty} \mu_{2n}^{-1/2n} = \infty \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} \frac{1}{\beta_n^*} = \infty.$$

Combining **Lemma B.2** (Page 100) with **Lemma B.1** (Page 97), it follows that $I(z;\alpha_1) - I(z;\alpha_2)$ is identically zero. Since $\alpha_1(x)$ and $\alpha_2(x)$ were two arbitrary full-line solutions, **Theorem 1.5** (Page 15) tells us that the Hamburger moment problem

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x),$$

is determinate in the sense of **Definition 1.2** (Page 7), whenever $\sum_{n=1}^{\infty} \mu_{2n}^{-1/2n} = \infty$. $\square$

APPENDIX C
APPENDIX TO CHAPTER 4

The purpose of this appendix is to provide a second, semi-formal spectral-theoretic proof of **Theorem 1.1**, which we restate here for the case of a Stieltjes moment problem. The argument presented closely follows the one given in [3] (see Pages 138 - 145).

Theorem 1.1 ([3], Page 45)**.** *If $\alpha(x)$ is the solution of a determinate Stieltjes moment problem, the set of all polynomials is dense in $L^2_\alpha(0,\infty)$.*

Proof. Given a solvable Stieltjes moment problem

$$\mu_n = \int_0^\infty x^n d\alpha(x), \qquad \forall n = 0, 1, 2, \cdots,$$

we construct an inner product on the space of all polynomials $\mathscr{P}$ by defining

$$(x^n, x^m) = \mu_{n+m}, \qquad \forall n, m = 0, 1, 2, \cdots,$$

for monomials and then extending linearly. Note that the positivity of the inner product follows from the assumed solubility of the Stieltjes moment problem because the existence of a solution necessarily implies

$$\sum_{j,k=0}^{n} a_j a_k \mu_{j+k} \geq 0$$

for each $n = 0, 1, 2, \cdots$, and all choices of $(a_0, \ldots, a_n) \in \mathbb{R}^{n+1}$ (see [36], Page 5; [3], Page 30).

Now, for a determinate moment problem, the multiplication operator A defined on $\mathscr{P}$ by $(AP)(x) = xP(x)$ has deficiency indices $(0, 0)$ (see [3], Pages 140 - 141) and, therefore, has a unique self-adjoint extension (see [29], Pages 138 - 141). Letting $\{E_x\}_{x\in\mathbb{R}}$ denote the spectral family of projection operators for A (see [17], Page 42), the unique solution to the given Stieltjes moment problem is equivalent (in the sense of **Definition 1.2**, see Page 7) to the non-decreasing function

$$\alpha(x) = (E_x \cdot 1, 1)$$

because, by the spectral theorem (see [17], Page 42),

$$\int_0^\infty x^n d\alpha(x) = \int_0^\infty x^n d(E_x \cdot 1, 1) = (A^n \cdot 1, 1) = (x^n, 1) = \mu_n.$$

The claim, now, is that $\mathscr{P}$ is dense in $L^2_\alpha(0,\infty)$. Indeed, let $\{P_n(x)\}_{n=0}^\infty$ denote the orthonormal basis for $\mathscr{P}$ obtained from the Gram-Schmidt process with respect the inner product $(\cdot,\cdot)$ (see [41], Page 26). Note that the family $\{P_n(x)\}_{n=0}^\infty$ here is precisely the one described in Chapter 2. Since $\{E_x\}_{x\in\mathbb{R}}$ is a family of projection operators, we know $E_x^2 = E_x$ for all $x \in \mathbb{R}$. Hence,

$$(E_x \cdot 1, 1) = (E_x^2 \cdot 1, 1) = (E_x \cdot 1, E_x \cdot 1), \qquad \forall x \in \mathbb{R}.$$

Since $\{P_n(x)\}_{n=0}^\infty$ is an orthonormal basis for $\mathscr{P}$, we can write

$$(E_t \cdot 1, 1) = (E_t \cdot 1, E_t \cdot 1) = \sum_{n=0}^{\infty}(E_t \cdot 1, P_n(x))(P_n(x), E_t \cdot 1), \tag{C.1}$$

for all $t > 0$. Using the spectral theorem, we compute the term $(E_t \cdot 1, P_n(x))$ appearing on the right side of (C.1) to obtain

$$\begin{aligned}(E_t \cdot 1, P_n(x)) &= (E_t \cdot 1, P_n(A) \cdot 1) \\ &= \int_0^\infty P_n(x) d_x (E_t \cdot 1, E_x \cdot 1) \\ &= \int_0^\infty P_n(x) d_x (E_x E_t \cdot 1, 1) \\ &= \int_0^t P_n(x) d_x (E_x E_t \cdot 1, 1) \\ &= \int_0^t P_n(x) d_x (E_x \cdot 1, 1) \\ &= \int_0^t P_n(x) d\alpha(x), \end{aligned} \tag{C.2}$$

where we have used

$$t \leq x \quad \Longrightarrow \quad E_x E_t = E_t \quad \Longrightarrow \quad d_x(E_x E_t \cdot 1, 1) = d_x(E_t \cdot 1, 1) = 0$$

and

$$x < t \quad \Longrightarrow \quad E_x E_t = E_x \quad \Longrightarrow \quad d_x(E_x E_t \cdot 1, 1) = d_x(E_x \cdot 1, 1) = d\alpha(x),$$

where each aforementioned property comes from the characteristics of the spectral family $\{E_x\}_{x \in \mathbb{R}}$ (see [17], Page 42). Similarly, we also have

$$(E_t \cdot 1, 1) = (E_t \cdot 1, A^0 \cdot 1) = \int_0^\infty d_x (E_t \cdot 1, E_x \cdot 1) = \int_0^t d\alpha(x) \tag{C.3}$$

Using (C.2) and (C.3) in (C.1), we obtain

$$\int_0^t d\alpha(x) = \sum_{n=0}^\infty \left[\int_0^t P_n(x) d\alpha(x) \right]^2. \tag{C.4}$$

Defining the step function $\theta_t(x)$ by

$$\theta_t(x) = \begin{cases} 1 & 0 \leq x \leq t \\ 0 & t < x, \end{cases}$$

we see that (C.4) can be written as

$$\int_0^\infty |\theta_t(x)|^2 \, d\alpha(x) = \sum_{n=0}^{\infty} \left[\int_0^\infty \theta_t(x) P_n(x) d\alpha(x) \right]^2,$$

which says that equality holds in Bessel's inequality for $\theta_t(x)$, so that, for any fixed $t > 0$, the step function $\theta_t(x)$ can be polynomially approximated to any degree of accuracy in $L^2_\alpha(0,\infty)$. One can then extend this result to all step functions by considering the differences $\theta_t(x) - \theta_\tau(x)$, where $\tau < t$. Therefore, since every step function can be polynomially approximated to an arbitrary degree of accuracy, the same is true of *all* functions in $L^2_\alpha(0,\infty)$. Hence, the space of polynomials $\mathscr{P}$ is dense in $L^2_\alpha(0,\infty)$ whenever $\alpha(x)$ is the unique solution to a Stieltjes moment problem. $\square$

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* 9th Printing. Washington D.C.: United States National Bureau of Standards, 1970.

[2] G. J. Aiello and W. N. Polyzou, Scattering Asymptotic Conditions in Euclidean Relativistic Quantum Theory, *Physical Review D,* Volume **93**, Issue 5 (2016).

[3] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis,* 1st Edition. New York: Hafner Publishing Company, 1965.

[4] K. E. Atkinson, *An Introduction to Numerical Analysis,* 2nd Edition. New York: John Wiley & Sons, Inc., 1989.

[5] K. E. Atkinson and W. Han, *Elementary Numerical Analysis,* 3rd Edition. New York: John Wiley & Sons, Inc., 2004.

[6] R. G. Bartle and D. R. Sherbert, *Introduction to Real Analysis,* 3rd Edition. New York: John Wiley & Sons, Inc., 2000.

[7] H. Baumgärtel and M. Wollenberg, *Mathematical Scattering Theory,* 1st Edition. Basel: Birkhäuser-Verlag, 1983.

[8] F. Coester, Scattering Theory for Relativistic Particles, *Helvetica Physica Acta,* Volume **38**, Issue 7 (1965), Pages 7 - 23.

[9] J. M. Cook, Convergence to the Møller Wave-Matrix, *Journal of Mathematics and Physics,* Volume **36**, Issue 1 - 4 (1957), Pages 82 - 87.

[10] E. De Faria and W. De Melo, *Mathematical Aspects of Quantum Field Theory,* 1st Edition. New York: Cambridge University Press, 2010.

[11] J. J. Dennis and H. S. Wall, The Limit-Circle Case for a Positive Definite J-Fraction, *Duke Mathematical Journal,* Volume **12**, Number 2 (1945), Pages 255 - 273.

[12] F. Gross, *Relativistic Quantum Mechanics and Field Theory,* 1st Edition. New York: John Wiley & Sons, Inc., 1993.

[13] B. C. Hall, *Quantum Theory for Mathematicians,* 1st Edition. New York: Springer Publishing, 2013.

[14] B. C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction,* 1st Edition. New York: Springer-Verlag Publishing, 2003.

[15] P. Henrici, *Applied and Computational Complex Analysis, Volume 1: Power Series, Integration, Conformal Mapping, and Location of Zeros,* 1st Edition. New York: John Wiley & Sons, Inc., 1974.

[16] P. Henrici, *Applied and Computational Complex Analysis, Volume 2: Special Functions, Integral Transforms, Asymptotics, and Continued Fractions,* 1st Edition. New York: John Wiley & Sons, Inc., 1977.

[17] T. F. Jordan, *Linear Operators for Quantum Mechanics,* Reprinted 1st Edition. Mineola, New York: Dover Publications, Inc., 2006.

[18] R. Jost, *The General Theory of Quantized Fields,* 1st Edition. Providence, Rhode Island: American Mathematical Society, 1965.

[19] B. D. Keister and W. N. Polyzou, Relativistic Hamiltonian Dynamics, *Advances in Nuclear Physics,* Volume **20**. New York: Plenum Press, 1991.

[20] A. Klein and L. J. Landau, Construction of a Unique Self-Adjoint Generator for a Symmetric Local Semigroup, *Journal of Functional Analysis,* Volume **44**, Issue 2 (1981), Pages 121 - 137.

[21] P. Kopp and W. N. Polyzou, Euclidean Formulation of Relativistic Quantum Mechanics, *Physical Review D,* Volume **85**, Issue 1 (2012).

[22] S. M. Lea, *Mathematics for Physicists,* 1st Edition. Belmont, CA: Thomson Brooks/Cole, 2004.

[23] M. E. Miller, The Origins of Schwinger's Euclidean Green's Functions, *Studies in History and Philosophy of Modern Physics,* Volume **50**, May 2015, Pages 5 - 12.

[24] K. Osterwalder and R. Schrader, Axioms for Euclidean Green's Functions, *Communications in Mathematical Physics,* Volume **31**, Issue 2 (1973), Pages 83 - 112.

[25] H. L. Pedersen, Stieltjes Moment Problems and the Friedrichs Extension of a Positive Definite Operator, *Journal of Approximation Theory,* Volume **83**, Issue 3 (1995), Pages 289 - 307.

[26] P. A. Perry, *Scattering Theory by the Enss Method,* 1st Edition. New York: Harwood Academic Publishers, 1983.

[27] W. N. Polyzou, Scattering and Reflection Positivity in Relativistic Euclidean Quantum Mechanics, *Physical Review D,* Volume **89**, Issue 7 (2014).

[28] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Volume I: Functional Analysis,* Revised Edition. New York: Academic Press, Inc., 1980.

[29] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Volume II: Fourier Analysis, Self-Adjointness* 1st Edition. New York: Academic Press, Inc., 1975.

[30] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Volume III: Scattering Theory,* 1st Edition. New York: Academic Press, Inc., 1979.

[31] H. Robbins, A Remark on Stirling's Formula, *The American Mathematical Monthly,* Volume **62**, Issue 1 (1955), Pages 26 - 29.

[32] W. Rudin, *Principles of Mathematical Analysis,* 3rd Edition. New York: McGraw-Hill, Inc., 1976.

[33] W. Rudin, *Real and Complex Analysis,* 3rd Edition. New York: McGraw-Hill Book Company, 1987.

[34] J. Schwichtenberg, *Physics from Symmetry,* 1st Edition. Cham, Switzerland: Springer International Publishing, 2015.

[35] J. Schwinger, On the Euclidean Structure of Relativistic Field Theory, *Proceedings of the National Academy of Sciences,* Volume **44**, Issue 9 (1958), Pages 956 - 965.

[36] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments,* 1st Edition. New York: American Mathematical Society, 1943.

[37] B. Simon, *Real Analysis: A Comprehensive Course in Analysis, Part 1,* 1st Edition. Providence, Rhode Island: American Mathematical Society, 2015.

[38] E. M. Stein and R. Shakarchi, *Princeton Lectures in Analysis 2: Complex Analysis,* 1st Edition. Princeton, New Jersey: Princeton University Press, 2003.

[39] J. Stewart, *Calculus,* 4th Edition. Pacific Grove, CA: Brooks/Cole Publishing Company, 1999.

[40] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That,* 1st Edition. New York: W. A. Benjamin, Inc., 1964.

[41] G. Szegö, *Orthogonal Polynomials,* 4th Edition. Providence, Rhode Island: American Mathematical Society, 1975.

[42] J. R. Taylor, *Classical Mechanics,* 1st Edition. University Science Books, 2005.

[43] G. Teschl, *Mathematical Methods in Quantum Mechanics with Applications to Schrödinger Operators,* 1st Edition. Providence, Rhode Island: American Mathematical Society, 2009.

[44] P. A. Tipler and R. A. Llewellyn, *Modern Physics,* 4th Edition. New York: W.H. Freeman and Company, 2003.

[45] J. S. Townsend, *A Modern Approach to Quantum Mechanics,* 1st Edition. Sausalito, California: University Science Books, 2000.

[46] L. W. Tu, *An Introduction to Manifolds,* 2nd Edition. New York: Springer Publishing, 2011.

[47] S. Weinberg, *The Quantum Theory of Fields, Volume I: Foundations,* 1st Edition. New York: Cambridge University Press, 1995.

[48] D. V. Widder, *The Laplace Transform,* 1st Edition. Princeton, New Jersey: Princeton University Press, 1941.

[49] E. Wigner, On Unitary Representations of the Inhomogeneous Lorentz Group, *Annals of Mathematics,* Volume **40**, Issue 1 (1939), Pages 149 - 204.

[50] D. R. Yafaev, *Mathematical Scattering Theory: General Theory,* 1st Edition. Providence, Rhode Island: American Mathematical Society, 1992.

[51] A. Zee, *Quantum Field Theory in a Nutshell,* 2nd Edition. Princeton, New Jersey: Princeton University Press, 2010.