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# Lowest terms in commutative rings

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# LOWEST TERMS IN COMMUTATIVE RINGS

by

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CERTIFICATE OF APPROVAL

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PH.D. THESIS

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This is to certify that the Ph.D. thesis of

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**ABSTRACT**

Putting fractions in lowest terms is a common problem for basic algebra courses, but it is rarely discussed in abstract algebra. In a 1990 paper, D.D. Anderson, D.F. Anderson, and M. Zafrullah published a paper called *Factorization in Integral Domains* [2], which summarized the results concerning different factorization properties in domains. In it, they defined an LT domain as one where every fraction is equal to a fraction in lowest terms. That is, for any $x/y$ in the field of fractions of $D$, there is some $a/b$ with $x/y = a/b$ and the greatest common divisor of $a$ and $b$ is 1. In addition, R. Gilmer included a brief exercise concerning lowest terms over a domain in his book *Multiplicative Ideal Theory* [5].

In this thesis, we expand upon those definitions. First, in Chapter 2 we make a distinction between putting a fraction in lowest terms and reducing it to lowest terms. In the first case, we simply require the existence of an equal fraction which is in lowest terms, while the second requires an element which divides both the numerator and the denominator to reach lowest terms. We also define essentially unique lowest terms, which requires a fraction to have only one lowest terms representation up to unit multiples. We prove that a reduced lowest terms domain is equivalent to a weak GCD domain, and that a domain which is both a reduced lowest terms domain and a unique lowest terms domain is equivalent to a GCD domain. We also provide an example showing that not every domain is a lowest terms domain as well as an example showing that putting a fraction in lowest terms is a strictly weaker condition

than reducing it to lowest terms.

Next, in Chapter 3 we discuss how lowest terms in a domain interacts with the polynomial ring. We prove that if $D[T]$ is a unique lowest terms domain, then $D$ must be a GCD domain. We also provide an alternative approach to some of the earlier results using the group of divisibility.

So far, all fractions have been representatives of the field of fractions of a domain. However, in Chapter 4 we examine fractions in other localizations of a domain. We define a necessary and sufficient condition on the multiplicatively closed set, and then examine how this relates to existing properties of multiplicatively closed sets.

Finally, in Chapter 5 we briefly examine lowest terms in rings with zero divisors. Because many properties of GCDs do not hold in such rings, this proved difficult. However, we were able to prove some results from Chapter 2 in this more general case.

## PUBLIC ABSTRACT


Basic algebra courses teach you to reduce fractions to lowest terms. For example, the fraction 15/20 reduces to 3/4. In a slightly more advanced course, you might see this concept for polynomials too. However, it is rarely a topic which occurs in advanced mathematics.

In the field of abstract algebra, a ring is something with addition, subtraction, and multiplication, but not necessarily division. The positive and negative integers are a common example. When you add, subtract, or multiply two integers, you get another integer. However, dividing two integers doesnt necessarily result in an integer. Ring theory is the study of these structures.

Whenever we have a ring, we can look at the fractions with the elements of the ring in the numerator and denominator. A fraction is in lowest terms if the numerator and denominator have no common factors. In this thesis, we state and prove conditions which force the fractions of a ring to have lowest terms.

We also discuss different ways to define lowest terms. First, there are many ways to talk about common factors. Second, there are two ways to change the representation of a fraction. We examine how these definitions relate to each other, and how they fit into existing properties of rings.

## TABLE OF CONTENTS

# CHAPTER 1
# INTRODUCTION

## 1.1 Motivation

In basic algebra courses, reducing fractions to lowest terms is one of the very first concepts introduced. While it remains useful through elementary mathematics classes, it's never discussed in the broader topic of abstract algebra. The usual definition of lowest terms is that the numerator and denominator of a fraction have no factors in common besides 1. This definition translates perfectly into rings, and we have a natural notion of fractions over a domain using the field of fractions. So this forms the starting point for our definition of lowest terms in a domain. A fraction $a/b$ is said to be in lowest terms if the GCD of $a$ and $b$ is 1. Of course, we have to be careful about distinguishing between elements of the field of fractions and their representation. However in practice this will not cause troubles because we are only very rarely working with the field of fractions as anything more than a formal pair of elements of the ring.

When students are taught lowest terms, it's typically done in two settings: First, it's introduced in $\mathbb{Q}$, and then seen again with rational functions. In either case, if it's proven at all it's usually done as follows. For $a, b, c$ either integers or polynomials with $b, c \neq 0$, we have

$$\frac{ac}{bc} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b} \cdot 1 = \frac{a}{b}.$$

This approach translates to an arbitrary domain $D$ and its field of fractions. However, we also have an alternative approach to show that $\frac{ac}{bc} = \frac{a}{b}$. Because the field of fractions is typically defined as the localization of $D$ at $D \setminus \{0\}$, we can use the equivalence relation from the definition of localization. Recall that for a ring $R$ (not necessarily a domain) and a multiplicatively closed set $S \subseteq R$ (with $0 \notin S$), $R_S$ is the set of all equivalence classes of ordered pairs $(r, s) \in R \times S$ with the equivalence relation $(r_1, s_2) \sim (r_2, s_2)$ if and only if there exists $s_3 \in S$ such that $s_3(s_2r_1 - s_1r_2) = 0$. Notice that when $R$ is a domain, because $s_3 \neq 0$ we must necessarily have $s_2r_1 - s_1r_2 = 0$, so that $s_2r_1 = s_1r_2$. Representing the equivalence class of $(r, s)$ as $\frac{r}{s}$, we quickly see that $\frac{ac}{bc} = \frac{a}{b}$ because $(ac)b = a(bc)$. While this approach requires far more machinery than would be taught to non-mathematicians, the fundamental concept of $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$ is certainly taught, and indeed could be used as a way to prove that we can reduce to lowest terms.

This second approach is in some sense a weaker condition. In the first we required that the numerator be written as $ac$ and the denominator as $bc$. However, the second approach didn't require this, and so we have a way to put a fraction $\frac{a'}{b'}$ into lowest terms as $\frac{a}{b}$, even if we can't write $a' = ca$ and $b' = cb$. While this distinction is meaningless in the rational numbers and rational functions, it turns out to be much more interesting in arbitrary domains and rings. We will see that in an arbitrary integral domain this difference matters, and there exist fractions that have a lowest terms representation, but cannot be reduced to lowest terms by canceling some common factor.

The other thing to notice about the elementary approach to lowest terms is that it's assumed to always exist. Both $\mathbb{Z}$ and $\mathbb{Q}[X]$ are GCD domains, so from the perspective of ring theory this is clear. In practice it wouldn't make sense for such a course to teach the difference between existence (putting something in lowest terms without finding a common factor) and construction (canceling a common factor), but it's certainly relevant to us. In chapter 2 we'll see an example where this distinction is made.

## 1.2 Overview

In Chapter 2 we will define a lowest terms domain and a number of related domains. We will prove a number of results about these structures and provide examples and counterexamples. These results have been published by the author and advisor in [3].

In Chapter 3 we will expand the results concerning lowest terms domains. We will examine the relationship between lowest terms and polynomial rings, as well as provide an alternate perspective on lowest terms.

In Chapter 4 we will formulate results on multiplicatively closed subsets of a ring. These results will then be applied to generalize the results of chapter 2.

In Chapter 5 we will briefly examine lowest terms in rings with zero divisors. We will examine some results which remain true in this case.

## 1.3 Notation and Conventions

We will use the following notation through all chapters. Any differences from these conventions will be clearly noted. As usual, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ will denote the integers, rational numbers, and real numbers respectively. We will use $\mathbb{Z}_{\geq 0}$ to denote the non-negative integers (i.e. $\{0, 1, 2, 3, \ldots\}$) and $\mathbb{Z}_{>0}$ for the positive integers (i.e. $\{1, 2, 3, \ldots\}$). $F$ and $k$ will be fields with no restriction on the characteristic unless otherwise noted. Typically $K$ will be used for the quotient field of a domain $D$, however it may be used as an arbitrary field as well.

All rings are assumed to be commutative with identity unless otherwise noted. A general ring will be called $R$, while a domain will be denoted by $D$. The set of all non-zero elements of $R$ is written $R^*$, $U(R)$ is the set of units of $R$, and $\mathrm{reg}(R)$ is the set of regular elements of $R$ (i.e. elements which are not zero divisors). The regular non-unit elements of $R$ will be denoted by $R^{\#}$. Note that in a domain $D$, $D^* = D \setminus \{0\}$ and $D^{\#} = D^* \setminus U(D)$. The GCD of two elements $a, b \in R$ is given by $\gcd(a, b)$ or $[a, b]$ and the LCM is given by $\mathrm{lcm}(a, b)$ or $]a, b[$ (if they exist). If two elements $a, b \in D$ are relatively prime (that is, the only common divisors of $a$ and $b$ are units), then $[a, b] = 1$. Even if the GCD of $a$ and $b$ does not exist, if they have any common non-unit divisors, we still write $[a, b] \neq 1$.

### 1.4 Background

Factorization has been a major topic in commmutative ring theory for quite some time. We begin with some of the well known definitions. For a domain $D$, we say that $D$ is *atomic* if every nonzero, nonunit element of $D$ is a finite product of irreducible elements (also called atoms). We say $D$ satisfies the *ascending chain condition on principal ideal (ACCP)* if every ascending chain of principal ideals stabilizes. A *bounded factorization domain (BFD)* is atomic domain where each nonzero, nonunit element has a bound on the length of its atomic factorizations. The domain $D$ is said to be a *half factorization domain (HFD)* if $D$ is atomic and for each nonzero, nonunit element of $D$, all of its atomic factorizations have the same length. We say $D$ is a *finite factorization domain (FFD)* if every nonzero, nonunit element of $D$ has only finitely many factorizations up to order and associate (or equivalently has only finitely many nonassociate divisors). Finally, $D$ is an *idf-domain* (for irreducible-divisor-finite) if each nonzero element of $D$ has at most finitely many irreducible divisors.

The following diagram shows the well known relationship between these types of domains.

$$\begin{tikzcd}
\text{UFD} \arrow[r] \arrow[rrd] & \text{HFD} \arrow[r] & \text{BFD} \arrow[r] & \text{ACCP} \arrow[r] & \text{atomic} \\
& & \text{FFD} \arrow[u] \arrow[r] & \text{idf-domain} &
\end{tikzcd}$$

In addition to these relationships, there are counterexamples which show that no further non-trivial implications can be drawn.

In 1990, D.D. Anderson, D.F. Anderson, and M. Zafrullah published a paper titled *Factorization in Integral Domains* [2]. Within it, they collected results about different factorization conditions on integral domains. They also defined a number of domains weaker than ACCP. First, a domain $D$ is *strongly atomic* if for all $a, b \in D^*$, there exists $a_1, \ldots, a_s, c, d \in D$ $(s \geq 0)$ such that $a = a_1 \cdots a_s c$, $b = a_1 \cdots a_s d$, $a_1, \ldots, a_s$ are irreducible, and $[c, d] = 1$. Next, $D$ is a *weak GCD domain* if for each $a, b \in D^*$, there exist $c, a', b' \in D$ such that $a = ca'$, $b = cb'$, and $[a', b'] = 1$. The element $c$ is called a *weak GCD for $a$ and $b$*. Third, $D$ is a *lowest terms domain (LT* *domain)* if for any $a, b \in D^*$ there exists $c, d \in D^*$ such that $a/b = c/d$ (i.e. $ad = bc$) and $[c, d] = 1$. This final definition is the one we explore most, although we'll see that the concept of a weak GCD is closely linked with that of lowest terms.

With these new definitions, we have the following relationships.

**Theorem 1.1** ([2, Theorem 1.3])**.** *Let $D$ be an integral domain. Then:*

*(a) If $D$ satisfies ACCP or $D[X]$ is atomic, then $D$ is strongly atomic.*

*(b) If $D$ is a weak GCD domain, then $D$ is an LT domain.*

*(c) $D$ is strongly atomic if and only if $D$ is an atomic weak GCD-domain.*

*Proof.* (a) First note that if $D$ satisfies ACCP, then $D[X]$ does as well. To see this, let $(f_1) \subseteq (f_2) \subseteq \cdots$ be a ascending chain of principal ideals in $D[X]$. Then $f_{i+1}|f_i$, so that $\deg(f_{i+1}) \leq \deg(f_i)$. Because degrees are bounded below by 0, the sequence of degrees must stabilize. Let $n$ be such that $\deg(f_n) = \deg(f_{n+i})$ for all $i \geq 0$. Then $f_{n+i+1}|f_{n+i}$ and $\deg(f_{n+i+1}) = \deg(f_{n+i})$ for all $i \geq 0$. Thus $f_{n+i} = r_i f_{n+i+1}$ for some $r_i \in D$ and all $i \geq 0$. If we let $a_i$ denote the leading coefficient of $f_{n+i}$, then $a_i = r_i a_{i+1}$, which gives an ascending chain in $D$: $(a_0) \subseteq (a_1) \subseteq \cdots$. By assumption, $D$ satisfies ACCP and so this chain stabilizes, say at $k$. Then $(a_k) = (a_{k+j})$ for all $j \geq 0$. Then we claim that $(f_{n+k}) = (f_{n+k+j})$ for all $j \geq 0$. Clearly $f_{n+k+j}|f_{n+k}$ and $\deg(f_{n+k+j}) = \deg(f_{n+k})$, so that $f_{n+k} = r f_{n+k+j}$ for some $r \in D$. By comparing coefficients, we see that $r a_{k+j} = a_k = u a_{k+j}$, where $u \in U(D)$ exists because $(a_k) = (a_{k+j})$. Because $D$ is a domain and $a_{k+j} \neq 0$, we obtain $r = u$ so that $f_{n+k+j} = r^{-1} f_{n+k+j}$, showing that $f_{n+k}|f_{n+k+j}$. Therefore $(f_{n+k}) = (f_{n+k+j})$ as desired.

Now in the case that $D$ satisfies ACCP, we have that $D[X]$ does as well, which means that $D[X]$ is atomic, and so it suffices to show that if $D[X]$ is atomic, then $D$ is strongly atomic. Let $a, b \in D^*$ and consider $aX + b \in D[X]$. Because $D[X]$ is atomic, we can write $aX + b = a_1 \cdots a_s(cX + d)$ where $cX + d$ and each $a_i$ is irreducible in $D[X]$. Then $a_1, \ldots, a_s$ are atoms in $D$ and $c$ and $d$ have no common factor, i.e. $[c, d] = 1$. This is precisely the condition for $D$ to be strongly atomic.

(b) Let $D$ be a weak GCD domain and let $a, b \in D^*$. Then there exists $c, a', b' \in D$ such that $a = ca'$, $b = cb'$, and $[a', b'] = 1$. But then $a/b = (ca')/(cb') = a'/b'$ with $[a', b'] = 1$ so that $D$ is an LT domain.

(c) First suppose that $D$ is strongly atomic and note that for $a, b \in D^*$, the definition of strongly atomic gives $a_1, \ldots, a_s, c, d \in D$ with $a_i$ irreducible, $[c, d] = 1$, $a = a_1 \cdots a_s c$, and $b = a_1 \cdots a_s d$. Taking $a' = c$, $b' = d$, and the $c$ from the weak GCD domain definition as $a_1 \cdots a_s$, we see that $D$ is a weak GCD domain. Now to show that $D$ is atomic, let $a \in D^\#$. Applying the definition of strongly atomic to $a$ and $a^2$, we get $a = a_1 \cdots a_s c$ and $a^2 = a_1 \cdots a_s d$ for some $a_1, \ldots, a_s \in D$ irreducible and $c, d \in D$ with $[c, d] = 1$. But then

$$a_1 \cdots a_s d = a^2 = (a_1 \cdots a_s c)^2 = a_1^2 \cdots a_s^2 c^2,$$

which shows that $d = a_1 \cdots a_s c^2$. However, because $[c, d] = 1$ we must have that $c$ is a unit. Therefore $a$ is a product of atoms, and thus $D$ is atomic.

For the other implication, assume that $D$ is an atomic weak GCD domain. Then for $a, b \in D^*$, there exists $c, a', b' \in D$ such that $a = ca'$, $b = cb'$, and $[a', b'] = 1$. Note that because $a, b \neq 0$, we get $c \neq 0$. If $c$ is a unit then we take no $a_i$'s in the definition of strongly atomic, and $a'$ and $b'$ take the place of $c$ and $d$ and we're done. If $c$ is a non-unit, then because $D$ is atomic we can write $c = a_1 \cdots a_s$ where each $a_i$ is an atom, and once again $a'$ and $b'$ take the place of $c$ and $d$, so that $D$ is strongly atomic.

$\square$

This allow us to extend the diagram above as follows:

$$\begin{tikzcd}
\text{UFD} \arrow[r] \arrow[drr] & \text{HFD} \arrow[r] & \text{BFD} \arrow[r] & \text{ACCP} \arrow[r] & \begin{array}{c}\text{strongly}\\ \text{atomic}\end{array} \arrow[r] \arrow[d] & \text{atomic} \\
 & & \text{FFD} \arrow[r] \arrow[u] & \text{idf-domain} & \begin{array}{c}\text{weak}\\ \text{GCD}\end{array} \arrow[r] & \begin{array}{c}\text{lowest}\\ \text{terms}\end{array}
\end{tikzcd}$$

However, unlike the previous diagram, we do not necessarily have examples showing no more arrows can be drawn. As we will see in Chapter 2, we were not able to find any LT domains where are not weak GCD domains. The example $A$ given later in this section is an atomic domain which is not a weak GCD domain, and thus not a strongly atomic domain. To see an example of a weak GCD domain which is not strongly atomic, first we note that an atomic GCD domain is a UFD. So to find such an example, we can take any GCD domain which is not a UFD. Let $F$ be a field and consider the monoid domain $D = F[X; \mathbb{Q}^+]$, i.e. the monoid domain formed by taking "polynomials" with coefficients in $F$ and exponents in $\mathbb{Q}^+$. A monoid domain $R[X; S]$ is a GCD domain if and only if $R$ is a GCD domain and $S$ is a torsion-free, cancellative GCD semigroup [6, Theorems 6.1 and 6.4]. In this case, $F$ is a field so certainly a GCD domain. The monoid $(\mathbb{Q}^+, +)$ is certainly torsion-free and cancellative. So all that remains is to show that $\mathbb{Q}^+$ is a GCD semigroup. Because we're considering $\mathbb{Q}^+$ additively, this definition looks slightly awkward. However, for $a, b \in \mathbb{Q}^+$ we say that $a|b$ if and only if there exists $c \in \mathbb{Q}^+$ such that $a + c = b$. So then in $\mathbb{Q}^+$, it is clear that $[a, b] = \min\{a, b\}$, so the GCD certainly exists. Therefore $D = F[X; \mathbb{Q}^+]$ is a GCD domain. However, it is clear that $D$ cannot be a UFD (e.g. $X = X^{1/2} \cdot X^{1/4} \cdot X^{1/8} \cdots$), so it cannot be atomic. Therefore $D$ is a weak GCD domain which is not strongly atomic.

To see an example of a domain which is strongly atomic but does not satisfy ACCP, we can use Example 5 in Chapter 2. The domain $A$ constructed there is an atomic weak GCD-domain, but $A[X]$ is not atomic. By the theorem above, an atomic weak GCD-domain is strongly atomic, and so $A$ is strongly atomic. However, because $A[X]$ is not atomic, it does not satisfy ACCP. A domain $D$ satisfies ACCP if and only if $D[X]$ does, so $A$ does not satisfy ACCP.

It is also worth discussing some of the background on atomic domains and domains satisfying ACCP here. First, notice that every domain satisfying ACCP must be atomic, but the converse does not hold. While the example from the previous paragraph works, we provide an alternative as well. The first counterexample to this converse was found by Anne Grams in [7]. While the details of this proof are quite technical, the example itself is fairly easy to understand. Let $F$ be a field and $T$ be the additive submonoid of $\mathbb{Q}^+$ generated by $\{1/3, 1/(2 \cdot 5), \ldots, 1/(2^j p_j), \ldots\}$ where $p_0 = 3$, $p_1 = 5, \ldots$ is the sequence of odd primes. Let $R = F[X; T]$ be the monoid domain as discussed above. Let $N = \{f \in R | f \text{ has non zero constant term}\}$. If we take $A = F[X;T]_N$, then $A$ is an atomic domain which does not satisfy ACCP. That it does not satisfy ACCP is easy to see. Take the ascending chain

$$(X) \subseteq (X^{1/2}) \subseteq \cdots \subseteq (X^{1/2^k}) \subseteq \cdots ,$$

which clearly does not stabilize. To show that $A$ is atomic, one first shows that every element of the form $X^{1/(2^k p_k)}$ is an atom, and use that to show than any $f/g \in A \setminus \{0\}$ with $f \in R$ and $g \in N$ can be written as a finite product of atoms. However, we will not include the details here.

Another major result concerning atomic domains was provided by Moshe Roitman in [9]. As shown above, if $D$ satisfies ACCP, then $D[X]$ does as well. However, this paper gave a counterexample which shows the same does not hold for atomic domains. That is, there exists a domain $D$ which is atomic while $D[X]$ is not atomic. This example will also prove to be very useful when looking for counterexamples related to lowest terms domains, so we will discuss it here. Let $D$ be a domain and $S \subseteq D \setminus \{0\}$. Let $\underline{X} = \{X_s | s \in S\}$ be a set of indeterminates. Then take $\mathscr{L}(D;S) = D[\underline{X} \cup \{s/X_s | s \in S\}]$. Intuitively, these are a subset of the Laurent polynomials where the coefficient of any terms with an inverse of $X_s$ must be a multiple of $s$. Now let $\mathrm{Red}(D)$ be the set of all reducible elements of $D$ and define $\mathscr{A}(D) = \mathscr{L}(D, \mathrm{Red}(D))$.

Now inductively define $\mathscr{A}^0(D) = D$ and $.\mathscr{A}^n(D) = \mathscr{A}(\mathscr{A}^{n-1}(D))$. By construction

$$D = \mathscr{A}^0(D) \subseteq \mathscr{A}^1(D) \subseteq \mathscr{A}^2(D) \subseteq \cdots ,$$

so it is well defined to take $\mathscr{A}^\infty(D) = \bigcup_{n=0}^{\infty} \mathscr{A}^n(D)$. Roitman gives a lot of properties of this construction, but two in particular are useful here. First, any reducible element of $\mathscr{A}^\infty(D)$ can be written as a product of two atoms, and second if $\mathscr{A}^\infty(D)$ is a weak GCD domain, then so is $D$. Much like the previous example, these require some work to prove, so it will not be done here. This construction gives us the tools needed to construct a domain $A$ where $A$ is atomic but $A[T]$ is not atomic.

Let $K$ be a field, let $X$, $Y$, and $Z$ be indeterminates, and let

$$D = F\left[Z, \left\{\frac{X}{Z^n}, \frac{Y}{Z^n}\middle| n \geq 0\right\}\right].$$

One can show that $X$ and $Y$ have no weak GCD in $D$ (and in fact we will in Example 1 in Chapter 2). Now let $A = \mathscr{A}^\infty(D)$. By the first fact about $\mathscr{A}^\infty$ stated above, every element of $A$ must be zero, a unit, an atom, or a product of two atoms. Thus $A$ is atomic. By the second fact, because $X$ and $Y$ have no weak GCD, we see that $D$ is not a weak GCD domain, and thus $A$ is not a weak GCD-domain. If $A[T]$ were atomic, then by Theorem 1.1 $A$ would be strongly atomic and hence a weak GCD domain. Thus $A[T]$ must not be an atomic domain and we have the desired example.

# CHAPTER 2
# LOWEST TERMS DOMAINS

## 2.1 Introduction and Notation

In 1990, D.D. Anderson, D.F. Anderson, and M. Zafrullah published a paper titled *Factorization in Integral Domains* [2]. Within it, they collected results about different factorization conditions on integral domains. One such property was a lowest terms (LT) domain. In grade school everyone is introduced to the concept of reducing a fraction to lowest terms. For example, given the fraction $4/12$, it can be reduced to $1/3$. We know that these two fractions are equal because $4 \cdot 3 = 1 \cdot 12$, just as we determine equality in a localization of an integral domain. Further, we know that $1/3$ cannot be simplified any further because $\gcd(1, 3) = 1$. In this example we can see why we need to make a distinction between the element as a member of the quotient field and it's representation as $a/b$. Despite being the same object in $\mathbb{Q}$, $4/12$ and $1/3$ are different representations.

Throughout this chapter, $K$ will be the quotient field of the domain $D$. Recall that in $K$, $a/b = c/d \Leftrightarrow ad = bc$. We will use $a/b$ to show a representation of an element of $K$, while if $b|a$ in $D$, then $\frac{a}{b}$ will denote the element of $D$ such that $b\frac{a}{b} = a$. Where possible we will use letters at the end of the alphabet (primarily $x$ and $y$) to denote fractions with any representation, while letters at the beginning of the alphabet (mostly $a$ and $b$) will denote fractions in a lowest terms representation. Recall that for a domain $D$ with quotient field $K$, a fractional ideal is a $D$-submodule of $K$. For a non-zero fractional ideal $I$ of $D$, the inverse of $I$, $I^{-1}$, is defined to be $\{x \in K | xI \subseteq D\}$ and $I_v := (I^{-1})^{-1} = \cap\{Dx | x \in K \text{ with } I \subseteq Dx\}$. This mapping $I \to I_v$ on fractional ideals has the following properties for all $a \in K \setminus \{0\}$ and all $A, B$ non-zero fractional ideals of $D$:

(1) $(a)_v = (a)$ and $(aA)_v = aA_v$,

(2) $A \subseteq A_v$ and $A \subseteq B \Rightarrow A_v \subseteq B_v$, and

(3) $(A_v)_v = A_v$.

Given $a, b \in D$, if $d \in D$ is a common divisor for $a$ and $b$ and $\left[\frac{a}{d}, \frac{b}{d}\right] = 1$, then $d$ is a *weak GCD of $a$ and $b$*. If every pair of elements in $D$ has a GCD (resp., weak GCD), then $D$ is a *GCD domain* (resp., *weak GCD domain*).

Before proceeding, we state two results concerning $v$-ideals and their relationship to the GCD and LCM.

**Proposition 2.1** ([1, §2]). *Let $a, b, d \in D$ such that $(a, b)_v = (d)$, then $[a, b] = d$. The converse does not necessarily hold.*

**Proposition 2.2** ([1, §2]). *Let $a, b \in D$. The following are equivalent:*

*(1)* $\operatorname{lcm}(a, b)$ *exists*

*(2)* $(a) \cap (b)$ *is principal*

*(3)* $(a, b)_v$ *is principal*

*In this case, $(aD \cap bD)(a, b)_v = abD$.*

Notice that this implies that $(a, b)_v = D$ if and only if $aD \cap bD = abD$. In this case we say that $a$ and $b$ are $v$-coprime.

To close this section, we provide a few more definitions. An element $a \in D^{\#}$ is said to be *irreducible* if $a = bc$ implies that $b$ or $c$ is a unit. Such elements are also known as *atoms*, and an *atomic* domain is one where every non-zero, non-unit element is a product of atoms. Finally, $D$ satisfies the *ascending chain condition on principal ideals (ACCP)* if every chain $(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \cdots$ of principal ideals stabilizes.

## 2.2 Definitions

In the example given above where 4/12 is rewritten as 1/3, we know that this can be done by "canceling the 4". However, in a general integral domain it may not be possible to arrive at a lowest terms representation in this manner. Thus in the definitions we make a distinction between putting a representation in lowest terms and reducing it to lowest terms. First, let's define exactly what we mean by "lowest terms." Because there are multiple ways to call two elements coprime, we define a different version of lowest terms for each. For now we will only care about 3 different ways (the GCD of $a$ and $b$, the $v$-ideal generated by $a$ and $b$, and the ideal generated by $a$ and $b$). Later in this chapter, we will extend these definitions to a more general concept.

**Definition 2.1.** Let $a, b \in D^*$. We say that $a/b$ is in *lowest terms (resp., strong lowest terms, absolute lowest terms)* if $[a, b] = 1$ (resp., $(a, b)_v = D$, $(a, b) = D$).

**Definition 2.2.** Let $x, y \in D^*$. We say that $x/y$ can be *put in the form* $a/b$ if $x/y = a/b$. Further, we say that $x/y$ can be *put in lowest terms (resp., strong lowest* *terms, absolute lowest terms)* if it can be put in the form $a/b$ where $a/b$ is in lowest terms (resp., strong lowest terms, absolute lowest terms). The integral domain $D$ is a *lowest terms (LT) domain* if every non-zero fraction $a/b$ $(a, b \in D^*)$ can be put in lowest terms.

**Definition 2.3.** Let $x, y \in D^*$. We say that $x/y$ can be *reduced to the form $a/b$* if there exists a common divisor $d \in D$ of $x$ and $y$ such that $a = \frac{x}{d}$ and $b = \frac{y}{d}$. We say $x/y$ can be *reduced to lowest terms (resp., strong lowest terms, absolute lowest terms* if it can be reduced to the form $a/b$ where $a/b$ is in lowest terms (resp., strong lowest terms, absolute lowest terms). That is, there exists some $d \in D^*$ such that $d|a$, $d|b$, and $\frac{a}{d}/\frac{b}{d}$ is in in lowest terms (resp. strong lowest terms, absolute lowest terms). The integral domain $D$ is a *reduced lowest terms (RLT) domain* if every non-zero fraction $a/b$ $(a, b \in D^*)$ can be reduced to lowest terms.

It is also of interest to know when the lowest terms representation is unique. However, because of the presence of units, it is more complicated than simply having the same numerator and denominator.

**Definition 2.4.** Let $x, y \in D^*$. We say that $x/y$ has *essentially unique (reduced) lowest terms* if $x/y$ can be put in (reduced to) lowest terms $a/b$, and if for any $c/d$ in (reduced) lowest terms with $x/y = c/d$, there exists a unit $u \in D$ such that $c = ua$ and $d = ub$. The integral domain $D$ is a *unique lowest terms (ULT) domain* if every non-zero fraction $a/b$ $(a, b \in D^*)$ has essentially unique lowest terms.

It is immediately clear that if a fraction can be reduced to lowest terms, it can be put in lowest terms. Thus every RLT domain is an LT domain. Similarly if a fraction has essentially unique lowest terms, it can be put in lowest terms. Thus every ULT domain is an LT domain.

**Remark.** There are many other domains we could have defined: unique reduced lowest terms domain, strong lowest terms domain, reduced strong lowest terms domain, unique strong lowest terms domain, and unique reduced strong lowest terms domain. However these all are equivalent to being a GCD domain as shown in Proposition 2.9.

Further, we could have defined absolute lowest terms domain, reduced absolute lowest terms domain, unique reduced lowest terms domain, and unique reduced absolute lowest terms domain. These conditions are all equivalent to being a Bézout domain by Proposition 2.11.

**Remark.** Notice that in these definitions we only considered fractions $a/b$ where $a, b$ are non-zero. Because we're working in the quotient field $K$, it is clear why $b$ needs to be non-zero. If $a = 0$, then because $[0, b] = b$ and $(0, b)_v = (b)$, $0/b$ can be reduced to $0/1$, which is essentially unique. Thus there's no loss of generality by requiring $a, b \in D^*$.

## 2.3 Results

**Lemma 2.3.** *For $x, y \in D^*$, $x/y$ can be put in lowest terms if and only if there exists an $s \in D^*$ such that $sx$ and $sy$ have a weak GCD. If $d$ is a weak GCD for $sx$ and $sy$, then $x/y = \left(\frac{sx}{d}\right) / \left(\frac{sy}{d}\right)$, and the latter is in lowest terms. Thus $D$ is an LT domain if and only if for all $x, y \in D^*$, there exists an $s \in D^*$ such that $sx$ and $sy$ have a weak*

*GCD.*

*Proof.* First assume there exists $s \in D^*$ such that $sx$ and $sy$ have a weak GCD $d \in D^*$. Then $\left[\frac{sx}{d}, \frac{sy}{d}\right] = 1$, and so

$$x/y = sx/sy = \left(\frac{sx}{d}\right) \Big/ \left(\frac{sy}{d}\right).$$

Therefore $x/y$ can be put in lowest terms.

Now assume that $x/y$ can be put in lowest terms. Then there exists $a, b \in D^*$ such that $x/y = a/b$ and $[a, b] = 1$, so that $xb = ya$. Take $s = a$, then $sx = xa$ and $sy = ay = xb$, with $[a, b] = 1$, which means $x$ is a weak GCD for $sx$ and $sy$.

The last statement immediately follows. $\square$

**Proposition 2.4.** *Let $x, y \in D^*$. The fraction $x/y$ can be reduced to lowest terms if and only if $x$ and $y$ have a weak GCD. In this case, if $d$ is the weak GCD of $x$ and $y$, then $x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ and the last fraction is in lowest terms. Therefore $D$ is a weak GCD domain if and only if it is an RLT domain*

*Proof.* Let $x, y \in D^*$. We know that $d$ is a weak GCD for $x$ and $y$ if and only if $d$ divides $x$ and $y$, and $\left[\frac{x}{d}, \frac{y}{d}\right] = 1$. But this is equivalent to $x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$, where the second fraction is in lowest terms. This proves the first statement, and the second immediately follows. $\square$

**Proposition 2.5.** *Let $x, y \in D^*$. The following are equivalent:*

*(1)* $[x, y]$ *exists,*

*(2)* *(a) If $c$ is a common divisor of $x$ and $y$, then $\left(\frac{x}{c}\right) / \left(\frac{y}{c}\right)$ can be reduced to lowest terms, and*

*(b) $x/y$ has essentially unique reduced lowest terms*

*Proof.* (1) $\Rightarrow$ (2): If $[x, y] = d$ exists, then for any common divisor $c$ of $x$ and $y$, $\left[\frac{x}{c}, \frac{y}{c}\right]$ exists, and thus is the unique weak GCD (up to units) of $\frac{x}{c}$ and $\frac{y}{c}$. Thus by Proposition 2.4, $\left(\frac{x}{c}\right) / \left(\frac{y}{c}\right)$ can be reduced to lowest terms, showing $(a)$. To see $(b)$, notice that $d = [x, y]$ is the unique (again, up to units) element of $D$ such that $\left[\frac{x}{d}, \frac{y}{d}\right] = 1$. Thus if $\left(\frac{x}{c}\right) / \left(\frac{y}{c}\right)$ and $\left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ are both reduced lowest terms representations of $x/y$, then $c$ and $d$ only differ by a unit: $d = uc$, which means that $\frac{x}{d} = u^{-1}\frac{x}{c}$ and $\frac{y}{d} = u^{-1}\frac{y}{c}$. Thus $x/y$ has essentially unique reduced lowest terms, i.e. $(b)$ holds.

(2) $\Rightarrow$ (1): Assume $(a)$ and $(b)$ and let $x, y \in D^*$. Then by taking $c = 1$ in $(a)$, $x/y$ can be reduced to lowest terms, which means that $x$ and $y$ have a weak GCD $d$ by Proposition 2.4. We claim that $[x, y] = d$. Clearly $d$ is a common divisor of $x$ and $y$, so let $c$ be any common divisor of $x$ and $y$. Then by $(b)$, $\left(\frac{x}{c}\right) / \left(\frac{y}{c}\right)$ can be reduced to lowest terms, which again by Proposition 2.4 implies that $\frac{x}{c}$ and $\frac{y}{c}$ have a weak GCD, say $e$. Further, this proposition says that $\left(\frac{x}{ce}\right) / \left(\frac{y}{ce}\right)$ is in lowest terms. But then

$$\left(\frac{x}{d}\right) \Big/ \left(\frac{y}{d}\right) = x/y = \left(\frac{x}{ce}\right) \Big/ \left(\frac{y}{ce}\right),$$

and by $(b)$, because $x/y$ has essentially unique reduced lowest terms, $\frac{x}{d} = u\frac{x}{ce}$ for some unit $u$. Thus $d = u^{-1}ce$, which shows that $c|d$, and therefore $d = [x, y]$. $\square$

The following lemma will prove useful any time we're working with the $v$-operation.

**Lemma 2.6.** *For $a, b \in D^*$, the following are equivalent:*

*(1)* $(a,b)_v = D$ *(or equivalently* $aD \cap bD = abD$*),*

*(2) For all* $x \in D^*$*, if* $a|bx$ *then* $a|x$*, and*

*(3) For all* $x, y \in D^*$*, if* $a/b = x/y$*, then there exists* $d \in D^*$ *such that* $x = da$*, and hence* $y = db$*.*

*Proof.* (1) $\Rightarrow$ (2): Assume that $(a,b)_v = D$, and let $x \in D^*$ such that $a|bx$. Then $bx \in (a) \subseteq (a)_v$, and certainly $ax \in (a)_v$. So then

$$(a) = (a)_v = (a, ax, bx)_v = (a, x(a,b)_v)_v = (a,x)_v,$$

thus $x \in (a,x)_v = (a)$, so $a|x$.

(2) $\Rightarrow$ (3): Because $a/b = x/y$, $ay = bx$, so $a|bx$, which by assumption means $a|x$. Thus there exists $d$ such that $x = ad$. Further, $ay = bx = bad$, so $y = bd$.

(3) $\Rightarrow$ (1): We will prove this by showing that $aD \cap bD = abD$. One inclusion, $abD \subseteq aD \cap bD$, is always true. For the other, let $r \in aD \cap bD$. Then there exist $x, y \in D^*$ such that $r = ay = bx$. Thus $a/b = x/y$. By (3) there exists $d \in D^*$ such that $y = db$, and thus $r = ay = abd \in abD$. Therefore $aD \cap bD = abD$, so $(a,b)_v = D$. $\square$

**Remark.** Notice that, as convenient as it would be, $[a,b] = 1$ is generally not a sufficient condition for (2). Consider as an example $F[X^2, X^3]$ where $F$ is a field and $X$ is an indeterminate over $F$. We have $[X^2, X^3] = 1$, and $X^3|X^2 \cdot X^4$, but $X^3 \nmid X^4$. As the theorem would imply, $(X^2, X^3)_v \neq D$ because $X^2D \cap X^3D \supsetneq X^5D$ (for example $X^6$ is in $X^2D \cap X^3D$ but not $X^5D$).

Recall that the proof of the fact that if $[a,b] = 1$ and $a|bx$ then $a|x$ in the integers, you write $bx = ac$ for some $c$, and then use the fact that if $[a,b] = 1$, there exist intergers $m$ and $n$ such that $ma+nb = 1$, so that $x = max+nbx = max+nac = a(mx+nc)$, which shows $a|x$. However, it is not true in general that such a linear combination exists. Domains where such a combination exists for every pair are called Bézout domains. Equivalently, a Bézout domain $D$ is one where $(a,b)$ is principal for all $a,b \in D$. Then if $(a,b) = (d)$, $(a,b)_v = (d)_v = (d)$, showing that $(a,b)_v$ is principal and so if $D$ is a Bézout domain then part (1) of the proposition holds for all $x,y \in D^*$.

**Proposition 2.7.** *Let $a,b,x,y \in D^*$. The fraction $a/b$ is in strong lowest terms if and only if for all $x,y \in D^*$ such that $a/b = x/y$, there exists some $d \in D^*$ with $x = da$ and $y = db$.*

*Proof.* This is simply (1) $\Leftrightarrow$ (3) in Lemma 2.6, noting that $a/b$ is in strong lowest terms if and only if $(a,b)_v = D$. $\square$

**Proposition 2.8.** *Let $x,y \in D^*$. Then the following are equivalent:*

*(1) $(x,y)_v$ is principal,*

*(2) $x/y$ can be reduced to strong lowest terms, and*

*(3) $x/y$ can be put in strong lowest terms*

*If $(x,y)_v = (d)$ then $x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ where the second fraction is in strong lowest terms. Further, this representation is unique in the sense that if $x/y = a/b$, where $[a,b] = 1$, then $a = u\left(\frac{x}{d}\right)$ and $b = u\left(\frac{y}{d}\right)$ with $u$ a unit of $D$. Thus $a/b$ is actually a strong lowest terms representation for $x/y$.*

*Proof.* (1) $\Rightarrow$ (2): Assume that $(x,y)_v = (d)$. Then $[x,y] = d$, and in particular $d|x$ and $d|y$. Then

$$\left(\frac{x}{d}, \frac{y}{d}\right)_v = \frac{1}{d}(x,y)_v = \frac{1}{d}(d) = D,$$

which shows that $x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ with the second fraction in strong lowest terms. Thus $x/y$ can be reduced to strong lowest terms. This also proves the second statement.

(2) $\Rightarrow$ (3): Trivially holds.

(3) $\Rightarrow$ (1): Assume that $x/y = a/b$ where $(a,b)_v = D$. Then $bx = ay$ and

$$(x) = xD = x(a,b)_v = (xa, xb)_v = (ax, ay)_v = a(x,y)_v,$$

which shows that $(x,y)_v = \left(\frac{x}{a}\right)$. (Notice that $a|x$ by Lemma 2.6, so that $\frac{x}{a} \in D$.)

It remains to show the final statement regarding uniqueness. Let $a, b \in D^*$ such that $x/y = a/b$ and $[a,b] = 1$. Because $\left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ is in strong lowest terms and $a/b = x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$, by Proposition 2.7, there exists some $c \in D^*$ such that $a = c\frac{x}{d}$ and $b = c\frac{y}{d}$. But then

$$1 = [a,b] = \left[c\frac{x}{d}, c\frac{y}{d}\right],$$

which means that $c$ must be a unit, completing the proof. □

**Proposition 2.9.** *For an integral domain $D$, the following are equivalent:*

*(1) $D$ is a GCD domain,*

*(2) Every non-zero fraction of $D$ can be reduced to strong lowest terms,*

*(3) Every non-zero fraction of $D$ can be put in strong lowest terms,*

*(4) Every non-zero fraction of $D$ has essentially unique reduced lowest terms, and*

*(5) $D$ is an RLT domain and a ULT domain.*

*Proof.* Note that an equivalent definition of a GCD domain is that for all $a, b \in D^*$, $(a, b)_v$ is principal. Thus the equivalence of $(1) - (3)$ follows directly from Proposition 2.8. Further $(1) \Rightarrow (4)$ follows from the final statement of Proposition 2.8, and $(4) \Rightarrow (5)$ is simply the definition of an RLT domain and a ULT domain. Finally, $(5) \Leftrightarrow (1)$ is precisely Proposition 2.5. □

Now we turn to absolute lowest terms. First, we provide a result analogous to Proposition 2.8:

**Proposition 2.10.** *Let $x, y \in D^*$. Then the following are equivalent.*

*(1) $(x, y)$ is principal,*

*(2) $x/y$ can be reduced to absolute lowest terms, and*

*(3) $x/y$ can be put in absolute lowest terms*

*If $(x, y) = (d)$, then $x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ where the second fraction is in strong lowest terms. Further, this representation is unique.*

*Proof.* $(1) \Rightarrow (2)$: If $(x, y) = (d)$, then $x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ and $\left(\frac{x}{d}, \frac{y}{d}\right) = (1) = D$, so that we've reduced $x/y$ to absolute lowest terms. This also proves the second statement.

$(2) \Rightarrow (3)$: Clear.

$(3) \Rightarrow (1)$: Assume that $x/y = a/b$ where $(a, b) = D$. Then $bx = ay$ and

$$(x) = xD = x(a, b) = (xa, xb) = (ax, ay) = a(x, y),$$

which shows that $(x, y) = \left(\frac{x}{a}\right)$. (Note that because $D = (a, b) \subseteq (a, b)_v \subseteq D$, we have $a|x$ by Lemma 2.6.)

For the uniqueness statement, we have $(a,b)_v = (d)_v = (d)$, and so the uniqueness follows from Proposition 2.9. □

Notice that $D$ is a Bézout domain if and only if $(a,b)$ is principal for every $a, b \in D^*$. Thus we have a characterization of Bézout domain following directly from Proposition 2.10:

**Proposition 2.11.** *For an integral domain $D$, the following are equivalent:*

*(1) $D$ is a Bézout domain*

*(2) Every non-zero fraction of $D$ can be reduced to absolute lowest terms*

*(3) Every non-zero fraction of $D$ can be put in absolute lowest terms*

We can generalize Propositions 2.8 and 2.10 to a single result. First a definition. Let $D$ be an integral domain and let $K$ be its quotient field. Let $F(D)$ be the set of nonzero fractional ideals of $D$. A *$\star$-operation* is a mapping $F(D) \to F(D)^\star$, which satisfies the following conditions for all $a \in K$ and all nonzero fractional ideals $A, B$:

(1) $(a)^\star = (a)$ and $(aA)^\star = aA^\star$

(2) $A \subseteq A^\star$ and if $A \subseteq B$ then $A^\star \subseteq B^\star$.

(3) $(A^\star)^\star = A^\star$

Notice that the $v$-operation defined above and the identity operation $A_d = A$ are both $\star$-operations.

We can say that $a/b$ is in *$\star$-lowest terms* if $(a,b)^\star = D$, and that $x/y$ can be *put in* (equivalently *reduced to*) *$\star$-lowest terms* if there exist $a, b \in D^*$ with $x/y = a/b$

where $a/b$ is in $\star$-lowest terms. Thus a *$\star$-lowest terms domain* is an integral domain where every nonzero fraction can be put in $\star$-lowest terms. It immediately follows that a $v$-lowest term domain is the same as a strong lowest terms domain and that a $d$-lowest terms domain is the same as an absolute lowest terms domain.

This leads to a generaliztion of Propositions 2.8 and 2.10. Its proof is similar to that of those two propositions with the appropriate modifications, so we omit it here.

**Proposition 2.12.** *Let $x, y \in D^*$. Then the following are equivalent.*

*(1) $(x, y)^\star$ is principal,*

*(2) $x/y$ can be reduced to $\star$-lowest terms, and*

*(3) $x/y$ can be put in $\star$-lowest terms.*

*If $(x, y)^\star = (d)$, then $x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ where the second fraction is in $\star$-lowest terms. Further, this representation is unique.*

Interestingly, Robert Gilmer considered lowest terms in a single exercise [5, Exercise 5, Page 183]. Here is the exercise:

> Each nonzero element of $K$ is representable in the form $a/b$ for some $a, b \in D \setminus \{0\}$. A representation $a/b$ is said to be in *canonical form* if $a/b = c/d$, where $c, d \in D \setminus \{0\}$, implies that there is an element $x$ in $D$ such that $c = ax$ and $d = bx$. The representation $a/b$ is said to be *irreducible* if the only common divisors of $a$ and $b$ in $D$ are units. Let $G$ be the group of divisibility of $D$. Prove:

(1) The representation $a/b$ of an element of $K \setminus \{0\}$ is in canonical form if and only if $aU$ and $bU$ are disjoint; the representation is irreducible if and only if $aU$ and $bU$ are weakly disjoint. Hence a representation in canonical form is also irreducible.

(2) Each nonzero element of $K$ admits a representation in canonical form if and only if $D$ is a GCD domain.

Notice that canonical form is equivalent to strong lowest terms via Proposition 2.8, while irreducible is clearly equivalent to lowest terms. In Section 3.3, we will examine the how the group of divisibility is related to lowest terms. We will discuss part (1) in more detail there. However, part (2) is (1) $\Leftrightarrow$ (3) of Proposition 2.9.

It turns out that RLT domains are very common, as the following theorem shows.

**Theorem 2.13.** *Let $D$ be an integral domain. If $D$ is a GCD domain or if $D$ satisfies ACCP, then $D$ is a weak GCD domain and hence an RLT domain.*

*Proof.* If $D$ is a GCD domain, then the result is clear. So assume that $D$ satisfies ACCP. One proof of this result can be found in [2, Theorem 1.3]. However, what follows is a proof without going through another property. Assume to the contrary that $D$ is not a weak GCD domain. Then there exists some pair of elements which does not have a weak GCD. Now let

$$S = \{(a) | a \in D^* \text{ and there exists some } b \in D^* \text{ where } a \text{ and } b \text{ have no weak GCD}\}.$$

Notice that this set is nonempty because of our assumption, and $D \notin S$ because 1 is a weak GCD for $u$ and $a$ where $u$ is a unit and $a$ is any element of $D$. Because $D$ satisfies ACCP, $S$ has a maximal element $(a)$ which is a proper principal ideal. Let $b \in D^*$ such that $a$ and $b$ have no weak GCD. In particular this means that $[a, b] \neq 1$, but there exists some non-unit common divisor $e$ of $a$ and $b$. Then $\frac{a}{e} \in D^*$, and because $e$ is a non-unit, $\left(\frac{a}{e}\right) \supsetneq (a)$. Because $\left(\frac{a}{e}\right)$ is properly larger it cannot be in $S$, and thus $\frac{a}{e}$ has a weak GCD with every element of $D^*$, in particular with $\frac{b}{e}$. Let $d$ be that weak GCD. This means that

$$\left[\frac{a}{de}, \frac{b}{de}\right] = \left[\frac{\frac{a}{e}}{d}, \frac{\frac{b}{e}}{d}\right] = 1.$$

But then $de$ is a weak GCD for $a$ and $b$, which is a contradiction. Therefore our assumption was false, so that $D$ is a weak GCD domain and thus an RLT domain. □

**Corollary 2.14.** *Let $D$ be an integral domain. $D$ is a UFD if and only if $D$ is a ULT domain which satisfies ACCP.*

*Proof.* This is mainly combining implications between various integral domains. First assuming that $D$ is a UFD, then it satisfies ACCP and it is a GCD domain. But by Proposition 2.9, this means it's also a ULT domain.

Now assume that $D$ satisfies ACCP and is a ULT domain. Because $D$ satisfies ACCP, it's an RLT domain by Theorem 2.13. Thus $D$ is an RLT domain and a ULT domain, which means $D$ is a GCD domain by Proposition 2.9. It is well known that a GCD domain which satisfies ACCP is a UFD. □

The introduction of LT domains and weak GCD domains was in the context of atomic factorization. If $D$ satisfies ACCP, then $D$ is atomic, but a counterexample to the converse exists [7]. Further, $D$ satisfies ACCP if and only if $D[X]$ does. However, it is not true that $D$ is atomic if and only if $D[X]$ is atomic. The counterexample was first provided by Moshe Roitman in [9]. Because of the ubiquity of RLT domains, having examples which do not satisfy ACCP will prove useful in the following section. We now examine how the strongly atomic property which was discussed in Chapter 1 is related to lowest terms.

**Theorem 2.15.** *Let $D$ be an integral domain. The following are equivalent.*

*(1) $D$ is an atomic RLT domain*

*(2) $D$ is an atomic weak GCD domain*

*(3) $D$ is strongly atomic*

*(4) Every linear polynomial in $D[X]$ is a product of atoms.*

*Proof.* (1) $\Leftrightarrow$ (2): This is simply Proposition 2.4.

(2) $\Rightarrow$ (3): This is Theorem 1.1(c).

(3) $\Rightarrow$ (4): Let $aX + b \in D[X]$ be a linear polynomial. This means that $a \in D^*$. If $b = 0$, then because $D$ is strongly atomic, it is atomic, so $a$ is a product of atoms in $D$. We note that atoms in $D$ remain atoms in $D[X]$ and that $X$ is an atom in $D[X]$, thus $aX$ is a product of atoms in $D[X]$. If $b \neq 0$, then because $D$ is strongly atomic, there exist $c, d \in D^*$ and atoms $a_1, \ldots, a_s \in D^*$ such that $a = a_1 \cdots a_s c$, $b = a_1 \cdots a_s d$, and $[c, d] = 1$. But then $aX + b = a_1 \cdots a_s(cX + d)$, where $cX + d$ is also an atom because $[c, d] = 1$. Thus $aX + b$ is a product of atoms as desired.

(4) $\Rightarrow$ (2): First, let $a \in D^*$ be a non-unit. Then $aX$ is a linear polynomial, so

it's a product of atoms: $aX = a_1 \cdots a_s X$, which means that $a = a_1 \cdots a_s$ is a product of atoms. Thus $D$ is atomic. Now let $a, b \in D^*$, so that $aX + b$ is a linear polynomial and thus a product of atoms: $aX + b = a_1 \cdots a_s(cX + d)$ where each $a_i$ is an atom and $[c, d] = 1$. Then $e = a_1 \cdots a_s$ is a weak GCD for $a$ and $b$ because $\left[\frac{a}{e}, \frac{b}{e}\right] = [c, d] = 1$. Therefore $D$ is a weak GCD domain. □

## 2.4 Examples

Many of these examples lie somewhere between $D$ and its quotient field. This makes notation regarding an element versus its representation particularly challenging. However, when discussing representations of elements, we will continue to use $a/b$ as before. While it's not surprising, it is important to know that not every domain is an LT domain. Here is one such example.

**Example 1.** *(An integral domain that is not an LT domain.)*

*Let $F$ be a field, let $X$, $Y$, and $Z$ be indeterminates over $F$, and let*

$$D = F[X, Y, Z]\left[\left\{\frac{X}{Z^n}, \frac{Y}{Z^n}\right\}_{n \geq 1}\right].$$

*First notice that for any $a \in D$, we can write $a$ as $\frac{f}{Z^n}$ for some $f \in F[X, Y, Z]$. Not every $f \in F[X, Y, Z]$ is possible, but every element of $D$ can be written this way.*

*Now we claim that $X/Y$ has no lowest terms representation. If we write $X/Y = a/b$ where $a/b \in D^*$, we can write $a = \frac{f}{Z^m}$ and $b = \frac{g}{Z^n}$ where $f, g \in F[X, Y, Z]^*$ and $m, n \in \mathbb{Z}_{\geq 0}$. Because $X/Y = a/b$,*

$$X\frac{g}{Z^n} = Xb = Ya = Y\frac{f}{Z^m},$$

*and thus* $XgZ^m = YfZ^n$. *This shows that* $X|f$ *and* $Y|g$ *in* $F[X,Y,Z]$. *Then* $\frac{X}{Z^m}|\frac{f}{Z^m}$ *and* $\frac{Y}{Z^n}|\frac{g}{Z^n}$ *in* $D$, *which is the same as saying* $\frac{X}{Z^m}|a$ *and* $\frac{Y}{Z^n}|b$ *in* $D$. *However, because* $Z$ *divides* $\frac{X}{Z^m}$ *and* $\frac{Y}{Z^n}$ *in* $D$, *this shows that* $Z$ *divides* $a$ *and* $b$. *Therefore* $[a,b] \neq 1$, *which means there can be no lowest terms representation of* $X/Y$.

By combining Theorem 2.13 and Corollary 2.14, we know that any domain which satisfies ACCP but is not a UFD must be an RLT domain which is not a ULT domain. Here is an example of such a domain

**Example 2.** *(An RLT domain that is not a ULT domain.)*

*Consider* $D = F[X^2, X^3]$ *where* $F$ *is a field and* $X$ *is an indeterminate over* $F$. *This domain is Noetherian, hence it satisfies ACCP, so it must be an RLT domain by Theorem 2.13. Further,* $X^3 \cdot X^3 = X^2 \cdot X^2 \cdot X^2$, *which shows that* $D$ *is not a UFD, hence it is not a ULT domain. One example showing that it is not a ULT domain is* $X^4/X^3 = X^3/X^2$, *where both* $[X^4, X^3] = 1$ *and* $[X^3, X^2] = 1$. *However, there is clearly no unit* $u$ *such that* $X^4 = uX^3$ *and* $X^3 = uX^2$.

*This also provides an example where* $X^5/X^4$ *can be reduced to lowest terms as* $\left(\frac{X^5}{X^2}\right) / \left(\frac{X^4}{X^2}\right) = X^3/X^2$, *however, it can only be put in lowest terms form as* $X^4/X^3$, *because there is no* $d \in D^*$ *such that* $\frac{X^5}{d} = X^4$ *and* $\frac{X^4}{d} = X^3$. *On the other hand,* $X^6/X^5$ *can be reduced to both* $\left(\frac{X^6}{X^2}\right) / \left(\frac{X^5}{X^2}\right) = X^4/X^3$ *and* $\left(\frac{X^6}{X^3}\right) / \left(\frac{X^5}{X^3}\right) = X^3/X^2$. *This is because both* $X^2$ *and* $X^3$ *are weak GCDs for* $X^5$ *and* $X^6$.

In the previous example $X^5/X^4$ cannot be reduced to $X^4/X^3$, but it can be reduced to $X^3/X^2$. This leads to the following question: Is there a fraction in some domain that can be put in lowest terms but cannot be reduced to lowest terms? The following example shows that the answer is yes.

**Example 3.** *(A fraction that can be put in lowest terms but not reduced to lowest terms.)*

*Let $F$ be a field and let $X$, $Y$, $Z$, and $T$ be indeterminates over $F$. Let*

$$D = F[X,Y,Z,T]\left[\frac{X}{T}, \frac{Y}{T}, \left\{\frac{X}{Z^n}, \frac{Y}{Z^n}\right\}_{n\geq 1}\right].$$

*Notice that $X/Y$ can be reduced to lowest terms $\left(\frac{X}{T}\right) / \left(\frac{Y}{T}\right)$. To check that $\left[\frac{X}{T}, \frac{Y}{T}\right] = 1$, let $d$ be a common divisor of $\frac{X}{T}$ and $\frac{Y}{T}$. Then $dT$ is a common divisor of $X$ and $Y$. However, up to units the only common divisors of $X$ and $Y$ are $1$, $T$, and powers of $Z$. Clearly $dT$ isn't a unit, and $dT$ can't be a unit multiple of $Z^n$, so $dT$ must be a unit multiple of $T$, i.e. $d$ must be a unit, showing that $\left[\frac{X}{T}, \frac{Y}{T}\right] = 1$.*

*Now consider $\left(\frac{X}{Z}\right) / \left(\frac{Y}{Z}\right)$. Notice that the set of divisors of $\frac{X}{Z}$ is*

$$\left\{uZ^n, u\frac{X}{Z^{n+1}} \,\middle|\, u \in F^*, n \in \mathbb{Z}_{\geq 0}\right\},$$

*and similarly the set of divisors of $\frac{Y}{Z}$ is*

$$\left\{uZ^n, u\frac{Y}{Z^{n+1}} \,\middle|\, u \in F^*, n \in \mathbb{Z}_{\geq 0}\right\}.$$

*Then the common divisors of $\frac{X}{Z}$ and $\frac{Y}{Z}$ are of the form $uZ^n$ for $u \in F^*$ and $n \in \mathbb{Z}_{\geq 0}$. Thus $\frac{X}{Z}$ and $\frac{Y}{Z}$ have no weak GCD, which means that $\left(\frac{X}{Z}\right) / \left(\frac{Y}{Z}\right)$ cannot be reduced to lowest terms. However it can be put in lowest terms $\left(\frac{X}{T}\right) / \left(\frac{Y}{T}\right)$.*

*We believe that this is in fact an LT domain, however we do not currently have a proof of this. If this is an LT domain, then it also provides an example where this* *property is not maintained by localization. Let* $S = \{T^n | n \in \mathbb{Z}_{\geq 0}\}$ *and localize to* $D_S$. *Then by the same argument as in Example 1,* $X/Y$ *has no lowest terms representation in* $D_S$. *Thus* $D_S$ *is not an LT domain, while* $D$ *is conjectured to be.*

Before discussing the next two examples, we provide some background. As mentioned above, Moshe Roitman provided an example of an atomic domain $A$ where $A[X]$ is not atomic [9]. In fact he provided a construction for such examples which is also useful here. For a subset $S \subseteq D \setminus \{0\}$, let $\underline{X} = \{X_s | s \in S\}$. Let $\mathscr{L}(D;S) = D\left[\underline{X} \cup \left\{\frac{s}{X_s} \middle| s \in S\right\}\right]$. Now define $\mathscr{A}(D) = \mathscr{L}(D; \mathrm{Red}(D))$ where $\mathrm{Red}(D)$ is the set of reducible elements of $D$. Inductively define $\mathscr{A}^0 = \mathscr{A}^0(D) = D$, $\mathscr{A}^n = \mathscr{A}^n(D) = \mathscr{A}(\mathscr{A}^{n-1}(D))$, and $\mathscr{A}^\infty(D) = \bigcup_{n=0}^{\infty} \mathscr{A}^n(D)$. Notice that $D \subseteq \mathscr{A}^n(D) \subseteq \mathscr{A}^{n+1}(D) \subseteq \mathscr{A}^\infty(D)$ for all $n \in \mathbb{Z}_{\geq 0}$. A more complete examination of the properties of $\mathscr{A}^\infty$ can be found in [9, §3]. Of particular interest to us are the following two facts:

(1) Every reducible element in $\mathscr{A}^\infty(D)$ is a product of two atoms. Hence $\mathscr{A}^\infty(D)$ is atomic.

(2) If $\mathscr{A}^\infty(D)$ is a weak GCD domain, then so is $D$.

**Example 4.** *(An integrally closed atomic domain that is not an LT domain.)*

*Let* $F$ *be a field, let* $X$, $Y$, *and* $Z$ *be indeterminates over* $F$, *and let*

$$D = F[X, Y, Z]\left[\left\{\frac{X}{Z^n}, \frac{Y}{Z^n}\right\}_{n \geq 1}\right]$$

*as in Example 1. From that example we know that if* $X/Y = a/b$ *for* $a, b \in D$, *then* $Z|a$ *and* $Z|b$. *Let* $A = \mathscr{A}^\infty(D)$. *Because* $D$ *is not an LT domain, it is not a weak GCD domain. Thus by the contrapositive to fact* (2) *above,* $A$ *is not a weak GCD* *domain and hence not an RLT domain. We claim that it is not even an LT domain.*

*To show the claim, we show that that if $X/Y = a/b$ for $a, b \in A$, then $Z|a$ and $Z|b$. Before showing this, we need a lemma:*

**Lemma.** *Let $R$ be an integral domain, let $a, b, s \in R^*$. Let $X$ be an indeterminate over $R$. Assume that there exists some $t \in R^*$ with the following property: if $a/b = c/d$ where $c, d \in R^*$, then $t|c$ and $t|d$. If $a/b = f/g$ for $f, g \in R\left[X, \frac{s}{X}\right]^*$, then $t|f$ and $t|g$.*

*Proof.* Write

$$f = r_{-n}\frac{s^n}{X^n} + \cdots + r_{-1}\frac{s}{X} + r_0 + r_1 X + \cdots + r_m X^m$$

$$g = t_{-n}\frac{s^n}{X^n} + \cdots + t_{-1}\frac{s}{X} + t_0 + t_1 X + \cdots + t_m X^m$$

where $r_i, t_i \in R$ for $-n \leq i \leq m$. Because $a/b = f/g$, $ag = bf$, so by comparing coefficients (and canceling the $s^i$ if necessary), $at_i = br_i$ for $-n \leq i \leq m$. Because $a, b \neq 0$, this means that $r_i = 0$ if and only if $t_i = 0$. In the case where $t_i, r_i \neq 0$, we have $a/b = r_i/t_i$, and hence by assumption $t|r_i$ and $t|t_i$. On the other hand, if $t_i, r_i = 0$, then clearly we have $t|r_i$ and $t|t_i$. In either case, $t|r_i$ and $t|t_i$, which shows that $t|f$ and $t|g$ because it divides each term of the sum. □

*Now we show the claim. First, notice that as shown in Example 1, $D$ satisfies the lemma with $a := X$, $b := Y$, $t := Z$. Next, we can apply the lemma inductively because by the lemma, each successive $R\left[X, \frac{s}{X}\right]$ satisfies the hypothesis. However, there exists some $\mathscr{A}^n(D)$ such that $a, b \in \mathscr{A}^n(D)$, and thus $a, b$ are simply polynomials in indeterminates of the form $X_s$ or $\frac{s}{X_s}$ for some $s \in \operatorname{Red}(\mathscr{A}^i(D))$ where $0 \leq i \leq n$.*

*However, because polynomials can only have a finite number of indeterminates, by applying the lemma inductively we show that $Z|a$ and $Z|b$, thus $a/b$ cannot be put in lowest terms. Therefore $A = \mathscr{A}^\infty(D)$ is an atomic domain which is not an LT domain.*

In addition to the $\mathscr{A}^n$ and $\mathscr{A}^\infty$ constructions, Roitman gave a second method to construct domains where $A$ is atomic but $A[X]$ is not atomic. It is a generalization of the first method. Let $D$ be a domain and let $\mathscr{I}$ be a family of ideals in $D$. Define

$$\mathscr{L}(D, \mathscr{I}) = D\left[\left\{X_I, \frac{c}{X_I} | I \in \mathscr{I} \text{ and } c \in I\right\}\right].$$

For $k \geq 1$, let $\mathscr{I}_k(D)$ be the family of ideals $\{I_\alpha\}_{\alpha\in\Lambda}$ where at least one of the following holds:

- $I_\alpha$ is generated by $k$ elements which have infinitely many non-associate common divisors in $D$, or
- $I_\alpha$ is principal and generated by a reducible element.

Let $\mathscr{L}_k(D) = \mathscr{L}(D, \mathscr{I}_k)$. For a given $k \geq 0$, inductively define $\mathscr{A}_{k,0}(D) = D$, $\mathscr{A}_{k,n} = \mathscr{A}_{k,n}(D) = \mathscr{L}_k(\mathscr{A}_{k,n-1}(D))$, and $\mathscr{A}_{k,\omega} = \mathscr{A}_{k,\omega}(D) = \bigcup_{n=0}^{\infty} \mathscr{A}_{k,n}(D)$.

**Remark.** Although not explicitly stated in [9], it turns out that $\mathscr{A}_{1,m} = \mathscr{A}^m$ (and thus $\mathscr{A}_{1,\omega} = \mathscr{A}^\infty$). To see this, first we show that as long as $a$ has at least two nonunit nonassociate divisors, then $a$ must be reducible. We do this via the contrapositive: If $a$ is irreducible, then write $a = b_1c_1$ and $a = b_2c_2$. Because $a$ is irreducible, $b_i$ or $c_i$ is a unit in each case. By relabeling, assume that $b_1$ and $b_2$ are units. So then $c_1 = b_1^{-1}a = (b_1^{-1}b_2)c_2$, so $c_1$ and $c_2$ are associates. Because these are arbitrary nonunit divisors, it follows that every nonunit divisor of $a$ is associate.

Because $\mathscr{I}_1$ consists entirely of principal ideals, for $I \in \mathscr{I}_1$, $I = (a)$. By the two properties which define $\mathscr{I}_1$, we have $a$ has infinitely many nonassociate common divisors, or $a$ is reducible. In either case $a \in \mathrm{Red}(D)$ by the previous paragraph. On the other hand, $a \in \mathrm{Red}(D)$ clearly means that $(a) \in \mathscr{I}_1$. Thus $a \in \mathrm{Red}(D) \Leftrightarrow (a) \in \mathscr{I}_1$. But then the correspondence between $D\left[\left\{X_s, \frac{s}{X_s} | s \in \mathrm{Red}(D)\right\}\right]$ and $D\left[\left\{X_I, \frac{c}{X_I} | I \in \mathscr{I}_1 \text{ and } c \in I\right\}\right]$ becomes clear. Although the second one seem as though it could be larger due to the $c \in I$, because $I = (d)$ is principal $c = ad$ for some $a$, any power of $a$ can be absorbed into the coefficient, meaning that $D\left[\left\{X_I, \frac{c}{X_I} | I \in \mathscr{I}_1 \text{ and } c \in I\right\}\right] = D\left[\left\{X_{(d)}, \frac{d}{x_{(d)}} | (d) \in \mathscr{I}_1\right\}\right]$.

This shows that $\mathscr{A}(D) = \mathscr{L}_1(D) = \mathscr{A}_{1,1}(D)$, and so by induction $\mathscr{A}^m(D) = \mathscr{A}_{1,m}(D)$.

Before proceeding to the next example, we need a few more definitions: For any set $S \subseteq D$, $m$ is said to be a maximal common divisor (MCD) of $S$ if $m|s$ for all $s \in S$ (i.e. $m$ is a common divisor of $S$) and for any $d$ which divides every element of $S$, if $m|d$ then $d$ is an associate of $m$. Equivalently, $m$ is an MCD of $S$ if and only if $(1/m)S = \{s/m | s \in S\}$ has GCD 1. This immediately shows that a weak GCD for $a$ and $b$ is the same as an MCD of $\{a, b\}$. A $k$-MCD domain is a domain $D$ where every subset of $D$ with exactly $k$ elements has an MCD and an MCD domain is a domain where every subset has an MCD. Thus a weak GCD domain is equivalent to a 2-MCD domain. Also note that every one element subset has an MCD (namely that element), so every domain is a 1-MCD domain.

As with $\mathscr{A}^m$, more properties can be found in [9]. The following facts are needed here:

(1) Every reducible element in $\mathscr{A}_{k,\omega}(D)$ is a product of two atoms. Hence $\mathscr{A}_{k,\omega}(D)$ is atomic, and

(2) $\mathscr{A}_{k,\omega}(D)$ is a $k$-MCD domain.

**Example 5.** *(An atomic RLT domain $A$ such that $A[T]$ is neither atomic nor an LT domain.)*

*Let $X_1$, $X_2$, $X_3$, and $Z$ be indeterminates over a field $F$. Let*

$$D = F\left[Z, \left\{\frac{X_1}{Z^n}, \frac{X_2}{Z^n}, \frac{X_3}{Z^n}\right\}_{n\geq 0}\right].$$

*Let $A = \mathscr{A}_{2,\omega}(D)$. By fact (1), every reducible element of $A$ is a product of two atoms, so that $A$ is atomic. By fact (2), $A$ is a 2-MCD (≡weak GCD≡RLT) domain. By [9, Lemmas 5.3 and 5.4], $\{X_1, X_2, X_3\}$ has no MCD in $A$, so that $X_1T + X_2$ and $X_3$ have no weak GCD in $A[T]$. But then $A[T]$ is not an RLT domain. This shows that reduced lowest terms is not preserved by polynmoial extensions. For discussion of why $A[T]$ is not atomic, see [9]. As an overview, $A$ is a 2-MCD domain, but not a 3-MCD domain. Thus it is not an MCD domain. This implies that $A[T]$ is not atomic.*

*We further claim that $A[T]$ is not even an LT domain. To show this, we show that if $(X_1T + X_2)/X_3 = a/b$ for $a, b \in A[T]$, then $Z|a$ and $Z|b$. First we need a lemma which is very similar to the one in Example 4:*

**Lemma.** *Let $D$ be an integral domain, let $a, b \in D^*$. Let $X$ be an indeterminate over*

*$D$. Assume that there exists some $t \in D^*$ with the following property: if $a/b = c/d$ where $c, d \in D^*$, then $t|c$ and $t|d$. Let $I$ be a non-zero ideal of $D$. If $a/b = f/g$ for $f, g \in D\left[X, IX^{-1}\right]^*$, then $t|f$ and $t|g$.*

*Proof.* The proof is nearly identical to that of the corresponding lemma in Example 4. $\square$

*Now let $a/b \in A[T]$. Much like in Example 4, there exists some $n$ such that $a, b \in \mathscr{A}_{2,n}(D)$, and we can inductively apply the lemma to reach the desired conclusion. In particular, if we let $S$ be a subring of $A$ which contains $D$ with the property that for any $a, b \in S[T]$ if $(X_1T + X_2)/X_3 = a/b$ then $Z|a$ and $Z|b$, and for any ideal $I$ of $S$ and indeterminate $Y$ we notice that*

$$S[Y, IY^{-1}] = S[T][Y, IS[T]Y^{-1}].$$

*So then we can apply the lemma starting with $D = S[T]$, $t = Z$, and $I$ being one of the ideals that shows up in the set of $I$s which correspond to indeterminates in either $a$ or $b$. Because the result of the lemma gives a ring which satisfies the hypothesis, we can repeat this process as many times as needed to get the desired result.*

The results and examples of this chapter can be summarized as follows.

$$\begin{tikzcd}
 & \text{UFD} \arrow[dl] \arrow[dr] & & \text{Bézout} \arrow[dl] \\
\text{ACCP} \arrow[d] & & \text{GCD} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\text{strongly}\\ \text{atomic}\end{array} \arrow[r] \arrow[d] & \text{RLT} \arrow[dr] & & \text{ULT} \arrow[dl] \\
\text{atomic} & & \text{LT} &
\end{tikzcd}$$

No further non-trivial arrows can be drawn, except for possibly the two we haven't provided counterexamples for: RLT$\Rightarrow$LT and GCD$\Rightarrow$LT. Both of these remain open questions.

We finish this chapter by giving brief examples of the instability of the various lowest terms domains under different ring constructions.

- Homomorphic images do not preserve any of these domains. For any set of indeterminates $\{X_a\}$, $\mathbb{Z}[\{X_a\}]$ is a UFD. However, for any integral domain $D$ there exists a set of indeterminates $\{X_a\}$ and a homomorphism $\phi : \mathbb{Z}[\{X_a\}] \to D$ such that $\phi(\mathbb{Z}[\{X_a\}]) = D$. So taking $D$ to be any of the examples above gives the proper counterexamples.
- Because fields satisfy the definitions of all of the lowest terms domains (for any fraction $a/b$ with $a, b \in F$, $a/b = ab^{-1}/1$, which is in lowest terms), these definitions are not preserved by subrings. Take $D$ to be an example above, let $K$ be it's quotient field, and then $D \subseteq K$ gives a counterexample.
- Example 1 shows that none of the properties are preserved by overrings. This is because $k[X,Y,Z]$ is a UFD, but $k[X,Y,Z]\left[\left\{\frac{X}{Z^n}, \frac{Y}{Z^n}\right\}_{n\geq 1}\right]$ is an overring which is not even an LT domain.
- If Example 3 is in fact an LT domain, then this gives an example of an LT domain whose localization is not an LT domain. Let $S$ be the multiplicatively closed set generated by $T$, so that

$$D_S = R[X,Y,Z]\left[\left\{\frac{X}{Z^n}, \frac{Y}{Z^n}\right\}_{n\geq 1}\right]$$

where $R$ is the ring $F[T, T^{-1}]$. Thus we know from Example 1 that $D_S$ is not an LT domain.

# CHAPTER 3
# MORE ON LT DOMAINS

## 3.1 Introduction

Within this chapter we examine other properties and generalizations of lowest terms domains. We'll first look at the relationship between lowest terms and the polynomial rings. Next we give an alternate characterization of lowest terms using groups of divisibility. Finally, we generalize lowest terms to localizations other than the field of fractions.

The notation follows that of the previous chapter.

## 3.2 Lowest Terms and the Polynomial Ring

Notably absent from the list at the end of the last section is how LT domains interact with the polynomial ring. We dedicate this section to investigating these relationships. Throughout the section, $T$ or $X$ is an indeterminate over $D$ or $F$ as indicated by context. First, it is well known that $D$ is a GCD domain if and only if $D[T]$ is a GCD domain. The remaining properties we're examining need more explanation. Before proceeding, we provide an example that initially seems counterintuitive.

**Example 6.** *Let $D = F[X^2, X^3]$ where $X$ is an indeterminate over a field $F$. As noted in Example 2, $X^3/X^2 = X^4/X^3$ shows that $D$ is not a ULT domain. Now consider $D[T]$. We have*

$$X^3/X^2 = \left(X^3 + X^4T\right) / \left(X^2 + X^3T\right).$$

*This can be checked by cross multiplying:*

$$X^3(X^2 + X^3T) = X^5 + X^6T = X^2(X^3 + X^4T).$$

*Further, in* $D[T]$, $[X^3 + X^4T, X^2 + X^3T] = 1$, *so that* $X^3/X^2$ *can be put in lowest terms as* $(X^3 + X^4T) \big/ (X^2 + X^3T)$. *To see this is in lowest terms, We'll prove a pair of more general results.*

**Lemma 3.1.** *Let* $a, b \in D^*$ *and let* $\sum_{i=0}^{n} a_iT^i, \sum_{i=0}^{n} b_iT^i \in D[T]$. *Then* $a/b = \left(\sum_{i=0}^{n} a_iT^i\right) \Big/ \left(\sum_{i=0}^{n} b_iT^i\right)$ *if and only if* $a/b = a_i/b_i$ *for all* $i = 0, \ldots, n$.

*Proof.* We have $a/b = \left(\sum_{i=0}^{n} a_iT^i\right) \Big/ \left(\sum_{i=0}^{n} b_iT^i\right)$ if and only if $\sum_{i=0}^{n} ab_iT^i = \sum_{i=0}^{n} ba_iT^i$ if and only if $ab_i = ba_i$ for all $i = 0, \ldots, n$ if and only if $a/b = a_i/b_i$ for all $i = 0, \ldots, n$. This completes the proof. □

**Lemma 3.2.** *Let* $a, b, c, d \in D^*$ *such that* $a/b = c/d$ *are two distinct lowest terms representations. That is,* $[a, b] = [c, d] = 1$ *and* $a \neq uc$ *and* $b \neq ud$ *for any unit* $u \in U(D)$. *Then* $a/b = (a + cT)/(b + dT)$ *where the second fraction is in lowest terms*

*Proof.* By the previous lemma, $a/b = (a + cT)/(b + dT)$. So all that remains is to show that the GCD of $a + cT$ and $b + dT$ is 1. Let $x + yT$ be a common divisor of $a + cT$ and $b + dT$. (Clearly any common divisor has degree at most one.) Then there exists $m_1, m_2, n_1, n_2 \in D^*$ such that

$$a + cT = (x + yT)(m_1 + n_1T) = xm_1 + (xn_1 + ym_1)T + yn_1T^2 \text{ and}$$

$$b + dT = (x + yT)(m_2 + n_2T) = xm_2 + (xn_2 + ym_2)T + yn_2T^2.$$

This means that $yn_1 = yn_2 = 0$, so that $n_1 = n_2 = 0$ or $y = 0$.

In the first case, $xm_1 + ym_1T = a + cT$ and $xm_2 + ym_2T = b + dT$, so that $a = xm_1$, $b = xm_2$, $c = ym_1$, $d = ym_2$. Thus $x|[a, b]$ and $y|[c, d]$, which means $x$ and $y$ are units. However, this means that $a = xm_1 = xy^{-1}c$, which is a contradiction because $a$ and $c$ are not unit multiples of each other. Thus it must be the case that $y = 0$.

When $y = 0$, $xm_1 + xn_1T = a + cT$ and $xm_2 + xn_2 = b + dT$. This means that $a = xm_1$ and $b = xm_2$, so that $x|[a, b]$. Thus $x$ is a unit, and so $x + yT = x + 0T = x$ is a unit, meaning that $x + yT|1$, and therefore the GCD of $a + cT$ and $b + dT$ is 1. $\square$

This lemma shows that $(X^3 + X^4T) \big/ (X^2 + X^3T)$ from the example above is in lowest terms.

We will begin by examining how RLT domains interact with polynomial rings.

**Proposition 3.3.** *If $D[T]$ is an RLT domain, then $D$ is an RLT domain. The converse is not generally true.*

*Proof.* Let $D[T]$ be an RLT domain and let $x, y \in D^* \subseteq D[T]^*$ then $x/y$ can be reduced to $\left(\frac{x}{d}\right) \Big/ \left(\frac{y}{d}\right)$ for some $d \in D[T]$, where $d|x, y$ and the second fraction is in lowest terms. But the only possible divisors of $x$ or of $y$ in $D[T]$ are in $D$, so that $d \in D$ and thus $x/y$ can be reduced to lowest terms in $D$. Therefore $D$ is an RLT domain.

Example 5 shows that the converse is not true. $\square$

Next, if $D[T]$ is a ULT domain, we in fact have something stronger than $D$

being a ULT domain, as this result shows.

**Proposition 3.4.** *Let $D$ be an integral domain and let $T$ be an indeterminate over $D$. If $D[T]$ is a ULT domain, then $D$ is a GCD domain.*

*Proof.* Let $a, b \in D^*$. First we show that if $a/b = f(T)/g(T)$ where $f(T)/g(T)$ is in lowest terms, then we have $f, g \in D^*$. If $D$ is finite, then it is a field and so the result holds trivially. So assume $D$ is infinite. Because $ag(T) = bf(T)$, we must have $\deg(f) = \deg(g)$. Assume to the contrary that this degree is greater than zero. We claim this means there exists some $r_0 \in D$ such that $f(T - r_0)$ is not associate to $f(T)$. Notice for any $r \in D$ such that $f(T - r) = uf(T)$ for some unit, if $s$ is a root of $f(T)$ in some algebraic closure of $D$ (which exists because $\deg(f) > 0$) then $f(s - r) = uf(s) = 0$. Thus $s - r$ is also a root. Further, for distinct $r_1, r_2$, we must have that $s - r_1$ and $s - r_2$ are distinct. Because $f$ has only finitely many roots, there can be only finitely many $r \in D$ for which $f(T - r)$ is associate to $f(T)$. Thus because $D$ is infinite, we can pick $r_0$ making $f(T - r_0)$ not associate to $f(T)$.

Notice that by substitution we also have $a/b = f(T-r)/g(T-r)$ for all $r \in D$. Further, we have $[f(T), g(T)] = d(T)$ if and only if $[f(T-r), g(T-r)] = d(T-r)$, so that $f(T-r_0)$ and $g(T-r_0)$ are relatively prime because $f(T)$ and $g(T)$ are. Because $D[T]$ is a ULT domain, this means that $f(T-r_0) = uf(T)$ and $g(T-r_0) = ug(T)$ for some unit $u \in U(D[T]) = U(D)$. This contradicts the choice of $r_0$ so that $f(T - r_0)$ and $f(T)$ are nonassociate. Thus $\deg(f) = \deg(g) = 0$, so that $f, g \in D^*$. In particular, this means that $D$ is a ULT domain.

Now let $a/b = c/d$ where $c, d \in D^*$ and $c/d$ is in lowest terms. Then $(aT +$

$c)/(bT+d) = a/b$, so there exists some nonunit common divisor $f$ of $aT+c$ and $bT+d$ (otherwise $(aT + c)/(bT + d)$ would be in lowest terms, contradicting uniqueness). Certainly $\deg f \leq 1$, so let $f = \alpha T + \beta$. If $\alpha = 0$, then there exist $r_1T + r_2, s_1T + s_2 \in D[T]$ such that $\beta(r_1T + r_2) = aT + c$, so that $\beta|c$ and $\beta(s_1T + s_2) = bT + d$, so that $\beta|d$. Thus $f = \beta|[c, d]$ and is therefore a unit, which is a contradiction.

Thus $\alpha \neq 0$, so there exist $r, s \in D$ such that $(\alpha T + \beta)r = aT + c$, so that $\beta|c$ and $(\alpha T + \beta)s = bT + d$, so that $\beta|d$. Then $\beta$ is a common divisor of $c$ and $d$, which means it must be a unit. So every common divisor has the form $\alpha T + u$ where $\alpha \in D^*$ and $u \in U(D)$. Thus we can pick $\alpha$ such that $\alpha T + 1$ is a common divisor of $aT + c$ and $bT + d$.

Then $(\alpha T+1)r = aT+c$ and $(\alpha T+1)s = bT+d$, which means that $r = c$ and $s = d$. Thus $\alpha c = a$ and $\alpha d = b$, meaning that $a/b$ can be reduced to lowest terms $\left(\frac{a}{\alpha}\right) / \left(\frac{b}{\alpha}\right) = c/d$. Therefore $D$ is a ULT domain and an RLT domain, which shows that $D$ is a GCD domain. □

This gives the following corollary.

**Corollary 3.5.** *Let $D$ be an integral domain and let $T$ be an indeterminate over $D$. The following are equivalent*

*(1) $D[T]$ is a ULT domain,*

*(2) $D$ is a GCD domain, and*

*(3) $D[T]$ is a GCD domain*

*Proof.* (1) $\Rightarrow$ (2) is the previous proposition, (2) $\Leftrightarrow$ (3) is a standard result, and

(3) $\Rightarrow$ (1) is Proposition 2.9. $\square$

Example 5 also shows that $D[T]$ is not necessarily an LT domain. Although at a first glance it seems as though the converse should be true ($D[T]$ LT $\Rightarrow$ $D$ LT), the example given at the beginning of this section casts some doubt on this. It seems possible that there could be a fraction $x/y$ which cannot be put in lowest terms in $D$, but could be put in lowest terms in $D[T]$. This remains an open question.

### 3.3 Lowest Terms Using the Group of Divisibility

An alternate perspective on lowest terms can be given through the group of divisibility. First recall the definition of the group of divisibility: $G(D) = K^*/U(D)$ where $K$ is the quotient field of $D$ and $U(D)$ is the group of units of $D$. Where there is no cause for confusion, we will write $G = G(D)$ and $U = U(D)$. $G$ can be given a partial order by $aU \leq bU$ if and only if $\frac{b}{a} \in D^*$. Another way to state this is that $a|b$ in $D$. Notice that if $a, b \in D^*$ then this is the same as saying that $(b) \subseteq (a)$ as ideals of $D$. Additionally, for $k \in K^*$ we have $1U \leq kU$ if and only if $k \in D^*$. It is customary to write the group of divisibility as an additive group, so that $aU + bU = abU$. The goal of this section is to find conditions on $G$ which are equivalent to the various types of lowest terms domains. The first is well known, but stated and proved here for completeness:

**Theorem 3.6.** *An integral domain $D$ is a GCD domain if and only if $G(D)$ is a lattice ordered group.*

*Proof.* First assume that $D$ is a GCD domain. To show that $G$ is a lattice ordered group, I must show that for any pair $a_1U, a_2U \in G$, the join $a_1U \vee a_2U$ (least upper bound) and the meet $a_1U \wedge a_2U$ (greatest lower bound) exist. Write $a_i = x_i/y_i$ where $x_i/y_i$ is in lowest terms, so $[x_i, y_i] = 1$. Because $D$ is a GCD domain, $[x_1, x_2]$ and $]y_1, y_2[$ exist, so let $d = \frac{[x_1,x_2]}{]y_1,y_2[} \in K$. We claim that $dU = a_1U \wedge a_2U$. Certainly

$$\frac{a_i}{d} = \frac{x_i/y_i}{[x_1, x_2]/]y_1, y_2[} = \frac{x_i}{[x_1, x_2]} \cdot \frac{]y_1, y_2[}{y_i} \in D,$$

so $dU \leq a_iU$. Now let $cU \leq a_1U$ and $cU \leq a_2U$. We must show that $cU \leq dU$. Because $c|(x_1/y_1)$ and $c|(x_2/y_2)$, we have $cy_1y_2|x_1y_2$ and $cy_1y_2|x_2y_1$. But then $cy_1y_2|[x_1y_2, x_2y_1]$. We now use a series of properties of the GCD to show

$$[x_1y_2, x_2y_1] = [x_1y_2, [x_2, x_1y_2][y_1, x_1y_2]] = [x_1y_2, [x_2, x_1][y_1, y_2]] = [x_2, x_1][y_1, y_2].$$

The first equality is because $[a, bc] = [a, [a, c][a, b]]$, the second is because $[x_i, y_i] = 1$, and the third is because $[x_2, x_1]$ and $[y_1, y_2]$ divide $x_1y_2$. Thus $cy_1y_2|[x_1, x_2][y_1, y_2]$, and so $c\big|[x_1, x_2]\frac{[y_1,y_2]}{y_1y_2}$, which is the same as $c\big|\frac{[x_1,x_2]}{]y_1,y_2[}$, i.e. $c|d$. Therefore $cU \leq dU$ and so $dU$ is the greatest lower bounds of $a_1U$ and $a_2U$: $a_1U \wedge a_2U = dU$.

A very similar argument shows that $a_1U \vee a_2U = \frac{]x_1,x_2[}{[y_1,y_2]}U$. Therefore the meet and the join exist so that $G$ is a lattice ordered group.

On the other hand, if $G(D)$ is a lattice ordered group, then for any pair $a, b \in D^*$, $aU \wedge bU$ exists, call it $dU$. We claim that $d = [a, b]$. Because $dU \leq aU$ and $dU \leq bU$, $d|a$ and $d|b$, so that it is a common divisor. Further, for any common divisor $c \in D^*$ of $a$ and $b$, because $cU \leq aU$ and $cU \leq bU$, $cU \leq dU$ because $dU$ is the greatest lower bound. Thus $c|d$, which shows that $d$ is the GCD of $a$ and $b$. Therefore any pair of nonzero elements of $D$ has a GCD. □

Now using Proposition 2.9, we have an equivalence for a "strong lowest terms domain" (which we never formally defined, as mentioned in a remark in Section 2.2, but it is clear) using the group of divisibility. For further inspiration, we turn to the exercise from Gilmer quoted above [5, Exercise 5, Page 183]. In particular:

> The representation $a/b$ of an element of $K\setminus\{0\}$ is in canonical form [strong lowest terms] if and only if $aU$ and $bU$ are disjoint; the representation is irreducible [in lowest terms] if and only if $aU$ and $bU$ are weakly disjoint. Hence a representation in canonical form is also irreducible.

In an additive partially ordered group $G$ (not necessarily a group of divisibility), two elements are disjoint if their meet (greatest lower bound) exists and is equal to the identity. Equivalently, two element are disjoint if their join (least upper bound) exists and is equal to their sum. Two elements $a, b \in G$ are weakly disjoint if $a, b \in G_+ = \{g \in G | 0_G \leq g\}$ and there is no element $c \in G$ such that $0_G < c \leq a$ and $0_G < c \leq b$.

We now state and prove this exercise as a proposition:

**Proposition 3.7.** *Let $D$ be an integral domain with group of divisibility $G$. Let $a, b \in D^*$.*

*(a) $a/b$ is in strong lowest terms if and only if $aU$ and $bU$ are disjoint.*

*(b) $a/b$ is in lowest terms if and only if $aU$ and $bU$ are weakly disjoint.*

*Proof.*

(a) First assume that $a/b$ is in strong lowest terms. We need to show that $aU \wedge bU = 1U$. Clearly $1|a$ and $1|b$, so that $1U$ is a lower bound for $aU$ and $bU$. Let

$cU \in G$ be some other lower bound. Then $\frac{a}{c}, \frac{b}{c} \in D^*$ and $a/b = \left(\frac{a}{c}\right) / \left(\frac{b}{c}\right)$. By Proposition 2.8, there exists some $d \in D^*$ such that $\frac{a}{c} = da$ and $\frac{b}{c} = db$. But then $1 = cd$, so that $d|1$ and therefore $dU \leq 1U$. Therefore $1U$ is the greatest lower bound for $aU$ and $bU$, which shows that $aU \wedge bU = 1U$, i.e. that $aU$ and $bU$ are disjoint.

On the other hand, if $aU$ and $bU$ are disjoint, then by the second equivalent definition of disjoint, $aU \vee bU = abU$. To show that $a/b$ is in strong lowest terms, we must show that $(a,b)_v = D$, which is equivalent to showing $aD \cap bD = abD$. Clearly $abD \subseteq aD \cap bD$, so let $x \in aD \cap bD$. Then there exist $c, d \in D^*$ such that $x = ac = bd$. This shows that $a|x$ and $b|x$, so that $aU \leq xU$ and $bU \leq xU$ which means that $xU$ is an upper bound for $aU$ and $bU$. Because $abU$ is the least upper bound, $abU \leq xU$. Thus $ab|x$, so that $x \in abD$. This shows that $aD \cap bD = abD$, and thus $a/b$ is in strong lowest terms.

(b) Assume that $a/b$ is in lowest terms. Let $cU \in G$ such that $1U \leq cU \leq aU$ and $1U \leq cU \leq bU$. Then $c|a$ and $c|b$. However, because $[a,b] = 1$, $c$ must be a unit, which means $cU = 1U$. Therefore $aU$ and $bU$ are weakly disjoint. Now assume that $aU$ and $bU$ are weakly disjoint and let $c$ be a common divisor of $a$ and $b$. Then $1U \leq cU \leq aU$ and $1U \leq cU \leq bU$. By the weakly disjoint condition, we have $cU = 1U$, so that $c$ is a unit. Thus the only common divisors of $a$ and $b$ are units, i.e. $[a,b] = 1$. Therefore $a/b$ is in lowest terms.

$\square$

While this proposition gives nice criteria to check for (strong) lowest terms, it doesn't give a way to check whether a domain is a (strong, reduced, unique) lowest terms domain. However, by extending this result, we can check for a lowest terms domain or a strong lowest terms domain (that is, a GCD domain).

**Proposition 3.8.** *Let $D$ be an integral domain with group of divisibility $G$. Then $D$ is an LT domain (resp., strong LT domain≡GCD domain) if and only if for all $kU \in G$ there exist $aU, bU \in G_+ = \{gU \in G | 1U \leq gU\}$ such that $aU$ and $bU$ are weakly disjoint (resp., disjoint) and $kU = aU - bU$.*

*Proof.* First assume that $D$ is an LT domain (resp., strong LT domain). Let $kU \in G$. Then $k \in K^*$, so that $k = x/y$ for some $x, y \in D^*$. Because $D$ is an LT domain (resp., strong LT domain), there exist $a, b \in D^*$ such that $k = x/y = a/b$ where $a/b$ is in lowest terms (resp., strong lowest terms). By Proposition 3.7(2) (resp., 3.7(1)), $aU$ and $bU$ are weakly disjoint (resp., disjoint). Further, $kU = (a/b)U = (ab^{-1}U) = aU - bU$, completing this implication.

Now assume that for all $kU \in G$, there exists $aU, bU \in G$ such that $aU$ and $bU$ are weakly disjoint (resp., disjoint) and $kU = aU - bU$. Let $x, y \in D^*$, so that $x/y \in K^*$. Then $(x/y)U \in G$, so there exist $aU$ and $bU$ fulfilling the condition. Because $aU$ and $bU$ are weakly disjoint (resp., disjoint), by Proposition 3.7(2) (resp., 3.7(1)) $a/b$ is in lowest terms (resp., strong lowest terms). Further, $(x/y)U = aU - bU = (a/b)U$, so $x/y = ua/b$. Therefore $x/y$ can be put in lowest terms (resp., strong lowest terms), which shows that $D$ is an LT domain (resp., strong LT domain). □

Combining this proposition with Theorem 3.6 gives an interesting condition to check if a group of divisibility is lattice-ordered: $G$ is lattice ordered if and only if for all $kU \in G$ there exist $aU, bU \in G_+ = \{gU \in G | 1U \leq gU\}$ such that $aU$ and $bU$ are disjoint and $kU = aU - bU$. A similar proposition gives a criterion for an RLT domain:

**Proposition 3.9.** *Let $D$ be an integral domain with group of divisibility $G$. Then $D$ is an RLT domain if and only if for all $kU \in G$ and all $aU, bU \in G_+$ with $kU = aU - bU$, there exists some $dU \in G_+$ such that $aU - dU$ and $bU - dU$ are weakly disjoint.*

*Proof.* First assume that $D$ is an RLT domain. Let $kU \in G$, and let $aU, bU \in G_+$ such that $kU = aU - bU$. Then $a, b \in D^*$, and so there exists some $d \in D^*$ (i.e. $dU \in G_+$) such that $\left(\frac{a}{d}\right) / \left(\frac{b}{d}\right)$ is in lowest terms. But then by Proposition 3.7(1), $\frac{a}{d}U = aU - dU$ and $\frac{b}{d}U = bU - dU$ are weakly disjoint.

On the other hand, assume that for all $kU \in G$ and all $aU, bU \in G_+$ with $kU = aU - bU$, there exists some $dU \in G_+$ such that $aU - dU$ and $bU - dU$ are weakly disjoint. Let $a, b \in D^*$, so that $aU, bU \in G_+$ and $kU = aU - bU \in G$. Thus there exists a $dU \in G_+$ such $\frac{a}{d}U = aU - dU$ and $\frac{b}{d}U = bU - dU$ are weakly disjoint. By Proposition 3.7(2), this means that $\left(\frac{a}{d}\right) / \left(\frac{b}{d}\right)$ is in lowest terms, and thus $a/b$ can be reduced to lowest terms. $\square$

It remains to state and prove an equivalence to ULT in terms of the group of divisibility:

**Proposition 3.10.** *Let $D$ be an integral domain with group of divisibility $G$. Then $D$*

*is an ULT domain if and only if for all $kU \in G$ there exists a unique pair $aU, bU \in G_+$ such that $aU$ and $bU$ are weakly disjoint and $kU = aU - bU$.*

*Proof.* First assume that $D$ is a ULT domain. The existence of such a pair is given by Proposition 3.8. Let $cU, dU \in G_+$ be any pair such that $cU$ and $dU$ are weakly disjoint and $kU = aU - bU$. Then $a/b = k = c/d$ where both $a/b$ and $c/d$ are in lowest terms by Proposition 3.7(2). But then $c = ua$ and $d = ub$ for some unit $u \in U$, and thus $cU = aU$ and $dU = bU$, meaning that the pair $aU$ and $bU$ is unique.

On the other hand, if for all $kU \in G$ there exists a unique pair $aU, bU \in G_+$ such that $aU$ and $bU$ are weakly disjoint and $kU = aU - bU$, then given $x, y \in D^*$, we can take $k = x/y$, so that there exists a unique weakly disjoint pair $aU, bU \in G_+$ with $kU = aU - bU$. Then $x/y = a/b$ with the second fraction in lowest terms Proposition 3.7(2). Let $c/d = a/b$ with $c/d$ in lowest terms. Then $cU$ and $dU$ are weakly disjoint and $cU - dU = kU$, which means $aU = cU$ and $bU = dU$. But then there exist units $u_1, u_2 \in U$ such that $a = u_1 c$ and $b = u_2 d$. However, $c/d = a/b = u_1 c / u_2 d$, so $u_1 cd = u_2 cd$, meaning that $u_1 = u_2$, which shows that $a/b$ has essentially unique lowest terms. $\square$

In summary, when treating lowest terms in the group of divisibility, a fraction $a/b$ corresponds to an arbitrary $kU \in G$. We then write $kU = aU - bU$, where $aU, bU \in G_+$. This correspondence allows us to work with only element in the positive cone of $G$, which makes it easy to translate our definitions from the previous chapter.

# CHAPTER 4
# MULTIPLICATIVELY CLOSED SETS AND LOWEST TERMS

## 4.1 Introduction

Thus far we have assumed that fractions $a/b$ lie in the quotient field of $D_{D\setminus\{0\}}$ of a domain. A natural generalization is to examine an arbitrary localization $D_S$ where $S$ is some multiplicatively closed set not containing 0. Within this chapter we will examine this generalization as well as prove some properties regarding multiplicatively closed sets.

We begin with some conventions. Any time that we refer to a multiplicatively closed set, we will require it to be non-empty and to not contain 0. It is common to assume that $1 \in S$. However we will not typically assume this. To construct multiplicatively closed sets, for any set $A$ contained in a commutative ring $R$, we will let $\langle A \rangle$ be the smallest multiplicatively closed subset of $R$ containing $A$. This is equivalent to saying

$$\langle A \rangle = \left\{ \prod_{i=1}^{n} a_i^{n_i} \middle| a_i \in A, n_i \in \mathbb{Z}_{>0} \right\}.$$

Notice that if $A$ is finite, this can be written

$$\langle a_1, \ldots, a_n \rangle = \{a_1^{n_1} \cdots a_n^{n_n} | n_i \in \mathbb{Z}_{\geq 0},\ n_i \text{ not all } 0\}.$$

Recall that $S$ is saturated if for any $ab \in S$, $a \in S$ and $b \in S$. We call $\overline{S} = \{a \in D | ab \in S \text{ for some } b \in D\}$ the *saturation of* $S$. Clearly $S \subseteq \overline{S}$ and $\overline{S}$ is a saturated multiplicatively closed subset of $D$. Further, $S$ is saturated if and only if $S = \overline{S}$. Following the definition for ring extensions, we say that $S$ is *inert* if for any $ab \in S$, there exists some $u \in U(D)$ such that $ua, u^{-1}b \in S$.

## 4.2 Lowest Terms over $S$

We begin with definitions for lowest terms in a localization $D_S$. These are very similar to the definitions from Section 2.2. First, we define what it means for a fraction to be in lowest terms over $S$.

**Definition 4.1.** Let $S$ be a multiplicatively closed set. Let $a \in D^*$, $b \in S$. We say that $a/b$ is in *lowest terms over $S$ (resp., strong lowest terms over $S$, absolute lowest terms over $S$)* if $[a, b] = 1$ (resp., $(a, b)_v = D$, $(a, b) = D$).

Next, we define putting something into lowest terms over $S$, and use that to define an $S$-lowest terms domain.

**Definition 4.2.** Let $S$ be a multiplicatively closed set. Let $x \in D^*$, $y \in S$. We say that $x/y$ can be *put in lowest terms over $S$ (resp., strong lowest terms over $S$, absolute lowest terms over $S$)* if it can be put in the form $a/b$ for some $a \in D^*$ and $b \in S$ where $a/b$ is in lowest terms over $S$ (resp., strong lowest terms over $S$, absolute lowest terms over $S$). The integral domain $D$ is an *$S$-lowest terms* (resp., *strong $S$-lowest terms*, *absolute $S$-lowest terms*) *domain* if every non-zero fraction $a/b$ ($a \in D^*$, $b \in S$) can be put in lowest terms (resp., strong lowest terms, absolute lowest terms) over $S$.

Just like in the original statements, we can draw a distinction between putting something into lowest terms and reducing it to lowest terms.

**Definition 4.3.** Let $S$ be a multiplicatively closed set. Let $x \in D^*$, $y \in S$. We say $x/y$ can be *reduced to lowest terms over $S$ (resp., strong lowest terms over $S$, absolute lowest terms over $S$* if it can be reduced to the form $a/b$ where $a/b$ is in lowest terms over $S$ (resp., strong lowest terms over $S$, absolute lowest terms over $S$). The integral domain $D$ is an *$S$-reduced lowest terms (S-RLT) domain* if every non-zero fraction $a/b$ ($a \in D^*$, $b \in S$) can be reduced to lowest terms over $S$.

Further, we can examine what makes a particular lowest terms representation over $S$ unique.

**Definition 4.4.** Let $S$ be a multiplicatively closed set. Let $x \in D^*$, $y \in S$. We say that $x/y$ has *essentially unique (reduced) lowest terms over $S$* if $x/y$ can be put in (reduced to) lowest terms over $S$ $a/b$, and if for any $c/d$ in (reduced) lowest terms over $S$ with $x/y = c/d$, there exists a unit $u \in S$ such that $c = ua$ and $d = ub$. The integral domain $D$ is a *$S$-unique lowest terms (ULT) domain* if every non-zero fraction $a/b$ ($a \in D^*$, $b \in S$) has essentially unique lowest terms over $S$.

And finally, we generalize lowest terms to any $\star$ ideal.

**Definition 4.5.** Let $S$ be a multiplicatively closed set. Let $a \in D^*$, $b \in S$. We say that $a/b$ is in *$\star$-lowest terms over $S$* if $(a, b)^\star = D$, and that $x/y$ can be *put in* (equivalently *reduced to*) *$\star$-lowest terms over $S$* if there exist $a \in D^*$ and $b \in S$ with $x/y = a/b$ where $a/b$ is in $\star$-lowest terms over $S$. Thus a *$S$-$\star$-lowest terms domain* is an integral domain where every nonzero fraction can be put in $\star$-lowest terms over $S$.

Notice that if $S$ is saturated (i.e., it contains all divisors of each of its elements), then for any $x/y$ with $x \in D$ and $y \in S$ and any common divisor $d$ of $x$ and $y$, we must have $d, \frac{y}{d} \in S$. Thus if $d$ is a weak gcd of $x$ and $y$, $\frac{x}{d}/\frac{y}{d}$ has been reduced to lowest terms over $S$. However, if $S$ is not saturated, this may not be the case. First we show a necessary and sufficient condition for $D$ to be an $S$-LT domain.

**Theorem 4.1.** *The following are equivalent:*

*(1) $D$ is an $S$-LT domain, and*

*(2) $D$ is an $\overline{S}$-LT domain and for any $ab \in S$, $ua \in S$ for some $u \in U(D)$.*

*Proof.* First assume that $D$ is an $S$-LT domain. Then for any $x \in D^*$, $y \in S$, there exists $a \in D^*$, $s \in S$ such that $x/y = a/s$ and $[a, s] = 1$. Let $xy \in S$. Then $y/xy$ is an element of $D_S$, so there exists $a \in D^*$ and $s \in S$ such that $y/xy = a/s$. Thus $ys = xya$ and so $s = xa$. Then $1 = [a, s] = [a, xa]$. Notice that the GCD is only uniquely defined up to units. So rather than having $a = 1$, we have that $[a, xa] = a$ is a unit. But then $xa = s \in S$ for some $a \in U(D)$.

Now we show that $D$ is an $\overline{S}$-LT domain. Let $x \in D^*$ and $y \in \overline{S}$. Then there exists some $r \in D$ such that $ry \in S$. By the previous paragraph, there exists some $u \in U(D)$ such that $uy \in S$. Notice that $x/y = ux/uy$, and thus there exists $a \in D^*$ and $s \in S$ such that $x/y = ux/uy = a/s$ and $[a, s] = 1$. Therefore $D$ is an $\overline{S}$-LT domain.

Next assume that $D$ is an $\overline{S}$-LT domain and for any $ab \in S$, $ua \in S$ for some $u \in U(D)$. Let $x \in D^*$ and $y \in S$. Because $S \subseteq \overline{S}$, we have $y \in \overline{S}$. Thus there exists $a \in D^*$ and $s \in \overline{S}$ such that $x/y = a/s$ and $[a, s] = 1$. Because $s \in \overline{S}$, there exists some $r \in D$ such that $rs \in S$ and thus there exists some $u \in U(D)$ such that $us \in S$.

Note that $1 = [a, s] = [ua, us]$ because $u$ is a unit. Then $x/y = a/s = ua/us$ and $[ua, us] = 1$. Therefore $D$ is an $S$-LT domain. □

With this theorem as motivation, we make the following definition.

**Definition 4.6.** Let $S \subseteq D$ be a multiplicatively closed set. We say that $S$ is *almost saturated* if for any $ab \in S$ there exists some $u \in U(D)$ such that $ua \in S$.

Using this definition, Theorem 4.2 can be stated as follows.

**Theorem.** *The following are equivalent:*

*(1) $D$ is an $S$-LT domain, and*

*(2) $D$ is an $\overline{S}$-LT domain and $S$ is almost saturated.*

We now turn our attention to developing some theory regarding multiplicatively closed sets before returning to LT domains.

## 4.3 Multiplicatively Closed Sets

From the definitions, it is clear that saturated $\Rightarrow$ inert $\Rightarrow$ almost saturated. In general neither of these implications can be reversed as the following examples show.

**Example 7.** *Let $S = \langle 1, 2\rangle = \{1, 2, 2^2, \dots\}$. By construction $S$ is multiplicatively closed. Further, it is inert: Consider $ab \in S$. Then $ab = 2^k$ for some $k \in \mathbb{Z}_{\geq 0}$. Then $a = \pm 2^i$ and $b = \pm 2^j$ where $a$ and $b$ have the same sign, $i, j \in \mathbb{Z}_{\geq 0}$, and $i + j = k$. Let $s = \operatorname{sgn}(a) = \operatorname{sgn}(b)$, so then $sa \in S$ and $s^{-1}b \in S$. However $S$ is not saturated because $(-2)(-2) = 4 \in S$, but $-2 \notin S$.*

*Now let $S = \langle 1, 2, -4 \rangle = \{1, 2, \pm 2^2, \pm 2^3, \dots\}$. Once again $S$ is multiplicatively closed. It is also almost saturated because for any $ab \in S$, $a$ must have the form $\pm 2^n$. If $a = 2^n$, then $1a \in S$, otherwise $-a \in S$. Thus $S$ is almost saturated. However $S$ is not inert because $-2 \cdot 2 = -2^2 \in S$, but $1 \cdot (-2) \notin S$ while $1^{-1} \cdot 2 \in S$ and $-1 \cdot (-2) \in S$ while $-1 \cdot 2 \notin S$. Because $\pm 1$ are the only units of $\mathbb{Z}$, this shows that $S$ is not inert.*

*Finally, $S = \langle 2 \rangle = \{2, 2^2, 2^3, \dots\}$ is an example of a multiplicatively closed set which is not almost saturated. Indeed, $1 \cdot 2 \in S$, but neither $1$ nor $-1$ is in $S$.*

When $S$ is saturated, $0 \in S$ implies that $S = R$ (because for all $r \in R$, $0r = 0 \in S$, so $r \in S$). However, this is not true in non-saturated multiplicatively closed sets. Consider $S = \mathbb{Z}_{\geq 0} \subseteq \mathbb{Z}$, which is not saturated because $(-1) \cdot 0 \in S$ but $-1 \notin S$. It is inert. To see this, let $ab \in S$. Then $a$ and $b$ have the same sign, say $\sigma$. Thus $\sigma a, \sigma^{-1} b \in S$ showing that $S$ is inert.

In [4], a multiplicatively closed set $S \subseteq R$ was said to be *weakly saturated* if for any $a, b \in R \setminus U(R)$, $ab \in S$ implies that $a, b \in S$. Clearly saturated implies weakly saturated, however $\{\pm 2, \pm 2^2, \dots\}$ is weakly saturated but not almost saturated (which can be seen by the following theorem) and $\{1, 2, 2^2, 2^3, \dots\}$ is inert but not weakly saturated. Thus there is no further relation between weakly saturated and the other properties discussed here. Because weakly saturated sets do not appear to be related to LT domains, we do not examine any further properties of them here.

There is clearly some relation between the types of multiplicatively closed sets and the units of the ring. We begin by examining the number of units contained in a multiplicatively closed set of each type.

**Theorem 4.2.** *Let $R$ be a commutative ring and let $S \subseteq R$ be a multiplicatively closed subset.*

*(a) If $S$ is almost saturated then $S$ contains at least one unit.*

*(b) If $S$ is inert then $1 \in S$ or $S$ contains at least two units.*

*(c) If $S$ is saturated then $U(R) \subseteq S$.*

*Proof.*

(a) Let $S$ be almost saturated and let $s \in S$. Then $1s = s \in S$ and so there exists $u \in U(R)$ such that $u = 1u \in S$. Therefore $S$ contains at least one unit.

(b) Let $S$ be inert. Assume that $1 \notin S$. Because inert $\Rightarrow$ almost saturated, $S$ contains at least one unit $u \in U(R)$ by part $(a)$. By assumption $u \neq 1$. Then we have $1u = u \in S$, so there exists $v \in U(R)$ such that $v = 1v \in S$ and $uv^{-1} \in S$. If $v = u$, then $uv^{-1} = 1$, which contradicts $1 \notin S$. Thus $u, v$ are distinct units contained in $S$. Therefore $1 \in S$ or $S$ contains at least two units.

(c) Let $S$ be saturated, let $u \in U(R)$, and let $s \in S$. Then $u \cdot u^{-1}s = s \in S$ and so $u \in S$. Therefore $U(R) \subseteq S$.

$\square$

For $R = \mathbb{Z}$ (or more generally any ring where for every unit $u \in U(D)$, $u^n = 1$ for some $n \in \mathbb{Z}_{>0}$), this theorem shows that an almost saturated multiplicatively closed subset contains 1. When working in $\mathbb{Z}$, we have the following characterizations of saturated and almost saturated multiplicatively closed sets.

**Proposition 4.3.** *Let $S \subseteq \mathbb{Z}$ be a multiplicatively closed set. Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$ be the sequence of positive primes in $\mathbb{Z}$.*

*(a) $S$ is saturated if and only if $S = \mathbb{Z} - \bigcup_{\alpha \in \Lambda}(p_\alpha)$ where $\Lambda \subseteq \mathbb{Z}_{>0}$.*

*(b) $S$ is almost saturated if and only if there exists some $\Lambda \subseteq \mathbb{Z}_{>0}$ and $\{q_n\}_{n\in\Lambda}$ where for each $n \in \Lambda$, $q_n$ is either $p_n$ or $-p_n$ such that $T \subset S \subseteq \overline{T}$, where $T = \langle\{1\} \cup \{q_n\}_{n\in\Lambda}\rangle$.*

*Proof.* (a) It is well known that a multiplicatively closed subset of a domain $D$ is saturated if and only if it is a complement of a union of prime ideals of $D$ [8, Theorem 2]. Applying that result to $D = \mathbb{Z}$, we see this result.

(b) Assume that $S$ is almost saturated and let $\Lambda = \{n \in \mathbb{Z}_{>0} | p_n \in S \text{ or } -p_n \in S\}$. For each $n \in \Lambda$, let $q_n$ be either $p_n$ or $-p_n$ so that $q_n \in S$. Let $T = \langle\{1\} \cup \{q_n\}_{n\in\Lambda}\rangle$. We claim that $T \subseteq S \subseteq \overline{T}$ as desired. Notice that by the discussion before this result, because $S \subseteq \mathbb{Z}$ is almost saturated, $1 \in S$. Further, by construction each $q_n \in S$. So clearly $T \subseteq S$. Now let $s \in S$. Factor $s$ as $p_{i_1}^{e_1} \ldots p_{i_n}^{e_n}$. Because each $p_i$ is either $q_i$ or $-q_i$, we can write $s$ as either $q_{i_1}^{e_1} \ldots q_{i_n}^{e_n}$ or $-q_{i_1}^{e_1} \ldots q_{i_n}^{e_n}$. But then either $s \in T$ or $-s \in T$, which in either case shows that $s \in \overline{T}$ by definition of the saturation. Therefore $T \subseteq S \subseteq \overline{T}$, as desired.

Now let $\Lambda \subseteq \mathbb{Z}_{>0}$, let $q_n$ be either $p_n$ or $-p_n$ for each $n \in \Lambda$, and let $T = \langle\{1\} \cup \{q_n\}_{n\in\Lambda}\rangle$. Assume that $T \subseteq S \subseteq \overline{T}$. Let $ab \in S$, we must show that $a \in S$ or $-a \in S$. Because $ab \in S \subseteq \overline{T}$, $a \in \overline{T}$. By definition of the saturation, this means that there exists some $c \in \mathbb{Z}$ with $ca \in T$. By construction of $T$, we can write $ca = q_{i_1}^{e_1} \cdots q_{i_n}^{e_n}$. But then we must have $a = q_{i_1}^{f_1} \cdots q_{i_n}^{f_n}$ or

$a = -q_{i_1}^{f_1} \cdots q_{i_n}^{f_n}$ for $0 \leq f_k \leq e_k$. Because each $q_i \in T \subseteq S$, either $a \in S$ or $-a \in S$, which shows that $S$ is almost saturated.

□

We also have the following result connecting multiplicatively closed sets and units.

**Theorem 4.4.** *Let $S, T \subseteq D$ be multiplicatively closed sets.*

*(a) $S$ is almost saturated if and only if $\overline{S} = U(D)S$.*

*(b) $S$ is saturated if and only if $S$ is almost saturated and $U(D) \subseteq S$.*

*(c) $U(D)S$ is almost saturated if and only if $U(D)S$ is saturated.*

*(d) If $S \subseteq T \subseteq \overline{S}$ and $S$ is almost saturated, then $T$ is almost saturated.*

*Proof.*

(a) First assume that $S$ is almost saturated. Let $a \in \overline{S}$ and let $b \in D$ such that $ab \in S$. Because $S$ is almost saturated, there exists $u \in U(D)$ such that $ua \in S$. Thus $a = u^{-1} \cdot ua \in U(D)S$. For the other containment, let $us \in U(D)S$. Then because $U(D) \subseteq \overline{S}$ and $S \subseteq \overline{S}$, $u \in \overline{S}$ and $s \in \overline{S}$. Therefore $us \in \overline{S}$ because $\overline{S}$ is multiplicatively closed. Thus $\overline{S} = U(D)S$

On the other hand, assume that $\overline{S} = U(D)S$ and let $ab \in S$. Then $a \in \overline{S}$ and so we can write $a = ua'$ for some $u \in U(D)$, and $a' \in S$. Thus $u^{-1}a = a' \in S$ which shows that $S$ is almost saturated.

(b) We've already noted that every saturated multiplicatively closed set is inert and hence almost saturated, and Theorem 4.4 shows that $U(D) \subseteq S$. So assume that $S$ is almost saturated and that $U(D) \subseteq S$. Let $ab \in S$, so that there exists some $u \in U(D)$ such that $ua \in S$. Because $u^{-1} \in U(D) \subseteq S$, we have $a = u^{-1} \cdot ua \in S$ as desired. Therefore $S$ is saturated.

(c) Notice that if $U(D)S$ is saturated, then it is certainly almost saturated. If $U(D)S$ is almost saturated, then by part (1), $\overline{U(D)S} = U(D)U(D)S = U(D)S$ and thus $U(D)S$ is saturated.

(d) Assume that $S \subseteq T \subseteq \overline{S}$ and that $S$ is almost saturated. Let $ab \in T$, so that $ab \in \overline{S}$. This means $a \in \overline{S}$, which is equal to $U(D)S$ by part (1). Write $a = ua'$ for some $u \in U(D)$ and $a' \in S$. Then $u^{-1}a = a' \in S \subseteq T$, which shows that $T$ is almost saturated.

$\square$

**Definition 4.7.** Let $D$ be an integral domain and let $S$ be a multiplicatively closed set.

(a) We say that $D$ is an *$n$-$S$-Bézout domain* if any $n$-generated ideal $I$ of $D$ with $I \cap S \neq \emptyset$ is principal.

(b) We say that $D$ is a *weak $n$-$S$-Bézout domain* if every ideal of the form $(a_1, \ldots, a_{n-1}, s$, $a_i \in R$, $s \in S$, is principal.

(c) We say that $D$ is an *$S$-Bézout domain* if it is an $n$-$S$-Bézout ring for all $n \in \mathbb{Z}_{>0}$. Equivalently, if any finitely generated ideal which has non-empty intersection with $S$ is principal.

(d) We say that $D$ is a *weak $S$-Bézout domain* if it is a weak $n$-$S$-Bézout ring for all $n \in \mathbb{Z}_{>0}$. Equivalently, if any finitely generated ideal where at least one generator lies in $S$ is principal.

It turns out that under the fairly weak condition of $S$ being almost saturated, these are all equivalent, as can be seen in the following theorem.

**Theorem 4.5.** *Let $S \subseteq D$ be a multiplicatively closed set and let $n \geq 2$. In the following,* $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$*:*

*(1) $D$ is an $S$-Bézout domain*

*(2) $D$ is a weak $S$-Bézout domain*

*(3) $D$ is an $n$-$S$-Bézout domain*

*(4) $D$ is a weak $n$-$S$-Bézout domain*

*(5) $D$ is an $(n-1)$-$S$-Bézout domain*

*Further, if $S$ is almost saturated, then every weak $2$-$S$-Bézout domain is an $S$-Bézout domain*

*Proof.* Clearly $n$-$S$-Bézout implies weak $n$-$S$-Bézout because any ideal of the form $(a_1, \ldots, a_{n-1}, s)$ is generated by $n$ elements and intersects non-trivially with $S$ and thus is principal. This shows both $(1) \Rightarrow (2)$ and $(3) \Rightarrow (4)$.

Next we show $(2) \Rightarrow (3)$. Let $D$ be a weak $S$-Bézout domain and let $(a_1, \ldots, a_n)$ be an ideal of $D$ such that $(a_1, \ldots, a_n) \cap S \neq \emptyset$. Let $s \in (a_1, \ldots, a_n) \cap S$. Then $(a_1, \ldots, a_n) = (a_1, \ldots, a_n, s)$ is principal because $D$ is a weak $S$-Bézout ring and in particular a weak $(n+1)$-$S$-Bézout ring. Similarly, if $D$ is a weak $n$-$S$-Bézout then for any $(a_1, \ldots, a_{n-1})$ with nonempty intersection with $S$ pick $s \in (a_1, \ldots, a_{n-1}) \cap S$, and thus $(a_1, \ldots, a_{n-1}) = (a_1, \ldots, a_{n-1}, s)$ is principal, which proves (4) $\Rightarrow$ (5).

All that remains is to show the final statement. Assume that $S$ is almost saturated and that $D$ is a weak 2-$S$-Bézout domain. We show that every finitely generated ideal of $D$ with nonempty intersection with $S$ is principal by induction on the number of generators. Let $(a, b)$ be an ideal such that $(a, b) \cap S \neq \emptyset$. Let $s \in (a, b) \cap S$. By assumption there exists $t \in D$ such that $(b, s) = (t)$. Notice that $t|s$, so because $S$ is almost saturated, there exists $u \in U(D)$ such that $ut \in S$. However, because $(t) = (ut)$, we can assume without loss of generality that $t \in S$. Then

$$(a, b) = (a, b, s) = (a, (b, s)) = (a, (t)) = (a, t),$$

which is principal by assumption.

Now assume that any $n - 1$-generated ideal $I$ of $D$ such that $I \cap S \neq \emptyset$ is principal. Let $(a_1, \ldots, a_n)$ be an ideal such that $(a_1, \ldots, a_n) \cap S \neq \emptyset$. Let $s \in (a_1, \ldots, a_n) \cap S$. By assumption there exists $t \in D$ such that $(a_n, s) = (t)$. As before this can be assumed to lie in $S$. Additionally, again by the assumption there exists $r \in D$ such that $(a_{n-1}, t) = (r)$, which can also be taken to lie in $S$. Then

$$\begin{aligned}(a_1, \ldots, a_n) &= (a_1, \ldots, a_n, s) = (a_1, \ldots, a_{n-1}, (a_n, s)) = (a_1, \ldots, a_{n-1}, (t)) \\ &= (a_1, \ldots, a_{n-1}, t) = (a_1, \ldots, a_{n-2}, (a_{n-1}, t)) = (a_1, \ldots, a_{n-2}, (r)) \\ &= (a_1, \ldots, a_{n-2}, r)\end{aligned}$$

This is an ideal generated by $n - 1$ elements which clearly has non-empty intersection with $S$, hence it is principal by the induction hypothesis. Therefore every finitely generated ideal $I$ of $R$ such that $I \cap S \neq \emptyset$ is principal, which shows that $D$ is $S$-Bézout. $\square$

It is also clear that any Bézout Domain must be $S$-Bézout. This theorem can be represented diagrammatically as

$$\begin{tikzcd}
S\text{-Bézout} \arrow[r] \arrow[d] & n\text{-}S\text{-Bézout} \arrow[r] \arrow[d] & (n-1)\text{-}S\text{-Bézout} \arrow[r] \arrow[d] & \cdots \arrow[r] & 3\text{-}S\text{-Bézout} \arrow[r] \arrow[d] & 2\text{-}S\text{-Bézout} \arrow[d] \\
\text{weak } S\text{-Bézout} \arrow[r] \arrow[ur] & \text{weak } n\text{-}S\text{-Bézout} \arrow[r] \arrow[ur] & \text{weak } (n-1)\text{-}S\text{-Bézout} \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & \text{weak } 3\text{-}S\text{-Bézout} \arrow[r] \arrow[ur] & \text{weak } 2\text{-}S\text{-Bézout}
\end{tikzcd}$$

It remains an open question whether or not these are all equivalent without the almost saturated condition. Our motivation in defining almost saturated was that for any $S$-LT domain, $S$ had to be almost saturated. As such, for all the domains we're concerned with these will all be equivalent.

### 4.4 Application to LT Domains

We've already seen the definitions concerning lowest terms over a multiplicatively closed set. With the results of the previous section we can also show some results concerning lowest terms over other multiplicatively closed sets. It turns out that many of the results from Chapter 2 hold as long as we make the appropriate modifications.

**Theorem 4.6.** *Let $S \subseteq D$ be an almost saturated multiplicatively closed set. If $D$ satisfies one of the following two conditions*

*(1) $D$ is an $S$-GCD domain, i.e., for every $a \in D^*$ and $s \in S$, $[a, s]$ exists*

*(2) $D$ satisfies $S$-ACCP, i.e., for every chain $(s_1) \subseteq (s_2) \subseteq (s_3) \subseteq \cdots$ with $s_i \in S$, there exists some $N$ such that $(s_N) = (s_{N+1}) = \cdots$.*

*then $D$ is an $S$-RLT domain*

*Proof.* First assume that $D$ is an $S$-GCD domain. Let $a \in D^*$ and $s \in S$. By assumption, $d = [a, s]$ exists. Because $S$ is almost saturated and $\frac{s}{d}d = s \in S$, there exists $u \in U(D)$ such that $u\frac{s}{d} \in S$. But then $1 = \left[\frac{a}{d}, \frac{s}{d}\right] = \left[u\frac{a}{d}, u\frac{s}{d}\right]$ and $a/s = \frac{a}{d}/\frac{s}{d} = u\frac{a}{d}/u\frac{s}{d}$

Now assume that $D$ satisfies $S$-ACCP. First, notice that if $T$ is a set of principal ideals of the form $(s)$ where $s \in S$, then $T$ has a maximal element. To see this, let $T$ be such a set and pick some $(s_1) \in T$. If this is maximal in $T$ we're done, otherwise there exists some $(s_2) \in T$ such that $(s_1) \subsetneq (s_2)$. Continue this process. Because $D$ satisfies $S$-ACCP, we cannot construct a strictly ascending infinite chain, meaning that we must eventually reach a maximal element as desired.

We now prove that $D$ is an $S$-RLT domain. Assume to the contrary, so that there exists some $r'/s'$ with $r' \in D^*$ and $s' \in S$ which cannot be reduced to lowest terms. Let $T$ be the set of all $(s)$ with $s \in S$ such that there exists $r \in D^*$ where $r/s$ cannot be reduced to $S$-lowest terms. By assumption this is non-empty because $(s') \in T$, and further for any unit $u \in S$ and all $a \in D^*$, $a/u$ is in lowest terms because $[a, u] = 1$. Thus $D \notin T$. By the previous paragraph, $T$ has a maximal element, call it $(s)$. Let $r \in D^*$ be an element such that $r/s$ cannot be reduced to $S$-lowest terms. In particular this means that $r/s$ is not already in lowest terms, i.e. $[r, s] \neq 1$. Then there exists some non-unit common divisor $c$ of $r$ and $s$. Notice that $\frac{s}{c}c = s \in S$, and so because $S$ is almost saturated there exists $u \in U(D)$ such that $u\frac{s}{c} \in S$. Then

$$r/s = \frac{r}{c}\Big/\frac{s}{c} = u\frac{r}{c}\Big/u\frac{s}{c}.$$

Further, $(s) \subsetneq \left(u\frac{s}{c}\right)$. Because this is a principal ideal generated by an element of $S$ which is properly larger than $(s)$, it cannot be in $T$. Thus $u\frac{r}{c}/u\frac{s}{c}$ can be reduced to lowest terms:

$$u\frac{r}{c}\Big/u\frac{s}{c} = \frac{u\frac{r}{c}}{d}\Big/\frac{u\frac{s}{c}}{d} = \frac{r}{u^{-1}cd}\Big/\frac{s}{u^{-1}cd}.$$

However, this shows that $r/s$ can be reduced to lowest terms as well, which is a contradiction. Therefore $D$ is an $S$-RLT domain. □

**Proposition 4.7.** *Let $S \subseteq D$ be an almost saturated multiplicatively closed set. Let $x \in D^*, y \in S$. Then the following are equivalent:*

*(1) $(x, y)^\star$ is principal,*

*(2) $x/y$ can be reduced to $\star$-lowest terms over $S$, and*

*(3) $x/y$ can be put in $\star$-lowest terms over $S$*

*If $(x, y)^\star = (d)$, then $x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ where the second fraction is in $S$-$\star$-lowest terms. Further, this representation is unique.*

*Proof.* (1) $\Rightarrow$ (2): Assume for $x \in D^*$ and $y \in S$ that $(x, y)^\star = (d)$. Then $[x, y] = d$, so that $d|x$ and $d|y$. Because $S$ is almost saturated and $y = d\frac{y}{d} \in S$, there exists $u \in U(D)$ such that $\frac{y}{u^{-1}d} = u\frac{y}{d} \in S$. Notice that

$$\left(\frac{x}{u^{-1}d}, \frac{y}{u^{-1}d}\right)^\star = \frac{u}{d}(x, y)^\star = \frac{u}{d}(d) = D,$$

and therefore $x/y$ can be reduced to strong lowest terms over $S$ as $\frac{x}{u^{-1}d}/\frac{y}{u^{-1}d}$.

(2) $\Rightarrow$ (3): This is clear.

(3) $\Rightarrow$ (1): Assume that $x/y = a/b$ where $x, a \in D^*$, $y, b \in S$, and $(a, b)^\star = D$. Then $bx = ay$ and

$$(x) = xD = x(a, b)^\star = (xa, xb)^\star = (ax, ay)^\star = a(x, y)^\star).$$

Notice that Lemma 2.6 still applies here, because $a, b \in D^*$, and so because $(a, b)^\star = D$ and $bx = ay$, we have $a|x$. Thus $\frac{x}{a} \in D$, and so $(x, y)^\star = \left(\frac{x}{a}\right)$ is principal.

Finally, we must show that if $(x, y)^\star = (d)$, where $d$ is taken such that $\frac{y}{d} \in S$, then $x/y = \frac{x}{d}/\frac{y}{d}$, with the second fraction in unique $S$-$\star$-lowest terms. Certainly the two fractions are equal and

$$\left(\frac{x}{d}, \frac{y}{d}\right)^\star = \frac{1}{d}(x, y)^\star = \frac{1}{d}(d) = D,$$

which shows that the second fraction is in lowest terms. To see that it is unique, let $a \in D^*$, $b \in S$ with $x/y = a/b$ and $(a, b)^\star = D$. Because $\frac{x}{d}/\frac{y}{d}$ is in $\star$-lowest terms and $a/b = \frac{x}{d}/\frac{y}{d}$, By again applying Lemma 2.6, there exists some $r$ such that $a = r\frac{x}{d}$ and $b = r\frac{y}{d}$. However

$$1 = [a, b] = \left[r\frac{x}{d}, r\frac{y}{d}\right] = r\left[\frac{x}{d}, \frac{y}{d}\right],$$

which means that $r$ must be a unit, meaning that the representation as $\frac{x}{d}/\frac{y}{d}$ is essentially unique. $\square$

The following corollary shows that an absolute $S$-lowest terms domain is precisely the same as an $S$-Bézout domain.

**Proposition 4.8.** *Let $S \subseteq D$ be an almost saturated multiplicatively closed set. The following are equivalent:*

*(1) For all $a \in D$, $s \in S$, $a/s$ can be put in absolute lowest terms over $S$,*

*(2) For all $a \in D$, $s \in S$, $a/s$ can be reduced to absolute lowest terms over $S$,*

*(3) For all $a \in D$, $s \in S$, $(a, s)$ is principal (i.e. $D$ is weak 2-$S$-Bézout),*

*(4) For $a, b \in D$, if $(a, b) \cap S \neq \emptyset$, then $(a, b)$ is principal (i.e. $D$ is 2-$S$-Bézout), and*

*(5) For $a_1, \ldots, a_n \in D$, if $(a_1, \ldots, a_n) \cap S \neq \emptyset$, then $(a_1, \ldots, a_n)$ is principal (i.e. $D$ is $S$-Bézout).*

*Proof.* The equivalence of (1) through (3) is Proposition 4.7 for the $d$-operation, while the equivalence of (3) through (5) is Theorem 4.5. □

**Theorem 4.9.** *Let $S \subseteq D$ be a multiplicatively closed set. Then the following are equivalent:*

*(1) $D$ is an absolute $S$-lowest terms domain*

*(2) $S$ is almost saturated and $D$ is an $S$-Bézout domain.*

*Proof.* (1) $\Rightarrow$ (2): If $D$ is an absolute $S$-lowest terms domain, then it is certainly an $S$-lowest terms domain, and thus by Theorem 4.2, $S$ is almost saturated. So then by Proposition 4.8, because every fraction $a/s$ for $a \in D$ and $s \in S$ can be put in absolute lowest terms, $D$ is an $S$-Bézout domain.

(2) $\Rightarrow$ (1): Because $S$ is almost saturated, Proposition 4.8 applies, and so this is simply (5) $\Rightarrow$ (1) in that result.

□

# CHAPTER 5
# LOWEST TERMS IN RINGS WITH ZERO DIVISORS

## 5.1 Introduction

In this section $R$ will be a commutative ring with an identity. We will use $\mathcal{Z}(R)$ to denote the set of zero divisors of $R$.

In integral domains, fractions have a natural home in the quotient field of the domain. That is, we can localize $D$ at $D \setminus \{0\}$ to obtain the quotient field. This means that the denominator can be anything other than zero. When a ring contains zero divisors, the setting is not quite so clear. Ideally we would like to allow the denominators of fractions to contain as many elements as possible. However, as soon as we localize at some multiplicatively closed set $S$ which contains a pair $s, t \in S$ such that $st = 0$, we must have $0 \in S$. But this leads to $D_S = 0$, which is not useful. Another possibility is to choose $S$ to include as many zero divisors as possible, without allowing pairs which multiply to zero. However, even in this case the standard homomorphism from $R$ to $R_S$ given by $x \mapsto x/1$ often fails to be injective, which means we can't treat $R$ as a subring of $R_S$.

Due to all this, we must use the total quotient ring: $T(R) := R_{R\setminus\mathcal{Z}(R)}$. Much like the quotient field is a formal way to give every nonzero element a multiplicative inverse, the total quotient ring gives a multiplicative inverse to every element that is not a zero divisor. It has the property that the map $R \to T(R)$ given by $x \mapsto x/1$ is injective, so that $R \subseteq T(R)$. Further, we still have $a/b = c/d$ if and only if $ad = bc$

because the set where we're localizing has no zero divisors.

The next problem arises from the greatest common divisor. While the definition of the GCD in a ring with zero divisors is the same, not all properties are maintained. For example, if the LCM of two elements exists in an integral domain, then the GCD of those elements also exists. However, this is not necessarily the case in a general commutative ring. For example, in $F[X^2, X^3]/(X^9, X^{10})$, we have $(\overline{X}^5) \cap (\overline{X}^6) = (\overline{X}^8)$, so that $]\overline{X}^5, \overline{X}^6[= \overline{X}^8$. However, $[\overline{X}^5, \overline{X}^6]$ does not exist.

Further problems are encountered when discussing the $v$-ideal. Because it is traditionally defined using the inverse of an ideal, we encounter problems with this definition. So instead define $I_v := \cap\{Rx | x \in T(R) \text{ with } I \subseteq Rx\}$. Notice that this is an equivalent definition of the $v$-ideal for integral domains.

## 5.2 Definitions

Many of the definitions from Section 2.2 can be restated for rings with zero divisors without any major changes. The following definitions correspond to the first three definitions of Chapter 2:

**Definition 5.1.** Let $a, b \in R$ with $b \notin \mathcal{Z}(R)$. We say that $a/b$ is in *lowest terms (resp., strong lowest terms, absolute lowest terms)* if $[a, b] = 1$ (resp., $(a, b)_v = R$, $(a, b) = R$).

**Definition 5.2.** Let $x, y \in R$ with $y \notin \mathcal{Z}(R)$. We say that $x/y$ can be *put in the form $a/b$* if $x/y = a/b$. Further, we say that $x/y$ can be *put in lowest terms (resp., strong lowest terms, absolute lowest terms)* if it can be put in the form $a/b$ where $a/b$

is in lowest terms (resp., strong lowest terms, absolute lowest terms). The ring $R$ is a *lowest terms (LT) ring* if every non-zero fraction $a/b$ ($a, b \in R \setminus \mathcal{Z}(R)$) can be put in lowest terms.

**Definition 5.3.** Let $x, y \in R$ with $y \notin \mathcal{Z}(R)$. We say that $x/y$ can be *reduced to the form* $a/b$ if there exists a common divisor $d \in R$ of $x$ and $y$ such that $a = \frac{x}{d}$ and $b = \frac{y}{d}$. We say $x/y$ can be *reduced to lowest terms (resp., strong lowest terms, absolute lowest terms* if it can be reduced to the form $a/b$ where $a/b$ is in lowest terms (resp., strong lowest terms, absolute lowest terms). The ring $R$ is a *reduced lowest terms (RLT) ring* if every non-zero fraction $a/b$ ($a, b \in R$, $b \notin \mathcal{Z}(R)$) can be reduced to lowest terms.

## 5.3 Results

The natural question is to ask which results from Chapter 2 remain true over general rings (with the necessary modifications). Unfortunately the answer remains unclear. However, some certainly do hold. For example, we still have much of Proposition 2.10:

**Proposition 5.1.** *Let $x \in R$, $y \in R \setminus \mathcal{Z}(R)$. Then the following are equivalent.*

*(1) $(x, y)$ is a regular principal ideal*

*(2) $x/y$ can be reduced to absolute lowest terms*

*(3) $x/y$ can be put in absolute lowest terms*

*If $(x, y) = (d)$, then $x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ where the second fraction is in strong lowest terms.*

The proof is identical to that of Proposition 2.10. However, notice that because of the problems with the GCD mentioned at the beginning of the chapter, it's unclear if the uniqueness statement would hold. (In fact, we haven't even defined uniqueness in a general commutative ring). In the same way as Chapter 2, we have an extension of Proposition 2.11:

**Proposition 5.2.** *For a ring $R$, the following are equivalent:*

*(1) Every fraction in $T(R)$ can be reduced to absolute lowest terms*

*(2) Every fraction in $T(R)$ can be put in absolute lowest terms*

*(3) For $a, b \in R$ with $b$ regular, $(a, b)$ is principal*

*(4) Every regular ideal of $R$ which can be generated by two elements is principal*

*(5) Every finitely generated regular ideal of $R$ is principal*

*Proof.* The equivalence of (1), (2), and (3) is simply Proposition 5.1. We also have that (5) $\Rightarrow$ (4) $\Rightarrow$ (3) is clear. To see (3) $\Rightarrow$ (4), notice let $(a, b)$ be a regular ideal and let $s \in (a, b)$ be regular. Then by assumption there exists some $t \in R$ with $(b, s) = (t)$. Notice that $t$ must be regular, because $s = rt$ for some $r \in R$, and if $t$ were not regular, there would exist $t' \in R^*$ such that $tt' = 0$, and then $st' = rtt' = r0 = 0$, but neither $s$ nor $t'$ are zero, contradicting that $s$ is regular. Then using a similar trick to what was done frequently in chapter 4,

$$(a, b) = (a, b, s) = (a, (b, s)) = (a, (t)) = (a, t).$$

By applying (3) again, this must be principal as desired. Finally, (4) $\Rightarrow$ (5) is simply induction just as in the case of a Bézout domain. $\square$

**Definition 5.4.** A ring $R$ where every finitely generated regular ideal of $R$ is called $r$-Bézout.

For the results, notice that the definition of a $\star$-operation can be transfered to rings with zero divisors simply by taking fractional ideals as the $R$ submodules of the total quotient ring instead of the quotient field. Then we can define a $\star$-operation using the same three properties as before. This give the following definitions for $x \in R$, $y \in R \setminus \mathcal{Z}(R)$: We say that $a/b$ is in *$\star$-lowest terms* if $(a, b)^\star = R$, and that $x/y$ can be *put in* (equivalently *reduced to*) *$\star$-lowest terms* if there exists $a \in R$, $b \in R \setminus \mathcal{Z}(R)$ with $x/y = a/b$ where $a/b$ is in $\star$-lowest terms. Thus a *$\star$-lowest terms ring* is an ring where every element of $T(R)$ can be put in $\star$-lowest terms. Now we have an analog of Proposition 2.12:

**Proposition 5.3.** *Let $x \in R$, $y \in R \setminus \mathcal{Z}(R)$. Then the following are equivalent.*

*(1) $(x, y)^\star$ is a regular principal ideal*

*(2) $x/y$ can be reduced to $\star$-lowest terms*

*(3) $x/y$ can be put in $\star$-lowest terms*

*If $(x, y)^\star = (d)$, then $x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ where the second fraction is in $\star$-lowest terms.*

Once again the proof is identical so we will omit it. By using the $v$-operation, we have a characterization of fractions which can be put into or reduced to strong lowest terms in a ring with zero divisors, just as in Proposition 2.8:

**Proposition 5.4.** *Let $x \in R$, $y \in R \setminus \mathcal{Z}(R)$. Then the following are equivalent:*

*(1) $(x, y)_v$ is principal*

*(2) $x/y$ can be reduced to strong lowest terms*

*(3) $x/y$ can be put in strong lowest terms*

*If $(x, y)_v = (d)$ then $x/y = \left(\frac{x}{d}\right) / \left(\frac{y}{d}\right)$ where the second fraction is in strong lowest terms.*

Thus there are two major pieces still missing: Lowest terms and uniqueness. Both of these depend on the GCD however. As mentioned at the beginning of the chapter, this has potential problems and has not yet been addressed.

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