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Atoms in quasilocal integral domains

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ATOMS IN QUASILOCAL INTEGRAL DOMAINS

by

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ABSTRACT

Let R be an integral domain. An *atom* is a nonzero nonunit x of R where $x = yz$ implies that either y or z is a unit. We say that R is an *atomic domain* if each nonzero nonunit is a finite product of atoms. An atomic domain with only finitely many nonassociate atoms is called a *Cohen-Kaplansky* (CK) domain. We will investigate atoms in integral domains R with a unique maximal ideal M. Of particular interest will be atoms that are not in M^2.

After studying the atoms in integral domains, we will narrow our focus to CK domains with a unique maximal ideal M. In this pursuit, we investigate atoms in M^2 for these CK domains. We will show that the minimal number of atoms needed to have an atom in M^2 is exactly eight. This disproves a conjecture given by Cohen and Kaplansky in 1946 that the minimal number would be ten. We then classify complete local CK domains with exactly three atoms.

PUBLIC ABSTRACT

Recall the integers: $\mathbb{Z} = \{0, 1, -1, 2, -2, \cdots\}$. Factorizations are an important concept in utilizing the integers. For example, a common example seen is breaking the number 14 down into smaller pieces called primes: $14 = 2 \cdot 7$. The numbers 2 and 7 cannot be factored further since they are prime numbers. Instead, we could call them *atoms* since they are irreducible pieces. Many mathematical systems have these irreducible pieces which we will call atoms. We will look at atoms in mathematical systems that have addition, subtraction, and multiplication, but not necessarily division. For example, polynomials. Factoring polynomials into atoms is a common skill taught in many math classes. We see that $X^2 - 1 = (X + 1)(X - 1)$. So here the polynomial $X^2 - 1$ can be factored into the atoms $X + 1$ and $X - 1$.

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CHAPTER 1
INTRODUCTION AND BACKGROUND

Unique factorization domains (UFDs) are common algebraic objects that have been studied extensively. Being able to factor elements into a unique product of irreducible elements is a useful property. However, we would often like to factor elements into irreducible elements even when unique factorization does not always hold. The most general example of this in the integral domain case is called an atomic domain. One of the simplest examples of an atomic domain, which turns out to be a UFD, is that of the integers $\mathbb{Z}$. Each integer has a unique factorization (up to multiplication by 1 or -1) into prime numbers. For example, $14 = 2 \cdot 7$ where 2 and 7 are prime numbers. This factorization property is convenient as the prime numbers are essentially the "building blocks" for all of the integers. With this observation, one might be tempted to call these prime elements *atoms* since they are the "fundamental pieces" for all of the integers. More formally, an *atom* (or irreducible element) of an integral domain R is a nonzero nonunit element $x \in R$ that is not a product of two nonzero nonunits. That is, if $x = ab$, then either a or b must be a unit. We note that the definition of a prime element differs from this definition. A nonzero nonunit element p in a commutative ring R is said to be *prime* if $p|ab$ implies that $p|a$ or $p|b$. For an integral domain, it is easily seen that every prime element is an atom. The two notions are equivalent when R is a unique factorization domain (UFD). (Or more generally, when R is a GCD domain.)

In the example above with the integers, it was hinted that factorization is unique up to multiplication by 1 or -1. If p is a prime number, then p and $-p$ are essentially "equivalent" atoms. To generalize this notion of two atoms being equivalent, we say two atoms x, y in an integral domain R are *associate*, written $x \sim y$, if there exists a unit $u \in R$ so that $y = ux$. Notice that $-1 \in \mathbb{Z}$ is a unit and so $-7 = (-1) \cdot 7$ yields that the two atoms are associate. When we talk about two distinct atoms, we mean two nonassociate atoms.

As mentioned earlier, the integers $\mathbb{Z}$ are a unique factorization domain (UFD). In UFDs, a given factorization into irreducibles is unique up to multiplication by a unit and reordering. D.D. Anderson, D.F. Anderson, and M. Zafrulah investigated a generalized notion of this in [2]. There are integral domains, called atomic domains, which have similar properties, but the factorizations are not necessarily unique. Formally, an *atomic domain* is an integral domain in which each nonzero nonunit has a factorization into a product of atoms. Note that this product is necessarily finite.

Example 1.1. Let $R := k[X^2, X^3]$ where k is a field. That is, the subring of $k[X]$ where each polynomial $f \in k[X]$ has zero linear term. For example, $f = 1 + X^2 + X^4 \in R$ while $1 + X \notin R$. Notice that X^2 and X^3 are atoms in R since any factorization of these elements must involve a unit multiple of X. In addition, these two atoms are clearly not associates of each other. However, observe that

$$X^6 = X^2X^2X^2 = X^3X^3.$$

Hence, X^6 does not have a unique factorization into a product of atoms. In fact, the number of atoms in the two factorizations are different. So the length of each factorization need not be a fixed length. This is actually an example of a finite factorization domain (FFD) as discussed in [2]. But more generally, R is an atomic domain.

I.S. Cohen and I. Kaplansky studied the specific case where these structures have only a finite number of irreducible elements in [7]. D.D. Anderson and J.L. Mott expanded upon their investigation in their paper [3]. A *Cohen-Kaplansky domain* (CK domain), named in honor of Cohen and Kaplansky, is an atomic domain which has a finite number of atoms (up to associates). We note that throughout this thesis, we will often omit the "up to associates" part when talking about a selection of atoms. This paper is mainly concerned with atomic domains with particular finiteness conditions on the atoms. We will in particular look at quasilocal integral domains and local CK domains, which will be formally discussed in Section 1.1.

In Chapter 2, we will seek to investigate atoms in as general context as possible. That is, we will look at atoms in quasilocal integral domains that may not necessarily even be atomic. Much of this work will be based on the prior work of Cohen and Kaplansky [7] who investigated atoms in the context of local CK domains. However, many of their ideas apply to this more general context. So we will elaborate on this work and expand upon their ideas. In Chapter 3, we will focus these ideas toward the local CK domain case which will allow us to answer some important questions. In Chapter 4, we will investigate some important special examples involving CK domains. In particular, we will classify complete local CK domains that have exactly three atoms.

1.1 Preliminary Definitions and Notation

Throughout this thesis, R will always denote a commutative ring with $1 \neq 0$. We will always mean a commutative ring when we simply state R is a ring. We will often further assume R to be an integral domain but will drop this condition when greater generality holds. Often we will refer to integral domains as simply a domain. The set of units of R will be denoted by $U(R) = \{x \in R \mid \exists y \in R \text{ so that } xy = 1\}$. The set R^* will denote the set of nonzero elements of R. If I and J are sets in some universal set, then

$$I \backslash J = \{x \in I \mid x \notin J\}.$$

The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the integers, the rational numbers, the real numbers, and the complex numbers, respectively.

Let R be a commutative ring. Suppose we have a chain of prime ideals:

$$P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n.$$

This chain of prime ideals is said to have length n. (Notice that the length is one less than the number of prime ideals.) The *Krull-dimension* or simply the *dimension* of a ring is the supremum of all the lengths of chains of prime ideals in R. We denote this value by $\dim R$, which we allow to be infinite. Let P be a prime ideal of R. The *height* or *rank* of P is the supremum of all the lengths of chains of prime ideals in R ending at P:

$$P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n = P.$$

Another way to view this is to see that the height or rank of P is the dimension of R_P. We will denote this value by $\operatorname{rank}(P)$.

A ring R is said to be *quasilocal* if it has a unique maximal ideal M. We often denote that a ring is quasilocal by writing (R, M) where R is the ring and M is its unique maximal ideal. In the quasilocal case, M is actually the set of nonunits of R. A ring R is said to be *local* if it is Noetherian and quasilocal. We note that many authors use local to mean our quasilocal. However, this distinction will be useful in the following chapters.

Recall that a *principal ideal domain* (PID) is an integral domain in which every ideal is principal. Any PID is also a UFD. A *discrete valuation ring* (DVR) is a local PID that is not a field. Notice that since a PID is a UFD, a DVR is necessarily local and atomic. A DVR has a unique atom since its unique maximal ideal is principal.

Let R be a ring and M be an R-module. A set of generators W of M is called a *minimal basis* of M if there is no proper subset of W that generates M. In the specific case where (R, M) is quasilocal, we have the following useful theorem:

Theorem 1.1 (Theorem 2.3 [12]). *Let (R, M) be a quasilocal ring. Let A be a finitely generated R-module. So A/MA is a finite-dimensional vector space over R/M. Let n be its dimension. Then given a basis $\{\overline{x_1}, \ldots, \overline{x_n}\}$ for A/MA, we can choose $x_i \in A$ so that $\{x_1, \ldots, x_n\}$ is a minimal basis of A. Conversely, each minimal basis for A is obtained in this way. In particular, each minimal basis has n elements.*

We will most often use the case where $A = M$ and so we are talking about M/M^2 being a vector space over R/M. The important fact to remember is that the dimension of this vector space corresponds to the minimal number of elements that generates the maximal idea. In this special case, we will denote the number of minimal elements by $V(R)$ following the notation of [11]. A local ring is said to be *regular* if $\dim R = V(R)$.

Suppose that R is a commutative ring. We define an *R-algebra* to be a ring T together with a ring homomorphism $f : R \to T$ so that $f(1_R) = 1_T$ and $f(R) \subseteq Z(T)$ where $Z(T)$ is the center of T. This is equivalent to T being a ring that is also an R-module together with an R-bilinear multiplication map.

Let R be a commutative ring and let T be an R-algebra. We say that $x \in T$ is *integral* over R if the element satisfies a polynomial equation with coefficients in R and highest degree coefficient 1:

$$x^n + a_{n-1}x^n + \cdots + a_0 = 0$$

where $a_i \in R$ for all $0 \leq i < n$. If all of the elements of T are integral, we say that T is integral. A more useful criterion is given by the following theorem:

Theorem 1.2 (Theorem 12 [11]). *Let R be a commutative ring, T an R-algebra, and $x \in T$. Then the following are equivalent:*

1) x is integral over R and

2) there exists a finitely generated R-submodule A of T so that $xA \subseteq A$.

It turns out that for a commutative R-algebra over R that the elements of T that are integral over R form a subring of T [11, Theorem 14]. We often call this subring the *integral closure* of R in T. We will usually denote this by R'.

A concept we utilize later is the concept of analytic independence. We follow the treatment in [13]. Let Q be an ideal of a ring R. We say that Q is a *primary ideal* if $ab \in Q$ and $a \notin Q$ implies that $b^n \in Q$ for some positive power n. Clearly all prime ideals are primary ideals.

Theorem 1.3 (Proposition 1.5.3 [13]). *Let Q be a primary ideal of a ring R and let*

$$P = \{x \in R \mid x^n \in Q \text{ for some positive integer } n\}.$$

Then P is a prime ideal that contains Q. Moreover, P is contained in every other prime ideal containing Q.

So if Q and P are ideals as in Theorem 1.3, then we say that Q is *P-primary.* Let (R, M) be a local ring of dimension $d \geq 1$. A *system of parameters* in R is a set of d elements which generates an M-primary ideal.. Let $\phi(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables with coefficients in R. We say that ϕ is a *form of degree s* $(s \geq 0)$ if each nonzero term has degree s.

Let $t_1, t_2, \ldots, t_n \in M$. We say that the set $\{t_i\}$ are *analytically independent* if whenever $\phi(t_1, t_2, \ldots, t_n) = 0$ for any form ϕ, then each coefficient of ϕ is also in M.

Theorem 1.4 (Theorem 4.4.3 [13]). *Let $t_1, t_2, \ldots, t_n$ be a system of parameters in a local ring (R, M). Then the t_i are analytically independent.*

1.2 Atomic Domains

In [2], D.D. Anderson, D.F. Anderson, and M. Zafrullah investigated factorization properties that were weaker than unique factorization for integral domains.

In addition to studying atomic domains, they investigated several stronger notions. Following their terminology, we say that R satisfies the *ascending chain condition on principal ideals* (ACCP) if each strictly ascending chain of principal integral ideals of R terminates. We say R is a *bounded factorization domain* (BFD) if R is atomic and if for each nonzero nonunit of R there is a bound on the lengths of factorizations of an element into a product of atoms. We say R is a *half-factorial domain* (HFD) if R is atomic and each factorization of an element into a product of atoms has uniform length. We say R is a *finite factorization domain* (FFD) if each nonzero nonunit has a finite number of nonassociate divisors. Finally a domain is called an *idf-domain* if each nonzero element of R has at most a finite number of nonassociate irreducible divisors.

$$\begin{tikzcd}
 & HFD \arrow[dr, Rightarrow] & & & \\
UFD \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & FFD \arrow[r, Rightarrow] \arrow[d, Rightarrow] & BFD \arrow[r, Rightarrow] & ACCP \arrow[r, Rightarrow] & Atomic \\
 & idf-domain & & &
\end{tikzcd}$$

Figure 1.1: Overall diagram of implications

Examples can show that none of these implications can be reversed and are provided in [2]. For completeness, we give an example provided in [9] which shows that the implication ACCP $\implies$ atomic cannot be reversed.

Example 1.2 (Grams [9]). Let F be a field and let T be the additive submonoid of $\mathbb{Q}^+$ which is generated by $\{1/3, 1/(2\cdot 5), \ldots, 1/(2^k p_k), \ldots\}$, where p_k are the sequence of odd primes where $p_0 = 3$. Let $R = F[X;T]$. (This is just the monoid domain that can be thought of as polynomials with coefficients in F and powers in T on the indeterminate X.) Let $N = \{f \in R \mid \text{F has nonzero constant term}\}$. Let $A = F[X;T]_N$. Then A is a quasilocal atomic domain not satisfying ACCP.

More examples of atomic domains that do not satisfy ACCP can be constructed using the following theorem:

Theorem 1.5 (Proposition 1.2 [2]). *Let $T = K + M$ be an integral domain where M is a nonzero maximal ideal of T and K is a subfield of T. Let $D \subseteq K$ be a subring and $R = D + M$. Then we have that:*

a) R is atomic if and only if T is atomic and D is a field,

b) R satisfies ACCP if and only if T satisfies ACCP and D is a field.

CHAPTER 2
ATOMS IN QUASILOCAL DOMAINS

I.S. Cohen and Irving Kaplansky first investigated whether atoms exist in M^2 for a local CK domain (R, M) in their paper [7]. One should note that their notation and choice of wording is dated and requires some translation to current notation. For example, the authors say "prime" when referring to an atom. We seek to translate their results to more modern vernacular and expand upon their results further. Let (R, M) be a quasilocal integral domain. We say that a power of the maximal ideal M^n is (*weakly*) *universal* if $M^n \subseteq Rx$ for each atom $x \in R$ ($x \in M\backslash M^2$). Cohen and Kaplansky first introduced the notion of M^n being universal in [7]. They specifically investigated this concept for CK domains, but we will investigate this for quasilocal domains in general since many of their ideas also apply in this context.

2.1 Universality and Atoms

We start by establishing a lemma that we will use frequently.

Lemma 2.1. *Let G be a group with subgroups H and K. If $G = H \cup K$, then $G = K$ or $G = H$.*

Proof. If not, then there is a $g_1 \in G\backslash H$ and a $g_2 \in G\backslash K$. Since $G = H \cup K$, $g_1 \in K$ and $g_2 \in H$. Now, $g_1 + g_2 \in G = H \cup K$. So $g_1 + g_2 \in H$ or $g_1 + g_2 \in K$. If the former, then $g_1 \in H$, a contradiction. The same goes for the latter. $\square$

In particular, if (R, M) is a quasilocal domain and if $M = I \cup J$ is a union of two ideals I and J, then $M = I$ or $M = J$. The next theorem was mostly proven by Cohen and Kaplansky in [7] for CK domains. But since their proof also applies to quasilocal integral domains, we restate their results and give a similar proof once again for completeness.

Theorem 2.2. *Let (R, M) be a quasilocal domain with $M \neq M^2$ and residue class field $\overline{R} := R/M$. Let $\{V_\alpha\}_{\alpha\in\Omega}$ be the collection of one-dimensional subspaces of M/M^2 as a $\overline{R}$ vector space. Write $V_\alpha = \overline{R}\overline{x_\alpha}$ where $x_\alpha \in M$.*

1) If $x \in M\backslash M^2$, then x is an atom.

2) If $x, y \in M$ with $x \sim y$, then $\overline{R}\overline{x} = \overline{R}\overline{y}$.

3) $\{x_\alpha\}_{\alpha\in\Omega}$ is a set of nonassociate atoms in $M\backslash M^2$ with cardinality $|\Omega|$.

4) Suppose $q \in M^2$ is an atom and $\{u_\beta\}_{\beta\in\Gamma}$ is a complete set of representatives of $\overline{R}$. Then $\{x_\alpha + u_\beta q\}_{(\alpha,\beta)\in\Omega\times\Gamma}$ is a set of nonassociate atoms in $M\backslash M^2$.

5) The following are equivalent:

a) M^2 is universal,

b) $\{x_\alpha\}_{\alpha\in\Lambda}$ is a complete set of nonassociate atoms in $M\backslash M^2$,

c) $\{x_\alpha\}_{\alpha\in\Lambda}$ is a complete set of nonassociate atoms of R, and

d) M^2 is weakly universal.

Proof. (1) This is clear. (2) If $x \sim y$, then $x = uy$ for some $u \in U(R)$. So we have that $\overline{x} = \overline{u}\,\overline{y}$ where $\overline{u} \in \overline{R}$ is a unit. Hence, $\overline{R}\overline{x} = \overline{R}\overline{y}$. (3) Since the $\overline{x_\alpha}$ are nonzero vectors in M/M^2, the x_α are nonassociate atoms in $M\backslash M^2$. Indeed, for $\overline{x_\alpha}$ to be nonzero in M/M^2, we must have that $x_\alpha \in M$ and $x_\alpha \notin M^2$. Since each $x_\alpha \in M\backslash M^2$, they are atoms. Now x_α and x_β are associate if and only if $x_\alpha = ux_\beta$ for some $u \in U(R)$. So we obtain that $\overline{x_\alpha} = \overline{u}\,\overline{x_\beta}$ where $\overline{u} \in \overline{R}$. Therefore, we have that $V_\alpha = V_\beta$.

(4) Note that each $x_\alpha + u_\beta q \in M \backslash M^2$ since $q \in M^2$ but $x_\alpha \notin M^2$. So each $x_\alpha + u_\beta q$ are atoms. Suppose that $x_\alpha + u_\beta q \sim x_{\alpha'} + u_{\beta'} q$. That is, $x_\alpha + u_\beta q = u(x_{\alpha'} + u_{\beta'} q)$ for some $u \in U(R)$. Then in M/M^2 we have that $\overline{x_\alpha} = \overline{u}\,\overline{x_{\alpha'}}$. Since these generate the same subspace, this means that $\alpha = \alpha'$ and $\overline{u} = \overline{1}$. So we have that $x_\alpha + u_\beta q = ux_\alpha + uu_{\beta'} q$. Or, $(1-u)x_\alpha = (uu_{\beta'} - u_\beta)q$. Since $x_\alpha \not\sim q$, we have that $uu_{\beta'} - u_\beta \in M$. So $\overline{u_{\beta'}} = \overline{u}\,\overline{u_{\beta'}} = \overline{u_\beta}$ in $\overline{R}$. Hence, $\beta' = \beta$.

(5) $(a) \implies (b)$. Take $x \in M \backslash M^2$ an atom. Then the subspace generated by $\overline{x}$ coincides with the subspace generated by some $\overline{x_\alpha}$. That is, $\overline{R}\overline{x} = \overline{R}\overline{x_\alpha}$. That means that $x = ux_\alpha + q$ for some $u \in U(R)$ and $q \in M^2$. Since M^2 is universal, then $M^2 \subseteq Rx \cap Rx_\alpha$. Hence, $x_\alpha \in Rx$ and $x \in Rx_\alpha$. So the atoms x and x_α are associate. $(b) \implies (c)$. Suppose there is an atom $q \in M^2$. Then by part (4), $x_\alpha + q \in M \backslash M^2$ are all atoms for each $\alpha \in \Lambda$ which are not associate to the x_α. But this contradicts that the x_α are a complete set of representatives of atoms in $M \backslash M^2$. $(c) \implies (a)$. Take $q \in M^2$. Again, each $x_\alpha + q \in M \backslash M^2$ for each $\alpha \in \Lambda$ are atoms. Hence, $x_\alpha + q \sim x_\beta$ for some β which depends on α. But this means that in M/M^2 that $\overline{x_\alpha} = \overline{u}\,\overline{x_\beta}$ for some $u \in U(R)$. Hence, $\beta = \alpha$ and so $x_\alpha + q = ux_\alpha$. So $q = (u-1)x_\alpha$. So $x_\alpha | q$ for each $\alpha \in \Lambda$. So M^2 is universal. $(a) \iff (d)$. The forward direction is clear. For the converse, suppose $q \in M^2$ is an atom. Then since M^2 is weakly universal, $q \in Rx_\alpha$ for each $\alpha \in \Lambda$, a contradiction. $\square$

With the prior details established in Theorem 2.2, we can now establish our first important result.

Theorem 2.3. *Let (R, M) be a quasilocal integral domain. Let $\overline{R} = R/M$. Then:*

1) *Every atom is in M^2 if and only if $M = M^2$.*

2) *If M has exactly $1 \leq m < \infty$ nonassociate atoms not in M^2, then M can be generated by $d := \lceil \frac{m}{2} \rceil$ elements. So M is finitely generated.*

3) *The following are equivalent:*

   a) *R is a DVR,*

   b) *R is atomic and M is principal,*

   c) *R is atomic, $dim_{\overline{R}} M/M^2 = 1$, and there are only finitely many atoms in $M \backslash M^2$ (up to associates),*

   d) *R is atomic and has only one atom (up to associates), and*

   e) *R is atomic and has only one atom in $M \backslash M^2$ (up to associates).*

4) *R cannot have exactly two atoms (up to associates) not in M^2.*

5) *If $\overline{R}$ has infinite cardinality, then $M \backslash M^2$ either has either 0 ($M = M^2$), 1 (M principal) or infinitely many nonassociate atoms.*

6) *Suppose $\overline{R}$ is finite. If $dim_{\overline{R}} M/M^2 = \infty$, there are infinitely many atoms in $M \backslash M^2$. If $dim_{\overline{R}} M/M^2 = 0$, then $M = M^2$ and there are no atoms in $M \backslash M^2$. Suppose $1 \leq dim_{\overline{R}} M/M^2 = k < \infty$ instead. Let $m := \frac{|\overline{R}|^k - 1}{|\overline{R}| - 1}$. Suppose there are n nonassociate atoms in $M \backslash M^2$. Then $n \geq m$. Moreover if there is an atom in M^2, then $n \geq m|\overline{R}|$. That is, there are at least $m|\overline{R}| + 1 = \frac{|\overline{R}|^{k+1} - 1}{|\overline{R}| - 1}$ atoms in R. Now suppose $n < \infty$. Then M can be generated by $\lfloor \log_{|\overline{R}|}(n) + 1 \rfloor$ elements.*

We also have that the following are equivalent:

a) M^2 *is universal,*

b) $n = m$*, and*

c) there are exactly m *nonassociate atoms in* $M \backslash M^2$*.*

Proof. 1) If $M = M^2$, then since every atom is in M, every atom is in M^2. Conversely, if $M \neq M^2$, then there is an $x \in M$ with $x \notin M^2$. Hence, x must be an atom or else $x \in M^2$.

2) Let $x_1, \ldots, x_m$ be the nonassociate atoms not in M^2. Then all of the atoms are either in $(x_1, \ldots, x_m)$ or M^2 by hypothesis. Hence,

$$M = (x_1, \ldots, x_m) \cup M^2.$$

By Lemma 2.1, we have that $M = (x_1, \ldots, x_m)$ since by part (1) $M \neq M^2$. So M is finitely generated. Define $N_1 := (x_1, \ldots, x_d)$ and $N_2 := (x_{d+1}, \ldots, x_m)$. Notice that $M = (N_1 + M^2) \cup (N_2 + M^2)$. So by Lemma 2.1 we have that either $M = N_1 + M^2$ or $M = N_2 + M^2$. By Nakayama's Lemma, $M = N_1$ or $M = N_2$. This proves the desired result.

3) (a) $\implies$ (b) If R is a DVR, then the result is obvious. (b) $\implies$ (c) Since $M = (p)$ is principal and clearly finitely generated, it's clear that a minimal basis for M contains one element. So $\dim_{\overline{R}} M/M^2 = 1$. So if $x \in M \backslash M^2$ is an atom, then $x = up$ for some $u \in R$. But since x is an atom, $u \in U(R)$. (c) $\implies$ (d) Since $M \backslash M^2$ has only finitely many atoms, we see that M is finitely

generated by (2). Hence by Theorem 1.1, we see that $\dim_{\overline{R}} M/M^2 = 1$ implies that M is principal, say $M = (p)$. Hence if $x \in R$ is an atom, then $x = up$ for some $u \in R$. But since x is an atom, we must have that $u \in U(R)$ and so $x \sim p$. (d) $\implies$ (e) Suppose p is the only atom of R (up to associates). Then since R is atomic, every element in M is divisible by p. Hence, $M = (p)$. So p is in a minimal basis for M and so $p \notin M^2$. (e) $\implies$ (a) Let x be the only atom not in M^2 (up to associates). Then by (2), $M = (x)$. If $y \in M$ is another atom, then $y = ux$ for some $u \in R$. Hence, u must be a unit. So there is only one atom up to associates. Since R is an atomic domain, every element is of the form ux^n for some $u \in U(R)$ and n a positive integer. Since each ideal of R must contain an element with a minimal power of x, it must be generated by that element. Hence, R is also a PID. So R is a DVR.

4) If x_1, x_2 are two such atoms not in M^2, then M must be generated by one of them by (2), say x_1. Then $x_2 \in (x_1) = M$ which is impossible.

5) Suppose $|\overline{R}|$ is infinite. If $\dim_{\overline{R}} M/M^2 = 0$, then $M = M^2$ and hence there are 0 atoms in $M \backslash M^2$. Suppose $\dim_{\overline{R}} M/M^2 = 1$ and there are only finitely many atoms in $M \backslash M^2$ (up to associates). So by (2) we have that M is finitely generated and hence $\dim_{\overline{R}} M/M^2 = 1$ implies that M has a minimal basis of one element. So M is principal and so there is only one atom in $M \backslash M^2$. (If there are infinitely many atoms in $M \backslash M^2$ in this case, then we are done.)

   Now suppose $\dim_{\overline{R}} M/M^2 > 1$. Then M/M^2 as a vector space over $\overline{R}$ has infinitely many one-dimensional subspaces. Indeed, since the dimension is at least two, we have two vectors $\overline{x_1}, \overline{x_2}$ where $x_1, x_2 \in M\backslash M^2$ that span distinct subspaces. But the linear combinations $\{\overline{x_1} + s\overline{x_2}\}_{s\in\overline{R}}$ all generate distinct subspaces. To see this, suppose $\overline{x_1} + s\overline{x_2}$ and $\overline{x_1} + s'\overline{x_2}$ generate the same subspace where $s, s' \in \overline{R}$ with $s \neq s'$. Then

$$\overline{x_1} + s\overline{x_2} = u(\overline{x_1} + s'\overline{x_2})$$

where $u \in \overline{R}^*$ and $u \neq 1$. Rewriting this we obtain that $(1-u)\overline{x_1} = (us' - s)\overline{x_2}$. Since $\overline{R}$ is a field, we have that $1 - u \in \overline{R}$ and $us' - s \in \overline{R}$ which is impossible. Hence there are infinitely many atoms in $M\backslash M^2$.

6) Suppose $|\overline{R}|$ is finite. If $\dim_{\overline{R}} M/M^2 = \infty$, then we have infinitely many atoms in $M\backslash M^2$. Indeed, we have infinitely many one-dimensional subspaces since the dimension is also infinite. The case of $\dim_{\overline{R}} M/M^2 = 0$ is clear. The rest of this proof is from [7, Theorem 8] but with the CK domain hypothesis removed (and some added argument because of this removed hypothesis). So suppose that $1 \leq \dim_{\overline{R}} M/M^2 = k < \infty$ and let n be the number of nonassociate atoms in $M\backslash M^2$. The one-dimensional subspaces of M/M^2 must be generated by atoms in $M\backslash M^2$. By Theorem 2.2 we have that distinct one-dimensional subspaces are represented by nonassociate atoms. Since there are

$$\frac{|\overline{R}|^k - 1}{|\overline{R}| - 1}$$

distinct one-dimensional subspaces, there must be at least this many nonassociate atoms in $M\backslash M^2$. Since there are n nonassociate atoms in $M\backslash M^2$, we thus have

$$m := \frac{|\overline{R}|^k - 1}{|\overline{R}| - 1} \leq n.$$

Now suppose there is an atom $q \in M^2$ and let $x_1, \ldots, x_m$ be nonassociate atoms representing the $m = (|\overline{R}|^k - 1)/(|\overline{R}| - 1)$ one-dimensional subspaces of M/M^2. Let $u_1, \ldots, u_{|\overline{R}|}$ be representatives in R of the elements of $\overline{R}$. Then $x_i + u_j q \in M\backslash M^2$ are all nonassociate atoms as shown in Theorem 2.2. Since m is the number of one-dimensional subspaces of M/M^2, we have at least $m|\overline{R}|$ total atoms in $M\backslash M^2$. Since we have an atom $q \in M^2$, we have at least

$$\begin{aligned} |\overline{R}|m + 1 &= \frac{|\overline{R}|(|\overline{R}|^k - 1)}{|\overline{R}| - 1} + 1 \\ &= \frac{|\overline{R}|(|\overline{R}|^k - 1)}{|\overline{R}| - 1} + \frac{|\overline{R}| - 1}{|\overline{R}| - 1} \\ &= \frac{|\overline{R}|^{k+1} - |\overline{R}| + |\overline{R}| - 1}{|\overline{R}| - 1} \\ &= \frac{|\overline{R}|^{k+1} - 1}{|\overline{R}| - 1} \end{aligned}$$

atoms of R. Now suppose $n < \infty$. By (2) we have that M is finitely generated. So M has a minimal basis of k atoms. Recall that we have

$$m := \frac{|\overline{R}|^k - 1}{|\overline{R}| - 1} \leq n.$$

Rearranging we obtain that $|\overline{R}|^k \leq n|\overline{R}| - (n-1)$. Hence,

$$
\begin{aligned}
k &\leq \log_{|\overline{R}|}\left(n\left(|\overline{R}| - \frac{n-1}{n}\right)\right) \\
&= \log_{|\overline{R}|} n + \log_{|\overline{R}|}\left(|\overline{R}| - \frac{n-1}{n}\right) \\
&\leq \log_{|\overline{R}|}(n) + 1.
\end{aligned}
$$

Hence, M can be generated by $\lfloor \log_{|\overline{R}|}(n) + 1 \rfloor$ elements.

For the last part, if M^2 is universal, then by Theorem 2.2, the set of atoms in $M \backslash M^2$ which determine each of the one-dimensional spaces is a complete set of atoms of R. Hence, $n = m$. If $n = m$, then it is clear there are exactly m nonassociate atoms in $M \backslash M^2$. Finally, if there are exactly m nonassociate atoms in $M \backslash M^2$, then these atoms are the complete set of atoms in Theorem 2.2.

□

Theorem 2.4. *Let (R, M) be a quasilocal domain.*

1) *If there are exactly $0 \leq n < \infty$ nonassociate atoms in $M \backslash M^2$, then M^n is weakly universal.*

2) *(Cohen and Kaplansky [7, Theorem 3]) Suppose $M \neq M^2$ and that there are exactly $2 \leq n < \infty$ nonassociate atoms in R. Then M^{n-1} is universal.*

Proof. We note that (2) was shown in [7, Theorem 3] but in the context of CK domains. However, the same proof applies for the quasilocal domain case. We modify their proof to prove (1).

The case $n = 0$ is trivial and the case $n = 1$ gives M is principal by Theorem 2.3. Then clearly in this case M is weakly universal. So assume $n > 1$. Note that even though R is not necessarily atomic that M is still generated by the n atoms $a_1, \ldots, a_n$ in $M \backslash M^2$ by the proof of Theorem 2.3. So let x be a product of n atoms which we may assume the atoms are in $\{a_1, \ldots, a_n\}$. (So it is in M^n.) Suppose by way of contradiction that a_1 does not divide x. Form the elements x_i by deleting successive atoms from the factorization of x. That is, $x_1 = x$ and $x_{n-1} = a_s a_k$ for some s and k other than 1. Form $x_i + a_1 \in M$. Note that $x_i \in M^2$ for $i = 1, \ldots, n-1$. Therefore, $x_i + a_1 \in M \backslash M^2$ for $i = 1, \ldots, n-1$. So $x_i + a_1$ is an atom in $M \backslash M^2$ and hence must be associate to one of the $a_1, \ldots, a_n$. Clearly a_1 does not divide $x_i + a_1$ or else $a_1 | x_i$ and hence $a_1 | x$ which contradicts our hypothesis. Notice that a_k does not divide $x_i + a_1$ for $i = 1, \ldots, n-1$ since if it did, then $a_1 = x_i - r a_k$ for some $r \in R$. Since $a_k | x_i$, then $a_k | a_1$, a contradiction. We must have that the $x_i + a_1$ are associate to one of $a_2, \ldots, a_n$ other than a_k. Since there are $n-1$ elements of the form $x_i + a_1$ and there are $n-2$ atoms in $M \backslash M^2$ distinct from a_1 and a_k, the pigeonhole principle states that two of these atoms must be associates, say $x_j + a_1$ and $x_l + a_1$ where $j < l$. Then $x_j + a_1 \sim x_l + a_1 \sim a_p$ for some $p \neq 1$. Hence,

$$x_l - x_j = x_l(1 - x_j/x_l)$$

is divisible by a_p. Notice that $1 - x_j/x_l \notin M$ so it is a unit. Hence, $a_p | x_l$. Since $a_p \sim x_l + a_1$, then $a_p | a_1$ which is a contradiction.

$\square$

CHAPTER 3
LOCAL CK DOMAINS

We investigate the occurrence of atoms inside of powers of the maximal ideal M of a local CK domain. From now on, let (R, M) be a local CK domain unless otherwise stated. Recall that M/M^2 may be regarded as a vector space over $\overline{R}$ and that minimal generating sets $\{a_1, \ldots, a_k\}$ for M correspond to vector space bases for M/M^2 by Theorem 1.1. Notice that if $k = 1$, then the number of atoms is one. Since the case of $k = 1$ means that in this particular instance that R is a DVR, a discrete valuation ring, the most interesting cases arise when $k > 1$. So we will often assume that R is not a DVR.

3.1 The Quotient Group V

Let (R, M) be a local CK domain, which is not a DVR, with quotient field K. Let $[M : M] := \{x \in K | xM \subseteq M\}$. This is clearly a ring containing R. Additionally, $R \subsetneq [M : M]$. Indeed, by Theorem 2.4, there is a smallest positive integer n so that M^n is universal. (This depends on the CK domain in question.) We note that the case $n = 1$ implies that R is a DVR. Let $a_1, \ldots, a_k$ be all of the atoms. Hence, there is an $s \in M^{n-1} \backslash M^n$ where some atom a_i does not divide s. Notice that $\frac{s}{a_i} M \subseteq M$. This is because any atom multiplied by s is in M^n, which is universal. Hence, $\frac{s}{a_i} \in [M : M]$. However, $\frac{s}{a_i} \notin R$ since a_i does not divide s.

Denote the integral closure of R by R'. We note that $[M : M]$ is an R-algebra. Indeed, $[M : M]$ is a ring together with the inclusion homomorphism. We note that

$M \subseteq [M : M]$ is still an ideal of $[M : M]$ since if $x \in [M : M] \subseteq K$ and $m \in M$, then $xm \in M$ by definition of $[M : M]$. Additionally, M is a finitely generated R-submodule of $[M : M]$. Hence, we see that $[M : M] \subseteq R'$ [11, Theorem 12]. So altogether we have that

$$R \subsetneq [M : M] \subseteq R'.$$

Define the quotient group

$$V := U([M : M])/U(R).$$

Recall that the *group of divisibility* is the quotient group $G(R) := K^*/U(R)$ where K^* denotes the group of nonzero elements of K. For a CK domain R, $G(R)$ is finitely generated [3]. It follows that $G(R) \cong G(R') \oplus U(R')/U(R)$ which yields that $U(R')/U(R)$ is finite [3, Theorem 3.1]. Since $[M : M] \subseteq R'$, we see then that this implies that V is also finite.

Lemma 3.1. *Let (R, M) be a local CK domain. Let $v \in U([M : M])$. Then we have:*

1) $z \in R$ is an atom if and only if $vz \in R$ is an atom.

2) $z \in M^s \backslash M^{s+1}$ if and only if $vz \in M^s \backslash M^{s+1}$ $(s \geq 1)$.

Proof. For (1), notice that $vz = xy$ if and only if $z = x(yv^{-1})$. Since $v \in U([M : M])$, we see that $yv^{-1} \in M$ still. So if one is reducible, so is the other. For (2), the idea is similar. If $z \in M^s$, then $z = \sum_{i=1}^{n} z_{i,1} \cdots z_{i,s}$ for some positive integer n and where each $z_{i,j} \in M$. So

$$\begin{aligned} vz &= \sum_{i=1}^{n} vz_{i,1} \cdots z_{i,s} \\ &= \sum_{i=1}^{n} (vz_{i,1}) z_{i,2} \cdots z_{i,s}. \end{aligned}$$

Since $v \in [M : M]$, this means that $vz_{i,1} \in M$ as well. So $vz \in M^s$. By a similar argument, we see that $vz \in M^s$ implies that $z \in M^s$. This is because $v \in U([M : M])$ has an inverse in $[M : M]$. □

With this, we see that the group V essentially divides the atoms into various cosets depending on the powers of M that they lie in. (There may be multiple cosets for various powers.)

Theorem 3.2. *Let (R, M) be a local CK domain which is not a DVR. Let $\overline{R} := R/M$. Then $|V| \geq |\overline{R}| = N$. Furthermore, if M is also the maximal ideal of $[M : M]$, then $|V| \geq |\overline{R}| + 1$.*

Proof. The first part of this proof is a rewording of [7, Theorem 10]. Let $u_1, \ldots, u_{N-1}$ be nonzero representatives of $\overline{R}$. Again by Theorem 2.4, there is smallest positive integer s so that M^{s+1} is universal but M^s is not. So there is a $y \in M^s \backslash M^{s+1}$ that is not divisible by an atom x. Since $y + u_i x \in M$, there is an atom r_i dividing $y + u_i x$ for each i. Note that each divisor must be distinct. If not, then r_i divides both $y + u_i x$ and $y + u_j x$ for some $j \neq i$. Hence, $r_i | (u_i - u_j)x$. Note that $u_i - u_j \notin M$ or else $\overline{u_i} = \overline{u_j} \in \overline{R}$. This means that r_i and x are associates. Hence, $r_i | y$ which implies that $x | y$, a contradiction.

So for each i, we have that $s_i r_i = y + u_i x$ for some $s_i \in R$. Let $q \in M$. We have that $qy + u_i qx = (s_i q) r_i$. Since $qy \in M^{s+1}$, then $r_i | u_i qx$. Hence, $r_i | qx$. As mentioned earlier, $r_i \nmid y$ for all i or else $x|y$. Hence, the argument given above also works with x being replaced by each r_i. Using the process above, we can form a sequence $w_1, w_2, \ldots$ where $w_{j+1} | w_j q$ for $q \in M$. However, since there are a finite number of atoms, some w_j must eventually be equivalent to a prior atom in the sequence. In the above argument for x, we generated $N-1$ new atoms. Hence, we can form a sequence where the first N are guaranteed to be distinct. Moreover, since there are finitely many atoms, eventually some atom must repeat. From this, we can form a subsequence that is cyclic. Say the first $w_1, \ldots, w_m$ are distinct from this subsequence where $m \geq N$. So we have that $w_j | w_k q$ for any $1 \leq j, k \leq m$.

Notice that $\frac{w_j}{w_k} q \in R$ by this property. Furthermore, if $\frac{w_j}{w_k} q = u \notin M$, then $w_j q = u w_k$, a contradiction. Hence, $\frac{w_j}{w_k} q \in M$ and so $\frac{w_j}{w_k} \in [M : M]$ for all $1 \leq j, k \leq m$. It follows that $\frac{w_j}{w_k} \in U([M : M])$. Notice that $\frac{w_1}{w_j}$ and $\frac{w_1}{w_k}$ for $j \neq k$ are two different representatives in V. If not, then $\frac{w_1}{w_j} = u \frac{w_1}{w_k}$ for some $u \in U(R)$. Hence, $w_j = u w_k$, a contradiction. Hence, we have at least $N = |\overline{R}|$ elements in V.

For the second part of this proof, suppose that M is the maximal ideal of $[M : M]$. We note that $[M : M]$ is an R-module and that $[M : M]/M$ is annihilated by M. Hence, $[M : M]/M$ is an $\overline{R}$ vector space. Since $R \subsetneq [M : M]$, we have that $\overline{R} = R/M \subsetneq [M : M]/M$ which means that

$$|[M : M]/M| > |\overline{R}| = N.$$

Hence, $\dim_N [M : M]/M > 1$. (Indeed, if $\overline{v} \in [M : M]/M$, then the span of $\overline{v}$ has N elements.) Hence, the number of one-dimensional subspaces is at least:

$$\frac{|\overline{R}|^2 - 1}{|\overline{R}| - 1} = \frac{(|\overline{R}| + 1)(|\overline{R}| - 1)}{|\overline{R}| - 1} = |\overline{R}| + 1.$$

If $\overline{v} \in [M : M]/M$ is nonzero, then $v \in [M : M]$ and $v \notin M$. Hence, $v \in U([M : M])$ since M is the unique maximal ideal of $[M : M]$. That is, $\overline{v} \in V$. Additionally, suppose $\overline{v_1}, \overline{v_2} \in [M : M]/M$ with $v_1 U(R) = v_2 U(R)$. That is, $v_1 = uv_2$ for representatives v_1, v_2 and $u \in U(R)$. That is, $u \notin M$. Hence, $\overline{v_1} = \overline{u}\,\overline{v_2}$ and $\overline{u} \in \overline{R}$ which is nonzero. Hence, $\overline{v_2} \in \text{span}\{\overline{v_1}\}$. Since there are at least $|\overline{R}| + 1$ of these one-dimensional subspaces which correspond to distinct elements of V, we have that $|V| \geq |\overline{R}| + 1$ as desired.

□

Anderson and Mott in [3] investigated the link between a quasilocal domain having M^2 universal and the ring $[M : M]$. We will need this result later:

Theorem 3.3 (Corollary 5.2. Anderson and Mott [3])**.** *Let (R, M) be a quasilocal domain. Then R is an atomic domain with M^2 universal if and only if $[M : M]$ is a DVR with maximal ideal M.*

Using this we can prove the following for a local CK domain:

Theorem 3.4. *Let (R, M) be a local CK domain that is not a DVR. Let R' be the integral closure of R. Then the following are equivalent:*

1) M^2 is universal.

2) M^2 *is weakly universal.*

3) The number of atoms is exactly the number of one-dimensional subspaces of M/M^2 *over* $\overline{R} := R/M$*. That is,* $\frac{N^k-1}{N-1}$ *where* $k = dim_{\overline{R}} M/M^2$ *and* $N = |\overline{R}|$*.*

4) All of the atoms of R *lie in one coset of* V*. (That is, given an atom* $x \in R$*, we have that each atom has the form* xr *where* $r \in U([M : M])$*.)*

5) All of the atoms of $M \backslash M^2$ *lie in one coset of* V*.*

6) $[M : M] = R'$ *and has unique maximal ideal* M*.*

Proof. $1 \iff 2 \iff 3$. These follow from Theorem 2.3.

$1 \implies 4$. Again suppose that M^2 is universal. By Theorem 3.3, $[M : M]$ is a DVR with unique maximal ideal M. So let $M = p[M : M]$. Since M^2 is universal, all of the atoms are in $M \backslash M^2$. So suppose p' is a distinct atom from p. Hence, $p' = rp$ where $r \in [M : M]$ but $r \notin U(R)$ since p and p' are distinct atoms. Note if $r \in M$, then $p' \in M^2$ which is a contradiction. Hence, $r \notin M$ and so $r \in U([M : M])$ since M is the unique maximal ideal of R'. That is, p and p' are in the same coset.

$4 \implies 1$. Suppose all of the atoms lie in the same coset. That is, all atoms are of the form rp where $r \in U([M : M])$ and p is an atom. Take any three atoms x_1, x_2, x_3. Then $x_i = r_i p$ where $r_i \in U([M : M])$. Notice that

$$x_2 x_3 = r_2 p x_3 = r_1 p r_2 (r_1)^{-1} x_3 = x_1 (r_2 (r_1)^{-1}) x_3.$$

Notice that $r_2(r_1)^{-1} \in [M : M]$ and so $(r_2(r_1)^{-1})x_3 \in M$. Hence, $x_1 | x_2 x_3$. It follows that M^2 is universal.

$1 \iff 5$. The forward direction is clear since (1) is equivalent to statement (4). For the converse, suppose that all of the atoms of $M \backslash M^2$ lie in one coset of V. By the proof of Theorem 2.3, we have that $M = (x_1, \ldots, x_n)$ where $\{x_i\}_{i=1}^n$ is a complete set of nonassociate atoms in $M \backslash M^2$. By hypothesis, we have that $x_i = v_{i,k}x_k$ where each $v_{i,k} \in U([M : M])$ for $1 \leq i, k \leq n$. So we have that $x_i x_j = v_{i,k}v_{j,k}x_k^2 = v_{i,k}v_{j,k}x_k x_k \in Rx_k$ for each k. Hence, $M^2 \subseteq Rx_k$ for each k. So M^2 is weakly universal. Hence, M^2 is universal.

$1 \implies 6$. Suppose that M^2 is universal. By Theorem 3.3, we have that M^2 universal implies that $[M : M]$ is a DVR with unique maximal ideal M. Recall that we have that $R \subsetneq [M : M] \subseteq R'$ where R' is the integral closure of R. Furthermore, the integral closure of R is the intersection of all valuation rings of the quotient field K of R that contain R [12, Theorem 10.4]. Since $[M : M]$ is such a ring, we see that $R' \subseteq [M : M]$. Hence, $[M : M] = R'$.

$6 \implies 1$. Suppose that $R' = [M : M]$ and that M is the unique maximal ideal of $[M : M]$. Since R is a CK domain, we have that $R' = [M : M]$ is a semilocal PID and hence is a local PID [3, Theorem 4.3]. That is, it is a DVR. By Theorem 3.3, we have that M^2 is universal. □

We note that the equivalence of (1) and (3) provided in the theorem was mentioned by Cohen and Kaplansky in [7]. However, we more explicitly provided the details of this claim. We get two corollaries from this theorem:

Corollary 3.5. *Let (R, M) be a local CK domain with exactly $n > 1$ atoms that is not a DVR. Let $\overline{R} := R/M$.*

1) We have n is a multiple of $|V|$. In particular, if n is prime, M^2 is universal.

2) The number of atoms in $M^s \backslash M^{s+1}$ is a multiple of $|V|$, possibly 0. Hence, if $M^s \backslash M^{s+1}$ contains an atom, there are at least $|\overline{R}|$ atoms in $M^s \backslash M^{s+1}$. Furthermore, if the number of atoms in $M \backslash M^2$ is prime, then M^2 is universal.

3) Suppose R has exactly $2p$ atoms where p is a prime and suppose that $|\overline{R}| \neq 2$, then no atoms lie in M^2.

Proof. For (1), Each coset contains $|V|$ classes of atoms by Lemma 3.1. Hence the total amount of atoms must be a multiple of $|V|$. That is, $n = k|V|$ for some positive integer k. For completeness, notice that if n is prime, then $k = 1$ and hence $n = |V|$. So by Theorem 3.4, we have that M^2 is universal.

For (2), by the second part of Lemma 3.1, we see that each coset containing an atom in $M^s \backslash M^{s+1}$ only contains atoms in $M^s \backslash M^{s+1}$. Since each coset contains $|V|$ distinct atoms, we see that the number of these atoms are a multiple of $|V|$, which may be zero. By Theorem 3.2, $|V| \geq |\overline{R}|$. Finally, by Theorem 3.4 we see that if the number of atoms in $M \backslash M^2$ is prime, then M^2 is universal.

Now for (3) notice that since $|\overline{R}| \neq 2$ means that $|V| \geq |\overline{R}| \geq 3$ by Theorem 3.2. By part (2), we have that $|V|$ must either be p or $2p$. If $|V| = 2p$, then since there are $2p$ total atoms, all atoms must lie in $M \backslash M^2$. If $|V| = p$, then there are two cases. The first case is that there are two cosets of atoms in $M \backslash M^2$ both of which have p atoms. In this case, all atoms lie in $M \backslash M^2$ and hence no atoms lie in M^2. In the second case, there are p atoms in $M \backslash M^2$ and p atoms in M^2. But by part (2), since the number of atoms in $M \backslash M^2$ is prime, then M^2 is universal. Hence, there cannot be any atoms in M^2. ◻

Corollary 3.6. *Let (R, M) be a local CK domain that's not a DVR. Let $\overline{R} = R/M$. If R has less than 2 $|\overline{R}|$ nonassociate atoms, then R has exactly $|\overline{R}| + 1$ atoms and M^2 is universal.*

Proof. By Theorem 3.2, we have that $|V| \geq |\overline{R}|$. By Corollary 3.5, the number of atoms is a multiple of $|V|$. Hence, the number of atoms must be equal to $|V|$. By Theorem 3.4, this implies that M^2 is universal and the number of atoms is equal to the number of one-dimensional subspaces of M/M^2 over $\overline{R}$. The smallest this can be is $|\overline{R}| + 1$ (where the dimension is 2). Notice that

$$\frac{|\overline{R}|^3 - 1}{|\overline{R}| - 1} = \frac{(|\overline{R}| - 1)(|\overline{R}|^2 + |\overline{R}| + 1)}{|\overline{R}| - 1} = |\overline{R}|^2 + |\overline{R}| + 1 \geq 2|\overline{R}|.$$

Since the other possible options are larger than this amount, this means that the dimension of M/M^2 over $\overline{R}$ must be 2 and hence the number of atoms is $|\overline{R}| + 1$ as desired. ◻

Additionally, since we later seek to investigate atoms in M^2, we have the following result:

Corollary 3.7. *Let (R, M) be a local CK domain with an atom in M^2. Let $\overline{R} = R/M$ and $k = dim_{\overline{R}} M/M^2$. Then the number of nonassociate atoms of R is at least $(|\overline{R}|^{k+1} - 1)/(|\overline{R}| - 1) + |\overline{R}| - 1$. If further M is the maximal ideal of $[M : M]$, then there are at least $(|\overline{R}|^{k+1} - 1)/(|\overline{R}| - 1) + |\overline{R}|$ nonassociate atoms of R.*

Proof. By Theorem 2.3 (6), if there is an atom in M^2, then the number of atoms in $M\backslash M^2$ is at least $(|\overline{R}|^{k+1} - 1)/(|\overline{R}| - 1) - 1$. Since there is also an atom in M^2 and since $|V| \geq |\overline{R}|$ by Theorem 3.2, by combining these with Corollary 3.5 we have that there are at least $(|\overline{R}|^{k+1} - 1)/(|\overline{R}| - 1) + |\overline{R}| - 1$ nonassociate atoms of R. The last part can similarly be argued by using Theorem 3.2. □

3.2 Examples

From this, Cohen and Kaplansky ask in [7] whether atoms exist in M^2. Anderson and Mott show in their paper [3] that such examples do exist:

Example 3.1 (Example 7.3. Anderson and Mott [3]). Let F be a finite field and let $s \geq 1$. Then there exists a complete local CK domain (R, M) with $R/M \cong F$ and an atom $f \in M^s\backslash M^{s-1}$. Moreover, no element of M^{s+1} is an atom.

Proof. We sketch their proof since the details will be needed later. Take $f_s \in F[Y]$ irreducible of degree s. Take a field extension L of F with $[L : F] = s + 1$. Form an F-basis $1, y, y^2, \ldots, y^s$ for L. Let

$$V_i = \{a_0 + a_1 y + \cdots + a_i y^i | a_j \in F \text{ for } 0 \leq j \leq i\}.$$

Define $R = F + V_1 X + \cdots + V_{s-1} X^{s-1} + L[[X]]X^s$. By [3, Theorem 7.1], R is a complete local CK domain. We also have that $f = f_s(y)X^s$ is irreducible. □

Suppose that R has an atom in M^2. It follows from Corollary 3.7 and using the minimal cases $\overline{R} \cong \mathbb{Z}_2$ and $k = 2$, that such a domain must have at least eight atoms. Cohen and Kaplansky claim in [7] that their results at the time ruled out the possibility of $n = 8$ or 9. We will provide an example of a CK domain with eight atoms. Two of these atoms are in M^2:

Example 3.2. Let $F = \mathbb{F}_2$, $L = \mathbb{F}_8$, and $s = 2$ as in Example 3.1. Take $f_2(Y) = Y^2 + Y + 1$. Let $V_1 = \{0, 1, y, 1+y\}$. Then $R = \mathbb{F}_2 + V_1X + \mathbb{F}_8[[X]]X^2$ is a complete local CK domain with eight atoms:

1) $f = (y^2 + y + 1)X^2 \in M^2 \backslash M^3$
2) $X \in M \backslash M^2$
3) $yX \in M \backslash M^2$
4) $(1 + y)X \in M \backslash M^2$
5) $X + (y^2 + y + 1)X^2 \in M \backslash M^2$
6) $yX + (y^2 + y + 1)X^2 \in M \backslash M^2$
7) $(1 + y)X + (y^2 + y + 1)X^2 \in M \backslash M^2$
8) $(y^2 + y + 1)X^2 + X^3 \in M^2 \backslash M^3$

Moreover, M^4 is universal, while M^3 is not.

Proof. We note that R was shown to be a complete local CK domain in [3, Example 7.3]. Additionally, atom (1), f, is the atom in $M^2 \backslash M^3$ in Example 3.1. To show each of these atoms are distinct, let $z = a_1X + a_2X^2 + a_3X^3 + \cdots$ where $a_1 \in V_1$ and $a_i \in \mathbb{F}_8$ for $i \geq 2$. We note that if $a_1 = a_2 = 0$, then $z \in M^3$ and so z is not an atom since an X can be factored out. So it suffices to only check the cases where at least one of these terms is nonzero.

	y	y^2	$y+1$	y^2+y	y^2+y+1	y^2+1
y	y^2	$y+1$	y^2+y	y^2+y+1	y^2+1	1
y^2	$y+1$	y^2+y	y^2+y+1	y^2+1	1	y
$y+1$	y^2+y	y^2+y+1	y^2+1	1	y	y^2
y^2+y	y^2+y+1	y^2+1	1	y	y^2	$y+1$
y^2+y+1	y^2+1	1	y	y^2	$y+1$	y^2+y
y^2+1	1	y	y^2	$y+1$	y^2+y	y^2+y+1

Table 3.1: Multiplication table for $\mathbb{F}_8^*$ which excludes 1

The multiplication table comes from reducing using the polynomial Y^3+Y+1. That is, $y^3 = y+1$. Now we break into cases. For the first case, we will explain the process more in depth:

Case 1. $a_1 = 0$.

Case 1a. $a_2 = 0$. Then $z \in M^3$ and so it is not an atom.

Case 1b. $a_2 = 1, y, y+1$. Then $z = X(a_2X + a_3X^3 + \cdots)$ and so z is not an atom.

Case 1c. $a_2 = y^2, y^2+y$. Then $z = yX(y^{-1}a_2X + y^{-1}a_3X^3 + \cdots)$ and so z is not an atom. Notice here that $y^{-1}a_2 \in V_1$.

Case 1d. $a_2 = y^2+1$. Then $z = (y+1)X((y+1)X + (y+1)^{-1}a_3X^3 + \cdots)$ and so z is not an atom. Notice here that $(y+1)^2 = y^2+1$.

Case 1e. $a_2 = y^2 + y + 1$. Notice that:

$$(y^2 + y + 1)X^2 + a_3X^3 + \cdots = (y^2 + y + 1)X^2(1 + (y^2 + y + 1)^{-1}a_3X + \cdots)$$

if and only if $(y^2 + y + 1)^{-1}a_3 \in V_1 = \{0, 1, y, y + 1\}$. That is, $a_3 \in \{0, y^2 + y + 1, y^2 + 1, y\}$. Furthermore, this case always yields an atom. Suppose that

$$(y^2 + y + 1)X^2 + a_3X^3 + \cdots = (\alpha X + \cdots)(\beta X + \cdots) = \alpha\beta X^2 + \cdots .$$

Then we have $\alpha\beta = y^2 + y + 1$ with $\alpha, \beta \in V_1$. But this is impossible. Hence, all of these must be atoms. From the above argument, this means that

$$(y^2 + y + 1)X^2 + X^3$$

is an atom in M^2 but cannot be associate to $f = (y^2 + y + 1)X^2$. Additionally,

$$\begin{aligned}(y^2 + y + 1)X^2 + a_3X^3 + \cdots &= ((y^2 + y + 1)X^2 + X^3)(1 + b_1X + \cdots) \\ &= (y^2 + y + 1)X^2 + X^3 + (y^2 + y + 1)b_1X^3 \\ &+ (b_1 + b_2(y^2 + y + 1))X^4 + \cdots .\end{aligned}$$

So we have $a_3 = 1 + (y^2 + y + 1)b_1$ and $a_n = b_{n-3} + b_{n-2}(y^2 + y + 1)$ for $n > 3$. So we may define: $b_1 = (y^2 + y + 1)^{-1}(a_3 + 1)$ and $b_n = (y^2 + y + 1)^{-1}(a_{n+2} + b_{n-1})$ recursively. For $b_1 \in V_1$, we must have $a_3 \in \{1, 1 + y, y^2, y^2 + y\}$. Notice that these are the remaining elements in $\mathbb{F}_8$ compared to the case with the atom $(y^2+y+1)X^2$. So all of these atoms must be associate to one of these two atoms.

Case 2. $a_1 = 1$.

Case 2a. $a_2 = 0, 1, y, y+1$. Then $z = X(1 + a_2X + \cdots)$ is associate to X.

Case 2b. $a_2 = y^2, y^2+1, y^2+y, ^2+y+1$. Then

$$\begin{aligned} z = X + a_2X^2 + a_3X^3 + \cdots &= (X + (y^2+y+1)X^2)(1 + b_1X + b_2X^2 + \cdots) \\ &= X + (b_1 + (y^2+y+1))X^2 \\ &+ (b_2 + b_1(y^2+y+1))X^3 + \cdots . \end{aligned}$$

with $b_1 = a_2 + y^2 + y + 1$ and $b_n = a_{n+1} + b_{n-1}(y^2+y+1)$ for $n > 1$ show that z is associate to $X + (y^2+y+1)X^2$.

Case 3. $a_1 = y$.

Case 3a. $a_2 = 0, y, y^2, y^2+y$. Then $z = yX(1 + y^{-1}a_2X + \cdots)$ is associate to yX.

Case 3b. $a_2 = 1, y+1, y^2+1, y^2+y+1$. Then

$$z = yX + a_2X^2 + a_3X^3 + \cdots = (yX + (y^2+y+1)X^2)(1 + b_1X + b_2X^2 + \cdots)$$

$$= yX + (yb_1 + (y^2 + y + 1))X^2$$

$$+ (yb_2 + b_1(y^2 + y + 1))X^3 + \cdots .$$

with $b_1 = y^{-1}(a_2 + y^2 + y + 1)$ and $b_n = y^{-1}(a_{n+1} + b_{n-1}(y^2 + y + 1))$ for $n > 1$ show that z is associate to $yX + (y^2 + y + 1)X^2$.

Case 4. $a_1 = 1 + y$.

Case 4a. $a_2 = 0, y+1, y^2+1, y^2+y$. Then $z = (1+y)X(1+(1+y)^{-1}a_2X+\cdots)$ is associate to $(1 + y)X$.

Case 4b. $a_2 = 1, y, y^2, y^2 + y + 1$. Then

$$z = (1 + y)X + a_2X^2 + a_3X^3 + \cdots = ((1 + y)X + (y^2 + y + 1)X^2)(1 + b_1X + b_2X^2 + \cdots)$$

$$= (1 + y)X + ((1 + y)b_1 + (y^2 + y + 1))X^2$$

$$+ ((1 + y)b_2 + b_1(y^2 + y + 1))X^3 + \cdots .$$

with $b_1 = (1+y)^{-1}(a_2+y^2+y+1)$ and $b_n = (1+y)^{-1}(a_{n+1}+b_{n-1}(y^2+y+1))$ for $n > 1$ show that z is associate to $(1 + y)X + (y^2 + y + 1)X^2$.

To prove the last part, we need to show that every atom divides all elements of M^4 and find an element in M^3 not divisible by an atom. For the latter, consider $X^3 \in M^3$. Suppose X^3 is divisible by the atom $f = (y^2 + y + 1)X^2$. Then

$$X^3 = (y^2 + y + 1)X^2(a_1X + a_2X^2 + \cdots)$$

where $a_1 \in V_1$ and $a_i \in \mathbb{F}_8$ for all $i \geq 2$. This implies that $a_1(y^2+y+1) = 1$ and hence $a_1 = (y^2+y+1)^{-1} = y^2 \notin V_1$, which is impossible. This shows that M^3 is not universal. To show that M^4 is universal, consider an arbitrary $z \in M^4$:

$$z = a_4X^4 + a_5X^5 + \cdots .$$

It obvious that $X, yX, (1+y)X, (y^2+y+1)X^2$ all divide z. To see why the other atoms also divide z, consider the atom $(y^2+y+1)X^2 + X^3$. (The rest are similar.) Notice that we can construct the other factor of z:

$$\begin{aligned} z &= a_4X^4 + a_5X^5 + \cdots \\ &= ((y^2+y+1)X^2 + X^3)(a_4(y^2+y+1)^{-1}X^2 + b_3X^3 + \cdots). \end{aligned}$$

By solving $a_4(y^2+y+1)^{-1} + (y^2+y+1)b_3 = a_5$ for b_3, we can recursively find each coefficient b_k. (As demonstrated more formally above earlier in this proof.) Since z is divisible by each atom, M^4 is universal.

□

We now illustrate the details with the group V with this example:

Example 3.3. Recall Example 3.2. Let $V_1 = \{0, 1, y, 1+y\}$. We defined

$$R = \mathbb{F}_2 + V_1X + \mathbb{F}_8[[X]]X^2$$

which was a CK domain with eight atoms:

1) $f = (y^2+y+1)X^2 \in M^2 \backslash M^3$

2) $X \in M \backslash M^2$

3) $yX \in M \backslash M^2$

4) $(1+y)X \in M \backslash M^2$

5) $X + (y^2+y+1)X^2 \in M \backslash M^2$

6) $yX + (y^2+y+1)X^2 \in M \backslash M^2$

7) $(1+y)X + (y^2+y+1)X^2 \in M \backslash M^2$

8) $(y^2+y+1)X^2 + X^3 \in M^2 \backslash M^3$

Notice atoms (1) and (8) are both in $M^2 \backslash M^3$. Notice that by Theorem 3.2, $|V| \geq 2$ since the class residue field must at least have 2 elements. However, by Corollary 3.5, the number of atoms in $M^2 \backslash M^3$ is a multiple of $|V|$. But the most that this number can be is 2. Hence, we have that $|R/M| = |V| = 2$. Since atoms (1) and (8) are the only distinct atoms in $M^2 \backslash M^3$, they must lie in the same coset. That is, they differ by a unit of $U([M:M])$. Actually, based on our construction in Theorem 3.2, we can determine this unit. Let:

$$v := \frac{(y^2+y+1)X^2 + X^3}{(y^2+y+1)X^2} = 1 + (y^2+y+1)^{-1}X = 1 + y^2 X.$$

First notice that $v \in [M:M]$. Indeed, multiplying by any of the atoms in $(1) - (8)$ yields an element in the maximal ideal $M = V_1 X + \mathbb{F}_8[[X]]X^2$. Its inverse is clearly

$$
\begin{aligned}
v^{-1} &= \frac{(y^2+y+1)X^2}{(y^2+y+1)X^2+X^3} \\
&= \frac{(y^2+y+1)X^2+X^3+X^3}{(y^2+y+1)X^2+X^3} \\
&= 1+\frac{X^3}{(y^2+y+1)X^2+X^3} \\
&= 1+y^2\frac{(y^2+y+1)X^2+X^3+X^3}{(y^2+y+1)X^2+X^3}X \\
&= 1+y^2X+\frac{y^2X^4}{(y^2+y+1)X^2+X^3}.
\end{aligned}
$$

Notice here that $\frac{y^2X^4}{(y^2+y+1)X^2+X^3} \in M = V_1X + \mathbb{F}_8[[X]]X^2$ as $y^2X^4 \in M^4$ and M^4 is universal. Notice that $v^{-1} \in [M : M]$ as well and so $v \in U([M : M])$ as claimed.

We can now compute which atoms are in the same cosets. Notice that we already determine which atoms are associate to each other in Example 3.2:

- $vX = (1+y^2X)X = X + y^2X^2 \sim X + (y^2+y+1)X^2$ (Case 2b in Example 3.2)
- $v(yX) = (1+y^2X)yX = yX + y^3X^2 = yX + (y+1)X^2 \sim yX + (y^2+y+1)X^2$ (Case 3b in Example 3.2)
- $v((1+y)X) = (1+y^2X)(1+y)X = (1+y+y^2X+y^3X)X = (1+y)X + (1+y+y^2)X^2$ (Already one of the atoms)

With this we know that the pairs are: $\{(1),(8)\}$, $\{(2),(5)\}$, $\{(3),(6)\}$, and $\{(4),(7)\}$. Our V group is simply $V = \{1, v\} \cong \mathbb{Z}/2$.

3.3 Bound on Generators

We now ask whether a bound on the number of generators of the maximal ideal M of a local CK domain implies there is a bound on the number of atoms. We will show that the answer to this is negative. To show this, we will use the $D+M$ construction as described in Theorem 1.5. In particular, we are interested in applying this proposition to the case where $T = F + XF[[X]] = F[[X]]$ and $R = K + XF[[X]]$ where $K \subset F$ are fields.

Theorem 3.8. *Let $K \subseteq F$ be arbitrary fields, not necessarily finite. Let $R = K + XF[[X]]$. Then R is a one-dimensional quasilocal atomic domain. Moreover, the number of nonassociate atoms has the cardinality of F^*/K^* and the maximal ideal $M = XF[[X]]$ has a minimal basis of cardinality $[F : K]$.*

Proof. Let T and R be as described above. Since T is atomic and K is a field, then by Theorem 1.5, R is atomic. Furthermore, the maximal ideal is clearly $M = XF[[X]]$ since all of the units have a nonzero constant term. Suppose $0 \subset P \subset M$ is a prime ideal. Let $f \in P$ be a nonzero nonunit element in P. Then

$$f = a_n X^n + a_{n+1} X^{n+1} + \cdots = a_n X^n (1 + a_{n+1} a_n^{-1} X + \cdots) \in P.$$

Now $a_n X \in P$ or $X \in P$ since P is prime. If $a_n X \in P$ then $a_n^{-1} X a_n X = X^2 \in P$ and so $X \in P$. For any $a \in F^*$, we then have that $a^2 XX = a^2 X^2 = aXaX \in P$. Since P is prime, then $aX \in P$. Since every atom (up to associates) is of this form, we see then that $P = M$. Hence, R is a one-dimensional quasilocal atomic domain.

Since atoms are of the form aX for $a \in F^*$, we see that two atoms $a_1 X$ and

a_2X are associate if and only if there exists a $k \in K^*$ so that $a_1 = ka_2$. That is, $a_1a_2^{-1} \in K^*$. Hence, the number of atoms is the cardinality of F^*/K^*.

Now let $\{a_i\}_{i\in I}$ be a basis for F over K. It's then clear that $\{a_iX\}_{i\in I}$ forms a generating set for the maximal ideal M in R. Moreover, this is a minimal generating set for if:

$$a_iX = b_1a_{j_1}X + \cdots + b_ma_{j_m}X$$

where $b_s \in K$, then $a_i = b_1a_{j_1} + \cdots + b_ma_{j_m}$ which contradicts the fact that the a_i are a basis. Hence, the number of generators for the maximal ideal M of R is $[F : K]$. □

To answer our original question, we will need a result. (We will investigate completions more in-depth in a later chapter. For now this detail is not crucial to the purpose of this chapter.)

Theorem 3.9 (Theorem 4.5. Anderson and Mott [3])**.** *Let $F_0 \subseteq F$ be finite fields and $s \geq 1$. Suppose that R is an integral domain with $F_0 + X^sF[[X]] \subseteq R \subseteq F[[X]]$. Then R is a complete local CK domain with residue field between F_0 and F.*

Corollary 3.10. *For positive integers n and k, let $K = GF(p^n)$ and $F = GF(p^{nk})$ as in Theorem 3.8. Then $R = K + XF[[X]]$ is a complete local CK domain with maximal ideal having a minimal basis of k elements and*

$$\frac{p^{nk} - 1}{p^n - 1}$$

nonassociate atoms.

Proof. By Theorem 3.8, the number of nonassociate atoms has the cardinality of F^*/K^* and the maximal ideal has a minimal basis of cardinality $[F : K]$. Moreover, by Theorem 3.9, R is a complete local CK domain. $\square$

Hence, it follows from the corollary that CK domains with maximal ideals with a minimal basis of k elements ($k > 1$) that the number of nonassociate atoms can be as large as we want. For example, take $n = 1$ and $k = 2$. Then R in this instance has a minimal basis of 2 elements. But it has

$$\frac{p^2 - 1}{p - 1} = \frac{(p + 1)(p - 1)}{p - 1} = p + 1$$

nonassociate atoms. It follows there is no bound on this number of nonassociate atoms based on the minimal amount of generators ($k > 1$) for the maximal ideal.

CHAPTER 4
CK DOMAINS WITH EXACTLY THREE ATOMS

In this chapter, we examine atomic domains with exactly three nonassociate atoms. Not much is known about the classification of CK domains. The cases of one or two atoms are trivial as we will quickly show. This chapter seeks to investigate the classification of CK domains with three atoms.

4.1 Preliminaries

We will begin by reducing the study of CK domains to the case of a complete local CK domain. To do so, we will provide the necessary background information that we will be needing as outlined in [3] and [7]. A CK domain is a one-dimensional semilocal domain. We can show this following a proof provided in [3]. Let $\{x_1, \ldots, x_n\}$ be the set of the nonassociate atoms of R. It follows that each prime ideal P has a basis which is a subset of this set. Indeed, if we have a product of atoms $y_1 y_2 \cdots y_s \in P$ where each $y_i \in R$, we see that one of the $y_i \in P$ since P is prime. Since a CK domain is atomic, every nonzero nonunit element in P is thus divisible by some atom in P. Since there are only finitely many atoms, it also follows that there are only finitely many prime ideals. Recall that Cohen's Theorem states that if every prime ideal is finitely generated, then the ring is Noetherian [11, Theorem 8]. Since there are only finitely many prime ideals, the dimension must be one [11, Theorem 144]. Hence, a CK domain is a one-dimensional semilocal domain.

We can reduce the study of factorization of a nonzero nonunit element into a product of atoms to the local case. To do so, we will need to prove some theorems from [7] which we will provide similar proofs to theirs here:

Theorem 4.1 (Theorem 1. Cohen and Kaplansky [7])**.** *Let R be a CK domain. Then two maximal ideals of R do not have an atom in common.*

Proof. Let $M_1, \ldots, M_s$ be the maximal ideals. We show the theorem for the maximal ideals M_1 and M_2. Suppose $y \in M_1 \cap M_2$ is an atom. If $M_1 \subseteq M_2 \cup \cdots \cup M_s$, then M_1 is contained in some M_k [11, Theorem 81]. So we can take an element, in fact an atom, that lies in M_1 but not any of the M_k for $k \neq 1$. Let $x_1, \ldots, x_n$ be all such (nonassociate) atoms. Let $a_i \in M_i \backslash M_1$ for $i \neq 1$. Define $a := a_2 \cdots a_s \in M_2 \cap \cdots \cap M_s$. We see that $a \notin M_1$ or else some $a_k \in M_1$. If $a + x_1 \cdots x_n \in M_1$, then $a \in M_1$ which is a contradiction. If $a + x_1 \cdots x_n \in M_k$ for some $k \neq 1$, then $x_1 \cdots x_n \in M_k$ which is again a contradiction. Hence, $u := a + x_1 \cdots x_n$ is a unit. Since $y \in M_1 \cap M_2$, then y and each x_k are nonassociates. Hence, $uy = ay + yx_1 \cdots x_n$ is not divisible by any x_k. So ay is not divisible by any x_k either. It again follows that $c := x_1 \cdots x_n + ya \in M_1$ is not divisible by any x_k. Notice $c \neq 0$ or else $x_1 \cdots x_n \in M_2$. So some $x_k \in M_2$, a contradiction. So since $c \in M_1$, it is divisible by an atom. Since $x_1, \ldots, x_n$ were all the nonassociate atoms in M_1 that aren't in any other maximal ideal, c must be divisible by an atom in M_k where $k \neq 1$. Hence, $x_1 \cdots x_n \in M_k$ which is a contradiction. □

Theorem 4.2 (Theorem 2. Cohen and Kaplansky [7])**.** *Let R be a CK domain and $M_1, \ldots, M_n$ be the maximal ideals of R. Suppose $a_i, b_i \in M_i$ are products of atoms in M_i with $a_1 \cdots a_n = b_1 \cdots b_n$. Then a_i and b_i are associates for all i.*

Proof. Following a similar idea to the previous theorem, $u := a_2 \cdots a_n + b_1$ is not in any maximal ideal and so is a unit. Similarly, $v := b_2 \cdots b_n + a_1$ is a unit. So $ua_1 = vb_1$. $\square$

Theorem 4.3 (Theorem 7. Cohen and Kaplansky [7]). *Let R be a CK domain and M be a maximal ideal in R. Then R_M is a local CK domain with its atoms being the atoms of R which are in M.*

Proof. Let $x_1, \ldots, x_n$ be the atoms in R where $x_1, \ldots, x_m$ are the atoms in M. We may write an element of R_M in the form $u\left(\prod_{i=1}^{n} x_i^{a_i}\right)/c$ where $a_i \in \mathbb{N}$, $c \notin M$, and $u \in U(R)$. Then we have that

$$u\left(\prod_{i=1}^{n} x_i^{a_i}\right)/c = \left(uc^{-1}\prod_{i=m+1}^{n} x_i^{a_i}\right)\prod_{i=1}^{m} x_i^{a_i}.$$

Notice that $uc^{-1}\prod_{i=m+1}^{n} x_i^{a_i} \in R_M$ is a unit. Hence, each nonzero nonunit element of R_M is associate to a product of the atoms from M. If one such atom, say x_1, were not an atom in R_M, then

$$x_1 = v\left(\prod_{i=2}^{m} x_i^{b_i}\right)/d$$

where $v \in U(R)$, $b_i \in \mathbb{N}$, and $d \notin M$. But this means that

$$dx_1 = v\left(\prod_{i=2}^{m} x_i^{b_i}\right)$$

which contradicts Theorem 4.2. Indeed, if $d \in U(R)$, then x_1 is either a product of atoms or associate to another atom, a contradiction. If instead d is in another maximal ideal, we contradict the theorem. $\square$

We also have the following useful theorem:

Theorem 4.4 (Theorem 2. D.D. Anderson [1]). *Let R be a commutative ring. Then the following are equivalent:*

(1) R is an atomic ring with a finite number of atoms.

(2) R is a finite direct sum of finite local rings, SPIRs (special principal ideal rings) and one-dimensional semi-local domains D with the property that for each non-principal maximal ideal M of D, D/M is finite and D_M is analytically irreducible.

(3) R is semi-quasilocal and every ideal of R is a finite union of principal ideals.

(4) R is semi-quasilocal and every prime ideal of R is a finite union of principal ideals.

Note: The definitions used in (1) here for the general case of a commutative ring coincide with our definition when R is an integral domain. A *SPIR* (*special principal ideal ring*) is a principal ideal ring which has a unique prime ideal and this prime ideal is nilpotent [10]. A domain D is *analytically irreducible* if its completion $\hat{D}$ is an integral domain. Completions will be explored more in depth below.

With these theorems in mind, we see that the study of factorization properties of a CK domain can be reduced to the local case. We will now provide the basic ideas for the *completion* $\widehat{R}$ of a ring R. A more detailed treatment of completions may be found in [12] as well as other excellent texts. We will follow the concise treatment as in Eisenbud [8].

Let R be a commutative ring and let

$$R = m_0 \supset m_1 \supset \dots$$

be a filtration of ideals. We define the *completion* $\widehat{R}$ of R *with respect* to the ideals m_i to be the *inverse limit* of the R/m_i. We may write this as:

$$\begin{aligned}\widehat{R} &= \lim_{\leftarrow} R/m_i \\ &= \{(x_1, x_2, \dots) \in \prod_i R/m_i \mid x_j \equiv x_i (\text{mod } m_i) \text{ for each } j > i\}.\end{aligned}$$

This is a ring which has a filtration by the ideals:

$$\widehat{m_i} = \{(x_1, x_2, \dots) \in \widehat{R} \mid x_j = 0 \text{ for each } j \leq i\}.$$

We have that $\widehat{R}/\widehat{m_i} \cong R/m_i$. Indeed, one may notice that $\widehat{m_i}$ is the kernel of the projection homomorphism $\widehat{R} \twoheadrightarrow R/m_i$. We are interested in the case where $m_i = m^i$ for an ideal m of R. This is called the m-adic filtration of R and from now on we will focus on the completion of R with respect to this filtration. In particular, if (R, M) is a local ring, we will take $m = M$ the maximal ideal. With this in mind, whenever we talk about the completion $\widehat{R}$, we will mean with respect to this filtration unless otherwise stated. A ring is *complete* if $\widehat{R} \cong R$. It is well-known that $\widehat{\widehat{R}} \cong \widehat{R}$.

Additionally, the completion of a local ring is again local [12]. Since, $\widehat{R}/\widehat{m_i} \cong R/m_i$ as noted above, the completion of a local ring has the same residue class field.

Example 4.1. Let K be a field, $R = K[X]$, and $M = (X)$. We investigate an element $f \in \widehat{R} = \widehat{R_M}$. So by definition each element can be written as follows:

$$f = (f_1 + (X), f_2 + (X^2), f_3 + (X^3), \ldots)$$

where each $f_i \in R$ and satisfying the necessary conditions. However, since f_i is being reduced mod (X^i), we can instead write this as:

$$f = (a_0 + (X), a_0 + a_1X + (X^2), a_0 + a_1X + a_2X^2 + (X^3), \ldots)$$

where $a_i \in K$. Indeed we must have this format since each term following the i-th term must be congruent to it mod (X^i). Hence, each term below X^i must always be the same for this condition to hold. We notice that we can write this information as a power series:

$$f = a_0 + a_1X + a_2X^2 + \cdots .$$

Indeed, one may notice that we can form a bijection between $\widehat{R}$ and $K[[X]]$. It turns out, the two are isomorphic. So we can represent the completion in this case as a power series.

Example 4.2. An important example is the completion $\widehat{\mathbb{Z}_{(p)}}$ called the *p-adic integers*. So we have that

$$\widehat{\mathbb{Z}_{(p)}} = \{(a_1, a_2, \ldots) \mid a_{m+1} \equiv a_m \bmod p^m\}.$$

For example, one can check that

$$(2 + 3\mathbb{Z}, 5 + 9\mathbb{Z}, 14 + 27\mathbb{Z}, \ldots) \in \widehat{\mathbb{Z}_{(3)}}.$$

Notice that we can actually think of the p-adic integers as a power series similar to how we thought of Example 4.1. We can do this by writing each element as:

$$a_0 + a_1 p + a_2 p^2 + \cdots$$

where $a_i \in \mathbb{Z}$ with $0 \leq a_i < p$. It is often useful to think of elements in the p-adic integers as power series. This example will turn out to be useful later. Some important facts about the p-adic integers will also be needed. It is known that the completion of a regular local ring is again a regular local ring [6]. It is also known that the dimension of a completion of a local ring is the same as the original ring [12]. So $\widehat{\mathbb{Z}_p}$ is a regular local ring of Krull dimension 1.

Notice that we can think of the p-adic integers as containing a copy of $\mathbb{Z}$ since we are in a way breaking a number down into its base p format. This means the characteristic of $\widehat{Z_p}$ is 0. But since $\widehat{Z_p}/\widehat{(p)} \cong \mathbb{Z}/p\mathbb{Z}$, we see that its residue field has characteristic p. This means $\widehat{\mathbb{Z}_p}$ and its residue field have different characteristics. We also see that $p \notin \widehat{(p)}^2$. As we will discuss later, this means that the p-adic integers are unramified.

Our reduction to this complete case comes from the following:

Theorem 4.5 (Anderson and Mott [3])**.** *Let (R, M) be a local CK domain. Then the mapping of ideals $I \to I\widehat{R}$ is a lattice isomorphism that preserves products, residuals, and principal ideals. Furthermore, we have that $a \in R$ is an atom if and only if $a \in \widehat{R}$ is an atom. Elements of $\widehat{R}$ have the form ru for $r \in R$ and $u \in U(\widehat{R})$.*

So we have from this theorem that from both a ideal or factorization point of view, in the local case we have that R and $\widehat{R}$ are basically the same. Since we will be reducing to the local case anyway for reasons stated above, we will therefore assume that R is a complete local CK domain for our classification at the end. If possible, we will present any necessary results in greater generality.

We will be utilizing Cohen's Structure Theorem on complete local rings as described in [6]. We will describe the necessary details outlined there. An important factor in the structure theorem for local rings is the characteristic of the ring R and its residue field $\overline{R} := R/M$. Since $\overline{R}$ is a field, its characteristic is either 0 or a prime p. However, the ring R may have characteristic k a positive integer power of a prime p. We cannot have $k = rs$ for $r > 1$,$s > 1$, and $(r, s) = 1$. This is because then $k = rs = 0 \in R$ which implies that r and s are zero-divisors and so $r, s \in M$. Since $(r, s) = 1$, then there are two positive integers a and b so that $1 = ar + bs \in M$. So we must either have characteristic 0 or a positive power of a prime number p. In the latter case, the residue field must have characteristic p. So we thus have four cases:

1 R and $\overline{R}$ both have characteristic zero (equal-characteristic).

2 R and $\overline{R}$ both have characteristic p (equal-characteristic).

3 R has characteristic zero and $\overline{R}$ has characteristic p (unequal-characteristic).

4 R has characteristic p^k ($k > 1$) and $\overline{R}$ has characteristic p (unequal-characteristic).

Since we are working with complete local domains, then we will not be concerned with the fourth possibility since it only occurs when R has zero-divisors. However, in the unequal-characteristic case, we have two distinctions to make. In this case, we have R has characteristic zero and $\overline{R}$ has characteristic p. It follows that $p \in M$. We will need to distinguish the case where R is *ramified* ($p \in M^2$) and R is *unramified* ($p \notin M^2$).

Of use is the following characterization of complete local CK domains:

Theorem 4.6 (Theorem 4.5. Anderson and Mott [3]). *(1) Let $F_0 \subseteq F$ be finite fields and let $n \geq 1$. Suppose that R is an integral domain with $F_0 + F[[X]]X^n \subseteq R \subseteq F[[X]]$. Then R is a complete local CK domain with residue field between F_0 and F.*

Conversely, suppose that (R, M) is a complete local CK domain with $\overline{R} = R/M$ finite and char $R =$ char $\overline{R}$. Let F_0 (resp., F) be a coefficient field for R (resp., R', the integral closure of R). Then there exists an $n \geq 1$ with $F_0 + F[[X]]X^n \subseteq R \subseteq F[[X]]$.

(2) Let $p > 0$ be prime and $\mathbb{Z}_p$ the p-adic integers and $\mathbb{Q}_p$ the field of rational p-adics, and let L be a finite field extension of $\mathbb{Q}_p$. Let $(\mathbb{Z}'_p, (\pi))$ be the integral closure of $\mathbb{Z}_p$ in L. So $\mathbb{Z}'_p$ is a complete DVR. Suppose that R is an integral domain with $\mathbb{Z}_p + \pi^n\mathbb{Z}'_p \subseteq R \subseteq \mathbb{Z}'_p$ for some n. Then (R, M) is a complete local CK domain with char $R = 0$ and char $\overline{R} = p > 0$.

Conversely, suppose that (R, M) is a complete local CK domain with char

$R = 0$ and $\overline{R}$ finite with char $\overline{R} = p > 0$. Let L be the quotient field of R and $\mathbb{Z}'_p$ the integral closure of $\mathbb{Z}_p$ in L. Then $\mathbb{Z}'_p = R'$ and there exists an $n \geq 1$ so that $\mathbb{Z}_p + \pi^n \mathbb{Z}'_p \subseteq R \subseteq \mathbb{Z}'_p$.

Recall that a *regular local ring* is a local ring whose Krull dimensional is equal to the number of generators in a minimal basis for the maximal ideal M. Cohen in [6] defines a *v-ring* to be a complete, discrete, unramified, valuation ring of characteristic zero with residue field of characteristic a prime p. It was shown that these rings are regular local domains of Krull dimension one. In particular, v-rings are unramified complete regular local rings:

Theorem 4.7 (Cohen [6])**.** *Any two unramified complete regular local rings having the same characteristic, dimension, and residue field are isomorphic.*

Additionally,

Theorem 4.8 (Cohen [6])**.** *Let R be a regular local ring with $M = (x_1, \ldots, x_n)$. Then $R/(x_1, \ldots, x_i)$ is a regular local ring for all $1 \leq i \leq n$.*

We may now list Cohen's Structure Theorem:

Theorem 4.9 (Theorems 9,12. Cohen [6]. Cohen's Structure Theorem)**.**

1 *Let (R, M) be a complete local ring of equal-characteristic with residue field $\overline{R}$. If M has a minimal basis of n elements, then R is a homomorphic image of a formal power series ring $\overline{R}[[x_1, \ldots, x_n]]$ via the natural mapping to a given minimal basis.*

2 Let (R, M) be a complete local ring of characteristic 0 and with residue field $\overline{R}$ of characteristic p. If M has a minimal basis of n elements, then R is a homomorphic image of a formal power series ring $V[[x_1, \ldots, x_n]]$, via the natural homomorphism, over a v-ring V whose residue field is $\overline{R}$. If R is unramified, then it may be taken over $n-1$ indeterminates instead. In fact, we may assume that $p \in M$ is part of the minimal basis.

4.2 Homomorphic Image

Let (R, M) be a local CK domain with exactly n (nonassociate) atoms. We have already seen from Theorem 2.3 that the case of $n = 1$ is exactly when R is a DVR. Additionally, we cannot have the case of $n = 2$. We thus focus on the case of $n = 3$. Notice that $M = (x_1) \cup (x_2) \cup (x_3) = (x_1, x_2) \cup (x_3)$. By Lemma 2.1, we must have that $M = (x_1, x_2)$ or else $M = (x_3)$. Clearly $M = (x_1, x_2) = (x_1, x_1 + x_2)$. We cannot have that $V(R) = 1$ or else the dimension of R would also be 1 and hence R would be a DVR, which is a contradiction. Hence, $x_1 + x_2 \notin M^2$ since $x_1 + x_2$ is in a minimal basis for M. Notice that $x_1 + x_2 \in M = (x_1) \cup (x_2) \cup (x_3)$ but $x_1 + x_2 \notin M^2$. Hence, $x_1 + x_2$ is associate to x_1, x_2 or x_3. We must have that $x_1 + x_2$ is associate to x_3. So for the case of $n = 3$, we can write the three atoms as x_1, x_2 and $x_1 + x_2$.

Lemma 4.10. *Let (R, M) be a local domain with precisely 3 nonassociate atoms. Then $\overline{R} = R/M \cong \mathbb{Z}/(2)$.*

Proof. By Theorem 2.3, the minimum number of atoms is

$$\frac{|\overline{R}|^k - 1}{|\overline{R}| - 1}$$

where $k = \dim_{\overline{R}} M/M^2$. Since the case of $k = 1$ yields a DVR, the minimal case for k is $k = 2$. And since the number of atoms is precisely 3, then

$$3 \geq \frac{|\overline{R}|^2 - 1}{|\overline{R}| - 1} = \frac{(|\overline{R}| - 1)(|\overline{R}| + 1)}{|\overline{R}| - 1} = |\overline{R}| + 1.$$

That is, $|\overline{R}| \leq 2$. Hence, $|\overline{R}| = 2$ since we cannot have $R = M$. □

We additionally offer another proof since a similar technique will be used later. Let x_1, x_2, x_3 be the three nonassociate atoms. Assume by contradiction that $|R/M| > 2$. If for all units $u, v \in R \backslash M$, we have that $u + v \in M$, then $u + M = u + (v - u) + M = v + M$. This would imply that $R/M \cong \mathbb{Z}/(2)$. So there are units $u, v \in U(R)$ so that $u, v, u+v \notin M$. Notice that $ux_1 + x_2 \in M = (x_1) \cup (x_2) \cup (x_3)$. If $ux_1 + x_2 \in (x_1)$, then $x_2 = (r - u)x_1$ for some $r \in R$. Hence, $x_2 \in (x_1)$, a contradiction. Similarly, we cannot have that $ux_1 + x_2 \in (x_2)$. Hence, $ux_1 + x_2 \in (x_3)$. We get then that $ux_1 + x_2 \in (x_3)$ and $vx_1 - x_2 \in (x_3)$. Therefore,

$$(u + v)x_1 = (ux_1 + x_2) + (vx_1 - x_2) \in (x_3)$$

and so $x_1 \in (x_3)$ since $u + v$ is a unit. This contradiction proves the desired result.

The techniques used here will prove important in the following theorem. We state the main idea for emphasis. Let (R, M) be as in Lemma 4.10. We have then that

$$x_1 + ux_2 \in (x_3)$$

for $u \in U(R)$ a unit. (In fact, elements of this form are associate to $x_3 = x_1 + x_2$.)

Similarly, notice that

$$x_1 + rx_2 \in (x_1)$$

for $r \in M$ a nonunit. Indeed, if $x_1 + rx_2 \in (x_2)$, then $x_1 = (s - r)x_2$ for some $s \in R$, a contradiction. If instead $x_1 + rx_2 \in (x_3) = (x_1 + x_2)$, then

$$x_1 + rx_2 = sx_1 + sx_2$$

for some $s \in R$. So $(1 - s)x_1 = (s - r)x_2$. If $s \in M$, then $(1 - s)$ is a unit and hence $x_1 \in (x_2)$. If $s \notin M$, then $s - r$ is a unit and so $x_2 \in (x_1)$. Hence, $x_1 + rx_2 \in (x_1)$. (In fact, elements of this form are associate to x_1.) The other possible cases are similar. An important detail used in the following theorem is that if $u \in U(R)$ is a unit, then $u = 1 + m$ where $m \in M$ (i.e. the difference of two units is in the maximal ideal).

Theorem 4.11. *Let (R, M) be a two-dimensional regular local domain with residue class field $\mathbb{Z}/(2)$. Let $M = (x_1, x_2)$. Then $R/(f)$, f a principal prime of R, has precisely three nonassociate atoms if and only if $f \equiv x_1^2 + x_1x_2 + x_2^2$ mod M^3.*

Proof. Let $\overline{R} := R/(f)$. We may take $\overline{x_1}$ and $\overline{x_2}$ to be atoms. Indeed, $\overline{R}$ has maximal ideal $\overline{M} = (\overline{x_1}, \overline{x_2})$. We have that $\overline{x_1}, \overline{x_2} \in \overline{M}\backslash\overline{M}^2$. Indeed, if $\overline{x_1} \in \overline{M}^2$, then $\overline{M}/(\overline{x_1})$ has a minimal basis consisting of two terms, a contradiction [11, Exercise 3.3.2]. If $\overline{x_1} = \overline{r_1}\,\overline{r_2}$ where $\overline{r_1}, \overline{r_2} \in \overline{M}$, then $\overline{x_1} \in \overline{M}^2$, a contradiction.

$\Longrightarrow$). Suppose that $\overline{R}$ has precisely three nonassociate atoms. Since $\overline{R}$ has precisely three nonassociate atoms, by the discussion prior to this theorem, we have that $\overline{x_1} + \overline{x_2}$ is the third atom. We show that $f \in M^2\backslash M^3$. Suppose $f \notin M^2$. Then

$\overline{R}$ is regular [11, Exercise 3.3.3]. Since the dimension of R is two and since (f) is a nonzero prime, we see that $\dim(\overline{R}) = V(\overline{R}) = 1$ where $V(\overline{R})$ is the V-dimension of $\overline{R}$. That is, the number of elements in a minimal basis for $\overline{M}$. Hence, $\overline{M}$ is principal. This implies that $\overline{R}$ is a DVR, which is a contradiction [11, Exercise 3.3.7]. So $f \in M^2$.

So suppose that $f \in M^3$. Then $M = (x_1, f) \cup (x_2, f) \cup (x_1 + x_2, f) \subseteq (x_1) + M^3 \cup (x_2) + M^3 \cup (x_1 + x_2) + M^3$. Consider $x_1 + x_2^2 \in M$. There are three cases:

Case 1. $x_1 + x_2^2 \in (x_1) + M^3$.

Then $x_1 + x_2^2 = rx_1 + \beta$ where $\beta \in M^3$ and $r \in R$. That is, $x_2^2 + r'x_1 - \beta = 0$ where $r' = 1 - r$. Since $\beta \in M^3$, then after arranging terms, we have something of the form

$$(r_1 + r')x_1 + (x_2 + r_2)x_2 = 0$$

where $r_1 \in M^2$ and $r_2 \in M^2$. Clearly x_1, x_2 form a system of parameters. Hence, they are analytically independent by Theorem 1.4. Now let $\psi(x, y) = (r_1 + r')x + (x_1 + r_2)y$. Then ψ is a form of degree one and $\psi(x_1, x_2) = 0$. If not all of the coefficients are in M, then $0 \notin M^2$ [13, p.67]. This is impossible and so $r_1 + r' \in M$. Hence, $r' \in M$. So we may write $x_2^2 + r'x_1 - \beta = 0$ in the form

$$(r_1 + r_1')x_1^2 + (r_2 + r_2')x_1x_2 + (1 + r_3)x_2^2 = 0$$

where $r_1, r_2, r_3 \in M$ and $r_1', r_2' \in R$. Let $\phi(x,y) = (r_1 + r_1')x^2 + (r_2 + r_2')xy + (1+r_3)y^2$. Then $\phi(x_1, x_2) = 0$ and ϕ is a form of degree 2. If not all of the coefficients are in M, then $0 \notin M^3$. Hence, $1 + r_3 \in M$. So $1 \in M$ which is impossible.

Case 2. $x_1 + x_2^2 \in (x_2) + M^3$.

As before, we have something of the form

$$x_1 + r'x_2 - \beta = 0$$

where $\beta \in M^3$ and $r' \in R$. So after arrangement, we may write

$$(r_1 + 1)x_1 + (r_2 + r')x_2 = 0$$

where $r_1, r_2 \in M^2$. By a similar argument as in the previous case, we have that $r_1 + 1 \in M$. Hence, $1 \in M$ which is impossible.

Case 3. $x_1 + x_2^2 \in (x_1 + x_2) + M^3$.

Then $x_1 + x_2^2 = rx_1 + rx_2 + \beta$ where $\beta \in M^3$ and $r \in R$. Rearranging, we may write this as

$$(1 - r - r_1)x_1 + (x_2 - r - r_2)x_2 = 0$$

where $r_1, r_2 \in M^2$. By a similar argument as in the previous cases, we see that $x_2 - r - r_2 \in M$ and so $r \in M$. Since $1 - r - r_1 \in M$ as well, we have that $1 \in M$

which is impossible. Hence, none of these three cases are possible. Therefore, $f \notin M^3$.

Hence, $f \in M^2 \backslash M^3$. We may write $f = a_1x_1^2 + a_2x_1x_2 + a_3x_2^2 + \beta$ where $\beta \in M^3$. If a coefficient is in M, say a_1, then $a_1x_1^2 \in M^3$. So we may assume without loss of generality that $a_1 = 0$. If a coefficient is a unit, say a_1, then $a_1 = 1 + m$ for some $m \in M$. So without loss of generality, we may assume that $a_1 = 1$. Hence,

$$f = a_1x_1^2 + a_2x_1x_2 + a_3x_2^2 + \beta$$

where $\beta \in M^3$ and $a_i \in \{0, 1\}$. Since $f \notin M^3$, not all of the a_i are zero. We have several cases:

Case 1. $a_1 = 1$ and $a_2 = a_3 = 0$.

We have here that $f = x_1^2 + \beta$. Hence, $\overline{x_1}^2 \in \overline{M}^3$. So we may write

$$\overline{x_1}^2 = \overline{b_1}\overline{x_1}^3 + \overline{b_2}\overline{x_1}^2\overline{x_2} + \overline{b_3}\overline{x_1}\ \overline{x_2}^2 + \overline{b_4}\overline{x_2}^3 \tag{4.1}$$

where $\overline{b_i} \in \overline{R}$ for $i \in \{1, 2, 3, 4\}$. Hence, $\overline{b_4}\overline{x_2}^3 = \overline{r}\ \overline{x_1}$ for some $\overline{r} \in \overline{M}$. Notice that $\overline{r} \in \overline{M}$ or else $\overline{x_1} \in (\overline{x_2})$. Since $\overline{R}$ is a domain, we obtain that

$$\overline{x_1} = \overline{b_1}\overline{x_1}^2 + \overline{b_2}\overline{x_1}\ \overline{x_2} + \overline{b_3}\overline{x_2}^2 + \overline{r}.$$

Case 1a. $\overline{r} \in (\overline{x_1})$. Then $\overline{r} = \overline{r}'\overline{x_1}$ for some $\overline{r}' \in \overline{R}$. So $\overline{b_4}\overline{x_2}^3 = \overline{r}'\overline{x_1}^2$. Hence, $\overline{x_1} | \overline{b_3}\overline{x_2}^2$ and so $\overline{b_3}\overline{x_2}^2 = \overline{s}\ \overline{x_1}$ for some $\overline{s} \in \overline{R}$. Canceling another $\overline{x_1}$ we obtain that

$$\overline{1} = \overline{b_1}\,\overline{x_1} + \overline{b_2}\,\overline{x_2} + \overline{s} + \overline{r}'.$$

If $\overline{s} \notin \overline{M}$ (a unit), then $\overline{x_1} \in (\overline{x_2})$ which is a contradiction. Hence, $\overline{r}' \notin \overline{M}$ or else $\overline{1} \in \overline{M}$. So $\overline{r}'\overline{x_1}^2 = \overline{b_4}\,\overline{x_2}^3$. For simplicity, we absorb the $\overline{r}'$ into $\overline{b_4}$. So $\overline{x_1}^2 = \overline{b_4}\,\overline{x_2}^3$. Notice that

$$\overline{x_1}(\overline{x_1} + \overline{x_2}) = \overline{x_1}^2 + \overline{x_1}\ \overline{x_2} = \overline{b_4}\,\overline{x_2}^3 + \overline{x_1}\ \overline{x_2} = \overline{x_2}(\overline{x_1} + \overline{b_4}\,\overline{x_2}^2) = \overline{u}\ \overline{x_1}\ \overline{x_2}$$

where $\overline{u} \notin \overline{M}$ is a unit. (See the discussion after the previous lemma.) Canceling we obtain that $(\overline{x_1} + \overline{x_2}) = \overline{u}\ \overline{x_2}$, a contradiction.

Case 1b. $\overline{r} \in (\overline{x_2})$. Then $\overline{r} = \overline{r}'\overline{x_2}$. Hence, $\overline{b_4}\,\overline{x_2}^3 = \overline{r}\ \overline{x_1} = \overline{r}'\overline{x_1}\ \overline{x_2}$. Canceling we obtain that $\overline{r}'\overline{x_1} = \overline{b_4}\,\overline{x_2}^2$. Hence, $\overline{r}' \in \overline{M}$ or else $\overline{x_1} \in (\overline{x_2})$. So equation 4.1 becomes

$$\begin{aligned} \overline{x_1} &= \overline{b_1}\,\overline{x_1}^2 + \overline{b_2}\,\overline{x_1}\ \overline{x_2} + \overline{b_3}\,\overline{x_2}^2 + \overline{r} \\ &= \overline{b_1}\,\overline{x_1}^2 + \overline{b_2}\,\overline{x_1}\ \overline{x_2} + (\overline{b_3}\,\overline{x_2} + \overline{r}')\overline{x_2}. \end{aligned}$$

Hence, $\overline{b_3}\,\overline{x_2}^2 + \overline{r} = \overline{s}\ \overline{x_1}$ where $\overline{s} \in \overline{M}$. Cancelling we obtain that

$$\overline{1} = \overline{b_1}\,\overline{x_1} + \overline{b_2}\,\overline{x_2} + \overline{s} \in \overline{M}.$$

This is a contradiction.

Case 1c. $\overline{r} \in (\overline{x_1} + \overline{x_2})$. Then $\overline{r} = \overline{r}'(\overline{x_1} + \overline{x_2})$. So equation 4.1 becomes

$$\begin{aligned}\overline{x_1} &= \overline{b_1}\overline{x_1}^2 + \overline{b_2}\overline{x_1}\ \overline{x_2} + \overline{b_3}\overline{x_2}^2 + \overline{r}'\overline{x_1} + \overline{r}'\overline{x_2} \\ &= (\overline{b_1}\overline{x_1} + \overline{b_2}\overline{x_2} + \overline{r}')\overline{x_1} + (\overline{b_3}\overline{x_2} + \overline{r}')\overline{x_2}\end{aligned}$$

Hence, $(\overline{b_3}\overline{x_2}+\overline{r}')\overline{x_2} = \overline{s}\ \overline{x_1}$ for some $\overline{s} \in \overline{R}$. Notice that $\overline{s} \in \overline{M}$ or else $\overline{x_1} \in (\overline{x_2})$. If $\overline{r}' \notin \overline{M}$ is a unit, then $\overline{x_2} \in (\overline{x_1})$, a contradiction. Hence, $\overline{m} = \overline{b_1}\overline{x_1} + \overline{b_2}\overline{x_2} + \overline{r}' \in \overline{M}$ and $\overline{s} \in \overline{M}$. So we obtain that

$$\overline{1} = \overline{m} + \overline{s} \in \overline{M}.$$

Again, we have a contradiction. Hence, Case 1 results in a contradiction.

Case 2. $a_1 = 1, a_2 = 1$ and $a_3 = 0$.

We have here that $f = x_1^2 + x_1x_2 + \beta$. Hence $\overline{x_1}^2 + \overline{x_1}\ \overline{x_2} \in \overline{M}^3$. So we may write

$$\overline{x_1}^2 + \overline{x_1}\ \overline{x_2} = \overline{b_1}\overline{x_1}^3 + \overline{b_2}\overline{x_1}^2\overline{x_2} + \overline{b_3}\overline{x_1}\ \overline{x_2}^2 + \overline{b_4}\overline{x_2}^3 \tag{4.2}$$

where $\overline{b_i} \in \overline{R}$ for $i \in \{1, 2, 3, 4\}$. Hence, $\overline{b_4}\overline{x_2}^3 = \overline{r}\ \overline{x_1}$ for some $\overline{r} \in \overline{M}$. Notice that $\overline{r} \in \overline{M}$ or else $\overline{x_1} \in (\overline{x_2})$. Rearranging equation 4.2 and canceling an $\overline{x_1}$, we obtain that

$$\begin{aligned}\overline{0} &= \overline{b_1}\overline{x_1}^2 + \overline{b_2}\overline{x_1}\,\overline{x_2} + \overline{b_3}\overline{x_2}^2 - \overline{x_1} - \overline{x_2} + \overline{r} \\ &= \overline{b_1}\overline{x_1}^2 + (\overline{b_2}\overline{x_2} - \overline{1})\overline{x_1} + (\overline{b_3}\overline{x_2} - \overline{1})\overline{x_2} + \overline{r}.\end{aligned} \tag{4.3}$$

Notice that $\overline{u} := \overline{b_2}\overline{x_2} - \overline{1}$ and $\overline{v} := \overline{b_3}\overline{x_1} - \overline{1}$ are units.

Case 2a. $\overline{r} \in (\overline{x_1})$. Then $\overline{v}\,\overline{x_2} \in (\overline{x_1})$, a contradiction.

Case 2b. $\overline{r} \in (\overline{x_2})$. So $\overline{r} = \overline{r}'\overline{x_2}$. Suppose $\overline{r}' \in \overline{M}$. Then $\overline{v} + \overline{r}'$ is a unit. Hence, $\overline{x_2} \in (\overline{x_1})$ by the equation 4.3. This is a contradiction. Hence, $\overline{r}' \notin \overline{M}$. So we obtain that

$$\overline{b_4}\overline{x_2}^3 = \overline{r}'\overline{x_2}\,\overline{x_1}.$$

Canceling an $\overline{x_2}$ we obtain that $\overline{x_1} \in (\overline{x_2})$, another contradiction.

Case 2c. $\overline{r} \in (\overline{x_1} + \overline{x_2})$. So $\overline{r} = \overline{r}'(\overline{x_1} + \overline{x_2})$. Equation 4.3 becomes

$$\begin{aligned}\overline{0} &= \overline{b_1}\overline{x_1}^2 + \overline{u}\,\overline{x_1} + \overline{v}\,\overline{x_2} + \overline{r}'\overline{x_1} + \overline{r}'\overline{x_2} \\ &= \overline{b_1}\overline{x_1}^2 + (\overline{u} + \overline{r}')\overline{x_1} + (\overline{v} + \overline{r}')\overline{x_2}.\end{aligned}$$

If $\overline{r}' \in \overline{M}$, then $\overline{x_2} \in (\overline{x_1})$, a contradiction. Hence, $\overline{r}'$ is a unit. So we obtain that

$$\overline{b_4}\overline{x_2}^3 = \overline{w}\,\overline{x_1}(\overline{x_1} + \overline{x_2})$$

where $\overline{w}$ is a unit. For simplicity, we absorb the unit into $\overline{b_4}$. So we obtain that $\overline{x_1}^2 + \overline{x_1}\,\overline{x_2} = \overline{b_4}\overline{x_2}^3$.

Notice now that

$$\begin{aligned}
\overline{x_1}^2 &= \overline{b_4}\overline{x_2}^3 - \overline{x_1}\,\overline{x_2} \\
&= \overline{x_2}(\overline{b_4}\overline{x_2}^2 - \overline{x_1}) \\
&= \overline{u}'\overline{x_1}\,\overline{x_2}
\end{aligned}$$

for some unit $\overline{u}' \in \overline{R}$. Hence, $\overline{x_1} = \overline{u}'\overline{x_2}$, a contradiction. Hence, Case 2 results in a contradiction.

Case 3. $a_1 = 1, a_3 = 1$ and $a_2 = 0$.

Notice that $M = (x_1, x_2) = (x_1, x_1 + x_2)$. Hence,

$$\begin{aligned}
f &= x_1^2 + x_2^2 + \beta \\
&= x_1^2 + x_2^2 + 2x_1x_2 - 2x_1x_2 + \beta \\
&= (x_1 + x_2)^2 + \alpha.
\end{aligned}$$

Here $2 = 1 + 1 \in M$ and hence $\alpha := -2x_1x_2 + \beta \in M^3$. So this case is similar to Case 1.

Case 4. $a_2 = 1$ and $a_1 = a_3 = 0$.

Notice that $M = (x_1, x_2) = (x_1, x_1 + x_2)$. Hence,

$$
\begin{aligned}
f &= x_1^2 + x_1x_2 + \beta \\
&= x_1(x_1 + x_2) + \beta.
\end{aligned}
$$

Hence, this case is similar to Case 2.

Case 5. $a_2 = 1, a_3 = 1$ and $a_1 = 0$.

This case is symmetric to Case 2 and hence will be omitted.

Case 6. $a_3 = 1$ and $a_1 = a_2 = 0$.

This case is symmetric to Case 1 and hence will be omitted.

The only other possible case left is that $f = x_1^2 + x_1x_2 + x_2^2 + \beta$ for some $\beta \in M^3$. This is the desired result.

$\Longleftarrow$). Suppose that $f \equiv x_1^2 + x_1x_2 + x_2^2 \mod M^3$. By the remark at the beginning, $\overline{x_1}$ and $\overline{x_2}$ are atoms. Notice that $\overline{x_1} + \overline{x_2}$ is also an atom. Indeed, if not, then $\overline{x_1} + \overline{x_2} \in \overline{M}^2$. Let $\widetilde{R} = \overline{R}/(\overline{x_1} + \overline{x_2})$ and similarly for $\widetilde{M}$. Then we have that

$V(\widetilde{R}) = V(\overline{R}) = V(R)$ which is impossible [11, Exercise 3.3.2]. So $\widetilde{M}$ has a minimal generating set of two elements. However, notice that $\overline{x_1} = (\overline{x_1} + \overline{x_2}) - \overline{x_2}$. Hence in $\widetilde{R}$, the projections of $\overline{x_1}$ and $\overline{x_2}$ are associates. This means that $\widetilde{M}$ can be generated by one element, a contradiction. Therefore, $\overline{x_1} + \overline{x_2}$ is an atom.

Notice that $\overline{x_1}$ and $\overline{x_2}$ are nonassociate. Indeed, if they were, then $\overline{R}$ would be a DVR. However, since $f \equiv x_1^2 + x_1x_2 + x_2^2 \bmod M^3$, then $f \in M^2$ and so $V(\overline{R}) = V(R)$ and hence this is impossible. Notice that $\overline{x_1} + \overline{x_2}$ is nonassociate to both $\overline{x_1}$ and $\overline{x_2}$. Indeed, if it were associate to say $\overline{x_1}$, then $\overline{x_1} = \overline{u}(\overline{x_1} + \overline{x_2})$ where $\overline{u}$ is a unit. Hence, $(1-\overline{u})\overline{x_1} = \overline{u}\,\overline{x_2} \in (\overline{x_1})$ which implies that $\overline{x_1}$ and $\overline{x_2}$ are associate. This contradiction proves the claim. This shows that $\overline{x_1}$, $\overline{x_2}$ and $\overline{x_1} + \overline{x_2}$ are three nonassociate atoms in $\overline{R}$.

We wish to show that $\overline{x_1}, \overline{x_2}$, and $\overline{x_1} + \overline{x_2}$ are the only (nonassociate) atoms of $\overline{R}$. To show this, we show that $\overline{M} = (\overline{x_1}) \cup (\overline{x_2}) \cup (\overline{x_1} + \overline{x_2})$. It is clear that $\overline{M} \supseteq (\overline{x_1}) \cup (\overline{x_2}) \cup (\overline{x_1} + \overline{x_2})$. So we need to prove that $\overline{M} \subseteq (\overline{x_1}) \cup (\overline{x_2}) \cup (\overline{x_1} + \overline{x_2})$. To show this, notice that any element in $\overline{M} = (\overline{x_1}, \overline{x_2})$ is of the form $\overline{r}\,\overline{x_1} + \overline{s}\,\overline{x_2}$ where $\overline{r}, \overline{s} \in \overline{R}$. Hence we have four cases:

Case 1. $\overline{r} \notin \overline{M}$ and $\overline{s} \in \overline{M}$.

Since $\overline{r}$ is a unit, we may assume without loss of generality that $\overline{r} = \overline{1}$. Since $\overline{s} \in \overline{M}$, then $\overline{s} = \overline{a}\,\overline{x_1} + \overline{b}\,\overline{x_2}$ and so

$$\overline{x_1} + \overline{s}\,\overline{x_2} = \overline{x_1} + \overline{a x_1}\,\overline{x_2} + \overline{b}\overline{x_2}^2.$$

If $\overline{b} = \overline{0}$, then we are done since our element is in $(\overline{x_1})$. We now show that $\overline{x_1} \mid \overline{x_2}^2$. Since $f \equiv x_1^2 + x_1 x_2 + x_2^2 \bmod M^3$, we have that

$$\overline{0} = \overline{x_1}^2 + \overline{x_1}\,\overline{x_2} - \overline{x_2}^2 + \overline{\beta}$$

where $\beta \in M^3$. (We note that $\overline{2}\overline{x_2}^2 \in \overline{M}^3$.) So

$$\overline{x_2}^2 = \overline{x_1}^2 + \overline{x_1 x_2} + \overline{\beta}. \tag{4.4}$$

Since $\beta \in M^3$, we write

$$\overline{\beta} = \overline{a_1}\,\overline{x_1}^3 + \overline{a_2}\,\overline{x_1}^2\overline{x_2} + \overline{a_3}\,\overline{x_1}\,\overline{x_2}^2 + \overline{a_4}\,\overline{x_2}^3.$$

So we rewrite equation 4.4 to obtain that

$$\overline{x_2}^2 = (\overline{1} + \overline{a_1}\,\overline{x_1} + \overline{a_2}\,\overline{x_2})\overline{x_1}^2 + (\overline{1} + \overline{a_3}\,\overline{x_2})\overline{x_1}\,\overline{x_2} + \overline{a_4}\,\overline{x_2}\,\overline{x_2}^2.$$

The coefficients on the $\overline{x_1}^2$ and $\overline{x_1}\,\overline{x_2}$ are units. We denote them by $\overline{u}$ and $\overline{v}$ respectively. So we obtain that

$$\begin{aligned}(\overline{1} - \overline{a_4}\,\overline{x_2})\overline{x_2}^2 &= \overline{u}\,\overline{x_1}^2 + \overline{v}\,\overline{x_1}\,\overline{x_2} \\ &= \overline{x_1}(\overline{u}\,\overline{x_1} + \overline{v}\,\overline{x_2}).\end{aligned}$$

Since $\overline{1} - \overline{a_4}\,\overline{x_2}$ is a unit, this means that $\overline{x_1} \mid \overline{x_2}^2$. Hence, we obtain that

$$\overline{x_1} + \overline{s}\,\overline{x_2} \in (\overline{x_1}).$$

Case 2. $\overline{r} \in \overline{M}$ and $\overline{s} \notin \overline{M}$.

This case is similar to Case 1. So in this case we obtain that $\overline{r}\,\overline{x_1} + \overline{s}\,\overline{x_2} \in (\overline{x_2})$.

Case 3. $\overline{r} \notin \overline{M}$ and $\overline{s} \notin \overline{M}$.

Since $\overline{r}$ and $\overline{s}$ are units, we may assume without loss of generality that $\overline{r} = \overline{1}$. Since $\overline{s} \notin \overline{M}$, notice that

$$\overline{x_1} + \overline{sx_2} = \overline{x_1} + \overline{x_2} + (\overline{s} - \overline{1})\overline{x_2}.$$

Here $\overline{s} - \overline{1} \in \overline{M}$. Since $\overline{M} = (\overline{x_1}, \overline{x_2}) = (\overline{x_1} + \overline{x_2}, \overline{x_2})$, notice how this case becomes similar to Case 1. So in this case, $\overline{x_1} + \overline{s}\,\overline{x_2} \in (\overline{x_1} + \overline{x_2})$.

Case 4. $\overline{r} \in \overline{M}$ and $\overline{s} \in \overline{M}$.

We have that

$$\overline{r}\,\overline{x_1} + \overline{s}\,\overline{x_2} = \overline{x_1} + \overline{x_2} + (\overline{r} - \overline{1})\overline{x_1} + (\overline{s} - \overline{1})\overline{x_2}.$$

Here $\overline{r} - \overline{1}, \overline{s} - \overline{1} \notin \overline{M}$. So by Case 3, we have that $\overline{r}\,\overline{x_1} + \overline{s}\,\overline{x_2} \in (\overline{x_1} + \overline{x_2})$. $\square$

4.3 Classification

We begin by classifying CK domains with three atoms for complete local domains. The following theorem is an application of results from Cohen's structure theorems on complete local rings [6].

Theorem 4.12. *Let (R, M) be a complete local domain. Let $\overline{R} = R/M$. Then R has precisely three atoms if and only if R is isomorphic to one of the following three forms of rings:*

1) $\mathbb{Z}/(2)[[X,Y]]/(f)$ *where* $f \equiv X^2 + XY + Y^2$ *mod* $(X,Y)^3$,

2) $\widehat{\mathbb{Z}_{(2)}}[[X]]/(f)$ *where* $f \equiv 4 + 2X + X^2$ *mod* $(2, X)^3$,

3) $\widehat{\mathbb{Z}_{(2)}}[[X,Y]]/(f,g)$ *where* $f \equiv 2$ *mod* $(2,X,Y)^2$ *and* $g \equiv X^2 + XY + Y^2$ *mod* $(X,Y)^3 + (f)$.

Proof. Suppose R has precisely three atoms. Then $\overline{R} \cong \mathbb{Z}/(2)$ by Lemma 4.10. Furthermore, $M = (x_1, x_2)$ where x_1, x_2 are two of the atoms. We have two cases: either char $R =$ char $\overline{R}$ or char $R = 0$ and char $\overline{R} = 2$. However in the latter case, we have two subcases: whether R is ramified or unramified. Note that char R cannot be 2^k for $k > 1$ since R is an integral domain.

Case 1. If the characteristics are equal, then by the structure theorem R is a homomorphic image of $D := \mathbb{Z}/(2)[[X,Y]]$ where $X \mapsto x_1$ and $Y \mapsto x_2$. We note that $\mathbb{Z}/(2)[[X,Y]]$ is a two-dimensional regular local domain. Indeed, if a ring is a regular local domain, then its power series ring is also a regular local domain [11, Exercise 3.3.5]. Recall that Krull's Height Theorem states that for a

Noetherian ring A and ideal $I = (a_1, \dots, a_n)$, if P is prime ideal minimal over I, then $\text{rank}(P) \leq n$ [12, Theorem 13.5]. Hence the rank of $M = (x_1, x_2)$ is at most 2. Since the rank is either one or two, assume that it is one. Then since D is a regular local domain, then its maximal ideal is principal, which is impossible.

Now since $R \cong D/\text{Ker}$, the kernel must be a prime ideal. Because D is two-dimensional, this ideal must have rank 1. Since D is a UFD ($\mathbb{Z}/(2)$ is a field) and since the kernel is a prime of rank 1, the kernel must be a principal ideal (f). Indeed, an equivalent characterization of UFDs is that every nonzero prime ideal contains a principal prime [11, Theorem 1.5]. Since the kernel is prime, it contains a principal prime. This principal prime generates another prime ideal contained in the kernel. If the kernel was not principal, then it would not have rank 1. By Theorem 4.11, we must have that $f \equiv X^2 + XY + Y^2 \bmod (X, Y)^3$.

Case 2. Suppose char $R = 0$ and char $\overline{R} = 2$ and that R is unramified ($2 \notin M^2$). We have then that R is a homomorphic image of $D := \widehat{\mathbb{Z}_{(2)}}[[X]]$ where $2 \mapsto x_1$ and $X \mapsto x_2$. Indeed, by Cohen's Structure Theorem, D is a power series ring over a *v-ring* V whose residue field is $\overline{R}$. Recall that a v-ring is an unramified complete regular local ring with characteristic zero, Krull dimension one, and whose residue field is of characteristic a prime p. Here the residue field is R/M and so the characteristic of the residue field is 2.

Recall from Example 4.2 that the 2-adic integers are an unramified complete regular local ring. The 2-adic integers have the same characteristic, dimension, and residue field as the v-ring V which can easily be verified using the details in Example 4.2. By Theorem 4.7, the two are isomorphic.

Now, $D := \widehat{\mathbb{Z}_{(2)}}[[X]]$ is a regular local domain and hence is a UFD [11, Exercise 3.3.5], [12, Theorem 20.3]. Furthermore, D has dimension two by the same reasoning as in the previous case. Now since $R \cong D/\mathrm{Ker}$ via the map $2 \mapsto x_1$ and $X \mapsto x_2$, the kernel must be a prime ideal. As in the previous case, the kernel must be a principal ideal (f). By Theorem 2.3, we have that $f \equiv 4 + 2X + X^2 \bmod (2, X)^3$.

Case 3. Suppose char $R = 0$ and char $\overline{R} = 2$ and that R is ramified ($2 \in M^2$). We have then that R is a homomorphic image of $D := \widehat{\mathbb{Z}_{(2)}}[[X, Y]]$ where $X \mapsto x_1$ and $Y \mapsto x_2$ by Cohen's Structure Theorem. As before, D is a regular local domain. Notice that the maximal ideal of D is $(2, X, Y)$. Since D is regular, then it is a 3-dimensional regular local domain and the kernel of the homomorphism must be a prime ideal of rank 2. Since $2 \mapsto 2 \in M^2$, then $2 = \overline{a}x_1^2 + \overline{b}x_1x_2 + \overline{c}x_2^2$ in R where $\overline{a}, \overline{b}, \overline{c} \in R$. That is,

$$2 = aX^2 + bXY + cY^2 + f$$

for some $a, b, c \in D$ and $f \in \mathrm{Ker}$. That is, $f \equiv 2 \bmod (2, X, Y)^2$. Hence, $(2, X, Y) = (f, X, Y)$ and $f \in \mathrm{Ker}$. By Theorem 4.8, we have that $D/(f)$ is a regular local domain of dimension 2. Hence, it is a UFD. Moreover, since $f \in \mathrm{Ker}$, we have that R is a homomorphic image of $D/(f)$. By a similar argument to Case 2, we see that $R \cong \overline{D}/(g)$ where $\overline{D} = D/(f)$ and $g \equiv X^2 + XY + Y^2 \bmod (X, Y)^3 + (f)$.

The converse is obvious by using the same arguments as before along with Theorem 4.11.

□

In the equal-characteristic case, we can actually say more. But first we will discuss a few more results. Examples of CK domains was discussed in [3]. Theorem 7.1 and Corollary 7.2 from [3] have small mathematical or typographical errors that can easily be corrected. The proof still works with these small corrections. We state the theorems with slight alterations and provide an elaborated version of their proof:

Theorem 4.13 (Anderson and Mott, Theorem 7.1 [3]). *Let $k \subseteq K$ be a pair of fields and for each i, $i = 1, \ldots, n-1$, let V_i be a k-vector subspace of K. Suppose further that for $1 \leq i, j \leq n-1$, $V_iV_j \subseteq V_{l+j}$, where $V_l = K$ for $n \leq l$. Then $R = k + V_1X + \cdots + V_{n-1}X^{n+1} + K[[X]]X^n$ is a complete quasilocal domain. We have R is Noetherian if and only if $[K : k] < \infty$. Furthermore, R is a CK domain if and only if K is finite or $k = K$.*

Theorem 4.14 (Anderson and Mott, Corollary 7.2 [3]). *Let $k \subseteq K$ be a pair of finite fields and let $n \geq 1$. Then $R = k + K[[X]]X^n$ is a complete local CK domain. Let $M = K[[X]]X^n$ be the maximal ideal of R. The irreducible elements of R have the form uX^i where $u \in U(K[[X]])$ and $n \leq i \leq 2n-1$. So R has $n|K^*/k^*||K|^{n-1}$ nonassociate atoms.*

Proof. As we can see from Theorem 4.13, R is a complete quasilocal domain. We notice that R is also Noetherian since k and K are finite fields. Since K is finite, R

is a CK domain. So R is a complete local CK domain. We note that the units of R are the elements with nonzero constant term. We see that the atoms of R must then be of the form $a_iX^i + a_{i+1}X^{i+1} + \cdots$ where $n \leq i \leq 2n-1$. (Notice if $i \geq 2n$, we can factor out X^n from this expression and hence it is not an atom. Since each coefficient is in a field, we do not have to worry about possible zero-divisors.) So factoring out an X^i term we obtain uX^i where $u = a_i + a_{i+1}X + \cdots$ is a unit in $K[[X]]$.

Now, if $uX^i \sim vX^j$ where $u, v \in U(K[[X]])$, then $uX^i = wvX^j$ where $w \in U(R)$. This is true if and only if $i = j$ and $u = wv$. Hence, we need that $uv^{-1} \in U(R)$. This means the number of atoms of the form uX^i for a fixed i is $|U(K[[X]])/U(R)|$. Now, define a function

$$U(K[[X]])/U(R) \to K^*/k^* \oplus K^{n-1}$$

by $(\sum_{s=0}^{\infty} a_s X^s)U(R) \mapsto (a_0k^*, a_0^{-1}a_1, \ldots, a_0^{-1}a_{n-1})$. Clearly this map is well-defined (which is where the a_0^{-1} part of the definition comes into play). This map is clearly surjective. It is injective since the constant terms must be equal. It follows from this that the other following $n-1$ terms in the power series must also be equivalent. But now this means that $|U(K[[X]])/U(R)| = |K^*/k^*||K|^{n-1}$. Since $n \leq i \leq 2n-1$, we thus have $n|K^*/k^*||K|^{n-1}$ nonassociate atoms. $\square$

Theorem 4.15. *Let (R, M) be a complete local CK domain with exactly three nonassociate atoms with char R = char $\overline{R}$. Then $R \cong GF(2) + GF(2^2)[[X]]X$.*

Proof. Notice that by the characterization outlined in Theorem 4.6, we see that $\mathrm{GF}(2) + \mathrm{GF}(2^2)[[X]]X$ is a complete local CK domain. Clearly char R = char $\overline{R} = 2$

in this instance. Using Theorem 4.14, we have $1|GF(2^2)^*/GF(2)^*||GF(2^2)|^0 = 3$ nonassociate atoms.

Now let (R, M) be a complete local CK domain with exactly three nonassociate atoms with char $R =$ char $\overline{R}$. By Theorem 3.5, M^2 is universal since 3 is prime. By Theorem 2.3, we cannot have $|\overline{R}| = \infty$. Hence it is finite. Then by Theorem 4.6, $F_0 + F[[X]]X^n \subseteq R \subseteq F[[X]]$ where F_0 is a coefficient field for R, F is a coefficient field for R' (the integral closure of R), and $n \in \mathbb{Z}^+$ is some positive integer.

Let R' be the integral closure of R. Again since M^2 is universal, by Theorem 3.4 we have that $R' = [M : M]$ is a DVR with unique maximal ideal M. The completion of a DVR is again a DVR [12]. Hence, $\widehat{R'_M} = \widehat{R}'$ is again a DVR. We can see that $R' \cong \widehat{R}'$. Indeed, define a map from R' to $\widehat{R}'$ by

$$s \mapsto (s + M, s + M^2, \cdots).$$

Since the completion is a DVR, we clearly see that its maximal ideal $\widehat{M}$ is generated by $(0 + M, p + M^2, p + M^3, \cdots)$ where $M = (p)$. Hence, this map is clearly an isomorphism. Hence, R' is a complete DVR with char $R' =$ char $\overline{R'}$ where $\overline{R'} = R'/M$. But by Cohen's Structure Theorem (Theorem 4.9), R' is a homomorphic image of $\overline{R'}[[X]]$ via the natural mapping to a given minimal basis. That is, the map $\overline{R'}[[X]] \to R'$ is given by $X \mapsto p$. But since their characteristics are equal, we see that this map is an isomorphism and so $R' \cong \overline{R'}[[X]]$.

But we know that M is the maximal ideal of R' and R. Notice that the maximal ideal of R' is still $\overline{R'}[[X]]X$. We have $R \cong \overline{R} + \overline{R'}[[X]]X$ since $M = \overline{R'}[[X]]X$.

Since the residue fields are finite, these are finite fields. Hence, $\overline{R} \cong \mathrm{GF}(p^n)$ and $\overline{R'} \cong \mathrm{GF}(p^{nk})$ for some prime p and some n, k with $k \geq 2$. Since R has 3 atoms, then using Theorem 4.14, we see that we are forced to have $p = 2$ and $k = 2$. So $R \cong \mathrm{GF}(2) + \mathrm{GF}(2^2)[[X]]X$ as desired.

$\square$

This means that in the characterization in Theorem 4.12, that form (1) is isomorphic to $\mathrm{GF}(2) + \mathrm{GF}(2^2)[[X]]X$ since this was the equal-characteristic case.

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