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Mechanics of the diffeomorphism field

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MECHANICS OF THE DIFFEOMORPHISM FIELD

by

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A thesis submitted in partial fulfillment
of the requirements for the
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To My Parents

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ABSTRACT

Coadjoint orbits of Lie algebras come naturally imbued with a symplectic two-form allowing for the construction of dynamical actions. Consideration of the coadjoint orbit action for the Kac-Moody algebra leads to the Wess-Zumino-Witten model with a gauge-field coupling. Likewise, the same type of coadjoint orbit construction for the Virasoro algebra gives Polyakov's 2D quantum gravity action with a coupling to a coadjoint element, $\mathcal{D}$, interpreted as a component of a field named the diffeomorphism field. Gauge fields are commonly given dynamics through the Yang-Mills action and, since the diffeomorphism field appears analogously through the coadjoint orbit construction, it is interesting to pursue a dynamical action for $\mathcal{D}$.

This thesis reviews the motivation for the diffeomorphism field as a dynamical field and presents results on its dynamics obtained through projective connections. Through the use of the projective connection of Thomas and Whitehead, it will be shown that the diffeomorphism field naturally gains dynamics. Results on the analysis of this dynamical theory in two-dimensional Minkowski background will be presented.

PUBLIC ABSTRACT

Einstein's theory of general relativity is the prevailing theory of gravity in four-dimensional spacetime. In four dimensions the theory determines how a fundamental field called the metric evolves in time. Consideration of general relativity in two dimensions leads to the conclusion that the time evolution of the metric is completely undetermined. This means gravity in two dimensions is not determined by general relativity. Despite this, the metric can gain dynamics due to quantum effects, thus generating gravity in two dimensions. Another field, called the diffeomorphism field, inherently appears in this two-dimensional context. This thesis describes a way to formulate a theory of the diffeomorphism field in the hope that it might prove relevant to gravitational physics.

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1 Introduction

For quite some time, it has been known that one can derive the Wess-Zumino-Witten (WZW) model [1] [3] and Polyakov's 2d quantum gravity action [5] [6] through construction of an action on coadjoint orbits of the Kac-Moody and Virasoro algebras, respectively. In [13] it was shown that the construction of the WZW model via coadjoint orbits of the Kac-Moody algebra gives rise to a coupling term to a gauge field A. In the same work, it was also shown that carrying out the same calculation using coadjoint orbits of the Virasoro algebra gives Polyzkov's 2d quantum gravity with a coupling to a term named the diffeomorphism field $\mathcal{D}$. The gauge field A is known to have dynamics provided through a Yang-Mills action and, since the diffeomorphism field $\mathcal{D}$ arises in an analogous way, there has been a search to find a natural dynamical action for $\mathcal{D}$ and thereafter study its gravitational implications. In the recent work [18] it was found that the same $\mathcal{D}$ coupling term arising through coadjoint orbit construction can be derived through the use of the projective connection of Thomas and Whitehead [21] [22]. This interpretation of $\mathcal{D}$ through projective connections allows for the natural construction of a dynamical action in terms of projective curvatures.

In this work, we will seek to describe the logical progression and motivation of a dynamical theory of the diffeomorphism field $\mathcal{D}$. We first detail the physical relevance of the WZW model (Section 2) and Polyakov 2D quantum gravity (Section 3). Both these actions arise with respect to anomalous quantum symmetries. In the WZW model's case, it will be shown to arise as the action encoding the anomalous breaking of chiral symmetry at the quantum level. Polyakov 2D quantum gravity will likewise be shown to encode anomalous breaking of Weyl symmetry at the quantum level. After doing fair justice to the physical relevance of these models, we will aim to show how they arise through construction of actions on coadjoint orbits. The relevant background on this construction will be reviewed in Section 4. The calculations for the case of Kac-Moody and Virasoro coadjoint orbits will be demonstrated in Section 5. Here, we will clearly demonstrate the analogy between the gauge field A and the diffeomorphism field $\mathcal{D}$. After this, we seek a formalism giving dynamics to $\mathcal{D}$, by analogy with the Yang-Mills action for the gauge field A. In Section 6 we will display how the projective connections of Thomas and Whitehead gives rise to the $\mathcal{D}$ coupling term demonstrated in Section 5.2. In Section 7.1 we will use curvature invariants defined through the Thomas-Whitehead projective connections to develop a dynamical action, called Thomas-Whitehead gravity, for the diffeomorphism field. To begin getting an understanding of the Thomas-Whitehead gravity action, we will look at some solutions to the equations of motion for the simple case of $1+1$-dimensional Minkowski space background (Section 7.2). In the last section (Section 7.3) of this thesis we will consider the Hamiltonian formulation of the theory in the same restricted case as before. After understanding the constraint algebra in the Hamiltonian formulation, quantization will be demonstrated through the use of the Dirac bracket.

2 Wess-Zumino-Witten Model

The goal of this section is to detail the derivation and motivation of the Wess-Zumino-Witten (WZW) model [1]. It is known that the classical Dirac action for massless fermions has both an axial and chiral symmetry. Classically, the chiral symmetry is broken by adding a mass term, mixing the left and right chiral fermions. Quantum mechanically, the chiral symmetry of this fermion theory is broken upon coupling to a gauge field A. This situation is known as anomalous symmetry breaking and can be traced, as we will show, to the transformation properties of the path integral measure. If one calculates the quantum effective action Γ, the generating functional of quantum vertex functions, then under a chiral transformation the effective action transforms into itself plus a piece representing the breaking of chiral symmetry. This extra piece is known as the WZ functional. The explicit evaluation of the WZ functional in 1+1 dimensions by Witten is what came to be known as the WZW model.

The main references for this section are [2], [3], [1].

2.1 Symmetries of Classical Fermions

Here we consider both the massless and massive case of the Dirac Lagrangian describing free fermions. Start with the Dirac Lagragian in natural units ($\hbar = c = 1$),

$$\begin{aligned} S_{Dirac} &= \int dt L_{Dirac} \\ &= \int dt \left(i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + m\bar{\psi}\psi \right) \end{aligned} \tag{2.1}$$

where $\bar{\psi} = \psi^{\dagger}\gamma^{0}$. This action is invariant under the global phase transformation

$$\psi \to e^{i\theta}\psi, \quad \bar{\psi} \to \bar{\psi}e^{-i\theta} \tag{2.2}$$

where θ is some constant. Since we have this continuous symmetry of the action, Noether's theorem tells us there exists a corresponding conserved current. In general, we can show this by considering an arbitrary transformation of whatever field our action is based on $\phi'(x) = \phi(x) + \alpha\Delta\phi(x)$. A transformation is a symmetry of the action if it gives rise to equivalent equations of motion. This means that we can have the Lagrangian transform into itself plus some total derivative term like $L \to L + \alpha\partial_{\mu}\mathcal{J}^{\mu}$. If we make a transformation of the field then the Lagrangian varies as

$$\begin{aligned} \alpha\Delta L &= \frac{\partial L}{\partial \phi}(\alpha\Delta\phi) + \frac{\partial L}{\partial(\partial_{\mu}\phi)}\partial_{\mu}(\alpha\Delta\phi) \\ &= \alpha\partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\phi)}\Delta\phi \right) + \alpha\left(\frac{\partial L}{\partial \phi} - \partial_{\mu}\frac{\partial L}{\partial(\partial_{\mu}\phi)} \right)\Delta\phi. \end{aligned} \tag{2.3}$$

The second term in the last line is the Euler-Lagrange equation. So if we are on-shell, the Euler-Lagrangian equation is satisfied, then this term vanishes. We may then write

$$\alpha\partial_{\mu}\left(\frac{\partial L}{\partial(\partial_{\mu}\phi)}\Delta\phi \right) = \alpha\partial_{\mu}\mathcal{J}^{\mu} \tag{2.4}$$

$$\Rightarrow \partial_\mu j^\mu = 0, \quad j^\mu \equiv \frac{\partial L}{\partial(\partial_\mu \phi)} \Delta\phi - \mathcal{J}^\mu \tag{2.5}$$

where j^μ is the conserved Noether current. Calculating the conserved current corresponding to the phase transformation symmetry 2.2 of the Dirac action gives

$$j^\mu = \frac{\partial L_{Dirac}}{\partial(\partial_\mu \psi)} = \psi^\dagger \gamma^0 \gamma^\mu \psi. \tag{2.6}$$

This current will be referred to hereafter as the vector current.

While the vector current is clearly conserved regardless of whether the fermions are massive or massless, there is an additional symmetry which only the massless fermions exhibit. To see this, we must define an additional gamma matrix called γ_{2n+1}. Here, n is the dimensionality of the spacetime in question. For our current purposes we will assume spacetime is $n = 4$ dimensional. The fifth gamma matrix is defined as

$$\gamma_5 \equiv \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \tag{2.7}$$

and has the properties

$$(\gamma_5)^\dagger = \gamma_5, \quad (\gamma_5)^2 = \mathbb{1}_{4x4}, \quad \{\gamma_5, \gamma^\mu\} = 0. \tag{2.8}$$

Using this fifth gamma matrix we can construct what we will call chiral projectors

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_5) \tag{2.9}$$

which, when acting on Dirac fermions, project out what we will call the left (ψ_-) and right (ψ_+) chiral components. It can be quickly checked that these chiral projectors satisfy usual projection relations

$$P_\pm + P_\mp = \mathbb{1}, \quad P_\pm P_\pm = P_\pm, \quad P_\pm P_\mp = 0. \tag{2.10}$$

We define the left and right projected fermions as $\psi_\pm \equiv P_\pm \psi$ such that they satisfy the γ_5 eigenvalue equation

$$\gamma_5 \psi_\pm = \pm \psi_\pm. \tag{2.11}$$

The field equations derived from $\delta S_{Dirac} = 0$ are

$$i\gamma^\mu \partial_\mu \psi - mc\psi = 0. \tag{2.12}$$

By applying the chiral projectors to the field equations we see that

$$P_+(i\gamma^\mu \partial_\mu \psi - mc\psi) = i\gamma^\mu \partial_\mu \psi_- - mc\psi_+ = 0 \tag{2.13}$$

$$P_-(i\gamma^\mu \partial_\mu \psi - mc\psi) = i\gamma^\mu \partial_\mu \psi_+ - mc\psi_- = 0 \tag{2.14}$$

so that the dynamics of left and right fermions are coupled to each other when $m \neq 0$ but decouple when $m = 0$. This means that in the massless case we can apply separate phase transformations to the left and right chiral fermions. This additional symmetry is called chiral symmetry. Due to this extra symmetry, it follows that we can define chiral currents which are separately conserved

$$\partial_\mu j_+^\mu = \partial_\mu(\psi_+^\dagger \gamma^0 \gamma^\mu \psi_+) = 0, \quad \partial_\mu j_-^\mu = \partial_\mu(\psi_-^\dagger \gamma^0 \gamma^\mu \psi_-) = 0. \tag{2.15}$$

Taking the difference of these currents, we can write the so-called chiral (axial) current

$$j_5^\mu \equiv j_+^\mu - j_-^\mu = \psi^\dagger \gamma^0 \gamma^\mu \gamma_5 \psi \tag{2.16}$$

which is conserved ($\partial_\mu j_5^\mu = 0$) due to chiral symmetry of classical massless fermions.

The final step before we discuss the anomalous breaking of chiral symmetry in the quantum theory is to add a gauge field coupling to the Dirac Lagrangian as

$$L_{Dirac} \to i\bar{\psi}\gamma^\mu D_\mu \psi, \quad D_\mu = \partial_\mu + iA_\mu. \tag{2.17}$$

It is straightforward to show that if we promote the global phase symmetry $\psi \to e^{i\theta}$ to a local symmetry, where $\theta \to \theta(x)$, then the gauge field must simultaneously transform as $A_\mu \to A_\mu - \partial_\mu \theta(x)$. Under this full transformation, the same results as before can be shown for both the vector and chiral symmetry. As we will see, this is not the case when the theory is quantized.

2.2 Quantum Fermions and Anomalous Chiral Symmetry

We are now in a position to discuss how chiral symmetry is broken in the quantum theory. Here we will discuss the quantum theory in the context of the functional (path-integral) formulation. Our general goal is to be able to use the path integral to compute Green's functions. If we have some set of fields ϕ^k then the generating functional of Green's functions is given by

$$Z[J] = \int \prod_k \mathcal{D}\phi^k \; e^{i(S[\phi] + \int dx \phi^k(x) J_k(x))} \tag{2.18}$$

where J represents some currents to which the fields couple. Since all Green's functions can be formed from so-called connected Green's functions, we can consider the generating functional of connected Green's functions

$$Z[J] = e^{iW[J]}. \tag{2.19}$$

From this, one can form an object called the quantum effective action through the Legendre transform

$$\Gamma[\phi^k] = W[J_\phi] - \int dx \phi^k(x) J_{\phi,k}(x), \quad \phi^k(x) = \frac{\delta W[J]}{\delta i J_k(x)} \equiv \langle \Phi^k(x) \rangle_J . \tag{2.20}$$

where $\langle \mathcal{O} \rangle_J$ denotes the expectation value of some operator $\mathcal{O}$ in the presence of sources J. Here $J_{\phi,k}$ is the current such that the expectation of the quantum operator $\Phi^k(x)$ equals a given value $\phi^k(x)$. It would seem straightforward that a symmetry of the action translates straightforwardly into a symmetry of the generating functional $Z[J]$ and therefore of the quantum effective action $\Gamma[\phi]$. This is not generally true though if the measure $\mathcal{D}\phi$ transforms non-trivially under this transformation. Specifically, in the case that the measure transforms non-trivially it can be shown that the variation of the quantum effective action is precisely the anomaly functional encoding the breaking of a classical symmetry. This is a form of the so-called anomalous Slavnov-Taylor identity. We will first show this in a general setting and then specify to a theory of massless fermions in 4-dimensions to show that the Wess-Zumino functional encodes anomalous breaking of chiral symmetry.

2.2.1 Anomalous Slavnov-Taylor Identity

Let's suppose again that we have some set of fields ϕ^k that transform under a general local symmetry as

$$\phi^k(x) \to \phi^k(x) + \epsilon F^k(x, \phi(x)) \tag{2.21}$$

such that the classical action is invariant

$$S[\phi^k + \epsilon F^k] = S[\phi^k]. \tag{2.22}$$

The situation we want to consider is the one where the path integral integration measure $\mathcal{D}\phi^k$ is not invariant under this transformation but instead transforms as

$$\prod_k \mathcal{D}\phi^k \to \prod_k \mathcal{D}(\phi^k + \epsilon F^k) = \prod_k \mathcal{D}\phi^k e^{i\epsilon \int d^4x \mathcal{A}(x)} \tag{2.23}$$

where $\mathcal{A}(x)$ is known as the anomaly function. We can use these transformations to show how the generating function of connected Green's functions $W[J]$ transforms. If we demand that $W[J]$ be invariant under this transformation we get

$$\begin{aligned} e^{iW[J]} &= \int \prod_k \mathcal{D}\phi^k e^{i\left(S[\phi^k] + \int \phi^k(x) J_k(x)\right)} = \int \prod_k \mathcal{D}\phi'^k e^{i\left(S[\phi'^k] + \int \phi'^k(x) J_k(x)\right)} \\ &= \int \prod_k \mathcal{D}\phi^k e^{i\left(S[\phi^k] + \int \phi^k(x) J_k(x)\right) + i\epsilon \int \left(\mathcal{A}(x) + J_k(x) F^k(x,\phi)\right)} \\ &= \int \prod_k \mathcal{D}\phi^k e^{i\left(S[\phi^k] + \int \phi^k(x) J_k(x)\right)} \left[1 + i\epsilon \int \left(\mathcal{A}(x) + J_k(x) F^k(x,\phi)\right)\right]. \end{aligned} \tag{2.24}$$

So we see that invariance requires

$$\int d^4x \left(\mathcal{A}(x) + J_k(x) \left\langle F^k(x,\Phi)\right\rangle_J\right) = 0. \tag{2.25}$$

This is known in the literature as the anomalous Slavnov-Taylor identity. Our goal is to write this requirement in terms of the quantum effective action $\Gamma[\phi]$ from 2.20. From the definition of $\Gamma[\phi]$ we have that $J_{\phi,k}(x) = -\frac{\delta\Gamma[\phi]}{\delta\phi^k(x)}$ and $\left\langle \Phi^k \right\rangle_{J_{\phi,r}} = \phi^k$. It is not necessarily true that we can write $\langle F(x,\Phi)\rangle_{J_{\phi,k}} = F(x,\phi)$ unless F is a linear function of ϕ. In the case this is true, we can choose $J = J_\phi$ and write the anomalous Slavnov-Taylor identity as

$$\int d^4x \left(\mathcal{A}(x) - F^k(x,\phi) \frac{\delta\Gamma[\phi]}{\delta\phi^k(x)}\right) = 0 \tag{2.26}$$

and since $\delta\phi^k = \epsilon F^r(x,\phi)$ we can write

$$\delta_\epsilon \Gamma[\phi] = \int d^4x\, \delta\phi^k(x) \frac{\delta\Gamma[\phi]}{\delta\phi^k(x)} = \int d^4x\, \epsilon \mathcal{A}(x). \tag{2.27}$$

This shows that if anomalies are present then the variation of the quantum effective action under the anomalous symmetry transformations is equal to the anomaly.

2.2.2 Chiral Symmetry and the Functional Measure

In the previous section we assumed the functional measure transformed anomalously as in 2.23. Now we will show explicitly how an anomalous situation like this arises through consideration of the free Dirac Lagrangian coupled to a gauge field as in 2.17. As previously described, the anomaly should arise through the transformation of the functional integration measure so consider

$$e^{iW[A]} \equiv \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, e^{i\int d^4x\, i\bar{\psi}\gamma^\mu D_\mu \psi}. \tag{2.28}$$

Formally, the above is a Gaussian integral over Grassmann variables so we can write $W[A]$ as

$$W[A] = \log \mathrm{Det}\ \gamma^\mu D_\mu. \tag{2.29}$$

Let us consider a generic unitary transformation of the fermionic field ψ such that

$$\psi(x) \to U(x)\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)\bar{U}(x) \tag{2.30}$$

$$\bar{U}(x) = i\gamma^0 U^\dagger(x) i\gamma^0. \tag{2.31}$$

Since ψ and $\bar{\psi}$ are Grassmann variables, the measure transforms as

$$\mathcal{D}\psi \to (\mathrm{Det}\ \mathcal{U})^{-1}\mathcal{D}\psi, \quad \mathcal{D}\bar{\psi} \to (\mathrm{Det}\ \bar{\mathcal{U}})^{-1}\mathcal{D}\bar{\psi} \tag{2.32}$$

where $\langle x|\mathcal{U}|y\rangle = U(x)\delta^4(x-y)$ and $\langle x|\bar{\mathcal{U}}|y\rangle = \bar{U}(x)\delta^4(x-y)$. Specifically, for the case of a unitary non-chiral transformation $U(x) = e^{i\epsilon^\alpha(x)t_\alpha}$, where t_α are generators of some gauge group, we will have

$$\bar{U}(x) = i\gamma^0 e^{-i\epsilon^\alpha(x)t_\alpha} i\gamma^0 = e^{-i\epsilon^\alpha(x)t_\alpha} = U^{-1}(x) \tag{2.33}$$

and therefore the measure transforms as

$$\begin{aligned} &\mathcal{D}\psi\,\mathcal{D}\bar{\psi} \to (\text{Det}\,\mathcal{U})^{-1}(\text{Det}\,\bar{\mathcal{U}})^{-1}\mathcal{D}\psi\,\mathcal{D}\bar{\psi} \\ &= (\text{Det}\,\mathcal{U})^{-1}(\text{Det}\,\mathcal{U})\mathcal{D}\psi\,\mathcal{D}\bar{\psi} = \mathcal{D}\psi\,\mathcal{D}\bar{\psi}. \end{aligned} \tag{2.34}$$

So we see the measure is invariant for a non-chiral unitary transformation of the fields. This means there is no anomaly corresponding to vector gauge transformations. Now consider the case of a chiral unitary transformation. The result will be markedly different due to the existence of a γ_5 in the transformation. Take $U(x) = e^{i\epsilon^\alpha(x)t_\alpha\gamma_5}$ so that

$$\bar{U}(x) = i\gamma^0 e^{-i\epsilon^\alpha(x)t_\alpha\gamma_5} i\gamma^0 = e^{i\epsilon^\alpha(x)t_\alpha\gamma_5} = U(x). \tag{2.35}$$

The measure will now transform as

$$\mathcal{D}\psi\,\mathcal{D}\bar{\psi} \to (\text{Det}\,\mathcal{U})^{-2}\mathcal{D}\psi\,\mathcal{D}\bar{\psi} \tag{2.36}$$

which is certainly not an expression of invariance. We wish to find an explicit expression for $(\text{Det}\,\mathcal{U})^{-2}$. First, we can write

$$\begin{aligned} \langle x|\mathcal{U}^2|y\rangle = \int d^4z\,\langle x|\mathcal{U}|z\rangle\,\langle z|\mathcal{U}|y\rangle = \int d^4z U(x)U(z)\delta^4(x-z)\delta^4(z-y) \\ = U^2(x)\delta^4(x-y). \end{aligned} \tag{2.37}$$

Since we want to write $(\mathrm{Det}\,\mathcal{U})^{-2}$ we can use the fact that $(\mathrm{Det}\,\mathcal{U})^{-2} = e^{-2\mathrm{Tr}\log\mathcal{U}}$ and calculate

$$\mathrm{Tr}\log\mathcal{U} = \int d^4x\, \langle x|\mathrm{tr}\log\mathcal{U}|x\rangle = \int d^4x\, \delta^4(0) i\epsilon^\alpha \mathrm{tr} t_\alpha \gamma_5. \tag{2.38}$$

This gives

$$(\mathrm{Det}\,\mathcal{U})^{-2} = e^{i\int d^4x\,\epsilon^\alpha a_\alpha(x)}, \quad a_\alpha(x) = -2\delta^4(0)\mathrm{tr} t_\alpha \gamma_5 \tag{2.39}$$

which is precisely the form of the measure transformation we assumed in 2.23. The $a_\alpha(x)$ is called the anomaly function and must be evaluated using a regularization scheme, since it is not well-defined as currently written. After evaluating this expression using some regularization scheme (see Section 3.3 of [2] for details) , one finds

$$a_\alpha(x) = -\frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr} t_\alpha F_{\mu\nu}(x) F_{\rho\sigma}(x) = -\frac{1}{16\pi^2}\mathrm{tr} t_\alpha F(\star F) \tag{2.40}$$

where $\star$ represents the Hodge star operator. This specific anomaly function encodes the breaking of chiral symmetry and the corresponding functional goes by the name of Wess-Zumino functional.

2.3 Wess-Zumino-Witten Model

The Wess-Zumino-Witten model arises from the explicit evaluation of the formal determinant $W[A] = \log\mathrm{Det}\,\gamma^\mu D_\mu$ in 1+1 dimensions with a non-Abelian gauge group. The quickest way to see how this arises is due to Polyakov and Wiegmann [3]. First, define the quantum current in the usual way

$$J_\mu = \frac{\delta W[A]}{\delta A_\mu}. \tag{2.41}$$

Now we will write relations stating the covariant conservation of the vector current $\psi^\dagger\gamma^0\gamma^\mu\psi$ and the covariant non-conservation of the chiral current $\psi^\dagger\gamma^0\gamma^\mu\gamma_5\psi$. Note that in 1+1 dimensions we have the identity $\bar{\psi}\gamma^\mu\gamma_5\psi = \epsilon^{\mu\nu}\bar{\psi}\gamma_\nu\psi$ so we can write covariant conservation and non-conservation of vector and axial currents, respectively, as

$$\partial_\mu J^\mu + [A_\mu, J^\mu] = 0 \tag{2.42}$$

$$\epsilon^{\mu\nu}\left(\partial_\mu J_\nu + [A_\mu, J_\nu]\right) = \frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{2.43}$$

Here we see the 1+1 dimensional anomaly function $F(\star F) \approx \epsilon^{\mu\nu}F_{\mu\nu}$ arise as the right hand side of 2.43. We can now use these relations to determine J^μ and thus $W[A]$. For this purpose, change to light-cone coordinates $x_\pm = x_0 \pm x_1$ and parametrize gauge fields with group elements g and h as

$$A_+ = g^{-1}\partial_+ g, \quad A_- = h^{-1}\partial_- h. \tag{2.44}$$

Equations 2.42 and 2.43 are then solved by

$$\begin{aligned} J_+ &= g^{-1}\partial_+ g - h^{-1}\partial_+ h \\ J_- &= h^{-1}\partial_- h - g^{-1}\partial_- g. \end{aligned} \tag{2.45}$$

If we choose the so-called axial gauge such that $h = \mathbb{1}$ and $A_- = 0$ then the variation of $W[g]$ becomes

$$\delta W[A] = \int d^2x \, \mathrm{tr} J_- \delta A_+ = \int d^2x \, \mathrm{tr}\partial_-(g^{-1}\partial_+ g)\delta g g^{-1} \tag{2.46}$$

The $W[g]$ giving this variation ends up being

$$W[g] = \frac{1}{2}\int_{S^2} d^2x \, \mathrm{tr}(\partial_\mu g^{-1}\partial_\mu g) + \frac{i}{8\pi^2}\int_Q d^3\xi \epsilon^{ABC}\mathrm{tr}(g^{-1}\partial_A g g^{-1}\partial_B g g^{-1}\partial_C g) \tag{2.47}$$

with $\partial Q = S^2$. The first term is seen to be the 2D WZ functional and the second term can be shown to count the number of times the mapping specified by g wraps around Q. The functional 2.47 is known as the WZW functional.

3 Polyakov 2D Quantum Gravity

Here we will review the derivation of Polyakov's 2D gravity action. Similar to the way the WZW model encoded the chiral anomaly, in the previous section, Polyakov's action encodes the Weyl anomaly. We first review classical Weyl symmetry in the context of the string action. Thereafter, we will present Polyakov's derivation of the effective action for the breaking of Weyl symmetry in two dimensions.
The main references for this section are [4], [5], [6], [7], [8].

3.1 Weyl Anomaly

Here we seek to describe how a classical Weyl symmetry ($g_{\mu\nu} \to e^{\Omega(x)} g_{\mu\nu}$), implying $g^{\mu\nu} T_{\mu\nu} = 0$, is broken quantum mechanically by the so-called Weyl (Trace) anomaly. Our discussion will take place in $1 + 1$-dimensions. In order to begin discussing an energy-momentum tensor we would need an action describing gravity coupled to some matter action like

$$S = S_{EH} + S_m \tag{3.1}$$

where S_{EH} represents the Einstein-Hilbert action and S_m is some matter action. In 2D, the Einstein-Hilbert action is simply a topological invariant proportional to the Euler-character (χ) of the surface

$$S_{EH} = \int d^2x \sqrt{-g} R \propto \chi. \tag{3.2}$$

For this reason, pure gravity in 2D is entirely trivial since the field equations just say that $0 = 0$. Upon quantization, 2D gravity can acquire non-trivial dynamics through the Weyl anomaly. To see this, we can consider any classically Weyl invariant matter action. We use the 2D string action functional,

$$S = \frac{1}{2}\int d^2x \sqrt{-g}g^{\mu\nu}\partial_\mu X_a \partial_\nu X^a \tag{3.3}$$

where the X_a describes the parametrization of the string surface. Classically, the energy-momentum tensor of a theory coupling to gravity through the metric $g^{\mu\nu}$ is defined by $T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$. Computing the energy-momentum tensor for the string functional 3.3 gives

$$T_{\mu\nu} = \partial_\mu X_a \partial_\nu X^a - \frac{1}{2}g_{\mu\nu}\partial^\rho X_a \partial_\rho X^a \tag{3.4}$$

and thus is traceless $g^{\mu\nu}T_{\mu\nu} = 0$ as well as divergenceless $\nabla^\mu T_{\mu\nu} = 0$. Vanishing of the trace of the energy-momentum tensor implies that the action is invariant under a Weyl transformation. This is seen by considering the transformation $g_{\mu\nu} \to e^{\Omega(x)}g_{\mu\nu}$, where $\Omega(x)$ is arbitrary for a Weyl transformation, and requiring $\delta S = 0$. Then

$$\begin{aligned} \delta S = \frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}}\Omega(x)g^{\mu\nu} \propto T_{\mu\nu}g^{\mu\nu}\Omega(x) = 0 \\ \Rightarrow T^\mu_\mu = 0. \end{aligned} \tag{3.5}$$

We will now review how this symmetry is broken at the quantum level such that $g^{\mu\nu}\langle T_{\mu\nu}\rangle \neq 0$. Note that these formulas are written after Wick rotation so the metric is of Euclidean signature. First, we define $\langle T_{\mu\nu}\rangle$ through the path integral

$$Z = \int d\mu e^{-S} \tag{3.6}$$

where $d\mu$ stands for the properly defined path integral measure for matter fields and the metric. The expectation of the energy-momentum tensor is defined as

$$\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{\mu\nu}}, \quad e^{-W} = \int d\mu e^{-S}. \tag{3.7}$$

Explicit evaluation requires the choice of an integration variable in the path integral. As detailed in [9], there exist two natural choices. The first keeps the integration of the $X_a(x)$ as it natural appears and the second is to instead integrate over $\tilde{X}_a(x) = g^{1/4}X_a(x)$. The first choice preserves Weyl invariance in the quantum theory while losing $\nabla^\mu \langle T_{\mu\nu}\rangle = 0$. The second choice allows one to preserve the divergence of the energy-momentum tensor while breaking Weyl invariance. It is the second choice of integration variable we will consider. After evaluating with a proper regularization scheme, one finds

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = \frac{D}{24\pi}(R(x) + \text{const}) \tag{3.8}$$

where D is the dimension of the spacetime the string surface is embedded in and $R(x)$ is the Ricci scalar of the string. For more details on the derivation of the trace anomaly see [9].

3.2 Polyakov Action

In [5], Polyakov takes the expression for the Weyl anomaly 3.8 and uses it to solve for W in non-local form as

$$g^{\mu\nu}\frac{\delta W}{\delta g^{\mu\nu}} = g^{\mu\nu}\langle T_{\mu\nu}\rangle \frac{D}{24\pi}(R + \text{const}) \tag{3.9}$$

$$\Rightarrow W = -\frac{D}{48\pi}\int d^2x d^2x' g^{1/2}(x) g^{1/2}(x') R(x) K(x,x')\, R(x') + \text{const}\int d^2x\sqrt{g} \tag{3.10}$$

where K is the Green's function for the Laplacian

$$\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_b)K(x,x') = \delta(x-x'). \tag{3.11}$$

In the case that conformal gauge is chosen ($g_{ab} = \rho\delta_{ab}$, $R = \rho^{-1}\partial^2\rho$) W becomes local and is known as 2D Liouville gravity

$$W[\rho] = -\frac{D}{96\pi}\int d^2x \left(\frac{1}{2}(\partial_a \log\rho)^2 + \mu^2\rho\right). \tag{3.12}$$

The evaluation of Polyakov's 2D gravity action W in lightcone gauge is of main interest to us. Rather than choose lightcone gauge and explicitly evaluate 3.9 using the gauge fixed metric

$$g^{(+-)}_{\mu\nu} = \begin{pmatrix} g_{++} & 1/2 \\ 1/2 & 0 \end{pmatrix} \tag{3.13}$$

with $R = \partial_-^2 g_{++}$, Polyakov works out the W in an analogous way to how the WZW model was derived in section 2.3. He writes the variation of W and the conformal anomaly operator relations, which are derived using conservation of the energy-momentum tensor and the expression for the Weyl anomaly, in lightcone gauge as

$$\delta W = \int T_{--}\delta g_{++} \tag{3.14}$$

$$\nabla_+ T_{--} \equiv \partial_+ T_{--} - g_{++}\partial_- T_{--} - 2(\partial_- g_{++})T_{--} = \frac{D}{24\pi}\partial_- R. \tag{3.15}$$

Polyakov introduces a field f defined through

$$\partial_+ f = g_{++}\partial_- f \tag{3.16}$$

and then is able to write the solution to 3.14 as

$$W[f] = \frac{D}{24\pi}\int d^2x \left(\frac{\partial_-^2 f\partial_+\partial_- f}{(\partial_- f)^2} - \frac{(\partial_-^2 f)(\partial_+ f)}{(\partial_- f)^3}\right). \tag{3.17}$$

This defines Polyakov's 2-dimensional quantum gravity action, which encodes the gravitational Weyl anomaly in lightcone gauge. We have seen how Polyakov 2D gravity encodes the Weyl anomaly in the same way that the WZW model encodes the chiral anomaly. For this reason and others, Polyakov's gravity is commonly referred to as the gravitational WZW model.

4 Mechanics on Coadjoint Orbits

In this section we will explain how one can generally formulate Hamiltonian mechanics on coadjoint orbits. We first review the basics of Lie algebra and group actions, with the intent of describing the coadjoint action. Then, using the coadjoint action we will construct coadjoint orbits with natural symplectic forms. The formulation of an action on these coadjoint orbits will be discussed.
The main references for this section are [10], [11].

4.1 The Coadjoint Representation

We first define the action of the Lie group G on itself for two arbitrary elements f and g by

$$g.f = gfg^{-1}. \tag{4.1}$$

For matrix Lie groups we have the exponential map between group and algebra elements

$$g = e^{tX} \tag{4.2}$$

where the X is a member of the Lie algebra $\mathcal{G}$ and t is some continuous parameter. The Lie algebra $\mathcal{G}$ is regarded as the tangent space about the identity of the Lie group since

$$e^{tX} = \mathbb{1} + tX. \tag{4.3}$$

The equality is due to the fact that the elements of the algebra are infinitesimal and thus only linear terms are kept in the expansion of the exponential map. A representation of a group G is a group action of G on some vector space, where the group action is given by invertible linear maps. Since the Lie algebra $\mathcal{G}$ is the tangent space about the identity of the Lie group, we may use its properties as a vector space to naturally define the adjoint representation of the Lie group G. To define the adjoint representation we take the group action from eqn 4.1 and linearize

$$Ad_g X \equiv \frac{d}{dt}(ge^{tX}g^{-1}) \mid_{t=0} = gXg^{-1} \tag{4.4}$$

This defines the adjoint representation of the Lie group G through its action on its Lie algebra $\mathcal{G}$. Once we have the adjoint representation of the group acting on the algebra, we also have an adjoint action of the algebra on itself. This is obtained by linearizing the adjoint group action

$$\begin{aligned} ad_Y X \equiv \frac{d}{dt}(Ad_{exp(tY)}X) \mid_{t=0} &= \frac{d}{dt}(e^{tY}Xe^{-tY}) \mid_{t=0} \\ = \frac{d}{dt}\big((1+tY)X(1-tY)\big) \mid_{t=0} &= YX - XY \end{aligned} \tag{4.5}$$

This shows that the adjoint action of two algebra elements is simply the commutator of the two elements

$$ad_Y X = [Y, X] = YX - XY. \tag{4.6}$$

Having defined the adjoint representation, in the context of Lie group and algebra elements, we are in a position to define the coadjoint representation. The Lie algebra naturally carries the structure of a vector space. This means we inherently have a notion of a dual vector space, given by the set of all linear functionals acting on the Lie algebra. Utilizing the bra-ket notation, we write the pairing of a vector, v, with its dual vector, b, as

$$\langle b|v\rangle \equiv b(v). \tag{4.7}$$

$\mathcal{G}$ is a vector space so, denoting the corresponding dual vector space by $\mathcal{G}^*$, we see that the pairing maps $\mathcal{G}^* \times \mathcal{G} \to \mathbb{R}$. Since any real number transforms trivially under the action of a Lie group, we may write

$$Ad_g\big(\,\langle b|v\rangle\,\big) = \langle b|v\rangle \tag{4.8}$$

for $g \in G$, $b \in \mathcal{G}^*$, and $v \in \mathcal{G}$. This statement expresses the invariance of the pairing under adjoint action of the group. If we apply the action to both vector and dual vector as

$$Ad_g\big(\,\langle b|v\rangle\,\big) = \langle Ad^*_g b|Ad_g v\rangle \tag{4.9}$$

it is obvious that we must have $Ad^*_g \equiv \big(Ad_{g^{-1}}\big)^*$ in order for the pairing to be invariant. This defines the coadjoint action and thus the coadjoint representation.

We now seek to define the coadjoint action of an algebra element. This comes easily since we have already defined the adjoint action of algebra elements. We simply carry out a similar procedure to the one we just did. The only change is that the algebra adjoint action acts as a derivation. Requiring the vanishing of the pairing under adjoint action

$$ad_w\big(\,\langle b|v\rangle\,\big) = 0 \tag{4.10}$$

$$\Rightarrow \langle ad^*_w b|v\rangle + \langle b|ad_w v\rangle = 0 \tag{4.11}$$

$$\Rightarrow \langle ad^*_w b|v\rangle + \langle b|[w,v]\rangle = 0 \tag{4.12}$$

$$\Rightarrow \langle ad^*_w b|v\rangle = -\,\langle b|[w,v]\rangle \tag{4.13}$$

which defines the coadjoint action of the algebra on another algebra element.

4.2 Symplectic Structure

A closed nondegenerate differential 2-form, also known as a symplectic structure, is at the heart of classical mechanics. Essentially, once we are given a smooth manifold and a symplectic structure, we are able to do classical mechanics. This is demonstrated by the fact that, written in some coordinate basis, we may write Hamiltons equations as

$$\omega_{ij}\dot{\xi}^j = \frac{\partial H}{\partial \xi^i} \tag{4.14}$$

where ω_{ij} are coefficients of the symplectic form and H is the Hamiltonian. Generally, Hamilton's equations are assumed to be derivable by setting the variation of an action to zero. This action is usually written as

$$S = \int_{\mathcal{C}} (\theta - H\,dt) \tag{4.15}$$

where $\mathcal{C}$ is the phase-space with coordinates ξ^μ parametrized by t, θ is the canonical one-form, and H is the Hamiltonian. Here θ is taken to be a globally-defined one-form that is related to the symplectic form via $\omega = d\theta$. We can write this in coordinates as

$$\theta = \theta_\mu(\xi)\,d\xi^\mu, \quad \omega = d\theta = -\frac{1}{2}\omega_{\mu\nu}(\xi)dx^\mu \wedge dx^\nu \tag{4.16}$$

$$\omega_{\mu\nu}(\xi) = \frac{\partial A_\nu(\xi}{\partial \xi^\mu} - \frac{\partial A_\mu(\xi}{\partial \xi^\nu}, \quad \det|\omega_{\mu\nu}(\xi)| \neq 0 \tag{4.17}$$

Assuming $H(\xi)$ is some globally defined Hamiltonian, we can write the action functional as

$$S = \int_{\mathcal{C}} (\theta_\mu(\xi)\,d\xi^\mu - H(\xi)\,dt)\,. \tag{4.18}$$

Taking the variation of this action gives

$$S = \int_{\mathcal{C}} \delta\xi^\mu(t)\left(\omega_{\mu\nu}(\xi)\dot{\xi}^\nu - \frac{\partial H(\xi)}{\partial \xi^\mu}\right)dt \tag{4.19}$$

and requiring this to vanish for arbitrary $\delta\xi^\mu(t)$ implies Hamilton's equations. For our purposes, the symplectic form will not be exact and it will not be possible to write an action like 4.15 which yields Hamilton's equations. This leads us to employ the results of [10] and [11] for non-exact symplectic mechanics.

Rather than considering the coadjoint orbit phase space, we will consider the space of paths on phase space. Assuming the phase space is path connected, we can make a straightforward mapping from phase space to path space by fixing a point, P_0, in phase space and constructing all possible paths to all possible points in phase space. Each path corresponds to a unique point in path space. We can write points in the path space as

$$\gamma^i \in \{\gamma^i(\sigma) \mid 0 \leq \sigma \leq 1, \gamma^i(0) = \xi_0^i, \gamma^i(1) = \xi^i\} \tag{4.20}$$

where ξ^i is any other point in phase space. In addition to the path parametrization, we can introduce time by writing $\gamma^i(\sigma,\tau)$ with $\gamma^i(0,\tau) = \xi_0^i$ and $\gamma^i(1,\tau) = \xi^i(\tau)$. If we define the action functional as

$$S = \int_{\tau_1}^{\tau_2} d\tau \int_0^1 d\sigma \omega_{ij} \partial_\sigma \gamma^i \partial_\tau \gamma^j - \int_{\tau_1}^{\tau_2} d\tau H\big(\xi(\tau)\big) \tag{4.21}$$

then one can show that setting its variation to zero, keeping fixed $\xi^i(\tau_1)$, $\xi^i(\tau_2)$ and $\gamma^i(\cdot,\tau_1)$, $\gamma^i(\cdot,\tau_2)$, gives Hamilton's equations. Thus, we have a way of writing an action yielding Hamilton's equations using only a symplectic form with no reference to a canonical one-form.

4.3 Action Functional on Coadjoint Orbits

Here we will show how the coadjoint action provides us with a natural notion of a symplectic structure. Kirillov [12] was the first to understand this fact. To start, we must define the orbit of a coadjoint algebra element. This will hereafter be simply called a coadjoint orbit. As with defining any orbit, we will fix a coadjoint algebra element b and define

$$W_b \equiv \{Ad_g^* b \mid g \in G, b \in \mathcal{G}^*\} \tag{4.22}$$

as the coadjoint orbit of b. So the coadjoint orbit is the set of coadjoint vectors that can be reached through coadjoint action on an initial vector b. We seek to define a symplectic structure on W_b that is invariant under action by the group G. Since we will regard the coadjoint orbit as the phase space our symplectic structure lives on, we know our symplectic form will act on tangent vectors of the orbit. Tangent vectors on the orbit are naturally given by infinitesimal group action on the coadjoint vector b. Thus, if we have our symplectic form acting on two tangent vectors a and a', we know these vectors should be obtainable as

$$a = ad_u^* b, \quad a' = ad_{u'}^* b \tag{4.23}$$

where u and u' are adjoint algebra elements. Since the symplectic form is antisymmetric, by definition, a natural guess for a symplectic form on the coadjoint orbit is

$$\omega(a, a') \equiv \langle b|[u, u']\rangle \tag{4.24}$$

This definition is made through the pairing, so it is naturally invariant under action by G. To show that this actually defines a symplectic structure, we are obliged to show that ω is both non-degenerate and closed. Non-degeneracy is easily shown to hold. If $a = ad_u^*$ is a nonzero coadjoint vector then b must be nonzero since ad^* is linear. There must be an adjoint vector v, which does not commute with u, so that $(ad_a^* b)v = -\langle b|[u, v]\rangle \neq 0$. Then ω is non-degenerate. To show that ω is closed we need to prove that $d\omega = 0$. For a general two-form λ acting on three vector fields u, v, w we have that

$$d\lambda(u,v,w) = (u\cdot\nabla)\lambda(v,w) + (v\cdot\nabla)\lambda(w,u) + (w\cdot\nabla)\lambda(u,v) \\ -\lambda([u,v],w) - \lambda([v,w],u) - \lambda([w,u],v). \tag{4.25}$$

Applying this general formula to ω with $a_1 = ad_u^* b$, $a_2 = ad_v^* b$, and $a_3 = ad_w^* b$ gives

$$d\omega(a_1,a_2,a_3) = (a_1\cdot\nabla)\omega(a_2,a_3) + (a_2\cdot\nabla)\omega(a_3,a_1) + (a_3\cdot\nabla)\omega(a_1,a_2) \\ - \langle b|[[u,v],w]\rangle - \langle b|[[v,w],u]\rangle - \langle b|[[w,u],v]\rangle . \tag{4.26}$$

The terms on the first line of 4.26 are adjoint actions on ω, which is defined through the pairing. Since the pairing has already been assumed invariant under group action, these terms all vanish. The last line can be expressed more concisely as

$$- \langle b|[[u,v],w] + [[v,w],u] + [[w,u],v]\rangle \tag{4.27}$$

which clearly vanishes due to the Jacobi identity inherent to the Lie algebra. This shows that ω, defined through 4.24, can be used to define a symplectic structure on coadjoint orbits.

Now that we have shown the coadjoint orbits naturally have a symplectic form, we may use the result 4.21 for writing an action in terms of an non-exact symplectic form. For current purposes we will be setting the Hamiltonian to zero and only consider a symplectic two-form given by the coadjoint orbit construction. This allows us to express the action functional as

$$S = \int_{W_b} d\lambda d\tau \, \langle b|[u_\tau, u_\lambda]\rangle . \tag{4.28}$$

where u_τ and u_λ are adjoint vectors parametrized by λ and τ and the integration is over the coadjoint orbit W_b also parametrized by λ and τ.

5 Coadjoint Actions

In this section we review the construction of coadjoint actions for the Kac-Moody and Virasoro algebra. It is shown precisely how the Yang-Mills gauge field couples to the WZW model in the same way that the diffeomorphism field couples to Polyakov's gravity action in $1+1$-dimensions. We first review the relevant properties of the Kac-Moody and Virasoro algebras and thereafter use the results of the previous section to write an action on their coadjoint orbits.
The main references for this section are [13], [14], [15].

5.1 Kac-Moody

We first describe how the Kac-Moody algebra arises as the central extension of the loop algebra. This will mostly mirror the discussion in [15]. First, let us define the set of elements of the circle as

$$S^1 \equiv \{z \in \mathbb{C} \bigm| |z| = 1\}. \tag{5.1}$$

We also define maps from S^1 to some Lie group G as $\gamma : z \to \gamma(z)$, with $\gamma(z) \in G$. Given two maps from the circle to the Lie group G we define their multiplication via the pointwise product

$$\gamma_1 \cdot \gamma_2(z) \equiv \gamma_1(z)\gamma_2(z). \tag{5.2}$$

The set of maps, γ, from the circle to G can now be seen to form a Lie group which is commonly referred to as the Loop group (LG)

$$LG = \{\gamma \mid \gamma : z \to \gamma(z), z \in \mathbb{C}, \gamma(z) \in G\}. \tag{5.3}$$

Now that we have defined LG we wish to write down the structure of its Lie algebra. If we restrict the Lie group G to only include its connected components, then we may write any element of LG via the product of exponential elements of its Lie algebra as

$$\gamma(z) = e^{-iT^a\theta_a(z)} \tag{5.4}$$

where the T^a form a basis for the Lie algebra $L\mathcal{G}$. These basis elements, also called group generators, have the generic commutation relations

$$[T^a, T^b] = if_c^{ab}T^c. \tag{5.5}$$

If we consider only infinitesimal deviations from the identity then we may write $\gamma(z)$ as

$$\gamma(z) = 1 - iT^a\theta_a(z). \tag{5.6}$$

Laurent expanding $\theta_a(z)$ about $z = 0$ allows us to rewrite this as

$$\gamma(z) = 1 - i\sum_{n=-\infty}^{+\infty} T^a\theta_a^n z^n \tag{5.7}$$

where the θ_a^n are the coefficients in the Laurent expansion. Making the redefinition $T^a z^n \equiv J_n^a$, and using the already defined basis from (5.5), we see that the commutation relations become

$$\begin{aligned} [T^a, T^b] = [J_n^a z^{-n}, J_m^b z^{-m}] = z^{-(m+n)}[J_n^a, J_m^b] \\ \Rightarrow [J_n^a, J_m^b] = [T^a, T^b]z^{m+n} = if_c^{ab}T^c z^{m+n} = if_c^{ab}J_{m+n}^c. \end{aligned} \tag{5.8}$$

This defines the Lie algebra $L\mathcal{G}$. The Kac-Moody algebra (KM) is then defined through the commutation relations of 5.8 plus a piece due to a central extension

$$0 \to \mathbb{R} \to KM \to L\mathcal{G} \to 0. \tag{5.9}$$

The KM commutation relations with central extension are

$$[J_n^a, J_m^b] = if^{ab}{}_c J_{m+n}^c + km\delta_{m+n,0}\gamma^{ab} I, \quad [J_m^a, I] = [I, I] = 0 \tag{5.10}$$

where the k is a constant, $\gamma^{ab} = f^{ac}{}_d f^{bd}{}_c$ is the Killing metric, and I is the generator of the central charge. This means that we can write a general element of the Kac-Moody algebra as $\Lambda_a^n J_n^a + \alpha k I$, where Λ_a^n and α are coefficients attached to the basis elements and summation is implied across repeated indices. Using the integral representation of the Kronecker delta function $\oint \frac{dz}{2\pi i} z^{p-1} = \delta_{p,0}$ we can write a general element of the Kac-Moody algebra as

$$\Lambda + \alpha k I = \Lambda_a^n J_n^a + \alpha k I = \oint \frac{dz}{2\pi i} \mathrm{Tr}\big(A(z) J(z)\big) + \alpha k I \tag{5.11}$$

where we use $\Lambda(z) = \Lambda^a(z) T^a$, $J(z) = J^a(z) T^a$, $\Lambda^a(z) = z^n \Lambda_n^a$, $J^a(z) = z^{-n-1} J_n^a$, and $\mathrm{Tr}\big(T^a T^b\big) = \gamma^{ab}$. We will now use this rewriting of a generic algebra element to explicitly form the commutator of two elements as

$$\begin{aligned}
&[\Lambda + \alpha k I, \Omega + \beta k I] \\
&= \left[\oint \frac{dz}{2\pi i} \mathrm{Tr}\big(\Lambda(z) J(z)\big) + \alpha k I, \oint \frac{dz'}{2\pi i} \mathrm{Tr}\big(\Omega(z') J(z')\big) + \beta k I\right] \\
&\equiv \oint \frac{dz}{2\pi i} \mathrm{Tr}\big([\Lambda(z), \Omega(z)]\big) J(z) + k \oint \frac{dz}{2\pi i} \mathrm{Tr}\big((\partial_z \Lambda(z)) \Omega(z)\big).
\end{aligned} \tag{5.12}$$

Upon integration, this definition for the commutator of two members of the centrally extended algebra can be shown to reproduce the correct commutation relations in (5.10). In general, we will use the notation

$$[(\Lambda(z), \alpha), (\Omega(z), \beta)] \equiv ([\Lambda(z), \Omega(z)], c(\Lambda(z), \Omega(z))] \tag{5.13}$$

for the commutator of centrally extended algebras. On the right side of this expression, the $c(\cdot, \cdot)$ is called a two-cocycle. For the case of the Kac-Moody algebra, we see that our choice of two-cocycle is

$$c(\Lambda(z), \Omega(z)) \equiv \oint \frac{dz}{2\pi i} \mathrm{Tr}\big((\partial_z \Lambda(z))\Omega(z)\big). \tag{5.14}$$

We have now defined the adjoint action of the Kac-Moody algebra on itself. The next step is to define the adjoint action of a group element on the algebra. Let us define a group element by

$$g = \exp(\Lambda + \alpha k I). \tag{5.15}$$

The adjoint action of the group on an algebra element is defined to be

$$Ad_g(\Lambda(z), \alpha) \equiv (g\Lambda(z)g^{-1}, \alpha + \oint \frac{dz}{2\pi i} \mathrm{Tr}\big((\partial_z g)\Lambda(z)g^{-1}\big) \tag{5.16}$$

where the action on the non-central part is seen as the typical adjoint action. Upon linearization of the group element, one can quickly show that this definition of the action group action gives back the adjoint algebra action we previously defined.

Having completely defined the adjoint action of the Kac-Moody algebra, we are in a position to define the coadjoint action. As usual, this will be done by requiring invariance of the pairing $\langle\cdot|\cdot\rangle : \mathcal{G}^* \times \mathcal{G} \to \mathbb{R}$ under group transformations. We select a coadjoint algebra element $(A(z), a)$ and write the pairing as

$$\langle (A(z), a) | (\Lambda(z), \alpha) \rangle = \oint \frac{dz}{2\pi i} \mathrm{Tr}\big(A(z)\Lambda(z)\big) + a\alpha. \tag{5.17}$$

If we require that

$$\begin{aligned} Ad_g \,\langle (A(z), a) | (\Lambda(z), \alpha) \rangle &= \langle Ad^*_g (A(z), a) | Ad_g (\Lambda(z), \alpha) \rangle \\ &= \langle (A(z), a) | (\Lambda(z), \alpha) \rangle \end{aligned} \tag{5.18}$$

then, making use of (5.16), we find that the coadjoint element must transform under the coadjoint action as

$$Ad^*_g \big(A(z), a\big) = \big(gAg^{-1} - a(\partial_z g)g^{-1}, a\big). \tag{5.19}$$

This is familiar since it is how a gauge connection would transform under group action.

Now that we have defined the coadjoint action we are free to employ the method from Section 4.3 in writing a geometric action on the coadjoint orbits. Following this procedure, we parametrize our group elements by τ and λ by writing $g = g(\lambda, \tau)$. We then define two parametrized adjoint algebra elements u_τ and u_λ as

$$\begin{aligned} u_\tau &\equiv g(\lambda, \tau)\partial_\tau g^{-1}(\lambda, \tau) \\ u_\lambda &\equiv g(\lambda, \tau)\partial_\lambda g^{-1}(\lambda, \tau) \end{aligned} \tag{5.20}$$

which are the Maurer-Cartan forms. We must also introduce a coadjoint element parametrized by λ and τ. The most natural way is to fix a generic coadjoint element $\big(A(z), a\big)$ and act on it via coadjoint action of the parametrized group element as

$$Ad^*_{g(\lambda,\tau)}\big(A(z), a\big) = \big(g(\lambda, \tau)Ag^{-1}(\lambda, \tau) - a(\partial_z g(\lambda, \tau))g^{-1}(\lambda, \tau), a\big). \tag{5.21}$$

Using the result of Section 4.3, we write the geometric action on coadjoint orbits of the Kac-Moody algebra

$$S_{KM} = \int d\lambda d\tau \bigg(\langle Ad^*_{g(\lambda,\tau)}\big(A(z),a\big)|[(u_\tau,c_\tau),(u_\lambda,c_\lambda)]\rangle \bigg). \tag{5.22}$$

Then, applying the definition of the algebra adjoint action and the pairing (eqn 5.17), we find that the action becomes

$$\begin{aligned} S_{KM} = \int d\lambda d\tau \bigg(\oint \frac{dz}{2\pi i} \mathrm{Tr}\big(gA(z)g^{-1}[g\partial_\tau g^{-1}, g\partial_\lambda g^{-1}]\big) \\ - \oint \frac{dz}{2\pi i} \mathrm{Tr}\big(a(\partial_z g)g^{-1}[g\partial_\tau g^{-1}, g\partial_\lambda g^{-1}]\big) \\ + \oint \frac{dz}{2\pi i} \mathrm{Tr}\big(a\partial_z(g\partial_\tau g^{-1})g\partial_\lambda g^{-1}\big)\bigg). \end{aligned} \tag{5.23}$$

If we write this action out as far as possible in terms of total z, τ, and λ derivatives we get

$$\begin{aligned} S_{KM} = \int d\lambda d\tau \oint \frac{dz}{2\pi i} \mathrm{Tr}\bigg(\partial_\lambda(A(z)g^{-1}\partial_\tau g) - \partial_\tau(A(z)g^{-1}\partial_\lambda g) \\ + \frac{a}{2}\partial_z g g^{-1}\partial_\lambda g g^{-1}\partial_\tau g g^{-1} - \frac{a}{2}\partial_z g g^{-1}\partial_\tau g g^{-1}\partial_\lambda g g^{-1} \\ + \partial_z(\partial_\tau g g^{-1}\partial_\lambda g g^{-1}) + \partial_\tau(\partial_z g g^{-1}\partial_\lambda g g^{-1}) \\ - \partial_\lambda(\partial_\tau g g^{-1}\partial_z g g^{-1})\bigg). \end{aligned} \tag{5.24}$$

Total z-derivatives vanish since there are no branch cuts and the integral is over a closed path. We also make the requirement that the parametrized group element, $g(z,\lambda,\tau)$, describes an embedding of a 3-ball into the group manifold with λ taken to be the radial coordinate and z, τ the two S^2 coordinates. With this assumption, we are also justified in forgetting the total τ-derivatives. According to our construction, we have no τ-dependence at the boundary

$\lambda = 0$ so $\partial_\tau g$ evaluated here will vanish. Thus, we can write 5.24 as

$$S_{KM} = \frac{a}{2} \int d\tau \oint \frac{dz}{2\pi i} \mathrm{Tr}\Big(\partial_\tau g(\lambda = 1) \partial_z g^{-1}(\lambda = 1) + \frac{1}{3} \int d\lambda \epsilon^{\alpha\beta\gamma} \partial_\alpha g g^{-1} \partial_\beta g g^{-1} \partial_\gamma g g^{-1} \Big) + \int d\tau \oint \frac{dz}{2\pi i} \mathrm{Tr}\big(A(z) g^{-1}(\lambda = 1) \partial_\tau(\lambda = 1) \big). \tag{5.25}$$

Observe that this action is proportional to the WZW action from 2.3. The difference here is that we see a coupling to the gauge field A. Dynamics for A is routinely given through a Yang-Mills action. In the next section we will detail how a field $\mathcal{D}$ couples similarly to Polyakov 2D gravity. This will motivate our hunt for an action giving dynamics to $\mathcal{D}$.

5.2 Virasoro

As in the previous section, we will first describe all the features of the Virasoro algebra relevant to construction of a coadjoint action. First, we start with the diffeomorphism algebra in one-dimension. This can be thought of as the algebra of vectors on a circle. This Lie algebra can be written using the definition of the Lie derivative as

$$\mathcal{L}_\xi \eta^a = \xi^b \partial_b \eta^a - \eta^b \partial_b \xi^a. \tag{5.26}$$

It is simple to see that this forms a Lie algebra since

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{\xi \circ \eta}. \tag{5.27}$$

In one-dimension we can write two arbitrary vectors on the circle as $\xi^\theta = \xi \frac{d}{d\theta}$, $\eta^\theta = \eta \frac{d}{d\theta}$ and their Lie bracket satisfies the commutation relation

$$\left[\xi \frac{d}{d\theta}, \eta \frac{d}{d\theta} \right] - (\xi \eta' - \xi' \eta) \frac{d}{d\theta}. \tag{5.28}$$

The Witt algebra is defined as the realization of these commutation relations through either of the sets of definitions

$$\xi^\theta = ie^{im\theta}\partial_\theta \equiv L_m, \quad \eta^\theta ie^{in\theta}\partial_\theta \equiv L_n \quad \text{(Angular Coordinates)} \tag{5.29}$$

$$\xi^\theta = -z^{m+1}\partial_z \equiv L_m, \quad \eta^\theta = -z^{n+1}\partial_z \equiv L_n \quad \text{(Complex Coordinates).} \tag{5.30}$$

Using either set of coordinates, we see that the commutation relations 5.28 become

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{5.31}$$

As we did for the Kac-Moody algebra, we can introduce a central extension of the Witt algebra such that

$$[(\mathcal{L}_{xi}; a), (\mathcal{L}_\eta; b)] = (\mathcal{L}_{\xi\circ\eta}; c(\xi,\eta)), \quad c(.,.) \,:\, \mathcal{G}\times\mathcal{G}\to\mathbb{R}. \tag{5.32}$$

In order for the resulting centrally-extended algebra to still satisfy the Jacobi identity, this cocycle must satisfy the 2-cocycle condition

$$c([\xi,\eta],\zeta) + c([\eta,\zeta],\xi) + c([\zeta,\xi],\eta) = 0. \tag{5.33}$$

The specific cocycle we use here is the Gelfand-Fuchs cocycle [16] defined by

$$c(\xi,\eta) \equiv \int \frac{d\theta}{2\pi}\,\xi\eta'''. \tag{5.34}$$

The centrally-extended bracket, from here on written as the normal bracket, now gives

$$[\xi,\eta] = (\xi\eta' - \xi'\eta)\frac{d}{d\theta} - \frac{ic}{48\pi}\int_0^{2\pi} d\theta\xi\eta''' \tag{5.35}$$

where the factor appearing in front of the centrally-extended contribution, with c a constant, is a convention. Using this central extension along with the definition for the L_m's from 5.29, we can write the commutation relation as

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}I \tag{5.36}$$

where I is the generator of central charge. We call the algebra defined by the commutation relation 5.36 the Virasoro algebra.

We now want to define how the Virasoro algebra and corresponding group acts on elements of the algebra through the adjoint and coadjoint action. First, we need to define an arbitrary group element. We know that we can write an arbitrary Virasoro algebra element as $A + a\frac{c}{12}I$ where a is a purely imaginary constant and $A = \sum A^n L_n$. Here, the A^n satisfy $(A^n)^* = -A^{-n}$. Elements of the Virasoro group will be defined via these exponentiated algebra elements so that

$$g_{Vir} = \exp\left(\sum A^n L_n + aI\right) = \exp\left(\oint \frac{dz}{2\pi i} A(z)L(z) + aI\right) \tag{5.37}$$

$$A(z) = \sum A^n z^{n+1}, \quad L(z) = \sum L_n z^{-n-2}. \tag{5.38}$$

From the definition 5.30, we can write

$$g_{Witt} = \exp\left(\sum A^n L_n\right) = \exp\left(-\sum A^n z^{n+1}\partial_z\right) = \exp\left(-A(z)\partial_z\right) \tag{5.39}$$

and we have that $\partial_z = -\frac{i}{z}\partial_\theta$ so this becomes

$$g = \exp\left(\frac{i}{z}A(z)\partial_\theta\right) = \exp\left(f(\theta)\partial_\theta\right). \tag{5.40}$$

Here the $f(\theta)$ are periodic over $\theta \in (0, 2\pi)$ so we have shown the group elements corresponding to the Witt algebra can be interpreted as diffeomorphisms of the circle. From here on we will identify the group elements g_{Witt}, the non centrally-extended part of the Virasoro group, with the diffeomorphisms $\tilde{Z}$ of the circle S^1 such that

$$g \to \tilde{Z}(\cdot), \quad \tilde{Z}(z) = \tilde{z}. \tag{5.41}$$

We now define the adjoint action of the Virasoro group on the Virasoro algebra as

$$g \,:\, (A(z); a) \to (A_g(z); a_g). \tag{5.42}$$

Knowing that the group transformation is none other than the familiar coordinate transformation

$$A_g(\tilde{z}) = \partial_z \tilde{Z}\, A(z) \tag{5.43}$$

let's us write

$$A_g(z) = \partial_z \tilde{Z} \left(\tilde{Z}^{-1}(z) \right) A \left(\tilde{Z}^{-1}(z) \right). \tag{5.44}$$

The central charge transforms as

$$g \,:\, a \to a_g = a + \oint \frac{dz}{2\pi i} S(z, \tilde{Z}(z)) A(z) \tag{5.45}$$

where $S(z, \tilde{Z})$ is known as the Schwarzian derivative

$$S(z, \tilde{z}) = \frac{\partial_z^3 \tilde{Z}}{\partial_z \tilde{Z}} - \frac{3}{2} \frac{(\partial_z^2 \tilde{Z})^2}{(\partial)_z \tilde{Z})^2}. \tag{5.46}$$

The central charge transformation can be shown to yield the correct algebra commutation relations upon taking an infinitesimal diffeomorphism.

Now that we have described the adjoint action of both the Virasoro group and algebra we will move on to defining the coadjoint representation. We will denote a general coadjoint element as $(\mathcal{D}^*(z), b^*)$ and define the pairing between coadjoint and adjoint element as

$$\langle (\mathcal{D}^*(z), b^*)\,, (A(z), a)\rangle = \oint \frac{dz}{2\pi i}\mathcal{D}^*(z)A(z) + b^*a. \tag{5.47}$$

As we have done before, we will determine the coadjoint action of a group element on $(\mathcal{D}^*(z), b^*)$ by requiring invariance of the pairing under this action. This results in the transformation

$$g\,:\,(\mathcal{D}^*(z), b^*) \to \left(\mathcal{D}^*_g(z), b^*_g\right) \tag{5.48}$$

$$\begin{aligned}\mathcal{D}^*_g(z) =& \mathcal{D}^*\left(\tilde{Z}^{-1}(z)\right)\left[\frac{\partial\tilde{Z}}{\partial z}\left(\tilde{Z}^{-1}(z)\right)\right]^{-2} \\ &- b^*S(\tilde{Z}^{-1}(z), z)\left[\frac{\partial\tilde{Z}}{\partial z}\left(\tilde{Z}^{-1}(z)\right)\right]^{-2}\end{aligned} \tag{5.49}$$

$$b^*_g = b* \tag{5.50}$$

This result looks much simpler if the argument z is replaced with $\tilde{z}$. Then the transformations are

$$\mathcal{D}^*_g(\tilde{z}) = \mathcal{D}^*(z)\left(\frac{\partial\tilde{Z}}{\partial z}(z)\right)^{-2} - b^*S(z, \tilde{Z}(z))\left(\frac{\partial\tilde{Z}}{\partial z}(z)\right)^{-2}. \tag{5.51}$$

From this expression it is clearer to see how $\mathcal{D}^*$ is transforming as a rank-two tensor, neglecting the central term. To construct the coadjoint action for the Virasoro algebra we now need to define adjoint and coadjoint vectors from the group elements. As detailed in Section 4, we require our group elements to be parametrized by two extra coordinates, in addition to their dependence on z. We do this by defining $g(\tau, \lambda)z = \tilde{Z}(\tau, \lambda)z \equiv \tilde{Z}(z, \tau, \lambda)$. Using these parametrized group elements, we need to define the adjoint elements u_τ and u_λ in the τ and λ directions, respectively. Using the differential operator representation, $-u_\tau\partial_\tau = g\partial_\tau g^{-1}$, and the fact that $\tilde{Z}(z) \equiv \tilde{z}$ we get that $u_\tau(\tilde{z}) = \partial_\tau \tilde{Z}(z,\lambda,\tau)$. Evaluating at z, this gives

$$u_\tau(z) = \partial_\tau \tilde{Z}(\tilde{Z}^{-1}(z,\tau,\lambda)) \tag{5.52}$$

and similarly for $u_\lambda(z)$. We must now evaluate the commutator of these two parametrized adjoint elements, this gives

$$[u_\tau, u_\lambda](\tilde{z}) = \partial_\tau \tilde{Z}(z,\lambda,\tau)\frac{\partial_\lambda\partial_z\tilde{Z}(z,\lambda,\tau)}{\partial_z\tilde{Z}(z,\lambda,\tau)} - \partial_\lambda\tilde{Z}(z,\lambda,\tau)\frac{\partial_\tau\partial_z\tilde{Z}(z,\lambda,\tau)}{\partial_z\tilde{Z}(z,\lambda,\tau)} \tag{5.53}$$

and for the central element we get

$$c(u_\tau, u_\lambda) = \oint \frac{dz}{2\pi i}\partial_z Z(z,\tilde{}\lambda,\tau)\left[((\partial_z\tilde{Z}(z,\lambda,\tau))^{-1}\partial_z)^3\partial_\tau s\right]\partial_\lambda\tilde{Z}(z,\lambda,\tau) \tag{5.54}$$

The only thing left to obtain is a parametrized coadjoint vector. We obtain this by taking a fixed element $(\mathcal{D}(z), b^*$ and acting on it by a parametrized group element, as before. This gives

$$(\mathcal{D}_g(\tilde{z}), b_g^*) = \left((\mathcal{D}(z) - b^* S(z,\tilde{Z}))(\partial_z\tilde{Z})^{-2}, b^*\right). \tag{5.55}$$

We now have all the necessary ingredients to write the coadjoint action as

$$S_{Virasoro} = \int d\tau\, d\lambda \oint \frac{d\tilde{z}}{2\pi i}\left(\mathcal{D}_g(\tilde{z})[u_\tau, u_\lambda](\tilde{z}) + b_g^* a\right). \tag{5.56}$$

Switching coordinates back to z and applying the same boundary condition we used for the Kac-Moody case one ends up finding

$$S_{Virasoro} = \int d\tau \oint \frac{dz}{2\pi i}\frac{\partial_\tau\tilde{Z}}{\partial_z\tilde{Z}}\mathcal{D}(z) - \frac{b^*}{2}\int d\tau \oint \frac{dz}{2\pi i}F_\tau(z,\tau,\lambda=1) \tag{5.57}$$

$$F_\tau = \frac{(\partial_z^2 \tilde{Z})^2 (\partial_z \tilde{Z})}{(\partial_z \tilde{Z})^3} - \frac{(\partial_z^2 \tilde{Z})(\partial_z \partial_\tau \tilde{Z})}{(\partial_z \tilde{Z})^2}. \tag{5.58}$$

Under the change of notation $\tilde{Z} \to f$, $\tau \to x^+$, $z \to x^-$ one sees that the second term in 5.57 is exactly Polyakov's 2D gravity action from 3.17. Here we see explicitly how the diffeomorphism field $\mathcal{D}$ is seen coupling to the Polyakov action in a way analogous to the gauge field A. For this reason, we are motivated to seek a dynamical action for $\mathcal{D}$.

6 Polyakov 2D Gravity from Projective Connections

In this section we will detail the use of projective connections in recovering the term in 5.57 coupling the diffeomorphism field to the metric in Polyakov's 2d gravity action. It will be shown how one can use the Thomas-Whitehead construction of a projective connection to form a projective Riemann curvature tensor. A projective Einstein-Hilbert term with the Polyakov metric inserted will be shown to recover the correct coupling term.
The main references for this section are [17], [18], [19], [20], [21], [22].

6.1 Projective Connections on $\mathbb{P}^m$

Two connections $\tilde{\Gamma}$ and Γ on a smooth manifold M related by the]projective transformation

$$\tilde{\Gamma}^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} + \delta^\mu_\nu \omega_\rho + \delta^\mu_\rho \omega_\nu \tag{6.1}$$

where ω_ρ is some 1-form, are said to be projectively equivalent. Weyl showed that, given two torsion-free linear connections, they will satisfy the projective equivalence relation 6.1 if and only if they have the same geodesics up to reparametrization. The set of connections satisfying this relationship on a smooth manifold M form a projective equivalence class. It is through the development of this line of study that Thomas and Whitehead [20] [21] [22] developed their version of projective connections. Through the use of a manifold M, having some family of geodesics, Thomas and Whitehead studied the space of projectively equivalent connections on M using a torsion-free linear connection defined on a manifold of one more dimension than that of M. To better understand the Thomas-Whitehead construction, we will first consider the construction of a projective connection on projective space $\mathbb{P}^m$.

The projective space $\mathbb{P}^m$ is defined to be the quotient of $\mathbb{R}^{m+1} - \{0\}$ under the multiplicative action of $\mathbb{R} - \{0\}$. The vector field generating the action is given by $\Upsilon = x^a \partial_a$ where x^a is the radial vector

$$x^a = \begin{pmatrix} x^0 \\ x^1 \\ \vdots \\ x^m \end{pmatrix}. \tag{6.2}$$

Functions on $\mathbb{P}^m$ should be projectively well-defined so we will define the set of functions $\mathcal{F}_\Upsilon$ on $\mathbb{P}^m$ as those functions on $\mathbb{R}^{m+1} - \{0\}$ such that

$$\Upsilon f = 0, \quad f(-x) = f(x) \Rightarrow f \in \mathcal{F}_\Upsilon. \tag{6.3}$$

Likewise, vector fields on $\mathbb{P}^m$ should be equivalent up to changes in the radial direction. Therefore we will define vector fields on $\mathbb{P}^m$ by considering the set of vector fields $\mathfrak{X}_\Upsilon$ on $\mathbb{R}^{m+1} - \{0\}$ such that

$$\mathcal{L}_\Upsilon X \propto \Upsilon, \quad X\, f(x) = X\, f(-x) \Rightarrow X \in \mathfrak{X}_\Upsilon. \tag{6.4}$$

Vectors $X, Y \in \mathfrak{X}_\Upsilon$ form an equivalence if $X - Y \propto \Upsilon$. We will use the notation X_E to refer to the equivalence class of a vector $X \in \mathfrak{X}_\Upsilon$. The set $\mathfrak{X}_{\Upsilon,E}$ of equivalence classes X_E for $X \in \mathfrak{X}_\Upsilon$ form a Lie algebra over $\mathcal{F}_\Upsilon$ with the bracket satisfying

$$[X_E, Y_E] = [X, Y]_E. \tag{6.5}$$

It is also true that given $X, Y \in \mathfrak{X}_\Upsilon$ then for any $f \in \mathcal{F}_\Upsilon$ that $Xf \in \mathcal{F}_\Upsilon$ and $Yf \in \mathcal{F}_\Upsilon$ if $X, Y \in X_E$. Thus it is well-defined to write $X_E f$ and $\mathfrak{X}_{\Upsilon,E}$ acts as derivations on $\mathcal{F}_\Upsilon$.

Our goal is to form a symmetric connection on $\mathfrak{X}_\Upsilon$ from the standard covariant derivative on $\mathbb{R}^{m+1}$. We first define a covariant derivative on $\mathfrak{X}_{\Upsilon,E}$ as a map $\nabla : \mathfrak{X}_{\Upsilon,E} \times \mathfrak{X}_{\Upsilon,E} \to \mathfrak{X}_{\Upsilon,E}$ such that

$$\nabla_{X_E}(fY_E) = f\nabla_{X_E} Y_E + (X_E f)Y_E \tag{6.6}$$

Also note that a covariant derivative is called symmetric if

$$\nabla_{X_E} Y_E - \nabla_{Y_E} X_E = [X_E, Y_E]. \tag{6.7}$$

We can now relate this covariant derivative operator to the standard covariant derivative operator D on $\mathbb{R}^{m+1}$, satisfying $D\Upsilon = \mathrm{id}$, by using a one-form ω to select a representative of each equivalence class. We define ω such that $\omega(\Upsilon) = 1$, $\mathcal{L}_\Upsilon \omega = 0$, and $j^*\omega = \omega$ (parity invariant). Given a vector field X we can now choose a representative $\tilde{X}$ from its equivalence class X_E by setting

$$\tilde{X} = X - \omega(X)\Upsilon \tag{6.8}$$

which satisfies $\omega(\tilde{X}) = 0$. We see that if $X, Y \in \mathfrak{X}_\Upsilon$ then $\tilde{X} = \tilde{Y}$ and also that $X \in \mathfrak{X}_\Upsilon$ implies $\tilde{X} \in \mathfrak{X}_\Upsilon$. Using the representative of equivalence classes defined through the use of the 1-from ω we can define a symmetric connection ∇^ω on $\mathfrak{X}_\Upsilon$ from the covariant derivative on $\mathbb{R}^{m+1}$ as

$$\nabla^\omega_{X_E} Y_E = \left(D_{\tilde{X}} \tilde{Y}\right)_E \tag{6.9}$$

6.2 Thomas-Whitehead Projective Connections

We will now develop the theory of Thomas-Whitehead (TW) for defining projective connections in the case of arbitrary smooth manifolds. For further details and proofs we recommend the review by M. Crampin and D. J. Saunders [17]. Consider some m-dimensional smooth manifold M with coordinates given by $x^\alpha \equiv (x^1, x^2, \ldots, x^m)$. The higher-dimensional manifold, on which we will construct the TW projective connection, will be defined as the volume bundle $V(M)$, which is the set of pairs $\{\pm\theta\}$ with $\theta \in \bigwedge^m T^*M$. The projection from $V(M)$ to M is given by $v : V(M) \to M$, defined as $v(\pm\theta) = x$ whenever $\pm\theta \in \bigwedge^m T^*_x M$. So if we let M have coordinates x^a then a possible coordinate for the one-dimensional fibre of v is $|v|$, with

$$\theta = v(\theta) \left(dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m\right)_x \tag{6.10}$$

where $v(\theta)$ is the coordinate with respect to the coordinate basis. Rather than use $|v|$ as the fibre coordinate, we will use the definition

$$\lambda \equiv |v|^{1/m+1} \tag{6.11}$$

since factors of $|v|^{m+1}$ would otherwise naturally appear. So we see explicitly that we have coordinates $x^a = (x^1, x^2, \ldots, x^m)$ on M and $x^\alpha = (\lambda, x^1, x^2, \ldots, x^m)$ on $V(M)$. From here on we will refer to the λ coordinate as the $\alpha = 0$ index of x^α. Notice how $V(M)$ has taken the place of $\mathbb{R}^{m+1}$ while M has taken the place of $\mathbb{P}^m$. By analogy with the previous section, we can define a natural scaling acting non-trivially only on the λ coordinate of $V(M)$. This scaling is analogous to the radial scaling from before and is generated through the vector field acting only on the λ coordinate

$$\Upsilon \equiv \lambda \frac{\partial}{\partial \lambda}. \tag{6.12}$$

Given a 1-form ω satisfying the same properties from before

$$\omega(\Upsilon) = 1, \quad \mathcal{L}_\Upsilon = 0, \quad j^*\omega = \omega \tag{6.13}$$

we can construct the TW connection via the requirement that $\nabla\Upsilon = \mathrm{id}$. Applying this condition in coordinates yields the following connection coefficients $\tilde{\Gamma}^\alpha{}_{\beta\rho}$ for the TW connection

$$\begin{aligned} \tilde{\Gamma}^0{}_{a0} = \tilde{\Gamma}^0{}_{0a} = 0, \quad \tilde{\Gamma}^0{}_{ab} = \lambda\mathcal{D}_{ab}, \quad \tilde{\Gamma}^\alpha{}_{00} = 0 \\ \tilde{\Gamma}^a{}_{0b} = \tilde{\Gamma}^a{}_{b0} - \lambda^{-1}\delta^a_b, \quad \tilde{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} \end{aligned} \tag{6.14}$$

where the Greek indices span the coordinates of $V(M)$, Latin indices span the coordintes of M, and $\Gamma^a{}_{bc}$ are the connection coefficients (Christoffel symbols) of the connection on M. Also note that $\mathcal{D}_{ab}$, forming part of the connection coefficient $\tilde{\Gamma}^0{}_{ab}$ is a tensor on the manifold M and will be identified as the diffeomorphism field in the next section.

6.3 Polyakov Coupling from Projective Curvature

The TW connection of the previous section has been well understood since the mid-1920's. The novel idea in the work of [18] is the application of TW connections to gravity. In Riemannian geometry one can form the Riemann curvature tensor as

$$[\nabla_a, \nabla_b]\kappa^c = R^c{}_{abd}\kappa^d. \tag{6.15}$$

Analogously, one can define a projective Riemann curvature tensor via the equivalent relation

$$[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta]\kappa^\rho = K^\rho{}_{\alpha\beta\delta}\kappa^\delta \tag{6.16}$$

which can be written, as usual, in terms of the connection coefficients as

$$K^\mu{}_{\nu\alpha\beta} = \partial_\alpha\tilde{\Gamma}^\mu{}_{\nu\beta} - \partial_\beta\tilde{\Gamma}^\mu{}_{\nu\alpha} + \tilde{\Gamma}^\rho{}_{\nu\beta}\tilde{\Gamma}^\mu{}_{\alpha\rho} - \tilde{\Gamma}^\rho{}_{\nu\alpha}\tilde{\Gamma}^\mu{}_{\beta\rho}. \tag{6.17}$$

Using the explicit TW connection coefficients from 6.14 one can calculate the only nonvanishing components of $K^{\mu}{}_{\nu\alpha\beta}$ to be

$$K^{a}{}_{bcd} = R^{a}{}_{bcd} + \delta_{[c}{}^{a}\mathcal{D}_{d]b} \tag{6.18}$$

$$K^{0}{}_{cab} = \lambda\partial_{[a}\mathcal{D}_{b]c} + \lambda\Gamma^{d}{}_{c[b}\mathcal{D}_{a]d} \equiv \lambda K_{cab} \tag{6.19}$$

It is interesting to note that the tensor K_{cab} satisfies the cyclic identity

$$K_{cab} + K_{abc} + K_{bca} = 0. \tag{6.20}$$

The goal is to now use this projective curvature to write a projective version of the Einstein-Hilbert action. In order to do this, we will need an $m+1$-dimensional metric to contract with the projective curvature. A natural method for building this metric is through the Dirac algebra on M

$$\{\gamma^{a}, \gamma^{b}\} = 2g^{ab}. \tag{6.21}$$

We can define the chiral Dirac matrix γ_{m+1} which is related to the coordinate λ as

$$\gamma(\lambda)_{m+1} = \frac{f(\lambda)}{m!} i^{\frac{m-2}{2}} \epsilon_{a_1 \ldots a_m} \gamma^{a_1} \ldots \gamma^{a_m}. \tag{6.22}$$

Using this extra Dirac matrix we can define an $(m+1)$-dimensional metric through

$$\{\gamma_{\alpha}, \gamma_{\beta}\} = 2G_{\alpha\beta}. \tag{6.23}$$

Explicitly, we may write the $(m+1)$-dimensional metric as

$$G_{\alpha\beta} = \begin{pmatrix} g_{ab} & 0 \\ 0 & f(\lambda)^2 \end{pmatrix}. \tag{6.24}$$

Through this we see that the volume will be given by

$$\sqrt{-\det(G_{\alpha\beta})} = \sqrt{-\det(g_{ab})} f(\lambda) \tag{6.25}$$

and thus, to maintain finite volume when integrating λ from 0 to ∞ we choose

$$f(\lambda) = e^{-2\frac{\lambda}{\lambda_0}}. \tag{6.26}$$

To avoid confusion, we should note that the TW connection $\tilde{\nabla}$ is not metric compatible with the $(m+1)$-dimensional metric $G_{\alpha\beta}$ but the reduction of both to M is metric compatible.

We are now in a position where we can write a well-defined projective version of the Einstein-Hilbert action (S_{PEH}) as

$$S_{PEH} = \int d^{m+1}x\sqrt{-G}G^{\mu\nu}K_{\mu\nu}. \tag{6.27}$$

To show that this produces the interaction term between the metric and diffeomorphism field found in [13] we will explicitly insert the Polyakov metric [6]

$$g_{ab} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & h_{\tau\tau}(\theta,\tau) \end{pmatrix} \tag{6.28}$$

where $h_{\tau\tau}(\theta,\tau) = \frac{\partial_\theta f(\theta,\tau)}{\partial_\tau f(\theta,\tau)}$, into S_{PEH}. For the $(m+1)$-dimensional metric, this gives

$$G_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & h_{\tau\tau} & 0 \\ 0 & 0 & f(\lambda)^2 \end{pmatrix}. \tag{6.29}$$

Inserting this metric, along with the projective Ricci tensor, into S_{PEH} gives

$$\begin{aligned} S_{PEH} &= \int d\lambda\, d\theta\, d\tau \frac{f(\lambda)}{\sqrt{2}}(\mathcal{D}_{\theta\theta}h_{\tau\tau} - \mathcal{D}_{\theta\tau} - \partial_\theta^2 h_{\tau\,\tau}) \\ &= \int d\lambda\, d\theta\, d\tau \frac{f(\lambda)}{\sqrt{2}}(\mathcal{D}_{\theta\theta}h_{\tau\tau} - \mathcal{D}_{\theta\tau} - \partial_\theta^2 h_{\tau\tau}). \end{aligned} \tag{6.30}$$

The only term giving non-trivial dynamics is the first term representing the coupling between the diffeomorphism field and the Polyakov metric. This is the same interaction term from 5.57, which was first discussed in [13].

7 Dynamical Diffeomorphism Field

Since the diffeomorphism field appears analogous to the gauge field A_i, which has dynamics through Yang-Mills, it is natural to pursue a dynamical action for $\mathcal{D}_{ij}$. We will now discuss how dynamics may be given to $\mathcal{D}_{ij}$ through the use of the projective connections of Thomas and Whitehead.

7.1 Thomas-Whitehead Gravity

One requirement on the dynamical action for $\mathcal{D}_{ij}$ should obviously be that the action is coordinate invariant. An easy way to achieve this is by writing an action based on the projective curvature tensor derived in the previous Section 6.3. The simplest combination of projective curvatures giving rise to dynamics for $\mathcal{D}_{ij}$ is given by the projective Kretschmann term

$$K_{\mu\nu\rho\sigma}K^{\mu\nu\rho\sigma} = \lambda^2 K_{ijk}K^{ijk} + R_{ijkl}R^{ijkl} - 4R_{ij}\mathcal{D}^{ij} + 2(m-1)\mathcal{D}_{ij}\mathcal{D}^{ij} \tag{7.1}$$

where the first term is the one providing dynamics. This term was defined in 6.19. Unfortunately, this projective Kretschmann term introduces higher-derivative terms of the metric. A natural way to evade this issue is by use of the generalized Gauss-Bonnet theorem which says for a manifold M of dimension $d \leq 4$, that

$$\int d^d x \sqrt{-g} \left(R^2 - 4R^{ij}R_{ij} + R^i{}_{jkl}R_i{}^{jkl} \right) = \chi(M) \tag{7.2}$$

where $\chi(M)$ is a topological invariant (Euler characteristic). Thus, we are motivated to write a projective version of the Gauss-Bonnet action in order to give dynamics to $\mathcal{D}_{ij}$ while avoiding higher-order derivative terms

$$\int d^{m+1} x \sqrt{-G} \left(K^2 - 4K^{ij}K_{ij} + K^i{}_{jkl}K_i{}^{jkl} \right). \tag{7.3}$$

By using the choice from section 6.3 of $f(\lambda) = e^{-2\lambda/\lambda_0}$ we can integrate over the λ-direction to arrive at the m-dimensional action describing the dynamics of the diff field

$$\begin{aligned} S = \frac{\beta_0 \lambda_0}{2} \int d^m x \bigg(& 4(2m-3)R_{ij}\mathcal{D}^{ij} - 2(m-1)R\mathcal{D} \\ & + (m-1)^2\mathcal{D}^2 - 2(2m-3)(m-1)\mathcal{D}_{ij}\mathcal{D}^{ij} + \frac{\lambda_0^2}{54}K_{ijk}K^{ijk} \bigg). \end{aligned} \tag{7.4}$$

In addition to having a dynamical action for the diff field, we also would like to include the projective Einstein-Hilbert term used to describe the coupling of the diff field to gravity. Interestingly, rather than inserting this term by hand we can get it for free by redefining the diff field as $\mathcal{D}_{ij} \to \mathcal{D}_{ij} + \mu_0 g_{ij}$. Under this field redefinition the action changes by

$$\Delta S = -\mu_0 \beta_0 (m-2)(m-3) \int d^m x \sqrt{-g} \left(R - \frac{\mu_0}{2} m(m-1) - (m-1)\mathcal{D} \right). \tag{7.5}$$

If we define μ_0 as

$$\mu_0 = -\frac{\kappa^2}{2\beta_0 (m-2)(m-3)} \tag{7.6}$$

we see that the dynamical diff action gains the terms

$$\Delta S = \frac{\kappa^2}{2} \int d^m x \sqrt{-g}\,(K + \Lambda_0)\,, \quad \Lambda_0 \equiv \frac{\kappa^2 m(m-1)}{8\beta_0(m-2)(m-3)} \tag{7.7}$$

and indeed the shift has generated the projective Einstein-Hilbert term plus a numerical constant. The κ^2 appearing in the definition of μ_0 is defined to be gravitational constant in four dimensions $\kappa^2 = \frac{c^4}{8\pi G}$. Note that this shift is only possible for $m \geq 4$ dimensions. Additionally, we can add a cosmological constant to the action and absorb the constant Λ_0 into it. The end result of these modifications is what we will refer to as Thomas-Whitehead (TW) gravity

$$\begin{aligned} S_{TW} = \frac{\kappa^2}{2} \int d^m x \sqrt{-g}\,(K + \Lambda) + \frac{\beta_0}{2} \int d^m x \sqrt{-g}\Big(& 4(2m-3)R_{ij}\mathcal{D}^{ij} - 2(m-1)R\mathcal{D} \\ & + (m-1)^2\mathcal{D}^2 - 2(2m-3)(m-1)\mathcal{D}_{ij}\mathcal{D}^{ij} + \frac{\lambda_0^2}{54}K_{ijk}K^{ijk}\Big). \end{aligned} \tag{7.8}$$

TW gravity is a theory of two rank-2 symmetry tensors g_{ij} and $\mathcal{D}_{ij}$ so we will have field equations corresponding to variation of the metric and diffeomorphism field. Calculating the field equations gives

$$\frac{\delta S_{TW}}{\delta g^{ij}} = 0 \Rightarrow R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = \frac{1}{\kappa^2}\Theta_{ij} \tag{7.9}$$

$$\begin{aligned} \frac{\delta S_{TW}}{\delta \mathcal{D}_{ij}} = 0 \Rightarrow \frac{\lambda_0^2}{27}\nabla_k K^{(ij)k} + 2(m-1)^2\mathcal{D}_{kl}g^{kl}g^{ij} - 4(m-1)(2m-3)\mathcal{D}_{kl}g^{ik}g^{jl} \\ = \frac{\kappa^2}{\beta_0}(m-1)g^{ij} + 2Rg^{ij} - 4(2m-3)R_{kl}g^{ik}g^{jl}. \end{aligned} \tag{7.10}$$

7.2 Solutions in 1+1 dim Minkowski Background

As a first step to understanding the TW gravitational theory we will study the diffeomorphism equations of motion for the simple case of $m = 2$ and $g_{ij} = \eta_{ij}$, where η_{ij} is the Minkowski metric. Recall that the shift, generating the projective Einstein-Hilbert term, only exists in $\geq$ 4-dimensions so the whole right hand side of the diffeomorphism equations of motion 7.10 vanishes. We are left with

$$\frac{\lambda_0^2}{27}\partial_k K^{(ij)k} + 2\mathcal{D}\eta^{ij} - 4\mathcal{D}^{ij} = 0. \tag{7.11}$$

Explicitly, we have the following three field equations for the three field equations for $\mathcal{D}_{00}$, $\mathcal{D}_{10}$, and $\mathcal{D}_{11}$

$$\frac{\lambda_0^2}{27}(\partial_x\partial_t\mathcal{D}_{10} - \partial_x^2\mathcal{D}_{00}) - 2(\mathcal{D}_{00} + \mathcal{D}_{11}) = 0 \tag{7.12}$$

$$\frac{\lambda_0^2}{54}(\partial_x^2\mathcal{D}_{10} - \partial_t^2\mathcal{D}_{10} + \partial_x\partial_t\mathcal{D}_{00} - \partial_x\partial_t\mathcal{D}_{11}) + 4\mathcal{D}_{10} = 0 \tag{7.13}$$

$$\frac{\lambda_0^2}{27}(\partial_t^2\mathcal{D}_{11} - \partial_x\partial_t\mathcal{D}_{10}) - 2(\mathcal{D}_{00} + \mathcal{D}_{11}) = 0. \tag{7.14}$$

By inspection, one can see that there exist two potentially simple classes of solutions for the case of $\mathcal{D}_{00} = \mathcal{D}_{11}$ or $\mathcal{D}_{00} = -\mathcal{D}_{11}$. Looking at the first case, when $\mathcal{D}_{00} = \mathcal{D}_{11}$, we see that $\mathcal{D}_{00}$ and $\mathcal{D}_{11}$ decouple from 7.13 and the equations of motion become

$$\frac{\lambda_0^2}{27}(\partial_x\partial_t\mathcal{D}_{10} - \partial_x^2\mathcal{D}_{00}) - 4\mathcal{D}_{00} = 0 \tag{7.15}$$

$$\frac{\lambda_0^2}{54}(\partial_x^2\mathcal{D}_{10} - \partial_t^2\mathcal{D}_{10}) + 4\mathcal{D}_{10} = 0 \tag{7.16}$$

$$\frac{\lambda_0^2}{27}(\partial_t^2\mathcal{D}_{00} - \partial_x\partial_t\mathcal{D}_{10}) - 4\mathcal{D}_{00} = 0. \tag{7.17}$$

Combining equations 7.15 and 7.17 and doing some rewriting, these equations become

$$(\Box + m^2)\mathcal{D}_{00} = 0, \quad (\Box + 2m^2)\mathcal{D}_{10} = 0, \quad m \equiv \frac{\sqrt{108}}{\lambda_0} \tag{7.18}$$

which are well-known Klein-Gordon equations for the components of $\mathcal{D}_{ij}$ with masses set by λ_0. We can also examine the class of solutions where $\mathcal{D}_{00} = -\mathcal{D}_{11}$. In this case, the field equations become

$$\frac{\lambda_0^2}{27}(\partial_x\partial_t\mathcal{D}_{10} - \partial_x^2\mathcal{D}_{00}) = 0 \tag{7.19}$$

$$\frac{\lambda_0^2}{54}(\partial_x^2\mathcal{D}_{10} - \partial_t^2\mathcal{D}_{10} + 2\partial_x\partial_t\mathcal{D}_{00}) + 4\mathcal{D}_{10} = 0 \tag{7.20}$$

$$\frac{\lambda_0^2}{27}(-\partial_t^2\mathcal{D}_{00} - \partial_x\partial_t\mathcal{D}_{10}) = 0. \tag{7.21}$$

By adding equations 7.19 and 7.21 and doing some rewriting, these equations become

$$(\partial_t^2 + \partial_x^2)\mathcal{D}_{00} = 0, \quad (\Box + m^2)\mathcal{D}_{10} = \rho(x,t) \tag{7.22}$$

$$\rho(x,t) \equiv -2\partial_x\partial_t\mathcal{D}_{00}. \tag{7.23}$$

We that in this case the Wick-rotated wave equation generates a source term for another Klein-Gordon equation.

Another nice way to analyze the equations of motion is by making the change to light-cone coordinates as

$$z = x - t, \quad \bar{z} = x + t \tag{7.24}$$

and then consider a solution ansatz of

$$\mathcal{D}_{ij} \sim \mathcal{D}_{ij}(z) + \mathcal{D}_{ij}(\bar{z}) \tag{7.25}$$

where the full solution is some linear combination of right and left movers. For now, consider only left-moving solutions $\mathcal{D}_{ij}(z)$. With this ansatz, the field equations become

$$\frac{\lambda_0^2}{27}\left(-\partial_z^2\mathcal{D}_{10}(z)-\partial_z^2\mathcal{D}_{00}(z)\right)-2\left(\mathcal{D}_{00}(z)+\mathcal{D}_{11}(z)\right)=0 \tag{7.26}$$

$$\frac{\lambda_0^2}{54}\left(\partial_z^2\mathcal{D}_{11}(z)-\partial_z^2\mathcal{D}_{00}(z)\right)+4\mathcal{D}_{10}(z)=0 \tag{7.27}$$

$$\frac{\lambda_0^2}{27}\left(\partial_z^2\mathcal{D}_{11}(z)+\partial_z^2\mathcal{D}_{10}(z)\right)-2\left(\mathcal{D}_{00}(z)+\mathcal{D}_{11}(z)\right)=0. \tag{7.28}$$

By specifying some explicit function for $\mathcal{D}_{ij}(z)$ we can see what a typical solution of this form looks like. Here we consider

$$\mathcal{D}_{ij}(z)=\begin{pmatrix} A_{00}\cosh(nz) & A_{10}\cosh(nz) \\ A_{10}\cosh(nz) & A_{11}\cosh(nz)\end{pmatrix}, \quad n\in\mathbb{Z}. \tag{7.29}$$

This yields an explicit solution to the field equations when

$$n=\frac{6\sqrt{3}}{\lambda_0}, \quad A_{00}=0, \quad A_{11}=-2A_{10}. \tag{7.30}$$

7.3 Hamiltonian Analysis in 1+1 dim Minkowski Background

We have shown that there exist explicit solution to the diffeomorphism field equations 7.10 for the case of $m=2$ and $g_{ij}=\eta_{ij}$. We now wish to do an in-depth analysis from the Hamiltonian perspective. This section will utilize the Dirac constraint algorithm presented in [23]. In order to pass from Lagrangian to Hamiltonian we define the conjugate momenta as

$$\begin{aligned}\Pi^{ij}(x) &= \int d^2x'\,\frac{\delta\mathcal{L}_{TW}(x)}{\delta(\partial_0\mathcal{D}_{ij}(x'))} \qquad (7.31)\\ &= \frac{\beta_0\lambda_0^2}{27}\begin{pmatrix} 0 & \dot{\mathcal{D}}_{10}(x)-\partial_x\mathcal{D}_{00}(x) \\ \dot{\mathcal{D}}_{10}(x)-\partial_x\mathcal{D}_{00}(x) & \partial_x\mathcal{D}_{10}(x)-\dot{\mathcal{D}}_{11}(x)\end{pmatrix}.\end{aligned}$$

Then, via the Legendre transform, we can define the Hamiltonian as

$$
\begin{aligned}
\mathcal{H}_{TW} &= \Pi^{ij}\dot{\mathcal{D}}_{ij} - \mathcal{L}_{TW} \\
&= \frac{27}{2\beta_0\lambda_0^2}\left(3(\Pi^{10})^2 - (\Pi^{11})^2\right) + \frac{\beta_0}{2}\left(-4(\mathcal{D}_{10})^2 + (\mathcal{D}_{00})^2 + (\mathcal{D}_{11})^2 + 2\mathcal{D}_{00}\mathcal{D}_{11}\right) \\
&+ 2\Pi^{10}\partial_x\mathcal{D}_{00} + \Pi^{11}\partial_x\mathcal{D}_{10} + \Pi^{00}\dot{\mathcal{D}}_{00}.
\end{aligned} \tag{7.32}
$$

Notice though that Π^{00} is constrained to vanish by definition of the momentum. Therefore we have what is known as a primary constraint $\phi^1 \equiv \Pi^{00}$. Another way to see the appearance of constraints is by finding the time-evolution of $\mathcal{D}_{00}$. To do this we must first define the Poisson-bracket as

$$
\{F(x), G(y)\} = \int dz \left(\frac{\delta F(x)}{\delta \mathcal{D}_{ij}(z)}\frac{\delta G(y)}{\delta \Pi^{ij}(z)} - \frac{\delta F(x)}{\delta \Pi^{ij}(z)}\frac{\delta G(y)}{\delta \mathcal{D}_{ij}(z)}\right). \tag{7.33}
$$

Now the time-evolution of some variable in phase-space is given by $\dot{F} = \{F, \mathcal{H}_{TW}\}$. Calculating the time-evolution of $\mathcal{D}_{00}$ gives $\dot{\mathcal{D}}_{00} = \dot{\mathcal{D}}_{00}$ so the evolution of $\mathcal{D}_{00}$ is completely unconstrained by evolution under the Hamiltonian. This signifies that $\dot{\mathcal{D}}_{00}$ is merely functioning as a Lagrange multiplier in the definition of the Hamiltonian $\mathcal{H}_{TW}$ enforcing the constraint $\Pi^{00} \approx 0$. Here and in the future we will use the notation $\approx$ to signify that a term only vanishes after working out a Poisson bracket all the way. This means, given a Poisson bracket $\{F, u_1\phi^1\}$, that we must evaluate the bracket to $\{F, u_1\}\phi^1 + u_1\{F, \phi^1\}$ before applying the vanishing of the constraint. This is a useful definition for us because the $\dot{\mathcal{D}}_{00}$ is not a function on the phase space and Poisson-brackets are only defined for phase space functions. Therefore, when calculating time-evolution of some operator F with the Hamiltonian $\mathcal{H}_{TW}$ we can evaluate the term containing $\dot{\mathcal{D}}_{00}$ as

$$
\{F, \dot{\mathcal{D}}_{00}\Pi^{00}\} = \{F, \dot{\mathcal{D}}_{00}\}\Pi^{00} + \dot{\mathcal{D}}_{00}\{F, \Pi^{00}\} = \dot{\mathcal{D}}_{00}\{F, \Pi^{00}\} \tag{7.34}
$$

which is well-defined.

We now wish to calculate the full algebra of constraints for $\mathcal{H}_{TW}$. First, we calculate Hamilton's equations of motion of the four dynamical variables

$$\dot{\mathcal{D}}_{11} = \{\mathcal{D}_{11}, \mathcal{H}_{TW}\} = -\frac{27}{\beta_0\lambda_0^2}\Pi^{11} + \partial_x\mathcal{D}_{10} \tag{7.35}$$

$$\dot{\mathcal{D}}_{10} = \{\mathcal{D}_{10}, \mathcal{H}_{TW}\} = \frac{3\cdot 27}{\beta_0\lambda_0^2}\Pi^{10} + 2\partial_x\mathcal{D}_{00} \tag{7.36}$$

$$\dot{\Pi}^{11} = \{\Pi^{11}, \mathcal{H}_{TW}\} = -\beta_0(\mathcal{D}_{11} + \mathcal{D}_{00}) \tag{7.37}$$

$$\dot{\Pi}^{10} = \partial_x\Pi^{11} + 4\beta_0\mathcal{D}_{10}. \tag{7.38}$$

Now we will calculate the algebra of constraints by requiring the primary constraint $\phi^1 \approx 0$ be conserved under time-evolution as

$$\begin{aligned}\dot{\phi}^1 &= \{\Pi^{00}, \mathcal{H}_{TW}\} \\ &= 2\partial_x\Pi^{10} - \beta_0(\mathcal{D}_{00} + \mathcal{D}_{11}) \approx 0\end{aligned} \tag{7.39}$$

This means we have a secondary constraint $\phi^2 = 2\partial_x\Pi^{10} - \beta_0(\mathcal{D}_{00} + \mathcal{D}_{11}) \approx 0$ whose time-evolution must also be calculated. Evolving this constraint gives

$$\begin{aligned}\dot{\phi}^2 &= \{2\partial_x\Pi^{10} - \beta_0(\mathcal{D}_{00} + \mathcal{D}_{11}), \mathcal{H}_{TW}\} \\ &= -9\beta_0\partial_x\mathcal{D}_{10} - 2\partial_x^2\Pi^{11} + \frac{27}{\lambda_0^2}\Pi^{11} - \beta_0\dot{\mathcal{D}}_{00}\end{aligned} \tag{7.40}$$

which, since it involves the Lagrange multiplier $\dot{\mathcal{D}}_{00}$, is not considered a constraint but rather a condition on the non-dynamical $\dot{\mathcal{D}}_{00}$. So in 2-dimensional Minkowski space we see that the Thomas-Whitehead theory gives two dynamical constraints (ϕ^1, ϕ^2). We now calculate their algebra to determine whether they are first-class (commuting) or second-class (non-commuting) according to Dirac's algorithm. Calculating their bracket gives

$$\begin{aligned}
\{\phi^1(x), \phi^2(y)\} &= \{\Pi^{00}(x), 2\partial_y \Pi^{10}(y) - \beta_0(\mathcal{D}_{00}(y) + \mathcal{D}_{11}(y))\} \\
&= \beta_0 \delta(x-y) \qquad (7.41)
\end{aligned}$$

so we see that we have two second-class constraints.

Since all we have are second-class constraints, we can define what is known as the Dirac bracket. We first define the matrix with entries

$$M^{ab} = \{\phi^a, \phi^b\} \qquad (7.42)$$

which explicitly evaluated gives

$$M^{ab} = \beta_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (7.43)$$

whose inverse is given by

$$\begin{aligned}
M_{ab} &= \beta_0^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (7.44) \\
&= -\beta_0^{-1} \epsilon_{ab} \qquad (7.45)
\end{aligned}$$

The Dirac bracket is defined as

$$\begin{aligned}
\{F, G\}_{DB} &= \{F, G\} - M_{ab}\{F, \phi^a\}\{\phi^b, G\} \\
&= \{F, G\} + \beta_0^{-1} \epsilon_{ab} \{F, \phi^a\}\{\phi^b, G\} \\
&= \{F, G\} + \beta_0^{-1}\{F, \phi^1\}\{\phi^2, G\} - \beta_0^{-1}\{F, \phi^2\}\{\phi^1, G\} \qquad (7.46)
\end{aligned}$$

which causes any brackets which were ≈ 0 to be promoted to full equality $= 0$. With the

Dirac bracket defined as 7.46, the only non-zero Dirac brackets between phase space variables are

$$\{\mathcal{D}_{ij}(x), \Pi^{ij}(y)\} = \delta(x-y), \quad i,j = (1,1),(1,0). \tag{7.47}$$

This simple bracket structure allows us to perform straightforward canonical quantization on the system via the identification

$$\{\mathcal{D}_{ij}(x), \Pi^{ij}(y)\} = \delta(x-y) \longrightarrow [\hat{\mathcal{D}}_{ij}(x), \hat{\Pi}^{ij}(y)] = i\hbar\delta(x-y) \tag{7.48}$$

where again i,j only take values $(1,1),(1,0)$ and the hatted phase space variables are identified as operators in our Hilbert space.

8 Conclusion

In this thesis we have outlined the emergence of a purely gravitational field $\mathcal{D}_{ij}$, the diffeomorphism field. Using the projective connections of Thomas and Whitehead, we were able to imbue this field with dynamics through the Thomas-Whitehead gravity action. The dynamical analysis in the simple case of two-dimensional Minkowski space showed that interesting solutions exist and straightforward canonical quantization is possible. Despite the huge progress in developing the theory of the diffeomorphism field through projective connections, there is still much to be done in terms of pinning down its physical relevance. In the future, a more extensive discussion of the quantum theory will be necessary. This will likely entail quantization in the physically relevant four-dimensional Minkowski space as well as extension to curved spacetime quantization. In addition to developing the quantum theory further, it will be interesting to study the relevance of the Thomas-Whitehead gravity theory to classical gravitational physics. Since the Thomas-Whitehead theory only couples gravitationally, it is a candidate for describing phenomena which are thought to only interact through gravity, such as dark matter and dark energy. Again, this research avenue will require the analysis of the Thomas-Whitehead gravity to be elevated to four-dimensions. One will need to derive the energy-momentum tensor corresponding to the Thomas-Whitehead gravity and insert classical solutions to understand potential physical relevance.

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