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Invariants of HOPF actions on path algebras of quivers

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INVARIANTS OF HOPF ACTIONS ON PATH ALGEBRAS OF QUIVERS

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

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ABSTRACT

The work of this thesis focuses primarily on non-commutative algebras and actions of Hopf algebras. Specifically, we study the possible H-module algebra structures which can be imposed on path algebras of quivers, for a variety of Hopf algebras, H, and then given a possible action, classify the invariant ring.

A Hopf algebra is a bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ together with an antipode $S : H \to H^{\text{op}}$ which is compatible with the counit, ε, of H. A quiver is a directed graph, and the path algebra $\Bbbk Q$ of a quiver Q is a vector space where all the paths of the quiver form a basis, and multiplication is given by concatenation of paths whenever possible, and zero otherwise. In their paper, [9], Kinser and Walton classify Hopf actions of a specific family of Hopf algebras called a Taft algebras, $T(n)$, on path algebras of loopless, finite, Schurian quivers. In this thesis, we extend their result to path algebras of any finite quiver and classify the invariant subring, $\Bbbk Q^{T(n)}$, in the case where the group like element $g \in T(n)$ acts transitively on Q_0.

In the future, we hope that the ideas presented in this work extend to a classification of quantum groups, such as $u_q(\mathfrak{sl}_2)$, acting on path algebras of finite quivers.

PUBLIC ABSTRACT

Group actions have been of interest in every field of mathematics for over a century, and particularly in geometry and algebra. We generally are interested in the objects that are fixed by the action of a certain group, and we investigate properties about these fixed objects such as whether the object is finitely generated as an algebra, and what is a minimal set of generators. Classically, mathematicians have been interested in the symmetries that arise from groups acting on certain objects, such as the symmetries of the n-gon, D_{2n}, or the symmetries of the plane by actions from the matrix algebras $SL_2(\mathbb{R})$ and $GL_2(\mathbb{C})$. More recently, we have extended these ideas by studying the actions of spaces which have a lot more structure than groups, such as Hopf algebras, on interesting mathematical objects, such as commutative domains [7]. The less structured the algebra, the fewer symmetries that we see; so in particular, non-commutative algebras yield an interesting area of exploration.

A Hopf algebra is an algebraic structure which is both an algebra and a coalgebra, has an endomorphism called an antipode, and all maps satisfy certain compatiblity relations [2]. This type of structure is pervasive, appearing naturally in algebraic topology, scheme theory, combinatorics, group theory and physics. Classically studied Hopf algebras include group algebras, universal enveloping algebras, one parameter deformations of Lie Algebras, etc. In this thesis, we focus on classifying the actions of Taft algebras on the path algebras of quivers, with the least number of conditions possible, building upon Kinser and Walton's previous work in [9]. Taft

algebras of order n, $T(n)$, are examples of Hopf algebras, and path algebras, $\Bbbk Q$, are non-commutative, so we have a rich theory to study when classifying possible actions of Taft algebras on $\Bbbk Q$.

The goal of this work is two-fold: we first classify the possible actions of Taft algebras acting on the path algebra of an arbitrary quiver, and then we describe the elements of the path algebra that were, in some sense, fixed by this action. This fixed subspace is the invariant subring of the path algebra, and understanding its structure gives us insight into the symmetry of the action, and helps us organize our findings by giving the action a focal point.

Invariant theory has its roots in groups acting on algebraic varieties. The main idea here is to describe the polynomial functions that do not change under the transformations from the group action. A classic question in the study of group actions is whether the invariant ring is finitely generated. In the case where a Hopf algebra is acting on algebras, invariants become increasingly complicated as we complexify the structure of the objects on which we act [3]. In this work, we explore the invariant ring, $\Bbbk Q^{T(n)}$, for a specific type of action, and give a description of the smallest generating set for the invariant ring.

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CHAPTER 1
PRELIMINARIES

The theorems and definitions in this section are provided for clarity, completeness, and to establish the notation and previous results which will be used throughout the paper. Proofs of most theorems and lemmas are omitted in this section, but a reference for each is provided.

1.1 Hopf Algebra Background

We follow the definitions and notation of [8]. We let $\mathbb{k}$ be an arbitrary field and all algebras are associative algebras over $\mathbb{k}$.

Definition 1.1.1 (Bialgebra)**.**

A bialgebra is a quintuple $(H, \mu, \eta, \Delta, \varepsilon)$ *such that*

- H *is a vector space;*
- *multiplication is given by a* $\mathbb{k}$*-linear map* $\mu : H \otimes H \to H$*;*
- *the unit* $\eta : \mathbb{k} \to H$ *is* $\mathbb{k}$*-linear;*
- *co-multiplication is given by a* $\mathbb{k}$*-linear map* $\Delta : H \to H \otimes H$*;*
- *the co-unit* $\varepsilon : H \to \mathbb{k}$ *is* $\mathbb{k}$*-linear;*
- *the maps* Δ *and* ε *are algebra morphisms, i.e.* $\Delta(ab) = \Delta(a)\Delta(b)$ *and* $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ *for all* $a, b \in H$*; and*
- *the following axioms (Ass), (Un), (Coass), (Coun) are satisfied:*

(Axioms) *The following diagrams commute:*

(Ass)
$$\begin{CD} H\otimes H\otimes H @>{id\otimes\mu}>> H\otimes H \\ @V{\mu\otimes id}VV @VV{\mu}V \\ H\otimes H @>{\mu}>> H \end{CD}$$

(Un)
$$\begin{array}{ccccc} \mathbb{k}\otimes H & \xrightarrow{\eta\otimes id} & H\otimes H & \xleftarrow{id\otimes\eta} & H\otimes \mathbb{k} \\ & \searrow^{\cong} & \downarrow{\scriptstyle\mu} & {}^{\cong}\swarrow & \\ & & H & & \end{array}$$

(Coass)
$$\begin{CD} H @>{\Delta}>> H\otimes H \\ @V{\Delta}VV @VV{id\otimes\Delta}V \\ H\otimes H @>{\Delta\otimes id}>> H\otimes H\otimes H \end{CD}$$

(Coun)
$$\begin{array}{ccccc} \mathbb{k}\otimes H & \xleftarrow{id\otimes\varepsilon} & H\otimes H & \xrightarrow{\varepsilon\otimes id} & H\otimes \mathbb{k} \\ & \nwarrow^{\cong} & \uparrow{\scriptstyle\Delta} & {}^{\cong}\nearrow & \\ & & H & & \end{array}$$

We use Sweedler notation for $\Delta(x)$ and write $\Delta(x) = \sum_{(x)} x' \otimes x''$.

Definition 1.1.2 (Hopf algebra)**.**

A Hopf algebra is a bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$, *together with a map* $S : H \to H^{\boldsymbol{op}}$ *called an antipode, such that for any* $x \in H$,

$$\sum_{(x)} S(x')x'' = \sum_{(x)} x'S(x'') = \eta \circ \varepsilon(x).$$

Definition 1.1.3 (Taft algebra)**.**

For every positive integer n, *the Taft algebra,* $T(n)$, *is a Hopf algebra generated by the* *elements* $g, x \in T(n)$ *satisfying the relations*

$$g^n = 1, x^n = 0, \text{ and } xg = \zeta gx, \text{ where } \zeta \text{ is a primitive } n\text{-th root of unity,}$$

and the comultiplication is given by $\Delta(g) = g \otimes g$, $\Delta(x) = 1 \otimes x + x \otimes g$.

The antipode S *takes* g *to* g^{-1} *and* x *to* $-g^{-1}x$.

The element g *is said to be* *group-like* *and the element* x *is* *$(1, g)$-skew primitive*. *Notice that the element* $g \in T(n)$ *generates a cyclic group of order* n, *i.e.* $G := \langle g \rangle \cong \mathbb{Z}/n\mathbb{Z}$.

Definition 1.1.4 (Hopf action)**.**

Given a Hopf algebra H *and an algebra* A, *we define a* *left action of* H *on* A *whenever,*

1. A *is a left* H*-module,*
2. $h \cdot (pq) = \sum_{(h)} (h' \cdot p)(h'' \cdot q)$, *and*
3. $h \cdot 1_A = \varepsilon(h) 1_A$.

We say that A *has a left* H*-module algebra structure, or* A *is a left* H*-module algebra.*

We now define path algebras of quivers, which will be the algebras that are acted on by the Taft algebras throughout this thesis.

1.2 Path Algebra Background

In this section, we follow [2] for definitions and background.

Definition 1.2.1 (Quiver)**.**

A quiver $Q = (Q_0, Q_1, s, t)$ *is a quadruple consisiting of a finite set of vertices* Q_0*, a finite set of arrows* Q_1*, and two functions* $s, t : Q_1 \to Q_0$ *producing the source and target of each arrow, respectively. We may visualize the arrows as follows:*

$$s(a) \xrightarrow{a} t(a).$$

Definition 1.2.2 (Schurian Quiver)**.**

A quiver Q *is said to be Schurian if given any pair of vertices* $i, j \in Q_0$ *there exists at most one arrow* $a \in Q_1$ *with* $s(a) = i$ *and* $t(a) = j$*.*

Definition 1.2.3 (Path algebra)**.**

The path algebra $\Bbbk Q$ *of a quiver* Q *is the* $\Bbbk$*-algebra whose underlying* $\Bbbk$*-vector space has as its basis the set of all paths* $(i \mid a_1, \ldots, a_l \mid j)$ *of length* $l \geq 0$ *in* Q *for each pair of vertices* $i, j \in Q_0$*, and such that the product of two basis vectors* $(i \mid a_1, \ldots, a_l \mid j)$ *and* $(k \mid b_1, \ldots, b_m \mid h)$ *of* $\Bbbk Q$ *is defined by*

$$(i \mid a_1, \ldots, a_l \mid j)(k \mid b_1, \ldots, b_m \mid h) = \delta_{jk}(i \mid a_1, \ldots, a_l, b_1, \ldots, b_m \mid h),$$

where δ_{jk} *denotes the Kronecker delta. In other words, the product of two paths* $a_1 \cdots a_l$ *and* $b_1 \cdots b_m$ *is equal to zero if* $t(a_l) \neq s(b_1)$ *and is equal to the composed path* $a_1 \cdots a_l b_1 \cdots b_m$ *if* $t(a_l) = s(b_1)$*.*

Therefore, $\Bbbk Q$ *can be written as a sum decomposition*

$$\Bbbk Q = \bigoplus_{l=0}^{\infty} \Bbbk Q_l$$

of the $\Bbbk$*-vector space* $\Bbbk Q$*, where, for each* $l \geq 0$*,* $\Bbbk Q_l$ *is the subspace of* $\Bbbk Q$ *generated by the set* Q_l *of all paths of length* l*.*

This decomposition defines a grading on $\Bbbk Q$*, so* $\Bbbk Q$ *is a graded* $\Bbbk$*-algebra.*

Definition 1.2.4 (Stationary path of $\Bbbk Q$)**.**

Given a quiver Q *with* $i \in Q_0$*, we define the stationary path,* $e_i \in \Bbbk Q$*, by* $e_i := (i \mid i)$*.*

Proposition 1.2.5 (Primitive orthogonal idempotents)**.**

The set $\{e_i \mid i \in Q_0\}$ *of all the stationary paths is a complete set of primitive orthogonal idempotents for* $\Bbbk Q$*.*

Observe that $\Bbbk Q_0$ is a basic, semi-simple algebra isomorphic to $\Bbbk^{|Q_0|}$. In particular, this means that the complete set of primitive orthogonal idemopotents is unique.

Lemma 1.2.6 (Conditions on path algebra $\Bbbk Q$)**.**

If Q *is a quiver and* $\Bbbk Q$ *is its path algebra, then*

1. $\Bbbk Q$ *is an associative algebra with a multiplicative identity* $1 = \sum_{i \in Q_0} e_i$*, and*
2. $\Bbbk Q$ *is finite dimensional if and only if* Q *is acyclic.*

1.3 Combinatorics Background

In this section, we follow [8], [12] and [13] for notation, definitions and background. The proofs of a few necessary lemmas are provided, as references could not be found.

Definition 1.3.1 (Symmetric Polynomials)**.**

1. *Let $e_k(x_1, \ldots, x_n)$ be the elementary symmetric polynomial of degree k in the variables $\{x_1, \ldots, x_n\}$, i.e. the sum of all products of k-subsets of the n variables:*

$$e_k(x_1, \ldots, x_n) := \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

2. *Let $h_k(x_1, \ldots, x_n)$ be the complete homogeneous symmetric polynomial of degree k in the variables $\{x_1, \ldots, x_n\}$, i.e. the sum of all monomials of total degree k in the given variables, allowing repetition:*

$$h_k(x_1, \ldots, x_n) := \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Definition 1.3.2 (q-integer)**.**

For a natural number n, the q-integer $[n]_q$ is defined by:

$$\begin{aligned} [n]_q &:= \frac{1 - q^n}{1 - q} \\ &= 1 + q + q^2 + \ldots + q^{n-1}. \end{aligned}$$

Observe that when $q = 1$, $[n]_q = n$, which inspires this definition as the q-analog of an integer.

Definition 1.3.3 (q-factorial)**.**

For a natural number n, we define $[0]_q! = [1]_q! := 1$, and $[n]_q!$ by:

$$[n]_q! := [n]_q[n-1]_q[n-2]_q \cdots [1]_q.$$

Definition 1.3.4 (q-Binomial Coefficient)**.**

The q-analog for the Binomial Coefficient is defined, as expected, by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

Definition 1.3.5 (q-Multinomial Coefficient)**.**

The q-analog for the Multinomial Coefficient is defined, as expected, by

$$\begin{bmatrix} m \\ m_1, m_2, \ldots, m_k \end{bmatrix}_q := \frac{[m]_q!}{\prod_{i=1}^{k} [m_i]_q!},$$

where $\displaystyle\sum_{i=1}^{k} m_i = m$.

If we define $m = (m_1, \ldots, m_k) \in \mathbb{N}^k$ and $|m| = \displaystyle\sum_{i=1}^{k} m_i$, then for brevity, denote the q-Multinomial Coefficient by

$$\begin{bmatrix} m \\ m_1, m_2, \ldots, m_k \end{bmatrix}_q = \left\langle \begin{matrix} |m| \\ m \end{matrix} \right\rangle_q.$$

Theorem 1.3.6 (q-Binomial formula)**.**

Given x, y such that $yx = qxy$, for some parameter q, then

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k.$$

Theorem 1.3.7 (q-Binomial Theorem)**.**

Given x, y such that $xy = yx$, i.e. x and y commute, define $(x+y)_q^n := (x+y)(x+qy)(x+q^2y)\dots(x+q^{n-1}y)$. *Then,*

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^{n-k} y^k.$$

Theorem 1.3.8 (The q-Multinomial recursive formula)**.**

Let $m = (m_1, m_2, \dots, m_k) \in \mathbb{N}^k$, $|m| = \sum_{i=1}^{k} m_i$, *and* $e_j \in \mathbb{Z}^k$ *be the* j*-th standard basis element of the free* $\mathbb{Z}$*-module* $\mathbb{Z}^k$. *Then we can express the q-Multinomial* $\left\langle \begin{matrix} |m| \\ m \end{matrix} \right\rangle_q$ *as:*

$$\left\langle \begin{matrix} |m| \\ m \end{matrix} \right\rangle_q = \sum_{i=1}^{k} q^{\sum_{i=1}^{j-1} m_i} \left\langle \begin{matrix} |m|-1 \\ m-e_j \end{matrix} \right\rangle_q.$$

Lemma 1.3.9 ($\zeta^{\binom{n}{2}}$ for ζ a primitive n-th root of unity)**.**

When ζ is a primitive n-th root of unity,

$$\zeta^{\binom{n}{2}} = \begin{cases} -1 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

$$= (-1)^{n+1}.$$

Proof. Notice that $\zeta^{\binom{n}{2}} = (\zeta^{(n-1)n/2})^2 = (\zeta^n)^{n-1} = 1$ so $\zeta^{(n-1)n/2}$ is either 1 or -1. We may assume $n > 0$ since if ζ is a primitive n-th root of unity, so is ζ^{-1}.

If $n = 2$, then $\zeta = -1$, and $(-1)^{(2-1)2/2} = (-1)$.

If n is even and greater than 2, then $\frac{n}{2} = b \in \mathbb{Z}_{>1}$, so $\zeta^{(n-1)n/2} = \zeta^{(n-1)}\zeta^b = \zeta^{-1}\zeta^b = \zeta^{b-1}$, and since $0 < b-1 < n$, and ζ is a primitive n-th root of unity, then $\zeta^{b-1} \neq 1$ which implies $\zeta^{(n-1)n/2} = -1$.

If n is odd, then $\frac{n-1}{2} = b \in \mathbb{Z}$, so $\zeta^{(n-1)n/2} = \zeta^n\zeta^{(n-1)/2} = (\zeta^n)^b = 1^b = 1$. □

The following lemmas are specializations of the q-Binomial Coefficient and the homogeneous symmetric polynomials in the case that $q = \zeta$ is a primitive n-th root of unity for some integer n.

Lemma 1.3.10 (Evaluation of $\begin{bmatrix} n \\ k \end{bmatrix}_\zeta$)**.**

$$\begin{bmatrix} n \\ k \end{bmatrix}_\zeta = \begin{cases} 1, & \text{if} \quad k = 0, n \\ 0, & \text{if} \quad 1 \leq k \leq n-1. \end{cases}$$

Proof. Let ζ be an n-th root of unity.

If $k = 0$ or $k = n$ then $\begin{bmatrix} n \\ k \end{bmatrix}_\zeta = 1$ by definition.

If $1 \leq k \leq n-1$, then

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_\zeta &= \frac{[n]_\zeta!}{[n-k]_\zeta![k]_\zeta!} \\ &= \frac{[n-1]_\zeta!(1+\zeta+\zeta^2+\ldots+\zeta^{n-1})}{[n-k]_\zeta![k]_\zeta!} \end{aligned}$$

$$= \frac{[n-1]_\zeta! \cdot 0}{[n-k]_\zeta![k]_\zeta!},$$

and since for every $k < n$ we have that $1 + \zeta + \ldots + \zeta^k \neq 0$, then $\begin{bmatrix} n \\ k \end{bmatrix}_\zeta = 0$ for all $1 \leq k \leq n$. □

Theorem 1.3.11 ($h_k(\{\zeta^i\})$ in terms of the q-Binomial Coefficient)**.**
Let $\begin{bmatrix} m \\ k \end{bmatrix}_q$ *be the q-Binomial Coefficient and* h_k *be defined as in Definition 1.3.1. Then*

$$h_k(1, \zeta, \ldots, \zeta^j) = \begin{bmatrix} k+j \\ k \end{bmatrix}_\zeta = \begin{bmatrix} k+j \\ j \end{bmatrix}_\zeta.$$

This theorem is a specific instance of Proposition 7.8.3 of [13], which summarizes the behavior of several symmetric polynomials under various specializations. This theorem specializes at $ps_{j+1}(h_\lambda)$, where $\lambda = (k)$, and $q = \zeta$ is a primitive n-th root of unity, so we obtain $ps_{j+1}(h_\lambda) = h_k(1, \zeta, \ldots, \zeta^j)$.

Lemma 1.3.12 (Properties of the q-Binomial Coefficient)**.**

1. $\begin{bmatrix} m \\ 0 \end{bmatrix}_q = \begin{bmatrix} m \\ m \end{bmatrix}_q = 1$;
2. $\begin{bmatrix} m \\ 1 \end{bmatrix}_q = [m]_q$;
3. $\begin{bmatrix} m \\ r \end{bmatrix}_q = \begin{bmatrix} m \\ m-r \end{bmatrix}_q$;
4. $\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{1-q^m}{1-q^{m-r}} \begin{bmatrix} m-1 \\ r \end{bmatrix}_q$.

1.4 Taft Actions Classification Background

Classifying Taft actions on path algebras in generality can be incredibly daunting, so Kinser and Walton in [9] restricted their focus to only actions which preserve the ascending filtration by path length, and then classified Hopf actions of Taft algebras, $T(n)$, on path algebras, $\Bbbk Q$, of loopless, finite, and Schurian quivers, over a field $\Bbbk$ such that $\Bbbk$ contains a primitive n-th root of unity, ζ.

To classify all possible $T(n)$ actions which preserve the ascending filtration by path length, we must answer the following questions:

1. Assuming an action does exist, how must g and x act on $\Bbbk Q_0$ and on $\Bbbk Q_1$? That is to say, if we assume we have non-trivial actions of $T(n)$ on $\Bbbk Q$, what are the necessary restrictions that are imposed on the action of g and x on a primitive idempotent $e_i \in \Bbbk Q_0$ and on an arbitrary arrow $a \in \Bbbk Q_1$?

   Since x and g are the generators of $T(n)$, and $\Bbbk Q$ is a graded algebra, then knowing how x and g act on a basis of $\Bbbk Q_0$ and a basis of $\Bbbk Q_1$ suffices to classify the action on $\Bbbk Q$ by virtue of the second axiom in the definition of a Hopf action in Definition 1.1.4.

2. Given a proposed formula for g and x to act on $\Bbbk Q_0$ and $\Bbbk Q_1$, does it indeed define a $T(n)$ action? In other words, if we impose the necessary restrictions we find when answering the above question, will these restrictions suffice in giving $\Bbbk Q$ a $T(n)$-module algebra structure?

   In order to answer this question, we need to check if the relations on x and g in

$T(n)$ are satisfied in the proposed action of $T(n)$ on $\Bbbk Q$. If so, then indeed we have an action.

Lemma 1.4.1 and Lemma 1.4.2 begin to answer the first question by examining the g action on $\Bbbk Q_0$ and $\Bbbk Q_1$ respectively. Later, Theorem 1.4.4 and Theorem 1.4.7 examine the x action on $\Bbbk Q_0$ and $\Bbbk Q_1$, respectively.

The lemmas and theorems presented below are slight variations of the original ones, which can be found in [9], along with the original proofs.

Lemma 1.4.1 (Action of g on $\Bbbk Q_0$)**.**

Let G be the subgroup of $T(n)$ generated by the group-like element g. Let Q be a finite quiver and $Q_0 = \{1, 2, \ldots, m\}$ be its set of vertices, so that $\{e_i\}_{i=1}^m$ are the corresponding primitive orthogonal idempotents in $\Bbbk Q_0$. Then g acts on $\Bbbk Q_0$ by permuting the set $\{e_i\}_{i=1}^m$.

In fact, we may think of G as acting by permuation on the set of vertices themselves. Therefore, for any $i \in Q_0$, if $\operatorname{orb}_g(i)$ denotes the orbit of i under the action of g, we have that $|\operatorname{orb}_g(i)| \mid n$.

Proof. Since g is group-like, it is invertible, and hence it must act on the semisimple algebra $\Bbbk Q_0$ as a $\Bbbk$-algebra automorphism. Furthermore, since $|Q_0| = m$, then $\Bbbk Q_0 \cong \Bbbk^m$, so for g to act as a $\Bbbk$-algebra automorphism it must send a complete collection of primitive orthogonal idempotents to another. In the case of $\Bbbk Q_0$, the set of primitive orthogonal idempotents is unique, and hence g permutes this set.

When we consider G as acting on the set Q_0 by permuting the vertices, then any such permutation gives rise to an action of $\Bbbk Q_0$ given by $g \cdot e_i = e_{g \cdot i}$. The fact that $|\operatorname{orb}_g(i)| \mid n$ follows immediately from $g^n = 1$. □

Lemma 1.4.2 (Action of g on $\Bbbk Q_1$)**.**

Let G be the subgroup of $T(n)$ generated by the group-like element g, and Q be a quiver which is loopless and Schurian. Suppose that g acts by a $\Bbbk$-algebra automorphism of $\Bbbk Q$, preserving the ascending filtration by path length. Then, for any arrow $a \in Q_1$, there exists a unique arrow $b \in Q_1$ and $\mu \in \Bbbk$ such that $g \cdot a = \mu b \in \Bbbk Q_1$. Furthermore, $s(g \cdot a) = g \cdot s(a)$ and $t(g \cdot a) = g \cdot t(a)$.

Proof. Since g is invertible, it must act as a $\Bbbk$-algebra automorphism on the path algebra $\Bbbk Q$. Every $\Bbbk$-algebra automorphism preserves the descending filtration by path length, i.e. the radical filtration, and hence $g \cdot \Bbbk Q_l \subset \bigoplus_{j \geq l} \Bbbk Q_j$. Since we are considering only actions that preserve the ascending filtration by path length, then we also have that $g \cdot \Bbbk Q_l \subset \bigoplus_{0 \leq j \leq l} \Bbbk Q_j$. Therefore, $g \cdot \Bbbk Q_l = \Bbbk Q_l$, which means g must act as a graded algebra automorphism. In particular, for any arrow $a \in Q_1$, $g \cdot a \in \Bbbk Q_1$. Since the arrows of Q form a basis for $\Bbbk Q_1$, which together with $\Bbbk Q_0$ generate $\Bbbk Q_l$ as an algebra, then the action of G on the arrows and on Q_0 defines the action of G on $\Bbbk Q$.

Furthermore, $g \cdot a = g \cdot (e_{s(a)} a) = (g \cdot e_{s(a)})(g \cdot a)$ since $\Delta(g) = g \otimes g$. By Lemma 1.4.1, $g \cdot e_{s(a)} = e_{g \cdot s(a)}$ so it follows that $s(g \cdot a) = e_{g \cdot s(a)}$. A nearly identical argument shows that $t(g \cdot a) = g \cdot t(a)$.

Lastly, since g is invertible, $g \cdot a \neq 0$, so in order for G to act on $\Bbbk Q$, there must exist an arrow b with $s(b) = g \cdot s(a)$ and $t(b) = g \cdot t(a)$, and since Q is Schurian, then b is unique. □

In [9], Kinser and Walton are able to provide this precise statement for the action of G on $\Bbbk Q$ by choosing the arrows in the quiver Q to be a basis for $\Bbbk Q_1$. In chapter 2 of this thesis, we extend the action of $T(n)$ to quivers which are not necessarily Schurian nor loopless, and we do so by eliminating this choice of basis.

Lemma 1.4.3 (Preliminary description for the action of x on $\Bbbk Q_0$)**.**

Let $e_i \in \Bbbk Q_0$ and $x \in T(n)$ be the $(1, g)$-skew primitive generator of $T(n)$. Then,

$$x \cdot e_i = \alpha_i e_i + \beta_i g \cdot e_i, \qquad \text{where } \alpha_i, \beta_i \in \Bbbk.$$

Proof. The relation $e_i^2 = e_i$ gives us that

$$x \cdot e_i = x \cdot e_i^2 = (e_i)(x \cdot e_i) + (x \cdot e_i)(g \cdot e_i).$$

We are assuming that the action preserves the ascending filtration by path length, so $x \cdot e_i$ is a linear combination of the $e_j \in \Bbbk Q_0$. Therefore, the above equality gives us that $x \cdot e_i \in \operatorname{span}\{e_i, g \cdot e_j\}$, as wanted. □

To simplify notation, we restrict ourselves to a single G-orbit of vertices and for each $e_i \in \Bbbk Q_0$, denote $g \cdot e_i = e_{i+1}$.

Theorem 1.4.4 (Action of x on $\Bbbk Q_0$)**.**

Let $Q_0 = \{1, 2, \ldots, m\}$ be the vertex set of a quiver, where m divides n, and g acts on $\Bbbk Q_0$ by $g \cdot e_i = e_{i+1}$. Here, subscripts are always interpreted modulo m.

1. *If $m < n$ (so the g-action on the complete set of primitive orthogonal idempotents of $\Bbbk Q_0$ is not faithful), then x acts on $\Bbbk Q_0$ by 0.*

2. *If $m = n$ (so g acts faithfully on the complete set of primitive orthogonal idempotents of $\Bbbk Q_0$), then the action of x on $\Bbbk Q_0$ is exactly of the form*

$$x \cdot e_i = \gamma\zeta^i(e_i - \zeta e_{i+1}) \quad \textit{for all } i, \tag{1.1}$$

where $\gamma \in \Bbbk$ can be any scalar and ζ is a primitive n-th root of unity.

In particular, we can extend the action of g on $\Bbbk Q_0$ to an inner faithful action of $T(n)$ on $\Bbbk Q_0$ if and only if $m = n$.

Proof. Assume that we have a $T(n)$-action on $\Bbbk Q_0$ extended from the g-action on $\Bbbk Q_0$ in the definition. By Lemma 1.4.3, we know that $x \cdot e_i = \alpha_i e_i + \beta_i e_{i+1}$ for some scalars $\alpha_i, \beta_i \in \Bbbk$. Therefore, we have that

$$0 = x \cdot 1 = x \cdot \sum_{i=1}^{m} e_i = \sum_{i=1}^{m} \alpha_i e_i + \beta_i e_{i+1} = \sum_{i=1}^{m} (\alpha_i + \beta_{i-1}) e_i, \tag{1.2}$$

which gives $\beta_{i-1} = -\alpha_i$. (Here, $\sum_{i=1}^{m} \beta_i e_{i+1} = \sum_{i=1}^{m} \beta_{i-1} e_i$ by reindexing.) Now the relation $xg = \zeta gx$ applied to e_i gives

$$\alpha_{i+1} e_{i+1} - \alpha_{i+2} e_{i+2} = \zeta(\alpha_i e_{i+1} - \alpha_{i+1} e_{i+2}), \tag{1.3}$$

so that $\alpha_{i+1} = \zeta\alpha_i$ for all i. Setting $\gamma := \alpha_1\zeta^{-1}$ gives $\alpha_i = \zeta^i\gamma$, so that (1.1) holds whenever a $T(n)$-action exists. We have assumed that m divides n, but on the other hand, $x \cdot e_i = x \cdot e_{i+m}$ implies that $\gamma\zeta^i = \gamma\zeta^{i+m}$. Thus, $\gamma = \gamma\zeta^m$. Hence, whenever $m < n = \mathrm{ord}(\zeta)$, we have $\gamma = 0$, and x acts by 0. This establishes (i).

On the other hand, suppose that $m = n$. We will show that Equation (1.1) defines a $T(n)$-action on $\Bbbk Q_0$ for any $\gamma \in \Bbbk$. A simple substitution verifies that $xg \cdot e_i = \zeta gx \cdot e_i$. The fact that the x-action preserves the relations $e_ie_j = \delta_{ij}e_i$ and $\sum_{i=1}^{n} e_i = 1$ is also easy to check by substitution.

It remains to show that x^n acts by 0, which we show after Lemma 1.4.5. $\square$

Lemma 1.4.5 (Action of x^m on $\Bbbk Q_0$ for $m \in \mathbb{N}$)**.**

Given a $T(n)$ action on $\Bbbk Q$, if x is the nilpotent generator in $T(n)$, $e_i \in \Bbbk Q_0$ is a primitive idempotent, and $x \cdot e_i \neq 0$, then for $m \in \mathbb{N}$ we have that

$$x^m \cdot e_i = \sum_{j=0}^{m} h_{m-j}(\alpha_i, \ldots, \alpha_{i+j})(\beta_i \cdots \beta_{i+j-1})e_{i+j},$$

where $\alpha_i = \gamma\zeta^i$, $\beta_i = -\gamma\zeta^{i+1}$, and $\gamma \in \Bbbk$ is a fixed scalar.

Proof. We prove the result by inducting on m.

Recall from Theorem 1.4.4 that $x \cdot e_i = \gamma\zeta^i e_i + \gamma\zeta^{i+1}e_{i+1}$ whenever $x \cdot e_i \neq 0$. So if $m = 1$, the result holds by Theorem 1.4.4, since

$$\gamma\zeta^i e_i + \gamma\zeta^{i+1}e_{i+1} = h_1(\alpha_i)(1)e_i + h_0(\alpha_i)\beta_1 e_{i+1}.$$

Suppose the result holds for $k \leq m$. Then $x^{m+1} \cdot e_i = x \cdot (x^m \cdot e_i)$, so by the inductive hypothesis, we can write

$$\begin{aligned} x^{m+1} \cdot e_i &= x \cdot (x^m \cdot e_i) \\ &= x \cdot \sum_{j=0}^{m} h_{m-j}(\alpha_i, \ldots, \alpha_{i+j})(\beta_i \cdots \beta_{i+j-1})e_{i+j} \\ &= \sum_{j=0}^{m} h_{m-j}(\alpha_i, \ldots, \alpha_{i+j})(\beta_i \cdots \beta_{i+j-1})\Big[\alpha_{i+j}e_{i+j} + \beta_{i+j}e_{i+j+1}\Big]. \end{aligned}$$

After we distribute, combine like terms, and reindex, we can rewrite the above expression so that we have a coefficient and a single term e_{i+j}. So we have,

$$\begin{aligned} x^{m+1} \cdot e_i = \sum_{j=0}^{m+1} \Big[& h_{m-j}(\alpha_i, \ldots, \alpha_{i+j})\alpha_{i+j}(\beta_i \cdots \beta_{i+j-1}) \\ & + h_{m-(j-1)}(\alpha_i, \ldots, \alpha_{i+j-1})(\beta_i \cdots \beta_{i+j-2})\beta_{i+j-1}\Big] e_{i+j} \end{aligned}$$

$$\begin{aligned} = \sum_{j=0}^{m+1} \Big[& h_{m-j}(\alpha_i, \ldots, \alpha_{i+j})\alpha_{i+j}(\beta_i \cdots \beta_{i+j-1}) \\ & + h_{(m+1)-j}(\alpha_i, \ldots, \alpha_{i+j-1})(\beta_i \cdots \beta_{i+j-1})\Big] e_{i+j} \end{aligned}$$

$$\begin{aligned} &= \sum_{j=0}^{m+1} (\beta_i \cdots \beta_{i+j-1})\Big[h_{m-j}(\alpha_i, \ldots, \alpha_{i+j})\alpha_{i+j} + h_{(m+1)-j}(\alpha_i, \ldots, \alpha_{i+j-1})\Big] e_{i+j} \\ &= \sum_{j=0}^{m+1} h_{(m+1)-j}(\alpha_i, \ldots, \alpha_{i+j})(\beta_i \cdots \beta_{i+j-1})e_{i+j}, \end{aligned}$$

which shows, by the principle of mathematical induction, our desired result.

Notice that this last equality follows from the definition of homogenous symmetric polynomial of degree k because:

$$h_{m-j}(\alpha_i, \ldots, \alpha_{i+j})\alpha_{i+j} + h_{(m+1)-j}(\alpha_i, \ldots, \alpha_{i+j-1}) = h_{(m+1)-j}(\alpha_i, \ldots, \alpha_{i+j}).$$

$\square$

Corollary 1.4.6 (Action of x^n on $\Bbbk Q_0$)**.**

If x is the nilpotent generator in $T(n)$, and $e_i \in \Bbbk Q_0$ a primitive idempotent, then $x^n \cdot e_i = 0$.

Proof. We can use Theorem 1.3.11 in the preliminaries section to simplify the expression for $x^m \cdot e_i$ as follows:

$$\begin{aligned} x^m \cdot e_i &= \sum_{j=0}^{m} h_{m-j}(\alpha_i, \ldots, \alpha_{i+j}) \cdot (\beta_i \cdots \beta_{i+j-1}) \cdot e_{i+j} \\ &= \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\zeta (\gamma\zeta^i)^{m-j} \cdot (-1)^j \gamma^j \zeta^{(i+1)j} \zeta^{j(j-1)/2} e_{i+j} \\ &= \gamma^m \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_\zeta (-1)^j \zeta^{im+j} \zeta^{j(j-1)/2} e_{i+j}. \end{aligned} \tag{1.4}$$

In particular, if $m = n$, then by Lemma 1.3.10, we have that $\begin{bmatrix} n \\ j \end{bmatrix}_\zeta = 0$ for $1 \leq j \leq n-1$. So we obtain:

$$\begin{aligned} x^n \cdot e_i = \gamma^n &\left(\begin{bmatrix} n \\ 0 \end{bmatrix}_\zeta (-1)^0 \zeta^{in+0} \zeta^0 e_i + \begin{bmatrix} n \\ n \end{bmatrix}_\zeta (-1)^n \zeta^{in+n} \zeta^{n(n-1)/2} e_{i+n} \right) \\ &\gamma^n \left(e_i + (-1)^n \zeta^{n(n-1)/2} e_{i+n} \right). \end{aligned} \tag{1.5}$$

By Lemma 1.3.9, we have that $(-1)^n \zeta^{n(n-1)/2} = (-1)^n (-1)^{n+1} = -1$, so $x^n \cdot e_i = e_i - e_i = 0$, which shows that x^n does in fact act by 0 on $\Bbbk Q_0$. $\square$

We now have a complete description of the action of $T(n)$ on the semisimple subalgebra $\Bbbk Q_0$, but we still need to understand the action of x on $\Bbbk Q_1$ in order to have a full description of the action on all of $\Bbbk Q$. This proves to be a bit more complicated, so we state the result below without proof and refer to [9] for the complete proof.

Theorem 1.4.7 (Action of x on $\Bbbk Q_1$)**.**

Let Q be a loopless Schurian quiver. Suppose we have an action of $T(n)$ on the path algebra $\Bbbk Q$, and let $a \in Q_1$ with $i^+ := s(a)$ and $j^- := t(a)$. Then, there exist scalars, $\alpha, \beta, \lambda \in k$ such that

$$x \cdot a = \alpha a + \beta(g \cdot a) + \lambda\sigma(a), \tag{1.6}$$

where $\sigma(a)$ is the unique path of length less than or equal to 1 (if it exists) with $s(\sigma(a)) = s(a)$ and $t(\sigma(a)) = t(g \cdot a)$. If such a path does not exist, then we set $\sigma(a) = 0$.

Moreover, α, β, λ, can be determined explicitly in specific cases.

The precise description of $x \cdot a$ found in [9] relies on choosing the set of arrows of Q as a basis for $\Bbbk Q_1$. In the next section, we prove an extension of this theorem which does not rely on a choice of basis, so we omit the precise description and instead refer the curious reader to [9] for more details.

To check if these conditions truly define an action, it is only necessary to check that x^n acts by 0 on $\Bbbk Q_1$. The computation showing this fact can be found in the appendix in [9], and we also provide a slightly different proof in Lemma 2.2.6.

We build upon Kinser and Walton's results from [9] in order to answer the following questions:

Problem 1. *Can this classification of actions be generalized to quivers with multiple edges and with loops? If so, we would obtain a classification of the action of $T(n)$ on arbitrary path algebras of quivers.*

Problem 2. *What is the invariant ring of this action and is it finitely generated as an algebra? In searching for an answer to this question, we find a minimal set which generates the invariant ring.*

We continue to consider only actions which preserve the ascending filtration by path length, and solve Problem 1 in Theorem 2.2.3. For Problem 2, we restrict to only actions which act transitively on the set of vertices of the quiver, and find a minimal set of generators in Theorem 3.2.12.

CHAPTER 2
EXTENSION OF THE CLASSIFICATION OF THE ACTION

Since $T(n)$ is generated by g and x, by the second axiom of Hopf actions (Definition 1.1.4) it suffices to know how g and x act on $\Bbbk Q_0$ and $\Bbbk Q_1$ to obtain a well defined action on all of $\Bbbk Q$. Therefore, we employ similar techniques to generalize the results in [9] to quivers which are not necessarily Schurian or loopless.

For the remainder of this thesis, Q is an arbitrary quiver, $G = \langle g \rangle$ is the subgroup of $T(n)$ generated by g, ζ is a primitive n-th root of unity, and $\{e_i\}_{i \in Q_0}$ is a complete set of primitive orthogonal idempotents.

2.1 Classification of the Action of g on $\Bbbk Q$

We begin, once again, by classifying the action of the group-like element, g, much like we did in the previous section. Observe that adding loops and multiple edges does not change the vertex set Q_0 of a quiver. Therefore, the action of G on the semisimple algebra $\Bbbk Q_0$ is still given by Lemma 1.4.1.

The first significant change we must address when adding multiple edges arises from the G action on $\Bbbk Q_1$. It is still true that for any path $a \in \Bbbk Q_1$, $s(g \cdot a) = g \cdot s(a)$ and $t(g \cdot a) = g \cdot t(a)$ as shown in Lemma 1.4.2. However, in [9], the restriction of considering solely Schurian, loopless quivers yielded a unique option, up to a scalar multiple, for the image of $g \cdot a$. If there are multiple edges between any two given vertices, this notion needs to be adjusted to reflect other options for $g \cdot a$. That said, the options are still limited by the requirement that the action preserve the ascending filtration by path length.

Observe that the quivers on which G acts must have a great deal of symmetry. In fact, define two types of quivers as follows:

Definition 2.1.1 (Type A quivers)**.**

A quiver Q is Type A if the permutation action of G on Q_0 is transitive.

See Figures 2.1 and 2.2.

Definition 2.1.2 (Type B quivers)**.**

A quiver Q is Type B, if the permutation action of G on Q_0 has two distinct orbits: one corresponding to the vertices which are all sources of arrows in Q, and one corresponding to the vertices which are all targets of arrows in Q.

See Figures 2.3 and 2.4.

Notice that there is a significant amount of symmetry occurring in the set of arrows of Q. In particular, Figures 2.1 and 2.2 seem to indicate that the number of arrows (or loops) of a specified color is always the same. This occurs because the g acts by an invertible map on the vector space $\Bbbk Q_1$. The symmetry in Figures 2.3 and 2.4 is also due to the fact that g is invertible.

We provide a few concrete examples to help clarify the G-action on $\Bbbk Q$.

Example 1. $T(2)$ *acts on the path algebra of Figure 2.1, with g acting on Q_0 via*

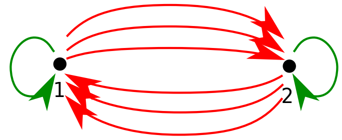


Figure 2.1: Type A quiver - single orbit with 2 vertices

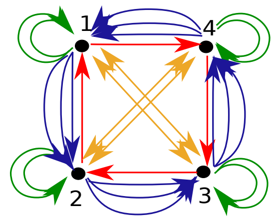


Figure 2.2: Type A quiver - single orbit with 4 vertices

the permutation $(1, 2)$*, and hence partitioning the set consisting of the four spaces* $\{e_i \Bbbk Q_1 e_j\}$*, for* $1 \leq i, j \leq 2$*, into two distinct orbits. In other words, the action of* g *preserves the color of the path in* $\Bbbk Q_1$*: i.e. red paths map to red paths and green paths map to green paths.*

Example 2. $T(4)$ *acts on the path algebra of Figure 2.2, with* g *acting on* Q_0 *via the permutation* $(1, 2, 3, 4)$*, and hence partitioning the set consisting of the sixteen spaces* $\{e_i \Bbbk Q_1 e_j\}$*, for* $1 \leq i, j \leq 4$*, into four distinct orbits. So again, the action of* g *preserves the color of the path in* $\Bbbk Q_1$*: i.e. red, blue, green and yellow paths map to red, blue, green and yellow paths respectively.*

Example 3. $T(6)$ *acts on the path algebra of Figure 2.3, with* g *acting on* Q_0 *via the permuation* $(1+, 2+)(1-, 2-, 3-)$*. Furthermore, since the* $\gcd(2, 3) = 1$*,* g *acts*

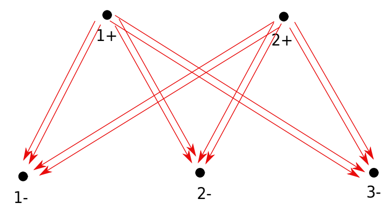


Figure 2.3: Type B quiver - source orbit with 2 vertices; target orbit with 3 vertices

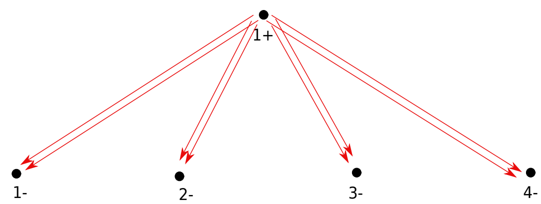


Figure 2.4: Type B quiver - source orbit with 1 vertex; target orbit with 4 vertices

transitively on the set consisting of the six spaces $\{e_{i+}\Bbbk Q_1 e_{j-}\}$*, for* $1 \leq i \leq 2$ *and* $1 \leq j \leq 3$.

Example 4. $T(4)$ *acts on the path algebra of Figure 2.4, with* g *acting on* Q_0 *via the permuation* $(1+)(1-, 2-, 3-, 4-)$*, and transitively on the set consisting of the four spaces* $\{e_{1+}\Bbbk Q_1 e_{j-}\}$*, for* $1 \leq j \leq 4$.

To generalize the concept of g acting on $\Bbbk Q_1$, we provide the following definition.

Definition 2.1.3 (Arrow Space)**.**

Let $\Bbbk Q$ *be the path algebra of a finite quiver* Q *with vertices labeled* $\{1, 2, \ldots, m\}$*. Define the* <u>*arrow space*</u>*,* V_j^i*, as the vector subspace of* $\Bbbk Q_1$ *generated by all arrows* $a \in Q_1$ *with source* $s(a) = i$ *and target* $t(a) = j$*, i.e.* $V_j^i := e_i \Bbbk Q_1 e_j$.

Instead of classifying the action of g on an arrow in Q_1, we classify the action of g as a linear map on V_j^i by defining it on an arbitrary element $a \in V_j^i$, i.e. we do not define it on a chosen basis. Furthermore, we extend the notion of source and target to an arbitrary element $a \in V_j^i$ by declaring $s(a) = i$ and $t(a) = j$.

Recall by Lemma 1.4.3, $g \cdot s(a) = s(g \cdot a)$ and $g \cdot t(a) = t(g \cdot a)$. Abusing the notation a little bit, we define the linear map g for $a \in \Bbbk Q_1$ by

$$g : V_j^i \to V_{g \cdot j}^{g \cdot i}$$
$$a \mapsto g \cdot a.$$

Using the fact that $\Delta(g) = g \otimes g$, we extend g to a linear map on $\Bbbk Q$ by

$$g : \Bbbk Q \to \Bbbk Q$$
$$p = a_1 \cdots a_l \mapsto g \cdot p = (g \cdot a_1) \cdots (g \cdot a_l),$$

for any path $p = (a_1 a_2 \cdots a_l) \in \Bbbk Q_l$, with $l \geq 0$.

We write $g(p)$ for $g \cdot p$, interpreting g as a linear map on $\Bbbk Q$.

In particular, since g acts by a $\Bbbk$-algebra automorphism of $\Bbbk Q$, we see that $V_j^i \cong V_{g \cdot j}^{g \cdot i}$, which explains why each of the colorful g-orbits have the same number of arrows.

We succinctly summarize the above observations in the following theorem:

Theorem 2.1.4 (G-action on $\Bbbk Q$)**.**

A G-action on a path algebra $\Bbbk Q$ acts as a permutation group on the complete set *of primitive orthogonal idempotents* $\{e_1, \ldots, e_m\} \in \Bbbk Q_0$, *and as a group of linear isomorphisms on the vector spaces* $V_j^i \subset \Bbbk Q_1$ *for all* $1 \leq i, j \leq m$.

Furthermore, a G*-action on a path algebra* $\Bbbk Q$ *determines a decomposition of the quiver* Q *into Type A and Type B subquivers, and the intersection of any of two of these is contained in* Q_0 *(possibly empty).*

2.2 Classification of the Action of x on $\Bbbk Q$

Given a G action on $\Bbbk Q$, we want to extend it to a $T(n)$ action on $\Bbbk Q$, so we need to classify the actions of x on $\Bbbk Q$ which preserve the ascending filtration by path length.

For any action of $T(n)$ extending a given G action on $\Bbbk Q$, the nilpotent generator x will act on $\Bbbk Q_0$ as described in Lemma 1.4.4. Again, adding multiple edges and loops to the quiver Q does not affect the structure of the semisimple algebra $\Bbbk Q_0$, so the previous results still hold. The modifications to the results in [9] occur when looking at the action of x on $\Bbbk Q_1$, which we once again do by analyzing the action of x on an arbitrary element $a \in V_j^i$ instead of a path.

Lemma 2.2.1 (Preliminary description for the action of x on $\Bbbk Q_1$)**.**

For an arbitrary element $a \in V_j^i$, *the action of* x *on* a *is given by*

$$x \cdot a = a\alpha + \beta g(a) + \sigma(a)$$

where $\alpha = x \cdot e_j$, $\beta = x \cdot e_i$, *and* $\sigma : \Bbbk Q_1 \to \Bbbk Q_0 \oplus \Bbbk Q_1$ *is a linear map such that*

$$\begin{cases} \sigma(a) \in V^i_{g\cdot j}, & \text{if} \quad g \cdot j \neq i \\ \sigma(a) \in e_i \Bbbk Q e_i \oplus V^i_i, & \text{if} \quad g \cdot j = i. \end{cases}$$

Proof. Since $a \in V^i_j$, we have that $a = e_i a$ and $a = a e_j$. Therefore, since $\Delta(x) = 1 \otimes x + x \otimes g$, we obtain:

$$x \cdot a = x \cdot (a e_j) = (1 \cdot a)(x \cdot e_j) + (x \cdot a) g(e_j) \quad \text{as well as}$$

$$x \cdot a = x \cdot (e_i a) = (1 \cdot e_i)(x \cdot a) + (x \cdot e_i) g(a).$$

Observe that by rearranging the terms of the expressions above, we can express $x \cdot a$ in terms of a and $g(a)$ as follows:

$$\begin{aligned} x \cdot a &= a(x \cdot e_j) + (x \cdot a)g(e_j) + (x \cdot e_i)g(a) - (x \cdot e_i)g(a) \\ &= a(x \cdot e_j) + (x \cdot e_i)g(a) + (x \cdot a)g(e_j) - (x \cdot e_i)g(a) \\ &= a(x \cdot e_j) + (x \cdot e_i)g(a) + \sigma_1(a) \end{aligned} \tag{2.1}$$

where $\sigma_1(a) := (x \cdot a)g(e_j) - (x \cdot e_i)g(a)$ or, alternatively, we could write

$$\begin{aligned} x \cdot a &= e_i(x \cdot a) + (x \cdot e_i)g(a) + a(x \cdot e_j) - a(x \cdot e_j) \\ &= a(x \cdot e_j) + (x \cdot e_i)g(a) + e_i(x \cdot a) - a(x \cdot e_j) \\ &= a(x \cdot e_j) + (x \cdot e_i)g(a) + \sigma_2(a) \end{aligned} \tag{2.2}$$

where $\sigma_2(a) := e_i(x \cdot a) - a(x \cdot e_j)$.

Comparing Equations (2.1) and (2.2), we see that $\sigma_1(a) = \sigma_2(a)$ and we denote it by $\sigma(a)$.

Since x and g both act as linear maps on $\Bbbk Q$, σ is also linear. It remains to show that, if $a \in V_j^i$ with $g \cdot j \neq i$, then $\sigma(a) \in V_{g\cdot j}^i$ and if $g \cdot j = i$ then $\sigma(a) \in e_i \Bbbk Q e_i \oplus V_i^i$.

By multiplying $\sigma(a)$ on the left by e_i and on the right by $e_{g\cdot j}$, we obtain:

$$\begin{aligned} e_i\sigma(a) &= e_i\Big(e_i(x \cdot a) - a(x \cdot e_j)\Big) \\ &= e_i^2(x \cdot a) - e_i a(x \cdot e_j) \\ &= e_i(x \cdot a) - a(x \cdot e_j) \\ &= \sigma(a), \end{aligned}$$

and

$$\begin{aligned} \sigma(a)e_{g\cdot j} &= \Big((x \cdot a)g(e_j) - (x \cdot e_i)g(a)\Big)e_{g\cdot j} \\ &= (x \cdot a)e_{g\cdot j}^2 - (x \cdot e_i)g(a)e_{g\cdot j} \\ &= (x \cdot a)e_{g\cdot j} - (x \cdot e_i)g(a) \\ &= \sigma(a). \end{aligned}$$

If $g \cdot j \neq i$, then we see that $\sigma(a) \in V_{g\cdot j}^i$, i.e. $\sigma(a)$ is homogeneous of degree 1.

If instead $g \cdot j = i$, then $\sigma(a)$ could be homogeneous as a stationary path, in which case $\sigma(a) \in e_i \Bbbk Q e_i$, homogeneous as a loop, in which case $\sigma(a) \in V_i^i$, or a non-homogeneous element of the form $\sigma(a) = \lambda e_i + b$, with $0 \neq \lambda \in \Bbbk$ and $0 \neq b \in V_j^i$. □

Notice that, in contrast with Theorem 1.4.7, α and β are elements of $\Bbbk Q_0$

instead of $\Bbbk$, and σ is defined as a map $\sigma : \Bbbk Q_1 \to \Bbbk Q_0 \oplus \Bbbk Q_1$.

We proceed to analyze the interaction between the maps σ and g.

Lemma 2.2.2 (Relation between σ and g)**.**

The map σ defined in Lemma 2.2.1 satisfies the following relation:

$$\sigma \circ g = \zeta g \circ \sigma.$$

Proof. If $e \in \Bbbk Q_0$, then $g(e) = \Bbbk Q_0$, so $\sigma(g(e)) = g(\sigma(e)) = 0$, so the relation holds.

If $a \in V_j^i$, then from Lemma 2.2.1, we have that $x \cdot a = a\alpha + \beta g(a) + \sigma(a)$, where $\alpha = (x \cdot e_j)$ and $\beta = (x \cdot e_i)$. Therefore,

$$\begin{aligned} g(x \cdot a) &= g\left(a\alpha + \beta g(a) + \sigma(a)\right) \\ &= g(a)g(\alpha) + g(\beta)g^2(a) + g(\sigma(a)) \\ \implies \zeta g(x \cdot a) &= \zeta g(a)g(\alpha) + \zeta g(\beta)g^2(a) + \zeta g(\sigma(a)). \end{aligned}$$

Lemma 2.2.1 also gives us the following relation:

$$x \cdot g(a) = g(a)\alpha' + \beta' g^2(a) + \sigma(g(a))$$

where

$$\alpha' = x \cdot e_{g \cdot j} = x \cdot g(e_j) = (\zeta g x) \cdot e_j = \zeta g(\alpha)$$

and

$$\beta' = x \cdot e_{g \cdot i} = x \cdot g(e_i) = (\zeta g x) \cdot e_i = \zeta g(\beta).$$

Substituting these values in for α' and β', we get:

$$x \cdot g(a) = \zeta g(a)g(\alpha) + \zeta g(\beta)g^2(a) + \sigma(g(a)).$$

The relation $xg = \zeta gx$ gives us that $\zeta g(x \cdot a) = x \cdot g(a)$, so we obtain:

$$\begin{aligned} &\zeta g(a)g(\alpha) + \zeta g(\beta)g^2(a) + \zeta g(\sigma(a)) \\ =&\zeta g(a)g(\alpha) + \zeta g(\beta)g^2(a) + \sigma(g(a)) \\ \implies& \zeta g(\sigma(a)) = \sigma(g(a)). \end{aligned}$$

Since $a \in V_j^i$ was arbitrary, we obtain that $\zeta g \circ \sigma = \sigma \circ g$, as wanted. □

For simplicity, from this point onward, assume that the vertices of Q are labeled such that $g \cdot i = i + 1 \mod m_i$ where $m_i = |\operatorname{orb}_g(i)|$, and $m_i \mid n$.

We summarize our results in the following theorem.

Theorem 2.2.3 (Action of x on $a \in V_j^i$)**.**

If $T(n)$ acts on the path algebra $\Bbbk Q$ of a quiver Q, and $a \in V_j^i$, then

$$x \cdot a = \begin{cases} \gamma_-\zeta^j a - \gamma_+\zeta^{i+1}g(a) + \sigma(a), & \text{if } |\operatorname{orb}_g(i)| = n \text{ and } |\operatorname{orb}_g(j)| = n \\ \gamma_-\zeta^j a + \sigma(a), & \text{if } |\operatorname{orb}_g(j)| < n \text{ and } |\operatorname{orb}_g(i)| = n \\ -\gamma_+\zeta^{i+1}g(a) + \sigma(a), & \text{if } |\operatorname{orb}_g(i)| < n \text{ and } |\operatorname{orb}_g(j)| = n \\ \sigma(a), & \text{if } |\operatorname{orb}_g(i)| < n \text{ and } |\operatorname{orb}_g(j)| < n \end{cases}$$

where $\gamma_-, \gamma_+ \in \Bbbk$, and σ is a linear map defined on $\Bbbk Q_1$ such that

$$\sigma(a) \in \begin{cases} V_{g\cdot j}^i, & \text{if} \quad g \cdot j \neq i \\ e_i \Bbbk Q e_i \oplus V_i^i, & \text{if} \quad g \cdot j = i. \end{cases}$$

$$\textit{and} \qquad \zeta g \circ \sigma = \sigma \circ g.$$

Notice that we are not claiming that any linear map, σ, as with the properties stated in Theorem 2.2.3 will define a valid $T(n)$ action on $\Bbbk Q$. Rather, we are saying that if we *do* have a valid $T(n)$ action defined, then the map σ must have these properties.

Proof. Suppose $T(n)$ acts on $\Bbbk Q$ and let $a \in V_j^i$. If $|\operatorname{orb}_g(i)| = |\operatorname{orb}_g(j)| = n$, then by Theorem 1.4.4, there exists $\gamma_+, \gamma_- \in \Bbbk$ such that

$$x \cdot e_i = \gamma_+ \zeta^i e_i - \gamma_+ \zeta^{i+1} e_{i+1} \quad \text{and} \tag{2.3}$$

$$x \cdot e_j = \gamma_- \zeta^j e_j - \gamma_- \zeta^{j+1} e_{j+1}. \tag{2.4}$$

By Lemma 2.2.1 and Lemma 2.2.2, we have that $x \cdot a = a\alpha + \beta g(a) + \sigma(a)$ for some linear map σ with

$$\sigma(a) \in \begin{cases} V^i_{g\cdot j}, & \text{if} \quad g \cdot j \neq i \\ e_i \Bbbk Q e_i \oplus V^i_i, & \text{if} \quad g \cdot j = i, \end{cases}$$

and $\zeta g \circ \sigma = \sigma \circ g$.

Therefore, substituting the values in Equations (2.3) and (2.4) into $x \cdot a = a\alpha + \beta g(a) + \sigma(a)$, we obtain:

$$\begin{aligned} x \cdot a &= a\alpha + \beta g(a) + \sigma(a) \\ &= a(\gamma_- \zeta^j e_j - \gamma_- \zeta^{j+1} e_{j+1}) + (\gamma_+ \zeta^i e_i - \gamma_+ \zeta^{i+1} e_{i+1}) g(a) + \sigma(a) \\ &= \gamma_- \zeta^j a - \gamma_+ \zeta^{i+1} g(a) + \sigma(a). \end{aligned}$$

If instead either $|\operatorname{orb}_g(i)| \neq n$ or $|\operatorname{orb}_g(j)| \neq n$, then $x \cdot e_i = 0$ or $x \cdot e_j = 0$ respectively, so we obtain the other three cases. In particular, if g does not act faithfully on the orbit of either i nor j, then x acts by σ. Also, if $\gamma_+ = \gamma_- = 0$, then again, x acts by σ.

Furthermore, in the case in which e_i and e_j belong to the same G-orbit, for example if Q is a Type A quiver, then $\gamma_+ = \gamma_- = \gamma$. □

The question now is whether there are any other conditions we must impose on the map σ in order to obtain an action of $T(n)$ on $\Bbbk Q$. In order to answer this question, we must see whether the relation $x^n = 0$ gives us any further restrictions. To do so, we introduce some new notation and a few lemmas which will help us further understand the action of x^n on an arbitrary element of $\Bbbk Q_1$.

Observation 1. *Observe that, as it stands right now, for any positive integer d the map σ^d is well defined if and only if $\sigma^k(a) \in \Bbbk Q_1$ for all $k < d$, and all $a \in \Bbbk Q_1$. However, since $\sigma(a)$ could possibly contain a degree 0 summand, we extend the map σ to $\Bbbk Q_0$ by setting $\sigma_{|\Bbbk Q_0} = 0$. In particular, if $\sigma^k(a)$ is homogeneous of degree 0 for any $k < d$, then $\sigma^d(a) = 0$. By the linearity of σ, we may now consider σ a map on $\Bbbk Q_0 \oplus \Bbbk Q_1$, and hence σ^d is well defined.*

Lemma 2.2.4 (Action of x on an arbitrary element $w_d^c \in \Bbbk Q_1$).

Suppose $T(n)$ acts on $\Bbbk Q$. Let a be an arbitrary element in V_j^i, and define $w_d^c := g^c \sigma^d(a)$.

Let $x \cdot e_i = \gamma_+ \zeta^i e_i - \gamma_+ \zeta^{i+1} e_{i+1}$ and $x \cdot e_j = \gamma_- \zeta^j e_j - \gamma_- \zeta^{j+1} e_{j+1}$, as given in Theorem 1.4.4, and declare $\gamma_- = 0$ if $|\operatorname{orb}_g(j)| < n$ and $\gamma_+ = 0$ if $|\operatorname{orb}_g(i)| < n$.

Then

$$x \cdot w_d^c = \alpha_d^c w_d^c + \beta^c w_d^{c+1} + \delta^c w_{d+1}^c,$$

where, $\alpha_d^c = \gamma_- \zeta^{j+c+d}$, $\beta^c = -\gamma_+ \zeta^{i+c+1}$ *and* $\delta^c = \zeta^c$.

Proof. Let $a \in V_j^i$. By definition of w_d^c, we may write $a = w_0^0$. We can clearly see from Theorem 2.2.3 that $x \cdot a = x \cdot w_0^0 = \alpha_0^0 w_0^0 + \beta^0 w_0^1 + \delta^0 w_1^0$ where $\alpha_0^0 = \gamma_- \zeta^j$, $\beta^0 = -\gamma_+ \zeta^{i+1}$ and $\delta^0 = \zeta^0 = 1$.

Observe that for $c, d \in \mathbb{N}$ and any pair $i, j \in \Bbbk Q_0$,

$$\sigma^d : V_j^i \to V_{j+d}^i \oplus e_i \Bbbk Q e_i \quad \text{and} \quad g^c : V_{j+d}^i \oplus e_i \Bbbk Q e_i \to V_{j+d+c}^{i+c} \oplus e_{i+c} \Bbbk Q e_{i+c}.$$

Therefore, $w_d^c \in V_{j+c+d}^{i+c} \oplus e_{i+c} \Bbbk Q e_{i+c}$. By the linearity of x, we consider the homogeneous cases independently, that is

$$\text{Case 1:} \quad w_d^c \in e_{i+c} \Bbbk Q e_{i+c} \quad \text{and}$$

$$\text{Case 2:} \quad w_d^c \in V_{j+c+d}^{i+c}.$$

Since we already classified the action of x on $\Bbbk Q_0$ in Theorem 1.4.4, we need to verify that if $w_d^c \in e_{i+c} \Bbbk Q e_{i+c}$, in particular if $w_d^c = e_{i+c}$, then the proposed action of x on w_d^c agrees with the one given in Theorem 1.4.4. So suppose $w_d^c = e_{i+c}$. Recall that

$$\sigma(a) \in \begin{cases} V_{g \cdot j}^i, & \text{if} \quad j+1 \neq i \\ e_i \Bbbk Q e_i \oplus V_i^i, & \text{if} \quad j+1 = i. \end{cases}$$

Therefore, if $w_d^c \in e_{i+c} \Bbbk Q e_{i+c}$ then $j + d = i$, and hence $\mathrm{orb}_g(j) = \mathrm{orb}_g(i)$, which implies that $\gamma_- = \gamma_+ = \gamma$. According to our claim,

$$x \cdot w_d^c = \alpha_d^c w_d^c + \beta^c w_d^{c+1} + \delta^c w_{d+1}^c$$

$$\implies x \cdot e_{i+c} = \gamma\zeta^{j+c+d} e_{i+c} - \gamma\zeta^{i+c+1} e_{i+c+1} + \delta^c \sigma(e_{i+c})$$

$$= \gamma\zeta^{i+c} e_{i+c} - \gamma\zeta^{i+c+1} e_{i+c+1}$$

$$= \gamma\zeta^{i+c}(e_{i+c} - \zeta e_{i+c+1}),$$

which is consistent with the result of Theorem 1.4.4.

For Case 2, we have that $w_d^c \in V_{j+c+d}^{i+c}$. Therefore, the source and target of the element w_d^c are given by $s(w_d^c) = i + c$ and $t(w_d^c) = j + c + d$, respectively. Notice that if $w_d^c \neq 0$, then $w_d^{c+1} \neq 0$, but w_{d+1}^c could still be zero or have a degree zero summand.

By Theorem 2.2.3, substituting a with w_d^c, we obtain

$$x \cdot w_d^c = \gamma_- \zeta^{j+c+d} w_d^c - \gamma_+ \zeta^{i+c+1} g(w_d^c) + \sigma(w_d^c)$$

$$= \gamma_- \zeta^{j+c+d} w_d^c - \gamma_+ \zeta^{i+c+1} w_d^{c+1} + \sigma(w_d^c).$$

Therefore, $\alpha_d^c = \gamma_- \zeta^{j+c+d}$ and $\beta^c = -\gamma_+ \zeta^{i+c+1}$.

Also, note that $\alpha_{d+r}^c = \gamma_- \zeta^{j+c+d+r} = \alpha_d^{c+r}$.

Lastly, by Lemma 2.2.2, $\sigma \circ g = \zeta g \circ \sigma$, and hence

$$\sigma(w_d^c) = \sigma \circ g^c \circ \sigma^d(a)$$

$$= \zeta^c g^c \circ \sigma \circ \sigma^d(a)$$

$$= \zeta^c g^c \sigma^{d+1}(a)$$

$$= \zeta^c w_{d+1}^c,$$

so, $\delta^c = \zeta^c$. □

We use Lemma 2.2.4 to compute $x^m \cdot w_d^c$ in the following theorem.

Theorem 2.2.5 (Action of x^m on $\Bbbk Q_1$)**.**

Given $m \in \mathbb{N}$ *and* w_d^c *as defined in Lemma 2.2.4, we have that*

$$x^m \cdot w_d^c = \sum_{r=0}^{m} \sum_{s=0}^{m-r} h_{m-(r+s)} \left(\alpha_d^c, \alpha_d^{c+1}, \ldots, \alpha_d^{c+(r+s)}\right) \cdot e_s \left(\beta^c, \beta^{c+1}, \ldots, \beta^{c+s-1}\right) h_r \left(\delta^c, \delta^{c+1}, \ldots, \delta^{c+s}\right) w_{d+r}^{c+s},$$

where h_k *and* e_k *are the symmetric polynomials given in Definition 1.3.1.*

Proof. We prove the result by inducting on m.

For $m = 1$, we have from Lemma 2.2.4 that $x \cdot w_d^c = \alpha_d^c w_d^c + \beta^c w_d^{c+1} + \delta_d^c w_{d+1}^c$, so we obtain:

$$\begin{aligned} x \cdot w_d^c &= \alpha_d^c w_d^c + \beta^c w_d^{c+1} + \delta^c w_{d+1}^c \\ &= h_1(\alpha_d^c) e_0(\beta^c) h_0(\delta^c) w_d^c + h_0(\alpha_d^c, \alpha_d^{c+1}) e_1(\beta^c) h_0(\delta^c, \delta^{c+1}) w_{d+1}^c \\ &\quad + h_0(\alpha_d^c, \alpha_d^{c+1}) e_0(\beta^c) h_1(\delta^c) w_{d+1}^c. \end{aligned}$$

Therefore, the result holds for $m = 1$.

Assume the result holds for $j \leq m$. Then

$$\begin{aligned} x^{m+1} \cdot w_d^c &= x \cdot (x^m \cdot w_d^c) \\ &= x \cdot \sum_{r=0}^{m} \sum_{s=0}^{m-r} h_{m-(r+s)} \left(\alpha_d^c, \ldots, \alpha_d^{c+(r+s)}\right) e_s \left(\beta^c, \ldots, \beta^{c+s-1}\right) h_r \left(\delta^c, \ldots, \delta^{c+s}\right) w_{d+r}^{c+s} \end{aligned}$$

by the inductive hypothesis

$$= \sum_{r=0}^{m} \sum_{s=0}^{m-r} h_{m-(r+s)} \left(\alpha_d^c, \ldots, \alpha_d^{c+(r+s)}\right) e_s \left(\beta^c, \ldots, \beta^{c+s-1}\right) h_r \left(\delta^c, \ldots, \delta^{c+s}\right)$$

$$\cdot \left[\alpha_d^{c+(r+s)} w_{d+r}^{c+s} + \beta^{c+s} w_{d+r}^{c+s+1} + \delta^{c+s} w_{d+r+1}^{c+s}\right].$$

After we distribute, combine like terms, and reindex, we can rewrite the above expression so that we have a coefficient and a single term w_{d+r}^{c+s}:

$$= \sum_{r=0}^{m+1} \sum_{s=0}^{m+1-r} \left[h_{m-(r+s)} \left(\alpha_d^c, \ldots, \alpha_d^{c+(r+s)}\right) \alpha_d^{c+(r+s)} e_s \left(\beta^c, \ldots, \beta^{c+s-1}\right) h_r \left(\delta^c, \ldots, \delta^{c+s}\right)\right.$$

$$+ h_{m-(r+(s-1))} \left(\alpha_d^c, \ldots, \alpha_d^{c+(r+(s-1))}\right) e_{s-1} \left(\beta^c, \ldots, \beta^{c+(s-1)-1}\right) \beta^{c+(s-1)} h_r \left(\delta^c, \ldots, \delta^{c+(s-1)}\right)$$

$$\left.+ h_{m-((r-1)+s)} \left(\alpha_d^c, \ldots, \alpha_d^{c+((r-1)+s)}\right) e_s \left(\beta^c, \ldots, \beta^{c+s-1}\right) h_{r-1} \left(\delta^c, \ldots, \delta^{c+s}\right) \delta^{c+s}\right] w_{d+r}^{c+s}.$$

Notice that a few terms are actually equal:

$$h_{m-(r+(s-1))} \left(\alpha_d^c, \ldots, \alpha_d^{c+(r+(s-1))}\right) = h_{m-((r-1)+s)} \left(\alpha_d^c, \ldots, \alpha_d^{c+((r-1)+s)}\right) \text{ and}$$

$$e_{s-1} \left(\beta^c, \ldots, \beta^{c+(s-1)-1}\right) \beta^{c+(s-1)} = e_s \left(\beta^c, \ldots, \beta^{c+s-1}\right),$$

so we can factor $e_s(\beta^c, \ldots, \beta^{c+s-1})$ from the coefficient of w_{d+r}^{c+s} to obtain:

$$= \sum_{r=0}^{m+1} \sum_{s=0}^{m+1-r} e_s \left(\beta^c, \ldots, \beta^{c+s-1}\right) \left[h_{m-(r+s)} \left(\alpha_d^c, \ldots, \alpha_d^{c+(r+s)}\right) \alpha_d^{c+(r+s)} h_r \left(\delta^c, \ldots, \delta^{c+s}\right)\right.$$

$$+ h_{(m+1)-(r+s)} \left(\alpha_d^c, \ldots, \alpha_d^{c+(r+(s-1))}\right) \left(h_r \left(\delta^c, \ldots, \delta^{c+(s-1)}\right)\right.$$

$$\left.\left.+ h_{r-1} \left(\delta^c, \ldots, \delta^{c+s}\right) \delta^{c+s}\right)\right] w_{d+r}^{c+s}$$

$$= \sum_{r=0}^{m+1} \sum_{i=0}^{m+1-r} e_s \left(\beta^c, \ldots, \beta^{c+s-1}\right) \left[h_{m-(r+s)} \left(\alpha_d^c, \ldots, \alpha_d^{c+(r+s)}\right) \alpha_d^{c+(r+s)} h_r \left(\delta^c, \ldots, \delta^{c+s}\right)\right.$$

$$\left.+ h_{(m+1)-(r+s)} \left(\alpha_d^c, \ldots, \alpha_d^{c+(r+(s-1))}\right) h_r \left(\delta^c, \ldots, \delta^{c+s}\right)\right] w_{d+r}^{c+s}.$$

This last equality follows from the definition of homogenous symmetric polynomial of degree r because:

$$h_r\left(\delta^c,\ldots,\delta^{c+(s-1)}\right)+h_{r-1}\left(\delta^c,\ldots,\delta^{c+s}\right)\delta_d^{c+s}=h_r\left(\delta^c,\ldots,\delta^{c+s}\right).$$

Also, we can now factor out $h_r\left(\delta^c,\ldots,\delta^{c+s}\right)$ and see that

$$=\sum_{r=0}^{m+1}\sum_{s=0}^{m+1-r}e_s\left(\beta^c,\ldots,\beta^{c+s-1}\right)h_r\left(\delta^c,\ldots,\delta^{c+s}\right)$$
$$\cdot\left[h_{m-(r+s)}\left(\alpha_d^c,\ldots,\alpha_d^{c+(r+s)}\right)\alpha_d^{c+(r+s)}\right.$$
$$\left.+h_{(m+1)-(r+s)}\left(\alpha_d^c,\ldots,\alpha_d^{c+(r+(s-1))}\right)\right]w_{d+r}^{c+s}.$$

Once again, using the definition of homogenous symmetric polynomials of a certain degree, we can equate the following:

$$h_{m-(r+s)}\left(\alpha_d^c,\ldots,\alpha_d^{c+(r+s)}\right)\alpha_d^{c+(r+s)}+h_{(m+1)-(r+s)}\left(\alpha_d^c,\ldots,\alpha_d^{c+(k+(s-1))}\right)$$
$$=h_{(m+1)-(r+s)}\left(\alpha_d^c,\ldots,\alpha_d^{c+(r+s)}\right),$$

so we finally have that

$$x^{m+1}\cdot w_d^c=\sum_{r=0}^{(m+1)}\sum_{s=0}^{(m+1)-r}h_{(m+1)-(r+s)}\left(\alpha_d^c,\ldots,\alpha_d^{c+(r+s)}\right)\cdot$$
$$\cdot e_s\left(\beta^c,\ldots,\beta^{c+s-1}\right)h_r\left(\delta^c,\ldots,\delta^{c+s}\right)w_{d+r}^{c+s},$$

as wanted. □

We can simplify the expression for $x^m\cdot w_d^c$ by using Theorem 1.3.11.

Therefore, if $m<n$,

$$x^m\cdot w_d^c=\sum_{r=0}^{m}\sum_{s=0}^{m-r}(\alpha_d^c)^{m-(r+s)}\begin{bmatrix}m\\r+s\end{bmatrix}_\zeta(\beta^c)^s\zeta^{(s-1)s/2}\zeta^{cr}\begin{bmatrix}r+s\\r\end{bmatrix}_\zeta w_{d+r}^{c+s},$$

so substituting α_d^c and β^c with their respective definitions and simplifying we have

$$x^m \cdot w_d^c = \sum_{r=0}^{m} \sum_{s=0}^{m-r} \gamma_-^{m-(r+s)} \gamma_+^s (-1)^s \frac{(\zeta^{i+1})^s \zeta^{(s-1)s/2}}{(\zeta^{j+d})^{r+s}} \begin{bmatrix} m \\ r+s \end{bmatrix}_\zeta \begin{bmatrix} r+s \\ r \end{bmatrix}_\zeta w_{d+r}^{c+s}. \tag{2.5}$$

This simplified expression for $x^m \cdot w_d^c$ allows us to compute the action of the n-th power of $x \in T(n)$ on $\Bbbk Q_1$.

Lemma 2.2.6 (Action of x^n on $\Bbbk Q_1$)**.**

The relation $x^n = 0$ imposes one extra condition on the map σ from the action of x on $a \in V_j^i$. If $i \notin \operatorname{orb}_g(j)$, then

$$\sigma^n = (\gamma_+^n - \gamma_-^n)I,$$

where I is the identity map, and γ_+ and γ_- come from $x \cdot e_i$ and $x \cdot e_j$ respectively as given in Theorem 1.4.4.

Furthermore, if $i \in \operatorname{orb}_g(j)$, then $\sigma^n = 0$.

Proof. First, observe that if $i \notin \operatorname{orb}_g(j)$, then $\sigma^n : V_j^i \to V_{j+n}^i = V_j^i$, i.e. σ^n acts by an endomorphism of the arrow space V_j^i.

Suppose $i \notin \operatorname{orb}_g(j)$. By Lemma 1.3.10, if $m = n$, the terms in equation (2.5) are 0 for all pairs (r, s) with $0 < r + s < n$, so the only remaining terms are those for which $(r, s) \in \{(0, 0), (0, n), (n, 0)\}$. Therefore, equation (2.5) simplifies to

$$x^n \cdot w_d^c = \gamma_-^n w_d^c + \gamma_+^n (-1)^n \zeta^{(n-1)n/2} w_d^{c+n} + w_{d+n}^c. \tag{2.6}$$

Now, $w_d^{c+n} = w_d^c$ since $g^n = 1$. Also, by Lemma 1.3.9 we may rewrite $(-1)^n \zeta^{(n-1)n/2} = (-1)^n(-1)^{n+1} = (-1)$. Lastly, $w_{d+n}^c = \sigma^n(w_d^c)$ since $\zeta^{-nc} = 1$. Therefore, we may further simplify Equation (2.6) to

$$x^n \cdot w_d^c = \Big((\gamma_-^n - \gamma_+^n)I + \sigma^n\Big)w_d^c. \tag{2.7}$$

Hence, since $x^n = 0$, we see that

$$\Big((\gamma_-^n - \gamma_+^n)I + \sigma^n\Big)w_d^c = 0$$

$$\implies (\gamma_+^n - \gamma_-^n)I = \sigma^n.$$

If we now suppose $i \in \text{orb}_g(j)$, then $\gamma_- = \gamma_+ = \gamma$, so this time, equation (2.6) simplifies to

$$x^n \cdot w_d^c = \sigma^n(w_d^c)$$

and hence $x^n = 0$ implies that $\sigma^n = 0$. □

This completes the classification of the possible Taft actions on a path algebra of a quiver Q. We summarize the results below for a quick reference.

2.3 Summary of $T(n)$ Action on $\Bbbk Q$

1. The action of g on each primitive idempotent $e_i \in \Bbbk Q_0$ is given by a choice of a permutation element of $\rho \in S_{|Q_0|}$ of order $m \mid n$.

2. Given that ρ has been chosen, the action of g on $\Bbbk Q_1$ is given by any choice of an invertible linear automorphism of $\Bbbk Q$, g, such that $g_{|V_j^i} : V_j^i \to V_{\rho(j)}^{\rho(i)}$.

3. Given the map ρ, the action of x on the primitive idempotent e_i is given by $\gamma_i(e_i - \zeta e_{\rho(i)})$ where γ_i can be any scalar in $\Bbbk$ if $|\operatorname{orb}_\rho(i)| = n$, and $\gamma_i = 0$ if $|\operatorname{orb}_\rho(i)| < n$. Furthermore, $\gamma_i = \gamma_{\rho^k(i)}$ for any $k \in \mathbb{Z}$.

4. Given the maps ρ and g and an appropriate choice of scalar γ_i for each representative e_i of a ρ orbit, the action of x on an element $a \in V_j^i$ is given by $\gamma_j \zeta^j a - \gamma_i \zeta^{i+1} g(a) + \sigma(a)$ such that σ is any linear endomorphism of $\Bbbk Q_0 \oplus \Bbbk Q_1$ with

   - $\sigma_{|\Bbbk Q_0} = 0$;
   - $\sigma(a) \in \begin{cases} V_{g\cdot j}^i, & \text{if} \quad \rho(j) \neq i \\ e_i \Bbbk Q e_i \oplus V_i^i, & \text{if} \quad \rho(j) = i. \end{cases}$
   - $\sigma \circ g = \zeta g \circ \sigma$;
   - $\sigma^n = (\gamma_i^n - \gamma_j^n) I_{V_j^i}$, where $I_{V_j^i}$ is the identity on V_j^i.

CHAPTER 3
THE INVARIANT RING OF THE ACTION

Now that we have a complete classification of the actions of $T(n)$ on the path algebra $\Bbbk Q$ of an arbitrary quiver Q, which preserve the ascending filtration by path length, we move on to understanding the invariant subring of a given action.

In this section, we restrict our attention to Type A quivers, i.e. quivers in which the action of $T(n)$ on a complete set of primitive idempotents of $\Bbbk Q_0$ is transitive. We begin by providing a few definitions and known theorems to familiarize the reader with Hopf Invariants and some useful combinatorics.

We follow the notation and definitions provided in [8] and use results found in [11].

3.1 Background and Definitions

Definition 3.1.1 (Invariant ring)**.**

The ring of invariants of a Hopf algebra, H, acting on an algebra, A, is the set:

$$A^H = \{a \in A \mid h \cdot a = \varepsilon(h)a\},$$

where ε is the counit of the Hopf algebra, H.

In the case $H = T(n)$, and $A = \Bbbk Q$, then

$$\Bbbk Q^{T(n)} = \{p \in \Bbbk Q \mid \ \ x \cdot p = \varepsilon(x)p = 0, \ \ g \cdot p = \varepsilon(g)p = p\}.$$

Definition 3.1.2 (Hopf integrals)**.**

The set of left integrals of a Hopf algebra is defined by:

$$\int_H^l := \{\omega \in H \mid h\omega = \varepsilon(h)\omega \ \textit{for all} \ h \in H\}.$$

Definition 3.1.3 (Weak compositions)**.**

For any two integers k, l, let $WC(k, l)$ be the set of weak compositions of k of length l, i.e.

$$WC(k,l) = \left\{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \,\middle|\, \sum_{i=1}^{l} \lambda_i = k \ \textit{and} \ \lambda_i \geq 0 \ \textit{for all} \ i\right\}.$$

Given a $\lambda \in WC(k, l)$, we write $|\lambda| = k$.

Theorem 3.1.4 (Hopf integrals generate invariants)**.**

If H is a finite dimensional Hopf algebra, $0 \neq \omega \in H$ is a left integral, and A is a left H-module algebra, the map

$$tr: \ A \to A^H, \quad a \mapsto \omega \cdot a$$

is a morphism of A^H-bimodules. The map tr is called the trace function, and if $\omega \cdot a = 1$ for some $a \in A$, then tr is surjective.

This result is stated as Corollary 6.2.17 in [4]. The proof that the trace is a morphism of A^H-bimodules follows from Proposition 6.2.16 in [4]. We prove below that $tr(A) \subseteq A^H$ and that if $\omega \cdot a = 1$ for some $a \in A$ then tr is surjective.

Proof. Suppose the Hopf algebra H acts on an algebra A. Let $a \in A$, and $\omega \in \int_H^l$ be a left Hopf integral of H. Then for any $h \in H$,

$$\begin{aligned} h \cdot (\omega \cdot a) &= (h\omega) \cdot a \\ &= (\varepsilon(h)\omega) \cdot a \\ &= \varepsilon(h)(\omega \cdot a) \\ \Longrightarrow \omega \cdot a &\in A^H && \text{by definition of } A^H \end{aligned}$$

Therefore, $tr(A) \subset A^H$.

Since tr is a morphism of A^H-bimodules, then if for some $a \in A$, $tr(a) = 1_A$, then given any $b \in A^H$ we have that

$$tr(ab) = tr(a)b = (1_A)b = b,$$

so tr is surjective. □

3.2 Description of Invariant Ring

The map tr is extremely useful in finding invariants of the ring, especially when it is surjective. We will use tr to find invariants in the case that $\gamma \neq 0$, so in particular G acts faithfully on the set of vertices of Q, i.e. $|Q_0| = n$. Before doing so, however, we compute the invariant ring in the case that $\gamma = 0$.

Proposition 3.2.1 ($\Bbbk Q^{T(n)}$ when $\gamma = 0$)**.**

Let $T(n)$ *act on* $\Bbbk Q$ *with* $\gamma = 0$*. Let* $m = |Q_0|$*, and* $c \in \mathbb{Z}_{\geq 1}$ *is such that* $cm = n$*. Then*

$$\Bbbk Q^{T(n)} = \ker\left(\left(1 + (\zeta g) + (\zeta g)^2 + \ldots + (\zeta g)^{(c-1)m}\right) \sigma \right).$$

In particular, if the action of G *is not faithful on* Q_0 *then* $\gamma = 0$*.*

Proof. If $\gamma = 0$, then by Theorem 2.2.3, x acts by σ. Therefore, since $\Bbbk Q^{T(n)} = \{p \in \Bbbk Q \mid g(p) = p \text{ and } x \cdot p = 0\}$, we see that $\Bbbk Q^{T(n)} = \Bbbk Q^G \cap \ker(\sigma)$.

Given an element $p \in \Bbbk Q$, we define

$$\overline{p} := p + g(p) + g^2(p) + \ldots + g^{n-1}(p).$$

It is clear that $\Bbbk Q^G = \langle \overline{p} \mid p \in \Bbbk Q \rangle$. In order for $\overline{p}$ to be a $T(n)$-invariant, we would need $\sigma(\overline{p}) = 0$. By the linearity of σ and Lemma 2.2.2, we have that

$$\begin{aligned}
\sigma(\overline{p}) &= \sigma\left(p + g(p) + g^2(p) + \ldots + g^{n-1}(p)\right) \\
&= \sigma(p) + \zeta g \sigma(p) + \zeta^2 g^2 \sigma(p) + \ldots + \zeta^{n-1} g^{n-1} \sigma(p) \\
&= \left((\zeta g) + (\zeta g)^2 + \ldots + (\zeta g)^{n-1} + (\zeta g)^n\right) \sigma(p) \\
&= \left(\sum_{k=1}^{m} (\zeta g)^k\right) \left(1 + (\zeta g)^m + \ldots + (\zeta g)^{(c-1)m}\right) \sigma(p). \qquad (3.1)
\end{aligned}$$

Since m is the smallest positive integer for which $g^m(e_i) = e_i$, then $\{(g)^k(q)\}_{k=1}^m$ is a set of linearly indendent elements of Q_1. Therefore, for any $0 \neq q \in \Bbbk Q_1$ we have that $\left(\sum_{k=1}^{m} (\zeta g)^k\right)(q) \neq 0$.

It follows that $\sigma(\overline{p}) = 0$ whenever $\left(1 + (\zeta g)^m + \ldots (\zeta g)^{(c-1)m}\right)\sigma(p) = 0.$ □

Assumption 1. *For the duration of this thesis, we assume $\gamma \neq 0$, so that G acts transitively and faithfully on the set of vertices of Q, and hence, $|Q_0| = n$.*

It is a well known result by Larson and Sweedler [10] that any finite dimensional Hopf algebra has a non-zero left integral. The lemma below states an explicit left integral of $T(n)$ and shows that it is onto $\Bbbk Q^{T(n)}$ in the case that G acts faithfully on the set Q_0.

Lemma 3.2.2 (Taft integral)**.**

Given the Hopf algebra $H = T(n)$, then $\omega = (1 + g + \ldots + g^{n-1})x^{n-1}$ is a left Hopf integral. Furthermore, the map $tr : p \to \omega \cdot p$ is surjective onto $\Bbbk Q^{T(n)}$.

Proof. To see that ω is a left integral, it suffices to show that $g\omega = \varepsilon(g)\omega = \omega$ and $x\omega = \varepsilon(x)\omega = 0$, since $T(n)$ is generated by x and g. So we have:

$$\begin{aligned} g\omega &= g(1 + g + \ldots + g^{n-1})x^{n-1} \\ &= (g + \ldots + g^{n-1} + g^n)x^{n-1} \\ &= (g + \ldots + g^{n-1} + 1)x^{n-1} \\ &= \omega, \qquad \text{and} \\ x\omega &= x(1 + g + \ldots + g^{n-1})x^{n-1} \\ &= (1 + \zeta g + \zeta^2 g^2 + \ldots + \zeta^{n-1}g^{n-1})x^n \\ &= 0, \qquad \text{since } x^n = 0. \end{aligned}$$

Therefore, ω is a left integral of $T(n)$.

Let $\{e_1, \dots, e_n\}$ be a set of complete primitive orthogonal idempotents of $\Bbbk Q_0$, with $g \cdot e_i = e_{i+1}$ with the subscripts taken modulo n. In chapter 1, we provided in equation (1.4) an explicit formula for an arbitrary power of x acting on an idempotent e_i. Therefore, for $\alpha = 1/\left(\zeta\gamma - \zeta^2\gamma\right)_\zeta^{n-1} \in \Bbbk$ we have:

$$\begin{aligned}
tr(\alpha e_1) &= (1 + g + \ldots + g^{n-1})x^{n-1}(\alpha e_1) \\
&= (1 + g + \ldots + g^{n-1}) \left(\gamma^{n-1} \sum_{j=0}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_\zeta (-1)^j \zeta^{n-1+j} \zeta^{j(j-1)/2} \alpha e_{1+j} \right) \\
&= \gamma^{n-1} \sum_{i=1}^{n} \sum_{j=0}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_\zeta (-1)^j \zeta^{j-1} \zeta^{j(j-1)/2} \alpha e_{i+j},
\end{aligned}$$

so if we collect coefficients by summing over all $i + j \equiv s \mod n$ for $1 \le s \le n$, then we may rewrite the above equation as:

$$\begin{aligned}
tr(\alpha e_1) &= \alpha\gamma^{n-1} \sum_{s=1}^{n} \sum_{j=0}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_\zeta (-1)^j \zeta^{j-1} \zeta^{j(j-1)/2} e_s \\
&= \alpha\gamma^{n-1} \left(\sum_{j=0}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_\zeta (-1)^j \zeta^{j-1} \zeta^{j(j-1)/2} \right) \left(\sum_{s=1}^{n} e_s \right) \\
&= \alpha\gamma^{n-1}\zeta^{-1} \left(\sum_{j=0}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_\zeta \zeta^{j(j-1)/2} (-\zeta)^j (1)^{n-1-j} \right) (1_{\Bbbk Q}) \\
&= \alpha \left(\zeta\gamma\right)^{n-1} (1 - \zeta)_\zeta^{n-1} (1_{\Bbbk Q}) \\
&= \alpha \left(\zeta\gamma - \zeta^2\gamma\right)_\zeta^{n-1} (1_{\Bbbk Q}) \\
&= 1_{\Bbbk Q}.
\end{aligned}$$

Therefore, tr is surjective, as wanted. □

The goal now is to describe the image, $tr(\Bbbk Q)$ in terms of a generating set $\mathcal{O} \subsetneq \Bbbk Q$, so that $tr(\Bbbk Q) = tr(\mathcal{O})$.

The image of the map tr is spanned by the set of elements of the form $\omega \cdot p$ for an arbitrary path $p = a_1 a_2 \cdots a_l$ of length l in $\Bbbk Q$. By definition of Hopf action provided in Definition 1.1.4, we have:

$$\begin{aligned}
\omega \cdot p &= \left((1 + g + \ldots + g^{n-1})x^{n-1}\right) \cdot p \\
&= (1 + g + \ldots + g^{n-1}) \cdot (x^{n-1} \cdot p) \\
&= (1 + g + \ldots + g^{n-1}) \cdot (x^{n-1} \cdot (a_1 a_2 a_3 \cdots a_l)).
\end{aligned}$$

In Chapter 1, Theorem 2.2.5 provides us with a description for an arbitrary power of x acting on a path of length 1. Now we need a description for an arbitrary power, k, of x acting on a path of length l, for $l > 1$. In other words, we need a description of $x^k \cdot (a_1 a_2 \cdots a_l)$.

We know that $\Bbbk Q$ is a left $T(n)$-module algebra, so $x^k \cdot (a_1 a_2 \cdots a_l)$ is well defined, and by the definition of a Hopf action in 1.1.4, and the co-associativity of Δ,

$$x^k \cdot (a_1 a_2 \cdots a_l) = \sum_{(x^k)} \left[(x^k)^{(1)} \cdot (a_1)\right] \left[(x^k)^{(2)} \cdot (a_2)\right] \cdots \left[(x^k)^{(l)} \cdot (a_l)\right],$$

where $\Delta^{l-1}(x^k) = \sum_{(x^k)} (x^k)^{(1)} \otimes (x^k)^{(2)} \otimes \cdots \otimes (x^k)^{(l)}$.

Our strategy will be to write $\Delta^{l-1}(x^k) = \sum_{(x^k)} (x^k)^{(1)} \otimes (x^k)^{(2)} \otimes \cdots \otimes (x^k)^{(l)}$ as an element of $T(n)^{\otimes l}$, and identifying $\Bbbk Q_l$ with $\Bbbk Q_1^{\otimes l}$ so we can use the action of $T(n)$ on $\Bbbk Q_1$ and extend it to $T(n)^{\otimes l}$ acting on $\Bbbk Q_1^{\otimes l}$ by acting on each factor in the natural way.

For simplicity, we omit the tensor notation in $\Bbbk Q_l$ and write $(a_1 a_2 \cdots a_l)$ for $(a_1 \otimes a_2 \otimes \cdots \otimes a_l)$.

Theorem 3.2.3 ($\Delta^{l-1}(x^k)$ and $\Delta^{l-1}(g^k)$ as elements of $T(n)^{\otimes l}$**).**

Given any two integers $l, k \geq 1$, the map $\Delta^{l-1} : T(n) \to T(n)^{\otimes l}$ is given by:

$$\Delta^{l-1}(g^k) = (g^k)^{\otimes l} \quad \textit{and}$$

$$\Delta^{l-1}(x^k) = \sum_{\substack{\lambda \in WC \\ (k,l)}} C_\lambda \, x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes g^{\lambda_1+\lambda_2} x^{\lambda_3} \otimes \ldots \otimes g^{\sum_{i=1}^{l-1} \lambda_i} x^{\lambda_l}$$

where $C_\lambda = \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_\zeta$ is the q-Multinomial Coefficient evaluated at $q = \zeta$.

Proof. The fact that $\Delta^l(g^k) = (g^k)^{\otimes l}$ follows pretty much immediately since

$$\begin{aligned} \Delta^l(g^k) &= (\Delta^l(g))^k && \text{because } \Delta \text{ is a morphism of algebras} \\ &= (g^{\otimes l})^k && \text{because } \Delta(g) = g \otimes g \\ &= (g^k)^{\otimes l} && \text{by definition of multiplication in } T(n)^{\otimes l}. \end{aligned}$$

We prove the result for $\Delta^{l-1}(x^k)$ by double induction on the tuple $(k, l) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.

For the tuple $(k, 1)$ the result holds, since $\lambda = (k)$ is the only partition of k of length 1: on one hand, $\Delta^{1-1}(x^k) = \Delta^0 x^k = x^k$, and on the other hand,

$$\sum_{\substack{\lambda \in WC \\ (k,l)}} C_\lambda \, x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes g^{\lambda_1+\lambda_2} x^{\lambda_3} \otimes \ldots \otimes g^{\sum_{i=1}^{l-1} \lambda_i} x^{\lambda_l}$$
$$= \sum_{\lambda = (k)} C_{(k)} \, x^k = \left\langle \begin{matrix} |k| \\ k \end{matrix} \right\rangle x^k = x^k.$$

In particular, the result holds for the tuple $(1, 1)$, which we take as our base case in our first inductive step.

Before proceeding with our induction, observe that since Δ is a morphism of algebras, then for any positive integer λ, $\Delta(x^\lambda) = \Delta(x)^\lambda = (1 \otimes x + x \otimes g)^\lambda$. Furthermore,

$$(1 \otimes x)(x \otimes g) = (x \otimes xg) = (x \otimes \zeta gx) = \zeta(x \otimes gx) = \zeta(x \otimes g)(1 \otimes x).$$

It follows from the q-Binomial Formula in Theorem 1.3.6, that:

$$\begin{aligned} \Delta(x^\lambda) = (1 \otimes x + x \otimes g)^\lambda &= \sum_{i=0}^{\lambda} \begin{bmatrix} \lambda \\ i \end{bmatrix}_\zeta (x \otimes g)^i (1 \otimes x)^{\lambda - i} \\ &= \sum_{i=0}^{\lambda} \begin{bmatrix} \lambda \\ i \end{bmatrix}_\zeta x^i \otimes g^i x^{\lambda - i}. \end{aligned} \tag{3.2}$$

Now, suppose for our first inductive hypothesis that the result holds for all tuples $(1, j)$ with $1 \leq j < l$ for some fixed integer l. Then,

$$\begin{aligned} \Delta^{l-1}(x) &= \left(id_{T(n)^{\otimes l-2}} \otimes \Delta\right) \circ \Delta^{l-2}(x) \qquad \text{by the co-associativity of } \Delta \\ &= \left(id_{T(n)^{\otimes l-2}} \otimes \Delta\right) \left(\sum_{\substack{\lambda \in WC \\ (1,l-1)}} C_\lambda \, x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes g^{\lambda_1 + \lambda_2} x^{\lambda_3} \otimes \ldots \otimes g^{\sum_{i=1}^{l-2} \lambda_i} x^{\lambda_{l-1}} \right) \\ &\qquad \text{by the inductive hypothesis} \\ &= \sum_{\substack{\lambda \in WC \\ (1,l-1)}} C_\lambda \, x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes g^{\lambda_1 + \lambda_2} x^{\lambda_3} \otimes \ldots \otimes \Delta\left(g^{\sum_{i=1}^{l-2} \lambda_i} \right) \Delta\left(x^{\lambda_{l-1}}\right) \\ &= \sum_{\substack{\lambda \in WC \\ (1,l-1)}} C_\lambda \, x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes \ldots \otimes \left(g^{\sum_{i=1}^{l-2} \lambda_i} \otimes g^{\sum_{i=1}^{l-2} \lambda_i} \right) \left(\sum_{j=0}^{\lambda_{l-1}} \begin{bmatrix} \lambda_{l-1} \\ j \end{bmatrix}_\zeta x^j \otimes g^j x^{\lambda_{l-1} - j} \right) \\ &\qquad \text{by equation (3.2)} \\ &= \sum_{\substack{\lambda \in WC \\ (1,l-1)}} C_\lambda \, x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes \ldots \otimes \left(\sum_{j=0}^{\lambda_{l-1}} \begin{bmatrix} \lambda_{l-1} \\ j \end{bmatrix}_\zeta g^{\sum_{i=1}^{l-2} \lambda_i} x^j \otimes g^{j + \sum_{i=1}^{l-2} \lambda_i} x^{\lambda_{l-1} - j} \right) \end{aligned}$$

by definition of multiplication in $T(n)^{\otimes 2}$.

Now, since $\lambda \in WC(1, l-1)$, it follows that $\lambda_{l-1} = 0$ for $l-2$ of the possible λ's, and $\lambda_{l-1} = 1$ exactly once. Notice that when $\lambda_{l-1} = 0$ then the sum $\sum_{i=1}^{l-2} \lambda_i = 1$. In particular, $g^{\sum_{i=1}^{l-2} \lambda_i} = g$.

Furthermore, $C_\lambda = 1$ for all $\lambda \in WC(1, l)$ for any integer l. We may therefore write:

$$= \sum_{\substack{\lambda \in WC \\ (1,l-1)}} x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes \ldots \otimes g^{\sum_{i=1}^{l-3} \lambda_i} x^{\lambda_{l-2}} \otimes \left(\sum_{j=0}^{\lambda_{l-1}} \begin{bmatrix} \lambda_{l-1} \\ j \end{bmatrix}_\zeta g^{\sum_{i=1}^{l-2} \lambda_i} x^j \otimes g^{j + \sum_{i=1}^{l-2} \lambda_i} x^{\lambda_{l-1} - j} \right)$$

$$= \sum_{\substack{\lambda \in WC(1,l-1) \\ \lambda_{l-1}=0}} x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes \ldots \otimes g^{\sum_{i=1}^{l-3} \lambda_i} x^{\lambda_{l-2}} \otimes g \otimes g$$

$$+ \underbrace{(1 \otimes 1 \otimes \ldots \otimes 1 \otimes 1 \otimes x)}_{\text{when } \lambda_{l-1} = 1 \text{ and } j = 0} \quad + \quad \underbrace{(1 \otimes 1 \otimes \ldots \otimes 1 \otimes x \otimes g)}_{\text{when } \lambda_{l-1} = 1 \text{ and } j = 1}$$

$$= \sum_{\substack{\lambda \in WC \\ (1,l)}} C_\lambda \, x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes \ldots \otimes g^{\sum_{i=1}^{l-2} \lambda_i} x^{\lambda_{l-1}} \otimes g^{\sum_{i=1}^{l-1} \lambda_i} x^{\lambda_l},$$

as wanted. Therefore, by the principle of mathematical induction, the result holds for all pairs $(1, j)$.

We take the result for $(1, j)$ as our base case, and assume for our second inductive hypothesis that the result holds for all pairs (i, j) with $i < k$ for a fixed integer k, and j an arbitrary integer greater than or equal to 1.

Furthermore, by considering the result for $(k, 1)$ as a second base case, we may further assume it holds for all pairs (k, j) with $j < l$ for a fixed integer l.

We now prove that the result holds for the tuple (k, l):

$$
\begin{aligned}
\Delta^{l-1}(x^k) &= \left(id_{T(n)^{\otimes l-2}} \otimes \Delta\right) \circ \Delta^{l-2}\left(x^k\right) \qquad \text{by the co-associativity of } \Delta \\
&= \left(id_{T(n)^{\otimes l-2}} \otimes \Delta\right) \left(\sum_{\substack{\lambda \in WC \\ (k,l-1)}} C_\lambda \, x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes \ldots \otimes g^{\sum_{i=1}^{l-2} \lambda_i} x^{\lambda_{l-1}} \right) \\
&= \sum_{\substack{\lambda \in WC \\ (k,l-1)}} C_\lambda \; x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes \ldots \otimes g^{\sum_{i=1}^{l-3} \lambda_i} x^{\lambda_{l-2}} \otimes \left(\sum_{j=0}^{\lambda_{l-1}} \begin{bmatrix} \lambda_{l-1} \\ j \end{bmatrix}_\zeta g^{\sum_{i=1}^{l-2} \lambda_i} x^j \otimes g^{j + \sum_{i=1}^{l-2} \lambda_i} x^{\lambda_{l-1}-j} \right) \\
&= \sum_{j=0}^{\lambda_{l-1}} \sum_{\substack{\lambda \in WC \\ (k,l-1)}} \left\langle \begin{matrix} k \\ \lambda \end{matrix} \right\rangle_\zeta \begin{bmatrix} \lambda_{l-1} \\ j \end{bmatrix}_\zeta \; x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes \ldots \otimes g^{\sum_{i=1}^{l-3} \lambda_i} x^{\lambda_{l-2}} \otimes g^{\sum_{i=1}^{l-2} \lambda_i} x^j \otimes g^{j + \sum_{i=1}^{l-2} \lambda_i} x^{\lambda_{l-1}-j}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\left\langle \begin{matrix} k \\ \lambda \end{matrix} \right\rangle_\zeta \begin{bmatrix} \lambda_{l-1} \\ j \end{bmatrix}_\zeta &= \frac{[k]_\zeta!}{[\lambda_1]_\zeta! [\lambda_2]_\zeta! \cdots [\lambda_{l-1}]_\zeta!} \cdot \frac{[\lambda_{l-1}]_\zeta!}{[j]_\zeta! [\lambda_{l-1} - j]_\zeta!} \\
&= \frac{[k]_\zeta!}{[\lambda_1]_\zeta! [\lambda_2]_\zeta! \cdots [\lambda_{l-2}]_\zeta! [\lambda_{l-1} - j]_\zeta! [j]_\zeta!} \\
&= C_\mu,
\end{aligned}
$$

where $\mu = (\lambda_1, \lambda_2, \ldots, \lambda_{l-2}, j, \lambda_{l-1} - j)$. Also, since λ_{l-1} varies between 0 and k and j varies between 0 and λ_{l-1}, it follows that j varies between 0 and k. We are therefore summing over all $\mu \in WC(k, l)$ since we split up λ_{l-1} into the sum of two terms: j and $\lambda_{l-1} - j$.

Therefore, we re-index the last double sum by letting

$$
\begin{aligned}
&\lambda_i = \mu_i \quad \text{for } 1 \le i \le l-2, \\
&j = \mu_{l-1} \quad \text{and,} \\
&\lambda_{l-1} - j = \mu_l,
\end{aligned}
$$

and we obtain

$$\sum_{j=0}^{\lambda_{l-1}} \sum_{\substack{\lambda \in WC \\ (k,l-1)}} \left\langle \begin{matrix} k \\ \lambda \end{matrix} \right\rangle_\zeta \begin{bmatrix} \lambda_{l-1} \\ j \end{bmatrix}_\zeta x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes \ldots \otimes g^{\sum_{i=1}^{l-3} \lambda_i} x^{\lambda_{l-2}} \otimes g^{\sum_{i=1}^{l-2} \lambda_i} x^j \otimes g^{j+\sum_{i=1}^{l-2} \lambda_i} x^{\lambda_{l-1}-j}$$

$$= \sum_{\substack{\mu \in WC \\ (k,l)}} C_\mu \; x^{\mu_1} \otimes g^{\mu_1} x^{\mu_2} \otimes \ldots \otimes g^{\sum_{i=1}^{l-2} \mu_i} x^{\mu_{l-1}} \otimes g^{\sum_{i=1}^{l-1} \mu_i} x^{\mu_l}.$$

We conclude by the principle of mathematical induction on the tuple (k, l) that

$$\Delta^{l-1}(x^k) = \sum_{\substack{\lambda \in WC \\ (k,l)}} C_\lambda \, x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes \ldots \otimes g^{\sum_{i=1}^{l-1} \lambda_i} x^{\lambda_l},$$

as wanted. □

Corollary 3.2.4 tells us how we apply Theorem 3.2.3 to the Taft integral $\omega = (1 + g + \cdots + g^{n-1})x^{n-1}$ on an arbitrary path p of length l.

Corollary 3.2.4 ($\Delta(\omega)$ as a map on $\Bbbk Q_l$)**.**

If $p = a_1 a_2 a_3 \cdots a_l \in \Bbbk Q_l$ *is a path of length* l, *and* $\omega = (1 + g + \ldots + g^{n-1})x^{n-1}$, *then*

$$\omega \cdot p = \sum_{k=0}^{n-1} \sum_{\substack{\lambda \in WC \\ (n-1,l)}} C_\lambda \prod_{m=1}^{l} g^{\sum_{i=1}^{m-1} \lambda_i + k} x^{\lambda_m} \cdot a_m.$$

Proof. The corollary follows since:

$$\Delta^{l-1}(\omega) = \Delta^{l-1}\left((1 + g + \ldots + g^{n-1})\, x^{n-1}\right)$$

$$= \Delta^{l-1}\left(x^{n-1} + gx^{n-1} + g^2 x^{n-1} + \ldots + g^{n-1} x^{n-1}\right)$$

$$
\begin{aligned}
&= \Delta^{l-1}\left(\sum_{k=0}^{n-1} g^k x^{n-1}\right) \\
&= \sum_{k=0}^{n-1} \Delta^{l-1}(g^k)\Delta^{l-1}(x^{n-1}) \\
&= \sum_{k=0}^{n-1} \left((g^k)^{\otimes l}\right) \left(\sum_{\substack{\lambda \in WC \\ (n-1,l)}} C_\lambda \, x^{\lambda_1} \otimes g^{\lambda_1} x^{\lambda_2} \otimes \cdots \otimes g^{\sum_{i=1}^{l-1} \lambda_i} x^{\lambda_l} \right)
\end{aligned}
$$

by Theorem 3.2.3

$$
= \sum_{k=0}^{n-1} \left(\sum_{\substack{\lambda \in WC \\ (n-1,l)}} C_\lambda \, g^k x^{\lambda_1} \otimes g^{\lambda_1 + k} x^{\lambda_2} \otimes \cdots \otimes g^{\sum_{i=1}^{l-1} \lambda_i + k} x^{\lambda_l} \right)
$$

$$
\begin{aligned}
\Longrightarrow \omega \cdot p &= \sum_{k=0}^{n-1} \sum_{\substack{\lambda \in WC \\ (n-1,l)}} C_\lambda \, (g^k x^{\lambda_1} \cdot a_1)(g^{\lambda_1 + k} x^{\lambda_2} \cdot a_2) \cdots (g^{\sum_{i=1}^{l-1} \lambda_i + k} x^{\lambda_l} \cdot a_l) \\
&= \sum_{k=0}^{n-1} \sum_{\substack{\lambda \in WC \\ (n-1,l)}} C_\lambda \prod_{m=1}^{l} g^{\sum_{i=1}^{m-1} \lambda_i + k} x^{\lambda_m} \cdot a_m.
\end{aligned}
$$

□

Recall that for a given $a \in \Bbbk Q$ we defined $w_d^c := g^c \sigma^d(a)$ in Lemma 2.2.4. To generalize this notation to a path $p = a_1 a_2 \cdots a_l$, define

$$
{}^m w_{d_m}^{c_m} := g^{c_m} \sigma^{d_m}(a_m) \quad \text{for } \ 1 \le m \le l.
$$

We use this new notation and Theorem 2.2.5 to rewrite the action of an arbitrary power, λ_m of x acting on the generalized element ${}^m w_{d_m}^{c_m}$. Then we will substitute ${}^m w_{d_m}^{c_m}$ for a_m in Corollary 3.2.4 to obtain a more general expression.

$$x^{\lambda_m} \cdot {}^m w_{d_m}^{c_m} = \sum_{r=0}^{\lambda_m} \sum_{s=0}^{\lambda_m - r} h_{\lambda_m-(r+s)}(\alpha_{d_m}^{c_m}, \dots, \alpha_{d_m}^{c_m+r+s})$$

$$\cdot\, e_s(\beta^{c_m}, \dots, \beta^{c_m+s-1}) h_r(\delta^{c_m}, \dots, \delta^{c_m+s}) {}^m w_{d_m+r}^{c_m+s} \quad (3.3)$$

where $\alpha_{d_m}^{c_m} := \gamma\zeta^{t(a_m)+c_m+d_m}$, $\beta^{c_m} := -\gamma\zeta^{s(a_m)+c_m+1}$ and $\delta^{c_m} := \zeta^{c_m}$.

For simplicity, define

$$f_{\lambda_m}(r, s, {}^m w_{d_m}^{c_m}) := h_{\lambda_m-(r+s)}(\alpha_{d_m}^{c_m}, \dots, \alpha_{d_m}^{c_m+r+s}) e_s(\beta^{c_m}, \dots, \beta^{c_m+s-1}) h_r(\delta^{c_m}, \dots, \delta^{c_m+s}),$$

and rewrite (3.3):

$$x^{\lambda_m} \cdot {}^m w_{d_m}^{c_m} = \sum_{r=0}^{\lambda_m} \sum_{s=0}^{\lambda_m - r} f_{\lambda_m}(r, s, {}^m w_{d_m}^{c_m}) {}^m w_{d_m+r}^{c_m+s}$$

$$\implies \quad (g^k x^{\lambda_m}) \cdot {}^m w_{d_m}^{c_m} = \sum_{r=0}^{\lambda_m} \sum_{s=0}^{\lambda_m - r} f_{\lambda_m}(r, s, {}^m w_{d_m}^{c_m}) {}^m w_{d_m+r}^{c_m+s+k}. \quad (3.4)$$

Furthermore, by Theorem 1.3.11, we can express $f_{\lambda_m}(r, s, {}^m w_{d_m}^{c_m})$ in terms of q-Binomial Coefficient and the q-Multinomial Coefficient:

$$f_{\lambda_m}(r, s, {}^m w_{d_m}^{c_m}) = \left(\gamma\zeta^{t(a_m)+c_m+d_m}\right)^{\lambda_m-(r+s)} \begin{bmatrix} \lambda_m \\ r+s \end{bmatrix}_\zeta$$

$$\cdot\, (-1)^s \gamma^s (\zeta^{s(a_m)+c_m+1})^s \zeta^{s(s-1)/2} (\zeta^{c_m})^r \begin{bmatrix} s+r \\ r \end{bmatrix}. \quad (3.5)$$

Our goal is to express $\omega \cdot p$ for an arbitrary path p of length l in terms of the maps g and σ from the $T(n)$ action. Theorem 3.2.10 gets us much closer to that goal, but first, we introduce some new notation and definitions.

Definition 3.2.5 (The path p_μ).

Let $p = a_1a_2\cdots a_l \in \Bbbk Q$ be a path of length l, with source and target given by $s(p) = s(a_1)$ and $t(p) = t(a_l)$, respectively.

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_l) \in WC(k,l)$ for some integer k, $M_1 := 0$ and for $2 \leq k \leq l$, let $M_k := \sum_{j=1}^{k-1} \lambda_j$. Then define p_μ by:

$$p_\mu := \,{}^1w_{\mu_1}^{M_1}\,{}^2w_{\mu_2}^{M_2}\cdots\,{}^lw_{\mu_l}^{M_l}.$$

Notice that $p_\mu = 0$ if and only if ${}^mw_{\mu_m}^{M_m} = 0$ for some $1 \leq m \leq l$, which is true if and only if $\sigma^{\mu_m}(a_m) = 0$ for some m.

Furthermore, let $t(p) - s(p) \equiv D \mod n$, with D taken to be a representative between 1 and n. If $\mu \in WC(n-D,l)$, then

$$\begin{aligned} s(p_\mu) &= s(p) \quad \text{and} \\ t(p_\mu) &= t(p) + |\mu| \\ &\equiv t(p) - D \mod n \\ &\equiv s(p) \mod n. \end{aligned}$$

Therefore, if $\mu \in WC(n-D,l)$, then p_μ is an oriented cycle of $\Bbbk Q_l$.

Definition 3.2.6 (Linear function χ)**.**

Define the linear function $\chi : \mathbb{R}^l \to \mathbb{R}^l$ by $\chi(a_1, a_2, \ldots, a_l) = (A_1, A_2, \ldots, A_l)$ where $A_1 := 0$ and $A_m := \sum_{m=1}^{l-1} a_m$.

By identifying the set $WC(k,l)$ with a subset of $\mathbb{R}^l$, the map χ restricts to $WC(k,l)$, and with this identification, operations on $WC(k,l)$ such as addition and dot product are well defined.

Recall from Proposition 3.2.1 we denoted by $\overline{p}$ the g-invariant element $(1+g+g^2+\ldots+g^{n-1})(p)$. We again use this notation and also write $\overline{\sigma^{d_m}(a_m)}$ for the element $(1+g+g^2+\ldots+g^{n-1})({}^m w^{c_m}_{d_m})$. Furthermore, note that $\overline{g^k(p)}=\overline{p}$ for any integer k.

Lemma 3.2.7 (Morphism)**.**

If $p=a_1a_2\cdots a_l\in \Bbbk Q$ is a path of length l, then

$$\overline{p}=\overline{(a_1a_2\cdots a_l)}=\overline{a_1}\cdot\overline{a_2}\ldots\overline{a_l}$$

In other words, the map $p\mapsto\overline{p}$ is a morphism of algebras.

Proof. Let $p=a_1a_2\cdots a_l\neq 0$. Then,

$$\overline{a_1}\,\overline{a_2}\ldots\overline{a_l}=\left(\sum_{k=0}^{n-1}g^k(a_1)\right)\left(\sum_{k=0}^{n-1}g^k(a_2)\right)\cdots\left(\sum_{k=0}^{n-1}g^k(a_l)\right)$$

$$=\sum_{k=0}^{n-1}g^k(a_1)g^k(a_2)\cdots g^k(a_l)$$

because if $k\not\equiv j \mod n$, then $g^k(a_m)g^j(a_{m+1})=0$

for all $1\le m\le l-1$, since G acts freely on the vertices of Q.

$$=\sum_{k=0}^{n-1}g^k\cdot(a_1a_2\cdots a_l)=\overline{(a_1a_2\cdots a_l)}=\overline{p}.$$

□

Lemma 3.2.8 (Linear independence of elements of $\Bbbk Q$)**.**

Let $p = a_1 a_2 \cdots a_l$ *be a path of length* l *in* $\Bbbk Q$*. Then*

$$\left\{ p_\lambda \mid \lambda \in WC(k,l) \right\}_{k=0}^{n-1}$$

is a set of linearly independent elements of $\Bbbk Q$*.*

The proof of this lemma is included in the Appendix. An immediate corollary of this lemma is that

$$\left\{ \overline{p_\lambda} \mid \lambda \in WC(k,l) \right\}_{k=0}^{n-1}$$

is also a set of linearly independent elements of $\Bbbk Q$.

The fact that these elements are all linearly independent allows us to compare expressions by comparing the coefficients of each path p_μ in each expression. This will ultimately be our strategy to prove Theorem 3.2.12. In Theorem 3.2.10, we give an expression for $\omega \cdot p$ in terms of these linearly independent pieces.

The following short proposition is only used in the proof of Theorem 3.2.10.

Proposition 3.2.9 (Equivalent sets of weak compositions)**.**

The set $WC(n-1,l)$ *is equal to the union of sets given below:*

$$\bigcup_{k=0}^{n-1} \left(\bigcup_{\substack{r+s=k \\ r,k \geq 0}} \{(\lambda_1 + r, \ldots, \lambda_l + s) \mid \lambda \in WC(n-1-k,l)\} \right)$$

Proof.

$$WC(n-1,l) = \Big\{(\lambda_1,\ldots,\lambda_l) \mid \lambda_1+\ldots+\lambda_l = n-1, 0 \le \lambda_i \le n-1\Big\}$$
$$= \Big\{(\lambda_1+r,\ldots,\lambda_l+s) \mid \lambda_1+\ldots+\lambda_l = n-1-k, r+s=k, 0\le r,s\le k,$$
$$\text{and} \quad 0 \le \lambda_i \le n-1-k\Big\}_{k=0}^{n-1}$$
$$= \Big\{(\lambda_1+r,\ldots,\lambda_l+s) \mid \lambda \in WC(n-1-k,l), r+s=k, 0\le r,s\le k\Big\}_{k=0}^{n-1}$$

□

Theorem 3.2.10 (Taft integral acting on an arbitrary path $p \in \Bbbk Q_l$).

Let $p = a_1a_2\cdots a_l \neq 0$, and $t(p) - s(p) \equiv D \mod n$, where D is a representative with $1 \le D \le n$. Then

$$\omega \cdot p = \sum_{k=0}^{D-1}\left(\sum_{\substack{\lambda\in WC\\(n-1-k,l)}} \alpha_\lambda(p)\overline{p_\lambda}\right)$$

such that for $0 \le k \le D-1$ and $\lambda \in WC(n-1-k,l)$, we have that

$$\alpha_\lambda(p) := \gamma^{n-1-|\lambda|}\begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_\zeta \left(\zeta^{s(p)+D} - \zeta^{s(p)+1}\right)_\zeta^{n-1-|\lambda|}.$$

Theorem 3.2.10 gives us a description of $\omega \cdot p$ in terms the linearly independent elements, $\overline{p_\lambda} = \prod_{i=1}^{l} \overline{\sigma^{\lambda_i}(a_i)}$, of σ-degree $|\lambda|$ (see Lemma 3.2.8). This description is key in understanding the set of elements in $tr(\Bbbk Q)$.

Proof. Let $p = a_1 a_2 \cdots a_l \neq 0 \in \Bbbk Q_l$ with $t(p) - s(p) \equiv D \mod n$, and $1 \leq D \leq n$. For each $\lambda \in WC(k,l)$, define $\Lambda_1 := 0$ and for $m \geq 2$, define $\Lambda_m := \sum_{i=1}^{m-1} \lambda_i$. By Corollary 3.2.4 and equation (3.4), we have that

$$\begin{aligned}
\omega \cdot p &= \sum_{k=0}^{n-1} \left(\sum_{\substack{\lambda \in WC \\ (n-1,l)}} C_\lambda \prod_{m=1}^{l} \left(g^{\Lambda_m + k} x^{\lambda_m}(a_m) \right) \right) \\
&= \sum_{k=0}^{n-1} \left(\sum_{\substack{\lambda \in WC \\ (n-1,l)}} C_\lambda \prod_{m=1}^{l} \left(\sum_{r_m=0}^{\lambda_m} \sum_{s_m=0}^{\lambda_m - r_m} f_{\lambda_m}(r,s,a_m)^m w_{r_m}^{s_m + \Lambda_m + k} \right) \right). \qquad (3.6)
\end{aligned}$$

Since $p \neq 0$, we know that $t(a_m) = s(a_{m+1})$ whenever $1 \leq m \leq l-1$.

Furthermore, $t({}^m w_{d_m}^{c_m}) = t(a_m) + c_m + d_m$ and $s({}^m w_{d_m}^{c_m}) = s(a_m) + c_m$ (see Lemma 2.2.4). Therefore, for all $1 \leq m \leq l-1$, we have

$$\begin{aligned}
&\left({}^m w_{r_m}^{s_m + \Lambda_m + k}\right) \left({}^{m+1} w_{r_{m+1}}^{s_{m+1} + \Lambda_{m+1} + k}\right) \neq 0 \\
\iff & t\left({}^m w_{r_m}^{s_m + \Lambda_m + k}\right) = s\left({}^{m+1} w_{r_{m+1}}^{s_{m+1} + \Lambda_{m+1} + k}\right) \\
\iff & t(a_m) + r_m + s_m + \Lambda_m + k = s(a_{m+1}) + s_{m+1} + \Lambda_{m+1} + k \\
\iff & r_m + s_m = s_{m+1} + \lambda_m. \qquad (3.7)
\end{aligned}$$

Now, observe that for all $1 \leq m \leq l$, we have that $r_m, s_m \geq 0$ and $r_m + s_m \leq \lambda_m$. Therefore, for all $1 \leq m \leq l-1$, equation (3.7) can only be true if $r_m + s_m = \lambda_m$ and $s_{m+1} = 0$. Notice that this implies that for $2 \leq m \leq l-1$, it must be the case that $r_m = \lambda_m$, and yet both r_1 and r_l are free to take on any value between 0 and λ_1 and 0 and λ_l, respectively. These conditions allow us to rewrite equation (3.6) as follows:

$$\omega \cdot p = \sum_{k=0}^{n-1} \Bigg[\sum_{\substack{\lambda \in WC \\ (n-1,l)}} C_\lambda \left(\sum_{r=0}^{\lambda_1} f_{\lambda_1}(r, \lambda_1 - r, a_1)^1 w_r^{\lambda_1 - r + k} \right) \cdot$$

$$\cdot \prod_{m=2}^{l-1} f_{\lambda_m}(\lambda_m, 0, a_m)^m w_{\lambda_m}^{\Lambda_m + k} \cdot \left(\sum_{r=0}^{\lambda_l} f_{\lambda_l}(r, 0, a_l)^l w_r^{\Lambda_l + k} \right) \Bigg]$$

$$= \sum_{\substack{\lambda \in WC \\ (n-1,l)}} C_\lambda \Bigg(\sum_{r=0}^{\lambda_1} \sum_{s=0}^{\lambda_l} f_{\lambda_1}(r, \lambda_1 - r, a_1) f_{\lambda_2}(\lambda_2, 0, a_2) \cdots$$

$$\cdot f_{\lambda_{l-1}}(\lambda_{l-1}, 0, a_{l-1}) f_{\lambda_l}(s, 0, a_l) \Bigg) \cdot$$

$$\cdot \overline{\sigma^r(a_1)}\ \overline{\sigma^{\lambda_2}(a_2)} \cdots \overline{\sigma^{\lambda_{l-1}}(a_{l-1})}\ \overline{\sigma^s(a_l)}, \qquad (3.8)$$

where the equality holds by Lemma 3.2.7.

We use Proposition 3.2.9 to reindex the sum in equation (3.8).

For $\lambda \in WC(k, l)$ and $r + s = n - 1 - k$, define

$$\lambda(r, s) := (\lambda_1 + r, \lambda_2, \ldots, \lambda_{l-1}, \lambda_l + s) \in WC(n-1, l)$$

so by Proposition 3.2.9, we have that equation (3.8) can be reindexed as:

$$\sum_{k=0}^{n-1} \sum_{\substack{\lambda \in WC \\ (n-1-k,l)}} \Bigg(\sum_{\substack{r+s=k \\ r,s \geq 0}} f_{\lambda_1 + r}(\lambda_1, r, a_1) f_{\lambda_2}(\lambda_2, 0, a_2)$$

$$\cdots f_{\lambda_{l-1}}(\lambda_{l-1}, 0, a_{l-1}) f_{\lambda_l + s}(\lambda_l, 0, a_l) \Bigg) C_{\lambda(r,s)}$$

$$\cdot \overline{\sigma^{\lambda_1}(a_1)}\, \overline{\sigma^{\lambda_2}(a_2)} \cdots \overline{\sigma^{\lambda_{l-1}}(a_{l-1})}\, \overline{\sigma^{\lambda_l}(a_l)}$$

$$= \sum_{k=0}^{n-1} \sum_{\substack{\lambda \in WC \\ (n-1-k,l)}} \Bigg(\sum_{\substack{r+s=k \\ r,s \geq 0}} f_{\lambda_1 + r}(\lambda_1, r, a_1) f_{\lambda_2}(\lambda_2, 0, a_2)$$

$$\cdots f_{\lambda_{l-1}}(\lambda_{l-1}, 0, a_{l-1}) f_{\lambda_l + s}(\lambda_l, 0, a_l) \Bigg) C_{\lambda(r,s)} \overline{p_\lambda}. \qquad (3.9)$$

It remains to show that the coefficient of the term $\overline{p_\lambda}$, which is

$$\Bigg(\sum_{\substack{r+s=k \\ r,s \geq 0}} f_{\lambda_1 + r}(\lambda_1, r, a_1) \cdots f_{\lambda_l + s}(\lambda_l, 0, a_l) \Bigg) C_{\lambda(r,s)},$$

is exactly equal to the expression for $\alpha_\lambda(p)$ as defined in the statement of the theorem.

Specializing equation (3.5) to our specific functions $f_{\lambda_m}(r, s, a_m)$, we obtain the following simplifications:

- $f_{\lambda_1+r}(\lambda_1, r, {}^1w_{d_1}^{c_1}) = \gamma^r(\zeta^{s(a_1)+c_m+1})^r(-1)^r\zeta^{r(r-1)/2}\begin{bmatrix}\lambda_1+r\\ \lambda_1\end{bmatrix}_\zeta \zeta^{c_1\lambda_1}$
- $f_{\lambda_m}(\lambda_m, 0, {}^mw_{d_m}^{c_m}) = \zeta^{c_m\lambda_m}$ for $2 \leq m \leq l-1$
- $f_{\lambda_l+s}(\lambda_l, 0, {}^lw_{d_l}^{c_l}) = \gamma^s(\zeta^{t(a_l)+c_l+d_l})^s\begin{bmatrix}\lambda_l+s\\ \lambda_l\end{bmatrix}_\zeta \zeta^{c_l\lambda_l}.$

Also, recall that we use the notation $\left\langle \begin{matrix} n-1\\ \lambda(r,s)\end{matrix}\right\rangle_\zeta$ to denote the q-Multinomial Coefficient $C_{\lambda(r,s)} = \begin{bmatrix} n-1\\ \lambda_1+r, \lambda_2, \ldots, \lambda_{l-1}, \lambda_l+s\end{bmatrix}_\zeta$. Hence, we may express the coefficient of $\overline{p_\lambda}$ as:

$$\left(\sum_{\substack{r+s=k\\ r,s\geq 0}} f_{\lambda_1+r}(\lambda_1, r, a_1)f_{\lambda_2}(\lambda_2, 0, a_2)\cdots f_{\lambda_{l-1}}(\lambda_{l-1}, 0, a_{l-1})f_{\lambda_l+s}(\lambda_l, 0, a_l)\right) C_{\lambda(r,s)}$$
$$= \left(\sum_{\substack{r+s=k\\ r,s\geq 0}} \gamma^r \left(\zeta^{s(a_1)+1}\right)^r (-1)^r\zeta^{r(r-1)/2}\begin{bmatrix}\lambda_1+r\\ \lambda_1\end{bmatrix}_\zeta \gamma^s\left(\zeta^{t(a_l)}\right)^s\begin{bmatrix}\lambda_l+s\\ \lambda_l\end{bmatrix}_\zeta \left\langle \begin{matrix} n-1\\ \lambda(r,s)\end{matrix}\right\rangle_\zeta\right)$$
$$= \left(\sum_{\substack{r+s=k\\ r,s\geq 0}} \gamma^k(-1)^r\zeta^{r(r-1)/2}\left(\zeta^{t(a_l)}\right)^{k-r}\left(\zeta^{s(a_1)+1}\right)^r\begin{bmatrix}\lambda_1+r\\ \lambda_1\end{bmatrix}_\zeta \begin{bmatrix}\lambda_l+s\\ \lambda_l\end{bmatrix}_\zeta \left\langle \begin{matrix} n-1\\ \lambda(r,s)\end{matrix}\right\rangle_\zeta\right). \tag{3.10}$$

Observe that the q-Binomial Coefficient and the q-Multinomial Coefficient can be rewritten to eliminate the parameter s as follows:

$$\begin{bmatrix}\lambda_1+r\\ \lambda_1\end{bmatrix}_\zeta \begin{bmatrix}\lambda_l+s\\ \lambda_l\end{bmatrix}_\zeta \left\langle \begin{matrix} n-1\\ \lambda(r,s)\end{matrix}\right\rangle_\zeta$$
$$= \frac{[\lambda_1+r]_\zeta![\lambda_l+s]_\zeta![n-1]_\zeta!}{[\lambda_1+r]_\zeta![\lambda_l+s]_\zeta![\lambda_1]_\zeta![\lambda_2]_\zeta!\cdots[\lambda_{l-1}]_\zeta![\lambda_l]_\zeta![r]_\zeta![s]_\zeta!}$$

$$= \frac{[n-1]_\zeta!}{[\lambda_1]_\zeta![\lambda_2]_\zeta!\cdots[\lambda_l]_\zeta![r]_\zeta![s]_\zeta!}$$
$$= \left(\frac{[k]_\zeta!}{[r]_\zeta![k-r]_\zeta!}\right)\left(\frac{[n-1]_\zeta!}{[k]_\zeta![n-1-k]_\zeta!}\right)\left(\frac{[n-1-k]_\zeta!}{[\lambda_1]_\zeta![\lambda_2]_\zeta!\cdots[\lambda_l]_\zeta!}\right)$$
$$= \begin{bmatrix} k \\ r \end{bmatrix}_\zeta \begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_\zeta, \qquad \text{since } n-1-k = |\lambda|.$$

So the coefficient of $\overline{p_\lambda}$ in equation (3.10) becomes:

$$\gamma^k \begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_\zeta \left(\sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}_\zeta (-1)^r \zeta^{r(r-1)/2} \left(\zeta^{t(a_l)}\right)^{k-r} \left(\zeta^{s(a_1)+1}\right)^r \right)$$
$$= \gamma^k \begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_\zeta \left(\zeta^{t(a_l)} - \zeta^{s(a_1)+1}\right)^k_\zeta \qquad \text{by Theorem 1.3.7.}$$
$$= \gamma^{n-1-|\lambda|} \begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_\zeta \left(\zeta^{t(a_l)} - \zeta^{s(a_1)+1}\right)^{n-1-|\lambda|}_\zeta.$$

Furthermore, since $t(p) = t(a_l)$, $s(p) = s(a_1)$, and $t(p) - s(p) \equiv D \mod n$, then $t(a_l) \equiv s(p) + D \mod n$, so we see that the coefficient of $\overline{p_\lambda}$ is

$$\alpha_\lambda(p) := \gamma^{n-1-|\lambda|} \begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_\zeta \left(\zeta^{s(p)+D} - \zeta^{s(p)+1}\right)^{n-1-|\lambda|}_\zeta. \tag{3.11}$$

Lastly, for $n-1-|\lambda| = k \geq D$, we can expand the term $\left(\zeta^{s(p)+D} - \zeta^{s(p)+1}\right)^k_\zeta$ to obtain

$$\left(\zeta^{s(p)+D} - \zeta^{s(p)+1}\right)^k_\zeta = \left(\zeta^{s(p)+D} - \zeta^{s(p)+1}\right)\cdots\left(\zeta^{s(p)+D} - \zeta^{s(p)+D}\right)\cdots\left(\zeta^{s(p)+D} - \zeta^{s(p)+k}\right),$$

which is 0 since $\zeta^{s(p)+D} - \zeta^{s(p)+D} = 0$. We conclude that

$$\omega \cdot p = \sum_{k=0}^{n-1} \sum_{\substack{\lambda \in WC \\ (n-1-k,l)}} \alpha_\lambda(p)\overline{p_\lambda} = \sum_{k=0}^{D-1} \sum_{\substack{\lambda \in WC \\ (n-1-k,l)}} \alpha_\lambda(p)\overline{p_\lambda},$$

as wanted. □

Observe that $\lambda \in WC(n-D,l)$ whenever $k = D-1$, so we have that $|\lambda|$ is minimized, and hence the σ-degree of p_λ is lowest. Furthermore, we showed in Definition 3.2.5 that for $\lambda \in WC(n-D,l)$ the element p_μ is an oriented cycle.

Corollary 3.2.11 (Taft integral acting on the path p_μ).

Let p_μ be defined as in Definition 3.2.5. Then,

$$\omega \cdot p_\mu = \sum_{k=0}^{n-1} \left(\sum_{\substack{\lambda \in WC \\ (n-1-k,l)}} \alpha_\lambda(p_\mu) \zeta^{\chi(\mu)\bullet\lambda}\, \overline{p_{\mu+\lambda}} \right)$$

where $\alpha_\lambda(p_\mu)$ is defined as in Theorem 3.2.10.

Notice that $D = n$ since p_μ is an oriented cycle, so in particular,

$$\alpha_\lambda(p_\mu) = \gamma^{n-1-|\lambda|} \begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_\zeta \left(\zeta^{s(p)} - \zeta^{s(p)+1} \right)_\zeta^{n-1-|\lambda|}.$$

Also, recall that $\chi(\mu) \bullet \lambda$ is the dot product, i.e. $\chi(\mu) \bullet \lambda = \sum_{m=1}^{l} M_m \lambda_m$.

Proof. Recall in the proof of Theorem 3.2.10, we had the following specialized formulas:

(1) $f_{\lambda_1+r}(\lambda_1, r, {}^1w_{d_1}^{c_1}) = \gamma^r (\zeta^{s(a_1)+c_m+1})^r (-1)^r \zeta^{r(r-1)/2} \begin{bmatrix} \lambda_1 + r \\ \lambda_1 \end{bmatrix}_\zeta \zeta^{c_1\lambda_1}$

(2) $f_{\lambda_m}(\lambda_m, 0, {}^m w_{d_m}^{c_m}) = \zeta^{c_m \lambda_m}$ for $2 \leq m \leq l-1$

(3) $f_{\lambda_l+s}(\lambda_l, 0, {}^l w_{d_l}^{c_l}) = \gamma^s (\zeta^{t(a_l)+c_l+d_l})^s \begin{bmatrix} \lambda_l + s \\ \lambda_l \end{bmatrix}_\zeta \zeta^{c_l\lambda_l}$,

which we evaluated at ${}^m w_{d_m}^{c_m} = a_m$, so $c_m = d_m = 0$. In this case, we evaluate these formulas at ${}^m w_{d_m}^{c_m} = {}^m w_{\mu_m}^{M_m}$, so $c_m = M_m$ and $d_m = \mu_m$. Therefore, $\zeta^{c_m\lambda_m}$ is not necessarily 1 for all m as was the case in Theorem 3.2.10.

Furthermore, since $d_m \neq 0$, we see that $\overline{{}^m w_{r_m+d_m}} = \overline{{}^m w_{\mu_m+\lambda_m}} = \overline{p_{\mu+\lambda}}$. Lastly, $s(p_\mu) = s(p)$ from the definition of p_μ, so the coefficient of the $\overline{p_{\mu+\lambda}}$ term in this case becomes:

$$\gamma^{n-1-|\lambda|} \begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_\zeta \left(\zeta^{s(p)} - \zeta^{s(p)+1} \right)_\zeta^{n-1-|\lambda|} \prod_{m=1}^{l} \zeta^{c_m \lambda_m}$$

$$= \alpha_\lambda(p_\mu) \zeta^{\sum\limits_{m=1}^{l} M_m \lambda_m}$$

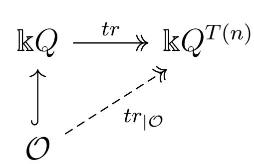
$$= \alpha_\lambda(p_\mu) \zeta^{\chi(\mu) \bullet \lambda}, \tag{3.12}$$

as wanted. □

The fact that p_μ is the lowest σ-degree term appearing in $\omega \cdot p$ leads us to the answer we have been searching for: oriented cycles generate the invariant space! The hope is that this description will help us learn whether or not the invariant ring is finitely generated. In certain cases, it can be finitely generated, and in particular, if there are no oriented cycles then the invariant ring of the action is trivial. However, the question of whether the invariant ring can always be finitely generated is more complicated, and while we do not provide an answer in this thesis, we are hopeful that the result of Theorem 3.2.12 will help in this eventual goal.

Theorem 3.2.12 (Oriented cycles generate $\Bbbk Q^{T(n)}$)**.**

Let $\mathcal{O}$ be the vector space generated by all oriented cycles of $\Bbbk Q$. Then the following diagram commutes:



i.e. the map tr restricted to the vector space $\mathcal{O}$ surjects onto the invariant ring.

What this theorem tells us is that any invariant $\omega \cdot p$ can be expressed as

$$\omega \cdot \left(\sum_k \beta_k p_k \right) = \sum_k \beta_k (\omega \cdot p_k)$$

where the p_k are oriented cycles.

Before we prove Theorem 3.2.12, we introduce some helpful notation and state a few quick lemmas. The proofs of these lemmas can be found in the Appendix since they are rather technical and tedious, but the lemmas are extremely helpful in the proof of Theorem 3.2.12.

Lemma 3.2.13 (Property of q-integers)**.**

Recall the definition of a q-integer: $[n]_q = \dfrac{1-q^n}{1-q}$.

Then for any $0 \leq a \leq n$, *and* ζ *an* n*-th root of unity,*

$$\zeta^a [n-a]_\zeta = -[a]_\zeta.$$

Definition 3.2.14 (Functions $P(|\lambda|)$ and $Q(|\lambda|)$)**.**

Given $n - D \leq k \leq n-1$, $\lambda \in WC(k,l)$, *and* $\mu \in WC(n-D,l)$, *we define the functions* $\underline{P(|\lambda|)}$ *and* $\underline{Q(|\lambda|)}$ *by:*

- $P(|\lambda|) := \left(\zeta^{s(p)+D} - \zeta^{s(p)+1}\right)_\zeta^{n-1-|\lambda|}$
- $Q(|\lambda|) := \left(\zeta^{s(p)} - \zeta^{s(p)+1}\right)_\zeta^{n-1-|\lambda|}$.

Observe that since $D = n - |\mu|$ then $\zeta^{s(p)+D} = \zeta^{s(p)+n-|\mu|} = \zeta^{s(p)-|\mu|}$ so we may also express $P(|\lambda|) = \left(\zeta^{s(p)-|\mu|} - \zeta^{s(p)+1}\right)_{\zeta}^{n-1-|\lambda|}$.

It follows that for any $\lambda \in WC(k,l)$, we may write the terms $\alpha_\lambda(p)$ and $\alpha_\lambda(p_\mu)$, which were defined in Theorem 3.2.10 and Corollary 3.2.11 as:

- $$\alpha_\lambda(p) = \gamma^{n-1-|\lambda|} \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_{\zeta} \begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_{\zeta} P(|\lambda|), \quad \text{and} \tag{3.13}$$
- $$\alpha_\lambda(p_\mu) = \gamma^{n-1-|\lambda|} \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_{\zeta} \begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_{\zeta} Q(|\lambda|). \tag{3.14}$$

Lemma 3.2.15 (Relationship between $P(|\lambda|)$ and $Q(|\lambda|)$)**.**

For $n - D \leq k \leq n-1$, $\lambda \in WC(k,l)$, and $\mu \in WC(n-D,l)$ then the following expressions are equivalent:

$$\begin{aligned}
&(1) \quad P(|\lambda|) \\
&(2) \quad \frac{Q(|\lambda|)}{\zeta^{|\mu|(n-1-|\lambda|)}} \begin{bmatrix} n-1-(|\lambda|-|\mu|) \\ |\mu| \end{bmatrix}_{\zeta} \\
&(3) \quad \frac{Q(|\lambda|-|\mu|)}{\zeta^{|\mu|(n-1-|\lambda|)} Q(n-1-|\mu|)}.
\end{aligned}$$

Lemma 3.2.15 allows us to compare $\alpha_\lambda(p)$ and $\alpha_\lambda(p_\mu)$ in terms of only the function $Q(x)$, while Lemma 3.2.16 provides us with an expression involving ζ-polynomials.

Lemma 3.2.16 (Formula for $Q(n-1-\mu)$).

For $n-D \leq k \leq n-1$, $\lambda \in WC(k,l)$, and $\mu \in WC(n-D,l)$

$$Q(n-1-|\mu|) = \left(\zeta^{s(p)} - \zeta^{s(p)+1}\right)^{|\mu|} [|\mu|]_\zeta!.$$

Notice that $\left(\zeta^{s(p)} - \zeta^{s(p)+1}\right)^{|\mu|} \neq \left(\zeta^{s(p)} - \zeta^{s(p)+1}\right)_\zeta^{|\mu|}$, which simplifies computations significantly.

Corollary 3.2.17 (Formula for $P(|\mu|)$).

For $n-D \leq k \leq n-1$, $\lambda \in WC(k,l)$, and $\mu \in WC(n-D,l)$

$$P(|\mu|) = \frac{Q(0)}{\zeta^{|\mu|(s(p)-|\mu|)}} \left(\frac{\zeta}{1-\zeta}\right)^{|\mu|} \frac{1}{[|\mu|]_\zeta!}.$$

Proof of Theorem 3.2.12. We will show that for any path $p \in \Bbbk Q$, with $t(p)-s(p) \equiv D \mod n$, then $\omega \cdot p = \sum_{\substack{\mu \in WC \\ (n-D,l)}} \frac{\alpha_\mu(p)}{\alpha_0} \omega \cdot p_\mu$, where $\alpha_0 := \gamma^{n-1} Q(0)$.

Recall that by Theorem 3.2.10 and Corollary 3.2.11, we have

- $\omega \cdot p = \sum_{k=0}^{D-1} \left(\sum_{\substack{\lambda \in WC \\ (n-1-k,l)}} \alpha_\lambda(p) \overline{p_\lambda} \right),$
- $\omega \cdot p_\mu = \sum_{k=0}^{n-1} \left(\sum_{\substack{\lambda \in WC \\ (n-1-k,l)}} \alpha_\lambda(p_\mu) \zeta^{\chi(\mu) \bullet \lambda} \, \overline{p_{\mu+\lambda}} \right),$
- $\alpha_\lambda(p) := \gamma^{n-1-|\lambda|} \begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_\zeta P(|\lambda|),$

- $\alpha_\lambda(p_\mu) := \gamma^{n-1-|\lambda|} \begin{bmatrix} n-1 \\ |\lambda| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\lambda| \\ \lambda \end{matrix} \right\rangle_\zeta Q(|\lambda|)$, and
- $\overline{p_\lambda} := \prod_{i=1}^{l} \overline{\sigma^{\lambda_i}(a_i)}$ and $\overline{p_{\lambda+\mu}} := \prod_{i=1}^{l} \overline{\sigma^{\lambda_i+\mu_i}(a_i)}$.

Fix $\nu \in WC(n-1-k, l)$ for some $0 \leq k \leq D-1$; in particular, $|\nu| \geq n-D$. We will compare the linearly dependent terms in each expression by comparing the coefficient of $\overline{p_\nu}$ in the term $\omega \cdot p$ with the coefficient of $\overline{p_\nu}$ in the term $\sum_{\substack{\mu \in WC \\ (n-D,l)}} \frac{\alpha_\mu(p)}{\alpha_0} \omega \cdot p_\mu$, and showing that they are equal.

The coefficient of $\overline{p_\nu}$ in the term $\omega \cdot p$ is simply $\alpha_\nu(p)$.

The coefficient of $\overline{p_\nu}$ in the term:

$$\sum_{\substack{\mu \in WC \\ (n-D,l)}} \frac{\alpha_\mu(p)}{\alpha_0} \omega \cdot p_\mu = \sum_{\substack{\mu \in WC \\ (n-D,l)}} \frac{\alpha_\mu(p)}{\alpha_0} \left(\sum_{k=0}^{n-1} \left(\sum_{\substack{\lambda \in WC \\ (n-1-k,l)}} \alpha_\lambda(p_\mu) \zeta^{\chi(\mu) \bullet \lambda} \, \overline{p_{\mu+\lambda}} \right) \right)$$

is seen by setting $\lambda = \nu - \mu$, whenever $\nu \geq \mu$, i.e. whenever $\nu_i \geq \mu_i$ for all $1 \leq i \leq l$.

If $\nu \ngeq \mu$, then the term $\alpha_{\nu-\mu} := 0$ since otherwise the q-Multinomial Coefficient $\left\langle \begin{matrix} |\nu-\mu| \\ \nu-\mu \end{matrix} \right\rangle_\zeta$ would have negative entries in the denominator, which is not defined. Hence, we do not need to concern ourselves with whether $\nu \geq \mu$, and we may simply take the sum over all μ. We therefore have that the coefficient of $\overline{p_\nu}$ in this latter term is:

$$\frac{1}{\alpha_0} \sum_{\substack{\mu \in WC \\ (n-D,l)}} \alpha_\mu(p) \alpha_{\nu-\mu}(p_\mu) \zeta^{\chi(\mu) \bullet (\nu-\mu)}.$$

Comparing the coefficients of $\overline{p_\nu}$ in the two terms, we see that the equality we need to prove is:

$$\alpha_0 \alpha_\nu(p) = \sum_{\substack{\mu \in WC \\ (n-D,l)}} \alpha_\mu(p) \alpha_{\nu-\mu}(p_\mu) \zeta^{\chi(\mu) \bullet (\nu-\mu)}. \tag{3.15}$$

We can express this equality in combinatorial terms as indicated in Lemma 3.2.18 below:

Lemma 3.2.18 (Combinatorial description of equation (3.15))**.**

$$\alpha_0 \alpha_\nu(p) = \sum_{\substack{\mu \in WC \\ (n-D,l)}} \alpha_\mu(p)\alpha_{\nu-\mu}(p_\mu)\zeta^{\chi(\mu)\bullet(\nu-\mu)}$$

if and only if

$$\begin{aligned} &\begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\nu| \\ \nu \end{matrix} \right\rangle_\zeta \zeta^{|\mu|(|\nu|-|\mu|)} \\ &\qquad = \sum_{\substack{\mu \in WC \\ (n-D,l)}} \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\nu|-|\mu| \\ \nu-\mu \end{matrix} \right\rangle_\zeta \zeta^{\chi(\mu)(\nu-\mu)}. \end{aligned} \tag{3.16}$$

The proof of this lemma simply relies on making substitutions from Lemma 3.2.15, Lemma 3.2.16, and Corollary 3.2.17. The description it yields allows us to prove Theorem 3.2.12 by induction. In an attempt to aid the reader through the steps of the proof, we highlight in red the moving pieces in each step.

Proof.

$$\begin{aligned} &\alpha_0 \alpha_\nu(p) = \sum_{\substack{\mu \in WC \\ (n-D,l)}} \alpha_\mu(p)\alpha_{\nu-\mu}(p_\mu)\zeta^{\chi(\mu)\bullet(\nu-\mu)} \\ \iff & \gamma^{n-1} Q(0) \gamma^{n-1-|\nu|} \left\langle \begin{matrix} |\nu| \\ \nu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta P(|\nu|) \\ &\quad = \sum_{\substack{\mu \in WC \\ (n-D,l)}} \gamma^{n-1-|\mu|} \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta P(|\mu|) \end{aligned}$$

$$\cdot\, \gamma^{n-1-(|\nu|-|\mu|)} \left\langle \begin{matrix} |\nu|-|\mu| \\ \nu-\mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta Q(|\nu|-|\mu|)\zeta^{\chi(\mu)\bullet(\nu-\mu)}$$

$$\iff Q(0)\left\langle \begin{matrix} |\nu| \\ \nu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta P(|\nu|)$$

$$= \sum_{\substack{\mu\in WC \\ (n-D,l)}} \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \frac{Q(0)}{\zeta^{|\mu|(s(p)-|\mu|)}} \left(\frac{\zeta}{1-\zeta}\right)^{|\mu|} \frac{1}{[|\mu|]_\zeta!}$$

$$\cdot \left\langle \begin{matrix} |\nu|-|\mu| \\ \nu-\mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta Q(|\nu|-|\mu|)\zeta^{\chi(\mu)\bullet(\nu-\mu)}$$

by Corollary 3.2.17

$$\iff \left\langle \begin{matrix} |\nu| \\ \nu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta P(|\nu|)$$

$$= \sum_{\substack{\mu\in WC \\ (n-D,l)}} \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \frac{1}{\zeta^{|\mu|(s(p)-|\mu|)}} \left(\frac{\zeta}{1-\zeta}\right)^{|\mu|} \frac{1}{[|\mu|]_\zeta!}$$

$$\cdot \left\langle \begin{matrix} |\nu|-|\mu| \\ \nu-\mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta \zeta^{|\mu|(n-1-|\nu|)}Q(n-1-|\mu|)P(|\nu|)\zeta^{\chi(\mu)\bullet(\nu-\mu)}$$

by Lemma 3.2.15

$$\iff \left\langle \begin{matrix} |\nu| \\ \nu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta = \sum_{\substack{\mu\in WC \\ (n-D,l)}} \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \frac{1}{\zeta^{|\mu|(s(p)-|\mu|)}} \left(\frac{\zeta}{1-\zeta}\right)^{|\mu|}$$

$$\cdot \frac{1}{[|\mu|]_\zeta!}\left\langle \begin{matrix} |\nu|-|\mu| \\ \nu-\mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta \zeta^{|\mu|(n-1-|\nu|)}[|\mu|]_\zeta!\left(\zeta^{s(p)}-\zeta^{s(p)+1}\right)^{|\mu|}\zeta^{\chi(\mu)\bullet(\nu-\mu)}$$

by Lemma 3.2.16

$$\iff \left\langle \begin{matrix} |\nu| \\ \nu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta = \sum_{\substack{\mu\in WC \\ (n-D,l)}} \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \frac{1}{\zeta^{|\mu|(s(p)-|\mu|)}} \left(\frac{\zeta}{1-\zeta}\right)^{|\mu|}$$

$$\cdot \left\langle \begin{matrix} |\nu|-|\mu| \\ \nu-\mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta \zeta^{|\mu|(s(p)-1-|\nu|)}(1-\zeta)^{|\mu|}\zeta^{\chi(\mu)\bullet(\nu-\mu)}$$

by canceling the $[|\mu|]_\zeta!$ term and factoring out $\zeta^{s(p)|\mu|}$

$$\iff \left\langle \begin{matrix} |\nu| \\ \nu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta$$

$$= \sum_{\substack{\mu \in WC \\ (n-D,l)}} \left\langle {|\mu| \atop \mu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \frac{1}{\zeta^{|\mu|(s(p)-|\mu|)}} \left\langle {|\nu|-|\mu| \atop \nu-\mu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta$$

$$\cdot \frac{\zeta^{|\mu|(s(p)-|\nu|)}}{\zeta^{|\mu|}} (1-\zeta)^{|\mu|} \zeta^{\chi(\mu)\bullet(\nu-\mu)} \frac{\zeta^{|\mu|}}{(1-\zeta)^{|\mu|}}$$

$$\iff \left\langle {|\nu| \atop \nu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta$$

$$= \sum_{\substack{\mu \in WC \\ (n-D,l)}} \left\langle {|\mu| \atop \mu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \frac{1}{\zeta^{|\mu|(|\nu|-|\mu|)}} \left\langle {|\nu|-|\mu| \atop \nu-\mu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta \zeta^{\chi(\mu)\bullet(\nu-\mu)}$$

$$\iff \left\langle {|\nu| \atop \nu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta \zeta^{|\mu|(|\nu|-|\mu|)}$$

$$= \sum_{\substack{\mu \in WC \\ (n-D,l)}} \left\langle {|\mu| \atop \mu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle {|\nu|-|\mu| \atop \nu-\mu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta \zeta^{\chi(\mu)\bullet(\nu-\mu)}$$

□

We proceed to prove equation (3.16) in Lemma 3.2.18 by inducting on $|\nu|$, where $n - D \leq |\nu|$.

If $|\nu| = n - D$, then $\nu \in WC(n-D,l)$, so ν is the unique composition in $WC(n-D,l)$ with $\nu \leq \nu$. Therefore,

$$\sum_{\substack{\mu \in WC \\ (n-D,l)}} \left\langle {|\mu| \atop \mu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle {|\nu|-|\mu| \atop \nu-\mu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta \zeta^{\chi(\mu)\bullet(\nu-\mu)}$$

$$= \sum_{\nu=\mu} \left\langle {|\mu| \atop \mu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle {|\mu|-|\mu| \atop \mu-\mu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu|-|\mu| \end{bmatrix}_\zeta \zeta^{\chi(\mu)\bullet(\mu-\mu)}$$

$$= \left\langle {|\mu| \atop \mu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta$$

$$= \left\langle {|\nu| \atop \nu} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta \zeta^{|\mu|(|\mu|-|\mu|)},$$

so the base case is true.

Suppose for our inductive hypothesis that for a fixed value $k > n - D$, and for all weak compositions ν with $n - D \le |\nu| < k$, we have

$$\zeta^{|\mu|(|\nu|-|\mu|)} \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\nu| \\ \nu \end{matrix} \right\rangle_\zeta = \sum_{\substack{\mu \in WC \\ (n-D,l)}} \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\nu|-|\mu| \\ \nu-\mu \end{matrix} \right\rangle_\zeta \zeta^{\chi(\mu)\bullet(\nu-\mu)}.$$

Let $|\hat{\nu}| \in WC(k,l)$, i.e. $|\hat{\nu}| = k$. Then for some $\nu \in WC(k-1,l)$ and $1 \le j \le l$, we can express $\hat{\nu}$ as $\nu + e_j$, where e_j is the standard j-th basis vector of $\mathbb{R}^l$. Recall that we have identificatied $WC(k,l)$ as a subset of $\mathbb{R}^l$.

Therefore,

$$\zeta^{|\mu|(|\hat{\nu}|-|\mu|)} \begin{bmatrix} n-1 \\ |\hat{\nu}| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\hat{\nu}| \\ \hat{\nu} \end{matrix} \right\rangle_\zeta = \zeta^{|\mu|}\zeta^{|\mu|(|\nu|-|\mu|)} \begin{bmatrix} n-1 \\ |\nu|+1 \end{bmatrix}_\zeta \left\langle \begin{matrix} |\nu|+1 \\ \nu+e_j \end{matrix} \right\rangle_\zeta$$

$$= \zeta^{|\mu|}\zeta^{|\mu|(|\nu|-|\mu|)} \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta \frac{[n-1-|\nu|]_\zeta}{[|\nu|+1]_\zeta} \left\langle \begin{matrix} |\nu|+1 \\ \nu+e_j \end{matrix} \right\rangle_\zeta$$

$$= \zeta^{|\mu|} \frac{[n-1-|\nu|]_\zeta}{[|\nu|+1]_\zeta} \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta \zeta^{|\mu|(|\nu|-|\mu|)} \left(\left\langle \begin{matrix} |\nu|+1 \\ \nu+e_j \end{matrix} \right\rangle_\zeta \right)$$

$$= \zeta^{|\mu|} \frac{[n-1-|\nu|]_\zeta}{[|\nu|+1]_\zeta} \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta \zeta^{|\mu|(|\nu|-|\mu|)} \left(\sum_{i=1}^{l} \zeta^{\sum_{k=1}^{i-1} \hat{v}_k} \left\langle \begin{matrix} |\nu| \\ \nu+e_j-e_i \end{matrix} \right\rangle_\zeta \right)$$

by Theorem 1.3.8

$$= \zeta^{|\mu|} \frac{[n-1-|\nu|]_\zeta}{[|\nu|+1]_\zeta} \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta \zeta^{|\mu|(|\nu|-|\mu|)} \left(\sum_{i=1}^{l} \zeta^{\chi(\hat{\nu})\bullet e_i} \left\langle \begin{matrix} |\nu| \\ \nu+e_j-e_i \end{matrix} \right\rangle_\zeta \right)$$

$$= \zeta^{|\mu|} \frac{[n-1-|\nu|]_\zeta}{[|\nu|+1]_\zeta} \left(\sum_{i=1}^{l} \zeta^{|\mu|(|\nu|-|\mu|)} \begin{bmatrix} n-1 \\ |\nu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\nu| \\ \nu+e_j-e_i \end{matrix} \right\rangle_\zeta \zeta^{\chi(\hat{\nu})\bullet e_i} \right)$$

$$= \zeta^{|\mu|} \frac{[n-1-|\nu|]_\zeta}{[|\nu|+1]_\zeta} \left(\sum_{i=1}^{l} \left[\sum_{\substack{\mu \in WC \\ (n-D,l)}} \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \right.\right.$$

$$\left.\left. \cdot \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\nu|-|\mu| \\ \nu+e_j-\mu-e_i \end{matrix} \right\rangle_\zeta \zeta^{\chi(\mu)\bullet(\nu+e_j-e_i-\mu)} \right] \zeta^{\chi(\hat{\nu})\bullet e_i} \right)$$

by the inductive hypothesis, since $|v+e_j-e_i| = |v| < k$

$$= \sum_{\substack{\mu \in WC \\ (n-D,l)}} \zeta^{|\mu|} \frac{[n-1-|\nu|]_\zeta}{[|\nu|+1]_\zeta} \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta$$

$$\cdot \left(\sum_{i=1}^{l} \left\langle \begin{matrix} |\nu|-|\mu| \\ \nu+e_j-\mu-e_i \end{matrix} \right\rangle_\zeta \zeta^{\chi(\mu)\bullet(\nu+e_j-\mu)} \zeta^{\chi(\mu)\bullet e_i} \right) \zeta^{\chi(\hat{\nu})\bullet e_i}$$

$$= \sum_{\substack{\mu \in WC \\ (n-D,l)}} \zeta^{|\mu|} \frac{[n-1-|\nu|]_\zeta}{[|\nu|+1]_\zeta} \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix}_\zeta \zeta^{\chi(\mu)\bullet(\hat{\nu}-\mu)}$$

$$\cdot \left(\sum_{i=1}^{l} \left\langle \begin{matrix} |\nu|-|\mu| \\ \hat{\nu}-\mu-e_i \end{matrix} \right\rangle_\zeta \zeta^{\chi(\hat{\nu}-\mu)\bullet e_i} \right) \qquad \text{by the linearity of } \chi$$

$$= \sum_{\substack{\mu \in WC \\ (n-D,l)}} \zeta^{|\mu|} \frac{[n-1-|\nu|]_\zeta}{[|\nu|+1]_\zeta} \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\nu|-|\mu| \end{bmatrix} \zeta^{\chi(\mu)\bullet(\hat{\nu}-\mu)}$$

$$\cdot \left(\left\langle \begin{matrix} |\nu|+1-|\mu| \\ \hat{\nu}-\mu \end{matrix} \right\rangle_\zeta \right) \qquad \text{by Theorem 1.3.8}$$

$$= \sum_{\substack{\mu \in WC \\ (n-D,l)}} \zeta^{|\mu|} \frac{[n-1-|\nu|]_\zeta}{[|\nu|+1]_\zeta} \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \zeta^{\chi(\mu)\bullet(\hat{\nu}-\mu)}$$

$$\cdot \begin{bmatrix} n-1 \\ |\nu|+1-|\mu| \end{bmatrix}_\zeta \frac{[|\nu|-|\mu|+1]_\zeta}{[n-1-(|\nu|-|\mu|)]_\zeta} \left\langle \begin{matrix} |\nu|+1-|\mu| \\ \hat{\nu}-\mu \end{matrix} \right\rangle_\zeta$$

$$= \sum_{\substack{\mu \in WC \\ (n-D,l)}} \zeta^{|\mu|} \frac{[n-(|\nu|+1)]_\zeta}{[|\nu|+1]_\zeta} \frac{[|\nu|-|\mu|+1]_\zeta}{[n-((|\nu|-|\mu|)+1)]_\zeta}$$

$$\cdot \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\hat{\nu}|-|\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\nu|+1-|\mu| \\ \hat{\nu}-\mu \end{matrix} \right\rangle_\zeta \zeta^{\chi(\mu)\bullet(\hat{\nu}-\mu)}$$

$$= \sum_{\substack{\mu \in WC \\ (n-D,l)}} \zeta^{|\mu|} \frac{-1}{\zeta^{|\nu|+1}} \frac{\zeta^{|\nu|-|\mu|+1}}{-1} \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\hat{\nu}|-|\mu| \end{bmatrix}_\zeta$$

$$\cdot \left\langle \begin{matrix} |\nu|+1-|\mu| \\ \hat{\nu}-\mu \end{matrix} \right\rangle_\zeta \zeta^{\chi(\mu)\bullet(\hat{\nu}-\mu)} \qquad \text{by Lemma 3.2.13}$$

$$= \sum_{\substack{\mu \in WC \\ (n-D,l)}} \frac{\zeta^{|\mu|+|\nu|-|\mu|+1}}{\zeta^{|\nu|+1}} \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\hat{\nu}|-|\mu| \end{bmatrix}_\zeta$$

$$\cdot \left\langle \begin{matrix} |\nu|+1-|\mu| \\ \hat{\nu}-\mu \end{matrix} \right\rangle_\zeta \zeta^{\chi(\mu)\bullet(\hat{\nu}-\mu)}$$

$$= \sum_{\substack{\mu \in WC \\ (n-D,l)}} \begin{bmatrix} n-1 \\ |\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\mu| \\ \mu \end{matrix} \right\rangle_\zeta \begin{bmatrix} n-1 \\ |\hat{\nu}| - |\mu| \end{bmatrix}_\zeta \left\langle \begin{matrix} |\nu| + 1 - |\mu| \\ \hat{\nu} - \mu \end{matrix} \right\rangle_\zeta \zeta^{\chi(\mu)\bullet(\hat{\nu}-\mu)}.$$

By the principle of mathematical induction, we have proved our desired result. □

We end on the conjecture that given an action of $T(n)$ on $\Bbbk Q$ in which g acts transitively and faithfully on the set of vertices of Q, we can produce a family of invariants that cannot be written as a sum of products over $\Bbbk$ of finitely many elements in $tr(\mathcal{O})$. In other words, we conjecture that there are invariant rings of specific actions which are not finitely generated as algebras. While we do not currently have a definitive answer to this, given our useful description of $\mathcal{O}$, we do have computational evidence suggesting that for $T(2)$ acting on the quiver Q in 3.1, either of the families given by

$$c_k = bg(a^k b) \quad \text{or}$$

$$d_k = g(b)(a^k b)$$

will generate invariants which cannot be written as a sum of products of finitely many elements in $\mathcal{O}$.

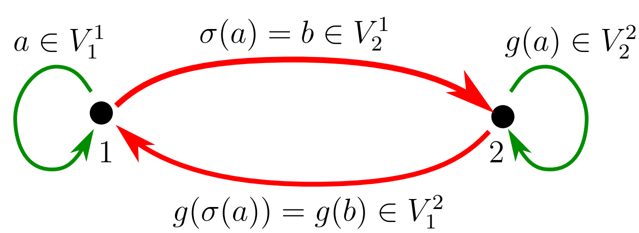


Figure 3.1: Action of $T(2)$ on the path algebra of the quiver Q

CHAPTER 4
FUTURE WORK

There are several other avenues still left to explore in the future with this work. The results presented in chapter 3 are specific to a type of quiver configuration, namely Type A quivers. In general, however, there can be invariants that are products of elements from two subquivers, possibly neither of which is Type A. What this suggests is that the decomposition which worked well to classify the action of Taft algebras on the path algebras of quivers, is not necessarily the most efficient to give an adequate description of the invariant ring. This leads to the following questions:

Problem 1. *Is there a decomposition of the quiver that is compatible with the invariant subspaces, in the sense that once we understand each subquiver, we can "glue" the results together to obtain a general result?*

One idea to pursue this question is to explore examples of small quivers which are neither Type A nor Type B, and see if a pattern forms which relates to the invariants found in the case of a Type A quiver. Even if this approach does not appear to be fruitful, it will likely lead to a deeper understanding and yield other ideas. This approach has the advantage that it is highly computational and can be explored by advanced undergraduate students and lead to interesting undergraduate research projects.

The second question opens up a different possibility for exploration.

Problem 2. *Since oriented cycles are necessary to produce invariants in Type A* *quivers, we can reduce a quiver by deleting edges that do not contribute cycles until its invariant ring is trivial. Is it possible to find an inductive argument for classifying the invariant ring by adding the edges back one at a time? Can we deduce anything from the relationship between the invariant rings of a quiver and its subquivers?*

The result that we are seeing here that cycles play a significant role in the invariant ring is ubiquitous. In fact, if Q is a quiver and n a dimension vector, then $GL(n)$ acts by conjugation on the set $\mathrm{Rep}(Q, n)$. The ring of invariants under this action is trivial if Q has no oriented cycles [5]. It would be interesting to explore what other actions have oriented cycles playing an important role in the invariant ring, and research other results using smash products which also seem to yield information about oriented cycles.

Another direction we can pursue is to generalize the action to larger Hopf algebras. In particular, there is a lot of interest in understanding the action of quantum groups such as $u_q(\mathfrak{sl}_2)$ on non-commutative algebras.

Problem 3. *Since $u_q(\mathfrak{sl}_2)$ is generated by two subalgebras isomorphic to Taft algebras, can we use similar techniques to classify the actions of $u_q(\mathfrak{sl}_2)$ on $\Bbbk Q$?*

Kinser and Walton began to explore this question in [9], and it yielded a more restricted result in terms of the possible actions of $u_q(\mathfrak{sl}_2)$ on $\Bbbk Q$ for Schurian, loopless quivers Q. It would be a short but worthwhile endeavor to generalize their exploration to arbitrary quivers, and investigate the invariant ring of this action. It might be the case that it yields a new perspective which generalizes what we saw in chapter 3.

Problem 4. *Similarly, could we continue this approach to classify $u_q(\mathfrak{sl}_n)$ actions?*

The quantum groups $u_q(\mathfrak{sl}_n)$ all arise by gluing copies of $u_q(\mathfrak{sl}_{n-1})$. There might be a pattern which we could prove by inducting on n to classify all possible such actions.

Problem 5. *Given the increasingly complicated structure that $u_q(\mathfrak{sl}_n)$ has for each larger value of n, is there a finite number n for which the only action of $u_q(\mathfrak{sl}_m)$ on $\Bbbk Q$ is trivial for all $m > n$?*

Even at the level of $u_q(\mathfrak{sl}_2)$, Kinser and Walton found more constraints to define an action [9]. In general, the more relations imposed on the generators, the more restrictive the actions will be. It seems plausible that given large enough n, the number of relations is too large to produce any non-trivial actions on $\Bbbk Q$.

We may also explore a different family of algebras on which to act. All of the above problems consider acting on path algebras of quivers, which are nice because of their hands-on nature despite their non-commutativity. However, any basic and connected finite dimensional algebra is Morita equivalent to the bound quiver algebra of some bound quiver (Q, I), where I is some admissible ideal of $\Bbbk Q$ [2]. Therefore, we raise the question:

Problem 6. *Can we classify the actions of $T(n)$ on $\Bbbk Q/I$ for some admissible ideal I of $\Bbbk Q$ in such a way that it extends the results where $I = 0$?*

In other words, we are looking for admissible ideals $I \subseteq \Bbbk Q$ which are $T(n)$-submodules. Since $T(n)$ is, as an algebra, a Nakayama algebra, and Nakayama algebras have well understood finite dimensional indecomposable modules, it seems reasonable that we can study the $T(n)$-submodules via Nakayama algebras.

Ultimately, the goal is to have a big picture understanding of the possible actions of $u_q(\mathfrak{g})$ for any semisimple Lie algebra. In [1], Andruskiewitsch and Schneider classified all finite dimensional pointed Hopf algebras, whose group of group-like elements is abelian and all prime divisors of the order of the group are greater than 7. This classification provides an axiomatic description that might be helpful in providing some insight as to how to classify actions of pointed Hopf algebras in general.

APPENDIX

Lemma 3.2.8 (Linear independence of elements of $\Bbbk Q$).

Let $p = a_1 a_2 \cdots a_l$ be a path of length l in $\Bbbk Q$. Then

$$\left\{ p_\lambda \mid \lambda \in WC(k,l) \right\}_{k=0}^{n-1}$$

is a set of linearly independent elements of $\Bbbk Q$.

Proof. Let $0 \le k, m \le n-1$ be integers with $k \ne m$. Then for any $\lambda \in WC(k,l)$ and $\mu \in WC(m,l)$, p_λ and p_μ are linearly independent since $t(p_\lambda) = t(p) + |\lambda| = t(p) + k \ne t(p) + m = t(p) + |\mu| = t(p_\mu)$.

If $\lambda, \mu \in WC(k,l)$ with $\lambda \ne \mu$, then there exists a minimal $1 \le j \le l$ such that $\lambda_j \ne \mu_j$. Therefore, $t((p_\lambda)_{j-1}) = t((p_\mu)_{j-1})$, while $t((p_\lambda)_j) = t((p_\lambda)_{j-1}) + |\lambda_j| \ne t((p_\mu)_{j-1}) + |\mu_j| = t((p_\mu)_j)$. Therefore, p_λ and p_μ belong to different Vector Subspaces of $\Bbbk Q$, and hence are linearly independent. □

Lemma 3.2.13 (Property of q-integers).

Recall the definition of a q-integer: $[n]_q = \dfrac{1-q^n}{1-q}$.

Then for any $0 \le a \le n$, and ζ an n-th root of unity,

$$\zeta^a [n-a]_\zeta = -[a]_\zeta.$$

Proof.

$$\begin{aligned}
\zeta^a[n-a]_\zeta &= \zeta^a \frac{1-\zeta^{n-a}}{1-\zeta} \\
&= \frac{\zeta^a - \zeta^n}{1-\zeta} \\
&= \frac{\zeta^a - 1}{1-\zeta} \quad \text{since } \zeta^n = 1 \\
&= -\frac{1-\zeta^a}{1-\zeta} \\
&= -[a]_\zeta
\end{aligned}$$

□

Lemma 3.2.15 (Relationship between $P(|\lambda|)$ and $Q(|\lambda|)$).

For $n-D \leq k \leq n-1$, $\lambda \in WC(k,l)$, and $\mu \in WC(n-D,l)$ then the following expressions are equivalent:

$$\begin{aligned}
&(1) \quad P(|\lambda|) \\
&(2) \quad \frac{Q(|\lambda|)}{\zeta^{|\mu|(n-1-|\lambda|)}} \begin{bmatrix} n-1-(|\lambda|-|\mu|) \\ |\mu| \end{bmatrix}_\zeta \\
&(3) \quad \frac{Q(|\lambda|-|\mu|)}{\zeta^{|\mu|(n-1-|\lambda|)} Q(n-1-|\mu|)}
\end{aligned}$$

Proof. $\underline{(1) = (2):}$

$$P(|\lambda|) = \left(\zeta^{s(p)-|\mu|} - \zeta^{s(p)+1}\right)_\zeta^{n-1-|\lambda|} \qquad \text{by definition of } P(|\lambda|)$$

$$= \left(\zeta^{s(p)-|\mu|}\right)^{n-1-|\lambda|} \left(1-\zeta^{|\mu|+1}\right)_\zeta^{n-1-|\lambda|}$$

$$= \left(\zeta^{s(p)-|\mu|}\right)^{n-1-|\lambda|} \left(1-\zeta^{|\mu|+1}\right)\left(1-\zeta^{|\mu|+2}\right)\cdots\left(1-\zeta^{|\mu|+n-1-|\lambda|}\right)$$

by definition of $\left(1-\zeta^{|\mu|+1}\right)_\zeta^{n-1-|\lambda|}$

$$= (\zeta^{s(p)-|\mu|})^{n-1-|\lambda|}\frac{\left(1-\zeta\right)\cdots\left(1-\zeta^{|\mu|}\right)\left(1-\zeta^{|\mu|+1}\right)\cdots\left(1-\zeta^{|\mu|+n-1-|\lambda|}\right)}{\left(1-\zeta\right)\cdots\left(1-\zeta^{|\mu|}\right)}$$

$$= (\zeta^{s(p)-|\mu|})^{n-1-|\lambda|}\frac{\left(1-\zeta\right)\cdots\left(1-\zeta^{n-1-(|\lambda|-|\mu|)}\right)}{(1-\zeta)^{n-1-(|\lambda|-|\mu|)}}\frac{(1-\zeta)^{|\mu|}}{(1-\zeta)\cdots(1-\zeta^{|\mu|})}\cdot$$

$$\cdot\,(1-\zeta)^{n-1-|\lambda|}$$

$$= (\zeta^{s(p)-|\mu|})^{n-1-|\lambda|}[n-1-(|\lambda|-|\mu|)]_\zeta!\frac{1}{[|\mu|]_\zeta!}(1-\zeta)^{n-1-|\lambda|}$$

$$= \frac{\zeta^{s(p)(n-1-|\lambda|)}}{\zeta^{|\mu|(n-1-|\lambda|)}}\frac{[n-1-(|\lambda|-|\mu|)]_\zeta!}{[|\mu|]_\zeta![n-1-|\lambda|]_\zeta!}[n-1-|\lambda|]_\zeta!(1-\zeta)^{n-1-|\lambda|}$$

$$= \frac{\zeta^{s(p)(n-1-|\lambda|)}}{\zeta^{|\mu|(n-1-|\lambda|)}}\begin{bmatrix} n-1-(|\lambda|-|\mu|) \\ |\mu| \end{bmatrix}_\zeta (1-\zeta)(1-\zeta^2)\cdots(1-\zeta^{n-1-|\lambda|})$$

$$= \frac{1}{\zeta^{|\mu|(n-1-|\lambda|)}}\begin{bmatrix} n-1-(|\lambda|-|\mu|) \\ |\mu| \end{bmatrix}_\zeta (\zeta^{s(p)}-\zeta^{s(p)+1})\cdots(\zeta^{s(p)}-\zeta^{s(p)+n-1-|\lambda|})$$

$$= \frac{Q(|\lambda|)}{\zeta^{|\mu|(n-1-|\lambda|)}}\begin{bmatrix} n-1-(|\lambda|-|\mu|) \\ |\mu| \end{bmatrix}_\zeta.$$

<u>(3) = (1) :</u>

$$\frac{Q(|\lambda|-|\mu|)}{\zeta^{|\mu|(n-1-|\lambda|)}Q(n-1-|\mu|)} = \frac{(\zeta^{s(p)}-\zeta^{s(p)+1})\cdots(\zeta^{s(p)}-\zeta^{s(p)+n-1-|\lambda|+|\mu|})}{\zeta^{|\mu|(n-1-|\lambda|)}Q(n-1-|\mu|)}$$

$$= \frac{(\zeta^{s(p)}-\zeta^{s(p)+1})\cdots(\zeta^{s(p)}-\zeta^{s(p)+|\mu|})}{Q(n-1-|\mu|)}\frac{(\zeta^{s(p)}-\zeta^{s(p)+|\mu|+1})\cdots(\zeta^{s(p)}-\zeta^{s(p)+|\mu|+n-1-|\lambda|})}{\zeta^{|\mu|(n-1-|\lambda|)}}$$

$$= \frac{(\zeta^{s(p)}-\zeta^{s(p)+1})\cdots(\zeta^{s(p)}-\zeta^{s(p)+|\mu|)})}{Q(n-1-|\mu|)}(\zeta^{s(p)-|\mu|}-\zeta^{s(p)+1})\cdots(\zeta^{s(p)-|\mu|}-\zeta^{s(p)+n-1-|\lambda|})$$

$$= \frac{(\zeta^{s(p)}-\zeta^{s(p)+1})\cdots(\zeta^{s(p)}-\zeta^{s(p)+n-1-(n-1-|\mu|)})}{Q(n-1-|\mu|)}P(|\lambda|)$$

$$= \frac{Q(n-1-|\mu|)}{Q(n-1-|\mu|)}P(|\lambda|)$$

$$= P(|\lambda|).$$

□

Lemma 3.2.16 (Formula for $Q(n-1-mu)$).

For $n-D \leq k \leq n-1$, $\lambda \in WC(k,l)$, *and* $\mu \in WC(n-D,l)$

$$\frac{Q(n-1-|\mu|)}{\left(\zeta^{s(p)}-\zeta^{s(p)+1}\right)^{|\mu|}} = [|\mu|]_\zeta!.$$

Proof.

$$\begin{aligned}\frac{Q(n-1-|\mu|)}{\left(\zeta^{s(p)}-\zeta^{s(p)+1}\right)^{|\mu|}} &= \frac{\left(\zeta^{s(p)}-\zeta^{s(p)+1}\right)_\zeta^{|\mu|}}{\left(\zeta^{s(p)}-\zeta^{s(p)+1}\right)^{|\mu|}} \\ &= \frac{\zeta^{s(p)|\mu|}\left(1-\zeta\right)_\zeta^{|\mu|}}{\zeta^{s(p)|\mu|}\left(1-\zeta\right)^{|\mu|}} \\ &= \frac{\left(1-\zeta\right)\left(1-\zeta^2\right)\ldots\left(1-\zeta^{|\mu|}\right)}{\left(1-\zeta\right)^{|\mu|}} \\ &= [|\mu|]_\zeta!.\end{aligned}$$

□

Corollary 3.2.17 (Formula for $P(|\mu|)$).

For $n-D \leq k \leq n-1$, $\lambda \in WC(k,l)$, *and* $\mu \in WC(n-D,l)$

$$P(|\mu|) = \frac{Q(0)}{\zeta^{|\mu|(s(p)-|\mu|)}}\left(\frac{\zeta}{1-\zeta}\right)^{|\mu|}\frac{1}{[|\mu|]_\zeta!}.$$

Proof. By the third expression in Lemma 3.2.15, if we let $\lambda = \mu$ then we may write:

$$\begin{aligned}P(|\mu|) &= \frac{Q(|\mu|-|\mu|)}{\zeta^{|\mu|(n-1-|\mu|)}Q(n-1-|\mu|)} \\ &= \frac{Q(0)}{\zeta^{|\mu|(n-|\mu|)}\zeta^{-|\mu|}Q(n-1-|\mu|)}\end{aligned}$$

$$= \frac{\zeta^{|\mu|}Q(0)}{\zeta^{|\mu|(n-|\mu|)}Q(n-1-|\mu|)},$$

and by Lemma 3.2.16 we have that

$$\begin{aligned}
P(|\mu|) &= \frac{\zeta^{|\mu|}Q(0)}{\zeta^{|\mu|(n-|\mu|)}\left(\zeta^{s(p)} - \zeta^{s(p)+1}\right)^{|\mu|}[|\mu|]_\zeta!} \\
&= \frac{\zeta^{|\mu|}Q(0)}{\zeta^{|\mu|(s(p)-|\mu|)}\,(1-\zeta)^{|\mu|}[|\mu|]_\zeta!} \\
&= \frac{Q(0)}{\zeta^{|\mu|(s(p)-|\mu|)}\,(1-\zeta)^{|\mu|}}\frac{\zeta^{|\mu|}}{(1-\zeta)^{|\mu|}}\frac{1}{[|\mu|]_\zeta!} \\
&= \frac{Q(0)}{\zeta^{|\mu|(s(p)-|\mu|)}}\left(\frac{\zeta}{1-\zeta}\right)^{|\mu|}\frac{1}{[|\mu|]_\zeta!}.
\end{aligned}$$

□

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