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Numerical analysis in energy dependent radiative transfer

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NUMERICAL ANALYSIS IN ENERGY DEPENDENT RADIATIVE TRANSFER

by

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requirements for the Doctor of Philosophy
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The University of Iowa

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Applied Mathematical and Computational Sciences at the December 2017 graduation.

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Soli Deo gloria.

ABSTRACT

The radiative transfer equation (RTE) models the transport of radiation through a participating medium. In particular, it captures how radiation is scattered, emitted, and absorbed as it interacts with the medium. This process arises in numerous application areas, including: neutron transport in nuclear reactors, radiation therapy in cancer treatment planning, and the investigation of forming galaxies in astrophysics. As a result, there is great interest in the solution of the RTE in many different fields.

We consider the energy dependent form of the RTE and allow media containing regions of negligible absorption. This particular case is not often considered due to the additional dimension and stability issues which arise by allowing vanishing absorption. In this thesis, we establish the existence and uniqueness of the underlying boundary value problem. We then proceed to develop a stable numerical algorithm for solving the RTE. Alongside the construction of the method, we derive corresponding error estimates. To show the validity of the algorithm in practice, we apply the algorithm to four different example problems. We also use these examples to validate our theoretical results.

PUBLIC ABSTRACT

The radiative transfer equation (RTE) is a mathematical description of the transport of radiation through a participating medium. As radiation travels through a medium, it collides with the particles comprising the medium. These collisions then result in various reactions which affect the transport of the radiation. For example, one can consider the transport of a photon through the human body. As the photon travels through the body it will interact with the underlying biological tissue which, in turn, will alter the course of the photon before it exits. The RTE accurately describes this transport of radiation through the human body. As a result, solutions of the RTE can be used in medical imaging applications. In fact, the underlying idea of transport through participating media has a broad range of applications: neutron transport in nuclear reactors, radiation therapy in cancer treatment planning, the investigation of forming galaxies in astrophysics. As a result, there is great interest in the solution of the RTE from many different fields. In our work, we develop an algorithm which provides solutions to the RTE.

This thesis provides a complete numerical analysis of the RTE. First, we establish the existence and uniqueness of the underlying boundary value problem. We then proceed to develop a stable numerical algorithm for solving the RTE. Alongside the construction of the method, we derive corresponding error estimates. To show the validity of the algorithm in practice, we apply the algorithm to four different example problems.

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CHAPTER 1
INTRODUCTION

1.1 Motivation

The radiative transfer equation is an integro-differential equation which mathematically describes the propagation of radiation in a medium which scatters, absorbs, and emits radiation. In many applications, this is viewed as the propagation of photons within a medium that participates. This can be the propagation of light through human tissue to its transport through stellar atmospheres. Although much of the early interest in radiative transfer stemmed from astrophysical applications ([13]), the same underlying form of the equation has been employed in various fields of physics and engineering (cf. [12, 17]). As a result, the number of applications is numerous. These include neutron transport ([5, 16, 29, 33, 34]), heat transfer ([39]), astrophysical applications ([13, 48, 44]), meteorology ([45, 49]), biomedical imaging ([3, 7, 6, 47]), and radiation therapy ([24, 46]).

Consider the case of neutron transport ([5, 16]). The realization of fission reactors in the 1940's provided motivation for understanding the transport of neutrons in various contexts. Since then, much work has evolved and one may consult [5, Ch. 8] for a through survey. In nuclear reactors, one goal is to obtain a steady nuclear fission chain reaction. As neutrons stream throughout a reactor they undergo collisions with other atomic nuclei. These collisions then result in scattering or absorption events, which may lead to the emission of new particles (e.g. fission). In particular, neutrons colliding with uranium can cause a fission reaction, which produces energy and new neutrons. These new neutrons then participate in the system, which can result in further fission reactions, producing a chain-reaction.

Knowledge of the distribution of neutrons throughout the reactor allows one to determine the rate at which nuclear reactions occur. Therefore, one usually considers the angular density $n(\mathbf{x}, \boldsymbol{\omega}, e)$, where $\mathbf{x} \in \mathbb{R}^3$ denotes the spatial coordinate, $\boldsymbol{\omega} \in \mathbb{R}^3$ the direction of travel, and $e \in \mathbb{R}$ the kinetic energy of the particle. Then the quantity

$$n(\mathbf{x}, \boldsymbol{\omega}, e) d\mathbf{x} d\boldsymbol{\omega} de$$

represents the number of neutrons centered about $\mathbf{x}$ in differential volume $d\mathbf{x}$, moving in the direction $\boldsymbol{\omega}$ into the solid angle $d\boldsymbol{\omega}$, with kinetic energy e in de. We also note that the quantity obtained for a differential volume is not exact, but denotes an expected value for the number of particles within the region. In this sense, the solutions of transport problems are statistical in nature. Additionally, we remark on the relation above to velocity space. Let m denote a particle's mass and $\mathbf{v}$ its velocity. By using the kinetic energy relation $e = \frac{1}{2} m |\mathbf{v}|^2$, the quantity $d\boldsymbol{\omega} de$ can be related to a differential volume in velocity space. We find particle speed $|\mathbf{v}| = \sqrt{2e/m}$, from which it follows that $\mathbf{v} = \boldsymbol{\omega} |\mathbf{v}|$. As a result, the energy dependent RTE is sometimes referred to as the multi-speed or multi-velocity RTE.

Another example arises in radiation therapy for cancer treatment planning ([24, 46]). In this example, the goal is to eliminate a cancerous tumor through the use of radiation. In an effort to eradicate the cancerous cells, charged particles are delivered to the site of the tumor where they deposit energy. The methodology by which this is done varies. In one methodology, a beam of charged particles, external to the body, is directed toward the tumor. As the particles travel through the body, they interact with the biological medium, begin scattered, absorbed, and in some cases collisions result in the emission of new particles. One challenge encountered in this application is that the scattering is extremely "forward peaked"; that is, the majority of the particles tend to scatter at small angles ([5, 46]). As a result, the scattering cross section appears singular near zero. Therefore, it is desirable to have an approximation that can accurately capture the behavior. In capturing the singular behavior, the Fokker-Plank approximation ([41]) is often used. The scattering cross section may then be split into small angle scattering and large angle scattering, using the appropriate approximations on each part. This refers to the Boltzmann-Fokker-Plank approximation ([11, 42]).

The previous two examples correspond to *direct* problems for the RTE. In direct problems, properties and conditions on the environment are given, and the goal is to determine the particle distribution or intensity. However, in many applications the *inverse* problem is of great importance; for example, medical imaging ([3, 7, 6]). In the preceding example of radiation therapy, one must first determine the location of the tumor, which may require the solution of an inverse problem.

We remark on two types of inverse problems: parameter estimation and inverse source. In both cases, data is given over the boundary of the domain. In parameter estimation, optical properties of the medium are determined using the boundary information. In this case, quantities such as the scattering and absorption cross sections are determined. In the inverse source problem, a particle emitting source is assumed to reside within the medium; the goal is to reconstruct the source. Given information over the boundary, properties of the internal source, such as location and intensity, are derived. For example, one method in molecular medical imaging works by injecting light emitting biochemical markers into the biological medium. These markers then emit light which can be measured on the boundary of the domain. Determination of the spatial concentration of the source allows one to derive further medically useful information on the internals. For some specific examples of these types of inverse problems one may consult [26, 31, 32].

In all of the above examples, the solution of the RTE or its approximation is required; however, the RTE is notoriously difficult to solve in many real-world applications ([43]). Analytical solutions for the equation are known only for a small subset of situations, which necessitates its numerical solution. Yet numerically, due to the high dimension and integro-differential form of the equation, the computational demands are significant, and there has been much investigation within each field (cf. [34, 36, 30, 27, 40]). Some techniques employed include approximations to the equation itself ([42, 11]), numerical techniques ([22, 25]), and high performance implementations ([21, 40]).

However, with respect to numerical analysis of the RTE, relatively little work has been done with the energy dependent RTE and cases which contain void regions where the absorption is considered negligible. In this work, we introduce a numerical method for the solution of the energy dependent RTE. We prove well-posedness, provide error estimates, and give a rigorous analysis of each obtained semi-discretization. Further, it is shown that the method maintains stability even in the case of vanishing absorption.

1.2 Radiative Transfer Equation

In this work, $X \subset \mathbb{R}^3$ with boundary $\partial X \in C^1$ will denote a spatial domain, Ω will be used to denote the unit sphere in $\mathbb{R}^3$, and $E \subset \mathbb{R}$ will denote the energy domain. The spatial and energy domains X and E are assumed to be bounded. Using these quantities, define $U = X \times \Omega \times E$ and $\Gamma = \partial X \times \Omega \times E$. We define the inflow and outflow portions of Γ by

$$\Gamma_- = \{(\mathbf{x}, \boldsymbol{\omega}, e) \in \Gamma \,|\, \boldsymbol{\omega} \cdot \boldsymbol{\nu}(\mathbf{x}) < 0\} \quad \text{and} \quad \Gamma_+ = \{(\mathbf{x}, \boldsymbol{\omega}, e) \in \Gamma \,|\, \boldsymbol{\omega} \cdot \boldsymbol{\nu}(\mathbf{x}) > 0\}, \tag{1.1}$$

respectively; here $\boldsymbol{\nu}(\mathbf{x})$ denotes the unit outward normal at the spatial variable $\mathbf{x} \in \partial X$. We consider the following boundary value problem (BVP) of the RTE (cf. [16, 2])

$$\boldsymbol{\omega} \cdot \nabla_x u(\mathbf{x}, \boldsymbol{\omega}, e) + \sigma_t(\mathbf{x}, e) u(\mathbf{x}, \boldsymbol{\omega}, e) - \mathcal{S}u(\mathbf{x}, \boldsymbol{\omega}, e) = f(\mathbf{x}, \boldsymbol{\omega}, e) \text{ in } U, \tag{1.2}$$

$$u(\mathbf{x}, \boldsymbol{\omega}, e) = g(\mathbf{x}, \boldsymbol{\omega}, e) \text{ on } \Gamma_-, \tag{1.3}$$

where ∇_x denotes the gradient with respect to the spatial variable $\mathbf{x} \in X$ and

$$\mathcal{S}u(\mathbf{x}, \boldsymbol{\omega}, e) = \int_E \int_\Omega \sigma_s(\mathbf{x}, e') \mathcal{P}(\mathbf{x}, \boldsymbol{\omega} \cdot \boldsymbol{\omega}', e, e') u(\mathbf{x}, \boldsymbol{\omega}', e') d\boldsymbol{\omega}' de'. \tag{1.4}$$

The function σ_t denotes the total cross section, $\sigma_t = \sigma_a + \sigma_s$, where σ_s and σ_a are the scattering and absorption cross sections, respectively. The functions f and g represent the source term and boundary data, respectively. The function $\mathcal{P}$ reflects the probability that particles scatter from direction $\boldsymbol{\omega}'$ with energy e' into direction $\boldsymbol{\omega}$ with energy e.

We let $I_0 := [-1, 1]$ be the range of the expression $\boldsymbol{\omega} \cdot \boldsymbol{\omega}'$. In the study of the boundary value problem (1.2)–(1.3) we make the following assumptions:

$$\sigma_t, \sigma_s \in L^\infty(X \times E),\ \sigma_t \geq \sigma_s \geq 0,\ \text{and } \sigma_t \geq \sigma_s', \tag{1.5}$$

$$\mathcal{P} \in L^\infty(X \times I_0 \times E^2) \text{ is non-negative and } \mathcal{P}(\mathbf{x}, \boldsymbol{\omega} \cdot \boldsymbol{\omega}', e, e') = 0 \text{ for } e' < e, \tag{1.6}$$

$$\mathcal{P} \text{ is normalized such that } \int_E \int_\Omega \mathcal{P}(\mathbf{x}, \boldsymbol{\omega} \cdot \boldsymbol{\omega}', e, e') d\boldsymbol{\omega} de = 1, \tag{1.7}$$

$$f \in L^2(X, C(\Omega \times E)), \quad g \in L^2(\Gamma_-) \text{ is continuous in } \boldsymbol{\omega} \text{ and } e, \tag{1.8}$$

where

$$\sigma_s'(\mathbf{x}, e) := \int_E \int_\Omega \sigma_s(\mathbf{x}, e') \mathcal{P}(\mathbf{x}, \boldsymbol{\omega} \cdot \boldsymbol{\omega}', e, e') d\boldsymbol{\omega}' de'.$$

Assumption (1.5) implies $\sigma_a = \sigma_t - \sigma_s \geq 0$, which permits the presence of regions with vanishing absorption in the formulation. The condition allowing a vanishing σ_a is not often considered in the literature due to stability issues in a-priori estimates. We note that the second portion of (1.6) is a no-upscatter condition. It states that a particle cannot gain energy due to a collision. The redistribution function contains the inner product $\boldsymbol{\omega} \cdot \boldsymbol{\omega}'$ as an argument. With the exception of Chapter 2, it will be notationally simpler to consider $\mathcal{P}$ as a function of the arguments $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ individually. Therefore, outside of Chapter 2, we replace $\mathcal{P}(\mathbf{x}, \boldsymbol{\omega} \cdot \boldsymbol{\omega}', e, e')$ by $\mathcal{P}(\mathbf{x}, \boldsymbol{\omega}, \boldsymbol{\omega}', e, e')$ but continue to enforce assumptions (1.6) and (1.7). These assumptions are used throughout this entire work.

1.3 Sobolev Spaces

Sobolev spaces are of great use in the analysis of partial differential equations. As a result, a basic introduction is given here and the reader is referred to [1] for a thorough introduction to the subject. Let X be a nonempty open set in $\mathbb{R}^n$ and let $1 \leq p < \infty$. Define the space of p-integrable functions by

$$L^p(X) := \{f : X \to \mathbb{R} \mid \int_X |f(\mathbf{x})|^p d\mathbf{x} < \infty\}.$$

This is the space of functions whose p^{th} power results in an integrable function. The standard norm over $L^p(X)$ is then

$$\|f\|_{L^p(X)} := \left(\int_X |f(\mathbf{x})|^p d\mathbf{x}\right)^{1/p}.$$

It holds that $L^p(X)$ is a Banach space and that when $p = 2$ it is a Hilbert space, where the inner product is defined by

$$(f, g)_{L^2(X)} := \int_X f(\mathbf{x})g(\mathbf{x})d\mathbf{x}.$$

The space comprised of almost everywhere bounded functions is defined by

$$L^\infty(X) := \{f : X \to \mathbb{R} \mid \operatorname*{ess\,sup}_{\mathbf{x}\in X} |f(\mathbf{x})| < \infty\},$$

with norm

$$\|f\|_{L^\infty(X)} := \operatorname*{ess\,sup}_{\mathbf{x}\in X} |f(\mathbf{x})|.$$

Define the space of continuous functions over $\overline{X}$ by

$$C(\overline{X}) := \{f : \overline{X} \to \mathbb{R} \mid \max_{\mathbf{x}\in\overline{X}} |f(\mathbf{x})| < \infty\}.$$

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ denote a multi-index with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Then denote the α^{th} derivative of a function f by

$$D^\alpha f(\mathbf{x}) = \frac{\partial^{|\alpha|} f(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Let m be a non-negative integer. Using multi-index notation, the space of functions with m continuous derivatives is defined by

$$C^m(\overline{X}) := \{f \in C(\overline{X}) \mid D^\alpha f \in C(\overline{X}), \text{ for } 0 \le |\alpha| \le m\}.$$

Denote the set of infinitely differentiable functions with compact support in X by $C_0^\infty(X)$. This may then be used to define the notion of weak derivative. Let f and g be locally integrable functions on X. The function g is called the α^{th} weak derivative of f, if

$$\int_X f(\mathbf{x}) D^\alpha \phi(\mathbf{x}) d\mathbf{x} = (-1)^{|\alpha|} \int_X g(\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x}, \quad \forall \phi \in C_0^\infty(X).$$

The weak derivative is defined such that the integration by parts relation between the two functions holds. Using the weak derivative, Sobolev spaces may be introduced. Define the Sobolev space

$$W^{m,p}(X) := \{f \in L^p(X) \mid D^\alpha f \in L^p(X), \text{ for } 0 \le |\alpha| \le m\},$$

where $D^\alpha f$ denotes the weak derivative of f. These are the functions with m weak derivatives, each of which reside in L^p. When $p = 2$ the notation $H^m(X) = W^{m,2}(X)$ will be used. The norm over $W^{m,p}$ is defined by

$$\|f\|_{W^{m,p}(X)} := \begin{cases} \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha f\|_{L^p(X)}^p \right)^{1/p}, & 1 \le p < \infty, \\ \max_{0 \le |\alpha| \le m} \|D^\alpha f\|_{L^\infty(X)}, & p = \infty. \end{cases}$$

It is also of use to define Sobolev spaces analogous to those used when dealing with time; the reader is referred to [20, Ch. 5] and references therein for more details. Let $f : I \times X \to \mathbb{R}$ with $X \subset \mathbb{R}^n$ and $I = [a,b] \subset \mathbb{R}$ for $a < b$. For $1 \leq p, q < \infty$, the space $L^p(I, L^q(X))$ is defined as the space of functions f satisfying

$$\|f\|_{L^p(I,L^q(X))} := \left(\int_I \|f(t)\|_{L^q(X)}^p \, dt \right)^{1/p} < \infty.$$

The space $W^{1,p}(I, L^q(X))$ is the space comprised of all $f \in L^p(I, L^q(X))$ such that

$$\|f\|_{W^{1,p}(I,L^q(X))} := \begin{cases} \left(\int_I \left(\|f(t)\|_{L^q(X)}^p + \|f'(t)\|_{L^q(X)}^p \right) dt \right)^{1/p} < \infty, & 1 \leq p < \infty \\ \operatorname{ess\,sup}_{t \in I} \left(\|f(t)\|_{L^q(X)} + \|f'(t)\|_{L^q(X)} \right) < \infty, & p = \infty, \end{cases}$$

where f' denotes the weak derivative of f with respect to t. Lastly, let $C(I, L^p(X))$ be defined as the space of functions satisfying

$$\|f\|_{C(I,L^p(X))} := \max_{t \in I} \|f(t)\|_{L^p(X)} < \infty.$$

CHAPTER 2
WELL-POSEDNESS

In this chapter the well-posedness of the BVP (1.2)–(1.3) is established. The well-posedness of the energy dependent BVP is derived in a manner analogous to the approach used in [18] and [19], which consider the energy independent problem. The chapter is organized as follows: appropriate function spaces are introduced and properties relevant to the well-posedness of (1.2)–(1.3) are discussed. The problem is then reformulated into a mixed variational problem. After which, a variant of Brezzi's theorem for the existence and uniqueness of saddle point problems is introduced and used to establish the existence and uniqueness of the mixed formulation. Solutions of the mixed formulation are then shown to contain sufficient regularity to satisfy (1.2)–(1.3). Much of the energy independent analysis carries over to the energy dependent problem. As a result, we follow [18] and [19] in much of our analysis.

2.1 Function Spaces

For functions in $C^\infty(\overline{U})$, define the inner product

$$(u,v)_W := (\boldsymbol{\omega}\cdot\nabla_x u, \boldsymbol{\omega}\cdot\nabla_x v)_{L^2(U)} + (u,v)_{L^2(U)} + (u,v)_{L^2_w(\Gamma)}$$

with associated norm $\|u\|_W := \sqrt{(u,u)_W}$, where

$$(u,v)_{L^2_w(\Gamma)} := (|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|u,v)_{L^2(\Gamma)} = \int_\Gamma uv|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|ds d\boldsymbol{\omega} de$$

with corresponding norm $\|u\|_{L^2_w(\Gamma)} := \sqrt{(u,u)_{L^2_w(\Gamma)}}$. Define $Q = L^2(U)$ and let W be defined as the completion of $C^\infty(\overline{U})$ with respect to the given norm. Further, let

$L^2_w(\Gamma)$ be the completion of $L^2(\Gamma)$ with respect to the norm induced by the inner product $(u,v)_{L^2_w(\Gamma)}$. Next we show that functions in W have a well defined trace over the boundary Γ.

Lemma 2.1. *Consider the linear operator $T : C^1(\overline{U}) \to L^2_w(\Gamma)$, defined by $Tu = u|_\Gamma$. The operator may be extended to a bounded linear operator $T : W \to L^2_w(\Gamma)$ satisfying $\|Tu\|_{L^2_w(\Gamma)} \le \|u\|_W$ for all $u \in W$.*

Proof. Let $u \in C^1(\overline{U})$. By definition $Tu = u|_\Gamma$, and, therefore,

$$\|u\|_W^2 = (\boldsymbol{\omega} \cdot \nabla_x u, \boldsymbol{\omega} \cdot \nabla_x u)_{L^2(U)} + (u,u)_{L^2(U)} + \|Tu\|^2_{L^2_w(\Gamma)}.$$

Hence, $\|Tu\|_{L^2_w(\Gamma)} \le \|u\|_W$ for all $u \in C^1(\overline{U})$. By construction $C^1(\overline{U})$ is dense in W, and the trace operator may be extended to $T : W \to L^2_w(U)$. □

By the previous lemma, functions $u \in W$ have a well defined trace in $L^2_w(\Gamma)$. In the definition of W, $u|_\Gamma$ is understood in the trace sense. In particular, the BVP (1.2)–(1.3) is well defined for functions $u \in W$. The space W may be characterized by

$$W := \left\{ u \in L^2(U) \mid \boldsymbol{\omega} \cdot \nabla_x u \in L^2(U) \text{ and } u|_\Gamma \in L^2_w(\Gamma) \right\}.$$

The analysis to come will rely on the orthogonal splitting of W and Q into functions of even and odd parity. Defining

$$W^\pm := \left\{ \frac{1}{2} \left(v(\mathbf{x}, \boldsymbol{\omega}) \pm v(\mathbf{x}, -\boldsymbol{\omega}) \right) \mid v \in W \right\},$$

the space W can be decomposed into even and odd parity functions resulting in the orthogonal splitting $W = W^+ \oplus W^-$. The splitting is orthogonal with respect to the standard L^2 inner product,

$$(u, v)_{L^2(U)} = (v, u)_{L^2(U)} = 0 \quad \forall\, u \in W^+,\, v \in W^-.$$

Then for any function $u \in W$, one has $u = u^+ + u^-$ with $u^+ \in W^+$ and $u^- \in W^-$. In the same way the spaces Q^+ and Q^- can be defined.

2.2 Mixed Formulation

It is useful for the presentation of the mixed formulation to introduce operator notation for the transport term. Define the transport operator

$$(\mathcal{T}u)(\mathbf{x}, \boldsymbol{\omega}, e) := \boldsymbol{\omega} \cdot \nabla_x u(\mathbf{x}, \boldsymbol{\omega}, e).$$

Using this notation, equation (1.2) may be written

$$\mathcal{T}u + \sigma_t u = Su + f, \tag{2.1}$$

where the variables have been omitted for simplicity. Decomposing the terms involving u in equation (2.1) into even and odd parts yields

$$(\mathcal{T}u)^+ + (\mathcal{T}u)^- + \sigma_t(u^+ + u^-) = (Su)^+ + (Su)^- + f. \tag{2.2}$$

Let $v^+ \in W^+$. Multiply (2.2) by v^+ and integrate over U. Noting the orthogonality with respect to the L^2 inner product, we obtain

$$((\mathcal{T}u)^+, v^+) + (\sigma_t u^+, v^+) = ((Su)^+, v^+) + (f, v^+), \quad \forall\, v^+ \in W^+, \tag{2.3}$$

where we have used the inner product notation $(\cdot,\cdot)$ to denote integration over U. From this point, the first variational equation is derived in two main steps. First, the parity of each operator is related to the parity of the function u. Second, standard integration by parts operations are carried out and rearranged appropriately.

1. Consider $(\mathcal{T}u)^+$ and its dependence on $\boldsymbol{\omega}$. We have

$$\begin{aligned}(\mathcal{T}u)^+(\boldsymbol{\omega}) &= \frac{1}{2}\left[(\mathcal{T}u)(\boldsymbol{\omega}) + (\mathcal{T}u)(-\boldsymbol{\omega})\right] \\ &= \frac{1}{2}\left[\boldsymbol{\omega}\cdot\nabla_x u(\boldsymbol{\omega}) + (-\boldsymbol{\omega}\cdot\nabla_x u(-\boldsymbol{\omega}))\right] \\ &= \frac{1}{2}\boldsymbol{\omega}\cdot\nabla_x\left[u(\boldsymbol{\omega}) - u(-\boldsymbol{\omega})\right] \\ &= (\mathcal{T}u^-)(\boldsymbol{\omega}). \end{aligned} \tag{2.4}$$

Therefore, equation (2.3) can be written as

$$(\mathcal{T}u^-, v^+) + (\sigma_t u^+, v^+) = ((Su)^+, v^+) + (f, v^+), \quad \forall\ v^+ \in W^+. \tag{2.5}$$

Similar to the transport term, consider $(Su)^+(\boldsymbol{\omega})$. We continue to omit all variables with the exception of $\boldsymbol{\omega}$ and the integration variables arising from the application of the scattering operator S. We have

$$(Su)^+(\boldsymbol{\omega}) = \frac{1}{2}\left[(Su)(\boldsymbol{\omega}) + (Su)(-\boldsymbol{\omega})\right]. \tag{2.6}$$

Make the change of variable $\boldsymbol{\omega}' \to -\boldsymbol{\omega}'$ for $(Su)(-\boldsymbol{\omega})$,

$$\begin{aligned}(Su)(-\boldsymbol{\omega}) &= \int_E\int_\Omega \sigma_s(e')\mathcal{P}(-\boldsymbol{\omega}\cdot\boldsymbol{\omega}', e')u(\boldsymbol{\omega}', e')d\boldsymbol{\omega}'de'. \\ &= \int_E\int_\Omega \sigma_s(e')\mathcal{P}(\boldsymbol{\omega}\cdot\boldsymbol{\omega}', e')u(-\boldsymbol{\omega}', e')d\boldsymbol{\omega}'de'.\end{aligned}$$

Returning to equation (2.6) yields

$$\begin{aligned}(Su)^+(\boldsymbol{\omega}) &= \int_E\int_\Omega \sigma_s(e')\mathcal{P}(\boldsymbol{\omega}\cdot\boldsymbol{\omega}', e')\frac{1}{2}\left[u(\boldsymbol{\omega}', e') + u(-\boldsymbol{\omega}', e')\right]d\boldsymbol{\omega}'de' \\ &= (Su^+)(\boldsymbol{\omega}).\end{aligned}$$

Equation (2.5) now reads

$$(\mathcal{T}u^-, v^+) + (\sigma_t u^+, v^+) = (Su^+, v^+) + (f, v^+), \quad \forall\, v^+ \in W^+. \tag{2.7}$$

2. Integration by parts is performed on the transport term of (2.7) to move the transport operator onto the test function. We have

$$\begin{aligned}(\mathcal{T}u^-, v^+) &= \int_U \boldsymbol{\omega} \cdot \nabla_x u^- v^+ dU \\ &= \int_\Gamma (\boldsymbol{\omega} \cdot \boldsymbol{\nu}) u^- v^+ d\Gamma - \int_U u^- (\boldsymbol{\omega} \cdot \nabla_x v^+) dU \end{aligned} \tag{2.8}$$

Consider the integration over Γ. Splitting into inflow and outflow portions of Γ and noting that $(\boldsymbol{\omega} \cdot \boldsymbol{\nu})u^-$ is an even function leads to

$$\begin{aligned}\int_\Gamma (\boldsymbol{\omega} \cdot \boldsymbol{\nu}) u^- v^+ d\Gamma &= \int_{\Gamma_+} (\boldsymbol{\omega} \cdot \boldsymbol{\nu}) u^- v^+ d\Gamma + \int_{\Gamma_-} (\boldsymbol{\omega} \cdot \boldsymbol{\nu}) u^- v^+ d\Gamma \\ &= \int_{\Gamma_-} (\boldsymbol{\omega} \cdot \boldsymbol{\nu}) u^- v^+ d\Gamma + \int_{\Gamma_-} (\boldsymbol{\omega} \cdot \boldsymbol{\nu}) u^- v^+ d\Gamma \\ &= -2 \int_{\Gamma_-} |\boldsymbol{\omega} \cdot \boldsymbol{\nu}| u^- v^+ d\Gamma. \end{aligned}$$

Applying the inflow boundary condition results in

$$\begin{aligned}\int_\Gamma (\boldsymbol{\omega} \cdot \boldsymbol{\nu}) u^- v^+ d\Gamma &= -2 \int_{\Gamma_-} |\boldsymbol{\omega} \cdot \boldsymbol{\nu}| (g - u^+) v^+ d\Gamma \\ &= -2 \int_{\Gamma_-} |\boldsymbol{\omega} \cdot \boldsymbol{\nu}| g v^+ d\Gamma + \int_\Gamma |\boldsymbol{\omega} \cdot \boldsymbol{\nu}| u^+ v^+ d\Gamma. \end{aligned}$$

Returning to equation (2.8) and using the inner product notation, we have

$$(\mathcal{T}u^-, v^+) = (u^+, v^+)_{L^2_w(\Gamma)} - (u^-, \mathcal{T}v^+) - 2(g, v^+)_{L^2_w(\Gamma_-)}.$$

The variational formulation is then given by

$$-(u^-, \mathcal{T}v^+) + (u^+, v^+)_{L^2_w(\Gamma)} + (\sigma_t u^+, v^+) = (Su^+, v^+) + (f, v^+) + 2(g, v^+)_{L^2_w(\Gamma_-)} \tag{2.9}$$

for all $v^+ \in W^+$.

The second variational equation is constructed using test functions in Q^-. In a similar fashion, decompose equation (2.1) into even and odd parts to arrive at equation (2.2). Multiply (2.2) by $q^- \in Q^-$ and integrate over U. Again noting the orthogonality with respect to the L^2 inner product,

$$((\mathcal{T}u)^-, q^-) + (\sigma_t u^-, q^-) = ((Su)^-, q^-) + (f, q^-), \quad \forall\, q^- \in Q^-.$$

By a similar decomposition of the transport and scattering terms the above becomes

$$(\mathcal{T}u^+, q^-) + (\sigma_t u^-, q^-) = (Su^-, q^-) + (f, q^-), \quad \forall\, q^- \in Q^-. \tag{2.10}$$

This results in two coupled variational equations

$$-(u^-, \mathcal{T}v^+) + (u^+, v^+)_{L^2_w(\Gamma)} + (\sigma_t u^+, v^+) = (Su^+, v^+) + (f, v^+) + 2(g, v^+)_{L^2_w(\Gamma_-)} \tag{2.11}$$

$$(\mathcal{T}u^+, q^-) + (\sigma_t u^-, q^-) = (Su^-, q^-) + (f, q^-) \tag{2.12}$$

for all $v^+ \in W^+$ and $q^- \in Q^-$. Equations (2.11) and (2.12) constitute the mixed formulation. More concisely, by defining the bilinear forms

$$a(u^+, v^+) := (u^+, v^+)_{L^2_w(\Gamma)} + (\sigma_t u^+, v^+) - (Su^+, v^+), \tag{2.13}$$

$$b(v^+, u^-) := -(\mathcal{T}v^+, u^-), \quad \text{and} \quad c(u^-, q^-) := (\sigma_t u^-, q^-) - (Su^-, q^-), \tag{2.14}$$

alongside linear functionals

$$l(v^+) := (f, v^+) + 2(g, v^+)_{L^2_w(\Gamma_-)} \quad \text{and} \quad p(q^-) := (-f, q^-), \tag{2.15}$$

the mixed variational problem can be stated as follows:

Problem 2.2. *Find* $(u^+, u^-) \in W^+ \times Q^-$ *such that*

$$a(u^+, v^+) + b(v^+, u^-) = f(v^+), \quad v^+ \in W^+$$

$$b(u^+, q^-) - c(u^-, q^-) = p(q^-), \quad q^- \in Q^-.$$

This problem solves for the even and odd portions of the solution $u = u^+ + u^-$ and is a mixed problem because the spaces W^+ and Q^- differ. In particular, it does not immediately hold that $u \in W$, which is required for satisfying (1.2)–(1.3). The conditions for existence and uniqueness of such a problem can be found in [8, Sect. 4.3.1] which is a variant of Brezzi's theorem [10] for the well-posedness of saddle point problems.

Theorem 2.3. *Let* Y *and* M *be Hilbert spaces and let* Y' *and* M' *denote their dual spaces, respectively. Further define the bilinear forms* $a : Y \times Y \to \mathbb{R}$, $b : Y \times M \to \mathbb{R}$, *and* $c : M \times M \to \mathbb{R}$. *Then if* a, b, *and* c *are continuous and*

1. $a(\cdot,\cdot)$ *is coercive over* $K = \{v \in Y \mid b(v,q) = 0 \ \forall q \in M\}$.
2. $b(\cdot,\cdot)$ *satisfies the inf-sup condition*

$$\inf_{q\in M} \sup_{v\in Y} \frac{b(v,q)}{\|q\|_M \|v\|_Y} \geq \beta.$$

3. $c(\cdot,\cdot)$ *non-negative.*

Then for $l \in Y'$ *and* $p \in M'$ *there exists a unique* $(u,h) \in Y \times M$ *such that*

$$a(u,v) + b(v,h) = l(v), \quad v \in Y$$

$$b(u,q) - c(h,q) = p(q), \quad q \in M.$$

In the context of Problem 2.2, the bilinear forms and linear functionals are defined as in (2.13)–(2.15) and the Hilbert spaces are $Y = W^+$ and $M = Q^-$. We will verify that the assumptions Theorem 2.3 are satisfied in order to establish the well-posedness of the mixed variational formulation. The first condition we show is the non-negativity of $c(\cdot,\cdot)$.

Lemma 2.4. *Assume that conditions (1.5)–(1.7) hold. Then the bilinear form $c(\cdot,\cdot)$ is non-negative. That is, there exists a $\gamma \geq 0$ such that*

$$c(q^-, q^-) \geq \gamma \left\| q^- \right\|_Q^2$$

for all $q^- \in Q^-$.

Proof. Let $q^- \in Q^-$. Consider the inner product $(Sq^-, q^-)_{E\times\Omega}$, where $(\cdot,\cdot)_{E\times\Omega}$ denotes integration over E and Ω. Then applying the Cauchy-Schwarz inequality and using the normalization assumption results in

$$\begin{aligned}
(Sq^-, q^-)_{E\times\Omega} &= \int_E \int_\Omega \int_E \int_\Omega \sigma_s(e')\mathcal{P}(\boldsymbol{\omega}\cdot\boldsymbol{\omega}', e, e')q^-(\boldsymbol{\omega}', e')q^-(\boldsymbol{\omega}, e)d\boldsymbol{\omega}'de'd\boldsymbol{\omega}de \\
&\leq c\left(\int_E \int_\Omega \int_E \int_\Omega \sigma_s(e')\mathcal{P}(\boldsymbol{\omega}\cdot\boldsymbol{\omega}', e, e')q^-(\boldsymbol{\omega}', e')^2 d\boldsymbol{\omega}'de'd\boldsymbol{\omega}de \right)^{1/2} \\
&\quad \left(\int_E \int_\Omega \int_E \int_\Omega \sigma_s(e')\mathcal{P}(\boldsymbol{\omega}\cdot\boldsymbol{\omega}', e, e')q^-(\boldsymbol{\omega}, e)^2 d\boldsymbol{\omega}'de'd\boldsymbol{\omega}de \right)^{1/2} \\
&\leq c\left(\int_E \int_\Omega \sigma_s(e')q^-(\boldsymbol{\omega}', e')^2 d\boldsymbol{\omega}'de' \right)^{1/2} \left(\int_E \int_\Omega \sigma_s'(e)q^-(\boldsymbol{\omega}, e)^2 d\boldsymbol{\omega}de \right)^{1/2}.
\end{aligned}$$

Return to the inner product notation and bound the multiplication by the sum of squares. This results in

$$(Sq^-, q^-)_{E\times\Omega} \leq \frac{c}{2}(\sigma_s q^-, q^-)_{E\times\Omega} + \frac{c}{2}(\sigma_s' q^-, q^-)_{E\times\Omega}.$$

Using assumption (1.5) results in

$$\begin{aligned}(\sigma_t q^-, q^-)_{E\times\Omega} - (Sq^-, q^-)_{E\times\Omega} &\geq (\sigma_t q^-, q^-) - (\frac{c}{2}(\sigma_s + \sigma_s')q^-, q^-)_{E\times\Omega} \\ &= \frac{c}{2}[((\sigma_t - \sigma_s)q^-, q^-)_{E\times\Omega} + ((\sigma_t - \sigma_s')q^-, q^-)_{E\times\Omega}] \\ &\geq \gamma \left\|q^-\right\|^2_{L^2(E\times\Omega)}.\end{aligned}$$

Integrating over X produces the claim. □

Next, we use that $\mathbf{x} \in X$ may be expressed by its offset from a point on the inflow boundary $\mathbf{x}_- \in \partial X$. That is, there exists a $\mathbf{x}_- \in \partial X$ and direction $\boldsymbol{\omega} \in \Omega$ such that $\mathbf{x} = \mathbf{x}_- + t\boldsymbol{\omega}$ for $\boldsymbol{\omega} \in \Omega$ and some $0 \leq t \leq \tau(\mathbf{x}, \boldsymbol{\omega})$, where $\tau(\mathbf{x}, \boldsymbol{\omega}) = \sup\{t \,|\, \mathbf{x}_- + t\boldsymbol{\omega} \in X\}$. Additionally, integration over the domain X may be rewritten as an integration over all maximal line segments integrated over the inflow boundary of the spatial domain. Formally,

$$\int_{\partial X_-^{\boldsymbol{\omega}}} \int_0^{\tau(\mathbf{x}_-, \boldsymbol{\omega})} v(\mathbf{x}_- + t\boldsymbol{\omega}) dt ds(\mathbf{x}_-) = \int_X v(\mathbf{x}) d\mathbf{x},$$

where $\partial X_-^{\boldsymbol{\omega}}$ denotes the inflow boundary of the spatial domain for a given direction $\boldsymbol{\omega}$. We will use this representation in the following lemma.

Lemma 2.5. *Assume that conditions (1.5)–(1.7) hold. Then the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition*

$$\inf_{q^- \in Q^-} \sup_{v^+ \in W^+} \frac{b(v^+, q^-)}{\left\|q^-\right\|_Q \left\|v^+\right\|_W} \geq \beta$$

for $\beta > 0$.

Proof. Let $q^- \in Q^-$. Let $t \in \mathbb{R}$ and $\boldsymbol{\omega} \in \Omega$ be such that $\mathbf{x} = \mathbf{x}_- + t\boldsymbol{\omega}$, where $\mathbf{x}_- \in \Gamma_-$. Then define $v \in W$ by

$$v(\mathbf{x}, \boldsymbol{\omega}, e) = -\int_0^t q^-(\mathbf{x}_- + s\boldsymbol{\omega}, \boldsymbol{\omega}, e)ds. \tag{2.16}$$

Omitting the variables for simplicity, note that $\mathcal{T}v = -q^-$, from which it follows that $\mathcal{T}v = (\mathcal{T}v)^- \in Q^-$. Following the argument leading to equation (2.4), we obtain $(\mathcal{T}v)^- = \mathcal{T}v^+$. As a result, $v = v^+ \in W^+$. Therefore, for any $q^- \in Q^-$ there exists $v^+ \in W^+$ such that $\mathcal{T}v^+ = -q^-$; that is, $\mathcal{T} : W^+ \to Q^-$ is a surjection as proved in [19]. Next, we aim to obtain an upper bound on $\|v^+\|_W$ in terms of q^-. By definition

$$\|v^+\|_W^2 = \|v^+\|_{L^2(U)}^2 + \|\mathcal{T}v^+\|_{L^2(U)}^2 + \|v^+\|_{L^2_w(\Gamma)}^2 .$$

By definition of v and the above argument, $\|\mathcal{T}v^+\|_{L^2(U)} = \|q^-\|_{L^2(U)}$. Next, by using equation (2.16) and the Cauchy-Schwarz inequality,

$$\|v^+\|_{L^2(U)}^2 = \int_U \left(\int_0^t q^-(\mathbf{x}_- + s\boldsymbol{\omega}, \boldsymbol{\omega}, e)ds \right)^2 d\boldsymbol{\omega} de d\mathbf{x} \le c \|q^-\|_{L^2(U)}^2 .$$

Similarly

$$\begin{aligned}
\|v^+\|_{L^2_w(\Gamma)}^2 &= \int_\Gamma \left(\int_0^t q^-(\mathbf{x}_- + s\boldsymbol{\omega}, \boldsymbol{\omega}, e)ds \right)^2 |\boldsymbol{\omega} \cdot \boldsymbol{\nu}| d\Gamma \\
&\le c \int_\Gamma \int_0^t q^-(\mathbf{x}_- + s\boldsymbol{\omega}, \boldsymbol{\omega}, e)^2 |\boldsymbol{\omega} \cdot \boldsymbol{\nu}| ds d\Gamma.
\end{aligned}$$

Since $\mathbf{x}_-$ is part of the inflow boundary, when integrating over Γ_- we have $t = 0$, which implies

$$\int_{\Gamma_-} \int_0^t q^-(\mathbf{x}_- + s\boldsymbol{\omega}, \boldsymbol{\omega}, e)^2 |\boldsymbol{\omega} \cdot \boldsymbol{\nu}| ds d\Gamma = 0.$$

Therefore,

$$\|v^+\|^2_{L^2_w(\Gamma)} \le c \int_{\Gamma_+} \int_0^t q^-(\mathbf{x}_- + s\boldsymbol{\omega}, \boldsymbol{\omega}, e)^2 |\boldsymbol{\omega} \cdot \boldsymbol{\nu}| ds d\Gamma$$
$$\le c \|q^-\|^2_{L^2(U)}.$$

Combining these three inequalities yields $\|v^+\|_W \le c\|q^-\|_Q$. The inf-sup condition is then obtained as follows:

$$b(v^+, q^-) = -(\mathcal{T} v^+, q^-) = \|q^-\|^2_Q \ge \frac{1}{c} \|v^+\|_W \|q^-\|_Q.$$

The result follows by rearranging terms. □

The following lemma is a Poincare type inequality for functions in W which we provide without proof. The inequality has been proved in [35] and used in [18, 19].

Lemma 2.6. *The inequality*

$$\|u\|_{L^2(U)} \le c\left(\|\mathcal{T} u\|_{L^2(U)} + \|u\|_{L^2_w(\Gamma)}\right)$$

holds for all $u \in W$.

This inequality is used in the proof of the following lemma.

Lemma 2.7. *Assume that (1.5)–(1.7) hold. Let*

$$K := \{v^+ \in W^+ \mid b(v^+, q^-) = 0 \ \forall q^- \in Q^-\}.$$

Then the bilinear form $a(\cdot,\cdot)$ *is coercive over* K.

Proof. Let $u^+ \in K$. Then

$$a(u^+, u^+) = (u^+, u^+)_{L^2_w(\Gamma)} + (\sigma_t u^+, u^+) - (S u^+, u^+).$$

Following the argument of Lemma 2.4 it follows that

$$(\sigma_t u^+, u^+) - (Su^+, u^+) \geq 0.$$

Then because $u^+ \in K$

$$\begin{aligned} a(u^+, u^+) &\geq (u^+, u^+)_{L^2_w(\Gamma)} \\ &= b(u^+, q^-) + (u^+, u^+)_{L^2_w(\Gamma)} \\ &= -(\mathcal{T}u^+, q^-) + (u^+, u^+)_{L^2_w(\Gamma)} \end{aligned}$$

for all $q^- \in Q^-$. Noticing that $\mathcal{T}u^+ = (\mathcal{T}u)^- \in Q^-$, by letting $q^- = \mathcal{T}u^+$ it follows that

$$a(u^+, u^+) \geq \|\boldsymbol{\omega} \cdot \nabla u^+\|^2_{L^2(U)} + \|u^+\|^2_{L^2_w(\Gamma)}.$$

Splitting up the summation and applying Lemma 2.6 then yields

$$\begin{aligned} a(u^+, u^+) &\geq \frac{1}{2C} \|u^+\|^2_{L^2(U)} + \frac{1}{2}\left(\|\boldsymbol{\omega} \cdot \nabla u^+\|^2_{L^2(U)} + \|u^+\|^2_{L^2_w(\Gamma)}\right) \\ &\geq c\|u^+\|^2_W \end{aligned}$$

for all $u^+ \in K$. $\square$

Lemma 2.8. *Assume that (1.5)–(1.7) hold. Then the bilinear forms* $a : W^+ \times W^+ \to \mathbb{R}$, $b : W^+ \times Q^- \to \mathbb{R}$, *and* $c : Q^- \times Q^- \to \mathbb{R}$ *are continuous.*

Proof. We prove each individually.

1. Let $u^+, v^+ \in W^+$. Applying the Cauchy-Schwarz inequality and using the upper bound on σ_t yields

$$\begin{aligned} a(u^+, v^+) &\leq (u^+, v^+)_{L^2_w(\Gamma)} + (\sigma_t u^+, v^+) + |(Su^+, v^+)| \\ &\leq \|u^+\|_{L^2_w(\Gamma)} \|v^+\|_{L^2_w(\Gamma)} + c\|u^+\|_{L^2(U)} \|v^+\|_{L^2(U)} + \|Su^+\|_{L^2(U)} \|v^+\|_{L^2(U)}. \end{aligned}$$

Consider the scattering term. Using the upper bounds on σ_s and $\mathcal{P}$ followed by the Cauchy-Schwarz inequality yields

$$\begin{aligned}\|Su^+\|^2_{L^2(U)} &= \int_U \left(\int_E \int_\Omega \sigma_s(\mathbf{x}, e')\mathcal{P}(\mathbf{x}, \boldsymbol{\omega}, \boldsymbol{\omega}', e, e') u^+(\mathbf{x}, \boldsymbol{\omega}', e') d\boldsymbol{\omega}' de' \right)^2 dU \\ &\leq c \int_X \left(\int_E \int_\Omega u^+(\mathbf{x}, \boldsymbol{\omega}', e') d\boldsymbol{\omega}' de' \right)^2 d\mathbf{x} \\ &\leq c \|u^+\|^2_{L^2(U)} .\end{aligned}$$

Returning to the bilinear form and using the definition of the norm $\|\cdot\|_W$ produces the result:

$$\begin{aligned}a(u^+, v^+) &\leq c \left(\|u^+\|_{L^2_w(\Gamma)} + \|u^+\|_{L^2(U)} \right) \left(\|v^+\|_{L^2_w(\Gamma)} + \|v^+\|_{L^2(U)} \right) \\ &\leq c \|u^+\|_W \|v^+\|_W .\end{aligned}$$

2. Let $u^+ \in W^+$ and $q^- \in Q^-$. Then applying the Cauchy-Schwarz inequality and the definition of $\|\cdot\|_W$ and $\|\cdot\|_Q$ results in

$$b(u^+, q^-) \leq |(\mathcal{T} u^+, q^-)| \leq \|\mathcal{T} u^+\|_{L^2(U)} \|q^-\|_{L^2(U)} \leq c \|u^+\|_W \|q^-\|_Q .$$

3. Let $q^- \in Q^-$. Similar to the proof in part 1 we have via the Cauchy-Schwarz inequality and the upper bounds on σ_t and $\mathcal{P}$

$$c(q^-, q^-) \leq c \|q^-\|^2_{L^2(U)} + \|Sq^-\|_{L^2(U)} \|q^-\|_{L^2(U)} \leq c \|q^-\|^2_Q .$$

□

Equipped with the previous lemmas we are now ready to discuss the existence and uniqueness of Problem 2.2.

Theorem 2.9. *Assume (1.5)–(1.7) hold. Then for* $f \in L^2(U)$ *and* $g \in L^2_w(\Gamma)$ *there exists a unique* $(u^+, u^-) \in W^+ \times Q^-$ *satisfying Problem 2.2.*

Proof. Lemmas 2.4, 2.5, 2.7, and 2.8 show that the bilinear forms in Problem 2.2 satisfy the conditions of Theorem 2.3. Therefore, the result follows from Theorem 2.3. □

The solution (u^+, u^-) of Problem 2.2 corresponds to the even and odd part of the solution $u = u^+ + u^-$. For u to satisfy the original BVP (1.2)–(1.3) it must be shown that $u^- \in W^-$, and that $u = u^+ + u^-$ satisfies equation (1.2) as well as the boundary conditions (1.3). This result is given by the next theorem. For its proof we refer the reader to the argument given in [18], which shows the regularity result for the energy independent problem.

Theorem 2.10. *Let* $(u^+, u^-) \in W^+ \times Q^-$ *satisfy Problem 2.2 for* $f \in L^2(U)$ *and* $g \in L^2_w(\Gamma)$*. Then* $u^- \in W^-$ *and* $u = u^+ + u^-$ *satisfies the original RTE BVP (1.2)–(1.3).*

In this chapter, the existence and uniqueness of a solution for the RTE BVP (1.2)–(1.3) has been established. This was obtained by rewriting the BVP as two coupled variational equations resulting in a mixed formulation. The well-posedness of the formulation was established by applying the existence and uniqueness theory for saddle point problems with a penalty term. Finally, the solution of (1.2)–(1.3) is obtained by noting that solutions of the mixed formulation have sufficient regularity to satisfy the original BVP.

CHAPTER 3
ENERGY DISCRETIZATION

In this chapter, an approximation of the energy variable in (1.2)–(1.3) is introduced alongside a rigorous analysis of the obtained semi-discretization. The solution existence and uniqueness for the semi-discretization is shown to follow from established monoenergetic theory; after which, the validity of the approximation is established by providing an error estimate for the error between solutions of the semi-discretization and solutions of the original RTE (1.2)–(1.3).

3.1 Energy Discretization

Begin by partitioning the energy domain, $E = [e_{\min}, e_{\max}]$, into N energy groups, E_i, so

$$E = \bigcup_{i=1}^{N} E_i,$$

here we have $e_{\min} = e_1 < e_2 < \cdots < e_N < e_{N+1} \equiv e_{\max}$. In discussing energy discretizations, the convention within the literature is to order the energy groups from higher energies to lower energies (cf. [16, 33, 34]). We follow this convention, defining $E_i = [e_{N-i+1}, e_{N-i+2}]$ for $i = 1, \ldots, N$; this results in $E_1 = [e_N, e_{N+1}]$ and $E_N = [e_1, e_2]$. For each E_i, let $|E_i|$ denote the length of the interval, and, therefore, $|E| = \sum_{i=1}^{N} |E_i|$. In each energy group, approximate the energy by a constant and let $u_i(\mathbf{x}, \boldsymbol{\omega})$ denote an approximation to the solution $u(\mathbf{x}, \boldsymbol{\omega}, e)$ for $e \in E_i$, see Figure 3.1. Additionally, denote by $f_i(\mathbf{x}, \boldsymbol{\omega})$ and $g_i(\mathbf{x}, \boldsymbol{\omega})$ approximations of $f(\mathbf{x}, \boldsymbol{\omega}, e)$ and $g(\mathbf{x}, \boldsymbol{\omega}, e)$ for $e \in E_i$, $i = 1, \ldots, N$.

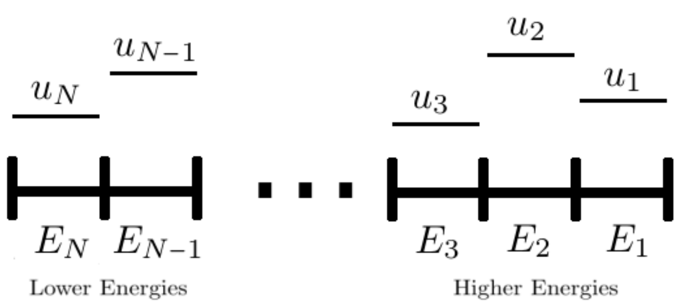

Figure 3.1: Energy group discretization.

For simplicity in the following analysis define $D = X \times \Omega$. Analogous to the definition of $\Gamma_\pm$ in (1.1) for U, also define $\partial D_\pm$ for D. Now, for each $i = 1, \ldots, N$, we introduce a BVP for $u_i(\mathbf{x}, \boldsymbol{\omega})$ by

$$\boldsymbol{\omega} \cdot \nabla_x u_i(\mathbf{x}, \boldsymbol{\omega}) + \sigma_{t,i}(\mathbf{x}) u_i(\mathbf{x}, \boldsymbol{\omega}) - \sum_{j=1}^{i} |E_j| S_{i,j} u_j(\mathbf{x}, \boldsymbol{\omega}) = f_i(\mathbf{x}, \boldsymbol{\omega}) \text{ in } D \tag{3.1}$$

$$u_i(\mathbf{x}, \boldsymbol{\omega}) = g_i(\mathbf{x}, \boldsymbol{\omega}) \text{ on } \partial D_- \tag{3.2}$$

where

$$S_{i,j} u(\mathbf{x}, \boldsymbol{\omega}) \coloneqq \sigma_{s,j}(\mathbf{x}) \int_\Omega \mathcal{P}_{i,j}(\mathbf{x}, \boldsymbol{\omega}, \boldsymbol{\omega}') u(\mathbf{x}, \boldsymbol{\omega}') d\boldsymbol{\omega}', \tag{3.3}$$

$1 \le j \le i, 1 \le i \le N$. Here, $\mathcal{P}_{i,j}(\mathbf{x}, \boldsymbol{\omega}, \boldsymbol{\omega}')$ is an approximation of $\mathcal{P}(\mathbf{x}, \boldsymbol{\omega}, \boldsymbol{\omega}', e, e')$ where $e \in E_i$ and $e' \in E_j$. The function $\sigma_{t,i}$ is the total cross section for energy group E_i and can be written in terms of the absorption and scattering cross sections, $\sigma_{t,i} = \sigma_{a,i} + \sigma_{s,i}$, for $1 \le i \le N$. The energy group parameter functions $\sigma_{t,i}, \sigma_{s,i}, \sigma_{a,i}$ and data functions f_i, g_i are defined by the average value of their energy dependent counterparts over each energy group. For example, the scattering cross section over energy group E_i is given by

$$\sigma_{s,i}(\mathbf{x}) := \fint_{E_i} \sigma_s(\mathbf{x}, e)de = \frac{1}{|E_i|}\int_{E_i} \sigma_s(\mathbf{x}, e)de,$$

for $1 \le i \le N$. Note that the notation $\fint$ is used to denote integration in the average sense. The energy group version of $\mathcal{P}$ is similarly defined and is given by

$$\mathcal{P}_{i,j}(\mathbf{x}, \boldsymbol{\omega}, \boldsymbol{\omega}') := \fint_{E_i}\fint_{E_j} \mathcal{P}(\mathbf{x}, \boldsymbol{\omega}, \boldsymbol{\omega}', e, e')de'de, \tag{3.4}$$

for $1 \le j \le i, 1 \le i \le N$. With the introduction of the discretization, we proceed by next introducing relevant function spaces for the analysis.

The solution existence and uniqueness of the energy semi-desciretization will rely on results presented in [19]; as a result, our choice of function spaces are analogous. With the boundary, associate the weighted L^2 space

$$L^2_w(\Gamma) := \left\{ v : \Gamma \to \mathbb{R} \mid \int_\Gamma v^2 |\boldsymbol{\omega}\cdot\boldsymbol{\nu}| ds(\mathbf{x}) d\boldsymbol{\omega} de \right\},$$

with inner product

$$(u,v)_{L^2_w(\Gamma)} := (|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|u, v)_{L^2(\Gamma)} = \int_\Gamma uv|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| ds(\mathbf{x}) d\boldsymbol{\omega} de$$

and norm $\|u\|_{L^2_w(\Gamma)} := \sqrt{(u,u)_{L^2_w(\Gamma)}}$. In a similiar manner, define $L^2_w(\partial D_-)$. Then we may define

$$V := \left\{ u \in L^2(D) \mid \boldsymbol{\omega}\cdot\nabla_x u \in L^2(D) \text{ and } u|_{\partial D_-} \in L^2_w(\partial D_-) \right\},$$

with corresponding inner product and norm

$$(u,v)_V := (\boldsymbol{\omega}\cdot\nabla_x u, \boldsymbol{\omega}\cdot\nabla_x v)_{L^2(D)} + (u,v)_{L^2(D)} + (u,v)_{L^2_w(\partial D_-)}$$

and $\|u\|_V := \sqrt{(u,u)_V}$. With these quantities defined, we may proceed with the analysis of the energy semi-discrete problem (3.1)–(3.2).

3.2 Existence and Uniqueness

The goal of this section is to establish the existence and uniqueness of a solution for the semi-discrete problem (3.1)–(3.2). The section begins by establishing two preliminary results. The first result establishes a recursive relation among sequences; this relation will be used extensively throughout this work. The second result establishes that the discretized scattering operator is a bounded linear operator on $L^2(D)$. Following this, the solution existence and uniqueness of the semi-discretization is established using the two preliminary results and a result from the monoenergetic RTE theory.

The recurrence below is form of discrete Gronwall's inequality which suites the purpose of this work. For a slightly stronger version, we refer the reader to [14].

LEMMA 3.1. *Suppose a_n, b_n, and γ_n are non-negative sequences. If*

$$a_n \le \gamma_n + \sum_{k=1}^{n-1} a_k b_k$$

for each n, then

$$a_n \le \gamma_n + \exp\left(\sum_{k=1}^{n} b_k\right)\left(\sum_{j=1}^{n-1} b_j \gamma_j\right).$$

Proof. Begin by unrolling the recurrence:

$$\begin{aligned} a_n &\le \gamma_n + a_{n-1}b_{n-1} + \sum_{k=1}^{n-2} a_k b_k \\ &\le \gamma_n + b_{n-1}\left(\gamma_{n-1} + \sum_{k=1}^{n-2} a_k b_k\right) + \sum_{k=1}^{n-2} a_k b_k \\ &\le \gamma_n + b_{n-1}\gamma_{n-1} + (b_{n-1}+1)\left(\sum_{k=1}^{n-2} a_k b_k\right). \end{aligned}$$

Explicitly applying the bound one more time yields

$$
\begin{aligned}
a_n &\le \gamma_n + b_{n-1}\gamma_{n-1} + (b_{n-1}+1)\left(a_{n-2}b_{n-2} + \sum_{k=1}^{n-3} a_k b_k\right) \\
&\le \gamma_n + b_{n-1}\gamma_{n-1} + (b_{n-1}+1)\left(b_{n-2}\left(\gamma_{n-2} + \sum_{k=1}^{n-3} a_k b_k\right) + \sum_{k=1}^{n-3} a_k b_k\right) \\
&\le \gamma_n + b_{n-1}\gamma_{n-1} + (b_{n-1}+1)b_{n-2}\gamma_{n-2} + (b_{n-1}+1)(b_{n-2}+1)\left(\sum_{k=1}^{n-3} a_k b_k\right).
\end{aligned}
$$

This may be continued, resulting in

$$
\begin{aligned}
a_n &\le \gamma_n + b_{n-1}\gamma_{n-1} + (b_{n-1}+1)b_{n-2}\gamma_{n-2} + \cdots + (b_{N-1}+1)(b_{N-2}+1)\cdots(b_4+1)b_3\gamma_3 \\
&\quad + \cdots + (b_{N-1}+1)(b_{N-2}+1)\cdots(b_3+1)(b_2a_2 + b_1a_1) \\
&\le \gamma_n + b_{n-1}\gamma_{n-1} + (b_{n-1}+1)b_{n-2}\gamma_{n-2} + \cdots + (b_{N-1}+1)(b_{N-2}+1)\cdots(b_4+1)b_3\gamma_3 \\
&\quad + \cdots + (b_{N-1}+1)(b_{N-2}+1)\cdots(b_3+1)(b_2(\gamma_2 + b_1a_1) + b_1a_1) \\
&\le \gamma_n + b_{n-1}\gamma_{n-1} + (b_{n-1}+1)b_{n-2}\gamma_{n-2} + \cdots + (b_{N-1}+1)(b_{N-2}+1)\cdots(b_3+1)b_2\gamma_2 \\
&\quad + (b_{N-1}+1)(b_{N-2}+1)\cdots(b_2+1)b_1a_1 \\
&\le \gamma_n + b_{n-1}\gamma_{n-1} + (b_{n-1}+1)b_{n-2}\gamma_{n-2} + \cdots + (b_{N-1}+1)(b_{N-2}+1)\cdots(b_3+1)b_2\gamma_2 \\
&\quad + (b_{N-1}+1)(b_{N-2}+1)\cdots(b_2+1)b_1\gamma_1.
\end{aligned}
$$

More compactly

$$
a_n \le \gamma_n + \sum_{k=1}^{n-1} \gamma_k b_k \Pi_{i=k+1}^{n-1}(b_i+1).
$$

Each product above has the bound

$$
\Pi_{i=k+1}^{n-1}(b_i+1) \le \exp\left(\sum_{i=k+1}^{n-1} b_i\right) \le \exp\left(\sum_{i=1}^{n-1} b_i\right).
$$

Using this, we may find

$$
a_n \le \gamma_n + \exp\left(\sum_{k=1}^{n} b_k\right)\left(\sum_{i=1}^{n-1} b_i\gamma_i\right).
$$

□

Lemma 3.2. *$S_{i,j} : L^2(D) \to L^2(D)$ defined by (3.3) is a bounded linear operator for $1 \le i \le N, 1 \le j \le N$.*

Proof. Let $v \in L^2(D)$. Using the boundedness of σ_s and $\mathcal{P}$ given by (1.5) and (1.6) and applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\|S_{i,j}v\|_{L^2(D)} &\le \left\| \sigma_{s,j}(\mathbf{x}) \int_\Omega \mathcal{P}_{i,j}(\mathbf{x},\boldsymbol{\omega},\boldsymbol{\omega}')v(\mathbf{x},\boldsymbol{\omega}')d\boldsymbol{\omega}' \right\|_{L^2(D)} \\
&\le \|\sigma_{s,j}\mathcal{P}_{i,j}\|_{L^\infty(D)} \left\| \int_\Omega |v(\mathbf{x},\boldsymbol{\omega}')| d\boldsymbol{\omega}' \right\|_{L^2(D)} \\
&\le C \|v\|_{L^2(D)}
\end{aligned}$$

where C is dependent on the upper bounds of σ_s and $\mathcal{P}$. □

The equations given by (3.1)–(3.2) constitute a coupled system of monoenergetic RTE BVPs. The well posedness of the monoenergetic form under conditions permitting vanishing absorption has been established in [19]. Further, the no up-scatter assumption (1.6) induces a lower triangular structure in the system of BVPs (3.1)–(3.2). This allows the existence and uniqueness of a solution for the semi-discretization to be established using a forward substitution argument and is detailed in the proof of the following proposition.

Proposition 3.3. *Given $f \in L^2(U)$ and $g \in L^2_w(\Gamma_-)$, there exists a unique $\{u_i\}_{i=1}^N \in (V)^N$ such that (3.1)-(3.2) holds a.e. Further, the bound*

$$\sum_{i=1}^N |E_i| \left(\|u_i^+\|_V^2 + \|u_i^-\|_{L^2(D)}^2 \right) \le C \sum_{i=1}^N |E_i| \left(\|f_i\|_{L^2(D)}^2 + \|g_i\|_{L^2_w(\partial D_-)}^2 \right).$$

holds, where $u_i = u_i^+ + u_i^-$.

Proof. First, the existence and uniqueness is established using a forward substitution argument. Consider the system of BVPs (3.1)–(3.2). For $i = 1$ we have

$$\boldsymbol{\omega} \cdot \nabla_x u_1 + \sigma_{t,1} u_1 - |E_1| S_{1,1} u_1 = f_1 \text{ in } D \tag{3.5}$$

$$u_1 = g_1 \text{ on } \partial D_-.$$

From the monoenergetic theory there exists a unique $u_1 \in V$ satisfying the BVP and the even and odd portions of the solution satisfy the bound

$$\|u_1^+\|_V + \|u_1^-\|_{L^2(D)} \leq C \left(\|f_1\|_{L^2(D)} + \|g_1\|_{L^2_w(\partial D_-)} \right),$$

where C is independent of the lower bounds on the cross sections ([19]). Upon squaring both sides one may obtain

$$\|u_1^+\|_V^2 + \|u_1^-\|_{L^2(D)}^2 \leq C \left(\|f_1\|_{L^2(D)}^2 + \|g_1\|_{L^2_w(\partial D_-)}^2 \right). \tag{3.6}$$

For $i = 2$ one has

$$\boldsymbol{\omega} \cdot \nabla_x u_2 + \sigma_{t,2} u_2 - |E_2| S_{2,2} u_2 - |E_1| S_{2,1} u_1 = f_2 \text{ in } D,$$

$$u_2 = g_2 \text{ on } \partial D_-.$$

Because u_1 is known, this may be rewritten as

$$\boldsymbol{\omega} \cdot \nabla_x u_2 + \sigma_{t,2} u_2 - |E_2| S_{2,2} u_2 = \tilde{f}_2 \text{ in } D, \tag{3.7}$$

$$u_2 = g_2 \text{ on } \partial D_-,$$

where $\tilde{f}_2 = |E_1| S_{2,1} u_1 + f_1$. The monoenergetic theory can again be employed to obtain a unique $u_2 \in V$ satisfying

$$\|u_2^+\|_V^2 + \|u_2^-\|_{L^2(D)}^2 \leq C \left(\left\|\tilde{f}_2\right\|_{L^2(D)}^2 + \|g_2\|_{L^2_w(\partial D_-)}^2 \right).$$

Substituting in the definition of $\tilde{f}_2$ and applying the triangle inequality results in

$$\|u_2^+\|_V^2 + \|u_2^-\|_{L^2(D)}^2 \le C\left(|E_1|^2 \|S_{2,1}(u_1^+ + u_1^-)\|_{L^2(D)}^2 + \|f_2\|_{L^2(D)}^2 + \|g_2\|_{L_w^2(\partial D_-)}^2\right). \quad (3.8)$$

Continuing in this way, existence and uniqueness of a solution for (3.1)–(3.2) can be established.

We next establish the bound presented in the claim. Building upon inequalities (3.6) and (3.8), the inequality for general i is given by

$$\|u_i^+\|_V^2 + \|u_i^-\|_{L^2(D)}^2 \le C\left(\left\|\sum_{j=1}^{i-1} |E_j| S_{i,j}(u_j^+ + u_j^-)\right\|_{L^2(D)}^2 + \|f_i\|_{L^2(D)}^2 + \|g_i\|_{L_w^2(\partial D_-)}^2\right) \quad (3.9)$$

for $i = 1, \ldots, N$. Applying the triangle inequality, Lemma 3.2, and using the bound

$$\left\|u_j^+\right\|_{L^2(D)} \le \left\|u_j^+\right\|_V,$$

yields

$$\begin{aligned}
\left\|\sum_{j=1}^{i-1} |E_j| S_{i,j}(u_j^+ + u_j^-)\right\|_{L^2(D)}^2 &\le \sum_{k=1}^{i-1} |E_k| \sum_{j=1}^{i-1} |E_j| \left\|S_{i,j}(u_j^+ + u_j^-)\right\|_{L^2(D)}^2 \\
&\le C \sum_{j=1}^{i-1} |E_j| \left(\left\|u_j^+\right\|_{L^2(D)}^2 + \left\|u_j^-\right\|_{L^2(D)}^2\right) \\
&\le C \sum_{j=1}^{i-1} |E_j| \left(\left\|u_j^+\right\|_V^2 + \left\|u_j^-\right\|_{L^2(D)}^2\right). \qquad (3.10)
\end{aligned}$$

Combining inequality (3.9) and (3.10) results in

$$\begin{aligned}
&\|u_i^+\|_V^2 + \|u_i^-\|_{L^2(D)}^2 \\
&\qquad \le C\left[\sum_{j=1}^{i-1} |E_j| \left(\left\|u_j^+\right\|_V^2 + \left\|u_j^-\right\|_{L^2(D)}^2\right) + \|f_i\|_{L^2(D)}^2 + \|g_i\|_{L_w^2(\partial D_-)}^2\right]. \qquad (3.11)
\end{aligned}$$

Applying Lemma 3.1 with

$$\gamma_i = \left(\|f_i\|_{L^2(D)}^2 + \|g_i\|_{L_w^2(\partial D_-)}^2\right),$$

$a_j = \left(\|u_j^+\|_V^2 + \|u_j^-\|_{L^2(D)}^2 \right)$, and $b_j = C|E_j|$, the inequality becomes

$$
\begin{aligned}
&\|u_i^+\|_V^2 + \|u_i^-\|_{L^2(D)}^2 \\
&\qquad \le C\left(\|f_i\|_{L^2(D)}^2 + \|g_i\|_{L^2_w(\partial D_-)}^2\right) + C\sum_{k=1}^{i-1}|E_k|\left(\|f_k\|_{L^2(D)}^2 + \|g_k\|_{L^2_w(\partial D_-)}^2\right). \qquad (3.12)
\end{aligned}
$$

Extending the summation from $i-1$ to N, multiplying by $|E_i|$ and summing over $1 \le i \le N$ yields

$$
\begin{aligned}
&\sum_{i=1}^{N}|E_i|\left(\|u_i^+\|_V^2 + \|u_i^-\|_{L^2(D)}^2\right) \\
&\qquad \le C\sum_{i=1}^{N}|E_i|\left(\|f_i\|_{L^2(D)}^2 + \|g_i\|_{L^2_w(\partial D_-)}^2\right) + C|E|\sum_{k=1}^{N}|E_k|\left(\|f_k\|_{L^2(D)}^2 + \|g_k\|_{L^2_w(\partial D_-)}^2\right).
\end{aligned}
$$

Grouping the summations and redefining the constant yields the claim. □

Next, through the following error analysis, it is shown that this semi-discrete system is a good approximation for the energy-dependent RTE.

3.3 Error Analysis

In this section, under additional regularity assumptions, we provide an error estimate for the error between the solution of the semi-discrete problem (3.1)–(3.2) and the solution of the original BVP (1.2)–(1.3). Corresponding to a partition $E = \bigcup_{i=1}^{N} E_i$, let $\{u_i\}_{i=1}^{N}$ denote the solution of the semi-discrete system (3.1)–(3.2). For $e_i \in E_i$, define $\varepsilon_i(\mathbf{x}, \boldsymbol{\omega}, e_i) := u(\mathbf{x}, \boldsymbol{\omega}, e_i) - u_i(\mathbf{x}, \boldsymbol{\omega})$, where we have the splitting $\varepsilon_i = \varepsilon_i^+ + \varepsilon_i^-$ with $\varepsilon_i^+ = u^+ - u_i^+$ and $\varepsilon_i^- = u^- - u_i^-$ for $1 \le i \le N$. Because u satisfies (1.2)–(1.3) and $\{u_i\}_{i=1}^{N}$ satisfies (3.1)–(3.2), after suppressing the spatial and angular variables, the following equality may be obtained

$$\boldsymbol{\omega} \cdot \nabla_x \left[u(e_i) - u_i \right] + \sigma_{t,i} \left[u(e_i) - u_i \right] \\ = \sum_{j=1}^{i} |E_j| S_{i,j} \left[u(e_j) - u_j \right] + \Theta_i(\{e_j\}_{j=1}^{i}) + \Psi_i(e_i)$$

where $u(e_i)$ denotes $u(\mathbf{x}, \boldsymbol{\omega}, e_i)$, $\Theta_i(\{e_j\}_{j=1}^{i})$ denotes

$$\Theta_i(\mathbf{x}, \boldsymbol{\omega}, \{e_j\}_{j=1}^{i}) = (\mathcal{S}u)(\mathbf{x}, \boldsymbol{\omega}, e_i) - \sum_{j=1}^{i} |E_j| (S_{i,j} u(e_j))(\mathbf{x}, \boldsymbol{\omega}), \tag{3.13}$$

and $\Psi_i(e_i)$ denotes

$$\Psi_i(\mathbf{x}, \boldsymbol{\omega}, e_i) = \left[\sigma_{t,i}(\mathbf{x}) - \sigma_t(\mathbf{x}, e_i) \right] u(\mathbf{x}, \boldsymbol{\omega}, e_i) + \left[f(\mathbf{x}, \boldsymbol{\omega}, e_i) - f_i(\mathbf{x}, \boldsymbol{\omega}) \right]. \tag{3.14}$$

Therefore, the following system of BVPs can be constructed for the error over each energy group

$$\boldsymbol{\omega} \cdot \nabla_x \varepsilon_i(e_i) + \sigma_{t,i} \varepsilon_i(e_i) - \sum_{j=1}^{i} |E_j| S_{i,j} \varepsilon_j(e_j) = \Theta_i(\{e_j\}_{j=1}^{i}) + \Psi_i(e_i) \text{ in } D, \tag{3.15}$$

$$\varepsilon_i(e_i) = g(e_i) - g_i \text{ on } \partial D_-. \tag{3.16}$$

In the above discretization, we have approximated various functions using their average value over an energy group. Integrability is a sufficient condition for the average value of a function over an open ball of decreasing radius to approach the function value; however, we wish to establish a bound in terms of the interval size. Therefore, next we establish the closeness of these approximations as a function of energy group size. We begin by proving one lemma in an abstract setting, which will be used frequently in the subsequent analysis.

Lemma 3.4. *Let $I = [a,b] \subset \mathbb{R}$ and let Y be a bounded subset of $\mathbb{R}^3$. Given a function $h(t,y) : I \times Y \to \mathbb{R}$, denote its average value over I by $\hat{h}(y) := ⨍_I h(s,y)ds$. If $h \in L^2(Y, H^1(I))$ then*

$$\left\|\hat{h} - h\right\|_{L^2(I\times Y)} \le |I| |h|_{L^2(Y,H^1(I))},$$

where $|\cdot|_{L^2(Y,H^1(I))}$ denotes the semi-norm of $L^2(Y, H^1(I))$.

Proof. We begin with

$$|⨍_I h(s,y)ds - h(t,y)| \le ⨍_I |\int_t^s h'(\xi,y)d\xi| ds \le \int_I |h'(\xi,y)|d\xi.$$

Squaring both sides and applying the Cauchy-Schwarz inequality results in

$$|⨍_I h(s,y)ds - h(t,y)|^2 \le |I| \int_I |h'(\xi,y)|^2 d\xi.$$

Integrating t over I and y over Y we have

$$\int_I \left\|\hat{h}(y) - h(t,y)\right\|^2_{L^2(Y)} dt \le |I|^2 \int_I \|h'(\xi,y)\|^2_{L^2(Y)} d\xi.$$

The claim then follows by taking the square root of both sides and using the definition of the semi-norm of $L^2(Y, H^1(I))$. □

The next two lemmas relate to the consistency error incurred by the energy approximation of the scattering operator. First, we prove a result similar to the above for the double average defined by equation (3.4). It will be necessary to make some additional regularity assumptions. Specifically, that $\mathcal{P}$ contains weak derivatives in both energy variables which are square integrable with respect to the independent variables. Formally, it is assumed that

$$\mathcal{P} \in L^2(X \times \Omega^2, H^1(E^2)). \tag{3.17}$$

Below the notation $\frac{\partial \mathcal{P}}{\partial e}$ and $\frac{\partial \mathcal{P}}{\partial e'}$ are used to denote the weak derivative of the function $\mathcal{P}(\mathbf{x}, \boldsymbol{\omega}, \boldsymbol{\omega}', e, e')$ with respect to the variables e and e', respectively.

Lemma 3.5. *Assume (3.17) holds. Then*

$$\|\mathcal{P} - \mathcal{P}_{i,j}\|_{L^2(X\times\Omega^2\times E_i\times E_j)} \le C h_e \left(\left\| \frac{\partial \mathcal{P}}{\partial e} \right\|^2_{L^2(X\times\Omega^2\times E_i\times E_j)} + \left\| \frac{\partial \mathcal{P}}{\partial e'} \right\|^2_{L^2(X\times\Omega^2\times E_i\times E_j)} \right)^{1/2}$$

for $1 \le j \le i$ *with* $1 \le i \le N$.

Proof. Let $e_i \in E_i$ and $e_j \in E_j$. Consider the difference

$$|\mathcal{P}(e_i, e_j) - \mathcal{P}_{i,j}| = \left| \fint_{E_i} \fint_{E_j} (\mathcal{P}(e_i, e_j) - \mathcal{P}(y_i, y_j))\, dy_j dy_i \right|.$$

Addition and subtraction within the integrand, followed by the triangle inequality, and simplifying we have

$$\begin{aligned} |\mathcal{P}(e_i, e_j) - \mathcal{P}_{i,j}| \le & \left| \fint_{E_i} (\mathcal{P}(e_i, e_j) - \mathcal{P}(y_i, e_j))\, dy_i \right| \\ & + \left| \fint_{E_i} \fint_{E_j} (\mathcal{P}(y_i, e_j) - \mathcal{P}(y_i, y_j))\, dy_j dy_i \right|. \end{aligned}$$

Applying an argument similar to that in the proof of Lemma 3.4 we arrive at

$$|\mathcal{P}(e_i, e_j) - \mathcal{P}_{i,j}| \le \int_{E_i} \left| \frac{\partial \mathcal{P}}{\partial e}(\xi_1, e_j) \right| d\xi_1 + \fint_{E_i} \int_{E_j} \left| \frac{\partial \mathcal{P}}{\partial e'}(y_i, \xi_2) \right| d\xi_2 dy_i.$$

Squaring both sides and using the Cauchy-Schwarz inequality leads to

$$\begin{aligned} |\mathcal{P}(e_i, e_j) - \mathcal{P}_{i,j}|^2 &\le C \left(\int_{E_i} \left| \frac{\partial \mathcal{P}}{\partial e}(\xi_1, e_j) \right| d\xi_1 \right)^2 + C \left(\fint_{E_i} \int_{E_j} \left| \frac{\partial \mathcal{P}}{\partial e'}(y_i, \xi_2) \right| d\xi_2 dy_i \right)^2 \\ &\le C|E_i| \int_{E_i} \left| \frac{\partial \mathcal{P}}{\partial e}(\xi_1, e_j) \right|^2 d\xi_1 + C|E_j| \fint_{E_i} \int_{E_j} \left| \frac{\partial \mathcal{P}}{\partial e'}(y_i, \xi_2) \right|^2 d\xi_2 dy_i. \end{aligned}$$

Integrating with respect to e_i over E_i and e_j over E_j we have

$$\int_{E_i}\int_{E_j} |\mathcal{P}(e_i,e_j) - \mathcal{P}_{i,j}|^2 de_j de_i$$
$$\leq C|E_i|^2 \int_{E_j}\int_{E_i} \left|\frac{\partial \mathcal{P}}{\partial e}(\xi_1, e_j)\right|^2 d\xi_1 de_j + C|E_j|^2 \int_{E_i}\int_{E_j} \left|\frac{\partial \mathcal{P}}{\partial e'}(y_i, \xi_2)\right|^2 d\xi_2 dy_i.$$

Bounding the measure of the energy groups by h_e and renaming the dummy variables within the integration, we have

$$\int_{E_i}\int_{E_j} |\mathcal{P}(e_i,e_j) - \mathcal{P}_{i,j}|^2 de_j de_i \leq Ch_e^2 \int_{E_i}\int_{E_j} \left(\left|\frac{\partial \mathcal{P}}{\partial e'}(e_i, e_j)\right|^2 + \left|\frac{\partial \mathcal{P}}{\partial e'}(e_i, e_j)\right|^2\right) de_j de_i.$$

Integrating with respect to $\mathbf{x}$ over X, $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ over Ω, and taking the square root yields the result. □

The above provides a first step in estimating the error between the scattering term (1.4) and its approximation (3.3). The next lemma provides this estimate. For simplicity in the following analysis, define the quantity

$$\hat{\Theta}_i^2(\mathbf{x}, \boldsymbol{\omega}) := \fint_{E_1} \cdots \fint_{E_i} \Theta_i^2\left(\mathbf{x}, \boldsymbol{\omega}, \{e_j\}_{j=1}^i\right) de_i \ldots de_1, \tag{3.18}$$

where each $e_j \in \{e_j\}_{j=1}^i$ has been integrated over the corresponding energy group E_j, in the average sense. It is assumed that $u(\mathbf{x}, \boldsymbol{\omega}, e)$ and $\sigma_s(\mathbf{x}, e)$ admit weak derivatives in the energy variable that are square integrable with respect to their independent variables and further that u is bounded almost everywhere. That is,

$$u \in L^\infty(U) \cap L^2(D, H^1(E)) \tag{3.19}$$

$$\sigma_s \in L^2(X, H^1(E)) \tag{3.20}$$

The next lemma provides a bound on the error between the scattering term in the original RTE BVP and the semi-discrete system.

Lemma 3.6. *Assume that (3.17), (3.19), and (3.20) hold. Then the bound*

$$\left(\sum_{i=1}^{N} |E_i| \left\|\hat{\Theta}_i\right\|^2_{L^2(D)}\right)^{1/2} \le C h_e \left(\|u\|^2_{L^\infty(U)} + |u|^2_{L^2(D,H^1(E))}\right)^{1/2}$$

holds.

Proof. Let

$$\begin{aligned}\Delta(\mathbf{x},\boldsymbol{\omega},\boldsymbol{\omega}',e_i,e',e_j) &:= \sigma_s(\mathbf{x},e')\mathcal{P}(\mathbf{x},\boldsymbol{\omega},\boldsymbol{\omega}',e_i,e')u(\mathbf{x},\boldsymbol{\omega}',e') \\ &\quad - \sigma_{s,j}(\mathbf{x})\mathcal{P}_{i,j}(\mathbf{x},\boldsymbol{\omega},\boldsymbol{\omega}')u(\mathbf{x},\boldsymbol{\omega}',e_j). \end{aligned} \tag{3.21}$$

Then equation (3.13) may be written as

$$\Theta_i\left(\mathbf{x},\boldsymbol{\omega},\{e_j\}_{j=1}^{i}\right) = \sum_{j=1}^{i}\int_{E_j}\int_{\Omega}\Delta(\mathbf{x},\boldsymbol{\omega},\boldsymbol{\omega}',e_i,e',e_j)d\boldsymbol{\omega}'de'.$$

Multiple applications of the Cauchy-Schwarz inequality and extending the summation from i to N results in the bound

$$|\Theta_i\left(\mathbf{x},\boldsymbol{\omega},\{e_j\}_{j=1}^{i}\right)| \le C\left(\sum_{j=1}^{N}\int_{E_j}\int_{\Omega}|\Delta(\mathbf{x},\boldsymbol{\omega},\boldsymbol{\omega}',e_i,e',e_j)|^2 d\boldsymbol{\omega}'de'\right)^{1/2}.$$

Squaring both sides, integrating $\mathbf{x}$ over X and $\boldsymbol{\omega}$ over Ω yields

$$\left\|\Theta_i\left(\{e_j\}_{j=1}^{i}\right)\right\|^2_{L^2(X\times\Omega)} \le C\sum_{j=1}^{N}\int_{E_j}\|\Delta(e_i,e',e_j)\|^2_{L^2(X\times\Omega^2)}de'. \tag{3.22}$$

Integrate each e_j over E_j for $1 \le j \le N$, in the average sense; we then have

$$\left\|\hat{\Theta}_i\right\|^2_{L^2(X\times\Omega)} \le C\sum_{j=1}^{N} ⨍_{E_j}\int_{E_j}\|\Delta(e_i,e',e_j)\|^2_{L^2(X\times\Omega^2)}de'de_j.$$

Then integrating e_i over E_i in the average sense leads to

$$\left\|\hat{\Theta}_i\right\|^2_{L^2(X\times\Omega)} \le C\sum_{j=1}^{N} ⨍_{E_i} ⨍_{E_j}\int_{E_j}\|\Delta(e_i,e',e_j)\|^2_{L^2(X\times\Omega^2)}de'de_jde_i. \tag{3.23}$$

Through addition and subtraction we have the following equality

$$\Delta(e_i, e', e_j) = [\mathcal{P}(e_i, e') - \mathcal{P}_{i,j}]\sigma_s(e')u(e') + [\sigma_s(e') - \sigma_{s,j}]\mathcal{P}_{i,j}u(e') + [u(e') - u(e_j)]\sigma_{s,j}\mathcal{P}_{i,j},$$

where the spatial and angular variables have been suppressed. Squaring both sides and applying the Cauchy-Schwarz inequality results in

$$|\Delta(e_i, e', e_j)|^2 = C[[\mathcal{P}(e_i, e') - \mathcal{P}_{i,j}]^2(\sigma_s(e')u(e'))^2 + [\sigma_s(e') - \sigma_{s,j}]^2(\mathcal{P}_{i,j}u(e'))^2 + [u(e') - u(e_j)]^2(\sigma_{s,j}\mathcal{P}_{i,j})^2].$$

Combining the inequality for $|\Delta(e_i, e', e_j)|^2$ with equation (3.23), and using the upper bounds on σ_s, $\mathcal{P}$ and u we may obtain

$$\begin{aligned}\left\|\hat{\Theta}_i\right\|^2_{L^2(X\times\Omega)} \le C\sum_{j=1}^{N} \fint_{E_i}\fint_{E_j}\int_{E_j}\Bigg[&\|\sigma_s u\|^2_{L^\infty(U)}\|\mathcal{P}(e_i, e') - \mathcal{P}_{i,j}\|^2_{L^2(X\times\Omega^2)} \\ &+ \|\mathcal{P}u\|^2_{L^\infty(X\times\Omega^2\times E^2)}\|\sigma_s(e') - \sigma_{s,j}\|^2_{L^2(X\times\Omega^2)} \\ &+ \|\sigma_s\mathcal{P}\|^2_{L^\infty(X\times\Omega^2\times E^2)}\|u(e') - u(e_j)\|^2_{L^2(X\times\Omega^2)}\Bigg]de'de_jde_i.\end{aligned}$$

Then extending the sum on the right hand side from i to N, multiplying by $|E_i|$ and summing over $1 \le i \le N$ we have

$$\begin{aligned}\sum_{i=1}^{N}|E_i|\left\|\hat{\Theta}_i\right\|^2_{L^2(X\times\Omega)} \le C\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{E_i}\fint_{E_j}\int_{E_j}\Bigg[&\|\sigma_s u\|^2_{L^\infty(U)}\|\mathcal{P}(e_i, e') - \mathcal{P}_{i,j}\|^2_{L^2(X\times\Omega^2)} \\ &+ \|\mathcal{P}u\|^2_{L^\infty(X\times\Omega^2\times E^2)}\|\sigma_s(e') - \sigma_{s,j}\|^2_{L^2(X\times\Omega^2)} \\ &+ \|\sigma_s\mathcal{P}\|^2_{L^\infty(X\times\Omega^2\times E^2)}\|u(e') - u(e_j)\|^2_{L^2(X\times\Omega^2)}\Bigg]de'de_jde_i.\end{aligned}$$

After simplifying, the above may be written as

$$\sum_{i=1}^{N}|E_i|\left\|\hat{\Theta}_i\right\|^2_{L^2(X\times\Omega)} \le \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 \tag{3.24}$$

where

$$\mathbb{I}_1 := C\,\|\sigma_s u\|^2_{L^\infty(U)} \sum_{i=1}^{N}\sum_{j=1}^{N} \int_{E_i}\int_{E_j} \|\mathcal{P}(e_i,e') - \mathcal{P}_{i,j}\|^2_{L^2(X\times\Omega^2)}\,de'de_i$$

$$\mathbb{I}_2 := C\,\|\mathcal{P}u\|^2_{L^\infty(X\times\Omega^2\times E^2)} \sum_{j=1}^{N} \int_{E_j} \|\sigma_s(e') - \sigma_{s,j}\|^2_{L^2(X)}\,de'$$

$$\mathbb{I}_3 := C\,\|\sigma_s\mathcal{P}\|^2_{L^\infty(X\times\Omega^2\times E^2)} \sum_{j=1}^{N} ⨍_{E_j}\int_{E_j} \|u(e') - u(e_j)\|^2_{L^2(X)}\,de'de_j.$$

The claim follows by estimating the terms on the right hand side of (3.24). We now bound each of the three terms.

1. First consider $\mathbb{I}_1$. Applying Lemma 3.5 and extending the summation from i to N we have

$$\begin{aligned}\mathbb{I}_1 &\le C\,\|\sigma_s u\|^2_{L^\infty(U)}\,h_e^2 \sum_{i=1}^{N}\sum_{j=1}^{N}\int_{E_i}\int_{E_j}\left(\left\|\frac{\partial\mathcal{P}}{\partial e'}(e_i,e')\right\|^2_{L^2(X\times\Omega^2)} + \left\|\frac{\partial\mathcal{P}}{\partial e'}(e_i,e')\right\|^2_{L^2(X\times\Omega^2)}\right)de'de_i \\ &\le C\,\|u\|^2_{L^\infty(U)}\,h_e^2\int_E\int_E\left(\left\|\frac{\partial\mathcal{P}}{\partial e'}(e_i,e')\right\|^2_{L^2(X\times\Omega^2)} + \left\|\frac{\partial\mathcal{P}}{\partial e'}(e_i,e')\right\|^2_{L^2(X\times\Omega^2)}\right)de'de_i.\end{aligned}$$

   Simplifying further and recalling that the terms involving $\frac{\partial\mathcal{P}}{\partial e}$ and $\frac{\partial\mathcal{P}}{\partial e'}$ are finite, we have

$$\begin{aligned}\mathbb{I}_1 &\le C\,\|u\|^2_{L^\infty(U)}\,h_e^2\left(\left\|\frac{\partial\mathcal{P}}{\partial e}\right\|^2_{L^2(X\times\Omega^2\times E^2)} + \left\|\frac{\partial\mathcal{P}}{\partial e'}\right\|^2_{L^2(X\times\Omega^2\times E^2)}\right) \\ &\le C\,\|u\|^2_{L^\infty(U)}\,h_e^2. \end{aligned} \tag{3.25}$$

   where the constant now depends on the functions σ_s and $\mathcal{P}$.

2. We next bound $\mathbb{I}_2$. Applying Lemma 3.4 with $Y = X$ and $I = E_j$ leads to

$$\begin{aligned}\mathbb{I}_2 &\le C\,\|\mathcal{P}u\|^2_{L^\infty(X\times\Omega^2\times E^2)}\sum_{j=1}^{N}|E_j|^2\int_{E_j}\left\|\frac{\partial\sigma_s}{\partial e}(e')\right\|^2_{L^2(X)}de' \\ &\le Ch_e^2\,\|\mathcal{P}u\|^2_{L^\infty(X\times\Omega^2\times E^2)}\int_E\left\|\frac{\partial\sigma_s}{\partial e}(e')\right\|^2_{L^2(X)}de'.\end{aligned}$$

Simpliying further we find

$$\mathbb{I}_2 \leq Ch_e^2 \left\|\mathcal{P}u\right\|^2_{L^\infty(X\times\Omega^2\times E^2)} \left\|\frac{\partial\sigma_s}{\partial e}\right\|^2_{L^2(X\times E)} \leq Ch_e^2 \left\|u\right\|^2_{L^\infty(U)}, \tag{3.26}$$

where C now depends on the functions $\mathcal{P}$ and σ_s.

3. Lastly, we consider $\mathbb{I}_3$. Again we employ Lemma 3.4, this time with $Y = X\times\Omega$ and $I = E_j$. We have

$$\begin{aligned}
\mathbb{I}_3 &= C\left\|\sigma_s\mathcal{P}\right\|^2_{L^\infty(X\times\Omega^2\times E^2)} \sum_{j=1}^N \fint_{E_j}\int_{E_j} \left\|u(e') - u(e_j)\right\|^2_{L^2(X\times\Omega)} de' de_j \\
&\leq C\left\|\sigma_s\mathcal{P}\right\|^2_{L^\infty(X\times\Omega^2\times E^2)} \sum_{j=1}^N |E_j|^2 \int_{E_j} \left\|\frac{\partial u}{\partial e}(e')\right\|^2_{L^2(X\times\Omega)} de' \\
&\leq Ch_e^2 |u|^2_{L^2(X\times\Omega, H^1(E))}.
\end{aligned} \tag{3.27}$$

where C depends on the upper bound of σ_s and $\mathcal{P}$.

Applying inequality (3.25), (3.26), and (3.27) to (3.24) and taking the square root produces the cliam. □

Therefore, we have an estimate on the error due to the approximation of the scattering term. In particular, it provides a bound for a portion of the right hand side of the system (3.15)–(3.16). The next result gives an error bound for the approximation of the semi-discrete solution to the solution for (1.2)–(1.3).

Theorem 3.7. *Assume that (3.17), (3.19), and (3.20) hold. Then if* $f \in L^2(D, H^1(E))$ *and* $g \in L^2_w(\partial D_-, H^1(E))$, *we have*

$$\begin{aligned}
&\sum_{i=1}^N \int_{E_i} \left(\left\|\varepsilon_i^+(e_i)\right\|^2_V + \left\|\varepsilon_i^-(e_i)\right\|^2_{L^2(D)}\right) de_i \\
&\qquad \leq Ch_e^2\left(\left\|u\right\|^2_{L^\infty(U)} + |u|^2_{L^2(D,H^1(E))} + |g|^2_{L^2_w(\partial D_-,H^1(E))} + |f|^2_{L^2(D,H^1(E))}\right).
\end{aligned}$$

where $\varepsilon_i(e_i)$ *denotes* $\varepsilon_i(\cdot,\cdot,e_i)$ *as found in equations (3.15)–(3.16) and has the splitting* $\varepsilon_i(e_i) = \varepsilon_i^+(e_i) + \varepsilon_i^-(e_i)$.

Proof. Note that $\varepsilon_i(e_i)$ satisfies the monoenergetic form of the RTE (3.15)–(3.16). We may, therefore, employ the same methods used to establish inequality (3.11) in Proposition 3.3 to obtain

$$\begin{aligned}\|\varepsilon_i^+(e_i)\|_V^2 + \|\varepsilon_i^-(e_i)\|_{L^2(D)}^2 \le C\sum_{j=1}^{i-1}|E_j|\left(\left\|\varepsilon_j^+(e_j)\right\|_V^2 + \left\|\varepsilon_j^-(e_j)\right\|_{L^2(D)}^2\right)&\\ + \left\|\Theta_i(\{e_j\}_{j=1}^i)\right\|_{L^2(D)}^2 + \|\Psi_i(e_i)\|_{L^2(D)}^2&\\ + \|g(e_i) - g_i\|_{L^2_w(\partial D_-)}^2.&\end{aligned}$$

For $k = 1,\ldots,i$, integrating with respect to e_k over E_k in the average sense results in

$$\begin{aligned}⨍_{E_i}\left(\|\varepsilon_i^+(e_i)\|_V^2 + \|\varepsilon_i^-(e_i)\|_{L^2(D)}^2\right)de_i \le C\sum_{j=1}^{i-1}|E_j|⨍_{E_j}\left(\left\|\varepsilon_j^+(e_j)\right\|_{L^2(D)}^2 + \left\|\varepsilon_j^-(e_j)\right\|_{L^2(D)}^2\right)de_j&\\ + \left\|\hat{\Theta}_i\right\|_{L^2(D)}^2 + ⨍_{E_i}\|\Psi_i(e_i)\|_{L^2(D)}^2\,de_i&\\ + ⨍_{E_i}\|g(e_i) - g_i\|_{L^2_w(\partial D_-)}^2\,de_i.&\end{aligned}$$

Using Lemma 3.1 with

$$a_j := ⨍_{E_j}\left(\left\|\varepsilon_j^+(e_j)\right\|_V^2 + \left\|\varepsilon_j^-(e_j)\right\|_{L^2(D)}^2\right)de_j, \quad b_j := C|E_j|$$

and

$$\gamma_i := \left\|\hat{\Theta}_i\right\|_{L^2(D)}^2 + ⨍_{E_i}\|\Psi_i(e_i)\|_{L^2(D)}^2\,de_i + ⨍_{E_i}\|g(e_i) - g_i\|_{L^2_w(\partial D_-)}^2\,de_i$$

we find

$$⨍_{E_i}\left(\|\varepsilon_i^+(e_i)\|_V^2 + \|\varepsilon_i^-(e_i)\|_{L^2(D)}^2\right)de_i \le \gamma_i + C\exp(C|E|)\sum_{j=1}^{i-1}|E_j|\gamma_j.$$

Multiplying by $|E_i|$, modifying the constant, and extending the summation from $i-1$ to N we obtain

$$\int_{E_i} \left(\|\varepsilon_i^+(e_i)\|_V^2 + \|\varepsilon_i^-(e_i)\|_{L^2(D)}^2 \right) de_i \le |E_i|\gamma_i + C|E_i| \sum_{j=1}^{N} |E_j|\gamma_j.$$

Finally, summing over $1 \le i \le N$, redefining the constant, and substituting in the definition of γ_i yields

$$\begin{aligned}\sum_{i=1}^{N} \int_{E_i} &\left(\|\varepsilon_i^+(e_i)\|_V^2 + \|\varepsilon_i^-(e_i)\|_{L^2(D)}^2 \right) de_i \\ &\le C \sum_{j=1}^{N} |E_j| \left(\left\|\hat{\Theta}_j\right\|_{L^2(D)}^2 + \fint_{E_j} \|\Psi_j(e_j)\|_{L^2(D)}^2 \, de_j + \fint_{E_j} \|g(e_j) - g_j\|_{L_w^2(\partial D_-)}^2 \, de_j \right).\end{aligned}$$

The claim follows from bounding each of the terms on the right hand side.

1. The first term follows from Lemma 3.6. We have

$$\sum_{j=1}^{N} |E_j| \left\|\hat{\Theta}_i\right\|_{L^2(D)}^2 \le h_e^2 \left(\|u\|_{L^\infty(U)}^2 + |u|_{L^2(D,H^1(E))}^2 \right).$$

2. Consider the second term on the right hand side. Using the definition of Ψ_j, the upper bound on u, and applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\sum_{j=1}^{N} \int_{E_j} &\|\Psi_j(e_j)\|_{L^2(D)}^2 \, de_j \\ &\le \sum_{j=1}^{N} \int_{E_j} \left(\|(\sigma_t(e_j) - \sigma_{t,j})u(e_j)\|_{L^2(D)} + \|f(e_j) - f_j\|_{L^2(D)} \right)^2 de_j \\ &\le C \sum_{j=1}^{N} \|u\|_{L^\infty(U)}^2 \int_{E_j} \|\sigma_t(e_j) - \sigma_{t,j}\|_{L^2(D)}^2 \, de_j + C \sum_{j=1}^{N} \int_{E_j} \|f(e_j) - f_j\|_{L^2(D)}^2 \, de_j.\end{aligned}$$

By methods analogous to those used to obtain (3.26), we find

$$\sum_{j=1}^{N} \int_{E_j} \|\sigma_t(e_j) - \sigma_{t,j}\|_{L^2(D)}^2 \, de_j \le Ch_e^2 \|u\|_{L^\infty(U)}^2, \tag{3.28}$$

where C depends on the function σ_t. Similarly we find

$$\sum_{j=1}^{N} \int_{E_j} \|f(e_j) - f_j\|^2_{L^2(D)} \, de_j \le Ch_e^2 |f|^2_{L^2(D,H^1(E))}.$$

3. The remaining term follows in the same way as the previous two

$$\sum_{j=1}^{N} \int_{E_j} \|g(e_j) - g_j\|^2_{L^2_w(\partial D_-)} \, de_j \le Ch_e^2 |g|^2_{L^2_w(\partial D_-,H^1(E))}.$$

Combining each of these estimates yields the claim. □

In this sense, a bound on the error between the semi-discrete solution and the true solution of the RTE has been established. We have established well posedness and an error estimate of the semi-discretization. In the next chapter, the angular discretization is introduced. We then give an error estimate between the solution of the semi-discretization in energy and angle and the solution of the RTE BVP (1.2)–(1.3).

CHAPTER 4
ANGULAR DISCRETIZATION

The angular discretization is motivated by the finite element based approach presented in [22, 23]. This work's treatment of the angular variable differs in two ways: first an isoparametric approach is taken when dealing with the triangulation of the sphere; second, the regularity assumptions in the angular variable are relaxed from $C^2(\Omega)$ to $H^2(\Omega)$ while maintaining the same convergence order. The chapter is organized as follows. The triangulation of the sphere proposed in [23] is presented and relevant properties are stated. Next, finite element inequalities for the curved elements are given. Finally, the inequalities are used in the error analysis of the energy-angular discretzation. However, before proceeding, we briefly define Sobolev spaces over Ω and refer the reader to [28] for a more in depth development.

Let α be a multi-index. For $k \geq 0$ an integer and $1 \leq p < \infty$, the Sobolev space

$$W^{k,p}(\Omega) := \{v \in L^p(\Omega) \mid D_S^\alpha v \in L^p(\Omega), \, |\alpha| \leq k\}$$

is a Banach space with norm $\|v\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k} \|D_S^\alpha v\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}$. When $p = 2$ it is a Hilbert space, and we use the canonical notation $H^k(\Omega) \equiv W^{k,2}(\Omega)$. Here, D_S^α is the tangential αth order partial derivative over the sphere.

4.1 Angular Discretization

The scheme generates a set of discrete directions, denoted by Ω_{h_a}, over which the RTE is solved. To handle the scattering operator, a piecewise linear finite element representation is used. The set Ω_{h_a} is constructed by projecting a uniformly triangulated triangular plane onto the unit sphere in each octant of $\mathbb{R}^3$, see Figure 4.1. The projection of the vertices contained in the plane gives rise to the set of discrete directions

$$\Omega_{h_a} = \{\boldsymbol{\omega}_l \in \Omega \,|\, 1 \leq l \leq L\}\,.$$

The set of nodes then partitions Ω into spherical triangles. Let $\mathcal{T}_{h_a}$ denote the partition. It then follows that

$$\Omega = \bigcup_{\Omega_K \in \mathcal{T}_{h_a}} \Omega_K,$$

where each Ω_K denotes a spherical element obtained from the projection.

Let $\mathbf{v}_1^K, \mathbf{v}_2^K$, and $\mathbf{v}_3^K$ denote the vertices of Ω_K and let K denote the corresponding planar triangle. Now, numerically, this triangulation has been shown to be quite uniform [23]. Therefore, we assume that the underlying polyhedral approximation obtained from the set Ω_{h_a} is shape regular. Lastly, define the meshsize of the triangulation $\mathcal{T}_{h_a}$ by

$$h_a = \max_{\Omega_K \in \mathcal{T}_{h_a}} \operatorname{diam}(\Omega_K).$$

In working directly with the curved elements, scattering cross sections and solutions with simple angular distributions are captured well. In this case, h_a need not be very small to capture the behavior. Whereas, a polyhedral approximation of the sphere will accrue more error simply due to the geometry. For more complex angular distributions multigrid can be used to obtain the needed accuracy, as is done in [23]. In particular, because coarsening and refining the angular domain is straightforward, the triangulation is particularly well suited for multigrid.

Consider the system of in-group RTEs (3.1)-(3.2), restricting $\boldsymbol{\omega}$ to Ω_{h_a}, the following system of hyperbolic equations in the spatial variable $\mathbf{x} \in X$ is obtained:

$$\boldsymbol{\omega}_l \cdot \nabla_x u_{i,l} + \sigma_{t,i} u_{i,l} = \sum_{j=1}^{i} |E_j| \sigma_{s,j} \int_{\Omega} \mathcal{P}_{i,j}(\boldsymbol{\omega}_l, \boldsymbol{\omega}') u_j(\boldsymbol{\omega}') d\boldsymbol{\omega}' + f_{i,l}, \text{ in } X \tag{4.1}$$

$$u_{i,l} = g_{i,l}, \text{ on } \partial X_-^l \tag{4.2}$$

for $1 \leq l \leq L$, $1 \leq i \leq N$, where $\partial X_-^l = \{\mathbf{x} \in \partial X \mid \boldsymbol{\omega}_l \cdot \nu(\mathbf{x}) < 0\}$ is the inflow portion of the boundary of X with respect to the direction $\boldsymbol{\omega}_l$. In the above, $u_{i,l}(\mathbf{x})$ is an approximation of $u_i(\mathbf{x}, \boldsymbol{\omega}_l)$, $f_{i,l}(\mathbf{x}) = f(\mathbf{x}, \boldsymbol{\omega}_l)$, and $g_{i,l}(\mathbf{x}) = g(\mathbf{x}, \boldsymbol{\omega}_l)$ where $\boldsymbol{\omega}_l \in \Omega_{h_a}$. However, the integration over the angular variable has yet to be handled. In an effort to treat this term, piecewise linear approximations of $\mathcal{P}_{i,j}$ and u_i are constructed.

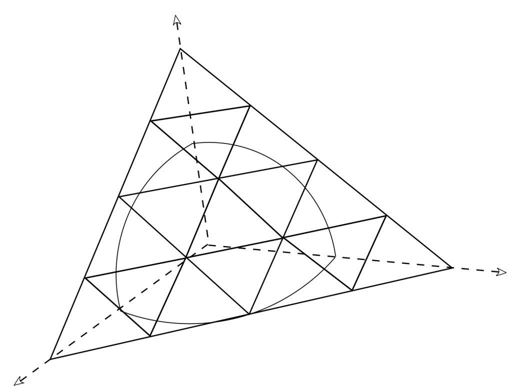

Figure 4.1: Projection of triangular plane on the unit sphere in the first octant.

For each $\Omega_K \in \mathcal{T}_{h_a}$ recall that K denotes the associated planar triangle in $\mathbb{R}^3$. The union over all K results in a polyhedral approximation to the sphere, denoted by $\Omega_\triangle$, where Ω and $\Omega_\triangle$ coincide at the nodes in Ω_{h_a}. Further, Ω_K and K may be related trough a radial projection map $P : \Omega_\triangle \to \Omega$, defined by $P(\boldsymbol{\xi}) = \boldsymbol{\xi} / \|\boldsymbol{\xi}\|$ which is

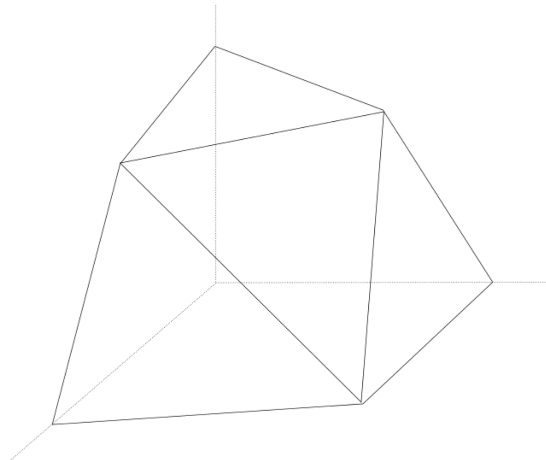

Figure 4.2: Example $\Omega_\triangle$ in the first octant.

a smooth bijection with a piecewise smooth inverse. Figure 4.2 gives an example of $\Omega_\triangle$ in the first octant and Figure 4.3 illustrates the relation between the planar triangles and the curved elements. For each planar triangle K, let $\left\{\hat{\lambda}_m^K : K \to \mathbb{R} \mid m = 1,2,3\right\}$ denote the standard piecewise linear Lagrange shape functions. The corresponding shape functions over the curved elements, $\Omega_K \in \mathcal{T}_{h_a}$, are defined by $\{\lambda_m^K : \Omega_K \to \mathbb{R} \mid \lambda_m^K = \hat{\lambda}_m^K \circ P^{-1},\ m = 1,2,3\}$. From the element-wise defined basis functions, let $\{\phi_l : \Omega \to \mathbb{R} \mid 1 \le l \le L\}$ denote their global representation. Then the finite element interpolant of $v \in C(\Omega)$ is given by

$$\tilde{v}(\boldsymbol{\omega}) := \sum_{l=1}^{L} \phi_l(\boldsymbol{\omega}) v(\boldsymbol{\omega}_l). \tag{4.3}$$

We may then construct the following finite element interpolants

$$\tilde{\mathcal{P}}_{i,j}(\mathbf{x}, \boldsymbol{\omega}, \boldsymbol{\omega}') := \sum_{l=1}^{L} \phi_l(\boldsymbol{\omega}') \mathcal{P}_{i,j}(\mathbf{x}, \boldsymbol{\omega}, \boldsymbol{\omega}_l), \quad 1 \le j \le i, 1 \le i \le N, \tag{4.4}$$

$$\tilde{u}_i(\mathbf{x}, \boldsymbol{\omega}) := \sum_{l=1}^{L} \phi_l(\boldsymbol{\omega}) u_{i,l}(\mathbf{x}), \quad 1 \le i \le N, \tag{4.5}$$

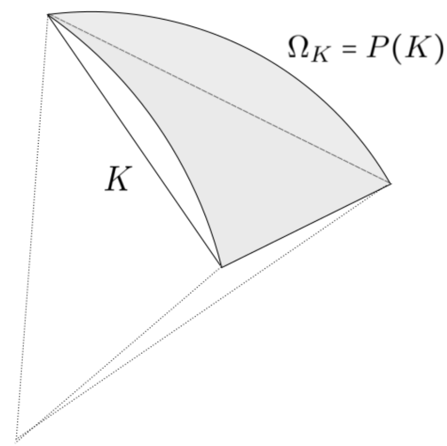

Figure 4.3: Relationship between a planar triangle K on $\Omega_\triangle$ and spherical element Ω_K on Ω.

The integration over Ω is approximated by replacing the functions in the integrand by their corresponding approximations. This leads to

$$\begin{aligned}\int_\Omega \mathcal{P}_{i,j}(\mathbf{x},\boldsymbol{\omega}_l,\boldsymbol{\omega}')u_i(\mathbf{x},\boldsymbol{\omega}')d\boldsymbol{\omega}' &\approx \int_\Omega \tilde{\mathcal{P}}_{i,j}(\mathbf{x},\boldsymbol{\omega}_l,\boldsymbol{\omega}')\tilde{u}_i(\mathbf{x},\boldsymbol{\omega}')d\boldsymbol{\omega}' \\ &= \sum_{k=1}^{L} w_{l,k}^{i,j}(\mathbf{x})u_{i,k}(\mathbf{x}) \end{aligned} \tag{4.6}$$

for $1 \le j \le i$, $1 \le i \le N$, and $1 \le l \le L$, where

$$w_{l,k}^{i,j}(\mathbf{x}) = \int_\Omega \phi_k(\boldsymbol{\omega}')\tilde{\mathcal{P}}_{i,j}(\mathbf{x},\boldsymbol{\omega}_l,\boldsymbol{\omega}')d\boldsymbol{\omega}'. \tag{4.7}$$

Replacing the integration in (4.1) with the above approximation (4.6), the full angular discretization of the problem is obtained; this results in L coupled equations per energy group, for a total of NL coupled hyperbolic problems in space. The system of BVPs obtained from the full discretizaton of the angular variable is given by

$$\boldsymbol{\omega}_l \cdot \nabla_x u_{i,l} + \sigma_{t,i} u_{i,l} = \sum_{j=1}^{i} |E_j| \sigma_{s,j} \sum_{k=1}^{L} w_{l,k}^{i,j} u_{j,k} + f_{i,l}, \text{ in } X \tag{4.8}$$

$$u_{i,l} = g_{i,l}, \text{ on } \partial X_-^l \tag{4.9}$$

for $1 \le i \le N$ and $1 \le l \le L$, where, again, the spatial variable is suppressed. For error analysis, we will use the norm

$$\left\| \{v_{i,l}\}_{i=1,l=1}^{N,L} \right\| := \left(\sum_{i=1}^{N} \sum_{l=1}^{L} w_l |E_i| \, \|v_{i,l}\|_{L^2(X)}^2 \right)^{1/2}, \quad w_l := \int_\Omega \phi_l(\boldsymbol{\omega}') d\boldsymbol{\omega}', \; 1 \le l \le L. \tag{4.10}$$

4.2 FEM Inequalites

We introduce the notation

$$\hat{v}(\boldsymbol{\xi}) = v(P(\boldsymbol{\xi})), \quad \boldsymbol{\xi} \in \Omega_\triangle \tag{4.11}$$

when parameterizing v from any of the planar elements $K \subset \Omega_\triangle$. The main tool in the analysis is the following lemma which relates the norm over planar elements to the norm over spherical elements in the triangulation. It is a restatement of Lemma 3.3 presented in [37] with a minor modification and we refer the reader there for the proof. In the following, $\|\cdot\|$ refers to the norm defined in (4.10), and h_e and h_a denote the energy and angular mesh sizes, respectively.

LEMMA 4.1. *There exist positive constants c_i, $1 \le i \le 5$, such that for each $\Omega_K \in \mathcal{T}_{h_a}$ and associated $K = P^{-1}(\Omega_K)$, we have*

$$c_1 \|\hat{v}\|_{L^2(K)} \le \|v\|_{L^2(\Omega_K)} \le c_2 \|\hat{v}\|_{L^2(K)} \quad \forall\, v \in L^2(\Omega_K),$$

$$c_3 \|\hat{v}\|_{H^1(K)} \le \|v\|_{H^1(\Omega_K)} \le c_4 \|\hat{v}\|_{H^1(K)} \quad \forall\, v \in H^1(\Omega_K),$$

and

$$|\hat{v}|_{H^2(K)} \le c_5 \|v\|_{H^2(\Omega_K)} \quad \forall\, v \in H^2(\Omega_K).$$

where $|\cdot|_{H^2(K)}$ *denotes the semi-norm of* $H^2(K)$ *and* $\hat{v}$ *is given by* (4.11).

The next set of lemmas provides a-priori error estimates for the finite element interpolation by employing Lemma 4.1 and the familiar reference element type technique.

LEMMA 4.2. *Assume* $v \in H^2(\Omega)$. *Then*

$$\|v - \tilde{v}\|_{L^2(\Omega)} \le c h_a^2 \|v\|_{H^2(\Omega)}.$$

where $\tilde{v}$ *is the finite element interpolant given in* (4.3).

Proof. Rewriting in terms of the planar elements and applying Lemma 4.1 yields

$$\|v - \tilde{v}\|^2_{L^2(\Omega)} = \sum_{\Omega_K} \|v - \tilde{v}\|^2_{L^2(\Omega_K)} \le c \sum_K \left\|\hat{v}_K - \hat{\tilde{v}}_K\right\|^2_{L^2(K)}.$$

Applying the standard finite element inequalities over the planar element then yields

$$\|v - \tilde{v}\|^2_{L^2(\Omega)} \le c h_a^4 \sum_K |\hat{v}_K|^2_{H^2(K)}.$$

The proof concludes by using the semi-norm inequality in Lemma 4.1. □

By integrating the previous result over X and E, the following corollary may be obtained.

COROLLARY 4.3. *Assume* $v \in L^2(X \times E, H^2(\Omega))$. *Then*

$$\|v - \tilde{v}\|_{L^2(U)} \le c h_a^2 \|v\|_{L^2(X \times E, H^2(\Omega))},$$

where $\tilde{v}$ *is the finite element interpolant of the angular variable given in* (4.3).

We have similar results for the interpolation of the function $\mathcal{P}$. The interpolation is done with respect to the second angular variable.

LEMMA 4.4. *Assume* $\mathcal{P}(\mathbf{x}, \boldsymbol{\omega}, \cdot, e, e') \in H^2(\Omega)$. *Then*

$$\left\|\mathcal{P}_{i,j} - \tilde{\mathcal{P}}_{i,j}\right\|_{L^2(\Omega)} \le ch_a^2 \left\|\mathcal{P}_{i,j}\right\|_{H^2(\Omega)},$$

where $\tilde{\mathcal{P}}_{i,j}$ *is the finite element interpolant.*

COROLLARY 4.5. *Let* $\boldsymbol{\omega} \in \Omega$. *Assume* $\mathcal{P}(\cdot, \boldsymbol{\omega}, \cdot, \cdot, \cdot) \in L^2(X \times E^2, H^2(\Omega))$. *Then*

$$\left\|\mathcal{P} - \tilde{\mathcal{P}}\right\|_{L^2(X\times\Omega\times E^2)} \le ch_a^2 \left\|\mathcal{P}\right\|_{L^2(X\times E^2, H^2(\Omega))},$$

where $\mathcal{P}$ *denotes* $\mathcal{P}(\cdot, \boldsymbol{\omega}, \cdot, \cdot, \cdot)$ *and* $\tilde{\mathcal{P}}$ *denotes the finite element interpolant of* $\mathcal{P}$ *at* $\boldsymbol{\omega}$.

The above results provide useful bounds on the error incurred by replacing the function by its finite element interpolant. The following two results build upon these. They will provide an estimate for the error due to the approximation in both energy and angle. The next two lemmas will act as intermediary results for obtaining bounds on the error incurred by approximating the scattering operator (1.4) by the approximation in (4.6). It will be useful to discuss the number of weak derivatives a function has in specific variables. Therefore, we introduce the spaces

$$H^{0,2,1}(X \times \Omega \times E) := \left\{v \in L^2(X \times \Omega \times E) \mid \partial_e v, D_S^\alpha v \in L^2(X \times \Omega \times E),\ |\alpha| \le 2\right\},$$

$$H^{0,2,1}(X \times \Omega \times E^2) := \left\{v \in L^2(X \times \Omega \times E^2) \mid \partial_e v, \partial_{e'} v, D_S^\alpha v \in L^2(X \times \Omega \times E^2),\ |\alpha| \le 2\right\}.$$

LEMMA 4.6. *Assume* $u \in H^{0,2,1}(X \times \Omega \times E)$. *Then for* $1 \le j \le N$,

$$\int_{E_j}\int_{E_j} \left\|u(\cdot,\cdot,e) - \tilde{u}(\cdot,\cdot,e')\right\|^2_{L^2(X\times\Omega)}\, de\, de'$$
$$\le c|E_j|\left(h_a^4 \left\|u\right\|^2_{L^2(X\times E_j, H^2(\Omega))} + h_e^2 \left\|u\right\|^2_{L^2(X\times\Omega, H^1(E_j))}\right).$$

Proof. For $1 \le j \le N$ and $e, e' \in E_j$, we write

$$|u(\mathbf{x},\boldsymbol{\omega},e) - \tilde{u}(\mathbf{x},\boldsymbol{\omega},e')|^2 \le 2\left(|u(\mathbf{x},\boldsymbol{\omega},e) - u(\mathbf{x},\boldsymbol{\omega},e')|^2 + |u(\mathbf{x},\boldsymbol{\omega},e') - \tilde{u}(\mathbf{x},\boldsymbol{\omega},e')|^2\right).$$

Using

$$u(\mathbf{x},\boldsymbol{\omega},e) - u(\mathbf{x},\boldsymbol{\omega},e') = \int_{e'}^{e} \frac{\partial u}{\partial e}(\mathbf{x},\boldsymbol{\omega},\xi)d\xi,$$

we can derive the bound

$$|u(\mathbf{x},\boldsymbol{\omega},e) - u(\mathbf{x},\boldsymbol{\omega},e')|^2 \le |E_j| \int_{E_j} \left|\frac{\partial u}{\partial e}(\mathbf{x},\boldsymbol{\omega},\xi)\right|^2 d\xi. \tag{4.12}$$

This results in

$$|u(\mathbf{x},\boldsymbol{\omega},e) - \tilde{u}(\mathbf{x},\boldsymbol{\omega},e')|^2 \le c|E_j| \int_{E_j} \left|\frac{\partial u}{\partial e}(\mathbf{x},\boldsymbol{\omega},\xi)\right|^2 d\xi + c\,|u(\mathbf{x},\boldsymbol{\omega},e') - \tilde{u}(\mathbf{x},\boldsymbol{\omega},e')|^2.$$

Integrating with respect to $\boldsymbol{\omega}$ over Ω, $\mathbf{x}$ over X, and e and e' over E_j, we have

$$\begin{aligned}&\int_{E_j}\int_{E_j} \|u(\cdot,\cdot,e) - \tilde{u}(\cdot,\cdot,e')\|^2_{L^2(X\times\Omega)}\,de\,de' \\ &\qquad \le c\,|E_j|\left(|E_j|^2\,\|u\|^2_{L^2(X\times\Omega,H^1(E_j))} + \|u-\tilde{u}\|^2_{L^2(X\times\Omega\times E_j)}\right).\end{aligned}$$

Applying Corollary 4.3 and bounding $|E_j|^2$ by h_e^2 yields the claim. □

LEMMA 4.7. *Assume that $\mathcal{P}(\cdot,\boldsymbol{\omega},\cdot,\cdot,\cdot) \in H^{0,2,1}(X\times\Omega\times E^2)$ and is continuous in $\boldsymbol{\omega} : \Omega \to H^{0,2,1}(X\times\Omega\times E^2)$. Then*

$$\left\|\mathcal{P} - \tilde{\mathcal{P}}_{i,j}\right\|^2_{L^2(X\times\Omega\times E_i\times E_j)} \le ch_a^4\,\|\mathcal{P}\|^2_{L^2(X\times E_i\times E_j,H^2(\Omega))} + ch_e^2\,\|\mathcal{P}\|^2_{L^2(X\times\Omega,H^1(E_i\times E_j))}$$

for $1 \le j \le i$, $1 \le i \le N$. Here $\mathcal{P}$ denotes $\mathcal{P}(\cdot,\boldsymbol{\omega},\cdot,\cdot)$ and $\tilde{\mathcal{P}}_{i,j}$ is the finite element interpolant at $\boldsymbol{\omega}$.

Proof. Fix $\boldsymbol{\omega} \in \Omega$. Through addition and subtraction we have

$$\left\|\mathcal{P} - \tilde{\mathcal{P}}_{i,j}\right\|_{L^2(X\times\Omega\times E_i\times E_j)} \le \left\|\mathcal{P} - \mathcal{P}_{i,j}\right\|_{L^2(X\times\Omega\times E_i\times E_j)} + \left\|\mathcal{P}_{i,j} - \tilde{\mathcal{P}}_{i,j}\right\|_{L^2(X\times\Omega\times E_i\times E_j)}. \quad (4.13)$$

The first term on the right hand side can be bounded as follows (Lemma 3.5):

$$\left\|\mathcal{P} - \mathcal{P}_{i,j}\right\|_{L^2(X\times\Omega\times E_i\times E_j)} \le ch_e^2 \left\|\mathcal{P}\right\|_{L^2(X\times\Omega, H^1(E_i\times E_j))}. \quad (4.14)$$

For the second term, using the Cauchy-Schwarz inequality we have

$$\begin{aligned} \left\|\mathcal{P}_{i,j} - \tilde{\mathcal{P}}_{i,j}\right\|_{L^2(X\times\Omega\times E_i\times E_j)} &\le c\left(|E_i||E_j| \fint_{E_i}\fint_{E_j} \left\|\mathcal{P}(e,e') - \tilde{\mathcal{P}}(e,e')\right\|^2_{L^2(X\times\Omega)} de'de\right)^{1/2} \\ &= c\left\|\mathcal{P} - \tilde{\mathcal{P}}\right\|^2_{L^2(X\times\Omega\times E_i\times E_j)}, \quad (4.15) \end{aligned}$$

where for simplicity, $\mathcal{P}(e,e')$ and $\tilde{\mathcal{P}}(e,e')$ stand for $\mathcal{P}(\cdot,\cdot,e,e')$ and $\tilde{\mathcal{P}}(\cdot,\cdot,e,e')$. The proof concludes by applying Corollary 4.5 to (4.15) and combining the resulting inequality with inequality (4.14) as an upper bound for (4.13). □

This section has provided the necessary preliminary lemmas which will be used to establish an error bound between the semi-discrete solution and the solution of the original BVP.

4.3 Error Analysis

The goal of this section is to establish an error estimate between the semi-discrete solution and the solution of the original BVP (1.2)–(1.3). For each energy group and angular direction we introduce the function $\varepsilon_{i,l} : X \times E_i \to \mathbb{R}$ defined by $\varepsilon_{i,l}(\mathbf{x}, e_i) :=$

$u(\mathbf{x}, \boldsymbol{\omega}_l, e_i) - u_{i,l}(\mathbf{x})$. We have

$$\boldsymbol{\omega}_l \cdot \nabla_x \varepsilon_{i,l}(\mathbf{x}, e_i) + \sigma_{t,i}(\mathbf{x})\varepsilon_{i,l}(\mathbf{x}, e_i) - \sum_{j=1}^{i} |E_j| \sigma_{s,j}(\mathbf{x}) \sum_{k=1}^{L} w_{l,k}^{i,j}(\mathbf{x})\varepsilon_{j,k}(\mathbf{x}, e_j)$$

$$= \Theta_{i,l}(\mathbf{x}, \{e_j\}_{j=1}^{i}) + \Psi_{i,l}(\mathbf{x}, e_i), \tag{4.16}$$

where

$$\Theta_{i,l}(\mathbf{x}, \{e_j\}_{j=1}^{i}) = (\mathcal{S}u)(\mathbf{x}, \boldsymbol{\omega}_l, e_i) - \sum_{j=1}^{i} |E_j| \sigma_{s,j}(\mathbf{x}) \sum_{k=1}^{L} w_{l,k}^{i,j}(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\omega}_k, e_j)$$

and

$$\Psi_{i,l}(\mathbf{x}, e_i) = [\sigma_{t,i}(\mathbf{x}) - \sigma_t(\mathbf{x}, e_i)] u(\mathbf{x}, \boldsymbol{\omega}_l, e_i) + [f_{i,l}(\mathbf{x}) - f(\mathbf{x}, \boldsymbol{\omega}_l, e_i)]. \tag{4.17}$$

In the next three lemmas we establish estimates relating to the right hand side of equality (4.16). In the following we use $\{\Psi_{i,l}\}_{i=1,l=1}^{N,L}$ to denote the collection of $\Psi_{i,l}$ for $i = 1, \ldots, N$ and $l = 1, \ldots, L$.

Lemma 4.8. *Assume $u \in L^\infty(U)$, $\sigma_s \in L^2(X, H^1(E))$, and $f \in C(\Omega, L^2(X, H^1(E)))$. Then*

$$\left\| \{\Psi_{i,l}\}_{i=1,l=1}^{N,L} \right\| \le c h_e \left[\|u\|_{L^\infty(U)} + \left(\sum_{l=1}^{L} w_l |f(\boldsymbol{\omega}_l)|^2_{L^2(X, H^1(E))} \right)^{1/2} \right].$$

Proof. Beginning from equation (4.17), applying $\|\cdot\|_{L^2(X)}$ on both sides, we have

$$\|\Psi_{i,l}\|_{L^2(X)} \le \|(\sigma_{t,i} - \sigma_t(e_i)) u(\boldsymbol{\omega}_l, e_i)\|_{L^2(X)} + \|f_{i,l} - f(\boldsymbol{\omega}_l, e_i)\|_{L^2(X)}.$$

Squaring both sides

$$\|\Psi_{i,l}\|^2_{L^2(X)} \le c \left(\|(\sigma_{t,i} - \sigma_t(e_i)) u(\boldsymbol{\omega}_l, e_i)\|^2_{L^2(X)} + \|f_{i,l} - f(\boldsymbol{\omega}_l, e_i)\|^2_{L^2(X)} \right).$$

Integrating over E_i and using the upper bound on u results in

$$\int_{E_i} \|\Psi_{i,l}\|^2_{L^2(X)}\, de \le c\Bigg[\|u\|_{L^\infty(U)} \int_{E_i} \|\sigma_{t,i} - \sigma_t(e)\|^2_{L^2(X)}\, de \\ + \int_{E_i} \|f_{i,l} - f(\boldsymbol{\omega}_l, e)\|^2_{L^2(X)}\, de \Bigg].$$

Using Lemma 3.4 we can obtain

$$\int_{E_i} \|\sigma_{t,i} - \sigma_t(e)\|^2_{L^2(X)}\, de \le ch_e^2 |\sigma_s|^2_{L^2(Y,H^1(I))}.$$

Recall $|f_{i,l}(\mathbf{x}) - f(\mathbf{x}, \boldsymbol{\omega}_l, e)| = |f_i(\mathbf{x}, \boldsymbol{\omega}_l) - f(\mathbf{x}, \boldsymbol{\omega}_l, e)|$; then given $f(\cdot, \boldsymbol{\omega}_l, \cdot) \in L^2(X, H^1(E))$ for each $\boldsymbol{\omega}_l \in \Omega_{h_a}$ the following bound may be obtained using Lemma 3.4

$$\int_{E_i} \|f_{i,l} - f(\boldsymbol{\omega}_l, e)\|^2_{L^2(X)}\, de \le ch_e^2 |f(\boldsymbol{\omega}_l)|^2_{L^2(X,H^1(E_i))}.$$

Combining the inequalities, we obtain

$$\int_{E_i} \|\Psi_{i,l}\|^2_{L^2(X)}\, de \le ch_e^2 \Bigg[\|u\|_{L^\infty(U)} |\sigma_s|^2_{L^2(Y,H^1(E))} + |f(\boldsymbol{\omega}_l)|^2_{L^2(X,H^1(E_i))} \Bigg].$$

Multiplying by w_l, summing over $1 \le l \le L$ and $1 \le i \le N$, and absorbing $|\sigma_s|_{L^2(X,H^1(E))}$ into the constant yields

$$\begin{aligned}
\left\| \{\Psi_{i,l}\}_{i=1,l=1}^{N,L} \right\|^2 &\le ch_e^2 \sum_{i=1}^{N} \sum_{l=1}^{L} w_l \Bigg[\|u\|_{L^\infty(U)} |\sigma_s|^2_{L^2(Y,H^1(E))} + |f(\boldsymbol{\omega}_l)|^2_{L^2(X,H^1(E_i))} \Bigg] \\
&\le ch_e^2 \Bigg[\|u\|_{L^\infty(U)} \sum_{i=1}^{N} |\sigma_s|^2_{L^2(Y,H^1(E_i))} + \sum_{i=1}^{N} \sum_{l=1}^{L} w_l |f(\boldsymbol{\omega}_l)|^2_{L^2(X,H^1(E_i))} \Bigg] \\
&\le ch_e^2 \Bigg[\|u\|_{L^\infty(U)} |\sigma_s|^2_{L^2(Y,H^1(E))} + \sum_{l=1}^{L} w_l |f(\boldsymbol{\omega}_l)|^2_{L^2(X,H^1(E))} \Bigg] \\
&\le ch_e^2 \Bigg[\|u\|_{L^\infty(U)} + \sum_{l=1}^{L} w_l |f(\boldsymbol{\omega}_l)|^2_{L^2(X,H^1(E))} \Bigg].
\end{aligned}$$

The claim follows by taking the square root of both sides. $\square$

As was done in the previous chapter, for simplicity we define a quantity analogous to (3.18). Define

$$\hat{\Theta}^2_{i,l}(\mathbf{x}) := ⨍_{E_1} \cdots ⨍_{E_i} \Theta^2_{i,l}\left(\mathbf{x}, \boldsymbol{\omega}_l, \{e_j\}_{j=1}^{i}\right) de_i \ldots de_1. \tag{4.18}$$

Lemma 4.9. *Assume* $\sigma_s \in L^2(X, H^1(E))$, $\mathcal{P}(\cdot, \boldsymbol{\omega}, \cdot, \cdot, \cdot) \in H^{0,2,1}(X \times \Omega \times E^2)$ *and is continuous in* $\boldsymbol{\omega} : \Omega \to H^{0,2,1}(X \times \Omega \times E^2)$. *Then if* $u \in H^{0,2,1}(X \times \Omega \times E)$ *and is bounded a.e.,*

$$\left\|\{\hat{\Theta}_{i,l}\}_{i=1,l=1}^{N,L}\right\| \leq C\left[h_e\left(\|u\|_{L^\infty(U)} + \|u\|_{L^2(X\times\Omega,H^1(E)))}\right) + h_a^2\left(\|u\|_{L^\infty(U)} + \|u\|_{L^2(X\times E,H^2(\Omega))}\right)\right]$$

where $\hat{\Theta}_{i,l}$ *is defined as in* (4.18).

Proof. Let

$$\begin{aligned}\Delta(\mathbf{x}, \boldsymbol{\omega}_l, \boldsymbol{\omega}', e_i, e', e_j) := {}& \sigma_s(\mathbf{x}, e')\mathcal{P}(\mathbf{x}, \boldsymbol{\omega}_l, \boldsymbol{\omega}', e_i, e')u(\mathbf{x}, \boldsymbol{\omega}', e') \\ & - \sigma_{s,j}(\mathbf{x})\tilde{\mathcal{P}}_{i,j}(\mathbf{x}, \boldsymbol{\omega}_l, \boldsymbol{\omega}')\tilde{u}(\mathbf{x}, \boldsymbol{\omega}', e_j),\end{aligned} \tag{4.19}$$

which, through addition as subtraction can be written as

$$\begin{aligned}\Delta(\mathbf{x}, \boldsymbol{\omega}_l, \boldsymbol{\omega}', e_i, e', e_j) = {}& [\mathcal{P}(\mathbf{x}, \boldsymbol{\omega}_l, \boldsymbol{\omega}', e_i, e') - \tilde{\mathcal{P}}_{i,j}(\mathbf{x}, \boldsymbol{\omega}_l, \boldsymbol{\omega}')]\sigma_{s,j}(\mathbf{x})u(\mathbf{x}, \boldsymbol{\omega}', e') \\ & + [\sigma_s(\mathbf{x}, e') - \sigma_{s,j}(\mathbf{x})]\mathcal{P}(\mathbf{x}, \boldsymbol{\omega}_l, \boldsymbol{\omega}', e_i, e')u(\mathbf{x}, \boldsymbol{\omega}', e') \\ & + [u(\mathbf{x}, \boldsymbol{\omega}', e') - \tilde{u}(\mathbf{x}, \boldsymbol{\omega}', e_j)]\sigma_{s,j}(\mathbf{x})\tilde{\mathcal{P}}_{i,j}(\mathbf{x}, \boldsymbol{\omega}_l, \boldsymbol{\omega}').\end{aligned}$$

Squaring both sides and applying the Cauchy-Schwarz inequality results in

$$|\Delta|^2 \leq C[(u - \tilde{u})^2(\sigma_{s,j}\tilde{\mathcal{P}}_{i,j})^2 + (\sigma_s - \sigma_{s,j})^2(\mathcal{P}u)^2 + (\mathcal{P} - \tilde{\mathcal{P}}_{i,j})^2(\sigma_{s,j}u)^2].$$

By methods identical to that used in Lemma 3.6 we may arrive at

$$\begin{aligned}\left\|\hat{\Theta}_{i,l}(\boldsymbol{\omega}_l)\right\|^2_{L^2(X)} \le C\sum_{j=1}^{N} ⨍_{E_i} ⨍_{E_j} \int_{E_j} \Bigg[&\|\sigma_s u\|^2_{L^\infty(U)} \left\|\mathcal{P}(\boldsymbol{\omega}_l, e_i, e') - \tilde{\mathcal{P}}_{i,j}(\boldsymbol{\omega}_l)\right\|^2_{L^2(X\times\Omega)} \\ &+ \|\mathcal{P}u\|^2_{L^\infty(X\times\Omega^2\times E^2)} \|\sigma_s(e') - \sigma_{s,j}\|^2_{L^2(X\times\Omega^2)} \\ &+ \|\sigma_s\mathcal{P}\|^2_{L^\infty(X\times\Omega^2\times E^2)} \|u(e') - \tilde{u}(e_j)\|^2_{L^2(X\times\Omega^2)} \Bigg] de' de_j de_i.\end{aligned}$$

Distributing and simplifying results in

$$\begin{aligned}\left\|\hat{\Theta}_{i,l}(\boldsymbol{\omega}_l)\right\|^2_{L^2(X)} \le\ & C\|\sigma_s u\|^2_{L^\infty(U)} \sum_{j=1}^{N} ⨍_{E_i} \int_{E_j} \left\|\mathcal{P}(\boldsymbol{\omega}_l, e_i, e') - \tilde{\mathcal{P}}_{i,j}(\boldsymbol{\omega}_l)\right\|^2_{L^2(X\times\Omega)} de' de_i \\ &+ C\|\mathcal{P}u\|^2_{L^\infty(X\times\Omega^2\times E^2)} \sum_{j=1}^{N} \int_{E_j} \|\sigma_s(e') - \sigma_{s,j}\|^2_{L^2(X\times\Omega^2)} de' \\ &+ C\|\sigma_s\mathcal{P}\|^2_{L^\infty(X\times\Omega^2\times E^2)} \sum_{j=1}^{N} ⨍_{E_j} \int_{E_j} \|u(e') - \tilde{u}(e_j)\|^2_{L^2(X\times\Omega^2)} de' de_j.\end{aligned}$$

Multiplying by $|E_i|$ and w_l, and summing over $1 \le i \le N$ and $1 \le l \le L$ we have

$$\sum_{i=1}^{N} |E_i| \sum_{l=1}^{L} w_l \left\|\hat{\Theta}_{i,l}\right\|^2_{L^2(X)} \le \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3, \tag{4.20}$$

where

$$\begin{aligned}\mathbb{I}_1 &:= C\|\sigma_s u\|^2_{L^\infty(U)} \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{L} w_l \int_{E_i}\int_{E_j} \left\|\mathcal{P}(\boldsymbol{\omega}_l, e_i, e') - \tilde{\mathcal{P}}_{i,j}(\boldsymbol{\omega}_l)\right\|^2_{L^2(X\times\Omega)} de' de_i, \\ \mathbb{I}_2 &:= C\|\mathcal{P}u\|^2_{L^\infty(X\times\Omega^2\times E^2)} \sum_{j=1}^{N} \int_{E_j} \|\sigma_s(e') - \sigma_{s,j}\|^2_{L^2(X\times\Omega^2)} de', \\ \mathbb{I}_3 &:= C\|\sigma_s\mathcal{P}\|^2_{L^\infty(X\times\Omega^2\times E^2)} \sum_{j=1}^{N} ⨍_{E_j} \int_{E_j} \|u(e') - \tilde{u}(e_j)\|^2_{L^2(X\times\Omega^2)} de' de_j.\end{aligned}$$

The claim follows by estimating the terms on the right hand side. We now bound each of the three terms.

1. Starting with $\mathbb{I}_1$, applying Lemma 4.7

$$\mathbb{I}_1 \leq C \left\| \sigma_s u \right\|^2_{L^\infty(U)} \sum_{l=1}^{L} w_l \Bigg[\sum_{i=1}^{N} \sum_{j=1}^{N} \Bigg(h_a^4 \left\| \mathcal{P}(\boldsymbol{\omega}_l) \right\|^2_{L^2(X\times E_i \times E_j, H^2(\Omega))} + h_e^2 \left\| \mathcal{P}(\boldsymbol{\omega}_l) \right\|^2_{L^2(X\times\Omega, H^1(E_i\times E_j))} \Bigg) \Bigg]$$
$$\leq C \left\| \sigma_s u \right\|^2 \sum_{l=1}^{L} w_l \Bigg(h_a^4 \left\| \mathcal{P}(\boldsymbol{\omega}_l) \right\|^2_{L^2(X\times E^2, H^2(\Omega))} + h_e^2 \left\| \mathcal{P}(\boldsymbol{\omega}_l) \right\|^2_{L^2(X\times\Omega, H^1(E^2))} \Bigg)$$

We then have

$$\mathbb{I}_1 \leq C \left\| u \right\|^2_{L^\infty(U)} (h_e^2 + h_a^4) \tag{4.21}$$

where the constant now depends on the functions σ_s and $\mathcal{P}$.

2. The estimate for $\mathbb{I}_2$ has been established in Lemma 3.6 and is given by inequality (3.26),

$$\mathbb{I}_2 \leq C h_e^2 \left\| u \right\|^2_{L^\infty(U)}. \tag{4.22}$$

3. Lastly, we have

$$\mathbb{I}_3 \leq C \left\| \sigma_s \mathcal{P} \right\|^2_{L^\infty(X\times\Omega^2\times E^2)} \sum_{j=1}^{N} \fint_{E_j} \int_{E_j} \left\| u(e') - \tilde{u}(e_j) \right\|^2_{L^2(X\times\Omega^2)} de' de_j$$
$$\leq C \sum_{j=1}^{N} \fint_{E_j} \int_{E_j} \left\| u(e') - \tilde{u}(e_j) \right\|^2_{L^2(X\times\Omega^2)} de' de_j$$

For each term in the summation, applying Lemma 4.6 yields

$$\fint_{E_j} \int_{E_j} \left\| u(e') - \tilde{u}(e_j) \right\|^2_{L^2(X\times\Omega^2)} de' de_j$$
$$= \frac{1}{|E_j|} \int_{E_j} \int_{E_j} \left\| u(e') - \tilde{u}(e_j) \right\|^2_{L^2(X\times\Omega^2)} de' de_j$$
$$\leq C \left(h_e^2 \left\| u \right\|^2_{L^2(X\times\Omega, H^1(E_j))} + h_a^4 \left\| u \right\|^2_{L^2(X\times E_j, H^2(\Omega))} \right)$$

Using this bound, we then obtain

$$\mathbb{I}_3 \le C \left\|\sigma_s \mathcal{P}\right\|^2_{L^\infty(X\times\Omega^2\times E^2)} \sum_{j=1}^{N} \left(h_e^2 \left\|u\right\|^2_{L^2(X\times\Omega,H^1(E_j))} + h_a^4 \left\|u\right\|^2_{L^2(X\times E_j,H^2(\Omega))}\right)$$
$$\le C \left(h_e^2 \left\|u\right\|^2_{L^2(X\times\Omega,H^1(E))} + h_a^4 \left\|u\right\|^2_{L^2(X\times E,H^2(\Omega))}\right). \tag{4.23}$$

Applying inequality (4.21), (4.22), and (4.23) to (4.20) yields

$$\left\|\{\hat{\Theta}_{i,l}\}_{i=1,l=1}^{N,L}\right\|^2 \le C\left[h_e^2\left(\left\|u\right\|^2_{L^\infty(U)} + \left\|u\right\|^2_{L^2(X\times\Omega,H^1(E))}\right) + h_a^4\left(\left\|u\right\|^2_{L^\infty(U)} + \left\|u\right\|^2_{L^2(X\times E,H^2(\Omega))}\right)\right],$$

from which the claim is obtained. □

We next rewrite equation (4.16) into an integral form obtained by integrating along characteristics. Given any $\mathbf{x} \in X$ and $\boldsymbol{\omega}_l \in \Omega_{h_a}$, there exists a point $\mathbf{x}_- \in \partial X_-$ such that $\mathbf{x} = \mathbf{x}_- + t\boldsymbol{\omega}_l$ for some $0 \le t \le \tau(\mathbf{x},\boldsymbol{\omega}_l)$, where $\tau = \tau(\mathbf{x},\boldsymbol{\omega}_l) = \sup\{t \,|\, \mathbf{x}_- + t\boldsymbol{\omega}_l \in X\}$. Replacing $\mathbf{x}$ in equation (4.16) by the aforementioned identity,

$$\frac{\partial}{\partial t}\varepsilon_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l) + \sigma_{t,i}(\mathbf{x}_- + t\boldsymbol{\omega}_l)\varepsilon_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l)$$
$$= \sum_{j=1}^{i} |E_j| \sigma_{s,j}(\mathbf{x}_- + t\boldsymbol{\omega}_l) \sum_{k=1}^{L} w_{l,k}^{i,j} \varepsilon_{j,k}(\mathbf{x}_- + t\boldsymbol{\omega}_l)$$
$$+ \Theta_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l) + \Psi_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l).$$

Multiplying both sides by $e^{\int_0^t \sigma_{t,i}(\mathbf{x}_- + r\boldsymbol{\omega}_l)dr}$, we obtain

$$\frac{\partial}{\partial t}\left(\varepsilon_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l) e^{\int_0^t \sigma_{t,i}(\mathbf{x}_- + r\boldsymbol{\omega}_l)dr}\right)$$
$$= e^{\int_0^t \sigma_{t,i}(\mathbf{x}_- + r\boldsymbol{\omega}_l)dr}\Bigg(\sum_{j=1}^{i} |E_j| \sigma_{s,j}(\mathbf{x}_- + t\boldsymbol{\omega}_l) \sum_{k=1}^{L} w_{l,k}^{i,j} \varepsilon_{j,k}(\mathbf{x}_- + t\boldsymbol{\omega}_l)$$
$$+ \Theta_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l) + \Psi_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l)\Bigg).$$

Integrating over t yields

$$\begin{aligned}
&\varepsilon_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l)e^{\int_0^t \sigma_{t,i}(\mathbf{x}_- + r\boldsymbol{\omega}_l)dr} \\
&\quad = \int_0^t e^{\int_0^s \sigma_{t,i}(\mathbf{x}_- + r\boldsymbol{\omega}_l)dr}\Bigg(\sum_{j=1}^{i}|E_j|\sigma_{s,j}(\mathbf{x}_- + s\boldsymbol{\omega}_l)\sum_{k=1}^{L} w_{l,k}^{i,j}\varepsilon_{j,k}(\mathbf{x}_- + s\boldsymbol{\omega}_l) \\
&\qquad + \Theta_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l) + \Psi_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l)\Bigg)ds,
\end{aligned}$$

which becomes

$$\begin{aligned}
&\varepsilon_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l) \\
&\quad = \int_0^t e^{-\int_s^t \sigma_{t,i}(\mathbf{x}_- + r\boldsymbol{\omega}_l)dr}\Bigg(\sum_{j=1}^{i}|E_j|\sigma_{s,j}(\mathbf{x}_- + s\boldsymbol{\omega}_l)\sum_{k=1}^{L} w_{l,k}^{i,j}\varepsilon_{j,k}(\mathbf{x}_- + s\boldsymbol{\omega}_l) \\
&\qquad + \Theta_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l) + \Psi_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l)\Bigg)ds. \qquad (4.24)
\end{aligned}$$

This will be the starting point for the error analysis. Next we derive an inequality of the form found in Lemma 3.1. For simplicity in the following proofs, let us define the following norm. For a function $\{v_l\}_{l=1}^L \in (L^2(X))^L$, we introduce the norm

$$\left\|\{v_l\}_{l=1}^L\right\|_{h_a} := \left(\sum_{l=1}^{L} w_l \left\|v_l\right\|_{L^2(X)}^2\right)^{1/2}, \qquad (4.25)$$

where w_l is defined as in (4.10).

Lemma 4.10. *For h_e sufficiently small the relation*

$$\left\|\{\varepsilon_{i,l}\}_{l=1}^L\right\|_{h_a} \le C\left[\sum_{j=1}^{i-1}|E_j|\left\|\{\varepsilon_{j,k}\}_{l=1}^L\right\|_{h_a} + \left\|\{\Theta_{i,l}\}_{l=1}^L\right\|_{h_a} + \left\|\{\Psi_{i,l}\}_{l=1}^L\right\|_{h_a}\right]$$

holds, for $1 \le i \le N$, where $\|\cdot\|_{h_a}$ is given in (4.25).

Proof. Consider equation (4.24). Taking the absolute value of both sides, followed by the triangle inequality, we obtain

$$
\begin{aligned}
&|\varepsilon_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l)| \\
&\quad \le \int_0^t e^{-\int_s^t \sigma_{t,i}(\mathbf{x}_- + r\boldsymbol{\omega}_l)dr} \Bigg(\sum_{j=1}^{i} |E_j| \sigma_{s,j}(\mathbf{x}_- + s\boldsymbol{\omega}_l) \sum_{k=1}^{L} w_{l,k}^{i,j}(\mathbf{x}_- + s\boldsymbol{\omega}_l) |\varepsilon_{j,k}(\mathbf{x}_- + s\boldsymbol{\omega}_l)| \\
&\qquad + |\Theta_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l)| + |\Psi_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l)| \Bigg) ds.
\end{aligned}
$$

Using the bound $e^{-\int_s^t \sigma_{t,i}(\mathbf{x}_- + r\boldsymbol{\omega}_l)dr} \le 1$, the bound

$$
w_{l,k}^{i,j} \le \|\mathcal{P}\|_{L^\infty(X\times\Omega^2\times E^2)} \int_\Omega \phi_k(\boldsymbol{\omega}')d\boldsymbol{\omega}' = \|\mathcal{P}\|_{L^\infty(X\times\Omega^2\times E^2)} w_k,
$$

as well as the upper bound on σ_s we obtain

$$
\begin{aligned}
|\varepsilon_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l)| \le \|\sigma_s \mathcal{P}\|_{L^\infty(X\times\Omega^2\times E^2)} \int_0^t \Bigg(& \sum_{j=1}^{i} |E_j| \sum_{k=1}^{L} w_k |\varepsilon_{j,k}(\mathbf{x}_- + s\boldsymbol{\omega}_l)| \\
& + |\Theta_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l)| + |\Psi_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l)| \Bigg) ds.
\end{aligned}
$$

Extending the integration for $0 \le s \le \tau(\mathbf{x}_-, \boldsymbol{\omega}_l)$ and squaring both sides lead to

$$
\begin{aligned}
|\varepsilon_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l)|^2 \le C \Bigg[\int_0^\tau \Bigg(& \sum_{j=1}^{i} |E_j| \sum_{k=1}^{L} w_k |\varepsilon_{j,k}(\mathbf{x}_- + s\boldsymbol{\omega}_l)| \\
& + |\Theta_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l)| + |\Psi_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l)| \Bigg) ds \Bigg]^2,
\end{aligned}
$$

where $\tau = \tau(\mathbf{x}_-, \boldsymbol{\omega}_l)$ and the dependence on σ_s and $\mathcal{P}$ is now contained within the constant. Repeated application of the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
|\varepsilon_{i,l}(\mathbf{x}_- + t\boldsymbol{\omega}_l)|^2 \le C \Bigg(& \sum_{j=1}^{i} |E_j| \sum_{k=1}^{L} w_k \int_0^\tau |\varepsilon_{j,k}(\mathbf{x}_- + s\boldsymbol{\omega}_l)|^2 ds \\
& + \int_0^\tau |\Theta_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l)|^2 ds + \int_0^\tau |\Psi_{i,l}(\mathbf{x}_- + s\boldsymbol{\omega}_l)|^2 ds \Bigg).
\end{aligned}
$$

Integrating with respect to $t \in [0, \tau]$ and $\mathbf{x}_- \in \partial X_-$ results in

$$
\|\varepsilon_{i,l}\|_{L^2(X)}^2 \le C \Bigg(\sum_{j=1}^{i} |E_j| \sum_{k=1}^{L} w_k \|\varepsilon_{j,k}\|_{L^2(X)}^2 + \|\Theta_{i,l}\|_{L^2(X)}^2 + \|\Psi_{i,l}\|_{L^2(X)}^2 \Bigg),
$$

where we have used the identity

$$\int_{\partial X_-}\int_0^{\tau(\mathbf{x}_-,\boldsymbol{\omega})} v(\mathbf{x}_- + t\boldsymbol{\omega})dtdS(\mathbf{x}_-) = \int_X v(\mathbf{x})d\mathbf{x}.$$

Multiplying by w_l, summing over $l = 1,\ldots,L$, and renaming indices on the right hand side results in

$$\sum_{l=1}^{L} w_l \left\|\varepsilon_{i,l}\right\|_{L^2(X)}^2 \le C\left(\sum_{j=1}^{i}|E_j|\sum_{l=1}^{L} w_l \left\|\varepsilon_{j,l}\right\|_{L^2(X)}^2 + \sum_{l=1}^{L} w_l \left\|\Theta_{i,l}\right\|_{L^2(X)}^2 + \sum_{l=1}^{L} w_l \left\|\Psi_{i,l}\right\|_{L^2(X)}^2\right).$$

Breaking up the summation over the energy groups we have

$$\sum_{l=1}^{L} w_l \left\|\varepsilon_{i,l}\right\|_{L^2(X)}^2 \le C|E_i|\sum_{l=1}^{L} w_l \left\|\varepsilon_{i,l}\right\|_{L^2(X)}^2 + C\Bigg(\sum_{j=1}^{i-1}|E_j|\sum_{l=1}^{L} w_l \left\|\varepsilon_{j,l}\right\|_{L^2(X)}^2 + \sum_{l=1}^{L} w_l \left\|\Theta_{i,l}\right\|_{L^2(X)}^2 + \sum_{l=1}^{L} w_l \left\|\Psi_{i,l}\right\|_{L^2(X)}^2\Bigg).$$

For $|E_i|$ sufficiently small, i.e. $|E_i| < 1/C$, we have

$$\sum_{l=1}^{L} w_l \left\|\varepsilon_{i,l}\right\|_{L^2(X)}^2 \le C\left(\sum_{j=1}^{i-1}|E_j|\sum_{l=1}^{L} w_l \left\|\varepsilon_{j,l}\right\|_{L^2(X)}^2 + \sum_{l=1}^{L} w_l \left\|\Theta_{i,l}\right\|_{L^2(X)}^2 + \sum_{l=1}^{L} w_l \left\|\Psi_{i,l}\right\|_{L^2(X)}^2\right).$$

Taking the square root of both sides and using subadditivity results in

$$\left(\sum_{l=1}^{L} w_l \left\|\varepsilon_{i,l}\right\|_{L^2(X)}^2\right)^{1/2} \le C\Bigg[\sum_{j=1}^{i-1}|E_j|\left(\sum_{k=1}^{L} w_k \left\|\varepsilon_{j,k}\right\|_{L^2(X)}^2\right)^{1/2} + \left(\sum_{l=1}^{L} w_l \left\|\Theta_{i,l}\right\|_{L^2(X)}^2\right)^{1/2} + \left(\sum_{l=1}^{L} w_l \left\|\Psi_{i,l}\right\|_{L^2(X)}^2\right)^{1/2}\Bigg].$$

The claim then follows from the definition of $\|\cdot\|_{h_a}$. □

The next result is the main result of the chapter. It establishes an error bound for the error between the semi-discrete solution and the solution of the original boundary value problem.

Theorem 4.11. *Assume $f \in C(\Omega, L^2(X, H^1(E)))$ and that the conditions of Lemma 4.9 hold. Then for h_e sufficiently small, the inequality*

$$\left\| \{\varepsilon_{i,l}\}_{i=1,l=1}^{N,L} \right\| \le C\Bigg[h_e \Bigg(\|u\|_{L^\infty(U)} + \|u\|_{L^2(X\times\Omega, H^1(E))} + \left(\sum_{l=1}^{L} w_l |f(\boldsymbol{\omega}_l)|^2_{L^2(X,H^1(E))} \right)^{1/2} \Bigg) + h_a^2 \Big(\|u\|_{L^\infty(U)} + \|u\|_{L^2(X\times E, H^2(\Omega))} \Big) \Bigg],$$

holds.

Proof. Apply Lemma 4.10 and the Cauchy-Schwarz inequality to obtain

$$\left\| \{\varepsilon_{i,l}\}_{l=1}^{L} \right\|_{h_a} \le C\left[\left(\sum_{j=1}^{i-1} |E_j| \left\| \{\varepsilon_{j,k}\}_{l=1}^{L} \right\|_{h_a}^2 \right)^{1/2} + \left\| \{\Theta_{i,l}\}_{l=1}^{L} \right\|_{h_a} + \left\| \{\Psi_{i,l}\}_{l=1}^{L} \right\|_{h_a} \right].$$

Recalling the dependence on $\{e_j\}_{j=1}^{i}$, squaring both sides and using the Cauchy-Schwarz inequality leads to

$$\left\| \{\varepsilon_{i,l}(e_i)\}_{l=1}^{L} \right\|_{h_a}^2 \le C\Bigg[\sum_{j=1}^{i-1} |E_j| \left\| \{\varepsilon_{j,l}(e_j)\}_{l=1}^{L} \right\|_{h_a}^2 + \left\| \{\Theta_{i,l}\left(\{e_k\}_{k=1}^{i}\}_{l=1}^{L}\right) \right\|_{h_a}^2 + \left\| \{\Psi_{i,l}(e_i)\}_{l=1}^{L} \right\|_{h_a}^2 \Bigg].$$

For $1 \le j \le i$ we integrate e_j over E_j in the average sense. Let us do this for each term explicitly. For the first term on the right hand side we have

$$\fint_{E_1} \cdots \fint_{E_i} \sum_{j=1}^{i-1} |E_j| \left\| \{\varepsilon_{j,k}(e_j)\}_{l=1}^{L} \right\|_{h_a}^2 de_i \cdots de_1 = \sum_{j=1}^{i-1} |E_j| \fint_{E_j} \left\| \{\varepsilon_{j,k}(e_j)\}_{l=1}^{L} \right\|_{h_a}^2 de_j.$$

The second term may be written as

$$\fint_{E_1} \cdots \fint_{E_i} \left\| \{\Theta_{i,l}(\{e_k\}_{k=1}^{i}\}_{l=1}^{L} \right\|_{h_a}^2 de_i \cdots de_1 = \left\| \{\hat{\Theta}_{i,l}\}_{l=1}^{L} \right\|_{h_a}^2$$

using the definition of $\hat{\Theta}_{i,l}$ given by equation (4.18). Finally, the left hand side and the last term on the right hand side depend only on e_i, which leads to the inequality having the form

$$\fint_{E_i} \left\| \{\varepsilon_{i,l}(e_i)\}_{l=1}^{L} \right\|_{h_a}^2 de_i \le C \Bigg[\sum_{j=1}^{i-1} |E_j| \fint_{E_j} \left\| \{\varepsilon_{j,k}(e_j)\}_{l=1}^{L} \right\|_{h_a}^2 de_j + \left\| \{\hat{\Theta}_{i,l}\}_{l=1}^{L} \right\|_{h_a}^2 + \fint_{E_i} \left\| \{\Psi_{i,l}(e_i)\}_{l=1}^{L} \right\|_{h_a}^2 de_i \Bigg]. \tag{4.26}$$

Applying Lemma 3.1 with

$$a_j = \fint_{E_j} \left\| \{\varepsilon_{j,k}(e_j)\}_{l=1}^{L} \right\|_{h_a}^2 de_j, \quad b_j = C|E_j|$$

and

$$\gamma_i = C\left(\left\| \{\hat{\Theta}_{i,l}\}_{l=1}^{L} \right\|_{h_a}^2 + \fint_{E_i} \left\| \{\Psi_{i,l}(e_i)\}_{l=1}^{L} \right\|_{h_a}^2 de_i \right)$$

we obtain

$$\fint_{E_i} \left\| \{\varepsilon_{i,l}(e_i)\}_{l=1}^{L} \right\|_{h_a}^2 de \le C\left(\gamma_i + \sum_{j=1}^{i-1} |E_j|\gamma_j \right).$$

Extending the summation from $i-1$ to N, multiplying both sides by $|E_i|$, and summing over $1 \le i \le N$

$$\begin{aligned} \sum_{i=1}^{N} \int_{E_i} \left\| \{\varepsilon_{i,l}(e_i)\}_{l=1}^{L} \right\|_{h_a}^2 de_i &\le C\left(\sum_{i=1}^{N} |E_i|\gamma_i + |E| \sum_{j=1}^{N} |E_j|\gamma_j \right) \\ &\le C \sum_{i=1}^{N} |E_i|\gamma_i. \end{aligned}$$

Substituting in the definition of γ_i yields

$$\sum_{i=1}^{n} \int_{E_i} \left\| \{\varepsilon_{i,l}(e_i)\}_{l=1}^{L} \right\|_{h_a}^2 de_i \le C \sum_{i=1}^{N} \left(|E_i| \left\| \{\hat{\Theta}_{i,l}\}_{l=1}^{L} \right\|_{h_a}^2 + \int_{E_i} \left\| \{\Psi_{i,l}(e_i)\}_{l=1}^{L} \right\|_{h_a}^2 de_i \right).$$

Taking the square root and using its subadditivity yields

$$\left\| \{\varepsilon_{i,l}\}_{i=1,l=1}^{N,L} \right\| \le \left\| \{\hat{\Theta}_{i,l}\}_{i=1,l=1}^{N,L} \right\| + \left\| \{\Psi_{i,l}\}_{i=1,l=1}^{N,L} \right\|.$$

The claim then follows by applying Lemmas 4.8 and 4.10. □

CHAPTER 5
SPATIAL DISCRETIZATION

Lastly, the spatial variable in (4.8)–(4.9) is dealt with using a discontinuous Galerkin (DG) method. DG methods have recently been used in the discretization of the spatial variable of the monoenergetic form of the RTE under the assumption of non-vanishing absorption ([23, 25]). Our assumptions differ in that we only require the total cross section of each energy group be non-zero. A more general treatment of DG methods as applied to first-order hyperbolic BVPs is found in [9].

5.1 Spatial Discretization

For ease of presentation assume X is a polyhedron. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of the spatial domain X and from this family consider the triangulation $\mathcal{T}_h$. For each $K \in \mathcal{T}_h$, let h_K denote the diameter of K and let $h := \max_{K \in \mathcal{T}_h} h_K$. Given $K \in \mathcal{T}_h$, ∂K denotes the faces of element K and with each face we associate an outward facing unit normal vector $\boldsymbol{\nu}_K$. On a face on the boundary ∂X, $\boldsymbol{\nu}_k$ is the unit outward normal vector. Let $\mathcal{E}_h$ denote the set of interior faces of the triangulation. For each face $\rho \in \mathcal{E}_h$, let K^+ and K^- denote adjacent elements sharing ρ; we then define $\boldsymbol{\nu}_\rho$ to be the normal vector pointing from K^- toward K^+. When dealing with a function v over an edge ρ, let $v^+ := v|_{K^+}$ and $v^- := v|_{K^-}$, denote the restriction of v over the corresponding adjacent elements. We denote a jump by $[\![v]\!] := v^+ - v^-$ and define the space

$$V_h = \left\{ v \in L^2(X) \mid \ v|_K \in P_r(K) \ \forall \, K \in \mathcal{T}_h \right\}, \tag{5.1}$$

where $r \in \mathbb{N}$ and $P_r(K)$ denotes polynomials on K of a degree less than or equal to r.

For $v \in V_h$, multiply equation (4.8) by v and integrating over X, we obtain

$$\int_X \boldsymbol{\omega}_l \cdot \nabla_x u_{i,l} v d\mathbf{x} + \int_X \sigma_{t,i} u_{i,l} v d\mathbf{x} = \int_X v \sum_{j=1}^{i} |E_j| \sigma_{s,j} \sum_{k=1}^{L} w_{l,k}^{i,j} u_{j,k} d\mathbf{x} + \int_X f_{i,l} v d\mathbf{x}. \quad (5.2)$$

Using the triangulation, (5.2) becomes

$$\sum_{K \in T_h} \int_K \boldsymbol{\omega}_l \cdot \nabla_x u_{i,l} v_K d\mathbf{x} + \sum_{K \in T_h} \int_K \sigma_{t,i} u_{i,l} v_K d\mathbf{x}$$
$$= \sum_{K \in T_h} \int_K v_K \sum_{j=1}^{i} |E_j| \sigma_{s,j} \sum_{k=1}^{L} w_{l,k}^{i,j} u_{j,k} d\mathbf{x} + \int_X f_{i,l} v d\mathbf{x}, \quad (5.3)$$

where, because $v \in V_h$, $v_K = v|_K \in P_r(K)$. Integration by parts applied to the first term of (5.3) yields

$$\int_K \boldsymbol{\omega}_l \cdot \nabla_x u_{i,l} v_K d\mathbf{x} = \int_{\partial K} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_K \hat{u}_{i,l} v_K d\mathbf{x} - \int_K u_{i,l} (\boldsymbol{\omega}_l \cdot \nabla_x v_K) d\mathbf{x}.$$

Here $\hat{u}_{i,l}$ is the numerical trace and defines the meaning of $u_{i,l}$ along the boundary of an element. We define it as

$$\hat{u}_{i,l}(\mathbf{x}) = \begin{cases} g_{i,l}(\mathbf{x}) & \text{if } (\mathbf{x}, \boldsymbol{\omega}_l) \in \partial X_-^l, \\ \lim_{\varepsilon \to 0^+} u_{i,l}(\mathbf{x} - \varepsilon \boldsymbol{\omega}_l) & \text{otherwise,} \end{cases}$$

where ∂X_+^l and ∂X_-^l denote the inflow and outflow portions of ∂X with respect to the direction $\boldsymbol{\omega}_l$, respectively, $1 \le l \le L$. Therefore, integrating by parts, (5.3) becomes

$$\sum_{K \in T_h} \int_{\partial K} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_K \hat{u}_{i,l} v_K d\mathbf{x} - \sum_{K \in T_h} \int_K u_{i,l} (\boldsymbol{\omega}_l \cdot \nabla_x v_K) d\mathbf{x} + \sum_{K \in T_h} \int_K \sigma_{t,i} u_{i,l} v_K d\mathbf{x}$$
$$- \sum_{K \in T_h} \int_K v_K \sum_{j=1}^{i} |E_j| \sigma_{s,j} \sum_{k=1}^{L} w_{l,k}^{i,j} u_{j,k} d\mathbf{x} = \int_X f_{i,l} v d\mathbf{x}.$$

Applying the definition of the trace along the inflow boundary and breaking up the summation over the energy groups leads to

$$\begin{aligned}\sum_{\rho\in\mathcal{E}_h}\int_\rho \boldsymbol{\omega}_l\cdot\boldsymbol{\nu}_\rho \hat{u}_{i,l}[\![v_l]\!]d\mathbf{x} - \sum_{K\in T_h}\int_K u_{i,l}(\boldsymbol{\omega}_l\cdot\nabla_x v_K)d\mathbf{x} + \int_{\partial X_+^l}\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}\hat{u}_{i,l}v_K ds \\ + \sum_{K\in T_h}\int_K \sigma_{t,i}u_{i,l}v_K d\mathbf{x} - \sum_{K\in T_h}\int_K v_K|E_i|\sigma_{s,i}\sum_{k=1}^{L} w_{l,k}^{i,i}u_{i,k}d\mathbf{x} \\ = \sum_{K\in T_h}\int_K v_K\sum_{j=1}^{i-1}|E_j|\sigma_{s,j}\sum_{k=1}^{L}w_{l,k}^{i,j}u_{j,k}d\mathbf{x} + \int_X f_{i,l}v d\mathbf{x} - \int_{\partial X_-^l}\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}g_{i,l}ds. \qquad (5.4)\end{aligned}$$

Denote an element $\{v_l\}_{l=1}^L \in (V_h)^L$, by $\mathbf{v} = \{v_l\}_{l=1}^L$. For $\mathbf{u},\mathbf{v}\in(V_h)^L$ define the following three bilinear forms

$$\begin{aligned}\mathbf{a}(\mathbf{u},\mathbf{v}) = \sum_{l=1}^{L} w_l\Bigg[\sum_{\rho\in\mathcal{E}_h}\int_\rho \boldsymbol{\omega}_l\cdot\boldsymbol{\nu}_\rho\hat{u}_l[\![v_l]\!]d\mathbf{x} - \sum_{K\in T_h}\int_K u_l(\boldsymbol{\omega}_l\cdot\nabla_x v_{l,K})d\mathbf{x} \\ + \int_{\partial X_+^l}\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}u_l v_l ds\Bigg],\end{aligned}$$

$$\mathbf{b}_i(\mathbf{u},\mathbf{v}) = \sum_{l=1}^{L} w_l\left[\sum_{K\in T_h}\int_K \sigma_{t,i}u_l v_{l,K}d\mathbf{x} - \sum_{K\in T_h}\int_K v_{l,K}|E_i|\sigma_{s,i}\sum_{k=1}^{L}w_{l,k}^{i,i}u_k d\mathbf{x}\right],$$

$$\mathbf{c}_{i,j}(\mathbf{u},\mathbf{v}) = \sum_{l=1}^{L} w_l\sum_{K\in T_h}\int_K v_{l,K}|E_j|\sigma_{s,j}\sum_{k=1}^{L}w_{l,k}^{i,j}u_k d\mathbf{x},$$

for $1\le j\le i$ with $1\le i\le N$, where $v_{l,K} = v_l|_K$. Recall that we have broken up the summation over the energy groups; as a result note that we can write $\mathbf{b}_i$ using the bilinear form $\mathbf{c}_{i,i}$. Lastly, for $1\le i\le N$ define the linear functional

$$\mathbf{l}_i(\mathbf{v}) = \sum_{l=1}^{L} w_l\left[\int_X f_{i,l}v_l d\mathbf{x} - \int_{\partial X_-^l}\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}g_{i,l}ds.\right] \qquad (5.5)$$

We obtain a coupled system of equations and the following problem:

$$\begin{cases} \text{Find } \mathbf{u}_i^h \in (V_h)^L \text{ s.t.} \\ \mathbf{a}(\mathbf{u}_i^h, \mathbf{v}) + \mathbf{b}_i(\mathbf{u}_i^h, \mathbf{v}) = \sum_{j=1}^{i-1} \mathbf{c}_{i,j}(\mathbf{u}_j^h, \mathbf{v}) + \mathbf{l}_i(\mathbf{v}), \quad \forall\, \mathbf{v} \in (V_h)^L, \end{cases} \tag{5.6}$$

for $1 \leq i \leq N$. Here, $\mathbf{u}_i^h = \{u_{i,l}^h\}_{l=1}^L \in (V_h)^L$ denotes the solution of the i^{th} fully discrete energy group equation in (5.6), for $1 \leq i \leq N$.

Remark 5.1. *Due to the no upscatter assumption (1.6), the fully discrete scheme can be seen as solving a sequence of DG schemes. The bilinear form $c_{i,j}(\cdot,\cdot)$ is then used to couple solutions over each energy group. As a result, the stability of the method can be approached by considering the method over each individual energy group.*

5.2 Stability

The stability and error estimates of the method are derived using the following norm:

$$|\!|\!|\mathbf{v}|\!|\!|_h := \left[\sum_{l=1}^{L} w_l \left(\|v_l\|_{L^2(X)}^2 + \sum_{\rho \in \mathcal{E}_h} \left\| |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho|^{1/2} [\![v_l]\!] \right\|_{L^2(\rho)}^2 + \int_{\partial X_+^l} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu} v_l^2 ds \right) \right]^{1/2}, \tag{5.7}$$

defined on $(H^1(X) + V_h)^L$. We first establish the coercivity of the left hand side of (5.6).

Lemma 5.2. *Assume there exists a constant $m_{\sigma_t} > 0$ such that $\sigma_t \geq m_{\sigma_t} > 0$. Then for h_e sufficiently small,*

$$\mathbf{a}(\mathbf{v}, \mathbf{v}) + \mathbf{b}_i(\mathbf{v}, \mathbf{v}) \geq c |\!|\!|\mathbf{v}|\!|\!|_h^2, \quad \forall\, \mathbf{v} \in (V_h)^L,$$

for $1 \leq i \leq N$.

Proof. For $\mathbf{v} \in (V_h)^L$, consider $\mathbf{a}(\mathbf{v}, \mathbf{v})$. We have

$$\begin{aligned}\mathbf{a}(\mathbf{v}, \mathbf{v}) = \sum_{l=1}^{L} w_l \Bigg[&\sum_{\rho \in \mathcal{E}_h} \int_\rho \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho \hat{v}_l [\![v_l]\!] d\mathbf{x} - \sum_{K \in T_h} \int_K v_{l,K} (\boldsymbol{\omega}_l \cdot \nabla_x v_{l,K}) d\mathbf{x} \\ &+ \int_{\partial X_+^l} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu} (v_l)^2 ds \Bigg]. \end{aligned} \tag{5.8}$$

Considering the second term in (5.8), application of divergence theorem and splitting the summation over the interior and physical boundaries leads to

$$\begin{aligned}\sum_{K \in T_h} \int_K v_{l,K} (\boldsymbol{\omega}_l \cdot \nabla_x v_{l,K}) d\mathbf{x} &= \frac{1}{2} \sum_{K \in T_h} \int_K \boldsymbol{\omega}_l \cdot \nabla_x v_{l,K}^2 d\mathbf{x} \\ &= \frac{1}{2} \sum_{K \in T_h} \int_{\partial K} (\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_K) v_{l,K}^2 d\mathbf{x} \\ &= \frac{1}{2} \sum_{\rho \in \mathcal{E}_h} \int_\rho \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho [\![v_l^2]\!] d\mathbf{x} + \frac{1}{2} \int_{\partial X^l} (\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}) v_l^2 d\mathbf{x}.\end{aligned}$$

Over the inflow boundary, $\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_K < 0$; therefore, omitting the term results in

$$\sum_{K \in T_h} \int_K v_{l,K} (\boldsymbol{\omega}_l \cdot \nabla_x v_{l,K}) d\mathbf{x} \leq \frac{1}{2} \sum_{\rho \in \mathcal{E}_h} \int_\rho \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho [\![v_l^2]\!] d\mathbf{x} + \frac{1}{2} \int_{\partial X_+^l} (\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}) v_l^2 d\mathbf{x}.$$

Return to equation (5.8) we have

$$\begin{aligned}\mathbf{a}(\mathbf{v}, \mathbf{v}) &\geq \sum_{l=1}^{L} w_l \Bigg[\sum_{\rho \in \mathcal{E}_h} \int_\rho \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho \hat{v}_l [\![v_l]\!] d\mathbf{x} - \frac{1}{2} \sum_{\rho \in \mathcal{E}_h} \int_\rho \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho [\![v_l^2]\!] d\mathbf{x} + \frac{1}{2} \int_{\partial X_+^l} (\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}) v_l^2 ds \Bigg] \\ &= \sum_{l=1}^{L} w_l \Bigg[\sum_{\rho \in \mathcal{E}_h} \int_\rho \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho \left(\hat{v}_l [\![v_l]\!] - \frac{1}{2} [\![v_l^2]\!] \right) d\mathbf{x} + \frac{1}{2} \int_{\partial X_+^l} (\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}) v_l^2 ds \Bigg]. \end{aligned} \tag{5.9}$$

When $\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho \geq 0$ the integration of over the interior edge becomes

$$\begin{aligned}\int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| \left(\hat{v}_l [\![v_l]\!] - \frac{1}{2} [\![v_l^2]\!] \right) d\mathbf{x} &= \int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| \left(v_l^+ [\![v_l]\!] - \frac{1}{2} [\![v_l^2]\!] \right) d\mathbf{x} \\ &= \int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| \left(\frac{1}{2} (v_l^+)^2 - v_l^+ v_l^- + \frac{1}{2} (v_l^-)^2 \right) d\mathbf{x} \\ &= \frac{1}{2} \int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| [\![v_l]\!]^2 d\mathbf{x}.\end{aligned}$$

When $\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho < 0$

$$\begin{aligned}\int_\rho \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho \left(\hat{v}_l [\![v_l]\!] - \frac{1}{2} [\![v_l^2]\!] \right) d\mathbf{x} &= - \int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| \left(v_l^- [\![v_l]\!] - \frac{1}{2} [\![v_l^2]\!] \right) d\mathbf{x} \\ &= - \int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| \left(-\frac{1}{2} (v_l^+)^2 + v_l^+ v_l^- - \frac{1}{2} (v_l^-)^2 \right) d\mathbf{x} \\ &= \frac{1}{2} \int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| [\![v_l]\!]^2 d\mathbf{x}.\end{aligned}$$

Therefore,

$$\sum_{\rho \in \mathcal{E}_h} \int_\rho \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho \left(\hat{v}_l [\![v_l]\!] - \frac{1}{2} [\![v_l^2]\!] \right) d\mathbf{x} = \frac{1}{2} \sum_{\rho \in \mathcal{E}_h} \int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| [\![v_l]\!]^2 d\mathbf{x},$$

which, upon substitution into inequality (5.9), yields

$$\mathbf{a}^h(\mathbf{v}, \mathbf{v}) \geq \frac{1}{2} \sum_{l=1}^{L} w_l \left[\sum_{\rho \in \mathcal{E}_h} \int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| [\![v_l]\!]^2 d\mathbf{x} + \int_{\partial X_+^l} |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}| v_l^2 ds \right]. \tag{5.10}$$

We next consider the bilinear form

$$\mathbf{b}_i(\mathbf{v}, \mathbf{v}) = \sum_{l=1}^{L} w_l \left[\sum_{K \in T_h} \int_K \sigma_{t,i} v_{l,K}^2 d\mathbf{x} - \sum_{K \in T_h} \int_K v_{l,K} |E_i| \sigma_{s,i} \sum_{k=1}^{L} w_{l,k}^{i,i} v_{k,K} d\mathbf{x} \right], \tag{5.11}$$

for $1 \leq i \leq N$. Then the following bound on the second term can be obtained

$$\int_K |E_i| \sigma_{s,i} \sum_{k=1}^{L} w_{l,k}^{i,i} v_{k,K} v_{l,K} d\mathbf{x} \leq \frac{|E_i|}{2} \int_K \sigma_{s,i} \sum_{k=1}^{L} w_{l,k}^{i,i} \left(v_{l,K}^2 + v_{k,K}^2 \right) d\mathbf{x}.$$

Recalling $w_k = \int_\Omega \phi_k(\boldsymbol{\omega}') d\boldsymbol{\omega}'$ it follows from (4.7) that

$$w_{l,k}^{i,i} = \int_\Omega \phi_k(\boldsymbol{\omega}') \tilde{\mathcal{P}}_{i,j}(\mathbf{x}, \boldsymbol{\omega}_l \cdot \boldsymbol{\omega}') d\boldsymbol{\omega}' \leq \|\mathcal{P}\|_{L^\infty(X \times I \times E^2)} w_k, \tag{5.12}$$

for $1 \leq l, k \leq L$ and $1 \leq i \leq N$. Therefore, using the bound on the weights, there exists a constant C, such that

$$\int_K |E_i| \sigma_{s,i} \sum_{k=1}^{L} w_{l,k}^{i,i} v_{k,K} v_{l,K} d\mathbf{x} \leq \frac{|E_i|}{2} C \int_K \sigma_{s,i} \sum_{k=1}^{L} w_k \left(v_{l,K}^2 + v_{k,K}^2 \right) d\mathbf{x}.$$

Substitution back into (5.11) yields the lower bound

$$\begin{aligned}
\mathbf{b}_i(\mathbf{v},\mathbf{v}) &\geq \sum_{l=1}^{L} w_l \left[\sum_{K\in T_h} \int_K \sigma_{t,i} v_{l,K}^2 d\mathbf{x} - \sum_{K\in T_h} \frac{|E_i|}{2} C \int_K \sigma_{s,i} \sum_{k=1}^{L} w_k \left(v_{l,K}^2 + v_{k,K}^2 \right) d\mathbf{x} \right] \\
&= \sum_{l=1}^{L} w_l \left[\sum_{K\in T_h} \int_K \sigma_{t,i} v_{l,K}^2 d\mathbf{x} - \sum_{K\in T_h} \frac{|E_i|}{2} C \int_K \sigma_{s,i} v_{l,K}^2 \left(\sum_{k=1}^{L} w_k \right) d\mathbf{x} \right] \\
&\quad - \left(\sum_{l=1}^{L} w_l \right) \sum_{K\in T_h} \frac{|E_i|}{2} C \int_K \sigma_{s,i} \sum_{k=1}^{L} w_k v_{k,K}^2 d\mathbf{x}.
\end{aligned}$$

Then, after renaming indices in the summation

$$\begin{aligned}
\mathbf{b}_i(\mathbf{v},\mathbf{v}) &\geq \sum_{l=1}^{L} w_l \left[\sum_{K\in T_h} \int_K \sigma_{t,i} v_{l,K}^2 d\mathbf{x} - \sum_{K\in T_h} \frac{|E_i|}{2} C \int_K \sigma_{s,i} v_{l,K}^2 d\mathbf{x} \right] \\
&\quad - \sum_{K\in T_h} \frac{|E_i|}{2} C \int_K \sigma_{s,i} \sum_{k=1}^{L} w_k v_{k,K}^2 d\mathbf{x} \\
&= \sum_{l=1}^{L} w_l \left[\sum_{K\in T_h} \int_K \sigma_{t,i} v_{l,K}^2 d\mathbf{x} - \sum_{K\in T_h} \frac{|E_i|}{2} C \int_K \sigma_{s,i} v_{l,K}^2 d\mathbf{x} \right] \\
&\quad - \sum_{l=1}^{L} w_l \sum_{K\in T_h} \frac{|E_i|}{2} C \int_K \sigma_{s,i} v_{l,K}^2 d\mathbf{x}.
\end{aligned}$$

After simplifying, the following bound is obtained

$$\mathbf{b}_i(\mathbf{v},\mathbf{v}) \geq \sum_{l=1}^{L} w_l \sum_{K\in T_h} \int_K \left[\sigma_{t,i} - |E_i| C \sigma_{s,i} \right] v_{l,K}^2 d\mathbf{x}.$$

Define $\mu = \min_{1\leq i\leq N} \{\sigma_{t,i} - |E_i| C \sigma_{s,i}\}$. Given $\sigma_t \geq m_{\sigma_t} > 0$, it follows that for h_e sufficiently small, $\mu > 0$ and, therefore,

$$\mathbf{b}_i(\mathbf{v},\mathbf{v}) \geq \mu \sum_{l=1}^{L} w_l \sum_{K\in T_h} \int_K v_{l,K}^2 d\mathbf{x}. \tag{5.13}$$

Combing inequalities (5.10) and (5.13) yields the claim. □

With the coercivity of the left hand side of (5.6) established, we proceed to showing that the bilinear form on the right hand side is bounded.

Lemma 5.3. *For every* $\mathbf{u}, \mathbf{v} \in V_h$, *the inequality*

$$\mathbf{c}_{i,j}(\mathbf{u}, \mathbf{v}) \leq C|E_j| \|\mathbf{u}\|_h \|\mathbf{v}\|_h$$

holds, for $1 \leq j \leq i$ *with* $1 \leq i \leq N$.

Proof. By definition

$$\mathbf{c}_{i,j}(\mathbf{u}, \mathbf{v}) = \sum_{l=1}^{L} w_l \sum_{K \in T_h} \int_K v_{l,K} |E_j| \sigma_{s,j} \sum_{k=1}^{L} w_{l,k}^{i,j} u_k d\mathbf{x}.$$

Using the upper bounds on σ_s and $\mathcal{P}$ and repeated applications of the Cauchy-Schwarz inequality

$$\begin{aligned}
\mathbf{c}_{i,j}(\mathbf{u}, \mathbf{v}) &\leq C|E_j| \sum_{l=1}^{L} \sum_{k=1}^{L} w_l w_k \sum_{K \in T_h} \int_K u_k v_{l,K} d\mathbf{x} \\
&\leq C|E_j| \sum_{l=1}^{L} \sum_{k=1}^{L} w_l w_k \sum_{K \in T_h} \|v_l\|_{L^2(K)} \|u_k\|_{L^2(K)} \\
&\leq C|E_j| \sum_{l=1}^{L} w_l \|v_l\|_{L^2(X)} \sum_{k=1}^{L} w_k \|u_k\|_{L^2(X)}
\end{aligned}$$

One more application of the Cauchy-Schwarz inequality followed by the definition of the norm $\|\cdot\|_h$ yields

$$\begin{aligned}
\mathbf{c}_{i,j}(\mathbf{u}, \mathbf{v}) &\leq C|E_j| \left(\sum_{l=1}^{L} w_l \|v_l\|_{L^2(X)}^2 \right)^{1/2} \left(\sum_{k=1}^{L} w_k \|u_k\|_{L^2(X)}^2 \right)^{1/2} \\
&\leq C|E_j| \|\mathbf{u}\|_h \|\mathbf{v}\|_h.
\end{aligned}$$

□

The coercivity and above result lead to an existence and uniqueness result. The next corollary follows from the two preceding lemmas and a forward substitution argument. Denote the solution of the coupled system of equations given in (5.6) by $\mathbf{u}_h = \{u_{i,l}^h\}_{i=1,l=1}^{N,L} \in (V_h)^{NL}$.

Corollary 5.4. *Given the assumptions of Lemma 5.2, the system given in (5.6) has a unique solution* $\mathbf{u}_h \in (V_h)^{NL}$.

Proof. Let $i = 1$. Lemma 5.2 implies that

$$\mathbf{a}(\mathbf{u}_1^h, \mathbf{v}) + \mathbf{b}_1(\mathbf{u}_1^h, \mathbf{v}) = \mathbf{l}_1(\mathbf{v}), \quad \forall\ \mathbf{v} \in (V_h)^L,$$

has a unique solution $\mathbf{u}_1^h = \{u_{1,l}^h\}_{l=1}^L$. Note the absence of the bilinear form $\mathbf{c}_{i,j}$ for the case $i = 1$. As a result, $\mathbf{u}_1^h$ is a known quantity; therefore,

$$\mathbf{c}_{2,1}(\mathbf{u}_1^h, \mathbf{v}) = \sum_{l=1}^L w_l \sum_{K \in T_h} \int_K v_{l,K} |E_1| \sigma_{s,1} \sum_{k=1}^L w_{l,k}^{2,1} u_{1,k} d\mathbf{x}$$

is valid source term for

$$\mathbf{a}(\mathbf{u}_2^h, \mathbf{v}) + \mathbf{b}_2(\mathbf{u}_1^h, \mathbf{v}) = \mathbf{c}_{2,1}(\mathbf{u}_1^h, \mathbf{v}) + \mathbf{l}_2(\mathbf{v}), \quad \forall\ \mathbf{v} \in (V_h)^L.$$

It then follows from Lemma 5.2 that the above equation has a unique solution $\mathbf{u}_2^h = \{u_{2,l}^h\}_{l=1}^L$. Proceeding in this fashion, we find a unique solution $\mathbf{u}_h = \{\mathbf{u}_i^h\}_{i=1}^N \in (V_h)^{NL}$ satisfying (5.6). □

5.3 Error Analysis

We are now ready to proceed with the error estimation. A bound between $\mathbf{u}_h \in (V_h)^{NL}$ and the solution, $\{u_{i,l}\}_{i=1,l=1}^{N,L}$, of the angular semi-discretization (4.1)–(4.2) will be established. The norm used for the error estimation is defined as

$$\left\| \{u_{i,l}\}_{i=1,l=1}^{N,L} - \mathbf{u}_h \right\|_h := \left[\sum_{i=1}^N |E_i| \left\| \{u_{i,l}(\cdot)\}_{l=1}^L - \mathbf{u}_i^h \right\|_h^2 \right]^{1/2}. \tag{5.14}$$

This section begins by first establishing some needed notation and standard finite element results. Let $\pi_h : L^2(X) \to V_h$, denote the L^2–projector onto V_h. By definition, recall that given $u \in L^2(X)$, the error, $u - \pi_h u$, is orthogonal to V_h with respect the standard inner product on L^2. Formally, given $u \in L^2(X)$ one has

$$\int_X (u - \pi_h u) v dx = 0 \tag{5.15}$$

for all $v \in V_h$. Let $u \in H^{r+1}(X)$. Then the error between u and its projection onto V_h, over any element $K \in \mathcal{T}_h$, has the bound

$$\|u - \pi_h u\|_{H^m(K)} \leq C h^{r+1-m} \|u\|_{H^{r+1}(K)}, \tag{5.16}$$

where r is the polynomial degree used in the definition (5.1) of V_h. Additionally, the following trace inequality can be obtained (cf. [9])

$$\|u - \pi_h u\|_{L^2(\rho)} \leq C h^{r+\frac{1}{2}} \|u\|_{H^{r+1}(K)}, \tag{5.17}$$

for all $\rho \in \mathcal{E}_h$. Motivated by these inequalities, consider the following definitions. For $\{u_{i,l}\}_{i=1,l=1}^{N,L}$ satisfying the angular semi-discretization (4.1)–(4.2) and $\{u_{i,l}^h\}_{i=1,l=1}^{N,L}$ satisfying the fully discrete problem (5.6), define

$$\boldsymbol{\delta}_i = \{\delta_{i,l}\}_{l=1}^{L} = \{u_{i,l} - \pi_h u_{i,l}\}_{l=1}^{L} \quad \text{and} \quad \boldsymbol{\eta}_i = \{\eta_{i,l}\}_{l=1}^{L} = \{u_{i,l}^h - \pi_h u_{i,l}\}_{l=1}^{L}.$$

Examining the right hand sides of (5.16) and (5.17), it is natural to impose the regularity assumption

$$u_{i,l} \in H^{r+1}(X) \tag{5.18}$$

for $1 \leq i \leq N$, $1 \leq l \leq L$. The next two lemmas then provide estimates for the bilinear forms $\mathbf{a}(\cdot,\cdot)$ and $\mathbf{b}_i(\cdot,\cdot)$ with the arguments $\boldsymbol{\delta}_i$ and $\boldsymbol{\eta}_i$.

Lemma 5.5. *Assume (5.18). Then the estimate*

$$\mathbf{a}\left(\boldsymbol{\delta}_i, \boldsymbol{\eta}_i\right) \le Ch^{r+\frac{1}{2}}\left(\sum_{l=1}^{L} w_l \left\|u_{i,l}\right\|_{H^{r+1}(X)}^2\right)^{1/2} \|\boldsymbol{\eta}_i\|_h$$

holds for $1 \le i \le N$.

Proof. By definition of the bilinear form

$$\begin{aligned}\mathbf{a}\left(\boldsymbol{\delta}_i, \boldsymbol{\eta}_i\right) = \sum_{l=1}^{L} w_l \Bigg[& \sum_{\rho\in\mathcal{E}_h} \int_\rho \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho \widehat{\delta_{i,l}} [\![\eta_{i,l}]\!] d\mathbf{x} \\ & - \sum_{K\in T_h} \int_K \delta_{i,l} (\boldsymbol{\omega}_l \cdot \nabla_x \eta_{i,l}) d\mathbf{x} + \int_{\partial X_+^l} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu} \delta_{i,l} \eta_{i,l} ds \Bigg].\end{aligned} \tag{5.19}$$

Consider the second term. Note that $\eta_{i,l} = u_{i,l}^h - \pi_h u_{i,l} \in V_h$, which implies that $\boldsymbol{\omega}_l \cdot \nabla_x \eta_{i,l} \in V_h$. Then from (5.15), we have

$$\sum_{K\in T_h} \int_K \delta_{i,l} (\boldsymbol{\omega}_l \cdot \nabla_x \eta_{i,l}) d\mathbf{x} = \int_X \left(u_{i,l} - \pi_h u_{i,l}\right) \left(\boldsymbol{\omega}_l \cdot \nabla_x \eta_{i,l}\right) d\mathbf{x} = 0.$$

Therefore,

$$\mathbf{a}\left(\boldsymbol{\delta}_i, \boldsymbol{\eta}_i\right) = \sum_{l=1}^{L} w_l \Bigg[\sum_{\rho\in\mathcal{E}_h} \int_\rho \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho \widehat{\delta_{i,l}} [\![\eta_{i,l}]\!] d\mathbf{x} + \int_{\partial X_+^l} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu} \delta_{i,l} \eta_{i,l} ds \Bigg].$$

Next, consider the integration over the interior edges. Applying the triangle and Cauchy-Schwarz inequalities,

$$\begin{aligned}\sum_{\rho\in\mathcal{E}_h} \int_\rho \boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho \widehat{\delta_{i,l}} [\![\eta_{i,l}]\!] d\mathbf{x} &\le \sum_{\rho\in\mathcal{E}_h} \int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| |\delta_{i,l}| |[\![\eta_{i,l}]\!]| d\mathbf{x} \\ &\le \sum_{\rho\in\mathcal{E}_h} \left(\int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| \left(\delta_{i,l}\right)^2 d\mathbf{x}\right)^{1/2} \left(\int_\rho |\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| [\![\eta_{i,l}]\!]^2 d\mathbf{x}\right)^{1/2}.\end{aligned}$$

Using that $|\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}_\rho| \le 1$, applying the trace inequality (5.17), and applying the Cauchy-

Schwarz inequality,

$$
\begin{aligned}
&\sum_{\rho\in\mathcal{E}_h}\int_\rho \boldsymbol{\omega}_l\cdot\boldsymbol{\nu}_\rho\widehat{\delta_{i,l}}[\![\eta_{i,l}]\!]d\mathbf{x}\\
&\qquad\le \sum_{\rho\in\mathcal{E}_h}\|\delta_{i,l}\|_{L^2(\rho)}\left\||\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}_\rho|^{1/2}[\![\eta_{i,l}]\!]\right\|_{L^2(\rho)}\\
&\qquad\le \left(\sum_{\rho\in\mathcal{E}_h}\|\delta_{i,l}\|^2_{L^2(\rho)}\right)^{1/2}\left(\sum_{\rho\in\mathcal{E}_h}\left\||\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}_\rho|^{1/2}[\![\eta_{i,l}]\!]\right\|^2_{L^2(\rho)}\right)^{1/2}\\
&\qquad\le Ch^{r+\frac{1}{2}}\left(\sum_{K\in T_h}\|u_{i,l}\|^2_{H^{r+1}(K)}\right)^{1/2}\left(\sum_{\rho\in\mathcal{E}_h}\left\||\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}_\rho|^{1/2}[\![\eta_{i,l}]\!]\right\|^2_{L^2(\rho)}\right)^{1/2}\\
&\qquad\le Ch^{r+\frac{1}{2}}\|u_{i,l}\|_{H^{r+1}(X)}\left(\sum_{\rho\in\mathcal{E}_h}\left\||\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}_\rho|^{1/2}[\![\eta_{i,l}]\!]\right\|^2_{L^2(\rho)}\right)^{1/2}.
\end{aligned}
$$

Multiplying by w_l, summing over $1\le l\le L$, and applying the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\sum_{l=1}^{L}w_l\sum_{\rho\in\mathcal{E}_h}\int_\rho \boldsymbol{\omega}_l\cdot\boldsymbol{\nu}_\rho\widehat{\delta_{i,l}}[\![\eta_{i,l}]\!]d\mathbf{x}\\
&\qquad\le Ch^{r+\frac{1}{2}}\sum_{l=1}^{L}w_l\|u_{i,l}\|_{H^{r+1}(X)}\left(\sum_{\rho\in\mathcal{E}_h}\left\||\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}_\rho|^{1/2}[\![\eta_{i,l}]\!]\right\|^2_{L^2(\rho)}\right)^{1/2}\\
&\qquad\le Ch^{r+\frac{1}{2}}\left(\sum_{l=1}^{L}w_l\|u_{i,l}\|^2_{H^{r+1}(X)}\right)^{1/2}\left(\sum_{l=1}^{L}w_l\sum_{\rho\in\mathcal{E}_h}\left\||\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}_\rho|^{1/2}[\![\eta_{i,l}]\!]\right\|^2_{L^2(\rho)}\right)^{1/2}.
\end{aligned}
$$

Using the definition of $|\!|\!|\cdot|\!|\!|_h$ given in (5.7), we obtain

$$
\sum_{l=1}^{L}w_l\sum_{\rho\in\mathcal{E}_h}\int_\rho \boldsymbol{\omega}_l\cdot\boldsymbol{\nu}_\rho\widehat{\delta_{i,l}}[\![\eta_{i,l}]\!]d\mathbf{x}\le Ch^{r+\frac{1}{2}}\left(\sum_{l=1}^{L}w_l\|u_{i,l}\|^2_{H^{r+1}(X)}\right)^{1/2}|\!|\!|\boldsymbol{\eta}_i|\!|\!|_h. \tag{5.20}
$$

Lastly, consider the integration over the boundary. Performing two applications of the Cauchy-Schwarz inequality and using definition of $|\!|\!|\cdot|\!|\!|_h$, we obtain

$$
\begin{aligned}
&\sum_{l=1}^{L}w_l\int_{\partial X_+^l}\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}\delta_{i,l}\eta_{i,l}ds\\
&\qquad\le\sum_{l=1}^{L}w_l\left(\int_{\partial X_+^l}\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}\left(\delta_{i,l}\right)^2ds\right)^{1/2}\left(\int_{\partial X_+^l}\boldsymbol{\omega}_l\cdot\boldsymbol{\nu}\left(\eta_{i,l}\right)^2ds\right)^{1/2}
\end{aligned}
$$

$$\leq \left(\sum_{l=1}^{L} w_l \int_{\partial X_+^l} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu} \left(\delta_{i,l}\right)^2 ds \right)^{1/2} \left(\sum_{l=1}^{L} w_l \int_{\partial X_+^l} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu} \left(\eta_{i,l}\right)^2 ds \right)^{1/2}$$
$$\leq \left(\sum_{l=1}^{L} w_l \int_{\partial X_+^l} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu} \left(\delta_{i,l}\right)^2 ds \right)^{1/2} \left|\!\left|\!\left| \{\eta_{i,l}\}_{l=1}^{L} \right|\!\right|\!\right|_h .$$

Noting $|\boldsymbol{\omega}_l \cdot \boldsymbol{\nu}| \leq 1$ and using the trace inequality (5.17),

$$\begin{aligned}
\int_{\partial X_+^l} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu} \left(\delta_{i,l}\right)^2 ds &= \sum_{\rho \in \partial X_+^l} \int_{\rho} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu} \left(\delta_{i,l}\right)^2 ds \\
&\leq \sum_{\rho \in \partial X_+^l} \left\| \delta_{i,l} \right\|_{L^2(\rho)}^2 \\
&\leq C h^{2r+1} \sum_{K \in T_h} \left\| u_{i,l} \right\|_{H^{r+1}(K)}^2 \\
&= C h^{2r+1} \left\| u_{i,l} \right\|_{H^{r+1}(X)}^2 .
\end{aligned}$$

This then leads to

$$\sum_{l=1}^{L} w_l \int_{\partial X_+^l} \boldsymbol{\omega}_l \cdot \boldsymbol{\nu} \delta_{i,l} \eta_{i,l} ds \leq C h^{r+\frac{1}{2}} \left(\sum_{l=1}^{L} w_l \left\| u_{i,l} \right\|_{H^{r+1}(X)}^2 \right)^{1/2} |\!|\!| \boldsymbol{\eta}_i |\!|\!|_h . \tag{5.21}$$

Combining inequalities (5.20) and (5.21) we obtain

$$\mathbf{a}\left(\boldsymbol{\delta}_i, \boldsymbol{\eta}_i\right) \leq C h^{r+\frac{1}{2}} \left(\sum_{l=1}^{L} w_l \left\| u_{i,l} \right\|_{H^{r+1}(X)}^2 \right)^{1/2} |\!|\!| \boldsymbol{\eta}_i |\!|\!|_h$$

□

Note that the bilinear form $\mathbf{a}(\cdot,\cdot)$ contains integration over the boundary of individual elements in the interior as well as the outflow boundary. As a result, the trace inequality (5.17) must be used instead of (5.16). Therefore, the upper bound obtained in Lemma (5.5) only provides $h^{r+\frac{1}{2}}$. The next lemma will show that, in the absence of these boundary terms, h^{r+1} is achieved.

Lemma 5.6. *Assume (5.18). Then the inequality*

$$\mathbf{b}_i\left(\boldsymbol{\delta}_i, \boldsymbol{\eta}_i\right) \leq Ch^{r+1}|E_i|\left(\sum_{l=1}^{L} w_l \left\|u_{i,l}\right\|^2_{H^{r+1}(X)}\right)^{1/2} \|\boldsymbol{\eta}_i\|_h$$

holds for $1 \leq i \leq N$.

Proof. By definition,

$$\mathbf{b}_i\left(\boldsymbol{\delta}_i, \boldsymbol{\eta}_i\right) = \sum_{l=1}^{L} w_l \left[\sum_{K \in T_h} \int_K \sigma_{t,i}\delta_{i,l}\eta_{i,l} d\mathbf{x} - \sum_{K \in T_h} \int_K \eta_{i,l}|E_i|\sigma_{s,i} \sum_{k=1}^{L} w^{i,i}_{l,k}\delta_{i,k} d\mathbf{x}\right]. \tag{5.22}$$

Begin by considering the second term. Using the bound of $w^{i,i}_{l,k}$ given in (5.12), there exists a constant C such that

$$\sum_{K \in T_h} \int_K \eta_{i,l}|E_i|\sigma_{s,i} \sum_{k=1}^{L} w^{i,i}_{l,k}\delta_{i,k} d\mathbf{x} \leq C|E_i| \sum_{K \in T_h} \int_K |\eta_{i,l}| \sum_{k=1}^{L} w_k |\delta_{i,k}| d\mathbf{x}.$$

By repeated application of the Cauchy-Schwarz inequality

$$\begin{aligned}
&\sum_{K \in T_h} \int_K \eta_{i,l}|E_i|\sigma_{s,i} \sum_{k=1}^{L} w^{i,i}_{l,k}\delta_{i,k} d\mathbf{x} \\
&\qquad \leq C|E_i| \sum_{K \in T_h} \int_K |\eta_{i,l}| \left(\sum_{k=1}^{L} w_k\right)^{1/2} \left(\sum_{k=1}^{L} w_k |\delta_{i,k}|^2\right)^{1/2} d\mathbf{x} \\
&\qquad \leq C|E_i| \sum_{K \in T_h} \left(\int_K |\eta_{i,l}|^2 d\mathbf{x}\right)^{1/2} \left(\int_K \sum_{k=1}^{L} w_k |\delta_{i,k}|^2 d\mathbf{x}\right)^{1/2} \\
&\qquad \leq C|E_i| \left(\sum_{K \in T_h} \|\eta_{i,l}\|^2_{L^2(K)}\right)^{1/2} \left(\sum_{k=1}^{L} w_k \sum_{K \in T_h} \|\delta_{i,k}\|^2_{L^2(K)}\right)^{1/2}.
\end{aligned}$$

Applying inequality (5.16),

$$\sum_{K \in T_h} \int_K \eta_{i,l}|E_i|\sigma_{s,i} \sum_{k=1}^{L} w^{i,i}_{l,k}\delta_{i,k} d\mathbf{x} \leq C|E_i|h^{r+1} \left\|\eta_{i,l}\right\|_{L^2(X)} \left(\sum_{k=1}^{L} w_k \left\|u_{i,l}\right\|^2_{H^{r+1}(X)}\right)^{1/2}.$$

Multiplying by w_l, summing over $1 \leq l \leq L$, applying the Cauchy-Schwarz inequality, and renaming indices

$$\sum_{l=1}^{L} w_l \sum_{K \in T_h} \int_K \eta_{i,l} |E_i| \sigma_{s,i} \sum_{k=1}^{L} w_{l,k}^{i,i} \delta_{i,k} d\mathbf{x}$$
$$\leq C|E_i| h^{r+1} \sum_{l=1}^{L} w_l \left\| \eta_{i,l} \right\|_{L^2(X)} \left(\sum_{k=1}^{L} w_k \left\| u_{i,l} \right\|_{H^{r+1}(X)}^2 \right)^{1/2}$$
$$\leq C|E_i| h^{r+1} \left(\sum_{l=1}^{L} w_l \right)^{1/2} \left(\sum_{l=1}^{L} w_l \left\| \eta_{i,l} \right\|_{L^2(X)}^2 \right)^{1/2} \left(\sum_{k=1}^{L} w_k \left\| u_{i,l} \right\|_{H^{r+1}(X)}^2 \right)^{1/2}$$
$$\leq C|E_i| h^{r+1} \left(\sum_{l=1}^{L} w_l \left\| \eta_{i,l} \right\|_{L^2(X)}^2 \right)^{1/2} \left(\sum_{l=1}^{L} w_l \left\| u_{i,l} \right\|_{H^{r+1}(X)}^2 \right)^{1/2}.$$

Using the definition of $\|\!|\cdot|\!\|_h$,

$$\sum_{l=1}^{L} w_l \sum_{K \in T_h} \int_K \eta_{i,l} |E_i| \sigma_{s,i} \sum_{k=1}^{L} w_{l,k}^{i,i} \delta_{i,k} d\mathbf{x}$$
$$\leq C|E_i| h^{r+1} \left(\sum_{l=1}^{L} w_l \left\| u_{i,l} \right\|_{H^{r+1}(X)}^2 \right)^{1/2} \|\!|\boldsymbol{\eta}_i|\!\|_h. \tag{5.23}$$

By similar techniques the remaining term in (5.22) can be dealt with,

$$\sum_{l=1}^{L} w_l \sum_{K \in T_h} \int_K \sigma_{t,i} \delta_{i,l} \eta_{i,l} d\mathbf{x} \leq C|E_i| h^{r+1} \left(\sum_{l=1}^{L} w_l \left\| u_{i,l} \right\|_{H^{r+1}(X)}^2 \right)^{1/2} \|\!|\boldsymbol{\eta}_i|\!\|_h. \tag{5.24}$$

Combining (5.23) and (5.24) yields

$$\mathbf{b}_i \left(\boldsymbol{\delta}_i, \boldsymbol{\eta}_i \right) \leq C h^{r+1} |E_i| \left(\sum_{l=1}^{L} \left\| u_{i,l} \right\|_{H^{r+1}(X)}^2 \right)^{1/2} \|\!|\boldsymbol{\eta}_i|\!\|_h.$$

$\square$

Therefore, the bilinear forms $\mathbf{a}(\boldsymbol{\delta}_i, \boldsymbol{\eta}_i)$ and $\mathbf{b}_i(\boldsymbol{\delta}_i, \boldsymbol{\eta}_i)$ are bounded from above in terms of the mesh size and the norm of $\boldsymbol{\eta}_i$ for each energy group, $1 \leq i \leq N$. In an effort to bound the error, a Galerkin orthogonality type result is established. Consider

$$\mathbf{a}(\mathbf{u}_i^h, \mathbf{v}) + \mathbf{b}_i(\mathbf{u}_i^h, \mathbf{v}) = \sum_{j=1}^{i-1} \mathbf{c}_{i,j}(\mathbf{u}_j^h, \mathbf{v}) + \mathbf{l}_i(\mathbf{v}), \quad \forall\ \mathbf{v} \in (V_h)^L, \tag{5.25}$$

$$\mathbf{a}(\mathbf{u}_i, \mathbf{v}) + \mathbf{b}_i(\mathbf{u}_i, \mathbf{v}) = \sum_{j=1}^{i-1} \mathbf{c}_{i,j}(\mathbf{u}_j, \mathbf{v}) + \mathbf{l}_i(\mathbf{v}), \quad \forall\ \mathbf{v} \in (V_h)^L, \tag{5.26}$$

for $1 \le i \le N$. Subtracting equation (5.25) from (5.26) results in

$$\mathbf{a}(\mathbf{u}_i - \mathbf{u}_i^h, \mathbf{v}) + \mathbf{b}_i(\mathbf{u}_i - \mathbf{u}_i^h, \mathbf{v}) - \sum_{j=1}^{i-1} \mathbf{c}_{i,j}(\mathbf{u}_j - \mathbf{u}_j^h, \mathbf{v}) = 0 \quad \forall\ \mathbf{v} \in (V_h)^L.$$

Addition and subtraction, followed by the triangle inequality yields

$$|\mathbf{c}_{i,j}(\mathbf{u}_i - \mathbf{u}_i^h, \mathbf{v})| \le |\mathbf{c}_{i,j}(\boldsymbol{\delta}_i, \mathbf{v})| + |\mathbf{c}_{i,j}(\boldsymbol{\eta}_i, \mathbf{v})|. \tag{5.27}$$

We then obtain

$$0 \le \mathbf{a}(\mathbf{u}_i - \mathbf{u}_i^h, \mathbf{v}) + \mathbf{b}_i(\mathbf{u}_i - \mathbf{u}_i^h, \mathbf{v}) + \sum_{j=1}^{i-1} (|\mathbf{c}_{i,j}(\boldsymbol{\delta}_j, \mathbf{v})| + |\mathbf{c}_{i,j}(\boldsymbol{\eta}_j, \mathbf{v})|), \tag{5.28}$$

for all $\mathbf{v} \in (V_h)^L$. Apply Lemma 5.2 with $\mathbf{v} = \boldsymbol{\eta}_i$:

$$c \|\boldsymbol{\eta}_i\|_h^2 \le \mathbf{a}(\boldsymbol{\eta}_i, \boldsymbol{\eta}_i) + \mathbf{b}_i(\boldsymbol{\eta}_i, \boldsymbol{\eta}_i). \tag{5.29}$$

Adding (5.28) and (5.29), again with $\mathbf{v} = \boldsymbol{\eta}_i$, then yields

$$c \|\boldsymbol{\eta}_i\|_h^2 \le \mathbf{a}(\boldsymbol{\delta}_i, \boldsymbol{\eta}_i) + \mathbf{b}_i(\boldsymbol{\delta}_i, \boldsymbol{\eta}_i) + \sum_{j=1}^{i-1} \left(|\mathbf{c}_{i,j}(\boldsymbol{\delta}_j, \boldsymbol{\eta}_i)| + |\mathbf{c}_{i,j}(\boldsymbol{\eta}_j, \boldsymbol{\eta}_i)|\right).$$

Applying Lemmas 5.5 and 5.6 result in

$$\|\boldsymbol{\eta}_i\|_h^2 \le c h^{r+\frac{1}{2}} |E_i| \left(\sum_{l=1}^{L} w_l \|u_{i,l}\|_{H^{r+1}(X)}^2 \right)^{1/2} \|\boldsymbol{\eta}_i\|_h + c \sum_{j=1}^{i-1} |E_j| \left(\|\boldsymbol{\eta}_j\|_h + \|\boldsymbol{\delta}_j\|_h \right) \|\boldsymbol{\eta}_i\|_h.$$

From which we obtain

$$\|\boldsymbol{\eta}_i\|_h \le c h^{r+\frac{1}{2}} |E_i| \left(\sum_{l=1}^{L} w_l \|u_{i,l}\|_{H^{r+1}(X)}^2 \right)^{1/2} + c \sum_{j=1}^{i-1} |E_j| \left(\|\boldsymbol{\eta}_j\|_h + \|\boldsymbol{\delta}_j\|_h \right).$$

Consider $\|\boldsymbol{\delta}_j\|_h$. From the definition of $\|\cdot\|_h$, application of inequalities (5.16) and (5.17) yields

$$\|\boldsymbol{\delta}_j\|_h \le c h^{r+\frac{1}{2}} \left(\sum_{l=1}^{L} w_l \|u_{j,l}\|_{H^{r+1}(X)}^2 \right)^{1/2}. \tag{5.30}$$

We then have

$$\|\boldsymbol{\eta}_i\|_h \leq ch^{r+\frac{1}{2}} \sum_{k=1}^{i} |E_k| \left(\sum_{l=1}^{L} w_l \, \|u_{k,l}\|^2_{H^{r+1}(X)} \right)^{1/2} + c \sum_{j=1}^{i-1} |E_j| \|\boldsymbol{\eta}_j\|_h.$$

Applying the discrete Gronwall's inequality and extending the summation from i to N yields

$$\|\boldsymbol{\eta}_i\|_h \leq ch^{r+\frac{1}{2}} \sum_{k=1}^{N} |E_k| \left(\sum_{l=1}^{L} w_l \, \|u_{k,l}\|^2_{H^{r+1}(X)} \right)^{1/2}.$$

Squaring both sides, multiplying by $|E_i|$, summing over $1 \leq i \leq N$, and taking the square root yields

$$\left(\sum_{i=1}^{N} |E_i| \|\boldsymbol{\eta}_i\|_h^2 \right)^{1/2} \leq ch^{r+\frac{1}{2}} \left(\sum_{i=1}^{N} |E_i| \sum_{l=1}^{L} w_l \, \|u_{i,l}\|^2_{H^{r+1}(X)} \right)^{1/2}. \tag{5.31}$$

The same procedure applied to inequality (5.30) results in

$$\left(\sum_{i=1}^{N} |E_i| \|\boldsymbol{\delta}_i\|_h^2 \right)^{1/2} \leq ch^{r+\frac{1}{2}} \left(\sum_{i=1}^{N} |E_i| \sum_{l=1}^{L} w_l \, \|u_{i,l}\|^2_{H^{r+1}(X)} \right)^{1/2}. \tag{5.32}$$

With inequalities (5.31) and (5.32) an estimate of the error between solutions of the angular semi-discretization and the fully discrete system is available.

Theorem 5.7. *(Fully-discrete to Semi-discrete)*

Assume that $\sigma_t \geq m_{\sigma_t} > 0$ *and that* (5.18) *holds. Let* $\{u_{i,l}\}_{i=1,l=1}^{N,L}$ *satisfy* (4.8)–(4.9) *and* $\mathbf{u}_h = \{u_{i,l}^h\}_{i=1,l=1}^{N,L}$ *satisfy* (5.6). *Then for* h_e *sufficiently small*

$$\left\| \{u_{i,l}\}_{i=1,l=1}^{N,L} - \mathbf{u}_h \right\|_h \leq ch^{r+\frac{1}{2}} \left(\sum_{i=1}^{N} |E_i| \sum_{l=1}^{L} w_l \, \|u_{i,l}\|^2_{H^{r+1}(X)} \right)^{1/2},$$

where the norm $\|\cdot\|_h$ *is defined in* (5.14).

Proof. Using the definition of $\boldsymbol{\delta}_i$ and $\boldsymbol{\eta}_i$ followed by the triangle inequality produces

$$\left(\sum_{i=1}^{N}|E_i|\left\|\{u_{i,l}\}_{l=1}^{L}-\{u_{i,l}^{h}\}_{l=1}^{L}\right\|_h^2\right)^{1/2}\le\left(\sum_{i=1}^{N}|E_i|\|\boldsymbol{\delta}_i\|_h\right)^{1/2}+\left(\sum_{i=1}^{N}|E_i|\|\boldsymbol{\eta}_i\|_h\right)^{1/2}.$$

The claim then follows from inequalities (5.31) and (5.32). □

Remark 5.8. *Note that we are restricted to $r+1/2$ for the convergence order as a consequence of the trace inequality (5.17). This can be seen by comparing the estimates obtained in Lemmas (5.5) and (5.6).*

With the results obtained here and section 4, the error between the original BVP (1.2)–(1.3) and the fully discrete method (5.6) can be obtained.

Theorem 5.9. *Assume the conditions of Theorems 4.11 and 5.7 hold. Then,*

$$\left\|\{u(\cdot,\boldsymbol{\omega}_l,e_i)\}_{i=1,l=1}^{N,L}-\mathbf{u}_h\right\|\le C(h_e+h_a^2+h^{r+\frac{1}{2}}),$$

where the dependence of the constant C on u and f is given in (5.33) *below.*

Proof. Application of the triangle inequality and using the definition of the norm $\|\cdot\|_h$ yield

$$\left\|\{u(\cdot,\boldsymbol{\omega}_l,e_i)\}_{i=1,l=1}^{N,L}-\mathbf{u}_h\right\|\le\left\|\{u(\cdot,\boldsymbol{\omega}_l,e_i)\}_{i=1,l=1}^{N,L}-\{u_{i,l}\}_{i=1,l=1}^{N,L}\right\|+\left\|\{u_{i,l}\}_{i=1,l=1}^{N,L}-\mathbf{u}_h\right\|_h.$$

Then applying Theorem 4.11 and 5.7 leads to

$$\begin{aligned}&\left\|\{u(\cdot,\boldsymbol{\omega}_l,e_i)\}_{i=1,l=1}^{N,L}-\mathbf{u}_h\right\|\\&\qquad\le c\Bigg[h_e\Bigg(\|u\|_{L^\infty(U)}+\|u\|_{L^2(X\times\Omega,H^1(E))}+\left(\sum_{l=1}^{L}w_l|f(\boldsymbol{\omega}_l)|_{L^2(X,H^1(E))}^2\right)^{1/2}\Bigg)\\&\qquad\quad+h_a^2\left(\|u\|_{L^\infty(U)}+\|u\|_{L^2(X\times E,H^2(\Omega))}\right)+h^{r+\frac{1}{2}}\left(\sum_{i=1}^{N}|E_i|\sum_{l=1}^{L}w_l\|u_{i,l}\|_{H^{r+1}(X)}^2\right)^{1/2}\Bigg].\end{aligned}\tag{5.33}$$

Given the regularity assumptions on u and f the result follows. $\square$

CHAPTER 6
NUMERICAL EXAMPLES

In this chapter we present numerical examples of the method proposed in the previous chapters. We begin by presenting an analytic scattering kernel describing the energy change due to a collision. We will denote this by $P(e_i \to e_f)$, where energy e_i and e_f denote the energy before and after a collision, respectively. A common example of such a function in the context of neutron transport (cf. [16, 33, 34]) is given by

$$P(e_i \to e_f) := \begin{cases} \frac{1}{(1-\alpha)e_i}, & e_i\alpha \le e_f \le e_i, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Here, $0 \le \alpha < 1$, and is given by

$$\alpha = \left(\frac{A-1}{A+1}\right)^2,$$

where A is the ratio of the nucleus and neutron masses. In our examples we will assume $\alpha = 0$. If we assume the scattering is isotropic, the function $\mathcal{P}$ can be written as

$$\mathcal{P}(\mathbf{x}, \boldsymbol{\omega} \cdot \boldsymbol{\omega}', e, e') := \frac{1}{4\pi} P(e' \to e)$$

or, equivalently

$$\mathcal{P}(\mathbf{x}, \boldsymbol{\omega} \cdot \boldsymbol{\omega}', e, e') := \begin{cases} \frac{1}{4\pi(1-\alpha)e'}, & e'\alpha \le e \le e', \\ \\ 0, & \text{otherwise,} \end{cases}$$

and we note that $\mathcal{P}$ satisfies assumptions (1.6) and (1.7).

In all four numerical examples below, the spatial and energy domains are $X = [0,1]^3$ and $E = [1,2]$. Further, the angular discretization will be fixed and chosen such that it is sufficiently fine to not interfere in the convergence orders in the spatial and energy mesh sizes; the angular discretization projects four triangles onto the sphere in each octant, the number of directions is then $|\Omega_{h_a}| = 18$. The resulting linear systems are solved using Guass-Seidel iteration. In all of the numerical experiments, the norm $\|\cdot\|$ as defined in (4.10) is used, with $\mathbf{u}_h = \{u_{i,l}^h\}_{i=1,l=1}^{N,L}$ denoting the solution of the fully discrete problem and u the solution of the original BVP. For a given level of discretization h, let h^+ denote the adjacent finer level of discretization. Then the numerical convergence order is computed by $\text{Order}(h) = \log_{\frac{h}{h^+}}\left(\frac{\text{Error}(h)}{\text{Error}(h^+)}\right)$. Here $\text{Error}(h)$ denotes the error computed with respect to the norm $\|\cdot\|$ defined in (4.10) and it is understood that the remaining discretization parameters are taken sufficiently small. Piecewise linear approximation is used in the DG scheme; this corresponds to $r = 1$ in definition (5.1).

6.1 Example 1

In this example, $\sigma_s(\mathbf{x}, e) = 2$ and $\sigma_a(\mathbf{x}, e) = 1$. The solution is given by $u(\mathbf{x}, \boldsymbol{\omega}, e) = \sin(x_1\pi)\sin(x_2\pi)\sin(x_3\pi)e$, f and g being determined by substitution of u into the BVP. Consulting Table 6.1 and Figures 6.1 and 6.2, it appears that the convergence with respect to the spatial discretization approaches quadratic while linear convergence with respect to the energy mesh is observed.

h	$\|u - \mathbf{u}_h\|$	Order(h)
$\frac{\sqrt{2}}{2}$	1.9961e-01	1.8313
$\frac{\sqrt{2}}{4}$	5.6092e-02	1.9038
$\frac{\sqrt{2}}{6}$	2.5921e-02	1.9095
$\frac{\sqrt{2}}{8}$	1.4965e-02	1.9545
$\frac{\sqrt{2}}{10}$	9.6753e-03	-

(a)

h_e	$\|u - \mathbf{u}_h\|$	Order(h_e)
$\frac{1}{4}$	9.0796e-02	1.0002
$\frac{1}{8}$	4.5392e-02	9.9917e-01
$\frac{1}{16}$	2.2709e-02	9.9740e-01
$\frac{1}{32}$	1.1375e-02	9.9264e-01
$\frac{1}{64}$	5.7166e-03	9.7796e-01
$\frac{1}{128}$	2.9023e-03	-

(b)

Table 6.1: The error and convergence order for Example 1 in both h_e and h. (a): Fixed energy mesh of $h_e = 1/1024$ with various mesh sizes h. (b): The spatial variable in the DG scheme uses cubic interpolation with $h = \sqrt{2}/5$.

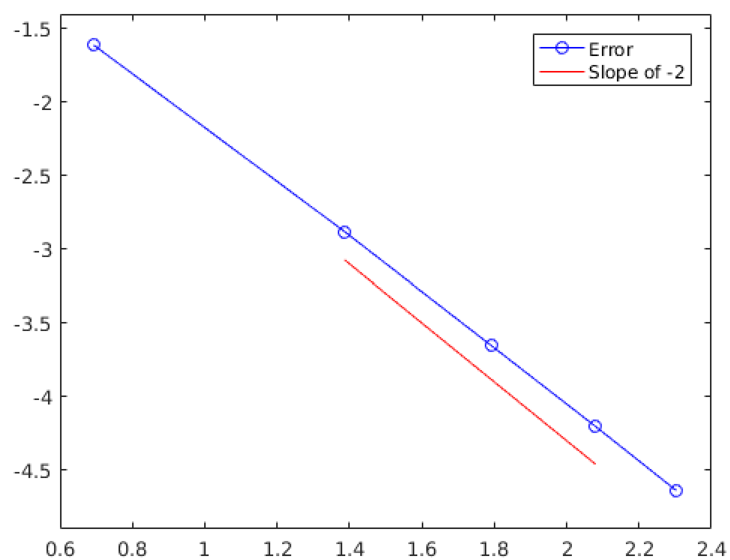

Figure 6.1: Log-log plot of the spatial error for Example 1.

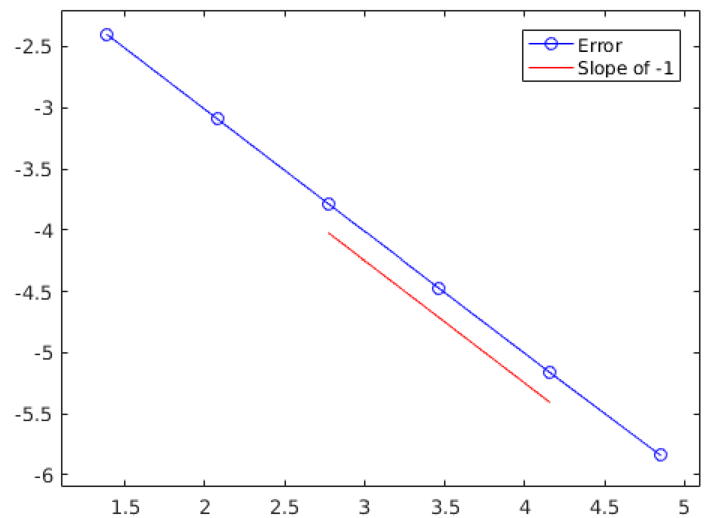

Figure 6.2: Log-log plot of the energy error for Example 1.

6.2 Example 2

Choose $u(\mathbf{x}, \boldsymbol{\omega}, e) = \sin(x_1)\sin(x_2)\sin(x_3)$ with cross sections $\sigma_s(\mathbf{x}, e) = 1$ and $\sigma_a(\mathbf{x}, e) = 0.5$, where f and g are determined through the BVP. The problem remains energy dependent due to the scattering term. To determine the convergence order of the energy discretization, the spatial discretization uses cubic interpolation with $h = \sqrt{2}/5$. Likely due to the minimal energy dependence, use of this spatial discretization significantly reduced the overall error and a finer resolution in the energy domain is needed for determining the convergence order in energy. This can be seen in Table 6.2.

In this example, it is clear from Figure 6.3 and Table 6.2:(a) that quadratic convergence with respect to the spatial mesh is obtained while Figure 6.4 and Table

6.2:(b) again indicate linear convergence with respect to the energy mesh.

h	$\|u - \mathbf{u}_h\|$	Order(h)
$\frac{\sqrt{2}}{2}$	4.0374e-02	2.0442
$\frac{\sqrt{2}}{4}$	9.7890e-03	2.0340
$\frac{\sqrt{2}}{6}$	4.2911e-03	1.8962
$\frac{\sqrt{2}}{8}$	2.4869e-03	1.9751
$\frac{\sqrt{2}}{10}$	1.6005e-03	-

(a)

h_e	$\|u - \mathbf{u}_h\|$	Order(h_e)
$\frac{1}{100}$	4.0750e-05	1.0267
$\frac{1}{200}$	2.0002e-05	1.0354
$\frac{1}{300}$	1.3144e-05	1.0340
$\frac{1}{400}$	9.7623e-06	1.0222
$\frac{1}{500}$	7.7713e-06	9.9973e-01
$\frac{1}{600}$	6.4764e-06	-

(b)

Table 6.2: The error and convergence order for Example 2 in both h_e and h. (a) Fixed energy mesh of $h_e = 1/256$ with various mesh sizes h. (b) The spatial variable in the DG scheme uses cubic interpolation with $h = \frac{\sqrt{2}}{5}$.

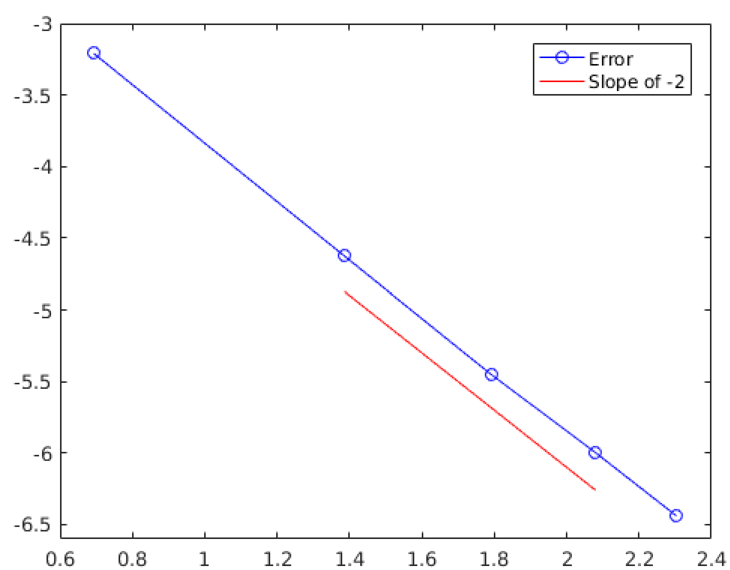

Figure 6.3: Log-log plot of the spatial error for Example 2.

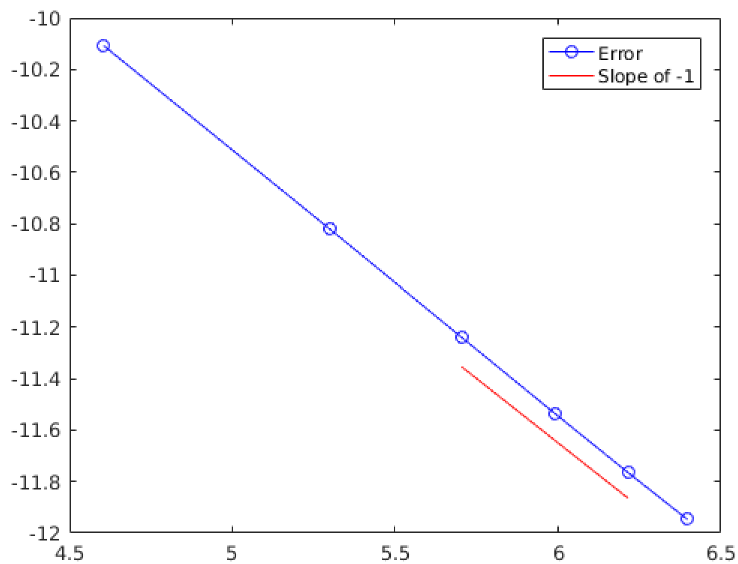

Figure 6.4: Log-log plot of the energy error for Example 2.

6.3 Example 3

In this example, the cross sections are allowed to depend on the energy variable. We use $\sigma_t(\mathbf{x}, e) = \ln(e)$ and $\sigma_s(\mathbf{x}, e) = 0.5\sigma_t(\mathbf{x}, e)$ and choose the true solution $u(\mathbf{x}, \boldsymbol{\omega}, e) = \ln(e) \exp(\omega_1 + \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3))$. The source and boundary data are determined through the BVP.

Consulting Table 6.3 and Figures 6.5 and 6.6 we find behavior similar to Example 1: quadratic convergence with respect to the spatial mesh and linear convergence with respect to the energy mesh.

h	$\|u - \mathbf{u}_h\|$	Order(h)
$\frac{\sqrt{2}}{2}$	1.1533e-01	1.6923
$\frac{\sqrt{2}}{4}$	3.5686e-02	1.9397
$\frac{\sqrt{2}}{6}$	1.6253e-02	1.9398
$\frac{\sqrt{2}}{8}$	9.3020e-03	1.9071
$\frac{\sqrt{2}}{10}$	6.0780e-03	-

(a)

h_e	$\|u - \mathbf{u}_h\|$	Order(h_e)
$\frac{1}{4}$	3.3669e-01	9.9765e-01
$\frac{1}{8}$	1.6862e-01	9.9942e-01
$\frac{1}{16}$	8.4344e-02	9.9979e-01
$\frac{1}{32}$	4.2178e-02	9.9979e-01
$\frac{1}{64}$	2.1092e-02	9.9959e-01
$\frac{1}{128}$	1.0549e-02	-

(b)

Table 6.3: The error and convergence order for Example 3 in both h_e and h. (a) Fixed energy mesh of $h_e = 1/1024$ with various mesh sizes h. (b) The spatial variable in the DG scheme uses cubic interpolation with $h = \sqrt{2}/5$.

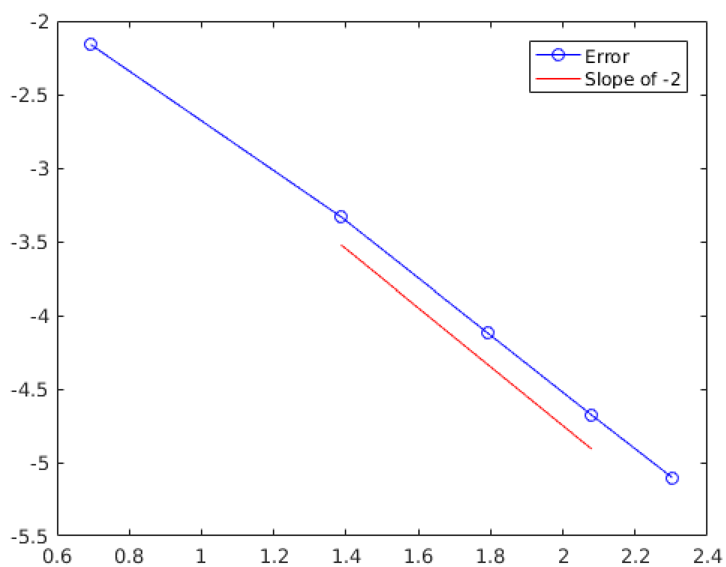

Figure 6.5: Log-log plot of the spatial error for Example 3.

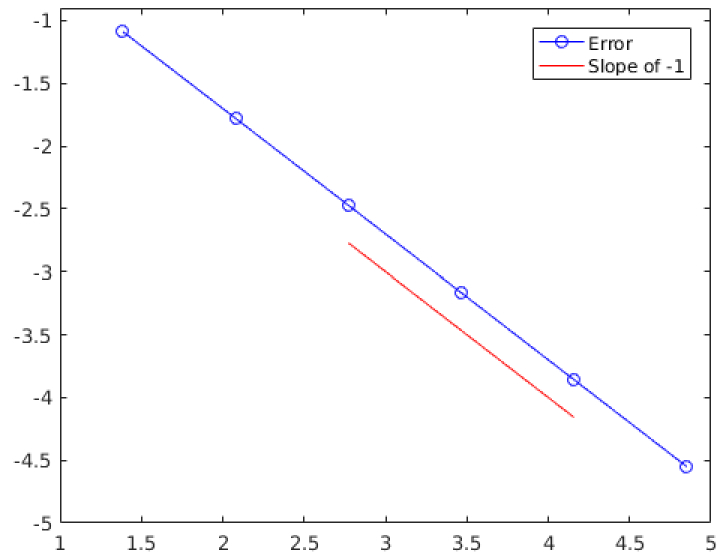

Figure 6.6: Log-log plot of the energy error for Example 3.

6.4 Example 4

In this example, the cross sections depend on the energy variable and $\sigma_a = 0$ when $e = 1.5$. We use $\sigma_t(\mathbf{x}, e) = (e - 1.5)^2 + 1$ and $\sigma_s(\mathbf{x}, e) = 1$ and choose the solution $u(\mathbf{x}, \boldsymbol{\omega}, e) = \ln(e)\sin(x_1)\sin(x_2)\sin(x_3)$. The source and boundary data are determined through the BVP.

Consulting Table 6.4 and Figures 6.7 and 6.8 we find good agreement with the previous examples.

h	$\lVert u - \mathbf{u}_h \rVert$	Order(h)
$\frac{\sqrt{2}}{2}$	1.7696e-02	2.0359
$\frac{\sqrt{2}}{4}$	4.3154e-03	2.0087
$\frac{\sqrt{2}}{6}$	1.9112e-03	1.9741
$\frac{\sqrt{2}}{8}$	1.0831e-03	1.9244
$\frac{\sqrt{2}}{10}$	7.0497e-04	-

(a)

h_e	$\lVert u - \mathbf{u}_h \rVert$	Order(h_e)
$\frac{1}{4}$	2.5725e-02	9.9793e-01
$\frac{1}{8}$	1.2881e-02	9.9933e-01
$\frac{1}{16}$	6.4435e-03	9.9949e-01
$\frac{1}{32}$	3.2229e-03	9.9870e-01
$\frac{1}{64}$	1.6129e-03	9.9614e-01
$\frac{1}{128}$	8.0861e-04	-

(b)

Table 6.4: The error and convergence order for Example 4 in both h_e and h. (a) Fixed energy mesh of $h_e = 1/1024$ with various mesh sizes h. (b) The spatial variable in the DG scheme uses cubic interpolation with $h = \sqrt{2}/5$.

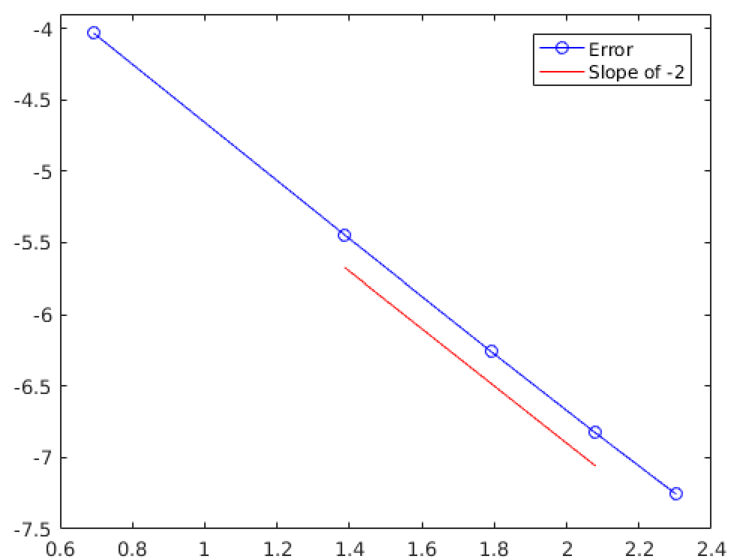

Figure 6.7: Log-log plot of the spatial error for Example 4.

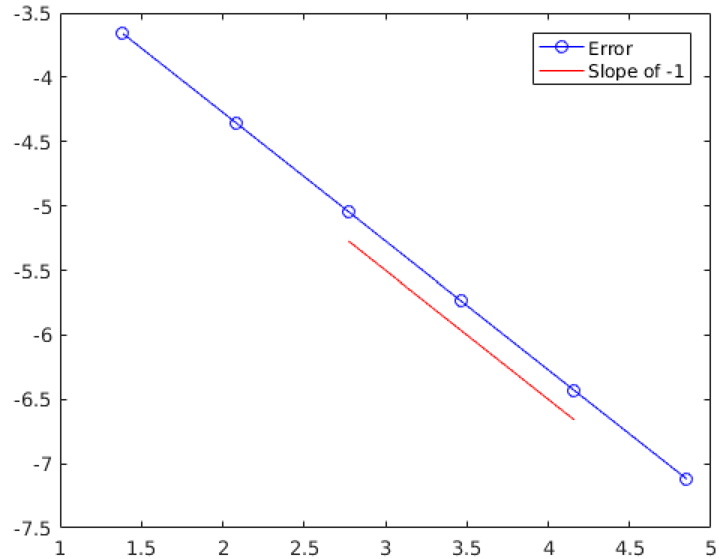

Figure 6.8: Log-log plot of the energy error for Example 4.

In this chapter, four numerical examples were presented. Through the numerical examples, first order convergence with respect to the energy mesh size h_e was observed. This was observed in all three examples which agrees with the theoretical results presented in the preceding chapters. Additionally, the theoretical results in chapter 5 showed a convergence order of $h^{3/2}$. In the numerical examples, however, second order convergence in the spatial mesh size h was observed. This is not completely unexpected; numerical methods employing discontinuous Galerkin methods to the monoenergetic RTE also present this behavior (cf. [23], [25]). Therefore, our results agree with previously used DG methods in the spatial variable.

CHAPTER 7
CONCLUSION

In this work a method for the numerical solution of the energy dependent RTE alongside a rigorous analysis of the obtained semi-discretizations was given. In dealing with the energy discretization, existence and uniqueness was established alongside a proof of the convergence of the semi-discretization to the true solution. As mentioned previously, much research has focused on the development of the monoenergetic RTE. It is our hope that this rigorous analysis has provided a solid starting point for applying methods developed for the monoenergetic case to the energy dependent RTE. Following the energy discretization, we introduced the angular discretization using FEM ideas presented in [23]. This work differs in that an isoparametric approach was taken when dealing with the triangulation of the sphere and we prove that the angular convergence order is maintained under relaxed angular regularity assumptions; that is, from $C^2(\Omega)$ in [23] to $H^2(\Omega)$. We provide an error estimate between the energy-angular discretization and the original RTE BVP. In order to obtain the full discretization, a DG method was used for the spatial variable. Existence and uniqueness of a solution for the full discretization was shown, as well as a bound on the error between solutions of the fully discrete system and the energy-angular semi-discrete system. Using these results, an error bound between the fully discrete solution and the original RTE BVP solution was established. Lastly, several numerical experiments were run with various problem parameters. These example problems were used to assess convergence order of both the energy and spatial mesh sizes. It was observed that the energy mesh matched the theoretical results, both agreeing on linear convergence. Considering the spatial variable, h^2 convergence was observed in the numerical experiments. In theoretical results, however, only $h^{3/2}$ was shown.

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