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Quiver representations and their dense orbits

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QUIVER REPRESENTATIONS AND THEIR DENSE ORBITS

by

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A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
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Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the May 2019 graduation.

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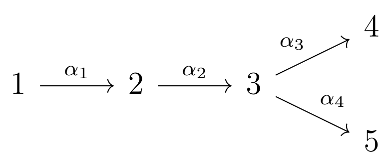
ABSTRACT

We can view quiver representations of a fixed dimension vector as an algebraic variety over an algebraically closed field K. There is an action of the product of general linear groups on each of these varieties where the orbits of the action correspond to isomorphism classes of quiver representation. A K-algebra A is said to have the dense orbit property if for each dimension vector, the product of the general linear group acts on each irreducible component of the module variety with a dense orbit. Under certain conditions, a K algebra A is representation finite if and only if it A has the dense orbit property. The implication representation finite implies the dense orbit property is always true. The converse is not true in general, as shown by Chindris, Kinser, and Weyman in [5]. Our main theorem of this thesis builds on their work to give a family of representation infinite algebras with the dense orbit property. We also give a conjectured classification of indecomposables with dense orbits.

In the future, we hope the work presented here can be used to find even more examples of representation infinite algebra with the dense orbit property to then develop deeper theory to classify algebras with the dense orbit property that are representation infinite.

PUBLIC ABSTRACT

A Quiver is a directed graph. In other words, it is a collection of a bunch of dots (or sometimes numbers) with directed arrows between them. We call the dots vertices and the arrows, well arrows. There is no restriction on the number of vertices and arrows. You can have finitely many or even infinitely many vertices and arrows! However, we only focus on finite quivers, which means we have a finite number of vertices and arrows. Here's an example of such an object.



If you place your finger on a vertex, a path is the collection of arrows you take as you move your finger along the arrows. You can do nothing, which we call a stationary path. You can be creative and take as long or short of a path as you want. But the rule is, you have to follow the direction of the arrows. You can also set some of the paths "equal to zero" and put other relations among them. The collection of all of these paths form an object called a bound quiver algebra over a field K.

The objects we are interested in are called quiver representations. You take a quiver and place vector spaces at the vertices and matrices on the arrows. The resulting object is called a quiver representation. There are two ways one can study these quiver representations, algebraically or geometrically.

From the algebraic perspective, a quiver representation corresponds to a very well known algebraic object called a module over an algebra. These modules have building blocks called indecomposable modules which you build all of your possible modules from. The algebras can have finitely or infinitely many building blocks.

From a geometric perspective, you first fix the vector spaces at the vertices. This "fixing" of the vector spaces is called the dimension vector. The matrices on the arrows are allowed to vary but they have to satisfy the relations of the bound quiver algebra. The collection of all possible matrices on the arrows is called the affine module variety of a fixed dimension vector. Long story short, this object can be described by the solutions of polynomials of multiple variables and you can use geometric tools to study quiver representations.

We are interested in algebras that have module varieties with a special property. For each dimension vector, the module variety must have a dense quiver representation in each of its irreducible components. A dense orbit can be thought of as an object that "fills up" the space it lives in. The types of algebras with such properties are called dense orbit algebras or algebras with the dense orbit property.

We can then relate both the geometric and algebraic perspectives together and ask, which algebras are dense orbit algebras? If we have finitely many building blocks, then it is very well know that the associated algebra is of dense orbit algebra. If we have a dense type algebra, does it mean we have finitely many building blocks? The answer is no! In some cases, this is true but not always. Our long term goal is to find more of these dense orbit algebras that have infinitely many building blocks to hopefully classify them.

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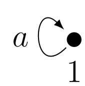
CHAPTER 1
INTRODUCTION

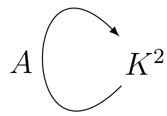
We work over a algebraically closed field K and let A be a finite dimensional basic K-algebra. A quiver is nothing more than a directed graph. (Definition 2.1) Taking all of the paths and vertices to be a K-basis for a vector space and defining multiplication as the concatenation of paths, we obtain a K-algebra from the quiver. Every finite dimensional basic K algebra is isomorphic to a quotient of a path algebra. (Theorem 2.12) We call these types of algebras bound quiver algebras. A representation of a quiver is the object we obtain when we insert K-vector spaces at the vertices and linear maps on the arrows. (Definition 2.13) This object is the visualization of modules of a given algebra. As a running example, consider the following quiver with relations:

Example 1.1.



$$a^n = 0 \quad n \in \mathbb{N}$$

we can label the vertex 1 and the arrow a. We have a trivial path at 1 which we will call ϵ_1 and paths that follow a multiple times. The relation $a^n = 0$ gives us a finite K basis of $\epsilon_1, a, a^2, \cdots, a^{n-1}$. Observe that the associated algebra to this quiver will be isomorphic to $K[T]/(T^n)$. We label this algebra B. A representation puts a vector space at a the vertex and a matrix on the arrow. For example:



We further require that the matrices satisfy the path relations on the quiver. That is, $A^n = 0$. So we can let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in our representation example.
We now present our main object of study. First we fix a dimension vector, that is we fix the dimensions of of all the vector spaces. In our example, we pick 2. If we consider all of the possible linear maps as a set that satisfy the associated relations of the quiver, this forms an affine variety (Definition 2.35). We let $\text{mod}(B, 2) := \{A \in \text{Mat}_{2\times 2}(K) : A^n = 0\}$. This set is an affine variety and it is called the module variety of an algebra B with dimension vector 2.

The algebraic group $GL_2(K)$ acts on this set by conjugation, where the action itself is a morphism of varieties:

$$GL_2(K) \times \text{mod}(B, 2) \to \text{mod}(B, 2)$$

The orbits of this action are isomorphism classes of quiver representations. The ones we are interested are the ones that are dense in the Zariski Topology. We say that an algebra has the dense orbit property if the following holds. If for each dimension vector d, the module variety has a dense orbit in each of its irreducible components (Definition 2.37).

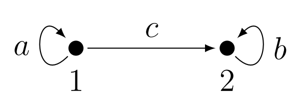
There are two main goals of this thesis. One is to give a proof method that shows when a infinite representation type algebra does not have the dense orbit property. Second, is to give an example of a representation infinite algebra that does. It is known that representation finite implies the dense orbit property. The converse is true under certain conditions on the algebra. For example, the converse is true if one of the following is true: the bound quiver algebra has a vanishing radical square [3], has a preprojective component [5], or the bound quiver algebra is a string algebra in [5]. In a paper by Chindris, Kinser and Weyman [5], the converse does not always hold as they give an example of one.

Hoshino and Miyachi in [7] give a list of 2-point tame algebras that gives us a good starting place to start looking for more algebras with the dense orbit property.

Our goal is to find more examples of algebras with the dense orbit property to help develop deeper theory with the dense orbit property.

We organize the rest of the thesis as follows. Chapter 2 is all prelimnary theory on quivers, algebraic geometry and some representation theory. Chapter 3 gives a classification of local algebras with the dense orbit property. Chapter 4 goes through some bound quiver algebras presented by Hoshino and Miyachi in [7]. Our main result is the following (Theorem 4.6).

Theorem. *Let* Γ *be the algebra given by the quiver*



with relations $ca - bc = b^n = a^3 = b^2c = 0$ *with* $n \in \mathbb{N}$. *Then* Γ *has the dense orbit property for all* $n \in \mathbb{N}$ *and is representation infinite for* $n \geq 6$.

CHAPTER 2
PRELIMINARIES

2.1 Background on Quivers

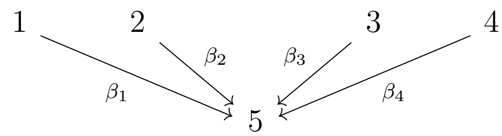
We let K denote an algebraically closed field for the rest of this paper. We begin by establishing important definitions, conventions, and results of the theory of quivers and their representations. For a more in depth background, [1] offers a complete introduction to quivers and their representations.

Definition 2.1. A **quiver** Q is a collection of 4 objects (Q_0, Q_1, s, t) where Q_0 is called the set of vertices, Q_1 is called the set of arrows and maps $s, t : Q_1 \to Q_0$ where for $\alpha \in Q_1$, $s(\alpha)$ is called the source of the arrow and $t(\alpha)$ is called the target.

In other words, a quiver is nothing more than a directed graph with no restrictions as to the number of arrows between two points, the existence of loops and so forth. One may ask why the term quiver is used rather than graph. This is to prevent any ambiguities since the word graph is used in many different contexts. That is, a graph can be oriented, may be without loops and etc. The word quiver was coined in to remove any preconceived notions from graph theory.

Usually an arrow $\alpha \in Q_1$ is denoted by $\alpha : a \to b$ with source $s(\alpha) = a$ and target $t(\alpha) = b$. When drawing a quiver, dots represent the vertices (or even just numbers can represent vertices) and arrows are drawn in whose tail is the source and the end point is the target. You can be creative or un-creative when drawing quivers.

For example,



is a drawing where $Q_0 = \{1, 2, 3, 4, 5\}$ is our set of vertices and $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ is our set of arrows. β_1 has source 1 and target 5.

We restrict ourselves to looking only at finite quivers, that is that the number of vertices and arrows are finite.

Definition 2.2. Given a quiver Q, we define a **path** to be a collection of arrows

$$a_n \cdots a_2 a_1$$

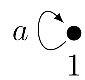
such that $t(a_i) = s(a_{i+1})$

We can then introduce notions of length of paths, which is the number of arrows it is composed of. We introduce trivial paths, those of length 0 to associate each vertex to a path. Arrows have length one. There are various familiar terms from graph theory that have the same meaning when talking about quivers. Things like cycles, loops, and so forth. With the definition of a path, we can now introduce the path algebra.

Definition 2.3. We define the **path algebra** KQ for a quiver Q to be the algebra with the set of all paths to be the basis for the K vector space and with multiplication on paths to be defined as follows. Let τ, σ be two paths then $\tau\sigma$ is the new path $\tau\sigma$ if $t(\sigma) = s(\tau)$ otherwise it is zero.

Consider the following quiver as an example:

Example 2.4. Consider the quiver



with one vertex and one loop. Paths are iterations of the same loop, $a, a^2, \cdots$ with the trivial path ϵ_1. The algebra given by this quiver is the set of polynomials of one variable. Note that this quiver gives an infinite dimensional algebra.

There is a lot of borrowed terminology from graph theory like walks, loops, cycles, connectedness, and paths which essentially mean the same thing for quivers. In the example, we obtained an infinite dimension path algebra. One can show that a path algebra is finite dimensional if and only if there are no cycles (acylic) in the quiver and that the quiver itself is finite dimensional. If we put some sort of restrictions or relations on the paths of the quiver from Example 2.4, like $a^4 = 0$, then we obtain a finite dimensional algebra. We make this more precise.

Let Q be a finite quiver. The path algebra KQ is an associative algebra with an identity and it is finite dimensional only when Q is acylic. We want to study finite dimensional quotients of the path algebra KQ when KQ itself may not be finite dimensional. These ideals we mod out by are called admissible which are defined below. The motivation is that every finite dimensional basic K algebra is isomorphic to some bound quiver algebra. For the rest of the paper, Q denotes a finite connected quiver unless otherwise stated.

Given a quiver Q, R_Q (or sometimes just denoted by R in this paper) denotes the arrow ideal of KQ. That is, it is the 2-sided ideal generated by the arrows of Q.

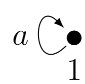
Definition 2.5. An ideal $I \subset KQ$ is said to be **admissible** if there exist $m \geq 2$ such that

$$R^m \subset I \subset R^2$$

It follows from the definition that an ideal I of KQ contained in R^2 is admissible if and only if it contains all paths whose length is large enough. In addition, we can also say an ideal is admissible if and only if for every cycle w in Q, there is some integer $s \geq 1$ such that $w^s \in I$. Thus, if Q is acyclic then any ideal I contained in R^2 is admissible.

Definition 2.6. Let I be an admissible ideal of KQ. The pair (Q, I) is said to be a **bound quiver**. The quotient algebra KQ/I is said to be the algebra of the bound quiver (Q, I) or simply stated, a **bound quiver algebra**.

Example 2.7. Consider the quiver and relation



$$a^n = 0 \quad n \in \mathbb{N}$$

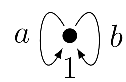
The path algebra for this quiver is $KQ \cong K[T]$ and the ideal $I = (a^n)$ is admissible. The bound quiver algebra is isomorphic to $K[T]/(T^n)$.

In the example, we obtained a finite dimensional algebra. This is always the case when we mod out by an admissible ideal.

Proposition 2.8. *Let Q be a finite quiver and I be an admissible ideal of KQ. The bound quiver algebra KQ/I is finite dimensional.*

The proof idea of this statement relies on the fact that $R^m \subset I \subset R^2$ for some m large enough. Thus it is sufficient to prove KQ/R^m is finite dimensional and since there exist a surjection $KQ/R^m \to KQ/I$, we have that KQ/I must be finite dimensional. If I is not admissible, then KQ/I need not always be finite dimensional. We present a classical example taken from [1] :

Example 2.9. Let Q be the quiver:



let $I = (ab, b^2)$. Note that I is not admissible since $a^m \notin I$ for any $m \in \mathbb{N}$. One can show that the associated bound quiver algebra KQ/I is not finite dimensional.

Definition 2.10. Let A be a K-algebra with a complete set $\{e_1, \cdots, e_n\}$ of primitive orthogonal idempotents. The algebra is called **basic** of $e_iA \ncong e_jA$ for $i \neq j$.

For an algebra A with a complete set of orthogonal idempotents, there is a basic algebra which has an equivalent of module category. This is the so called basicazation of the algebra A. So we assume all of our algebras are basic finite dimensional K algebras.

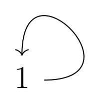
For a finite dimensional basic K algebra A with a complete set of primitive orthogonal idempotents $\{e_1, \cdots, e_n\}$ we can construct the associated quiver Q_A (which to we refer simply as Q unless otherwise noted) as follows:

(a) the set of vertices Q_0 are the numbers $1, 2, \cdots, n$ that correspond to the primitive orthogonal idempotents.

(b) If we take two points $a, b \in Q_0$, the arrows $\alpha : a \to b$ are in a one to one correspondence with the vectors of the basis of the K vector space $e_a(\text{rad } A/\text{rad }^2 A)e_b$. A is finite dimensional and hence so is every vector space of that bizarre looking form above. Therefore we can justify that the set of arrows, Q_1, is finite.

We illustrate this process with an example.

Example 2.11. Let $A = K[T]/(T^n)$ for some $n \in \mathbb{N}$. Our only idempotent is 1. So our set of vertices is $Q_0 = \{1\}$, the single vertex. Our radical is the ideal $(\bar{T})$ (where $\bar{T} = T + (T^n)$) and so $\text{rad }^2 A = (\bar{T}^2)$ and thus $\text{rad } A/\text{rad }^2 A$ is one dimensional and thus our associated quiver is



.

Note that the path algebra of the associated quiver need not be isomorphic to the algebra itself. But there is an admissible ideal I such that $A \cong KQ/I$. This leads to the well known result that lets us think of finite dimensional basic K algebras as bound quiver algebras.

Theorem 2.12. *Let A be a basic and connected finite dimensional K-algebra. Let Q be the associated quiver to A. There exist an admissible ideal I of KQ that $A \cong KQ/I$.*

Quivers give a way to sort of visualize the algebras themselves. So we usually refer to the associated quiver with some relations whenever we talk about algebras in this paper.

What about the modules of an algebra? We introduce the concept of a quiver representation which at the end of the day, is a pseudo visualization of a module. Asking questions about the algebras and their associated modules can be transformed into asking questions about quiver path algebras and their associated quiver representations.

A lot of prior motivation to the study of quivers involved the exploration of quiver representations. You take a quiver, put vectors spaces at each of the vertices with linear maps on the arrows and violà! You have yourself a quiver representation. Here's the formal way to define this object:

Definition 2.13. A K-**linear representation** M of a quiver Q is defined as follows.

- To each point $a \in Q_0$ there is an associated K vector space M_a.
- To each arrow $\alpha : a \to b$ there is an associated K linear map $\varphi_\alpha : M_a \to M_b$.

It is called finite dimensional if each vector space M_a is finite dimensional. We denote a representation M as $M = (M_a, \varphi_\alpha)$. A map $d : Q_0 \to \mathbb{N}$ is called a **dimension vector** for Q. The associated dimension vector d for a representation M is defined as $d(a) := \dim(M_a)$ for all $a \in Q_0$.

Definition 2.14. A **sub-representation** $M' \subset M = (M_a, \varphi_\alpha)$ is a collection of vector subspaces $M'_a \subset M_a$ with linear maps φ'_α which are defined to be the restrictions of the maps φ_α to M'_a.

Definition 2.15. Suppose we have a representation $M = (M_a, \varphi_\alpha)$ for a quiver Q. For any non trivial path $w = \alpha_n \cdots \alpha_1$ in Q, an **evaluation** of M on the path is the composition of maps defined as

$$\varphi_w = \varphi_{\alpha_n} \cdots \varphi_{\alpha_1}$$

This definition can be extended to K linear combinations of paths with a common source and a common target. Let ρ be such a combination. That is

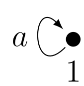
$$\rho = \sum \lambda_i w_i$$

where $\lambda_i \in K$ and $\{w_i\}$ is a collection of paths with a common source and target in Q. Then

$$\varphi_\rho = \sum \lambda_i \varphi_{w_i}$$

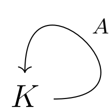
Definition 2.16. A quiver representation, $M = (M_a, \varphi_\alpha)$ of Q, is said to be **bound by** I or said to **satisfy the relations** of I if we have $\varphi_\rho = 0$ for all relations $\rho \in I$.

Example 2.17. We start with our favorite quiver with relation



$a^2 = 0$

A representation M that is bound by the relation $a^2 = 0$ is



where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Moving away from the example, quiver representations were originally introduces to treat problems from linear algebra. For example, the classification of tuples of subsapces of a prescribed vector space. The reader can refer to the introduction of Brion's notes [4] for a greater background in the origins of quiver representations.

We denote $\mathrm{Rep}_K(Q, I)$ (or $\mathrm{Rep}(Q, I)$)to be the full category of K linear quiver representations for a quiver Q that are bound by I. We denote $\mathrm{rep}_K(Q, I)$ (or $\mathrm{rep}(Q, I)$) to be the category of finite dimensional quiver representations that are bound by I.

There is a one to one correspondence between quiver representations of a quiver Q that are bound by I to modules of the bound quiver algebra $A \cong KQ/I$. Formally put:

Theorem 2.18. *Let $A = KQ/I$ where Q is a finite connected quiver and I is an admissible ideal of KQ. There exist a K linear correspondence of categories*

$$F : \mathrm{mod}\, A \to rep(Q, I)$$

that restricts to an equivalence of categories $F : \mathrm{mod}\, A \to rep(Q, I)$ where $\mathrm{mod}\, A$ is the category of finitely generated left A modules.

To illustrate why this is true, we describe the proof method by the way of an example.

Example 2.19. Consider the algebra given by the quiver

$$1 \xrightarrow{\alpha} 2$$

The path algebra is generated by the elements $\epsilon_1, \epsilon_2, \alpha$. Then $A = K(\epsilon_1, \epsilon_2, \alpha)$. We first choose a quiver represention and illustrate how it is "translated" to a module. Let M be the representation

$$K^2 \xrightarrow{A} K^2$$

where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Let $G(M) = K^2 \oplus K^2$, essentially the direct sum of the vertices. The action of the left A module structure on $G(M)$ is defined as follows: To define a module structure is it sufficient to define the products of the form $w \cdot x$ where $x = (x_1, x_2) \in G(M)$ and w is a path of the quiver. By cases, consider the case when w is a stationary path. In our example, we have only two. Then:

$$\epsilon_1 \cdot x = (x_1, 0)$$

$$\epsilon_2 \cdot x = (0, x_2)$$

It is the composition of an projection map with an inclusion. If we take our only path α from 1 to 2 we get

$$\alpha \cdot x = 0 \oplus Ax_1.$$

This gives $G(M)$ the structure of a KQ module. Next given a KQ module N, say $N = (\epsilon_1 + \alpha)$, that is the left KQ module generated by $\epsilon_1 + \alpha$, we will show that there is an associated quiver representation. The vector space that corresponds to the vertext one 1 is the space $M_1 = \epsilon_1 N$ and likewise for $M_2 = \epsilon_2 N$. So far we have

$$(\epsilon_1) \to (\alpha)$$

The map is given as follows:

If I take $x \in M_1$, define $\varphi_\alpha : M_1 \to M_2$ as follows:

$$\varphi_\alpha(x) = \alpha \cdot x = \epsilon_2(\alpha \epsilon_1 x) \in M_2$$

and since N is an A module this is a linear map. This method is important as it makes working with modules of a "complicated" algebra to matrices and vector spaces and gives ability to apply the plethora of linear algebra techniques.

Moving on, we know that quiver representations are identified with modules. We offer analogous definitions for simple representations and indecomposable representations that correspond to the definitions of simple and indecomposable for modules.

Definition 2.20. A representation $M \in \text{Rep}(Q, I)$ is said to be **irreducible** or **simple** if has no non trivial subrepresentations. That is, the only sub representations are V and 0. This corresponds to the definition of a simple module.

Definition 2.21. Let $M, N \in \text{Rep}(Q, I)$ where $M = (M_a, f_\alpha)$ and $N = (N_a, g_\alpha)$ We define the direct sum $M \oplus N$ as the representation with vector spaces $M_a \oplus N_a$ for $a \in Q_0$ and direct sum of maps $f_\alpha \oplus g_\alpha$ over each arrow $\alpha \in Q_1$

$$f_\alpha \oplus g_\alpha : M_{s(\alpha)} \oplus N_{s(\alpha)} \xrightarrow{\begin{pmatrix} f_\alpha & 0 \\ 0 & g_\alpha \end{pmatrix}} M_{t(\alpha)} \oplus N_{t(\alpha)}$$

Definition 2.22. A representation $M \in \text{Rep}(Q, I)$ is said to be **indecomposable** if whenever $M = M_1 \oplus M_2$ for subrepresentations $M_1, M_2 \subset M$ then either $M_1 = 0$ or $M_2 = 0$. This corresponds to the definition of an indecomposable module.

Definition 2.23. Given two quiver representations, $M, N \in \text{Rep}(Q, I)$, we define $\text{Hom}(M, N)$ to be the collection of all quiver morphism from M to N. The **endomoprhism** ring of a representation M is denoted by $\text{End}(M) := \text{Hom}(M, N)$

Theorem 2.24. *A representation $M \in Rep(Q, I)$ is indecomposable if and only if $End(M)$ is a local ring.*

2.2 Background on Algebraic Geometry

For the rest of this paper, we only consider bound quiver algebras $A \cong KQ/I$ and finite dimensional quiver representations, rep(Q, I), unless otherwise stated.

In this subsection, we present various key definitions and results from algebraic geometry to precisely define our object of interest, the affine module variety. For a more in depth introduction to algebraic geometry, [10] offers a complete background in basic algebraic geometry in Appendix A along with a complete background in algebraic groups in Chapter II. Or one can explore algebraic geometry in [6] in Chapter 15 for a more algebraic perspective.

Recall that given a polynomial $f \in K[x_1, \cdots, x_n]$, can be regarded as a K valued function on K^n in the usual way by "plugging in values for the variables." Or formally put:

$$a = (a_1, \cdots, a_n) \mapsto f(a) = f(a_1, \cdots, a_n)$$

These are called regular functions on K.

Definition 2.25. A K valued function $f : V \to K$ for V an n dimensional K vector space is said to be **regular** if it is given in terms of some polynomial with respect to a basis. That is, given a basis $v_1, \cdots, v_n$ of V and a vector $v = \sum av_i$

$$f(v) = f(\sum a_i v_i) = p(a_1, \cdots, a_n)$$

for a suitable polynomial p. The algebra of regular functions is denoted $O(V)$.

Definition 2.26. let $f \in O(V)$. The zero set of f is defined to be

$$Z(f) := \{v \in V : f(v) = 0\}$$

which can be generalized to a given set $S \subset O(V)$

$$Z(S) := \{v \in V : f(v) = 0 : \forall f \in S\}$$

The collection $\mathcal{T}$ consisting of all of the zero sets of V satisfies all of the three axioms for a topological space.

Definition 2.27. The **Zariski** topology on V is the topology where the zero sets are defined to be closed sets to form our topology.

Definition 2.28. A set Z together with a K algebra $O(Z)$ of K valued functions on Z is called an **affine variety** if there is a closed subset $X \subset K^n$ for some n and a bijection $\varphi : Z \to X$ which identifies $O(X)$ with $O(Z)$. That is, $\varphi^* : O(X) \to O(Z)$ given by $f \to f \circ \varphi$ is an isomorphism.

The functions from $O(Z)$ are called the regular functions and the algebra $O(Z)$ is called the coordinate ring of Z. And again, we give this set the zariski topology. The closed sets are the zero sets defined as:

$$V_Z(I) = \{z \in Z : f(z) = 0 \text{ for all } f \in I\}$$

where I is an ideal in $O(Z)$. Note that every closed subset of K^n is an affine variety.

Any closed subset of K^n is an affine variety. Our module variety below is an example of an affine variety.

Recall that topological space is said to be irreducible if it cannot be written as the union of two proper closed sets.

Theorem 2.29. *Every affine variety X can be written as a finite union of irreducible closed subsets X_i:*

$$X = X_1 \cup \cdots \cup X_n$$

For a given affine variety X, if it is irreducible then $O(X)$ is a domain. The field of fractions of $O(X)$ is denoted by $o(X)$. Recall the notion of transcendence degree. K is our field with commutative K-algebra A. A collection $a_1, \cdots, a_n \in A$ are said to be algebraically independent if they do not satisfy a non trivial polynomial equation, $F(a_1, \cdots, a_n)$, for $F \in K[x_1, \cdots, x_n]$. Or we can equivalently say that the canonical homomophism of K algebras $K[x_1, \cdots, x_n] \to A$ defined by $x_i \to a_i$ is injective. The transcendence degree is defined to be the degree of the field extension L/M. That is, is is the maximal number of algebraically independent elements in L over the field M.

Definition 2.30. Let X be an irreducible affine variety and $o(X)$be the field of rational functions. The dimension of X is defined by

$$\dim X = \mathrm{tdeg}_K o(X)$$

If X is reducible with $X = \cup X_i$ the irreducible decomposition, then

$$\dim X = \mathrm{max}_i \dim X_i$$

Example 2.31. If V is a n dimensional K vector space, $\dim V = n$. If X is an affine variety and $U \subset X$ is an open subset, $\dim U = \dim X$.

Definition 2.32. The Zariski tangent space is defined as follows. A tangent vector v at a point $x \in X$ for an affine variety X is a K linear map $v : O(X) \to K$ that satisfies:

$$v(f \cdot g) = f(x)v(g) + g(x)v(f)$$

for all $f, g \in O(X)$. This map is called a derivation of $O(X)$ in $x \in X$. Observe that for any polynomial $F \in K[x_1, \cdots, x_n]$:

$$v(F(f_1, \cdots, f_m)) = \sum_j \frac{\partial F}{\partial x_i}(f_1(x), \cdots, f_,(x)) \cdot v(f_j).$$

This implies that a derivation at $x \in X$ is completely determined by the values on a generating set of the algebra $O(X)$ and that a linear combination of derivations is a derivation. Thus the set of derivations at $x \in X$ forms a finite dimensional subspace of $\mathrm{Hom}(O(X), K)$. $T_x(X)$ is the set of all tangent vectors in x and is called the Zariski Tangent space at $x \in X$ of a variety X.

If we restrict to ideals $I \subset O(V)$, the zero set $V(I)$ has properties that allow us to define it as a closed set and construct the so called Zariski topology on V.
Our variety will be the collection of all possible representations bound by an admissible ideal I of a fixed dimension vector. Recall the definition of the general linear group.

$$\mathrm{GL}_n = \{A \in M_n(K) : \det(A) \neq 0\}$$

with coordinate ring

$$O(\mathrm{GL}_n) = K[x_{i,j}]_{det}$$

Definition 2.33. We say that a group G is an algebraic group if it a closed subgroup of the general linear group GL_n.

Definition 2.34. Let G be an algebraic group and X an affine variety. An **algebraic action** of the group G on X is a group action $G \times X \to X$ that is also a morphism of varieties.

Our group will be a product of general linear groups that will act on a module variety, a set that contains all representations of a given dimension vector.

Definition 2.35. Let $d : Q_0 \to \mathbb{N}$ be a dimension vector. Let A be an $K-$algebra and Q the associated quiver such that $A \cong KQ/I$ for an admissible ideal I. The **affine module variety** $\mathrm{mod}(A, d)$ is

$$\mathrm{mod}(A,d) := \left\{ M \in \prod_{a \in Q_1} Mat_{d_{s(a)} \times d_{t(a)}}(K), | M(r) = 0 \; \forall r \in I \right\}$$

The algebraic group $\mathrm{GL}(d) = \prod \mathrm{GL}(d_i)$ acts on the module by conjugation as follows. Consider the object $(x_\alpha)_\alpha$. We define the action on x_α. If $\alpha : a \to b$ and $g \in \mathrm{GL}(d)$ then

$$g \cdot x_\alpha := g_b x_\alpha g_a^{-1}$$

We illustrate the definitions with yet another example.

Example 2.36. Consider the quiver

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

with no relations. Let A denote the associated path algebra. Consider the dimension vector $d = (2, 2, 2, \cdots, 2)$ Then a arbitrary representation is of the form $(A_1, \cdots, A_n)$ where $A_i \in \mathrm{Mat}_{2\times 2}(K)$. Our module variety $\mathrm{mod}(A, d)$ is then the collection of all n tuples of 2×2 matrices. An element $g = (g_1, \cdots, g_n) \in GL(d)$ acts on $x = (A_1, \cdots, A_n) \in \ \mathrm{mod}\ (A, d)$ as follows:

$$g \cdot x = (g_2 A_1 g_1^{-1}, \cdots g_n A_n g_{n-1}^{-1})$$

Moving away from this example, this looks very familiar. Recall that two representations $M = (M_a, \varphi_\alpha)$, $N = (N_a, \phi_\alpha)$ of a quiver Q are isomorphic if there exist a collection of invertible maps (f_a) such that for arrow path $\alpha : a \to b$ the diagram below commutes, or that $\varphi = f_a \phi_\alpha f_b^{-1}$.

$$\begin{array}{ccc} M_a & \xrightarrow{\varphi_\alpha} & M_b \\ {\scriptstyle f_a}\downarrow & & {\scriptstyle f_b}\downarrow \\ N_a & \xrightarrow{\phi_\alpha} & N_b \end{array}$$

The orbits of the action are precisely the set of isomorphism classes of quiver representations. We can then use study representations geometrically.

2.3 Background on Representation Theory

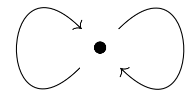
In this section, we offer key results and definitions from both a geometric approach and an algebraic approach to quivers and their representations. We also present well established results on algebras based on invariant theorectic properties presented in [5] and [3]. There are some great notes by Michel Brion in [4] that offer both an algebraic approach and a geometric approach to quivers and their representations for a more in depth background. There are also some great notes by Ryan Kinser in [9] for a brief introduction to the geometry of representations of algebras.

Definition 2.37. We say an algebra A has the **dense orbit property** if for each dimension vector d, $\mathrm{GL}(d)$ acts on each irreducible component of $\mathrm{mod}(A, d)$ with a dense orbit.

Finding which algebras have the dense orbit property is our main subject of interest. With the definitions given, the representations of an algebra A can be studied geometrically by studying the affine module variety $\mathrm{mod}(A, d)$ of a fixed dimension vector d under the actions of the general linear groups $\mathrm{GL}(d)$. The orbits of the action correspond to isomorphism classes of representations. Therefore, we can use some the tools of algebraic geometry to study quiver representations. Our "big picture" interest involves an attempt to classify algebras whose module varieties satisfy certain invariant theoretic properties so that we can compare them to classical representation theoretic properties. But we mainly focus on the dense orbit property.

We give off three key definitions of quivers that can be extended to bound quiver algebras.

Definition 2.38. We say that a quiver Q is of **finite type** if it has finitely many isomorphism classes of indecomposables. A bound algebra A is of **finite representation type** if it has finitely many isomorphism classes of indecomposable modules.

Definition 2.39. We say that a quiver Q is of **wild type** if there is an exact embedding rep(Ω) $\to$ rep(Q) where Ω is the double loop quiver  . Likewise a bound algebra $A \cong KQ/I$ is **wild** if there is an exact embedding mod($K\Omega$) $\to$ mod(A) where $K\Omega$ is the path algebra of the double loop quiver Ω.

Definition 2.40. We say that a quiver Q is of **tame type** if it is neither finite type or wild type. A bound algebra $A \cong KQ/I$ is said to be **tame** if it is neither finite representation type or wild.

It is known that a quiver is either of finite type, tame, or wild. There is a wonderful theorem of Gabriel that classifies quivers of finite type.

Theorem. *A quiver is of finite type if and only if the underlying undirected graph is a union of Dynking graphs of type A, D, E. [4].*

What interests us is the proof method in proving Gabriel's Theorem. In [4], Brion presents the following:

Theorem 2.41. *Let G be an algebraic group acting on a affine variety X algebraically. Then for $x \in X$, let $G \cdot x = \{g \cdot x : g \in G\}$ denote the orbit of $x \in X$ and $G_x = \{g \in G : g \cdot x = x\}$ denote the stabilizer of $x \in X$.*

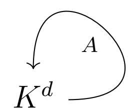
1. $\dim \overline{G \cdot x} = \dim G \cdot x = \dim G - \dim G_x$.

2. *$G \cdot x$ is dense in $\overline{G \cdot x}$*

3. *$\overline{G \cdot x} \setminus G \cdot x$ is the union of orbits of strictly smaller dimension.*

We give an example of a bound quiver algebra of finite representation type.

Example 2.42. We given an example of a representation finite algebra. Once again, we use our favorite algebra, $A = K[T]/(T^n)$. We already know the associated quiver with its associated admissible ideal. Thus consider an arbitrary representation M:



where $A^n = 0$. A is just a nilpotent matrix! Using linear algebra, we know that its eigenvalues are only zero, we know that we can find an invertible matrix P that gives us the associated Jordan form of A which we denote by J_A and write $PAP^{-1} = J_A$. In terms of quiver representations this just means that the representations (K^d, A) and (K^d, J_A) are isomorphic as quiver representations. In terms of the affine module variety and the algebraic action $\mathrm{GL}(n) \times \mathrm{mod}(A, n) \to \mathrm{mod}(A, n)$ defined as $(g, x) \to gxg^{-1}$, this implies that (K^d, A) and (K^d, J_A) lie in the same orbit.

The $r \times r$ Jordan blocks with eigenvalue zero are denoted as $J_r(0)$. Thus we can decompose our arbitrary quiver representation in terms of Jordan blocks

$$(K, J_1(0)), \cdots, (K^n, J_n(0)).$$

Thus A is of finite representation type in this example.

One notes the following:

Proposition 2.43. *A representation finite implies A has the dense orbit property.*

Proof. We prove the statement by the way of contradiction. Assume for contradiction that we do not have the dense orbit property. Then there exist a dimension vector d and an irreducible component $C \subset \text{mod}(A, d)$ without a dense orbit. Next observe that being representation finite implies that A has finitely many orbits for any dimension vector. Let $\{G_i\}_i$ be the finite collection of orbits of C. C is the union of these finitely many orbits and since none of them are dense, we must have that $\dim G_i < \dim C$ for all i. By properties of dimension, this cannot be true. Therefore we have a contradiction. Thus representation finite implies the dense orbit property for any algebra A. □

The converse of Proposition 2.43 is not true. In [5], Theorem 4.1 , the authors give an example of an algebra of infinite representation type (which is tame under certain conditions and wild under others) that has the dense orbit property. Building on their work, we ask the following: Can we find more examples of representation infinite algebras with the dense orbit property? Our main theorem says that yes we can!

There are a few results presented in [5] types of algebras for which the dense orbit property is equivalent to representation finiteness. We give a few brief key definitions and present some of those results with references.

Definition 2.44. Let A be a bound quiver algebra, $A \cong KQ/I$. The quiver $\Gamma(\text{mod}(A))$ of $\text{mod}\, A$ is defined as follows.

(a) The points of $\Gamma(\text{mod}\, A)$ are the isomorphism classes $[X]$ of indecomposable modules X in $\operatorname{mod} A$.

(b) Let $[M], [N]$ be points of $\Gamma(\operatorname{mod}(A))$ corresponding to the indecomposable modules $M, N \in \operatorname{mod} A$. The arrows $[M] \to [N]$ are in bijective correspondence with the vectors of a basis of the K vector space $\operatorname{Irr}(M, N) := \operatorname{rad}_A(M, N)/\operatorname{rad}_A^2(M, N)$ where $\operatorname{rad}_A(M, N)/\operatorname{rad}_A^2(M, N)$ "measures" the number of irreducible morphism from M to N.

The quiver $\Gamma(\operatorname{mod} A)$ of the module category $\operatorname{mod} A$ is called the **Auslander-Reiten quiver** of A.

Definition 2.45. Suppose we have a bound quiver algebra $A \cong KQ/I$. The algebra A is is said to have a **preprojective component** if it has a connected component in which every indecomposable can be taken to a projective by repeated application of the Auslander-Reiten translation, and that the component has no oriented cycles. We refer the read to [1] for a more extensive background.

Definition 2.46. A bound quiver algebra $A \cong KQ/I$ is said to be a **string algebra** if I can generated by a set of relations R that satisfy the following conditions.

1. each vertex of Q is the tail of at most two arrows, and the head of at most two arrows;

2. each relation in R is just a monomial in the arrows of Q.

3. for each arrows $b \in Q_1$, there is at most one arrow $a \in Q_1$ with $t(a) = s(b)$ and at most one arrow $c \in Q_1$ with $t(b) = s(c)$ such that $ab \notin R$ and $bc \notin R$.

Now we present some results. The authors in [5] present the following theorem

Theorem 2.47. *Let A be a connected bound quiver algebra with a preprojective component. Then A is representation finite if and only if A has the dense orbit property.*

There is another interesting result for algebras with a vanishing radical square. The authors in [3] present the following theorem:

Theorem 2.48. *Suppose A is a finite dimensional algebra and suppose $J^2 = 0$ where J denotes the radical of A. The following conditions are equivalent*

1. *A has the dense orbit property*
2. *For every dimension vector d, any irreducible component of $\mathrm{mod}(A, d)$ which contains an indecomposable module has a dense orbit*
3. *A has finite representation type*

We are mostly interested in finding more examples of infinite representation type algebras with the dense orbit property. The proof process is mostly elementary but may shed some light in finding a more theoretic approach.

Dense orbits have been explored from different perspectives. For example, the authors in [2] explore the complement of the dense orbit for a quiver of type $\mathbb{A}$. There has also been results in determining the irreducible components of the module variety in [8].

Our approach is to first start off with one point algebras which are defined precisely as:

Definition 2.49. A bound quiver algebra $A \cong KQ/I$ is said to be a n point algebra if it has exactly n simple module isomorphism classes. That is, n vertices of Q.

The proof method gives us an elementary approach in proving when an infinite representation type algebra does not have the dense orbit property.

CHAPTER 3
ONE POINT ALGEBRAS

3.1 A Sequence of Lemmas

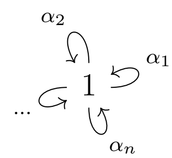
In this section, we prove that local algebras have the dense orbit property if and only if they are representation finite. As a summary, we first prove that local algebras are equivalent to one point algebras. That is, that their associated quiver only has one vertex. We then demonstrate that if the local algebra has at most one loop in its associated quiver then it is equivalent to being representation finite. We use these statements along with key lemmas to prove our main result of this section: Local one point algebras are representation finite if and only if they have the dense orbit property.

Proposition 3.1. *A is local algebra if and only if the associated quiver to A has one vertex.*

Proof. If the associated quiver has one vertex, then A is local. Suppose A is local and by way of contradiction, suppose the associated quiver to A, Q has more than one vertex. Let a and b denote two distinct vertices and e_a and e_b their associated stationary paths. A is local so $A/\text{rad}\ (A)$ is one dimensional. However $e_a, e_b \in A/\text{rad}\ (A)$ imply that the dimension is at least 2. Thus there can only be one vertex. □

Therefore one point algebras and local algebras are equivalent. The next step in the reduction is to relate finite representation-ness of a local algebra with its number of arrows. It turns out, having a most one arrow is equivalent to being representation finite. First we present a lemma to help us prove that representation finite implies at most one arrow in the local case:

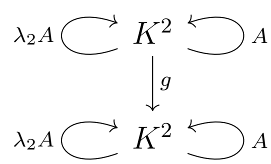
Lemma 3.2. *Let A be the path algebra obtained by the quiver*



mod the ideal R^2, the ideal of all paths of length 2 and $n \geq 2$. That is, $A \cong k < x_1, \cdots, x_n > /R^2$. Let $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Define $\lambda := (1, \lambda_2, \cdots, \lambda_n)$ where $\lambda_i \in K - \{0\}$ for every i and let $M_\lambda = (A, \lambda_2 A, \cdots, \lambda_n A)$ for a fixed λ. Then $M_\lambda \in \text{mod}(A, 2)$ is an indecomposable representation and $M_\lambda \cong M_\tau$ if and only if $\lambda_i = \tau_i$ for every i.

Proof. We proceed by induction on the number of arrows. We start off with our base case on 2 arrows.

We first show that the representations M_λ are indecomposable. This is equivalent to showing $\text{End}(M_\lambda)$ is local. An object $g \in \text{End}(M_\lambda)$ satisfies:

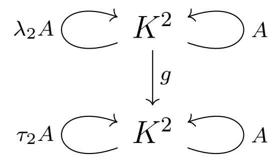


- $gA = Ag$
- $g\lambda_2 A = \lambda_2 A g$

The relation $gA = Ag$ forces g to be of the form $g = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$. The second relation doesn't give us anything more. Thus $\text{End}(M_\lambda) = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) : \ b \in K \quad a \in K^*\}$. In fact, $\text{End}(M_\lambda)$ is the same, regardless of the number of arrows we have.

We next prove the other part of the statement using induction. We start off with the base case $n = 2$ of two arrows.

Two representations $M_\lambda = (A, \lambda_2 A)$ and $(A, \tau_2 A) = M_\tau$ are isomorphic if there exist an invertible matrix g



such that it satisfies the relations:

- $gA = Ag$
- $g\lambda_2 A = \tau_2 A g$

The relation $gA = Ag$ forces g to be of the form $g = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ The relation $g\lambda_2 A = \tau_2 A g$ is then true only when $\lambda_2 = \tau_2$, which is generalized to any number of arrows. □

Proposition 3.3. *Let A be a finite dimensional local algebra. The associated quiver algebra then has one vertex. A is finite representation type if and only if the associated quiver has at most one arrow.*

Proof. The forward implication is always true as demonstrated in the previous section. For the converse consider the following: If there are no arrows, our algebra is a field. If there is one arrow, $A \cong k[x]/(x^n)$ and our corresponding quiver is the loop quiver. The representations are matrices such that $B^n = 0$, which we can decompose using the appropriate Jordan blocks. Let $J_i(\lambda)$ denote an i by i Jordan block with eigenvalue λ Then $J_1(0)$, $J_2(0) \cdots J_n(0)$ are the indecomposable representations.

If there are more than one arrow, we give an explicit example of an infinite family of non isomorphic indecomposables. First consider the case when $A \cong K < x_1, \cdots, x_n > /R^2$ where R^2 is the ideal generated by paths of length 2. Denote $\lambda = (1, \lambda_2, \cdots, \lambda_n)$ for $\lambda_i \in K - \{0\}$and

$$M_\lambda = (A, \lambda_2 A, \cdots, \lambda_n A) \in \text{mod}(A, 2)$$

where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Then $\{M_\lambda\}_\lambda$ is an infinite family of non isomorphic indecomposable representations. This is justified by Lemma 3.2. In the general case when $A \cong K < x_1, \cdots, x_n > /I$ where I is admissible. Thus we have a surjection $B/I \to B/R^2$. Since any indecomposable modules of B/R^2 is also an indecomposable B/I module, B/I also has infinitely many indecomposables and hence not representation finite. □

The infinite family of non isomorphic indecomposable representations in the proof above is used again in our main problem of this section. We want to show that when a one point algebra has more than one arrow in its associated quiver, it will fail to have the dense orbit. This family of indecompsables gave rise to the following sequence of lemmas in an attempt to prove the main result in this section.

Lemma 3.4. *Let X be a variety with an algebraic action of the group G. Let $Y \subset X$ be a locally closed irreducible sub variety and assume:*

1. *for all $x \in X$, the intersection $Y \cap G \cdot x$ is just one point*

2. there is an open subset $U \subset Y$ such that $\dim \frac{T_y(X)}{T_y(G \cdot y)} \leq \dim Y$ for all $y \in U$.

Then $\overline{G \cdot Y}$ is an irreducible component of X.

Proof. Let C be an irreducible component of X containing $G \cdot Y$. The claim is that $\dim C = \dim G \cdot Y$ to imply that $C = G \cdot Y$ since they are both irreducible closed sub varieties of X. Consider the action of the map $\varphi : G \times Y \to G \cdot Y \subset X$ defined as $(g, y) \to g \cdot y$. By theorems on the dimension of the image and fiber, $\dim(G \cdot Y) = \dim(G \times Y) - \dim \varphi^{-1}(y)$ for all y in some open dense subset of $Y \subset G \cdot Y$. Using the action of G to move the points, this is true for all y in an open dense subset $Y \subset G \cdot Y$. With the first assumption we have that $\varphi^{-1}(y) = G_y \times \{y\}$ so that $\dim(\varphi^{-1}(y)) = \dim(G_y)$.

Let y be a fixed point in the dense open subset of Y and let U be from the second assumption. We string a dimension inequalities.

$$\begin{aligned}
\dim G \cdot Y &= \dim(G \times Y) - \dim(G_y) \\
&= \dim(G) + \dim(Y) - \dim(G_y) \\
&\text{and by orbit statbilizer theorem 2.41} \\
&= \dim(Y) + \dim(G \cdot y) \\
&\text{and since } G \cdot y \text{ is a nonsingular variety} \\
&= \dim(Y) + \dim T_y(G \cdot y) \\
&\text{and by the second assumption} \\
&\geq \dim T_y(X) \geq \dim T_y(C) \geq \dim(C).
\end{aligned}$$

and thus $\dim G \cdot Y \geq \dim C$. Moreover since $G \cdot Y \subset C$, we have the reverse inequality.

Therefore $G \cdot Y = C$. and hence $\overline{G \cdot Y}$ is an irreducible component of X. □

We retain the same assumptions as the previous lemma.

Lemma 3.5. *If* $\dim G \cdot y < \dim G + \dim Y - \dim G_y$ *for all* $y \in Y$*, then the irreducible component* $\overline{G \cdot Y}$ *does not have a dense orbit.*

Proof. Suppose there was a dense orbit. That is, there is some $x \in \overline{G \cdot Y}$ such that $G \cdot x$ is dense in $\overline{G \cdot Y}$. In terms of dimensions, this means that $\dim G \cdot x = \dim \overline{G \cdot Y}$. From our assumptions, we know that $G \cdot x \cap Y$ only has one point. We call this point y. and from our assumption, $\dim G \cdot x = \dim G \cdot y < \dim G + \dim Y - \dim G_y = \dim G \cdot Y$ from the previous lemma. Hence there is no dense orbit. □

Our next lemma is a well known result that is key in as it connects back to our representation theory and module varieties. A proof can be found in [[**?**]].

Lemma 3.6. *(Artin-Voigt Lemma) For any representation* M *of an algebra* $A = KQ/I$ *of dimension vector* d*, we have*

$$\dim Ext^1_A(M, M) \geq \dim \frac{T_M(\text{mod}(A, d))}{T_M(GL(d) \cdot M)}$$

Our Y will consist of the representations M_λ illustrated in Lemma 3.2. A first attempt of a proof of our original problem, showing that a local algebra is representation finite if and only if it has the dense orbit property, was true in a particular case only. That is when our algebra was KQ/I and I was exactly the paths of length two. In any case, we know for sure that for any admissible ideal I, $I \subset R^2$, any representation M that is bound by R^2 will for sure be bound by I and hence mod $(KQ/R^2, d) \subset$ mod $(KQ/I, d)$ for each dimension vector d. Equality happens in a particular case when rad $^2(M) = 0$ which is demonstrated in the following lemma.

Lemma 3.7. *Let A be a finite dimensional algebra with associated quiver Q so that $A \cong KQ/I$ for an appropriate admissible ideal I. Let $d : Q_0 \to \mathbb{N}$ be an arbitrary dimension vector and let $M \in$ mod (A, d). That is a quiver representation bounded by I. If rad $^2(M) = 0$, then $M \in$ mod $(KQ/R^2, d)$*

Proof. $0 = \text{rad}\ ^2(M) = \text{rad}\ ^2(kQ/I)M \cong (\text{rad}\ (kQ)/I)^2\, M$

implies that rad $^2(kQ)M = 0$ Thus M is a quiver representation bounded by rad $^2(kQ)$ and thus $M \in$ mod $(kQ/R^2, d)$ □

This will give us a family of indecomposables as we vary λ to help us construct our Y from Lemma 3.4. We have all of our tools ready to prove the following theorem.

3.2 Classifying One Point Algebras by the Dense Orbit Property

Theorem 3.8. *Let $A \cong kQ/I$ be a finite dimensional local algebra. A is representation finite if and only if A has the dense orbit property.*

Proof. The forward implication, representation finite implies the dense orbit property, has been shown to be true by Proposition 2.43. The converse statement, A has has the dense orbit property implies A is representation finite, will be proved by proving the contrapositive statement; A not representation finite implies A does not have the dense orbit property. That is, we show that there is a dimension vector d such that mod(A, d) fails to have a dense orbit in some irreducible component.

Suppose A is not representation finite. Then by Proposition 3.3, we have that being representation infinite implies that the associated quiver for A has at least 2 arrows. So we assume the that the associated quiver has n number of arrows where $n \geq 2$. Therefore, we can say that $A \cong k < x_1, \cdots, x_n > /I$ where I is some admissible ideal. We fix our dimension vector to be $d = 2$. By Lemma 3.7, it is sufficient only consider the case when $I = \text{rad }^2(KQ)$ since for any representation M in $\text{mod}(A, 2)$, $\text{rad}^2(M) = 0$. So that we can say that $\text{mod}(A, 2) = \text{mod}(KQ/\text{rad }^2(KQ), 2)$.

We now construct a family of non isomorphic indecomposables in our module variety. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Denote $\lambda = (1, \lambda_2, \cdots, \lambda_n)$ where $\lambda_i \in K$ for every i and define $M_\lambda = (A, \lambda_2 A, \cdots, \lambda_n A)$ for a fixed λ. Note that $M_\lambda \in \text{mod}(A, 2)$ since $A^2 = 0$. Also observe that $M_{\lambda'} \cong M_\lambda$ only when $\lambda'_i = \lambda_i$ for all i.

$\{M_\lambda\}_\lambda$ is then a family of non isomorphic quiver representation for A as shown by Lemma 3.2. Let $Y = \cup_\lambda M_\lambda$. We make the following observations:

1. Y is locally closed irreducible subvariety since it is the zero set of a collection of linearly independent polynomials and closed.

2. for all $x \in \text{mod}(A, 2)$, $Y \cap G \cdot x$ is at most one point by construction.

To satisfy the second condition of Lemma 3.4, we can use Lemma 3.6 and only show that $\dim Y \geq \text{Ext}^1_A(M_\lambda, M_\lambda)$. This will show that $\overline{\text{GL}(2) \cdot Y}$ is an irreducible component of our module variety $\text{mod}(A, 2)$.

To show that it does not have a dense orbit, we to prove the inequality in

Lemma 3.5. That is:

$$\dim G \cdot y < \dim G + \dim Y - \dim G_y$$

for all $y \in Y$.

First we prove that $\dim Y \geq \mathrm{Ext}^1_A(M_\lambda, M_\lambda)$. Observe that $\dim Y = n - 1$. Next, fix an element of Y, $y = M_\lambda \in Y$.

Consider the short exact sequence

$$0 \leftarrow M_\lambda \stackrel{f}{\leftarrow} A \leftarrow \ker(f) \leftarrow 0$$

The $f : A \to M_\lambda$ is defined as follows. Let e_1 and e_2 denote the standard basis of K^2 for our quiver representation M_λ. Define f as follows. $f(1) = e_1$ and $f(x_i) = \lambda_i e_2$ for all $i \geq 2$. The the kernel of the map f is equal to $\ker(f) = (\{\lambda_i x_1 - x_i\}_{i=2}^n)$. For simplicity, we denote $N_\lambda := \ker(f)$. With this information we can form the following long exact sequence:

$$\begin{aligned} 0 \to &\mathrm{Hom}_A(M_\lambda, M_\lambda) \to \mathrm{Hom}_A(A, M_\lambda) \to \mathrm{Hom}_A(N_\lambda, M_\lambda) \\ \to &Ext^1_\lambda(M_\lambda, M_\lambda) \to Ext^1_A(A, M_\lambda) \to \cdots \end{aligned}$$

Since A is projective we have that $Ext^1_A(A, M_\lambda) = 0$. Thus we obtain the following exact sequence

$$\begin{aligned} 0 \to &\mathrm{Hom}_A(M_\lambda, M_\lambda) \to \mathrm{Hom}_(A, M_\lambda) \\ \to &\mathrm{Hom}_A(N_\lambda, M_\lambda) \to Ext^1_\lambda(M_\lambda, M_\lambda) \to 0 \end{aligned}$$

The sum of the dimensions with alternating signs must be zero from homological algebra theory. First, observe that $\mathrm{Hom}_A(A, M_\lambda) \cong M_\lambda$ and thus it must have dimension 2. Next we will demonstrate that $\text{Hom}_A(M_\lambda, M_\lambda)$ has dimension 2 as well. This is the endomorphism ring of the quiver representation M_λ. In Lemma 3.2, we explicitly found this endormorophism ring and thus we can justify to say it is 2 dimensional.

Therefore, the dimension of $\text{Ext}^1(M_\lambda, M_\lambda)$ is equal to the dimension of $\text{Hom}_A(N_\lambda, M_\lambda)$.

For simplicity, we will construct the associated bounded quiver representation for N_λ. The vertex of our quiver A is denoted by a and arrows α_i for all i. The stationary path is denoted by ϵ_a. The basis that spans the vector space at vertex a is the basis that is spanned by $M_a = \epsilon_a \cdot N_\lambda = N_\lambda$. So our basis elements are $\{\lambda_i x_1 - x_i\}_{i=2}^n$. The linear maps $\varphi_{\alpha_i} : M_a \to M_a$ can be defined on the basis vectors as follows. $\varphi_{\alpha_i}(\lambda_j x_1 - x_j) = \alpha_i \cdot (\lambda_j x_1 - x_j) = x_i \cdot (\lambda_j x_1 - x_j) = 0$. They are all zero! Thus N_λ is isomorphic to the direct sum of simple modules, $S(a)$. That is, $N_\lambda \cong \oplus_{i=1}^{n-1} S(a)$. Using properties of the Hom functor, we have the following string of isomorphisms

$$\begin{aligned}
\text{Hom}_A(N_\lambda, M_\lambda) &= \text{Hom}_A(N_\lambda, M_\lambda) \\
&\cong \text{Hom}_A(\oplus_{i=1}^{n-1} S(1), M_\lambda) \\
&\cong \oplus_{i=1}^{n-1} \text{Hom}_A(S(1), M_\lambda)
\end{aligned}$$

$\text{Hom}_A(S(1), M_\lambda)$ is the collection of maps of the form $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ that satisfy the following relations:

- $A \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) 0$
- $\lambda_i A \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) 0$ for all i

which tells us that $\mathrm{Hom}_A(S(1), M_\lambda)$ is the set of all maps of the form $\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ for $x \in K$ and thus $\mathrm{Hom}_A(S(1), M_\lambda) \cong K$. Therefore,

$$\mathrm{Hom}_A(N_\lambda, M_\lambda) \cong \oplus_{i=1}^{n-1} \mathrm{Hom}_A(S(1), M_\lambda) \cong K^{n-1}$$

and thus $\dim \mathrm{Ext}^1(M_\lambda, M_\lambda) = n - 1 = \dim Y$. Therefore $\overline{\mathrm{GL}(2) \cdot Y}$ is an irreducible component of our module variety $\mathrm{mod}(A, 2)$ by Lemma 3.4 and Lemma 3.6.

Also observe the following. For $y = M_\lambda \in Y$,

- $\dim G = \dim \mathrm{GL}(2) = 4$
- $\dim G_y = 2$
- $\dim G \cdot y = \dim G - \dim G_y = 4 - 2$

and since $n \geq 2$

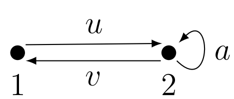
$$\dim G_y = 2 < 4 + n - 1 - 2 = \dim G + \dim Y - \dim G_y$$

and by Lemma 3.5, $\overline{G \cdot Y}$ does not have a dense orbit. Therefore A does not have the dense orbit property if and only if A is not representation finite. □

As a summary, we have shown that representation finite is equivalent to the dense orbit property when our algebra is local (or equivalent one point). Of course we could of just cited Theorem 2.48 but we use this proof method for other algebras whose radical squared may not be zero.

CHAPTER 4
TWO POINT ALGEBRAS

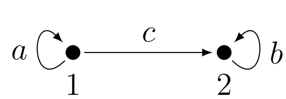
In this chapter we explore two point algebras and prove one of our main results. This thesis takes a lot of inspiration from Module Varieties and Representation Type of Finite-Dimensional Algebras by Chindris, Kinser, and Weyman [5]. In that paper, they prove that a certain two point representation infinite algebra has the dense orbit property. In [7], Hoshino and Miyachi take interest in two point algebras. Their goal in their paper was to classify two point algebras of certain classes according to their representation types. In particular, they consider two classes of of two point algebras. One is the class of the triangular matrix algebras, namely those of the form $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with A and B local and the other of the distributive algebras of the ordinary quiver



They give a list of isomorphism classes of such algebras. We are strongly interested in which of these tame bound quiver algebras have the dense orbit property.

4.1 Summary of Results

Consider the quiver below.



From Hoshino and Miyachi in [7] the bound quiver algebras with the relations given below are tame.

- $a^2 = b^2 = 0$

- $bca = a^2 = b^2 = 0$
- $ca - bc = b^q c = b^6 = a^3 = 0$ for $q = 2, 3$
- $ca - bc = b^q c = b^4 = a^4 = 0$ for $q = 2, 3, 4$

We essentially went through each of the bound quiver algebras to determine which of algebras have the dense orbit property. We generalized the relations when possible. We summarize our results below.

1. If the relations are $a^2 = b^2 = bca = 0$, the associated algebra does not have the dense orbit property. This is Theorem 4.4 and is proved in Section 4.3
2. If the relations are $a^2 = b^2 = 0$, then the associated algebra does not have the dense orbit property. This is Theorem 4.5 and it is proved in section 4.3.
3. If the relations are $bc - ac = b^2 c = b^n = a^3 = 0$, then the associated path algebra does have the dense orbit property for all $n \in \mathbb{N}$. This is our main result! This is our main theorem, Theorem 4.6. It also appears that if we generalize the relation a^3 to a^m, several examples strongly imply that it might have the dense orbit property. A generalized result appears to be close but at the moment proves to be a little difficult to work with. This will be furthered explored in the next chapter.

4.2 Overview of Method

A lot of the ”behind the scenes” work involves a lot of computations with matrices. If we recall, an finite dimensional K algebra A has the dense orbit property if and only if for each dimension vector d of A, $\mathrm{GL}(d)$ acts on each irreducible component of $\mathrm{mod}(A,d)$ with a dense orbit. It proves a little difficult to work with because our module variety need not be irreducible in general, and we don't always know what the irreducible components look like at each dimension vector. Lucky for us, we don't need this information. If we let $C \subset \mathrm{mod}(A,d)$ be an irreducible component, we say C is indecomposable if C has a dense open subset of indecomposable modules.

Lemma 4.1. *An algebra has the dense orbit property if and only if each of its indecomposable irreducible components has a dense orbit.*

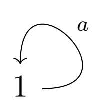
Proof. As stated in [5], any irreducible component $C \subset \mathrm{mod}(A,d)$ satisfies a Krull Schmidt type of decomposition

$$C = \overline{C_1 \oplus \cdots \oplus C_r}$$

for some indecomposable irreducible components $C_i \subset \mathrm{mod}(A,d_i)$ with $\sum d_i = d$. We call this the generic decomposition of C. If each C_i has a dense orbit, then this will force C to have a dense orbit. □

That is, we can take some general element of an irreducible component and show that either has a dense orbit or that it is decomposable.

Example 4.2.

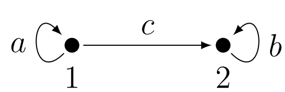


with relation $a^3 = 0$ for $n \in \mathbb{N}$. The associated algebra $KQ/I = K[T]/(T^3)$. The indecomposables in this case are all the $n \times n$ Jordan blocks of eigenvalue zero for

$n \leq 3$. Let us start off with the simplest example of determining the existence of a dense orbit. Consider the dimension vector (1). So that a general representation is of the form $[x]$ where $x \in K$ where $x^3 = 0$. So there is only one representation $x = 0$. We have a single point and hence a dense orbit in this case. That was pretty anticlimactic.

If we move on to dimension vector 2, a general representation $A \in \mathrm{mod}(A, 2)$ has the form $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We don't know what the irreducible components of our module variety are, but we can just let $C \subset \ \mathrm{mod}\ (A, 2)$ be some irreducible component. We let A be in that irreducible component. The action $g \cdot A := gAg^{-1}$ tells us we can choose our matrix g to reduce A down to its Jordan blocks. We may not know exactly which form it may have but we do know it's always possible. So that we have a dense orbit in this case.

If we move to something slightly more exotic, like the path algebra B given by the quiver with relations:



$$a^2 = b^2 = bca = 0 \quad n \in \mathbb{N}$$

Then the computations become a little more difficult to work with. We fix our dimension vector to be $(2, 1)$. Let C be some mysterious irreducible component contained in $\mathrm{mod}(B, (2, 1))$. A general representation in C is of the form (A, B, C) with the condition that $A^2 = B^2 = BCA = 0$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $C = \begin{bmatrix} e & f \end{bmatrix}$, $B = 0$. An object $(G, H) \in \mathrm{GL}(2) \times \mathrm{GL}(1)$ acts on (A, B, C) by $(G, H) \cdot (A, B, C) := (GAG^{-1}, 0, HCG^{-1})$. In this case, we can choose a proper G so that A is in its Jordan blocks. There are two choices for A in this case. We can disregard cases when we can decompose simple summands of A at our first vertex. so we choose $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

To prevese A, we can hit C on the right by matrices that preserve A. These are just matrices of the form $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$. With proper transformations

$$C = \begin{bmatrix} e & f \end{bmatrix} \to \begin{bmatrix} 1 & f' \end{bmatrix} \to \begin{bmatrix} 1 & 0 \end{bmatrix}$$

so that in this dimension vector, we do have a dense orbit.

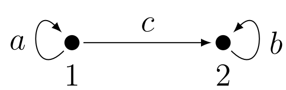
Moving from from this example, we essentially try to eliminate our variables using the group action. If we can't eliminate a variable, this strongly suggest that there may not be a dense orbit in a particular dimension vector. Playing with matrices becomes tedious and computationally inefficient for larger examples. Lucky for us, some of the relations on our quivers let us re frame the problem in terms of modules and polynomials. There is still a lot of computations but they are easier to work with.

We now present various results on dense algebras without the dense orbit property.

4.3 Examples of Algebras Without the Dense Orbit Property

In this section, one of our algebras fails to have the dense orbit property. The proof method is similar to the proof method for local algebras [include reference]. From out collection of Theorems, we can't quite apply them to this case. First we begin with a lemma that shows a certain family of bound quiver representation are indecomposable and non isomorphic.

Lemma 4.3. *Let* Λ *be the algebra generated by the quiver with relations:*



$$a^2 = b^2 = bca = 0 \qquad n \in \mathbb{N}$$

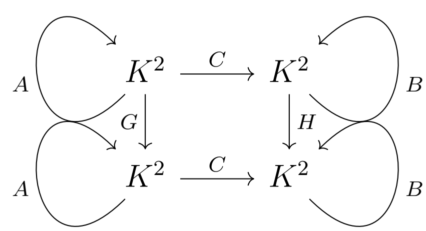
Consider the representation $M_\lambda = (A, B, C) \in \text{mod}(A, (2, 2))$ *with*

$$A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

for $\lambda \neq 0$*. Then* M_λ *is indecomposable for each* λ *and* $M_\lambda \cong M_\tau$ *if and only if* $\tau = \lambda$*.*

Proof. First fix some representation M_λ. We construct the Endomorphism ring $\text{End}_A(M_\lambda, M_\lambda)$ and show that it is local to show that M_λ is indecomposable. The maps in the endomorphism ring are pairs of maps (G, H) that satisfy the following conditions.

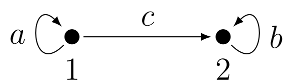


$$GA = AG \qquad\qquad CG = GC$$

$$BH = HB \qquad\qquad CH = HC$$

The first two relations, $GA = AG$ and $BH = HB$, imply that (G, H) must be of the form $\left(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ 0 & e \end{smallmatrix}\right)\right)$ The second relations imply that $\mathrm{End}_A(M_\lambda, M_\lambda) = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} a & \lambda b \\ 0 & a \end{smallmatrix}\right) : a, b \in K\}$. From here, $M_\lambda \cong M_\tau$ only when $\tau = \lambda$. □

Theorem 4.4. *Let A be the algebra generated by the quiver with relations:*



$$a^2 = b^2 = bca = 0 \quad n \in \mathbb{N}$$

Then A does not have the dense orbit in some irreducible component in $\mathrm{mod}(A, (2,2))$.

Proof. This bound quiver algebra is taken from the list in [7] of representation infinite algebras.

We will prove that it fails to have the dense orbit property in dimension vector $d = (2,2)$. We will use our sequence of lemmas from the previous chapter to demonstrate this. First we will find a family of non isomorphic indecomposable bound quiver representations M_λ to construct our Y as in Lemma 3.4. We will then use Lemma 3.6 to show that it is sufficient to show $\dim Y \geq \mathrm{Ext}^1_A(M_\lambda, M_\lambda)$ to prove that $\overline{G \cdot Y}$ is an irreducible component of $\mathrm{mod}(A, (2,2))$ where $G := \mathrm{GL}(2) \times \mathrm{GL}(2)$. Lastly, we will show the inequality from Lemma 3.5,

$$\dim G \cdot y < \dim G + \dim Y - \dim G_y$$

for all $y \in Y$ to show that $\overline{G \cdot Y}$ does not have a dense orbit.

First we find our irreducible component. Consider the representation $M_\lambda = (A, B, C) \in \mathrm{mod}(A, (2,2))$ with $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$. By Lemma 4.3, this is a family of non isomorphic indecomposable modules.

Let $Y = \cup_\lambda M_\lambda$. Note that $\dim(Y) = 1.$. Fix $y = M_\lambda \in Y$. We will compute $\dim \mathrm{Ext}^1_A(M_\lambda, M_\lambda)$ and show that $\dim \mathrm{Ext}^1_A(M_\lambda, M_\lambda) \leq \dim Y$.

We write the basis for our algebra as follows. Let $\{1, 2\}$ denote the vertices as above and let ϵ_1, ϵ_2 represent the stationary paths respectively. The ideal $I = (a^2, b^2, bca)$ gives us the bound quiver algebra KQ/I with basis elements $\{\bar{\epsilon_1}, \bar{\epsilon_2}, \bar{a}, \bar{b}, \bar{c}, \bar{ca}, \bar{ba}\}$. where $\bar{\epsilon_1} = \epsilon_1 + I$ and so forth. For notational simplicity, we omit the bars. We construct a short exact sequence as follows:

$$0 \leftarrow M_\lambda \leftarrow A \leftarrow \ker(\Theta) = N_\lambda$$

where $\Theta : A \to M_\lambda$ is defined as follows. Let e_1, e_2 denote the standard basis of K^2 for our vertex 1 and f_1, f_2 denote the standard basis of K^2 for our vertex 2. We know that $Ae_1 = 0$, $Ae_2 = e_1$, $Bf_1 = 0$, and $Bf_2 = f_1$. From this information we can construct our map Θ as follows.

$$\Theta(\epsilon_1) = e_2$$

$$\Theta(\epsilon_2) = f_2$$

$$\Theta(a) = e_1$$

$$\Theta(b) = f_1$$

$$\Theta(c) = \lambda f_2$$

$$\Theta(ca) = f_1$$

$$\Theta(bc) = \lambda f_1$$

and observe that $\ker(\Theta) = (\lambda\epsilon_2 - c, \lambda ca - bc, \lambda b - bc)$

We can then obtain a long exact sequence of the form

$$0 \to \mathrm{Hom}_A(M_\lambda, M_\lambda) \to \mathrm{Hom}_A(A, M_\lambda) \to \mathrm{Hom}_A(N_\lambda, M_\lambda)$$
$$\to Ext^1_A(M_\lambda, M_\lambda) \to Ext^1_A(A, M_\lambda) \to \cdots$$

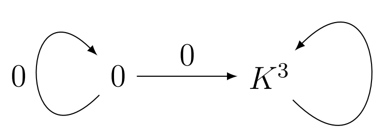
But since A is a projective, $Ext^1_A(A, M_\lambda) = 0$ and thus we can obtain the following exact sequence:

$$0 \to \mathrm{Hom}_A(M_\lambda, M_\lambda) \to \mathrm{Hom}_A(A, M_\lambda) \to \mathrm{Hom}_A(N_\lambda, M_\lambda) \to Ext^1_A(M_\lambda, M_\lambda) \to 0$$

We know that the sum of the dimensions of alternating sign must be zero. Note that the $\dim \mathrm{Hom}(M_\lambda, M_\lambda) = 2$ by Lemma 4.3 where we computed the endomorphism ring of M_λ. In addition, $\mathrm{Hom}_A(A, M_\lambda) \cong M_\lambda$ and so it must have dimension equal to 4.

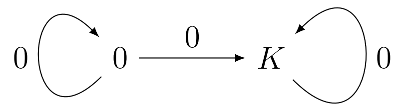
Next, we write out the associated quiver representation for N_λ to help us compute the dimension of $\mathrm{Hom}_A(N_\lambda, M_\lambda)$. The vector space in the first vertex is equal to $N_1 = \epsilon_1 \cdot N_\lambda = (\{\lambda\epsilon_2 - c, \lambda ca - bc, \lambda b - bc\}) = 0$. So there is a zero space in the first vertex.

The vector space in the second vertex is equal to $N_2 = \epsilon_2 N_\lambda = N_\lambda$ So it is generated by the K linearly independent elements of N_λ, which are $\lambda\epsilon_2 - c, \lambda ca - bc, \lambda b - bc$. We identify them with the canonical basis of K^3 and the action of b gives us the linear map $\varphi_b : N_2 \to N_2$ which under the canonical basis yields the following:

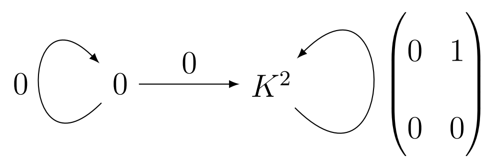


$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is isomorphic to the direct sum of: $N_\lambda^1 =$



and $N_\lambda^2 =$



Thus $\mathrm{Hom}(N_\lambda, M_\lambda) \cong \mathrm{Hom}(N_\lambda^1, M_\lambda) \oplus \mathrm{Hom}_A(N_\lambda^2, M_\lambda)$. From here we can explicitly compute the desired rings.

Observe that

$$\mathrm{Hom}(N_\lambda^1, M_\lambda) \cong \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in K \right\}$$

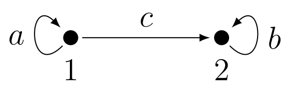
and that

$$\mathrm{Hom}_A(N_\lambda^2, M_\lambda) \cong \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in K \right\}$$

Thus $\dim(\mathrm{Hom}_A(N_\lambda, M_\lambda) = 3$ and thus $\dim Ext_A^1(M_\lambda, M_\lambda) = 1$. Thus by Lemma 3.4, $\overline{G \cdot Y}$ is an irreducible component.

And so that the conditions of Lemma 3.5 are satisfied and thus this algebra fails to have a dense orbit in dimension vector $d = (2, 2)$. □

Theorem 4.5. *Let Γ_1 be the algebra given by the quiver*

$Q =$ 

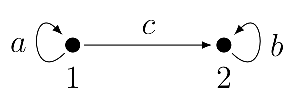
with the relations, $a^2 = b^2 = 0 \quad n \in \mathbb{N}$. *This algebra does not have a dense orbit on dimension vector* $(2, 2)$

Proof. This is an example of a representation infinite string algebra and it does not have the dense orbit property by Proposition 4.4 in [5]. □

4.4 Example of a Dense Orbit Algebra

Our main result is the following:

Theorem 4.6. *Let Γ be the algebra given by the quiver*



with relations $ca - bc = b^n = a^3 = b^2c = 0$. *Then Γ has the dense orbit property.*

Before we proceed to the proof, we will give a series of simplifications. We outline the method here.

1. We will reframe the problem in terms of matrices over the polynomial ring $K[T]$
2. We will use Lemma 4.1 to justify that it is sufficient to take a general object in $\mathrm{mod}(\Gamma, d)$ and show that it is either has a dense orbit or show that it is decomposable

4.4.1 A Series of Reductions

We first re-frame our problem in terms of matrices over the polynomial ring $K[T]$. Let $\mathbf{d} = (d_1, d_2)$ be a dimension vector for Γ and fix an irreducible component W of $\text{mod}(\Gamma, \mathbf{d})$. Let (A, B, C) be a representation in $\text{mod}(\Gamma, \mathbf{d})$ such that it satisfies the relations $CA - BC = B^n = A^3 = B^2C = 0$. There is an open subset of W on which A and B have a constant Jordan type. We fix representatives A and B of general Jordan type and let $Z_A \subset \text{GL}(d_1)$ and $Z_B \subset \text{GL}(d_2)$ be the centralizers of A and B respectively. W has a dense $Gl(\mathbf{d})$ orbit if and only if $\text{Hom}_K(V_1, V_2)$ has a dense $H := Z_A \times Z_B$ orbit.

We can identify (A, K^{d_1}) as a $K[T]/(T^3)$-module and (B, K^{d_2}) as a $K[T]/(T^n)$-module. Let X and Y be the corresponding modules for (A, d_1) and (B, d_2) respectively.

$$X \cong \bigoplus_{i=1}^{3} \tau_i J_i$$

$$Y \cong \bigoplus_{i=1}^{n} \lambda_i J_i$$

where $J_i := K[T]/(T^i)$ and λ_i and τ_i are the multiplicity of J_i. To help simplify our notation, we let $X_i := \tau_i J_i$ and $Y_j := \lambda_j J_j$ for all i, j.

The relation $ca - bc = 0$ implies that C can be identified as a $K[T]$ module morphism from X to Y. As a matrix, the rows of C correspond the the summands of Y and the columns correspond to the summands of X. We order the rows by decreasing dimension from top to bottom. The columns are organized by decreasing dimension from left to right. As a small example to illustrate what we are trying to describe, we can let $X = J_3 \oplus J_1$ and $Y = J_6 \oplus J_2$. Then C is of the form

$$C = \begin{array}{c} \\ Y_6 \\ Y_2 \end{array} \begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \left(\begin{array}{cc} \quad & \quad \\ \quad & \quad \end{array} \right) \end{array}$$

where the entries are polynomials in terms of T. We can make further reductions on the entries of C. For example, consider $C : J_3 \to J_6$. Since it is a $K[T]$ module, it is completely determined by where we send 1. Thus we can write

$$C = \left(a_1 + a_2 T + \cdots + a_5 T^5 \right)$$

where $a_i \in K$. But note that all the elements of J_3 are annihilated by T^3 and since C is a $K[T]$ module, this forces C to be of the form $C = (a_3 T^3 + a_4 T^4 + a_5 T^5)$. Lastly, the relation from the quiver comes into play, $b^2 c = 0$. In module language, this means that the entries of C are annihilated by T^2 as well. Thus a general $K[T]$ module homomorphism $CJ_3 \to J_6$ has the form

$$C = (aT^4 + bT^5)$$

where $a, b \in K$. We can generalize the form for all the entries as follows:

$$\begin{array}{c} \\ Y_n \\ \vdots \\ Y_5 \\ Y_4 \\ Y_3 \\ Y_2 \\ Y_1 \end{array}
\begin{array}{c} \begin{array}{ccc} X_3 & X_2 & X_1 \end{array} \\
\begin{pmatrix}
aT^{n-2} + bT^{n-1} & aT^{n-2} + bT^{n-1} & aT^{n-1} \\
\vdots & \vdots & \vdots \\
aT^3 + bT^4 & aT^3 + bT^4 & aT^4 \\
aT^2 + bT^3 & aT^2 + bT^3 & aT^3 \\
aT^1 + bT^2 & aT^1 + bT^2 & aT^2 \\
a + bT & a + bT & aT \\
a & a & a
\end{pmatrix} \end{array}$$

Note that the entries in row Y_i and column X_j are as large as the corresponding multiplicities of Y_i and X_j to avoid any confusion.

The centralizer of A and B are identified with $\mathrm{Aut}_{K[T]}(X)$ and $\mathrm{Aut}_{K[T]}(Y)$ respectively with the action of H as follows:

1. Row operations of the form add f times a row in block Y_i to a row in block Y_j where $f \in K[T]$, $j \leq i$ and $f \in (T^{j-i})$ for $j > i$.

2. Column operations of the form add g times a column in block X_i to a column in block X_j where $g \in K[T]$ $j \leq i$ and $g \in (T^{j-i})$ for $i < j$.

Thus we can describe a representation of Γ by a pair of $K[T]$ modules X, Y and decomposing a representation of Γ means decomposing C into a direct sum of matrices.

For clarification, consider the following example. Let $X = J_3 \oplus J_2$ and $Y = J_6 \oplus J_2$. Then $C : X \to Y$ has the form

$$C = \begin{pmatrix} a_1T^4 + a_2T^5 & a_3T^4 + a_4T^5 \\ a_5 + a_6T & a_7 + a_8T \end{pmatrix}$$

for $a_i \in K$. We assume C is injective, otherwise we have simple summands at our vertices of our quiver and decompose C into a smaller matrix. . The automorphism are described by polynomials whose constant entry is non zero. We multiply the second column by $\frac{1}{a_7}(1 - \frac{a_8}{a_8}T)$ to obtain

$$C = \begin{pmatrix} a_1'T^4 + a_2'T^5 & a_3'T^4 + a_4'T^5 \\ a_5' + a_6'T & 1 \end{pmatrix}$$

and performing row and column operations yields:

$$C = \begin{pmatrix} a_1''T^4 + a_2''T^5 & 0 \\ 0 & 1 \end{pmatrix}$$

to reduce down to:

$$C = \begin{pmatrix} T^4 & 0 \\ 0 & 1 \end{pmatrix}$$

which we can decompose C into a direct sum $(T^4) \oplus (1)$.

Therefore in this case we do have a dense orbit. There is one more clarification to make.

Proposition 4.7. *Fix a dimension vector d for Γ, and $\tau = (\tau_i)_{i=1}^3$ and $\lambda = (\lambda_i)_{i=1}^n$ be lists of nonnegative integers such that*

$$\tau_1 + \tau_2 + \tau_3 = d_1 \qquad \text{and} \qquad \lambda_1 + \cdots + \lambda_n = d_2.$$

Let $\operatorname{mod}(\Gamma, d)(\tau, \lambda)$ *be the locally closed subvariety of* $\operatorname{mod}(\Gamma, d)$ *consisting of points* (A, B, C) *such that the associated* $K[T]$*-modules* X *and* Y *are of the form*

$$X \cong \bigoplus_{i=1}^{3} \tau_i J_i \qquad \text{and} \qquad Y \cong \bigoplus_{i=1}^{n} \lambda_i J_i.$$

Then $\operatorname{mod}(\Gamma, d)(\tau, \lambda)$ *is irreducible. Furthermore, every irreducible component of* $\operatorname{mod}(A, d)$ *is the closure of some* $\operatorname{mod}(\Gamma, d)(\tau, \lambda)$*.*

Proof. Fix matrices A_0, B_0 such that $(A_0, K^{d_1}) \cong X$ and $(B_0, K^{d_2}) \cong Y$ as $K[T]$-modules, and let

$$H = \{C \in \operatorname{Mat}_{d_1 \times d_2} \mid (A_0, B, C_0) \in \operatorname{mod}(\Gamma, d)\}.$$

Since H can be identified with the K-subspace of $\operatorname{Hom}_{K[T]}(X, Y)$ consisting of homomorphisms that are annihilated by T^2, it is irreducible. Now consider the morphism of varieties

$$\operatorname{GL}(d_1) \times H \times \operatorname{GL}(d_2) \to \operatorname{mod}(\Gamma, d) \qquad (g_1, C, g_2) \mapsto (g_1 A_0 g_1^{-1},\ g_2 C g_1^{-1},\ g_2 B_0 g_2^{-1}).$$

The image is exactly $\operatorname{mod}(\Gamma, d)(\tau, \lambda)$. Since the domain is irreducible, the image is irreducible as well.

For the "furthermore" statement, we note that $\operatorname{mod}(\Gamma, d)$ is a finite disjoint union of the irreducible varieties $\operatorname{mod}(\Gamma, d)(\tau, \lambda)$ as τ, λ vary over the possibilities as in the proposition statement. Now for each irreducible component of $\text{mod}(\Gamma, d)$, there must exist (τ, λ) such that $\text{mod}(\Gamma, d)(\tau, \lambda)$ is dense in that irreducible component, and is therefore dense in that irreducible component. □

Corollary 4.8. *If* $\text{mod}(\Gamma, d)(\tau, \lambda)$ *has a dense orbit for all dimension vectors* d *and* (τ, λ) *as in the proposition statement, then* Γ *has the dense orbit property.*

Proposition 4.7 and Lemma 4.1 justifies that it is enough to take a general element of our module variety and show that it has a dense orbit or that it decomposes. We now proceed to prove our Main Theorem

4.4.2 Proof of the Main Theorem

Proof of Theorem 4.6.

Proof. Proposition 4.7 and Lemma 4.1 justifies that it is enough to take a general element of our module variety and show that it has a dense orbit or that it decomposes. . We let C be our $K[T]$ module morphism. We assume C is injective in all cases else we can split of a direct summand from the kernel. We also assume that we cannot obtain a row of zeroes in C in our reduction else we can split of a direct summand from the tangent space.

We start with our four main cases that that depend on the "zero-ness" of X_1 and Y_1. Case 3 and Case 4 described below have a lot of other conditions to consider that depend on the $\tau_i's$. All of the possible cases and subcases are described below.

Case 1 $X_1 = 0$ and $Y_1 = 0$

Case 2 $X_1 \neq 0$ and $Y_1 \neq 0$.

Case 3 $X_1 \neq 0$ and $Y_1 = 0$.

Case 3.1 $\tau_1 > \tau_3$

Case 3.2 $\tau_1 < \tau_3$

Case 3.2 $\tau_1 = \tau_3$

Case 4 $X_1 =$ and $Y_1 \neq 0$.

Case 4.1 $\lambda_1 < \tau_2$

Case 4.2 $\lambda_1 = \tau_2$

Case 4.2.1 $\tau_2 = \tau_3$

Case 4.2.2 $\tau_2 > \tau_3$

Case 4.2.3 $\tau_2 < \tau_3$

Case 4.3 $\lambda_1 > \tau_2$

Case 4.3.1 $\tau_3 + \tau_2 \leq \lambda_1$

Case 4.3.2 $\tau_3 + \tau_2 > \lambda_1$

Case 4.3.2.1 $\tau_2 < \tau_3$

Case 4.3.2.2 $\tau_2 = \tau_3$

Case 4.3.2.3 $\tau_2 > \tau_3$

We now begin with our very first case.

Case 1: $X_1 = 0$ and $Y_1 = 0$.

In this case we look at the bottom most and right most entry of our matrix C that is row J_i and column J_j. as shown below.

$$\begin{array}{c} \\ \vdots \\ J_i \end{array} \begin{array}{c} \begin{array}{ccc} \cdots & \cdots & J_j \end{array} \\ \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & aT^{i-2} + bT^{i-1} \end{pmatrix} \end{array}$$

Applying the appropriate column and row operations yield

$$\begin{array}{c} \\ \vdots \\ \vdots \\ J_i \end{array} \begin{array}{c} \begin{array}{ccc} \cdots & \cdots & J_j \end{array} \\ \begin{pmatrix} \cdots & \cdots & 0 \\ \cdots & \cdots & \vdots \\ 0 & \cdots & T^{i-2} \end{pmatrix} \end{array}$$

and hence we can pull off $\begin{array}{c} \\ J_i \end{array} \begin{array}{c} J_j \\ \begin{pmatrix} T^{i-2} \end{pmatrix} \end{array}$ to decompose C in this case.

Case 2: $X_1 \neq 0$ and $Y_1 \neq 0$.

This is similar to case 1, our entry of interest in our matrix C is the bottom most and right most entry. We pull out a $\begin{array}{cc} & J_1 \\ J_1 & \begin{pmatrix} 1 \end{pmatrix} \end{array}$ since we can reduce our matrix to the form:

$$\begin{array}{cc} & \begin{array}{ccc} \cdots & \cdots & J_1 \end{array} \\ \begin{array}{c} \vdots \\ \vdots \\ J_1 \end{array} & \begin{pmatrix} \cdots & \cdots & 0 \\ \cdots & \cdots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \end{array}$$

Case 3: $X_1 \neq 0$ and $Y_1 = 0$. We first make one reduction. If $X_2 \neq 0$, then we pull out a $\begin{array}{cc} & J_2 \\ J_j & \begin{pmatrix} T^{j-2} \end{pmatrix} \end{array}$ to decompose C as follows.

Let j denote the smallest j such that $Y_j \neq 0$

Then C has the form

$$\begin{array}{cc} & \begin{array}{ccc} X_3 & X_2 & X_1 \end{array} \\ \begin{array}{c} \vdots \\ Y_j \\ \vdots \end{array} & \begin{pmatrix} \cdots & \cdots & \\ \cdots & \cdots & \\ \cdots & \cdots & \end{pmatrix} \end{array}$$

Then we can pick any entry in the X_2 and Y_j block and can apply row and column operations to reduce the matrix to the form:

$$\begin{array}{c} \\ J_j \end{array}\begin{array}{c} \begin{array}{ccccccc} & \cdots & & J_2 & & \cdots & \end{array} \\ \begin{pmatrix} & & & 0 & & & \\ & & & \vdots & & & \\ & & & 0 & & & \\ 0 & \cdots & 0 & T^{j-2} & 0 & \cdots & 0 \\ & & & 0 & & & \\ & & & \vdots & & & \\ & & & 0 & & & \end{pmatrix} \end{array}$$

and thus we can pull out $\begin{array}{cc} & J_2 \\ J_j & \begin{pmatrix} T^{j-2} \end{pmatrix} \end{array}$.

Therefore we can assume $X_2 = 0$ in this case.

Our matrix C then has the form:

$$\begin{array}{cc} X_3 & X_1 \end{array}$$
$$\begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$$

In this case, we cannot pull out any 1 by 1 matrix. The difference of degrees prevents us from doing that. So what we will do is diagonalize from right to left on each of the column blocks X_3 and X_1. There are some cases to consider that depend on the relations between τ_3 and τ_1 have with each other.

Since C is injective, there is a pivot position in each of the columns. That is, there are at least $\min\{\tau_1, \tau_3\}$ rows. We diagonalize as much as we can in each of the column blocks of X_1 and X_3 using column operations to obtain these square matrices which we call P_1 and P_3 that are define below. Let i_1 denote the index of the target space J_{i_1} for the last row in our matrix C.

Let $(i_1, \cdots, i_{\tau_1})$ denote the sequence of the indices of the target spaces, $(J_{i_1}, \cdots, J_{i_{\tau_1}})$, for the rows going from bottom to top in the X_1 block Then we define P_1 to be the square τ_1 by τ_1 diagonal matrix to be:

$$(P_1)_{r,s} := \begin{cases} T^{(i_{\tau_1+1-r})-1} & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases}$$

for all integers r, s such that $1 \leq r, s \leq \tau_1$.

We define P_3 in a similar manner. Let $(i_1, \cdots, i_{\tau_3})$ be the sequence of numbers that correspond the indices of the target spaces, $(J_{i_1}, \cdots, J_{i_{\tau_3}})$, for the rows of C going from bottom to top in our X_3 block. We define P_3 to be the τ_3 by τ_3 diagonal matrix to be:

$$(P_3)_{r,s} = \begin{cases} T^{(i_{\tau_3+1-r})-2} & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases}$$

for all integers r, s such that $1 \leq r, s \leq \tau_3$.

Thus we can reduce C to one of the three following forms that depend on the number of rows P_3 and P_1 have in comparison with each other. That is, we have three subcases to consider that depend on the relations τ_3 and τ_1 have with each other.

$$\text{if } \tau_1 > \tau_3 \quad \begin{array}{c} \\ \vdots \end{array} \overset{\begin{array}{cc} X_3 & X_1 \end{array}}{\begin{pmatrix} * & & * \\ 0 & P_1' & 0 \\ P_3 & 0 & P_1'' \end{pmatrix}} \text{ where } P_1 = \begin{pmatrix} P_1' & 0 \\ 0 & P_1'' \end{pmatrix}$$

$$\text{if } \tau_1 < \tau_3 \quad \begin{array}{c} \\ \vdots \end{array} \overset{\begin{array}{cc} X_3 & X_1 \end{array}}{\begin{pmatrix} & * & * \\ P_3' & 0 & 0 \\ 0 & P_3'' & P_1 \end{pmatrix}} \text{ where } P_3 = \begin{pmatrix} P_3' & 0 \\ 0 & P_3'' \end{pmatrix}$$

$$\text{if } \tau_1 = \tau_3 \quad \begin{array}{c} \\ \vdots \end{array} \overset{\begin{array}{cc} X_3 & X_1 \end{array}}{\begin{pmatrix} * & * \\ P_3 & P_1 \end{pmatrix}}$$

Case 3.1: P_1 has more rows than P_3. That is, $\tau_1 > \tau_3$

First we clear our the terms above P_3. This will "mess up" some of the terms in P_1 but we perform column operations withing the X_1 to restore it back

$$\begin{array}{c} \\ \vdots \end{array} \overset{\begin{array}{cc} X_3 & X_1 \end{array}}{\begin{pmatrix} * & & * \\ 0 & P_1' & 0 \\ P_3 & 0 & P_1'' \end{pmatrix}} \to \begin{array}{c} \\ \vdots \end{array} \overset{\begin{array}{cc} X_3 & X_1 \end{array}}{\begin{pmatrix} 0 & & * \\ 0 & P_1' & * \\ P_3 & 0 & P_1'' \end{pmatrix}} \to \begin{array}{c} \\ \vdots \end{array} \overset{\begin{array}{cc} X_3 & X_1 \end{array}}{\begin{pmatrix} 0 & & * \\ 0 & P_1' & 0 \\ P_3 & 0 & P_1'' \end{pmatrix}}$$

i_{τ_1} is the index for the target space $J_{i_{\tau_1}}$ of the first row on P_1. Our goal is to factor out $J_{i_{\tau_1}} \overset{J_1}{\begin{pmatrix} T^{j-1} \end{pmatrix}}$ Our entry of interest is the first entry in P_1. We use row operations to clear out all terms above the entry. Since P_1 has more rows that P_3, this doesn't affect the zero block above P_3. Thus we can pull out $J_{i_{\tau_1}} \overset{J_1}{\left(T^{j-1} \right)}$ to decompose C in this case.

Case 3.2:$\tau_3 > \tau_1$. That is, P_3 has more rows than P_1

This is similar to the case 3.1 but now we zero out the terms above our P_1 block instead. This will mess up some terms in our P_3 block but using columns operations restores it.

$$
\begin{array}{c} \\ \vdots \end{array}
\begin{pmatrix} & \overset{X_3}{*} & \overset{X_1}{*} \\ P_3' & 0 & 0 \\ 0 & P_3'' & P_1 \end{pmatrix}
\rightarrow
\begin{array}{c} \\ \vdots \end{array}
\begin{pmatrix} & \overset{X_3}{*} & \overset{X_1}{0} \\ P_3' & * & 0 \\ 0 & P_3'' & P_1 \end{pmatrix}
\rightarrow
\begin{array}{c} \\ \vdots \end{array}
\begin{pmatrix} & \overset{X_3}{*} & \overset{X_1}{0} \\ P_3' & 0 & 0 \\ 0 & P_3'' & P_1 \end{pmatrix}
$$

i_{τ_3} is the index of the target space $J_{i_{\tau_3}}$ for the first row in the P_3 block. $J_{i_{\tau_3}} \overset{J_3}{\left(T^{i_{\tau_3}-2} \right)}$ is the desired term to be factored out. We can start by starting on the left most entry on our P_3 block and perform appropiate row operations to clear out all the terms above this entry. This doesn't affect the zero block above P_1 since P_3 has more rows that P_1. Then we can pull off $J_{i_{\tau_3}} \overset{J_3}{\left(T^{i_{\tau_3}-2} \right)}$ to decompose C.

Case 3.3: $\tau_3 = \tau_1$. That is, P_1 has the same number of rows as P_3.

Let r denote the index of the target space J_r for the first row above the matrices P_1 and P_3. Let $s = \tau_3 = \tau_1$. We first clear out all terms above the P_1 block using row operations.

$$
\begin{array}{cc}
 & \begin{array}{cc} X_3 & X_2 \end{array} \\
\begin{array}{c} \vdots \\ \vdots \\ J_r \\ \\ \end{array} &
\begin{pmatrix} * \cdots * & 0 \cdots 0 \\ \vdots & \vdots \\ * \cdots * & 0 \cdots 0 \\ P_3 & P_1 \end{pmatrix}
\end{array}
$$

We take the left most entry on the row J_r of our matrix C as denoted in the matrix above and perform the appropriate column to clear all terms above it. We then perform row operations to clear all terms to the right. In doing so we "mess up" P_3 but row operations from below "fixes" P_3 restores it. This "messes up" P_1 but column operations restores it P_1. That is,

$$
\begin{array}{cc}
 & \begin{array}{cc} X_3 & X_1 \end{array} \\
\begin{array}{c} \vdots \\ \vdots \\ J_r \\ \\ \end{array} &
\begin{pmatrix} * \cdots * & 0 \cdots 0 \\ \vdots & \vdots \\ * \cdots * & 0 \cdots 0 \\ P_3 & P_1 \end{pmatrix}
\end{array}
\rightarrow
\begin{array}{cc}
 & \begin{array}{cc} X_3 & X_1 \end{array} \\
\begin{array}{c} \vdots \\ \vdots \\ J_r \\ \\ \end{array} &
\begin{pmatrix} 0 \cdots * & 0 \cdots 0 \\ \vdots & \vdots \\ T^{r-2} \cdots 0 & 0 \cdots 0 \\ * & P_1 \end{pmatrix}
\end{array}
\rightarrow
\begin{array}{cc}
 & \begin{array}{cc} X_3 & X_1 \end{array} \\
\begin{array}{c} \vdots \\ \vdots \\ J_r \\ \\ \end{array} &
\begin{pmatrix} 0 \cdots * & 0 \cdots 0 \\ \vdots & \vdots \\ T^{r-2} \cdots 0 & 0 \cdots 0 \\ P_3 & * \end{pmatrix}
\end{array}
\rightarrow
$$

$$\begin{array}{c} \\ \vdots \\ \vdots \\ J_r \\ \\ \end{array} \begin{pmatrix} X_3 & X_1 \\ 0 \cdots * & 0 \cdots 0 \\ \vdots & \vdots \\ T^{r-2} \cdots 0 & 0 \cdots 0 \\ P_3 & P_1 \end{pmatrix}$$

Therefore we can pull off $\begin{array}{c} \\ J_r \\ J_s \end{array} \begin{pmatrix} J_3 & J_1 \\ T^{r-2} & 0 \\ T^{s-2} & T^{s-1} \end{pmatrix}$ and decompose C in this case.

Case 4: $X_1 = 0$ and $Y_1 \neq 0$. This case shares similar properties to the case before. But the same method cannot be applied. There is no row operations that can eliminate terms above the Y_1 block. For example, If we had the matrix $\begin{array}{c} \\ J_3 \\ J_1 \end{array} \begin{pmatrix} J_3 \\ * \\ * \end{pmatrix}$ we can reduce down to $\begin{array}{c} \\ J_3 \\ J_1 \end{array} \begin{pmatrix} J_3 \\ T \\ 1 \end{pmatrix}$ but we can't perform row operations from below since the element must come from (T^{3-1}). This makes this case the most difficult case compared relatively to all the cases before. There are 3 subcases that depend on comparing the multiplicities of X_2 and Y_1 and one of those cases has even more subcases to consider. Before we proceed, we define matrices U_2 and U_3 that are defined similarly to the matrices P_1 and P_3 as before. We let j_1 denote the index of the target space J_{j_1} of the first row above the Y_1 block.

Let $(j_1, \cdots, j_{\tau_2})$ denote a sequence of numbers that correspond to the indices of the target spaces of $(J_{j_1}, \cdots, J_{j_{\tau_2}})$ for the rows going from bottom to top in the X_2 block. Then we define our matrix U_2 as follows:

$$(U_2)_{r,s} := \begin{cases} T^{(j_{\tau_2+1-r})-2} & \text{if } r = s \\ \\ 0 & \text{if } r \neq s \end{cases}$$

for all integers r, s that satisfy $1 \leq r, s \leq \tau_2$.

Let $(j_1, \cdots, j_{\tau_3})$ denote a sequence of numbers that correspond to the indices of the target spaces of $(J_{j_1}, \cdots, J_{j_{\tau_3}})$ for the rows going from bottom to top in the X_3 block. Then we define our matrix U_3 as follows:

$$(U_3)_{r,s} := \begin{cases} T^{(j_{\tau_3+1-r})-2} & \text{if } r = s \\ \\ 0 & \text{if } r \neq s \end{cases}$$

for all integers r, s that satisfy $1 \leq r, s \leq \tau_3$.

Case 4.1:

$\lambda_1 < \tau_2$ That is, we have more columns that rows in the Y_1 and X_2 block.

We can diagonalize from right to left in the X_2 block and clear our all terms to the left to obtain:

$$\begin{array}{cc} & \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ Y_1 \end{array} & \begin{pmatrix} * & * \\ \vdots & \vdots \\ * & * \\ 0 & \begin{pmatrix} 0 & I_{\lambda_1} \end{pmatrix} \end{pmatrix} \end{array}$$

j_1 is the index of the target space J_{j_1} for the first row above the Y_1 block. We pick the left most entry in our X_2 block that lies in this J_{j_1} row so that it does not lie above I_{λ_1}. We can then perform row and column operations to obtain:

$$\begin{array}{cc} & \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ Y_1 \end{array} & \begin{pmatrix} * & * \\ \vdots & \vdots \\ * & * \\ 0 & \begin{pmatrix} 0 & I_{\lambda_1} \end{pmatrix} \end{pmatrix} \end{array} \rightarrow \begin{array}{cc} & \begin{array}{cc} X_3 & \cdots J_2 \cdots \end{array} \\ \begin{array}{c} \vdots \\ \vdots \\ J_{j_1} \\ Y_1 \end{array} & \begin{pmatrix} * & 0 & * \\ * & \vdots & * \\ 0 & T^{j_1-1} & \cdots 0 \\ 0 & \cdots 0 \cdots & I_{\lambda_1} \end{pmatrix} \end{array}$$

and thus we can pull out $\begin{array}{cc} & J_2 \\ J_{j_1} & \begin{pmatrix} T^{j_1-1} \end{pmatrix} \end{array}$

Case 4.2:

Suppose now $\tau_2 = \lambda_1$. Then we take C and reduce it down to:

$$\begin{array}{c} \\ \vdots \\ \vdots \\ \vdots \\ Y_1 \end{array} \begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} * & * \\ \vdots & \vdots \\ * & * \\ 0 & I_{\lambda_1} \end{pmatrix} \end{array}$$

We then take a similar approach as in case 3 where we diagonalize within the X_2 block to obtain our matrix U_2 as defined above. This ”messes” up our identity matrix below but using Gaussian elimination with strictly smaller rows, restores it. We then perform column operations to obtain U_3 in our X_3 block.

We obtain 3 cases that depend on the number of rows U_2 and U_3 have with each other. That is, 3 cases that depend on the the relations between τ_2 and τ_3. We can reduce C to one the following three forms:

$$\text{if } \tau_2 = \tau_3 \; \begin{array}{c} \\ \vdots \\ \vdots \\ Y_1 \end{array} \begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} * & * \\ U_3 & U_2 \\ 0 & I_{\lambda_1} \end{pmatrix} \end{array}$$

$$\text{if } \tau_2 > \tau_3 \; \begin{array}{c} \\ \vdots \\ \vdots \\ Y_1 \end{array} \begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} * & * \\ \begin{bmatrix} 0 \\ U_3 \end{bmatrix} & \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \\ 0 & I_{\lambda_1} \end{pmatrix} \end{array} \text{ where } U_2 = \begin{pmatrix} U_2' & 0 \\ 0 & U_2'' \end{pmatrix}$$

$$\text{if } \tau_2 < \tau_3 \quad \begin{array}{c} \\ \vdots \\ \vdots \\ Y_1 \end{array} \begin{array}{c} X_3 \quad\quad X_2 \\ \left(\begin{array}{cc} * & * \\ \begin{bmatrix} U_3' & 0 \\ 0 & U_3'' \end{bmatrix} & \begin{bmatrix} 0 \\ U_2 \end{bmatrix} \\ 0 & I_{\lambda_1} \end{array} \right) \end{array} \text{ where } U_3 = \begin{pmatrix} U_3' & 0 \\ 0 & U_3'' \end{pmatrix}$$

We now go through each of the 3 subcases

Case 4.2.1 $\tau_2 = \tau_3$. That is, U_2 and U_3 have the same number of rows.

Let r denote the index of J_r for the target space for the first row above U_2 and U_3. Let $s = \tau_2 = \tau_3$ so that the index for the target space for the first row in P_2 and P_3, J_s. Starting with the left most entry in row J_r, we can perform column operations to clear all terms above. Then perform the appropriate row operations to clear all terms to the right. This will mess up some the terms in U_3 but performing Gaussian elimination again from strictly lower rows restores it. In turn, this messes up U_2 but column operations restores it. This messes up the identity matrix in the bottom right but row operations fixes this.

$$\begin{array}{c} \\ \vdots \\ J_r \\ \vdots \\ Y_1 \end{array} \begin{array}{c} J_3 \cdots \quad X_2 \\ \begin{pmatrix} * \cdots * & 0 \\ T^{r-2} \cdots & 0 \\ U_3 & U_2 \\ 0 & I_{\lambda_1} \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \\ \vdots \\ J_r \\ \vdots \\ Y_1 \end{array} \begin{array}{c} J_3 \cdots J_3 \quad X_2 \\ \begin{pmatrix} 0 * \cdots * & 0 \\ T^{r-2} 0 \cdots 0 & 0 \\ * & U_2 \\ 0 & I_{\lambda_1} \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \\ \vdots \\ J_r \\ \vdots \\ Y_1 \end{array} \begin{array}{c} J_3 \cdots J_3 \quad X_2 \\ \begin{pmatrix} 0 * \cdots * & 0 \\ T^{r-2} 0 \cdots 0 & 0 \\ U_3 & * \\ 0 & I_{\lambda_1} \end{pmatrix} \end{array} \rightarrow$$

$$\begin{array}{c} \\ \vdots \\ J_r \\ \vdots \\ Y_1 \end{array}\begin{array}{c} \begin{array}{cc} J_3\cdots J_3 & X_2 \end{array} \\ \begin{pmatrix} 0*\cdots* & 0 \\ T^{r-2}0\cdots 0 & 0 \\ U_3 & U_1 \\ 0 & * \end{pmatrix}\end{array} \to \begin{array}{c} \\ \vdots \\ J_r \\ \vdots \\ Y_1 \end{array}\begin{array}{c} \begin{array}{cc} J_3\cdots J_3 & X_2 \end{array} \\ \begin{pmatrix} 0*\cdots* & 0 \\ T^{r-2}0\cdots 0 & 0 \\ U_3 & U_1 \\ 0 & I_{\lambda_1} \end{pmatrix}\end{array}$$

So that we may pull out $\begin{array}{c} \\ J_r \\ J_s \\ J_1 \end{array}\begin{array}{c} \begin{array}{cc} J_3 & J_2 \end{array} \\ \begin{pmatrix} T^{r-2} & 0 \\ T^{s-2} & T^{s-2} \\ 0 & 1 \end{pmatrix}\end{array}$ to decompose C in this case.

Case 4.2.2 $\tau_2 > \tau_3$. That is, U_2 has more rows than U_3.

In this case C has the following form.

$$\begin{array}{c} \\ \vdots \\ \vdots \\ Y_1 \end{array}\begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} * & * \\ \begin{bmatrix} 0 \\ U_3 \end{bmatrix} & \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \\ 0 & I_{\lambda_1} \end{pmatrix}\end{array}$$

We clear out the terms above the U_2 block using row operations.

$$\begin{array}{c} \\ \vdots \\ \vdots \\ Y_1 \end{array}\begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} * & * \\ \begin{bmatrix} 0 \\ U_3 \end{bmatrix} & \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \\ 0 & I_{\lambda_1} \end{pmatrix}\end{array} \to \begin{array}{c} \\ \vdots \\ \vdots \\ Y_1 \end{array}\begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} * & 0 \\ \begin{bmatrix} 0 \\ U_3 \end{bmatrix} & \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \\ 0 & I_{\lambda_1} \end{pmatrix}\end{array}$$

j_{τ_2} is the index of $J_{j_{\tau_2}}$ for the target space of the first row on P_2. Then we can pull out

$$\begin{array}{c c} & J_2 \\ J_{j_{\tau_2}} & \left(\begin{array}{c} T^{j_{\tau_2}-2} \\ \\ 1 \end{array}\right) \\ J_1 & \end{array}$$

to decompose C in this case.

Case 4.2.3 $\tau_2 < \tau_3$. That is, U_2 has less rows than U_3.

Then C in this case has the form.

$$\begin{array}{c} \vdots \\ \vdots \\ Y_1 \end{array} \begin{pmatrix} X_3 & X_2 \\ * & * \\ \begin{bmatrix} U_3' & 0 \\ 0 & U_3'' \end{bmatrix} & \begin{bmatrix} 0 \\ U_2 \end{bmatrix} \\ 0 & I_{\lambda_1} \end{pmatrix}$$

We can clear out terms above the U_3 block using row operations:

$$\begin{array}{c} \vdots \\ \vdots \\ Y_1 \end{array} \begin{pmatrix} X_3 & X_2 \\ * & * \\ \begin{bmatrix} U_3' & 0 \\ 0 & U_3'' \end{bmatrix} & \begin{bmatrix} 0 \\ U_2 \end{bmatrix} \\ 0 & I_{\lambda_1} \end{pmatrix} \to \begin{array}{c} \vdots \\ \vdots \\ Y_1 \end{array} \begin{pmatrix} X_3 & X_2 \\ 0 & * \\ \begin{bmatrix} U_3' & 0 \\ 0 & U_3'' \end{bmatrix} & \begin{bmatrix} 0 \\ U_2 \end{bmatrix} \\ 0 & I_{\lambda_1} \end{pmatrix}$$

j_{τ_3} denotes the index of the target space $J_{j_{\tau_3}}$ for the first row in U_3.We can then pull out

$$\begin{array}{c c} & J_3 \\ J_{j_{\tau_3}} & \left(T^{j_{\tau_3}-2} \right) \end{array}$$

to decompose C in this case. This ends our 3 subcases for case 4.2. We now move on to case 4.3.

Case 4.3 $\lambda_1 > \tau_2$. That is, we have more rows in our Y_1 block that than number of columns in our X_2 block. When then diagonalize in our X_2 and Y_1 block to obtain:

$$\begin{matrix} & X_3 & X_2 \\ \vdots & * & * \\ \vdots & * & * \\ Y_1 & \begin{pmatrix} * \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ I_{\tau_1} \end{pmatrix} \end{matrix} \rightarrow \begin{matrix} & X_3 & X_2 \\ \vdots & * & * \\ \vdots & * & U_2 \\ Y_1 & \begin{pmatrix} * \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ I_{\tau_1} \end{pmatrix} \end{matrix}$$

Case 4.3 We have two cases that depend on the sum of τ_3 and τ_2 in comparison with λ_1. The one case we can "ignore" is the case when $\tau_3 + \tau_2 < \lambda_1$ else we obtain a row of zeroes so that a direct summand splits off the target space. So we have two subcases:

$\tau_3 + \tau_2 = \lambda_1$

$\tau_3 + \tau_2 > \lambda_1.$

Case 4.3.1 Suppose $\tau_3 + \tau_2 = \lambda_1$

C is reduced to the form:

$$\begin{matrix} & X_3 & & X_2 \\ \vdots & * & & 0 \\ \vdots & * & & U_2 \\ Y_1 & & I_{\tau_3+\tau_1} & \end{matrix} = \begin{matrix} & X_3 & X_2 \\ \vdots & * & 0 \\ \vdots & * & U_2 \\ Y_1 & \begin{pmatrix} I_{\tau_3} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ I_{\tau_1} \end{pmatrix} \end{matrix}$$

where we cleared terms above the U_2 block.

We can also clear terms to the left of U_2. This will mess up some of the terms below I_{τ_3} but Gaussian elimination using rows operations restores it.

$$\begin{array}{c} \\ \vdots \\ \vdots \\ Y_1 \end{array} \begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} * & 0 \\ * & U_2 \\ \begin{pmatrix} I_{\tau_3} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ I_{\tau_1} \end{pmatrix} \end{pmatrix} \end{array} \to \begin{array}{c} \\ \vdots \\ \vdots \\ Y_1 \end{array} \begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} * & 0 \\ 0 & U_2 \\ \begin{pmatrix} I_{\tau_3} \\ * \end{pmatrix} & \begin{pmatrix} 0 \\ I_{\tau_1} \end{pmatrix} \end{pmatrix} \end{array} \to \begin{array}{c} \\ \vdots \\ \vdots \\ Y_1 \end{array} \begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} * & 0 \\ 0 & U_2 \\ \begin{pmatrix} I_{\tau_3} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ I_{\tau_1} \end{pmatrix} \end{pmatrix} \end{array}$$

j_{τ_2} denotes the index the target space, $J_{j_{\tau_2}}$, for the first row in U_2. We can then pull out $\begin{array}{c} \\ J_{j_{\tau_2}} \\ J_1 \end{array} \begin{array}{c} J_2 \\ \begin{pmatrix} T^{j_{\tau_2}-2} \\ 1 \end{pmatrix} \end{array}$ to decompose C in this case.

Subcase 4.3.2 Suppose $\tau_3 + \tau_2 > \lambda_1$

attempting to reduce C in this case yields:

$$\begin{array}{c} \\ \vdots \\ \vdots \\ Y_1 \end{array} \begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} & * & * \\ & * & * \\ 0 & I_{\lambda_1-\tau_2} & \begin{matrix} 0 \\ I_{\lambda_1} \end{matrix} \end{pmatrix} \end{array}$$

In this case, we continue the diagonalize "all the way", using column operations to obtain something of the form:

$$\begin{array}{c} \\ \vdots \\ \vdots \\ \\ \end{array} \begin{pmatrix} X_3 & X_2 \\ * & * \\ * & * \\ & D & \end{pmatrix}$$

We define it precisely as follows. As before, recall that j_1 was the index of the target space J_{j_1} for the first row above Y_1. We construct a next collection of indices. Counting from bottom to top in our matrix C, Let k_1 denote the index of the target space J_{k_1} for the $\tau_2 + 1$-th row starting from the bottom.

We let $k_1, k_2, \cdots, k_{\tau_3}$ denote the indices of the target spaces $J_{k_1}, \cdots, J_{k_{\tau_3}}$ counting from the bottom to top of our matrix C where k_1 is as defined as above. We define D to be the $\tau_3 \times \tau_3$ matrix defined as follows:

$$(D)_{r,s} := \begin{cases} T^{(k_{\tau_3+1-r})-2} & \text{if } r = s \\ \\ 0 & \text{if } r \neq s \end{cases}$$

For all integers r, s such that $1 \leq r, s \leq \tau_3$.

We can then write it as:

$$D = \begin{array}{c} \\ \vdots \\ Y_1 \end{array} \begin{pmatrix} J_3 \cdots J_3 & J_3 \cdots J_3 \\ D_2 & 0 \\ 0 & D_1 \end{pmatrix}$$

Where D_1 is the $\lambda_1 - \tau_2$ by $\lambda_1 - \tau_2$ square matrix that lies in the Y_1 block and D_2

is the $\tau_3 + \tau_2 - \lambda_1$ square matrix that lies out of the Y_1 block. We then have our last three cases to consider that again depend on the relations between τ_3 ,τ_2 and λ_1. That is, how height differences of D and U_2 compare with each other. In particular, how the height differences between D_2 and U_2 have with each other. That is, how $\tau_3 + \tau_2 - \lambda_1$ and τ_2 have in relation with each other.

We can reduce C the following one of three forms. We have arrived at our last 3 cases that depend on the "height" differences between D_2 and U_2.

$$\text{if } \tau_3+\tau_2-\lambda_1=\tau_2 \qquad \begin{array}{c} \\ \\ \vdots \\ \\ Y_1 \end{array} \begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} * & 0 \\ \begin{bmatrix} D_2 & 0 \end{bmatrix} & U_2 \\ \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix} \end{pmatrix} \end{array}$$

$$\text{if } \tau_3+\tau_2-\lambda_1<\tau_2 \qquad \begin{array}{c} \\ \\ \vdots \\ \\ Y_1 \end{array} \begin{array}{c} \begin{array}{cc} X_3 & X_2 \end{array} \\ \begin{pmatrix} * & * \\ \begin{bmatrix} * & * \\ D_2 & 0 \end{bmatrix} & \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \\ \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix} \end{pmatrix} \end{array}$$

$$\text{if } \tau_3+\tau_2-\lambda_1>\tau_2 \quad \begin{array}{c} \vdots \\ Y_1 \end{array} \begin{pmatrix} \begin{array}{c} X_3 \\ * \\ \begin{bmatrix} D_2' & 0 & 0 \\ 0 & D_2'' & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & D_1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} & \begin{array}{c} X_2 \\ * \\ \begin{bmatrix} 0 \\ U_2 \end{bmatrix} \\ \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix} \end{array} \end{pmatrix}$$

Case 4.3.2.1 D_2 has more rows than U_2. That is $\tau_3+\tau_2-\lambda_1>\tau_2$

Then C has the form:

$$\begin{array}{c} \vdots \\ Y_1 \end{array} \begin{pmatrix} \begin{array}{c} X_3 \\ * \\ \begin{bmatrix} D_2' & 0 & 0 \\ 0 & D_2'' & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & D_1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} & \begin{array}{c} X_2 \\ * \\ \begin{bmatrix} 0 \\ U_2 \end{bmatrix} \\ \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix} \end{array} \end{pmatrix}$$

k_{τ_3} denotes the index of the target space $J_{k_{\tau_3}}$ for the first row of D_2'. Then we can simply start off in the first position of this matrix to clear entries above and then pull out $\begin{array}{c} \\ J_{k_{\tau_3}} \end{array} \begin{pmatrix} J_3 \\ T^{k_{\tau_3}-2} \end{pmatrix}$ and thus we can decompose C in this case.

subsubcase 4.3.2.2 D_2 has the same rows as U_1. That is, $\tau_3+\tau_2-\lambda_1=\tau_2$.

Then C in this case the form:

$$\begin{array}{c} \\ \vdots \\ \\ Y_1 \end{array} \begin{array}{c} \quad X_3 \qquad\qquad X_2 \\ \begin{pmatrix} * & 0 \\ \begin{bmatrix} D_2 & 0 \end{bmatrix} & U_2 \\ \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix} \end{pmatrix} \end{array}$$

Let r denote the index of the target space J_j for the first row above blocks D_2 and U_2. Let $s = k_{\tau_3}$. Then s is the index of the target space J_s for the first row of the D_2 and U_2 blocks. Our starting position will be the first entry in the J_r row described to clear our all terms above.

$$\begin{array}{c} \\ \vdots \\ J_r \\ \vdots \\ \\ Y_1 \end{array} \begin{array}{c} \quad X_3 \qquad\qquad X_2 \\ \begin{pmatrix} 0 * \cdots * & \\ \vdots * \cdots * & \\ T^{r-2} * \cdots * & 0 \\ \begin{bmatrix} D_2 & 0 \end{bmatrix} & U_2 \\ \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix} \end{pmatrix} \end{array}$$

Next, we can perform column operations to clear out all terms to the right. This however ”messes up” entries in our D_2 block. Using row operations we can restore it. This messes up our U_2 block but using column operations restores it but ”messes up” our identity block $I_{\lambda_1-\tau_2}$ Gaussian elimination restore it.

$$
\begin{array}{c}
\\ \\ \vdots \\ J_r \\ \vdots \\ Y_1
\end{array}
\begin{array}{c}
\begin{array}{cc} X_3 & X_2 \end{array} \\
\begin{pmatrix}
0 * \cdots * & \\
\vdots * \cdots * & \\
T^{r-2} * \cdots * & 0 \\
\begin{bmatrix} D_2 & 0 \end{bmatrix} & U_2 \\
\begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix}
\end{pmatrix}
\end{array}
\rightarrow
\begin{array}{c}
\\ \\ \vdots \\ J_r \\ \vdots \\ Y_1
\end{array}
\begin{array}{c}
\begin{array}{cc} X_3 & X_2 \end{array} \\
\begin{pmatrix}
0 * \cdots * & \\
\vdots * \cdots * & \\
T^{r-2} 0 \cdots 0 & 0 \\
\begin{bmatrix} * & * \end{bmatrix} & U_2 \\
\begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix}
\end{pmatrix}
\end{array}
\rightarrow
$$

$$
\begin{array}{c}
\\ \\ \vdots \\ J_r \\ \vdots \\ Y_1
\end{array}
\begin{array}{c}
\begin{array}{cc} X_3 & X_2 \end{array} \\
\begin{pmatrix}
0 * \cdots * & \\
\vdots * \cdots * & \\
T^{r-2} 0 \cdots 0 & 0 \\
\begin{bmatrix} D_2 & * \end{bmatrix} & * \\
\begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix}
\end{pmatrix}
\end{array}
\rightarrow
\begin{array}{c}
\\ \\ \vdots \\ J_r \\ \vdots \\ Y_1
\end{array}
\begin{array}{c}
\begin{array}{cc} X_3 & X_2 \end{array} \\
\begin{pmatrix}
0 * \cdots * & \\
\vdots * \cdots * & \\
T^{r-2} 0 \cdots 0 & 0 \\
\begin{bmatrix} D_2 & * \end{bmatrix} & U_2 \\
\begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ * \end{bmatrix}
\end{pmatrix}
\end{array}
\rightarrow
$$

$$\to \begin{array}{c} \\ \vdots \\ J_r \\ \vdots \\ Y_1 \end{array} \overset{\begin{array}{cc} X_3 & X_2 \end{array}}{\begin{pmatrix} 0 * \cdots * & \\ \vdots * \cdots * & \\ T^{r-2} 0 \cdots 0 & 0 \\ \begin{bmatrix} D_2 & * \end{bmatrix} & U_2 \\ \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1 - \tau_2} \end{bmatrix} \end{pmatrix}}$$

We still have undersired terms next to our D_2 block that prevents us from pulling anything out. We use the terms in the U_2 to kill those terms there. This puts undesired terms under out D_1 block but row operations restores it.

$$\to \begin{array}{c} \\ \vdots \\ J_r \\ \vdots \\ Y_1 \end{array} \overset{\begin{array}{cc} X_3 & X_2 \end{array}}{\begin{pmatrix} 0 * \cdots * & \\ \vdots * \cdots * & \\ T^{r-2} 0 \cdots 0 & 0 \\ \begin{bmatrix} D_2 & 0 \end{bmatrix} & U_2 \\ \begin{bmatrix} 0 & D_1 \\ 0 & * \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1 - \tau_2} \end{bmatrix} \end{pmatrix}} \to \begin{array}{c} \\ \vdots \\ J_r \\ \vdots \\ Y_1 \end{array} \overset{\begin{array}{cc} X_3 & X_2 \end{array}}{\begin{pmatrix} 0 * \cdots * & \\ \vdots * \cdots * & \\ T^{r-2} 0 \cdots 0 & 0 \\ \begin{bmatrix} D_2 & 0 \end{bmatrix} & U_2 \\ \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1 - \tau_2} \end{bmatrix} \end{pmatrix}}$$

Now we can pull out $\begin{array}{c} J_r \\ J_s \\ J_1 \end{array} \overset{\begin{array}{cc} J_3 & J_2 \end{array}}{\begin{pmatrix} T^{r-2} & 0 \\ T^{s-2} & T^{s-2} \\ 0 & 1 \end{pmatrix}}$ to decompose C in this case.

subsubcase 4.3.2.3D_2 has less rows than U_2. That is $\tau_3 + \tau_2 - \lambda_1 < \tau_2$.

Then C has the form

$$\begin{array}{c c c} & X_3 & X_2 \\ \vdots & \begin{array}{c} * \\ \begin{bmatrix} * & * \\ D_2 & 0 \end{bmatrix} \end{array} & \begin{array}{c} 0 \\ \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \end{array} \\ Y_1 & \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1 - \tau_2} \end{bmatrix} \end{array}$$

j_{τ_2} is the index for the target space $J_{j_{\tau_2}}$ for the first row of block U_2. We can begin by clear out terms above D_2. Column operations restores U_2 and row operations restores $I_{\lambda_1-\tau_2}$.

$$\begin{array}{c c c} & X_3 & X_2 \\ \vdots & \begin{array}{c} * \\ \begin{bmatrix} * & * \\ D_2 & 0 \end{bmatrix} \end{array} & \begin{array}{c} 0 \\ \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \end{array} \\ Y_1 & \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1 - \tau_2} \end{bmatrix} \end{array} \rightarrow \begin{array}{c c c} & X_3 & X_2 \\ \vdots & \begin{array}{c} * \\ \begin{bmatrix} 0 & * \\ D_2 & 0 \end{bmatrix} \end{array} & \begin{array}{c} 0 \\ \begin{bmatrix} U_2' & * \\ 0 & U_2'' \end{bmatrix} \end{array} \\ Y_1 & \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1 - \tau_2} \end{bmatrix} \end{array} \rightarrow$$

$$\begin{array}{c c c} & X_3 & X_2 \\ \vdots & \begin{array}{c} * \\ \begin{bmatrix} 0 & * \\ D_2 & 0 \end{bmatrix} \end{array} & \begin{array}{c} 0 \\ \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \end{array} \\ Y_1 & \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ * \end{bmatrix} \end{array} \rightarrow \begin{array}{c c c} & X_3 & X_2 \\ \vdots & \begin{array}{c} * \\ \begin{bmatrix} 0 & * \\ D_2 & 0 \end{bmatrix} \end{array} & \begin{array}{c} 0 \\ \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \end{array} \\ Y_1 & \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1 - \tau_2} \end{bmatrix} \end{array}$$

We can then clear out the missing terms to the left of U_2'. This will add undesired terms below D_1 but row operations gets rid of them.

$$\begin{array}{cc} & \begin{array}{cc} X_3 & \quad X_2 \end{array} \\ \begin{array}{c} \vdots \\ \\ Y_1 \end{array} & \begin{pmatrix} * & 0 \\ \begin{bmatrix} 0 & * \\ D_2 & 0 \end{bmatrix} & \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \\ \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix} \end{pmatrix} \end{array} \to \begin{array}{cc} & \begin{array}{cc} X_3 & \quad X_2 \end{array} \\ \begin{array}{c} \vdots \\ \\ Y_1 \end{array} & \begin{pmatrix} * & 0 \\ \begin{bmatrix} 0 & 0 \\ D_2 & 0 \end{bmatrix} & \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \\ \begin{bmatrix} 0 & D_1 \\ 0 & * \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix} \end{pmatrix} \end{array} \to$$

$$\begin{array}{cc} & \begin{array}{cc} X_3 & \quad X_2 \end{array} \\ \begin{array}{c} \vdots \\ \\ Y_1 \end{array} & \begin{pmatrix} * & 0 \\ \begin{bmatrix} 0 & 0 \\ D_2 & 0 \end{bmatrix} & \begin{bmatrix} U_2' & 0 \\ 0 & U_2'' \end{bmatrix} \\ \begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{\lambda_1-\tau_2} \end{bmatrix} \end{pmatrix} \end{array}$$

Now we can pull out $\begin{array}{cc} & J_2 \\ \begin{array}{c} J_{j_{\tau_2}} \\ J_1 \end{array} & \begin{pmatrix} T^{j_{\tau_2}-2} \\ 1 \end{pmatrix} \end{array}$.

Those are all of the possible cases. Thus we have shown that in any $\mathrm{mod}(\Gamma, d)$, the locus where A and B have a fixed Jordan type has a dense orbit, thus each irreducible component of $\mathrm{mod}(\Gamma, d)$ has a dense orbit. □

4.4.3 Summary of Possible Indecomposables

A lot of the development of the proof strategy for the Main Theorem was knowing which matrices we can actually pull out. A lot of the off hand computations involved the following. We took some arbitrary matrix of some sizes and performed Gaussian elimination to see what we could pull out.

Example 4.9. For example, consider the arbitrary matrix: $C = \begin{array}{c} \\ J_3 \\ J_2 \\ J_2 \end{array} \begin{array}{c} \begin{array}{ccc} J_3 & J_1 & J_1 \end{array} \\ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \end{array}$

Then with appropriate row and column operations yields the following:

$$\begin{array}{c} \\ J_3 \\ J_2 \\ J_2 \end{array} \begin{array}{c} \begin{array}{ccc} J_3 & J_1 & J_1 \end{array} \\ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \\ J_3 \\ J_2 \\ J_2 \end{array} \begin{array}{c} \begin{array}{ccc} J_3 & J_1 & J_1 \end{array} \\ \begin{pmatrix} * & * & * \\ * & T & 0 \\ 1 & 0 & T \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \\ J_3 \\ J_2 \\ J_2 \end{array} \begin{array}{c} \begin{array}{ccc} J_3 & J_1 & J_1 \end{array} \\ \begin{pmatrix} 0 & * & * \\ 0 & T & * \\ 1 & 0 & T \end{pmatrix} \end{array} \rightarrow$$

$$\begin{array}{c} \\ J_3 \\ J_2 \\ J_2 \end{array} \begin{array}{c} \begin{array}{ccc} J_3 & J_1 & J_1 \end{array} \\ \begin{pmatrix} 0 & 0 & * \\ 0 & T & 0 \\ 1 & 0 & T \end{pmatrix} \end{array}$$

and in this example we can pull out $\begin{array}{c} \\ J_2 \end{array} \begin{array}{c} J_1 \\ \begin{pmatrix} T \end{pmatrix} \end{array}$.

We gradually varied the dimension and what we obtained was a finite list of matrices listed below to choose from.

If the dimension vector is $d = (1, 1)$

$$\begin{array}{cc} & J_1 \\ J_j & \begin{pmatrix} T^{j-1} \end{pmatrix} \end{array} \qquad \begin{array}{cc} & J_1 \\ J_1 & \begin{pmatrix} 1 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & J_{i\geq 2} \\ J_{j\geq 2} & \begin{pmatrix} T^{j-2} \end{pmatrix} \end{array}$$

If $d = (1, 2)$

$$\begin{array}{cc} & \begin{matrix} J_2 & J_1 \end{matrix} \\ J_{j\geq 2} & \begin{pmatrix} T^{j-2} & T^{j-1} \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} J_3 & J_1 \end{matrix} \\ J_j & \begin{pmatrix} T^{j-2} & T^{j-1} \end{pmatrix} \end{array}$$

If $d = (2, 1)$;

$$\begin{array}{cc} & J_{r\geq 2} \\ \begin{matrix} J_{i\geq 2} \\ J_1 \end{matrix} & \begin{pmatrix} T^{i-2} \\ 1 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & J_1 \\ \begin{matrix} J_{i\geq 2} \\ J_1 \end{matrix} & \begin{pmatrix} T^{i-1} \\ 1 \end{pmatrix} \end{array}$$

If $d = (2, 2)$:

$$\begin{array}{cc} & \begin{matrix} J_{r\geq 2} & J_1 \end{matrix} \\ \begin{matrix} J_i \\ J_{j\geq 2} \end{matrix} & \begin{pmatrix} T^{i-2} & 0 \\ T^{j-2} & T^{j-1} \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} J_r & J_{s\geq 2} \end{matrix} \\ \begin{matrix} J_{i\geq 2} \\ J_1 \end{matrix} & \begin{pmatrix} T^{i-2} & T^{i-1} \\ 0 & 1 \end{pmatrix} \end{array}$$

If $d = (3, 2)$

$$\begin{array}{c c} & \begin{array}{cc} J_r & J_{s\geq 2} \end{array} \\ \begin{array}{c} J_j \\ J_{i\geq 2} \\ J_1 \end{array} & \begin{pmatrix} T^{j-2} & 0 \\ T^{i-2} & T^{i-2} \\ 0 & 1 \end{pmatrix} \end{array}$$

In all other cases, we can always decompose C by pulling off one of the matrices listed above. This leads to the following corollary:

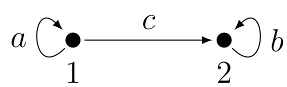
Corollary 4.10. *There are only finitely many indecomposable represenations of* (Γ, I) *whose orbits are dense in their irreducible components.*

We conjecture that the ones in the list above are all actually all of the indecomposable representations whose orbits are dense.

CHAPTER 5
FUTURE WORK

Here we list some suggested topics for further discussion. We can first start by having a more general version of our main theorem.

Conjecture 5.1. *Let A be the algebra given by the*



with relations $ca - bc = b^n = a^m = b^q c = 0$ where $n, m \in \mathbb{N}$ with $q \leq m$. Then A has the dense orbit property for all n, m and is representation infinite for $n \geq 6$.

In our proof of the main theorem, the conditions that mostly dictated on what we could pull out from out matrix C where the number of X_i blocks and the "zeroness" of the Y_1 block. If we change the relation $a^3 = 0$ to $a^4 = 0$, we already have 4 X_i blocks to consider and some computations suggest that there might be larger indecomposables. Our conjecture is the the sizes of the indecomposables are dependant on the size of the exponent of the relation $a^m = 0$. There is promising evidence that suggest that the conjecture is true.

We had finitely many indecomposables types with dense orbits in the Main Theorem. Finding out exactly which ones are they requires proof. However, there is evidence that suggest the representation types listed in section 4.4.3 are all of the indecomposable representations types from the Main Theorem with a dense orbit.

Conjecture 5.2. *The 10 types of summands which split off in the cases of the main* *theorem, the ones listed in section 4.4.3, are all of the indecomposables with dense orbits.*

We know that we have finitely many from Corollary 4.10. Proving that a representation is indecomposable is equivalent to showing that the associated endomorphism ring for the representation is local. This however proves to be computationally tedious for the representations from the list in section 4.4.3. In particular, the representation for dimension vector $d = (3, 2)$ proves to be very tedious. This then leads to a conjecture for dense orbit algebras in general.

Conjecture 5.3. *There are only finitely many indecomposable representations of a bound quiver algebra* $A \cong KQ/I$ *whose orbits are dense in their irreducible components.*

So far, evidence suggest that the conjecture is true. We will explore other types of 2-point tame algebras from the list by Hoshino and Miyachi [7] for further evidence.

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