Summer 2019

Positive definite kernels, harmonic analysis, and boundary spaces: Drury-Arveson theory, and related

Aqeeb A Sabree
University of Iowa

Follow this and additional works at: https://ir.uiowa.edu/etd
Part of the Mathematics Commons

POSITIVE DEFINITE KERNELS, HARMONIC ANALYSIS, AND BOUNDARY SPACES: DRURY-ARVESON THEORY, AND RELATED.

by

Aqeeb A. Sabree

A thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Mathematics
in the Graduate College of
The University of Iowa

August 2019

Thesis Supervisor: Professor Palle E.T. Jørgensen

Graduate College
The University of Iowa
Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Aqeeb A. Sabree

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the August 2019 graduation.

Thesis committee: ______________________________
Palle Jørgensen, Thesis Supervisor

____
Paul Muhly

____
Victor Camillo

____
Richard Baker

____
Ionut Chifan

I begin with the name of God, Most Gracious, Most Merciful; God has blessed me beyond measure.

To my parents, Haazim and Djameela Sabree. Thank you both for your work behind establishing the environment that shaped me. Each of you have been a great example of a believer, spouse, parent, teacher, and hard-worker, to name only a few. Through your examples I grew to appreciate learning in general and mathematics in particular. Most importantly, through your leadership I was introduced to Imam Warith Deen Mohammed, which propelled my critical thinking and logical reasoning ability to new heights.

To my eight siblings, As'ad, Nasheed, Makeen, Albertha, Shakoora, Haseena, Tareef, and Jawwaad. Thank you for your love and support. I am blessed with each one of you; and I want you all to know two things: I am proud to be one with you and you have helped me write this dissertation more than words can express.

To my children Halimah and Aliyah. This was done with you in mind every day.

Finally, to my wife, Dr. Ruqiah Muhammd. I love you! Everyday with you has been wonderful. Your hugs always managed to erase any hardship or confusion. Thank you for being with me on this amazing journey. Marrying you is (my) proof that I am a smart person. We made a powerful husband and wife team as undergraduate and graduate students. Indeed, with every difficulty there is relief: We simultaneously completed the mathematics doctoral program at the University of Iowa; congratulations!

ACKNOWLEDGEMENTS

First, I would like to acknowledge my advisor, Palle Jørgensen, who is a very patient, kind, precise, and intelligent advisor. Anybody who knows Jørgensen knows that statement is not an exageration but an understatement. Thank you Jørgensen for choosing to be my advisor and always being there for me. Also, thank you to my thesis committee; I have learned a lot through our interactions, trips to the Iowa-Nebraska Functional Analysis Seminar (INFAS), departmental seminars, classes, and office discussions.

I would especially like to thank my graduate school mentors Philip Kutzko, Paul Muhly, and Colleen Mitchell. You each have had a great impact on my life, and consequently the lives of my family members. Specifically, Dr. Phil and the Math Alliance are at the foundation of me applying to doctoral programs and working through the doctoral program at the University of Iowa. With that in mind, I would like to thank the entire faculty and staff in the math department; and I would like to thank the graduate college. The education that I received at the University of Iowa expanded my understanding of mathematics. The courses were rigorous and often times fast-paced; yet, the challenge was bearable because of the helpful professors and the atmosphere within the department.

Thank you to all of my family and friends; you are my support and each of you have been instrumental to this development. It would take too many pages to provide (almost) a full list of names, but your imprint is on my heart and soul.

Lastly, I would like to ackowledge some members of my family and friends (in no particular order) that helped me throughout this doctoral program. Thank you J. Edmonds, R. Clark-Edmonds, M. King, W. Bungula, R. Alexander, and R. Muhammad for making my life in Iowa one to remember. Also, I would be remiss if I did not thank Texas Southern University and it's math department faculty and staff. A special thank you goes to Ms. Tolbert, Dr. Holmes, Dr. Taylor, Dr. Nehs, Dr. Saydam, Dr. Evans, and Dr. Wilson. You all saw my potential at an early stage and you developed it. My spirit to teach, mentor, research, and learn would not be the same without you.

ABSTRACT

A reproducing kernel Hilbert space (RKHS) is a Hilbert space $\mathscr{H}$ of functions with the property that the values $f(x)$ for $f \in \mathscr{H}$ are reproduced from the inner product in $\mathscr{H}$. Recent applications are found in stochastic processes (Ito Calculus), harmonic analysis, complex analysis, learning theory, and machine learning algorithms. This research began with the study of RKHSs to areas such as learning theory, sampling theory, and harmonic analysis. From the Moore-Aronszajn theorem, we have an explicit correspondence between reproducing kernel Hilbert spaces (RKHS) and reproducing kernel functions—also called positive definite kernels or positive definite functions. The focus here is on the duality between positive definite functions and their boundary spaces; these boundary spaces often lead to the study of Gaussian processes or Brownian motion. It is known that every reproducing kernel Hilbert space has an associated generalized boundary probability space. The Arveson (reproducing) kernel is $K(z,w) = \frac{1}{1-<z,w>_{\mathbb{C}^d}}$, $z, w \in \mathbb{B}_d$, and Arveson showed, [6], that the Arveson kernel does not follow the boundary analysis we were finding in other RKHS. Thus, we were led to define a new reproducing kernel on the unit ball in complex n-space, and naturally this lead to the study of a new reproducing kernel Hilbert space. This reproducing kernel Hilbert space stems from boundary analysis of the Arveson kernel. The construction of the new RKHS resolves the problem we faced while researching "natural" boundary spaces (for the Drury-Arveson RKHS) that yield boundary factorizations:

$$K(z,w) = \int_{\mathcal{B}} K_z^{\mathcal{B}}(b)\overline{K_w^{\mathcal{B}}(b)}d\mu(b), \quad z,w \in \mathbb{B}_d \text{ and } b \in \mathcal{B} \qquad \textit{(Factorization of K).}$$

Results from classical harmonic analysis on the disk (the Hardy space) are generalized and extended to the new RKHS. Particularly, our main theorem proves that, relaxing the criteria to the contractive property, we can do the generalization that Arveson's paper showed (criteria being an isometry) is not possible.

PUBLIC ABSTRACT

A central theme in mathematics is the concept of a function. There are many examples and applications of functions. They are a fundamental topic of high school and collegiate math courses. Often times, a calculus course is a student's first introduction to an in-depth study of functions. The universality of functions is witnessed by viewing the impact calculus has on education. Functions connect many areas of mathematics such as Topology, Analysis, Abstract Algebra (for instance Galois Theory), and Differential Equations. A function space is simply a collection of functions; you can make this collection as you please, but some collections (of functions) are heavily studied because of their shared features—and the properties arising on the space. For example, the collection of complex numbers forms a function space. The stock market predictions, an airplane's ability to fly, and the human body are just three (of many) areas where the study of function spaces are useful. The more common phrase is mathematical modeling.

The following dissertation studies function spaces specific to Harmonic Analysis. Specifically, we study functions that live inside of functions spaces called reproducing kernel Hilbert spaces. The title of this dissertation was "Reproducing Kernel Hilbert Spaces:, Harmonic Analysis, and Boundary Spaces: Drury-Arveson Theory and Related". I say that to stress the importance of the phrase "Reproducing Kernel Hilbert Spaces;" this phrase alone provides an avenue research on the background or applications of this dissertation. The Drury-Arveson space is an example of a function

space, which is also a reproducing kernel Hilbert space. The polynomial functions that are studied in a high school algebra course can be viewed in a (function) space of polynomials $\mathscr{P}$; and this space of polynomials $\mathscr{P}$ lives inside of the Drury-Arveson space. Moreover, $\mathscr{P}$ is foundational for studying properties of the Drury-Arveson RKHS, for example. What's more useful than a polynomial when it comes to these function spaces? The function that we call *the positive definite kernel* or *reproducing kernel* that lives in the reproducing kernel Hilbert space. Every reproducing kernel Hilbert space has this *kernel function.* Most importantly, every RKHS has exactly one (an identifier of sorts) positive definite kernel; such that changing spaces changes the reproducing kernel. My research studies qualities, similarities, and differences between function spaces and factorizations of their positive definite kernels.

TABLE OF CONTENTS

LIST OF TABLES

Table

Chapter 1
INTRODUCTION

Hilbert space theory serves as a prime example in mathematics of a synergy between symbolic manipulation and visual reasoning. A positive definite function is synonomous with what we call a reproducing kernel, and the existence of a reproducing kernel implies the existence of a reproducing kernel Hilbert space (RKHS). Reproducing kernel Hilbert spaces are Hilbert spaces $\mathscr{H}$ of functions with the property that the values $f(x)$ for $f \in \mathscr{H}$ are reproduced from the inner product in $\mathscr{H}$. Positive definite functions and reproducing kernel Hilbert spaces play an important role in many aspects of pure and applied mathematics. Direct applications are found in such areas as probability theory, stochastic processes, representation theory, harmonic analysis, complex analysis, approximation theory, information theory, and machine learning. In chapter two, we give examples of positive definite functions with their respective RKHSs and applications. The following dissertation helps to understand RKHSs and their boundary spaces. Moreover, we construction a new RKHS related to that is boundedly contained in the Drury-Arveson space; and results from classical harmonic analysis on the disk are generalized and extended to this new RKHS.

As we begin, it is important to be aware of the different definitions in the literature for the terms *positive definite* and *positive semidefinite*. The definitions of the terms (in different literatures) are not the same. Some areas use the term positive definite function to refer to kernel functions, while others use the term positive semidefinite to refer to kernel functios. Here we use the term positive definite function synonomously with kernel functions or reproducing kernels. The difference between the terminology can be seen through the lens of matrices.

Definition 1.1. Let $M = (a_{i,j})$ be an $n \times n$ complex matrix. We say that M is a **Positive Matrix** if and only if for every $\{c_1, \ldots, c_n\} \subset \mathbb{C}$ we have that

$$\sum_{i,j=1}^{n} c_i \overline{c_j} a_{i,j} \geq 0.$$

When M is a positive matrix we write $M \geq 0$.

Remark. In the literatures, sometimes the terminology for the above definition is a Positive Semidefinite Matrix or simply a Nonnegative Matrix. The approach that we take here is most common among functional anaysis and operator theory. Note that throughout this dissertation the following terms are synonomous: Kernel Function, Positive Definite Kernel, Positive Definite Function, and Reproducing Kernel.

Definition 1.2. Let X be any set and let $K : X \times X \longrightarrow \mathbb{C}$ be a function of two variables. Then K is called a **Kernel Function** or a **Positive Definite Function** provided that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} K(x_i, x_j) \geq 0,$$

for all $\{x_1, x_2, \ldots, x_n\} \subseteq X$ and all $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathbb{C}$.

The correspondence between the terms *Positive Definite Function* and *Reproducing Kernel* is credited to the Moore-Aronszajn theorem; the two terms have a logical equaivalence not identical definitions. To emphasize this importance we present this result with the detailed definitions.

Definition 1.3. Let X be any set and let $\mathscr{H} \subset \mathscr{F}(X, \mathbb{C})$ be a Hilbert function space with inner product $< \cdot, \cdot >_{\mathscr{H}}$. The function $K : X \times X \longrightarrow \mathbb{C}$ is called a **Reproducing Kernel** for $\mathscr{H}$ provided that

1. for every $x \in X$, all functions of the form K_x are in $\mathscr{H}$, where

$$K_x : y \longmapsto K(x, y);$$

2. for every $x \in X$ and $f \in \mathscr{H}$ the reproducing property holds:

$$f(x) =< f, K_x >_{\mathscr{H}} .$$

The important thing to have in mind is that every RKHS $\mathscr{H}$ has exactly one reproducing kernel for $\mathscr{H}$. Also, the existence of a reproducing kernel for a function space $\mathscr{H}$ implies that $\mathscr{H}$ is a RKHS. Furthermore, the reproducing kernel determines a unique RKHS; that is, the function space itself (all functions in $\mathscr{H}$) can be reconstructed via the the reproducing kernel for $\mathscr{H}$. The study and the construction of the reproducing kernel Hilbert space associated to a positive definite kernel is of great importance to this dissertation.

Theorem 1.1 (Moore-Aronszajn)**.** *For any set S, $K : S \times S \longrightarrow \mathbb{C}$ is a positive definite function (or kernel function) if and only if it is a reproducing kernel.*

Proof Outline: Assume that $K : X \times X \longrightarrow \mathbb{C}$ is a positive definite function. The goal is to show that there is a unique RKHS, namely $\mathscr{H}(K)$, with K as its reproducing kernel. We have from the definition that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} K(x_i, x_j) \geq 0,$$

for all $\{x_1, x_2, \dots, x_n\} \subset X$ and all $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathbb{C}$. For every $x \in X$, define $k_x \in \mathscr{F}(X, \mathbb{C})$ as follows

$$k_x : X \longrightarrow \mathbb{C} \text{ such that } k_x(y) = K(x, y).$$

- **(Pre-Hilbert Space)** Define $\mathscr{H}_0 = \text{span}\{k_x : x \in X\}$ and take two functions in $\mathscr{H}_0$, $f = \sum_i \alpha_i k_{x_i}$ and $g = \sum_j \alpha_j k_{x_j}$. Then the function

$$< \cdot, \cdot >_{\mathscr{H}_0} : \mathscr{H}_0 \times \mathscr{H}_0 \longrightarrow \mathbb{C} \text{ defined by}$$

$$< f, g >_{\mathscr{H}_0} = \left\langle \sum_{i=1}^{n} \alpha_i k_{x_i}, \sum_{j=1}^{n} \alpha_j k_{x_j} \right\rangle_{\mathscr{H}_0} = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} K(x_i, x_j)$$

  defines a well-defined inner product on $\mathscr{H}_0$. It then follows that,

  (i) $\|f\|^2_{\mathscr{H}_0} = \sum_{i=1}^{n} \alpha_i \overline{\alpha_j} K(x_i, x_j) \geq 0$,

  (ii) $< f, k_x >_{\mathscr{H}_0} = \sum_{i=1}^{n} \alpha_i K(x_i, x) = \sum_{i=1}^{n} \alpha_i k_{x_i}(x) = f(x)$.

- **(RKHS)** Next we complete the space $\mathscr{H}_0$ to obtain a Hilbert space $\mathscr{H}$.

A reproducing kernel Hilbert space is characterized as a Hilbert space $\mathscr{H}$ of functions (defined on a predescribed set) in which point-evaluation is a continuous linear functional; so continuity is required to hold with respect to the norm in $\mathscr{H}$. These Hilbert spaces (RKHS) have a vast amount of applications to complex analysis, harmonic analysis, and quantum mechanics. Research in functional analysis combines different settings, which is why RKHSs have been instrumental in recent research in functional analysis. In this dissertation, we place the focus on understanding reproducing kernel Hilbert spaces, the kernel functions, and boundary space constructions. In functional analysis, "boundary" spaces for a RKHS can be understood beyond the existence of a topological boundary. Therefore, we are interested in understanding reproducing kernel boundary factorizations over the "natural" boundary space and comparisons bewteen the different boudary spaces. We prioritize making precise a variety of notions of boundary and boundary representations for the Arveson kernel and our new positive definite kernel.

We use the results that we have for RKHSs in the literature to answer similar questions for the Drury-Arveson space. The Drury-Arveson space is a generalization of the Hardy space, the RKHS of functions on the unit disk in $\mathbb{C}$. The Drury-Arveson space is obtained by generalizing to holomorphic functions on the unit ball in $\mathbb{C}^d$. Throughout this dissertation, we denote the set of boundary spaces for a given given positive definite kernel K as $\mathcal{M}(K)$. Using the theory of Gaussian processes, we know that there is always a generalized boundary for any positive deinite kernel. For example, as an element in $\mathcal{M}(K)$, we can take a "measure" boundary to be the Guassian process having K as its covariance kernel; such a construction exists by Kolmogorov's consistency theorem. Moreover, we know that the set of boundary spaces $\mathcal{M}(K)$ for a given kernel function K has minimal elements, and this minimal boundary space that carries a factorization is not unique. Different boundary spaces provide different factorizations for the kernel function K. The motivation behind this dissertation was to come up with factorizations for the Drury-Arveson space kernel function, referred to as the *Arveson Kernel*. We will present the general theory on factorizations of positive definite functions and provide examples to familiarize the concept.

Since Arveson showed, [6], that the Drury-Arveson space is not isometric to a space $L^2(\mu)$ where μ is a positive measure on $\mathbb{C}^d$, the following results are provided to introduce an approach to understanding the boundary analysis for the Arveson kernel. In general, Carleson measures are a central theme in the framework of boundary analysis and applications of kernel theory. Thus, we began our boundary analysis of the Arveson kernel by studying Carleson measures for the Drury-Arveson Hardy space, [4]. In this setting, the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ is discretized via a Bergman tree construction, and Carleson measures for H_d^2 are characterized in terms of the associated Bergman tree $\mathcal{T}_d$. Stegenga, [13], established a characterization of Carleson's classical theorem for the Hardy space; more specifically, he characterized the Carleson measures for the more general Dirichlet spaces. Furthermore, Carleson measures for Hardy spaces on trees were studied in [3].

Chapter 2
HILBERT FUNCTION SPACES AND BOUNDARY SPACES

The 1950 paper by Aronszajn, [5], on the Theory of Reproducing Kernels developed the general theory of reproducing kernel Hilbert spaces (RKHSs). Reproducing kernels were studied and utilized in reference to their applications before Aronszajn. Yet, this paper formulated the definitions and classifications of reproducing kernels and their respective RKHSs, and it increased research on reproducing kernels and their applications. Recently the interest in reproducing kernels has focused heavily on their applications to statistics, data science, and particulary (machine learning algorithms) learning theory. Parzen, [9], highlights the role of RKHS in statistics, specifically to time series analysis. He lists the following applications for which the theory of reproducing kernels are used:

1. Estimation of parameters of linear models.
2. Regression analysis and design of experiments.
3. Relations of time series analysis to approximation theory.
4. Probability density functionals of Gaussian processes.
5. Minimum variance unbiased estimation.
6. Representation of stochastic processes.
7. Properties of the probability measures on linear spaces induced by Gaussian processes.
8. Limit theorems for stochastic processes.

Applications of reproducing kernel Hilbert spaces far exceed the above list from Parzen. Along with providing the background, this chapter presents some examples of reproducing kernel Hilbert spaces that arose in our research leading up to this dissertation. This provides a framework for some applications beyond the current focus; though there are interesting applications and questions that stem from the focus of this dissertation.

A recurrent theme in functional analysis is the interplay between the theory of positive definite functions, or reproducing kernels, on the one hand, and Gaussian stochastic processes, on the other. This central theme is motivated by a host of applications. In this chapter, we define reproducing kernel Hilbert spaces and positive definite kernels, and we show the correspondence between the two. Next we introduce fundamental examples for factorizations of positive definite kernels. The intersection with the theory of Gaussian stochastic processes requires its own development. Thus, this is presented in chapter three when we discuss boundary spaces for RKHS. The examples we present provide a foundation for understanding RKHSs, and provide a framework to understand the results in chapter four.

2.1 Reproducing Kernel Hilbert Spaces and Positive Definite Kernels

A reproducing kernel Hilbert space is a Hilbert space $\mathscr{H}$ consisting of functions on some space X, such that for every $x \in X$ the point evaluation $f \longmapsto f(x)$ is a bounded linear functional on $\mathscr{H}$.

Definition 2.1. Let X be a set. We say that a subset $\mathscr{H} \subset \mathscr{F}(X, \mathbb{C})$ is a

Reproducing Kernel Hilbert Space (RKHS) of functions on X if the following conditions hold:

1. $\mathscr{H}$ is a vector subspace of $\mathscr{F}(X, \mathbb{C})$
2. $\mathscr{H}$ is endowed with an inner product $<,>_{\mathscr{H}}$ which makes $\mathscr{H}$ a Hilbert space
3. $\forall x \in X$, the linear evaluation functional, $E_x : \mathscr{H} \longrightarrow \mathbb{C}$, defined by $E_x(f) = f(x)$, is bounded.

As a consequence of the Riesz Representation Theorem, given that $\mathscr{H}$ is a RKHS of functions on X, we know that for every $x \in X$ there exists a unique vector $k_x \in \mathscr{H}$ such that

$$E_x(f) =< f(\cdot), k_x(\cdot) >_{\mathscr{H}} . \tag{2.1}$$

We call k_x the **Reproducing Kernel For The Point $x \in X$.**

Definition 2.2. Let $\mathscr{H}$ be a RKHS on a set X. Let $K : X \times X \longrightarrow \mathbb{C}$ be the function defined by

$$K(x, y) = k_x(y);$$

we call this function the **Reproducing Kernel For $\mathscr{H}$.**

Remark. In the literature, reproducing kernel Hilbert spaces are sometimes simply called Hilbert function spaces. The name reproducing kernel Hilbert spaces captures the identifying property of the Hilbert function spaces.

Proposition 2.1. *Given a reproducing kernel $K : X \times X \to \mathbb{C}$ for the RKHS $\mathscr{H}$, then the following conditions hold:*

1. $K(x,y) =< k_x(\cdot), k_y(\cdot) >_H$,

2. $K(x,x) \geq 0$,

3. $K(x,y) = \overline{K(y,x)}$,

4. $|K(x,y)|^2 \leq K(x,x) \cdot K(y,y)$,

for all $x, y \in X$.

Proof. Assume that $\mathscr{H}$ is a RKHS with reproducing kernel K and let $x, y \in X$.

1. It follows from the definition and (2.1) that

$$K(x,y) = k_x(y) \tag{2.2}$$

$$= E_y(k_x) \tag{2.3}$$

$$=< k_x(\cdot), k_y(\cdot) >_{\mathscr{H}}, \tag{2.4}$$

2. $K(x,x) =< k_x(\cdot), k_x(\cdot) >_{\mathscr{H}} = \|k_x\|^2 \geq 0$,

3. $K(x,y) =< k_x(\cdot), k_y(\cdot) >_{\mathscr{H}} = \overline{< k_y(\cdot), k_x(\cdot) >_{\mathscr{H}}} = \overline{K(y,x)}$,

4. It follows from the definition and the Cauchy-Schwartz inequality that

$$|K(x,y)|^2 = | < k_x, k_y > |^2 \tag{2.5}$$

$$\leq \|k_x\|^2 \cdot \|k_y\|^2 \qquad \text{(Cauchy-Schwartz)}$$

$$= K(x,x) \cdot K(y,y). \tag{2.6}$$

□

Proposition 2.2. *Let* $\mathscr{H}$ *be a RKHS on the set* X *with reproducing kernel* K*. Then the linear span of the functions* $k_y(\cdot) = K(\cdot, y)$ *is dense in* $\mathscr{H}$*.*

Proof. Let $\mathscr{H}$ be a RKHS on the set X with reproducing kernel K, and define $S = \text{span}\{k_y \in \mathscr{H} | y \in X\}$. Assume that $f \in S^{\perp}$. It follows that $f(y) =< f, k_y >= 0$ for every $y \in X$, and we have that $f = 0$. □

Lemma 2.3. *Let $\mathscr{H}$ be a RKHS on X and let $\{f_n\} \subseteq \mathscr{H}$. If*

$$\lim_{n \to \infty} \|f_n - f\| = 0,$$

then

$$f(x) = \lim_{n \to \infty} f_n(x), \quad \forall x \in X.$$

Proof. Let $\mathscr{H}$ be a RKHS on X and let $\{f_n\} \subseteq \mathscr{H}$. Assume that

$$\lim_{n \to \infty} \|f_n - f\| = 0.$$

Then it follows that

$$\lim_{n \to \infty} |f_n(x) - f(x)| = \lim_{n \to \infty} | < f_n - f, k_x > | \tag{2.7}$$

$$\leq \lim_{n \to \infty} \|fn - f\| \cdot \|k_x\| \tag{2.8}$$

$$= 0. \tag{2.9}$$

Therefore, we conclude that

$$f(x) = \lim_{n \to \infty} f_n(x), \quad \forall x \in X.$$

□

The following corollary is a direct consequence of the Moore-Aronszajn theorem.

Corollary 2.4. *Let $\mathscr{H}(K_1)$ and $\mathscr{H}(K_2)$ be two RKHSs on X with reproducing kernels K_1 and K_2, respectively. If $K_1(x,y) = K_2(x,y)$ for all $(x,y) \in X \times X$, then $\mathscr{H}(K_1) = \mathscr{H}(K_2)$ and $\|f\|_{\mathscr{H}(K_1)} = \|f\|_{\mathscr{H}(K_2)}$ for every f.*

Next we define band-limited functions, so named because they are functions for which the support of their Fourier transform is "band-limited" to a finite interval.

Definition 2.3. We say that a function (or time series) is a **Band-limited Function** if its Fourier transform is restricted to a finite range of frequencies or wavelenghts.

Theorem 2.5 (The Shannon Sampling Theorem)**.** *The Shannon sampling theorem states that if $f(t)$ can be represented as*

$$f(t) = \int_{-A}^{A} \hat{f}(\omega) e^{i\omega t} d\omega$$

where $\int_{[-A,A]} |\hat{f}(\omega)|^2 d\omega < \infty$, then

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\tau) \frac{\sin[(\pi/\tau)(t - n\tau)]}{(\pi/\tau)(t - n\tau)},$$

where $\tau \leq \frac{\pi}{A}$, and the sampling series converges uniformly on any bounded interval.

The space of band-limited functions, often called the Paley-Weiner space, is a RKHS of continuous functions on $\mathbb{R}$. This is an important RKHS used in the study of sampling theory and image processing.

Example 2.6 (Paley-Weiner Space)**.** The Paley-Weiner space is defined as the space of band-limited L^2-functions f defined in the whole real line $\mathbb{R}$. The terminology of band-limited function refers to the Fourier transform of f

$$\hat{f}(\omega) := \int_{\mathbb{R}} f(x) e^{-i\omega x} dx, \forall \omega \in \mathbb{R}$$

being supported in a frequency band, i.e. a compact interval such as $[-\pi, \pi]$. Next we show that the Paley-Weiner space $\mathcal{PW}_{[-\pi,\pi]}$ is a RKHS,

$$\mathcal{PW}_{[-\pi,\pi]} = \left\{ f \in L^2(\mathbb{R}) \,\middle|\, \operatorname{supp}(\hat{f}) \subset [\pi, \pi] \right\}.$$

One can show that if $f \in \mathcal{PW}_{[-\pi,\pi]}$, then

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{ix\omega} d\omega$$

for some $\omega \in L^2[-\pi, \pi]$. It then follows that

$$\begin{aligned} |f(x)| &\leq \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} 1 d\omega} \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)|^2 d\omega} && \text{(Cauchy-Schwartz)} \\ &= \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx} && \text{(Plancherel's Theorem)} \\ &= \|f\|_{L^2(\mathbb{R})}. && (2.10) \end{aligned}$$

This shows that the evaluation functional is bounded and thus $\mathcal{PW}_{[-\pi,\pi]}$ is a RKHS. The Paley-Weiner RKHS has as its kernel function

$$K(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)}, \ \forall (x, y) \in \mathbb{R} \times \mathbb{R}.$$

Remark. Note that $K(\cdot - y)$ is the band-limited version of the Dirac delta distribution and that $K(\cdot - y)$ converges to $\delta(\cdot - y)$ in the weak sense.

Example 2.7 (Sobolev Space on $[0, 1]$)**.** The condition on a function f to belong to the Sobolev Space on $[0, 1]$ is

$$\|f\|^2 = \int_0^1 |f'(x)|^2 < \infty,$$

where $f' = \frac{df}{dx}$. In this RKHS, the reproducing kernel is

$$K(x, y) = x \wedge y = \min(x, y), \forall (x, y) \in [0, 1] \times [0, 1].$$

This RKHS is also associated with the study of Brownian motion. In this particular application the reproducing kernel is extend to $\mathbb{R}_+ \times \mathbb{R}_+$,

$$K(x,y) = \min\{x,y\},\ x,y \in \mathbb{R}_+ \tag{2.11}$$

Remark. The Hilbert space $L^2[0,1]$ is not a reprocuing kernel Hilbert space.

Example 2.8 (Hilbert Energy Space)**.** The Hilbert Energy space $\mathscr{H}_E$ is the Hilbert space of all finite-energy functions defined on the vertices of some weighted graph (G,c). A weighted graph (G,c) is a system of two sets: vertices $G^{(0)}$ and edges $G^{(1)}$, where $G^{(1)} \subset G^{(0)\times(0)}$ and $c : G^{(1)} \longrightarrow \mathbb{R}_+$ (called the conductance) is defined such that

1. $c(x,y) = c(y,x), \forall (x,y) \in G^{(1)}$, and
2. $\forall x \in G^{(0)}$ the set $\{y \in G^{(0)} | c(x,y) > 0\}$ is finite.

   If $(x,y) \in G^{(1)}$ satisfies (ii), then we write $x \sim y$.

   For functions u on $G^{(0)}$ we set

$$\|u\|_E^2 = \frac{1}{2} \sum_{x \sim y} \sum c(x,y) |u(x) - u(y)|^2.$$

We say that $u \in \mathscr{H}_E := H(G,c)$ if $\|u\|_E < \infty$. The RKHS has the following *relative reproducing property:*

$$\forall (x,y) \in G^{(1)}, \exists K_{xy} \in \mathscr{H}_E \text{ such that}$$

$$u(x) - u(y) = < u(\cdot), K_{xy}(\cdot) > .$$

Definition 2.4. Let X be a set, and let $K : X \times X \longrightarrow \mathbb{C}$ be a function. We say that K is a **Positive Definite Kernel** if $\forall N \in \mathbb{N}$, $\forall x_1, \cdots, x_N \in X$, $\forall c_1, \cdots, c_N \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{N} c_i \overline{c_j} K(x_i, x_j) \geq 0.$$

Theorem 2.9 (Moore)**.** *Let X be a set, and let $K : X \times X \longrightarrow \mathbb{C}$ be a function. If K is a positive definite kernel, then there exists a reproducing kernel Hilbert space $\mathscr{H}$ of functions on X such that K is the reproducing kernel of $\mathscr{H}$.*

Example 2.10 (Positive Definite Kernels induced by Positive Matrices)**.** Given a square complex matrix $P = (p_{i,j})_{i,j=1}^{n}$, if we let $X = \{1, \ldots, n\}$, then we may identify P with a function

$$K : X \times X \to \mathbb{C} \quad \text{such that} \quad K(i, j) = p_{i,j}.$$

With this identification, we see that K is a positive definite function if and only if P is a positive definite matrix.

The following example has been utilized within probability and statistics; specifically, machine learning algorithms that involve kernel method techniques. In data analysis applications, we sometimes work with large data sets that are not "good" data sets. An example can be a data set that makes it difficult to apply separation techniques because it requires non-linear methods. Given data from a set X, we can deal with non-linear classification problems through the framework of a reproducing kernel Hilbert space.

Definition 2.5. A **Feature Map** Φ is an embedding

$$\Phi : X \longrightarrow \mathfrak{L}$$

which maps the set X where the data resides into a (possibly infinite-dimensional) Hilbert space.

Making use of feature maps is very common in applications, because Φ transforms the data and induces a reproducing kernel $K(x,y) =< \Phi(x), \Phi(y) >_{\mathfrak{L}}$, which can be used to simplify classifications through optimization techniques. Depending on the application, we can choose Φ and $\mathfrak{L}$ to generate different RKHSs $\mathscr{H}(K_\Phi)$, which becomes the RKHS of functions on X also known as the predictors.

Example 2.11 (Positive Definite Kernels induced by Inner Products)**.** Let $\mathfrak{L}$ be a Hilbert space with inner product $< \cdot, \cdot >_{\mathfrak{L}}$ and let $K : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathbb{C}$ be defined by $K(x,y) =< x, y >_{\mathfrak{L}}$. Then $\mathscr{H}(K)$ is a reproducing kernel Hilbert space with reproducing kernel K. To see that K is a positive definite kernel, let $\{x_1, \ldots, x_n\} \subset \mathfrak{L}$ and $\{c_1, \ldots, c_n\} \subset \mathbb{C}$. It follows that

$$\sum_{i,j=1}^{n} c_i \overline{c_j} K(x_i, x_j) = \sum_{i,j=1}^{n} c_i \overline{c_j} < x_i, x_j >_{\mathfrak{L}} = \sum_{i,j=1}^{n} < c_i x_i, c_j x_j >_{\mathfrak{L}} = \left\| \sum_{i=0}^{n} c_i x_i \right\|_{\mathfrak{L}}^2 \geq 0.$$

Remark. The following theorem by Aronszajn is important to recall for proving that functions belong to the new RKHS in chapter four. The Moore-Aronszajn theorem gives the necessary and sufficient condition for a function $f \in \mathscr{F}(X, \mathbb{C})$ to be an element of a RKHS $\mathscr{H}$ on X.

Theorem 2.12 (Moore-Aronszajn)**.** *Let K be a positive definite kernel on $X \times X$ and let $f \in \mathscr{F}(X, \mathbb{C})$. Then $f \in \mathscr{H}(K)$ if and only if there exists a constanct C depending on f such that*

$$|f(x)|^2 \leq C \cdot \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} K(x_i, x_j),$$

for any set of points $\{x_1, \dots, x_n\} \subseteq X$ *and any numbers* $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{C}$.

Lemma 2.13 (Reproducing Kernel Induced by Feature Maps)**.** *Let* X *be any set, let* K *be a kernel function on* $X \times X$*, and let* $\Phi : X \longrightarrow \mathscr{L}$ *be a feature map from definition 2.5. Define* $F : \mathscr{H}(K) \longrightarrow \mathscr{L}$ *such that* $K(x, y) = \langle \Phi(x), \Phi(y) \rangle_{\mathscr{L}}$*. Then the adjoint map* $F^* : \mathscr{L} \longrightarrow \mathscr{H}(K)$ *is defined by*

$$(F^*\ell)(x) = \langle \ell, \Phi(x) \rangle_{\mathscr{L}},$$

where ℓ *is a random process in* $\mathscr{L}$*.*

Proof. Following the assumptions in Lemma 2.13, consider $\ell \in \mathscr{L}$. Then it follows that

$$(F^*\ell)(x) = \langle (F^*\ell)(\cdot), k_x(\cdot) \rangle_{\mathscr{H}(K)} \tag{2.12}$$

$$= \langle \ell(\cdot), F(k_x)(\cdot) \rangle_{\mathscr{L}} \tag{2.13}$$

$$= \langle \ell, \Phi(x) \rangle_{\mathscr{L}}. \tag{2.14}$$

□

Example 2.14 (Positive Definite Kernels induced by Polynomials)**.** Let $p(x) = \sum_{n=0}^{N} a_n x^n$ be a polynomial with $a_n \geq 0$ for all n. Then the function $K_p : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by $K_p(z, w) = p(z\overline{w}) = \sum_{n=0}^{N} a_n (z\overline{w})^n$ is a positive definite kernel function. To see this, choose points $\{z_1, \dots, z_m\} \subset \mathbb{C}$ and let Q be the $m \times m$ rank one positive matrix $Q = (z_i \overline{z_j})$. Then the $m \times m$-matrix $(K_p(z_i, z_j))$ satisfies

$$(K_p(z_i, z_j)) = \sum_{n=0}^{N} a_n Q^{\circ N}, \quad \textit{(Schur product)}$$

where $Q^{\circ 0}$ is the matrix of all ones and $Q^{\circ N}$ denotes the Schur product of Q with itself N times. Since $a_n \geq 0, Q^{\circ 0} \geq 0$, and $Q \geq 0$, by the Schur product theorem each term $a_n Q^{\circ 0}$ is positive definite and, hence, the sum is positive definite. Thus, K_p is a positive definite kernel function on $\mathbb{C}$.

Example 2.15 (Positive Definite Kernels induced by Covariance Functions)**.** Let $(\Omega, \mathscr{F}, \mu)$ be a probability space and let $S \subset \mathbb{R}^n$. Let X be a stochastic process on S, i.e. X is a measurable function $X : S \times \Omega \longrightarrow \mathbb{R}$. Assume that for any $s \in S$, we have that $X(s) \in L^2(\mu)$. We define the expectation E of X at s to be the quantity

$$E(X(s)) = \int_\Omega X(s) d\mu.$$

The covariance function $C : S \times S \longrightarrow \mathbb{R}$ is given by

$$C(s,t) = E[(X_s - E(X_s))(X_t - E(X_t))],$$

and the covariance function is a positive definite function.

2.2 Factorizations of Positive Definite Kernels

Starting with a given positive definite kernel K on $X \times X$, we look at boundary space factorizations of K,

$$K(s,t) = \int_{\mathcal{B}} K_s^{\mathcal{B}}(b) \overline{K_t^{\mathcal{B}}(b)} d\mu(b), \quad s, t \in \mathbb{B}_d \text{ and } b \in \mathcal{B}. \tag{2.15}$$

The probability space $(\mathcal{B}, \mathfrak{B}, \mu)$ is a measure theoretic "boundary". We know that there is always a factorization like 2.15 associated to any positive definite kernel. So, we focus on whether there exists such a factorization for the "natural" boundary of X; often times thought of as the topological boundary.

Example 2.16 (Factorization of $K(s,t) = t \wedge s$)**.** *Let* $[0,1]$ *be the closed unit interval and set*

$$K(s,t) = t \wedge s, \quad s,t \in [0,1].$$

Set

$$\varphi_k(t) = \begin{cases} \sqrt{2}\frac{sin(k\pi t)}{k\pi} & t \in \mathbb{N}, \\ 0 & t = 0. \end{cases} \tag{2.16}$$

Then we have that

1. $\tau(t) := (\varphi_k(t))_{k\in\mathbb{N}_0}$ *satisfies*

$$\|\tau(t) - \tau(s)\|_{\ell^2}^2 = |t-s|, \quad t,s \in [0,1].$$

The reproducing kernel Hilbert space associated to K *is*

$$\mathscr{H}(K) = \{f \in L^2(0,1) | f' \in L^2(0,1) \text{ and } f(0) = 0\},$$

where the Hilbert norm in $\mathscr{H}(K)$ *is*

$$\|f\|_{\mathscr{H}(K)}^2 = \int_0^1 |f'(x)|^2 dx.$$

One easily checks that the function system $(\varphi_k)_{k\in\mathbb{N}_0}$ *is an orthonormal basis in* $\mathscr{H}(K)$*. Indeed, for* $j,k \in \mathbb{N}$*,*

$$< \varphi_j, \varphi_k >_{\mathscr{H}(K)} = 2\int_0^1 \cos(j\pi x)\cos(k\pi x)dx = \delta_{j,k}.$$

In this example the Gaussian process associated with (K,I) *is the Brownian motion. Hence,*

$$\|\tau(t) - \tau(s)\|_{\ell^2}^2 = \mathbb{E}(|X_t - X_s|^2) = |t-s|, \;\; t,s \in I,$$

which is (i)*.*

Example 2.17 (Factorization of $K(z,w) = \frac{1}{1-z\overline{w}}$)**.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{ix} : x \in (-\pi, \pi]\}$. Set

$$K(z,w) = \prod_{t=0}^{\infty} \left(1 + (z\overline{w})^{4^l}\right), \quad (z,w) \in \mathbb{D} \times \mathbb{D},$$

and identifying $e(x) := e^{ix}$ we get

$$K^{\partial\mathbb{D}}(z,x) = \prod_{t=0}^{\infty} \left(1 + (ze^{-ix})^{4^l}\right), \quad (z, e(x)) \in \mathbb{D} \times \partial\mathbb{D}.$$

Then (2.15) holds for the case when $\mu_{\frac{1}{4}}$ is defined to be the $\frac{1}{4}$−Cantor measure on $\partial\mathbb{D}$. Set

$$\Lambda_4 = \left\{ \sum_{i=0}^{n} b_i 4^i | b_i \in \{0,1\}, n \in \mathbb{N} \right\} \tag{2.17}$$

$$= \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \cdots\} \tag{2.18}$$

then

$$\prod_{t=0}^{\infty} \left(1 + t^{4^l}\right) = \sum_{\lambda \in \Lambda} t^{\lambda}, \quad |t| < 1.$$

The desired conclusion

$$K(z,w) = \int_{C_{\frac{1}{4}}} \overline{K_{C_{\frac{1}{4}}}(z,x)} K_{C_{\frac{1}{4}}}(w,x) d\mu_{\frac{1}{4}}(x)$$

follows from the fact that $\{e^{i\lambda \cdot x} | \lambda \in \Lambda_4\}$ is an orthonormal basis in $L^2(C_{\frac{1}{4}}, \mu_{\frac{1}{4}})$.

Example 2.18 (Factorization of $K(z,w) = \frac{1}{1-z\overline{w}}$)**.** Similar to example 2.17, let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in (-\pi, \pi]\}$. The reproducing kernel for the Hardy space $H^2(\mathbb{D})$ is

$$K(z,w) = \frac{1}{1 - z\overline{w}}. \tag{Szego Kernel}$$

We define $K^{\partial\mathbb{D}}$ on $\mathbb{D} \times \partial\mathbb{D}$ such that

$$K^{\partial\mathbb{D}}(z, e^{i\theta}) = \frac{1}{1 - ze^{-i\theta}}.$$

It then follows that the boundary factorization

$$\frac{1}{1 - z\overline{w}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{1 - ze^{-i\theta}} \frac{1}{1 - \overline{w}e^{i\theta}} d\theta$$

satisfies (2.15).

Example 2.19 (Factorization of $K(z, w) = \frac{1}{(1 - z\overline{w})^2}$)**.** The Bergman space is named after Stefan Bergman, who is credited in the literature as one of the first to study reproducing kernel Hilbert spaces. Bergman's work involved RKHS theory for function spaces from complex analysis. The Bergman space can be defined on any open connected subset of $\mathbb{C}$ but we focus this example on the unit disk in $\mathbb{C}$. We have that the *normalized* Bergman space $B^2(\mathbb{D})$ is the space of analytic functions on the disk with the condition that given $f \in B^2(\mathbb{D})$

$$\|f\|^2 = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 dz < \infty,$$

using the area integral, $dxdy = \frac{1}{2i}(dz \wedge d\overline{z})$. The RKHS $B^2(\mathbb{D})$ is a subspace of $L^2(\mathbb{D})$; specifically

$$B^2(\mathbb{D}) := Hol(\mathbb{D}) \cap L^2(\mathbb{D}, dz)$$

where $\frac{1}{\pi}dz$ denotes the *normalized* area measure on the unit disk. The reproducing kernel for $B^2(\mathbb{D})$ is given by

$$K(z, w) = \frac{1}{(1 - z\overline{w})^2},$$

and we call this the Bergman kernel. The holomorphic monomials for an orthonormal basis

Example 2.20 (Factorization of $K(z,w) = e^{<z,w>}$)**.** The Segal-Bargmann space is the analog of the $B^2(\mathbb{D})$ extended to complex n-space, with the Gaussian measure

$$d\mu(z) = \pi^{-n} e^{-|z|^2} dz, \ z \in \mathbb{C}^d.$$

This RKHS is associated with the Fock space because Bargmann credited his work to Fock; thus, the notation $\mathfrak{F}_n$ is used for the Segal-Bargmann. Similar to the definition of $B^2(\mathbb{D})$, the Segal-Bargmann space can defined as

$$\mathfrak{F}_n := Hol(\mathbb{C}^d) \cap L^2(\mathbb{C}^d, \mu) \quad \textit{also denoted sometimes as } L^2_a(\mathbb{C}^d).$$

$\mathfrak{F}_n$ consists of the entire analytic functions on $\mathbb{C}^d$ subject to

$$\|f\|^2_{\mathfrak{F}_n} = \frac{1}{\pi^n} \int_{\mathbb{C}^d} |f(z)|^2 e^{-|z|^2} dz < \infty.$$

The reproducing kernel for $\mathfrak{F}_n$ is $K(z,w) = e^{<z,w>}, z, w \in \mathbb{C}^d$, and has a boundary factorization given by the Segal-Bargmann transform $S : L^2(\mathbb{R}^n) \xrightarrow{\cong} \mathfrak{F}_n$ defined by

$$Sf(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-(z-x)^2} dx.$$

2.3 Metric Duality: Positive Definite Kernels and Generalized Boundary Processes

Starting with a given positive definite kernel K on $S \times S$, we introduce generalized boundaries for the set S that carry K. It is a measure theoretic "boundary" in the form of a probability space, but it is not unique. The set of measure boundaries will be denoted $M(K)$. Indeed, there exists such a generalized probability space associated to any positive definite kernel.

Palle Jørgensen, among other authors, has studied representations of positive definite kernels K in the general case. We reference some of the many papers that focus on their applications to harmonic analysis, to metric geometry, and to realizations of certain stochastic processes. The overall framework established in Jørgensen's papers proves results for the most general positive definite kernel. Next the results are specialized to the aforementioned applications and other applications beyond what we cover in this dissertation. This chapter establishes some of the general results from different papers which share the essential seamline for establishing boundary spaces for reproducing kernel Hilbert spaces. In essence, the following results created questions for the Drury-Arvseson RKHS, and this chapter provides the context for the questions to be answered in chapter four.

Given a positive definite kernel K on $S \times S$ where S is a fixed set, we first study families of factorizations of K. By a factorization (or representation) we mean a probability space (B, μ) and an associated stochastic process indexed by S which has K as its covariance kernel. For each realization we identify a co-isometric transorm from $L^2(\mu)$ onto $\mathscr{H}(K)$, where $\mathscr{H}(K)$ denotes the reproducing kernel Hilbert space of K. In some cases, this entails a certain renormalization of K. Our emphasis is on such realizations which are minimal in a sense we make precise. By minimal we mean roughly that B may be realized as a certain K-boundary of the given set S.

We prove existence of minimal realizations in a general setting.

(I) $K : S \times S \longrightarrow \mathbb{C}$ is a given positive definite kernel defined on a fixed set S, i.e. $\forall N \in \mathbb{N}, \forall \{s_i\}_{i=1}^N, s_i \in S, \forall \{\xi_i\}_{i=1}^N, \xi_i \in \mathbb{C}$, we have

$$\sum_i \sum_j \xi_i \overline{\xi}_j K(s_i, s_j) \geq 0$$

(II) Measure space $(B, \mathscr{B}, \mu)$ where B is a set equipped with a σ-algebra $\mathscr{B}$ of subsets, and μ is a probability measure defined on $\mathscr{B}$. In particular, μ satisfies $\mu(\emptyset) = 0, \mu(B) = 1, \mu(F) \geq 0 \forall F \in \mathscr{B}$, and if $\{F_i\}_{i \in \mathbb{N}} \subset \mathscr{B}, F_i \cup F_j = \emptyset, i \neq j$ in $\mathbb{N}$, then $\mu(\cup_i F_i) = \sum_i \mu(F_i)$.

Definition 2.6. Let K be a p.d. kernel as in (I). We shall denote by $\mathscr{H}(K)$ the corresponding reproducing kernel Hilbert space with respect to the inner product

$$\left\langle \sum \xi_i K_{s_i}, \sum \xi_j K_{s_j} \right\rangle_{\mathscr{H}(K)} := \sum \sum \xi_{s_i} \overline{\xi_{s_j}} K(s_i, s_j).$$

The following property holds:

$$f(s) =< f, K(\cdot, s) >_{\mathscr{H}(K)}, \forall s \in S, \forall f \in \mathscr{H}(K)$$

Definition 2.7. Given K as in (I), and (B, μ) as in (II), we say that $(B, \mu) \in \mathscr{M}(K)$ if there is a function $k : S \longrightarrow L^2(\mu)$ such that

$$< K(x, s), K(x, t) >_{\mathscr{H}(K)} = K(s, t) = \int_B k_s(b) \overline{k_t(b)} d\mu(b) =< k(b, s), k(b, t) >_{L^2(\mu)}$$

holds for all $(s, t) \in S \times S$. We say that (B, μ) is *tight* if and only if the span of $\{k_s; s \in S\}$ is dense in $L^2(B, \mu)$.

Given K as in (I) then the duality problem always has a solution in a discrete measure space relative to the counting measure. Nonetheless, in the interesting solutions $(B, \mathscr{B}, \mu)$ we aim to achieve B as a "boundary space" to the given set S from (I).

Definition 2.8. We shall say that a Hilbert space $\mathscr{H}$ is **separable** if there is an orthonormal basis $\{\beta_n\}_{n\in\mathbb{N}}$ indexed by $\mathbb{N}$; that is, we have

1. $< \beta_n, \beta_m >_{\mathscr{H}} = \delta_{n,m}$, and
2. $\|f\|^2 = \sum_{n\in\mathbb{N}} | < f, \beta_n >_{\mathscr{H}} |^2, f \in \mathscr{H}$.

If only (ii) holds, we say that $\{\beta_n\}_{n\in\mathbb{N}}$ is a *Parseval frame.* In both cases, vectors $f \in \mathscr{H}$ always have the representation

$$f = \sum_{n\in\mathbb{N}} < f, \beta_n >_{\mathscr{H}} \beta_n$$

where the summation converges in the norm $\| \cdot \|_{\mathscr{H}}$ of $\mathscr{H}$.

Lemma 2.21. *Let K be given and assumed positive definite on $S \times S$, where S is a set. Let $\mathscr{H}(K)$ be the corresponding RKHS, assumed separable. Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a Parseval frame, and set*

$$k_s(n) := \beta_n(s) = < \beta_n, K(\cdot, s) >_H .$$

Then the system is a solution to the duality question, with the measure space $(\mathbb{N}, \sum, \mu_{\mathbb{N}})$.

Proof. The existence of a Parseval frame $\{\beta_n\}_{n\in\mathbb{N}}$ is assumed, so (1.5)-(1.6) hold for the RKHS $\mathscr{H}$. Now, for all pairs $(s,t) \in S \times S$, we have

$$\begin{aligned} K(s,t) &=< K(\cdot,s), K(\cdot,t) >_{\mathscr{H}} \\ &= \sum_{n\in\mathbb{N}} < K(\cdot,s), \beta_n >_{\mathscr{H}} < \beta_n, K(\cdot,t) >_{\mathscr{H}} \\ &= \sum_{n\in\mathbb{N}} \beta_n(s)\overline{\beta_n(t)} \\ &= \sum_{n\in\mathbb{N}} k_s(n)\overline{k_t(n)} \end{aligned} \tag{2.19}$$

which is the desired conclusion. □

Proposition 2.22. *Let K be given as in Lemma 3.1 above, and let $\{\beta_n\}_{n\in\mathbb{N}}$ be a Parseval frame, set $k_s(n) := \beta_n(s)$; then this is a minimal solution, i.e., $\{k_s(\cdot)\}$ is dense in $l^2(\mathbb{N})$.*

Let $K : S \times S \longrightarrow \mathbb{C}$ be a given p.d. kernel, as specified in (I) above. Solutions $(B, \mu, \{k_s\}_{s\in S}$ to the problem are called factorizations.

Proposition 2.23. *Let K on $S \times S$ be given, and let $(B, \mu, \{k_s\}_{s\in S})$ be a solution to the factorization problem. Then the assignment*

$$W(K(\cdot,s)) := k_s \in L^2(\mu)$$

extends by linearity to an isometry, denoted $W : \mathscr{H}(K) \longrightarrow L^2(\mu)$, and its adjoint $V := W^* : L^2(\mu) \longrightarrow \mathscr{H}(K)$ is the following transform

$$(Vf)(s) = \int_B f(x)\overline{k_s(x)}d\mu(x),$$

and we have

$$W^*W = VW = I_{\mathscr{H}(K)},$$

while $WW^* = WV$ *is a projection in the Hilbert space* $L^2(\mu)$*.*

Remark. Proposition 2.23 is a nice application of the feature map construction and Lemma 2.13.

Given (K, S) *as in (I), let* $\mathscr{M}(K)$ *be the boundary space consisting of all measure spaces* (B, μ) *satisfying the duality problem. Here we will introduce an order relation on* $\mathscr{M}(K)$ *and show that there is always a minimal element in* $\mathscr{M}(K)$*.*

Definition 2.9. Suppose $(B_i, \mathscr{B}_i, \mu_i) \in \mathscr{M}(K), i = 1, 2$. We say that

$$(B_1, \mathscr{B}_1, \mu_1) \leq (B_2, \mathscr{B}_2, \mu_2)$$

if there exists a map $\varphi : B_2 \longrightarrow B_1$ such that

$$\mu_2 \circ \varphi^{-1} = \mu_1, \text{ and}$$

$$\varphi^{-1}(\mathscr{B}_1) = \mathscr{B}_2.$$

Definition 2.10. Let (K, S) be a fixed positive definite kernel, and let $(M, \nu) \in \mathscr{M}(K)$ be a boundary space. We say that (M, ν) is minimal in $\mathscr{M}(K)$ if given $(B, \mu) \in \mathscr{M}(K)$ and $(B, \mu) \leq (M, \nu)$, then we must have that $(B, \mu) \cong (M, \nu)$; i.e., $(M, \nu) \leq (B, \mu)$.

Summary of Results for Generalized Positive Definite Kernels:

1. For every positive definite kernel K, we define a "measure theoretic boundary space" $\mathcal{M}(K)$. Set
$$\mathcal{M}(K) := \{(B, \mathcal{F}, \mu)\},$$
where $(B, \mathcal{F}, \mu)$ is a measure space which yields a factorization for K. This set $\mathcal{M}(K)$ generalizes other notions of "boundary" used in the literature for networks, and for more general positive definite kernels, and their associated reproducing kernel Hilbert spaces (RKHSs).

2. For any positive definite kernel K, the corresponding $\mathcal{M}(K)$ is always non-empty. The natural Gaussian process path-space with covariance kernel K, and Wiener measure μ is in $\mathcal{M}(K)$.

3. Given K, let $\mathscr{H}(K)$ be the associated RKHS. Then for every $\mu \in \mathcal{M}(K)$ there is a canonical isometry W_μ mapping $\mathscr{H}(K)$ into $L^2(\mu)$.

4. The isometry W_μ in (3) generally does not map onto $L^2(\mu)$.

5. Using the isometries from (3), we can turn $\mathcal{M}(K)$ into a partially ordered set. Then, using Zorn's lemma, one shows that $\mathcal{M}(K)$ always contains minimal elements. The minimal elements are not unique.

6. And even is μ is chosen minimal in $\mathcal{M}(K)$, the corresponding isometry W_μ still generally does not map onto $L^2(\mu)$. A case in point: the Szego kernel, and $\mu =$ Lebesgue measure on a period interval. ($W_\mu : \mathbb{H}_2 \longrightarrow L^2([0,1], \mu_{Leb})$ is isometric but not onto.)

2.4 The Hardy Space

Example 2.24 (Hardy Space on $\mathbb{D}$)**.** The Hardy space of the unit disk $H^2(\mathbb{D})$ plays a key role in function theory, operator theory and in the theory of stochastic processes. To construct $H^2(\mathbb{D})$, we first consider formal complex power series,

$$f \sim \sum_{n=0}^{\infty} a_n z^n$$

such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. Using the usual definitions for sums and scalar multiples, the set of all such power series clearly forms a vector space. Given another such power series $g \sim \sum_{n=0}^{\infty} b_n z^n$ we define the inner product,

$$< f, g >= \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

Thus, we have that $\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2$. The map $L : H^2(\mathbb{D}) \longrightarrow \ell^2(\mathbb{Z}_+)$, defined by $L(f) = (a_n)_{n \in \mathbb{Z}_+}$ is a linear inner product preserving isomorphism. Hence we see that $H^2(\mathbb{D})$ can be defined with the Hilbert space, $\ell^2(\mathbb{Z}_+)$. Thus, we see that (ii) in the definition of a RKHS is met.

Next we show that every power series in $H^2(\mathbb{D})$ converges to define a function on the disk. To see this note that if $w \in \mathbb{D}$, then

$$|E_w(f)| = \left| \sum_{n=0}^{\infty} a_n w^n \right| \tag{2.20}$$

$$\leq \sum_{n=0}^{\infty} |a_n||w|^n \tag{2.21}$$

$$\leq \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} |w|^{2n} \right)^{1/2} \tag{2.22}$$

$$= \|f\| \cdot \frac{1}{\sqrt{1 - |w|^2}}. \tag{2.23}$$

Thus, each power series defines a function on $\mathbb{D}$. (...) The above inequality also shows that the map E_z is bounded with $\|E_z\| \leq \frac{1}{\sqrt{1-|z|^2}}$ and so $H^2(\mathbb{D})$ is a RKHS on $\mathbb{D}$. We now compute the kernel function for $H^2(\mathbb{D})$. Let $w \in \mathbb{D}$ and let $k_w \in H^2(\mathbb{D})$ denote the kernel function for the point w. Since $k_w \in H^2(\mathbb{D})$ we can write $k_w = \sum_{n=0}^{\infty} b_n z^n$. Let $f = \sum_{n=0}^{\infty} a_n z^n \in H^2(\mathbb{D})$. We have

$$f(w) = \sum_{n=0}^{\infty} a_n w^n =< f, k_w >= \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

It then follows that $b_n = \overline{w^n}$. Therefore, $k_w(z) = \sum_{n=0}^{\infty} \overline{w^n} z^n$. Hence, the kernel function of the Hardy space can now be computed in closed form

$$K(z, w) =< k_w, k_z >= k_w(z) = \sum_{n=0}^{\infty} \overline{w^n} z^n = \frac{1}{1 - \overline{w} z}.$$

The kernel for the Hardy space is called the **Szego Kernel** on the disk.

Among the applications of stochastic processes, the theory of "boundaries" is noteworthy. Common to these is the need for representations of functions on some set, say T, as integrals over some measure boundary space arising as a limiting operation derived from the points in the initial set T. As an example of this is the Hardy space $H^2(\mathbb{D})$, which is the reproducing kernel Hilbert space with kernel the Szego kernel

$$K(z, x) = \frac{1}{1 - z\overline{w}}, \quad z, w \in \mathbb{D}.$$

If $< \cdot, \cdot >_{H^2(\mathbb{D})}$ is the inner product of $H^2(\mathbb{D})$ we have

$$f(w) =< f(\cdot), K_w(\cdot) >_{H^2(\mathbb{D})}, \quad f \in H^2(\mathbb{D}), \quad w \in \mathbb{D}.$$

In this example we have the following factorization.

$$\begin{aligned} K(z,w) &= \frac{1}{1-z\overline{w}} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1-ze^{-i\theta}} \frac{1}{1-\overline{w}e^{i\theta}} d\theta. \end{aligned} \tag{2.24}$$

The classical Hardy space H^2 consists of those holomorphic functions f defined on $\mathbb{D}$ satisfying

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_0^1 |f(re^{2\pi ix})|^2 dx < \infty.$$

It is well-known that an equivalent description of H^2 is as the space of holomorphic functions on $\mathbb{D}$ with square-summable coefficients:

$$H^2 = \left\{ \sum_{n=0}^{\infty} c_n z^n \,\middle|\, \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\},$$

where the norm is then equivalently given by

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |c_n|^2.$$

In addition, for each $f \in H^2$, there exists a (unique) function $f^* \in L^2(\mathbb{T})$, which we shall call the Lebesgue boundary function of f, such that

$$\lim_{r\to 1^-} \int_0^1 |f(re^{2\pi ix}) - f^*(e^{2\pi ix})|^2 dx = 0.$$

In fact, $\lim_{r\to 1^-} f(re^{2\pi ix}) = f^*(e^{2\pi ix})$ pointwise for almost every x.

Remark. The next result shows that the Szego kernel reproduces itself with respect to what is, by some definition, its boundary. Professor Jørgensen has shown that, among *the functions in the Hardy space, there are a host of other kernels that reproduce with respect to their boundaries.*

Proposition 2.25. *Let $K(z,w)$ be Szego kernel, then $L^2(\mathbb{T},\mu) \in \mathscr{M}(K)$, where μ is a singular measure on $\mathbb{T}$, is a minimal solution.*

Proof. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open disk in the complex plane $\mathbb{C}$, and let

$$K(z,w) = \frac{1}{1 - z\overline{w}}, (z,w) \in \mathbb{D} \times \mathbb{D},$$

be the Szego kernel. Let μ be a singular measure on $\mathbb{T} = \partial\mathbb{D} \cong [0,1]$. We use the isomorphism $[0,1] \cong \mathbb{T}$, given by $[0,1] \ni x \longrightarrow e(x) = e^{i2\pi x} \in \mathbb{T}$. In this case,

$$k : \mathbb{T} \times \mathbb{D} \longrightarrow \mathbb{C} \text{ is defined by } k(e(x), z) = \frac{1}{1 - e(x)\overline{z}}, x \in [0,1],$$

and we set $k_z(x) = k(x,z) := k(e(x), z)$. Then it follows that $k_z(\cdot) \in L^2(\mathbb{T},\mu)$ and

$$K(z,w) = \frac{1}{1 - z\overline{w}} = \int_{\mathbb{T}} \frac{1}{1 - e(x)\overline{z}} \frac{1}{1 - \overline{e(x)}w} =< k_z, k_w >_{L^2(\mathbb{T},\mu)} .$$

To see that $L^2(\mathbb{T},\mu)$ is a minimal solution, suppose that $f \in L^2(\mathbb{T},\mu)$ satisfies

$$< f, k_z >_{L^2(\mathbb{T},\mu)} = 0, \forall z \in \mathbb{D}.$$

It follows that,

$$\int_0^1 e(nx) f(x) d\mu(x) = 0, \forall n \in \mathbb{N}_0.$$

□

Chapter 3
BOUNDARY SPACES FOR DRURY-ARVESON SPACE

The Drury-Arveson space $H^2_d(\mathbb{B}_d)$, referenced below as H^2_d is a multivariable generalization of the Hardy space on the disk, $H^2_1(\mathbb{B}_1) = H^2(\mathbb{D})$. There have been comparisons of the Drury-Arveson space with the Multivariable Hardy space, which is a different generalization of the Hardy space. The Drury-Arveson space is named after S.W. Drury, who is credited for introducing the study of this RKHS into multivariable operator theory, and named after W.B. Arveson, who moved the study of this RKHS to the center stage with his paper titled, “Subalgebras of C^*-Algebras III: Multivariable Operator Theory”. The Drury-Arveson space examines function theory and operator theory as it relates to the unit ball $\mathbb{B}_d$ in complex d-dimensional space $\mathbb{C}^d$, $d \in \mathbb{N}$, with

$$\mathbb{B}_d = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d : \|z\|_{\mathbb{C}^d} < 1\},$$

where $\|z\|_{\mathbb{C}^d}$ denotes the norm associated with the usual inner product on $\mathbb{C}^d$,

$$\|z\|_{\mathbb{C}^d} = |z_1|^2 + \cdots + |z_d|^2.$$

The Drury-Arevson space H^2_d is the reproducing kernel Hilbert space of functions on $\mathbb{B}_d$ determined by the kernel

$$K(z, w) = \frac{1}{1 - < z, w >}.$$

Definition 3.1 (The Multiplier Algebra of H^2_d)**.** The **Multiplier Algebra of H^2_d** is defined as

$$\mathcal{M}_d := \{f : \mathbb{B}_d \to \mathbb{C} | fh \in H^2_d \text{ for all } h \in H^2_d\}.$$

Definition 3.2 (The Ball Algebra $A(\mathbb{B}_d)$)**.** The **Ball Algebra** $A(\mathbb{B}_d)$ is the algebra of continuous functions on the closed ball

$$\overline{\mathbb{B}_d} := \{z = (z_1, \dots, z_d) \in \mathbb{C}^d : \|z\|_{\mathbb{C}^d} \leq 1\}$$

which are analytic functions on $\mathbb{B}_d$.

The Drury-Arveson space with dimension $d = 1$ yields the familiar Hardy space, H^2. In this setting, the multiplier algebra $\mathcal{M}_1$ is equivalent to $H^\infty(\mathbb{D})$, the space of bounded holomorphic functions on $\mathbb{D}$ with $\|f\|_\infty = \sup_{|z|<1}|f(z)|$; and the multiplier norm is equaivalent to the sup norm. These results are no londer true for the Drury-Arveson space when $d \geq 2$. In the following sections, we present these essential results by Arveson: For $d \geq 2$,

- The norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\mathcal{M}_d}$ are not comparible on $\mathcal{M}_d$;
- There is a strict containment $\mathcal{M}_d \subsetneq H^\infty(\mathbb{B}_d)$;
- The d-tuple M_z is not subnormal.

3.1 The Drury-Arveson Space, $H^2_d(\mathbb{B}_d)$

We begin with the general framework of H^2_d then we follow with the construction and proof that it is indeed a reproducing kernel Hilbert space.

Definition 3.3. The **Drury-Arveson Space** H^2_d is defined as the space of holomorphic functions $f : \mathbb{B}_d \longrightarrow \mathbb{C}$ which have a power series $f(z) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha z^\alpha$ such that

$$\|f\|^2_{H^2_d} := \sum_{\alpha \in \mathbb{N}_0^d} |c_\alpha|^2 \frac{\alpha!}{|\alpha|!} < \infty.$$

Remark. The **Drury-Arveson Space** H_d^2 *is also understood as the Hilbert space obtained by completing* $(\mathcal{P}, \|\cdot\|_{\mathcal{P}})$*, where* $\mathcal{P}$ *denotes the algebra of all complex holomorphic polynomials on* $\mathbb{B}_d$*.*

The above remark is important and is the process Arveson used to introduce H_d^2. Let $\mathcal{P}$ denote the algebra of all complex holomorphic polynomials f in the variable $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$. Every such polynomial f has a unique expansion into a finite series

$$f(z) = f_0(z) + f_1(z) + \cdots + f_n(z), \quad \textit{called the Taylor series of } f \tag{3.1}$$

where f_k is a homogeneous polynomial of degree k. To define the norm $\|\cdot\|_{\mathcal{P}}$, let $\mathbf{E} = \mathbb{C}^d$ where $\mathbf{E}$ has the usual inner product

$$< z, w >_{\mathbf{E}} = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_d}, \quad z_i, w_i \in \mathbb{C},\ i = 1, 2, \ldots, d.$$

For each $n \in \mathbb{N}_0$, we write $\mathbf{E}^{\otimes n} = \underbrace{\mathbf{E} \otimes \mathbf{E} \otimes \cdots \otimes \mathbf{E}}_{n-times}$ for the n-fold tensor product of $\mathbf{E}$; with $\mathbf{E}^{\otimes 0} := \mathbb{C}$. Hence, $\mathbf{E}^{\otimes n}$ is the Hilbert space obtained after completion of the pre-Hilbert space

$$\mathbf{span}\{Z_1 \otimes Z_2 \otimes \cdots \otimes Z_n |\ Z_i \in \mathbf{E},\ i = 1, \ldots, n\},$$

with the inner product on $\mathbf{E}^{\otimes n}$ defined as follows

$$< Z_1 \otimes Z_2 \otimes \cdots \otimes Z_n, W_1 \otimes W_2 \otimes \cdots \otimes W_n >_{\mathbf{E}^{\otimes n}} := < Z_1, W_1 >_{\mathbf{E}} \cdots < Z_n, W_n >_{\mathbf{E}} .$$

For $Z_1, \ldots, Z_n \in \mathbf{E}$ we define the symmetric tensor product

$$Z_1 \circ \cdots \circ Z_n = \frac{1}{n!} \sum_{\sigma \in S_n} Z_{\sigma(1)} \otimes \cdots \otimes Z_{\sigma(n)},$$

where S_n is the symmetric group of permutations on $\{1,2,\ldots,n\}$. The closed subspace generated by $Z_1 \circ \cdots \circ Z_n$ will be denoted $\mathbf{E}^n \subset \mathbf{E}^{\otimes n}$, and is called the n-fold symmetric tensor product of $\mathbf{E}$. We are now ready to define the symmetric (or bosonic) Fock space over $\mathbf{E}$.

Definition 3.4 (The Symmetric Fock Space)**.** We define the

Symmetric Fock space $\mathfrak{F}_+(\mathbf{E})$ as

$$\mathfrak{F}_+(\mathbf{E}) = \bigoplus_{n=0}^{\infty} \mathbf{E}^n.$$

Let $z \in \mathbf{E}$ be a fixed vector; we will use the notation $z^n = z^{\otimes n} \in \mathbf{E}^n$ for the n-fold tensor product of copies of z (with the identification that $z^0 = 1$).

Proposition 3.1. For $n \in \mathbb{N}_0$, the space $\mathbf{E}^n$ is linearly spanned by the set $\{z^n | z \in \mathbf{E}\}$.

Proof. Let $z \in \mathbf{E}$ be a fixed vector and declare $\mathcal{S} = \mathbf{span}\{z^n | z \in \mathbf{E}\} \subset \mathbf{E}^n$. Then it follows that

$$\begin{aligned}\mathcal{S}^{\perp} &= \{W \in \mathbf{E}^n | < W, z^n >_{\mathbf{E}^n} = 0 \ \forall z \in \mathbf{E}, n \in \mathbb{N}_0\} \\ &= \{W \in \mathbf{E}^n | < W_1, z >_{\mathbf{E}} \cdot < W_2, z > \cdots < W_n, z > = 0 \ \forall z \in \mathbf{E}, n \in \mathbb{N}_0\}.\end{aligned}$$

This implies that $W_i = 0$ for some $i \in \{1,\ldots,n\}$. Hence, $W = 0$; and we conclude that $\mathbf{span}\{z^n | z \in E\} = \mathbf{E}^n$. □

Proposition 3.2. Every homogeneous polynomial $g : \mathbf{E} \longrightarrow \mathbb{C}$ of degree k determines a unique linear functional $\widetilde{g}$ on $\mathbf{E}^k$ by

$$g(z) = \widetilde{g}(z^k), \quad z \in \mathbf{E}.$$

Proof. Let $g : \mathbf{E} \longrightarrow \mathbb{C}$ be a homogeneous polynomial of degree k. It follows that

$$g(z) = \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} c_\alpha z^\alpha, \quad z \in \mathbf{E}$$

$$= \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} < z^k, e^\alpha >_{\mathbf{E}^k}, \quad e \in \mathbf{E}.$$

We make the identification $\widetilde{g}(z^k) :=< z^k, \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha|=k}} e^\alpha >_{\mathbf{E}^k}$ and $\widetilde{g}$ is a linear functional on $\mathbf{E}^k$. The uniqueness of $\widetilde{g}$ follows from proposition 3.1. □

We have that the Taylor series of f (3.1) can be written in the form

$$f(z) = \sum_{k=0}^{n} \widetilde{f}_k(z^k), \quad z \in \mathbf{E}$$

where $\widetilde{f}_k$ is a uniquely determined linear functional on $\mathbf{E}^k$ for each $k = 0, 1, \ldots, n$. We have that $(\mathbf{E}, < \cdot, \cdot >_{\mathbf{E}})$ is the usual d-dimensional complex Hilbert space $\left(\mathbb{C}^d, < \cdot, \cdot >_{\mathbb{C}^d}\right)$. So, $\left(\mathbf{E}^{\otimes k}, < \cdot, \cdot >_{\mathbf{E}^{\otimes k}}\right)$ becomes a d^k-dimensional complex Hilbert space given the inner products defined earlier. Thus, the subspace $\mathbf{E}^k \subset \mathbf{E}^{\otimes k}$ is also a finite-dimensional Hilbert space in a natural way. The Riesz representation theorem can then be applied to $\mathbf{E}^k$ and we have that there is a unique vector $\xi_k \in \mathbf{E}^k$ such that

$$\widetilde{f}_k(z^k) =< z^k, \xi_k >_{\mathbf{E}^k}, \quad z \in \mathbf{E}.$$

It follows that the Taylor series for f takes the form

$$f(z) = \sum_{k=0}^{n} \widetilde{f}_k(z^k) = \sum_{k=0}^{n} < z^k, \xi_k >_{\mathbf{E}^k}, \quad z \in \mathbf{E}.$$

Going back to our last remark, we define the norm on $\mathcal{P}$ as

$$\|f\|_{\mathcal{P}}^2 = \sum_{k=0}^{n} \|\xi_k\|_{\mathbf{E}^k}^2,$$

and we have that $H^2_d = \overline{< \mathcal{P}, \| \cdot \|_{\mathcal{P}} >}^{\|\cdot\|_{\mathcal{P}}}$.

Proposition 3.3. *The Drury-Arveson space, H^2_d, can be identified with the Symmetric Fock space, $\mathfrak{F}_+(\mathbf{E})$, by*

$$\Phi : (\mathscr{P}, \| \cdot \|_{\mathscr{P}}) \longrightarrow \mathfrak{F}_+(\mathbf{E})$$

$$f(z) = \sum_{k=0}^{n} < z^k, \xi_k >_{E^k} \longmapsto (\xi_0, \xi_1, \dots),$$

where $\xi_0, \xi_1, \dots$ is the sequence of Taylor coefficitents, continued so that $\xi_k = 0$ for $k > n$. Then Φ extends uniquely to an anti-unitary operator mapping H^2_d onto $\mathfrak{F}_+(\mathbf{E})$.

Proposition 3.4. *The elements in the Drury-Arveson space, H^2_d, can be realized as analytic functions in $\mathbb{B}_d$ having a power series expansion of the form*

$$f(z) = \sum_{k=0}^{\infty} < z^k, \xi_k >_{E^k}, \quad z = (z_1, \dots, z_d) \in \mathbb{B}_d, \tag{3.2}$$

where the H^2_d-norm of f is given by

$$\|f\|^2_{H^2_d} = \sum_{k=0}^{\infty} \|\xi_k\|^2_{E^k} < \infty.$$

Proof. From Proposition 3.2, the elements of H^2_d can be identified with the formal power series

$$f(z) = \sum_{k=0}^{\infty} < z^k, \xi_k >_{\mathbf{E}^k},$$

where

$$\sum_{k=0}^{\infty} \|\xi_k\|^2_{E^k} = \|f\|^2_{H^2_d} < \infty. \tag{3.3}$$

It then follows that

$$|f(z)| \leq \sum_{k=0}^{\infty} | < z^k, \xi_k > | \tag{3.4}$$

$$\leq \left(\sum_{k=0}^{\infty} \|z\|_{\mathbf{E}^k}^{2k} \right)^{1/2} \left(\sum_{k=0}^{\infty} \|\xi_k\|_{\mathbf{E}^k}^{2} \right)^{1/2} \tag{3.5}$$

$$= \frac{\|f\|}{\sqrt{1 - \|z\|^2}}. \tag{3.6}$$

These calculations show that (3.2) converges in $\mathbb{B}_d$ to define an analytic function on B_d. □

We also have that the calculations in the proof of Proposition 3.4 show that the evaluation functional for H_d^2 is bounded. Hence, H_d^2 is a reproducing kernel Hilbert space on $\mathbb{B}_d$.

Proposition 3.5. *The reproducing kernel for* $z \in \mathbb{C}^d$ *is* $k_z(w) \in H_d^2$*, where* $\xi_k = z^k$ *in (3.2). Therefore, the reproducing kernel for* H_d^2 *is*

$$K(z, w) = \frac{1}{1 - < z, w >}.$$

Proof. Let $f \in H_d^2$. Then it follows that

$$f(z) = \sum_{k=0}^{\infty} < z^k, \xi_k > \tag{3.7}$$

$$= \left\langle \sum_{k=0}^{\infty} < w^k, \xi_k >, \sum_{k=0}^{\infty} < w^k, z^k > \right\rangle_{H_d^2} \tag{3.8}$$

$$=< f(w), k_z(w) > \tag{3.9}$$

□

3.2 A Negative Result for $d \geq 2$

The following construction is a result directly from Arveson, [6]; we only present the case $d = 2$. Assume that $z = (z_1, z_2)$ is in the closed unit ball $\overline{\mathbb{B}_2} \subset \mathbb{C}^2$, and let $\{c_n\}_{n \in \mathbb{N}_0^d}$ be a sequence of complex numbers having the following properties:

(i) $\sum_{n=0}^{\infty} |c_n| = 1$,

(ii) $\sum_{n=0}^{\infty} |c_n|^2 \sqrt{n} = \infty$.

Define the function

$$f(z) = \sum_{n=0}^{\infty} c_n \left(\frac{z_1 z_2}{\sup |z_1 z_2|} \right)^n , \tag{3.10}$$

where the sup norm is over $\mathbb{B}_2$,

$$\sup_{|z_1|^2 + |z_2|^2 \leq 1} |z_1 z_2| = \frac{1}{2}.$$

It follows that the power series (3.10) converges uniformly over the closed unit ball to a function f satisfying $\|f\|_\infty \leq 1$. Yet, the restriction of f to $\mathbb{B}_2$ does not belong to the Drury-Arveson space, H_2^2:

$$\sup_{N \in \mathbb{N}_0^d} \|f_N(z)\|_{H_2^2}^2 = \sum_{n=0}^{\infty} |c_n|^2 \frac{\|(z_1 z_2)^n\|_{H_2^2}^2}{\|(z_1 z_2)^n\|_\infty^2} = \infty. \tag{3.11}$$

Using (ii) in the definiton of $\{c_n\}$, we is established the existence of a positive constant A such that

$$\frac{\|(z_1 z_2)^n\|_{H_2^2}^2}{\|(z_1 z_2)^n\|_\infty^2} \geq A\sqrt{n}, \ \forall n = 1, 2, \ldots . \tag{3.12}$$

The last result (3.12) was shown by Arveson using the identification of H_d^2 with the symmetric Fock space over $\mathbb{C}^d$.

Remark. Arveson proved the above result for $d \geq 2$; and f serves as an explicit *example of a function that is in the ball algebra $A(\mathbb{B}_d)$, but is not in the multiplier algebra $\mathcal{M}_d$ of H^2_d.*

Arveson concludes that there does not exist a factorization of the Arveson kernel on a boundary subspace of $\mathbb{C}^d$ for $d \geq 2$.

Theorem 3.6 (Arveson)**.** *There is no positive measure μ on $\mathbb{C}^d$, $d \geq 2$, with the property that*

$$\|f\|^2_{H^2_d} = \int_{\mathbb{C}^d} |f(z)|^2 d\mu(z)$$

for every polynomial f.

Since there does not exist a factorization of the Arveson kernel on a boundary subspace of $\mathbb{C}^d$ for $d \geq 2$, the research turned to investigating our duality correspondence for a relative reproducing kernel Hilbert space. Moreover, we establish the anlysis for boudary spaces much larger than $\mathbb{C}^d$.

3.3 The Drury-Arveson Space and Boundary Analyses

Upon generalizing from the Hardy space $H^2(\mathbb{D})$ to the Drury-Arveson space H^2_d, we naturally seek to understand which properties are preserved. For example, the functions in the Hardy space on the unit disk have an extension to boundary functions on the circle; that is for $f \in H^2(\mathbb{D})$, $f(z) \sim f_r(e(t))$, we have

$$\lim_{r \to 1^-} f_r(e(t)) = \tilde{f}(e(t)),\ \tilde{f} \in L^2(\mathbb{T}).$$

Moreover, we have the following isometric boundary extension for the Szego kernel

$$K(z,w) = \frac{1}{1-\overline{w}z} \mapsto \frac{1}{1-\overline{e(t)}z} = K^{\mathbb{T}}(z,t).$$

From Arveson's result when $d > 1$, we know that there exists a function $f \in H_d^2$ such that

$$\|f\|_{H_d^2}^2 \neq \|f\|_{L^2(\mathbb{C}^d,\mu)}^2,$$

for every positive measure μ on $\mathbb{C}^d$. So, in the setting of the Drury-Arveson RKHS, there does not exist an isometric boundary extension for functions in the Drury-Arveson RKHS to a subspace of $\mathscr{F}(\mathbb{C}^d, \mathbb{C})$. Therefore, analogues of this boundary extension must be realized given other boundary spaces.

Definition 3.5. Fix a positive definite kernel $K : X \times X \longrightarrow \mathbb{C}$. Let $\mathcal{M}(K)$ be the set of all probability spaces. We say that $(B, \mathcal{F}, \mu) \in \mathcal{M}(K)$ if and only if there exist an extension

$$K^B : X \times B \longrightarrow \mathbb{C}, \text{ and}$$

$$\int_B K^B(x,b)\overline{K^B(y,b)}d\mu(b) = K(x,y), \tag{3.13}$$

for all $(x, y) \in X \times X$. We say that (4.24) is a **Boundary Factorization** of the reproducing kernel K.

Definition 3.6. We say $(B, \mathfrak{F}, \mu) \in GC$ is a **Generalized Carleson Measure** if and only if there exists a constant C_μ such that

$$\|\tilde{f}\|_{L^2(\mu)}^2 \leq C_\mu \|f\|_{\mathscr{H}(K)}^2, \; \forall f \in \mathscr{H}(K),$$

where $\tilde{f}$ is defined via the extension

$$\tilde{f}(b) := < f, K_b^B >_{\mathscr{H}(K)}, \; b \in B, f \in \mathscr{H}(K).$$

Definition 3.7. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces. We say that $\mathscr{H}_1$ is **Boundedly Contained** in $\mathscr{H}_2$ if and only if

(i) $\mathscr{H}_1 \subset \mathscr{H}_2$ (as a subset) and

(ii) the inclusion map $i : \mathscr{H}_1 \hookrightarrow \mathscr{H}_2$ defined by $h \mapsto h$ is bounded. That is, there exists $C < \infty$ such that for all $h \in \mathscr{H}_1$,

$$\|h\|_{\mathscr{H}_2} \leq C\|h\|_{\mathscr{H}_1}.$$

Given a positive definite kernel K and a measure space $(B, \mathfrak{F}, \mu)$, we say that $(B, \mathfrak{F}, \mu) \in GC$ if and only if $H(K)$ is boundedly contained in $L^2(B, \mathfrak{F}, \mu)$. Our boundary results for the Drury-Arveson space led us to investigate Carleson measures for H^2_d.

Theorem 3.7. *Let K be the reproducing kernel for the Drury-Arveson space,*

$$K(z, w) = \frac{1}{1- < z, w >_{\mathbb{B}_d}}.$$

Then for every $z \in \mathbb{B}_d$ there does not exists a boundary subspace of $(\mathbb{C}^d, \mu)$ $(d > 1)$ such that

$$\|k_z\|^2_{H^2_d} = \int_{\mathbb{C}^d} |k_z(b)|^2 \, d\mu(b).$$

That is, for every measure μ on $\mathbb{C}^d$ $(d > 1)$,

$$\frac{1}{1- < z, w >_{\mathbb{B}_d}} \neq \int_{\mathbb{C}^d} \frac{1}{1- < z, b >_{\mathbb{C}^d}} \cdot \frac{1}{1- < b, w >_{\mathbb{C}^d}} d\mu(b).$$

Proof. Let K be the reproducing kernel for the Drury-Arveson space H^2_d with $d > 1$. For the sake of contradiction, assume that we do have

$$\|K\|^2_{H^2_d} = \int_{\mathbb{C}^d} |K(b)|^2 \, d\mu(b). \tag{3.14}$$

Let $f \in H^2_d$ and $\{k_z | z \in \mathbb{B}_d\}$ be a subset of H^2_d such that $k_z(w) = K(z, w)$ for every $w \in \mathbb{B}_d$. We know from Theorem 4.21 that $\text{span}\{k_z | z \in \mathbb{B}_d\}$ is dense in H^2_d.

Case 1: Assume that $f \in \text{span}\{k_{z_j} | z_j \in \mathbb{B}_d\}$. Then we have that $f(w) = \sum_{j \in \mathbb{N}_0^d} c_j k_{z_j}(w)$ for every $w \in \mathbb{B}_d$. It then follows that

$$\|f\|^2_{H^2_d} = \left\| \sum_{j \in \mathbb{N}_0^d} c_j k_{z_j} \right\|^2_{H^2_d} \tag{3.15}$$

$$= \int_{\mathbb{C}^d} \left| \sum_{j \in \mathbb{N}_0^d} c_j k_{z_j}(w) \right|^2 d\mu(w) \tag{3.16}$$

$$= \int_{\mathbb{C}^d} |f(w)|^2 d\mu(w). \tag{3.17}$$

Case 2: Assume that $f \notin \text{span}\{k_{z_j} | z_j \in \mathbb{B}_d\}$. Then there exists a sequence of functions $(f_n) \subseteq \text{span}\{k_{z_j} | z_j \in \mathbb{B}_d\}$ with $\|f - f_n\|_{H^2_d} \longrightarrow 0$. We see that $f(z) = \lim_n f_n(z_j)$, for every $z_j \in \mathbb{B}_d$.

$$\|f\|^2_{H^2_d} = \lim_{n \to \infty} \|f_n\|^2_{H^2_d} \tag{3.18}$$

$$= \lim_{n \to \infty} \left\| \sum_{j \in \mathbb{N}_0^d} c_j K_n(z_j, \cdot) \right\|^2_{H^2_d} \tag{3.19}$$

$$= \lim_{n \to \infty} \int_{\mathbb{C}^d} \left| \sum_{j \in \mathbb{N}_0^d} c_j K_n(z_j, w) \right|^2 d\mu(w) \tag{3.20}$$

$$= \int_{\mathbb{C}^d} \lim_{n \to \infty} |f_n(w)|^2 d\mu(w). \tag{3.21}$$

$$= \int_{\mathbb{C}^d} |f(w)|^2 d\mu(w). \tag{3.22}$$

□

Proposition 3.8. *Let K be the reproducing kernel for the Drury-Arveson space H^2_d and let μ be a positive measure on $\mathbb{C}^d$. Then H^2_d is boundedly contained in $L^2(\mathbb{C}^d, \mu)$.*

Moreover,

$$\int_{\mathbb{C}^d} K(z,z)d\mu(z) < \infty$$

is a Carleson measure for the Drury-Arveson RKHS.

Proof. Let f be a function in the Drury-Arveson space H^2_d. Then it follows that

$$\|f\|^2_{L^2(\mathbb{C}^d,\mu)} = \int_{\mathbb{C}^d} |f(z)|^2 d\mu(z) \tag{3.23}$$

$$\leq \int_{\mathbb{C}^d} \|f\|^2_{H^2_d} \cdot \|K_z\|^2_{H^2_d} d\mu(z) \tag{3.24}$$

$$= \|f\|^2_{H^2_d} \int_{\mathbb{C}^d} \cdot K(z,z) d\mu(z) \tag{3.25}$$

$$\tag{3.26}$$

Therefore, we can define $C_\mu := \int_{\mathbb{C}^d} \frac{1}{1-\|z\|^2} d\mu(z) < \infty$, and we get that

$$\|f\|^2_{L^2(\mathbb{C}^d,\mu)} < C_\mu \cdot \|f\|^2_{H^2_d},\ \forall f \in H^2_d. \tag{3.27}$$

□

3.3.1 Factorizations for the Arveson Kernel

Let $K(z,w)$ be the Arveson kernel such that the associated mapping

$$E_{\mathbb{B}_d} : \mathbb{B}_d \longrightarrow \mathscr{F}(\mathbb{B}_d, \mathbb{C})$$

is one-to-one. Then we know from [P.D. Kernels and Boundary Spaces, Jorgensen] that we can form the probability space $(\mathscr{F}(\mathbb{B}_d,\mathbb{C}), Cyl(\mathscr{F}(\mathbb{B}_d,\mathbb{C})), \mu_{\mathscr{F}(\mathbb{B}_d,\mathbb{C})})$, where

- $\mathscr{F}(\mathbb{B}_d,\mathbb{C})$ is the set of all functions from $\mathbb{B}_d \to \mathbb{C}$,
- $\mathrm{Cyl}(\mathscr{F}(\mathbb{B}_d,\mathbb{C}))$ is the σ-algebra of cylinder subsets of $\mathscr{F}(\mathbb{B}_d,\mathbb{C})$,

- $\mu_{\mathscr{F}(\mathbb{B}_d,\mathbb{C})}$ is the associated measure on $\mathscr{F}(\mathbb{B}_d, \mathbb{C})$ established by the Kolmogorov extension theorem.

The referenced theorem by Jørgensen proves the existence of generalized boundaries for any positive definite kernel K on a set S. Along with this existence theorem, Jørgensen proved that there exists minimal factorizations for the positive definite kernel.

Example 3.9 (Gel'fand Triple Construction)**.** Using the theory of Gel'fand triples we constructed a (probability) measure space to serve as the boundary of $\mathbb{B}_d$. This approach was used to find a smaller boundary space than $(\mathscr{F}(\mathbb{B}_d, \mathbb{C}), Cyl(\mathscr{F}(\mathbb{B}_d, \mathbb{C})), \mu_{\mathscr{F}(\mathbb{B}_d,\mathbb{C})})$. The corresponding boundary space becomes the space of hyperfunctions. From the literature we know that no Hilbert space of functions is sufficient to support a Gaussian probability measure. To form the Gel'fand triple we begin with a dense sebspace of the Drury-Arveson space. It was Gel'fand's idea to formalize this construction abstractly using a system of nuclearity axioms. The construction here is adapted from Wiener and the triple $(A(\mathbb{T}), L^2([0,1]), A(\mathbb{T})')$, where $A(\mathbb{T})$ is the space of absolutely continuous Fourier series and $A(\mathbb{T})'$ is the space of psuedo-measures, $PM(\mathbb{T})$.

An element $(\xi_k) \in \mathfrak{F}_+(\mathbf{E})$ is said to be rapidly decreasing if

$$\lim_{k\to\infty} c^k \|(\xi_k)\|_{\mathfrak{F}_+(\mathbb{C}^d)} = 0, \ \forall c \in \mathbb{N}.$$

It follows that for every $k \in \mathbb{N}$ there exist $N_k \in \mathbb{N}$ such that

$$|\xi_k| \leq \frac{N_k}{c^k}.$$

We conclude that (ξ_k) is a rapidly decreasing sequence of Taylor coefficients in $\mathfrak{F}_+(\mathbf{E})$, and $S(H_d^2)$ is a nuclear space of entire analytic functions;

$$\sum_{k=0}^{\infty} |\xi_k z^k| \leq \sum_{k=0}^{\infty} N_k \left(\frac{|z|}{c}\right)^k, \text{ where } |z| < c.$$

The duality between $S(H_d^2)$ and $S'(H_d^2)$ allows for the extension of the inner product on $\mathfrak{F}_+(\mathbb{C}^d)$ to a pairing of $S(H_d^2)$ and $S'(H_d^2)$. In other words, one obtains a Fourier-type duality restricted to S. The construction of this Gel'fand triple, $S(H_d^2) \subseteq \mathfrak{F}_+(\mathbb{C}^d) \subseteq S'(H_d^2)$, carries a Gaussian probability measure $\mathbb{P}$ on $S'(H_d^2)$, and yields the following three outcomes:

1. *An isometric embedding of* $\mathfrak{F}_+(\mathbb{C}^d)$ *into* $L^2(S'(H_d^2), \mathbb{P})$,
2. *A representation of the boundary as a certain class of distributions,*
3. *A boundary integral representation for the Arveson kernel.*

Example 3.10 (Natural Generalized Boundary Space)**.** Let $(e_\alpha)_{\alpha \in \mathbb{Z}_+^d}$ be the orthonormal basis in H_d^2 where

$$e_\alpha : \mathbb{B}_d \longrightarrow \mathbb{C} \quad \textit{is defined by} \quad e_\alpha(z) = \left(\frac{|\alpha|!}{\alpha!}\right)^{\frac{1}{2}} z^\alpha,$$

$$< \sqrt{\frac{|\alpha|!}{\alpha!}} z^\alpha, \sqrt{\frac{|\beta|!}{\beta!}} z^\beta >_{H_d^2} = \begin{cases} 0 & \alpha \neq \beta, \\ 1 & \alpha = \beta. \end{cases}$$

Therefore, we know that

- $K(z, w) = \sum_{\alpha \in \mathbb{N}_0^d} \sum_{\beta \in \mathbb{N}_0^d} e_\alpha(z) \overline{e_\beta(w)}, \forall z, w \in \mathbb{B}_d,$
- $\sum_{\alpha \in \mathbb{N}_0^d} |e_\alpha(z)|^2 = K(z, z) < \infty,$
- $f(w) = \sum_{\alpha \in \mathbb{N}_0^d} < f, e_\alpha >_{H_d^2} e_\alpha(w), \forall f \in H_d^2.$

We conclude that there are infinitely many boundary factorizations for H_d^2 via the following construction;

$$\begin{array}{ccc} H_d^2 & \overset{i}{\hookrightarrow} & \ell^2(\mathbb{Z}_+^d) \\ & {\scriptstyle J} \searrow & \downarrow {\scriptstyle W_\mu \ (isometry)} \\ & & L^2(\Omega, \mu) \end{array}$$

The above diagram commutes when we have

- $i : H_d^2 \hookrightarrow \ell^2(\mathbb{Z}_+^d)$ *defined by* $i(f) = (e_\alpha)_{\alpha \in \mathbb{Z}_+^d} \in \ell^2(\mathbb{Z}_+^d)$,
- $W_\mu : \ell^2(\mathbb{Z}_+^d) \longrightarrow L^2(\Omega, \mu)$ *(an isometry) defined by* $W_\mu e_\alpha(w) = \psi(w, \cdot)$,
- $J^* : L^2(\mu) \longrightarrow H_d^2$ *where* $J^*\psi(w, \cdot) =< i^* e_\alpha(z), K_w(z) >_{H_d^2} = f(w)$.

Thus, we can form a boundary factorization of the Arveson kernel $K(z, w)$

$$\frac{1}{1- < z, w >_{\mathbb{B}_d}} = \int_\Omega J(K_z)(b)\overline{J(K_w)(b)} d\mu(b)$$

where $J(K(z, \cdot)) = \psi(z, \cdot) \neq K(z, \cdot)$ and

$$\|f\|^2_{H_d^2} = \|Jf\|^2_{L^2(\mu)}.$$

Among the boundary spaces, we focus on a natural generalization of the Hardy space factorization. Define $\tau : \mathbb{B}_d \longrightarrow \ell^2(\mathbb{Z}_+^d) \subsetneq \mathbf{s}'$ by

$$\tau(z) = (\varphi_\alpha(z))_{\alpha \in \mathbb{Z}_+^d}, \quad z \in \mathbb{B}_d. \tag{3.28}$$

It follows that τ is one-to-one and we may identify points $z \in \mathbb{B}_d$ with their image in $\mathbf{s}'$. Set $\tau(\mathbb{B}_d) = \{\tau(z) | z \in \mathbb{B}_d\}$ and let $\text{clo}_K(\mathbb{B}_d)$ be its closure in $\mathbf{s}'$. Here by closure we mean the weak*-topology in $\mathbf{s}'$ defined by the duality between $\mathbf{s}$ and $\mathbf{s}'$. Finally, set

$$\text{bd}_K(\mathbb{B}_d) = \text{clo}_K(\mathbb{B}_d) \setminus \tau(\mathbb{B}_d),$$

and for $b \in \mathrm{bd}_K(\mathbb{B}_d)$ define

$$X_z(b) = \sum_{\alpha \in \mathbb{N}_0^d} (\overline{\varphi_\alpha(z)})\pi_\alpha(b) = \sum_{\alpha \in \mathbb{N}_0^d} b_\alpha(\overline{\varphi_\alpha(z)}).$$

Now, let $\mathbb{P}_K$ by the measure on $\mathrm{bd}_K(\mathbb{B}_d)$. We then get

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_K}(X_z(\cdot)\overline{X_w(\cdot)} &= \mathbb{E}_{\mathbb{P}_K}\left(\left(\sum_{\alpha \in \mathbb{Z}_+^d} \varphi_\alpha(w)\overline{\pi_\alpha}\right)\left(\sum_{\beta \in \mathbb{Z}_+^d} \varphi_\beta(w)\overline{\pi_\beta(z)}\right)\right) \\
&= \sum_{\alpha \in \mathbb{Z}_+^d} \varphi_\alpha(z)\overline{\varphi_\alpha(w)} \\
&= K(z, w), \quad \forall z, w \in \mathbb{B}_d.
\end{aligned}$$

Chapter 4
P.D. KERNEL RELATIVE TO THE ARVESON KERNEL

In this chapter we define a new positive definite function on the unit ball in $\mathbb{C}^d$ and this leads to the study of a new reproducing kernel Hilbert space. This new reproducing kernel, labeled K^{new}, stems from the boundary analysis of the Arveson kernel; and this new RKHS resolves the negative result from section (3.2) in reference to the Drury-Arveson space. Recall that my research question stems from the following duality correspondence.

Positive Definite Functions	**Boundary Spaces**
We say that K is a positive definite function on $\mathbb{B}_d$ when given $K : \mathbb{B}_d \times \mathbb{B}_d \longrightarrow \mathbb{C}$, we have $\forall N \in \mathbb{N},\ \{x_i\}_{i=1}^N \subset \mathbb{B}_d,\ \{c_i\}_{i=1}^N \subset \mathbb{C}$ $\sum_i \sum_j c_i \overline{c}_j K(x_i, x_j) \geq 0.$	We say that $(\mathcal{B}, \mathfrak{B}, \mu)$ is a measure space, where $\mathcal{B}$ is a set equipped with a σ-algebra $\mathfrak{B}$ of subsets, and μ is a probability measure defined on $\mathfrak{B}$. *We call this a boundary space for a given positive definite function on $\mathbb{B}_d$ when $\mathcal{B}$ is understood to be some sort of boundary set containing $\mathbb{B}_d$.*

Table 4.1: Duality: Postive Definite Functions and Boundary Spaces

4.1 The New RKHS, $\mathcal{H}_d(\mathbb{B}_d)$

We have discussed examples of factorizations of reproducing kernels over natural boundary space; that is their natural topological boundary. Moreover, we showed that there always exists a general boundary space that yields a factorization for the reproducing kernel. In this section we define a new reproducing kernel on the unit ball in complex d-space, and we establish the natural boundary factorization. Moreover, we prove results for this new reproducing kernel Hilbert space. This reproducing kernel Hilbert space stems from boundary analysis of the Arveson kernel. The construction of the new RKHS resolves the problem we faced while researching "natural" boundary spaces for the Drury-Arveson RKHS in higher dimensions.

Recall the definition for a boundary factorization. Given K and $(\mathcal{B}, \mathfrak{B}, \mu)$ as in the table above, we shall say that $(\mathcal{B}, \mathfrak{B}, \mu) \in \mathcal{M}(K)$ if there is a function $K^{\mathcal{B}} : \mathbb{B}_d \to L^2(\mu)$ such that

$$K(z,w) = \int_{\mathcal{B}} K_z^{\mathcal{B}}(b)\overline{K_w^{\mathcal{B}}(b)}d\mu(b), \quad z,w \in \mathbb{B}_d, b \in \mathcal{B} \quad (Factorization\ of\ K). \qquad (4.1)$$

Since Arveson, [6], proved that there does not exist a factorization of the Arveson kernel on a boundary subspace of $\mathbb{C}^d$ for $d \geq 2$, we created a positive definite kernel relative to the Arveson kernel which satisfies 4.1. A primary motivation was to make sense of the Drury-Arveson space boundary results. There are three primary components that need to be understood from 4.1: the boundary $\mathcal{B}$, the boundary function $K^{\mathcal{B}}(z,b)$, and the positive measure μ on $\mathcal{B}$.

We define $\mathcal{B}$ to be the boundary space $\partial\mathbb{B}_d := \{b \in \mathbb{C}^d : \|b\|_{\mathbb{C}^d} = 1\}$. This is the natural generalization of the boundary of the complex unit ball in one-dimension.

The below proposition established the connection that the boundary space $\partial\mathbb{B}_d$ has to quantum mechanics, in particular the subject of quantum states.

Two known results from Lie theory are that

1. The unitary group $U(d)$ is transitive on $\partial\mathbb{B}_d$ for $d \geq 2$, and

2. The isotropy group $U(d)_{e_d}$ of the standard basis vector $e_d = (0, \dots, 0, 1)$ is

$$U(d)_{e_d} = \left\{ \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} : M \in U(d-1) \right\} \cong U(d-1).$$

Hence, it follows that the boundary space $\partial\mathbb{B}_d$ is diffeomorphic to ${}^{U(d)}/_{U(d-1)}$. Let $\mu := \mu_d$ be the normalized Haar measure on $U(d)$; passing the Haar measure μ on $U(d)$ to the quotient ${}^{U(d)}/_{U(d-1)}$, we use this to generalize results from Arveson's paper via the space $L^2({}^{U(d)}/_{U(d-1)}, \mu)$. In this chapter $L^2({}^{U(d)}/_{U(d-1)}, \mu)$ will be by $L^2(\mu)$, where

$$\|\widehat{f}\|^2_{L^2(\mu)} = \int_{U(d)/U(d-1)} |\widehat{f}(b)|^2 d\mu(b).$$

Since $\partial\mathbb{B}_d$ and ${}^{U(d)}/_{U(d-1)}$ are diffeomorphic, we can view μ as a measure on $\partial\mathbb{B}_d$.

Proposition 4.1. *Let $z \in \mathbb{B}_d$ and define $k_z^{\partial\mathbb{B}_d} : \partial\mathbb{B}_d \longrightarrow \mathbb{C}$ such that*

$$k_z^{\partial\mathbb{B}_d}(b) = \frac{1}{1- < z, b >}.$$

Then

$$k_z^{\partial\mathbb{B}_d}(b) = \lim_{r \to 1^-} K_{DA}(z, rb),$$

and we have that $k_z^{\partial\mathbb{B}_d} \in L^2(\mu)$.

Proof. Let $z \in \mathbb{B}_d$ and define $k_z^{\partial\mathbb{B}_d} : \partial\mathbb{B}_d \longrightarrow \mathbb{C}$ such that

$$k_z^{\partial\mathbb{B}_d}(b) = \frac{1}{1- < z, b >}.$$

It is clear that

$$\begin{aligned} k_z^{\partial \mathbb{B}_d}(b) &= \frac{1}{1- < z, b >} \\ &= \lim_{r \to 1^-} \frac{1}{1- < z, rb >} \\ &= \lim_{r \to 1^-} K_{DA}(z, rb). \end{aligned}$$

Moreover, fix $z \in \mathbb{B}_d$. Then we have that

$$\begin{aligned} \int_{U(d)/U(d-1)} \left|k_z^{\partial \mathbb{B}_d}(b)\right|^2 d\mu(b) &= \int_{U(d)/U(d-1)} \left| \frac{1}{1- < z, b >} \right|^2 d\mu(b) \\ &= \int_{U(d)/U(d-1)} \left| \sum_{k=0}^{\infty} < z, b >^k \right|^2 d\mu(b) \\ &= \int_{U(d)/U(d-1)} \sum_{k,l=0}^{\infty} < z, b >^k \overline{< z, b >}^l d\mu(b) \\ &= \int_{U(d)/U(d-1)} \sum_{k=0}^{\infty} |< z, b >|^{2k} d\mu(b) \qquad \text{(Lemma 4.3)} \\ &\le \left(\sum_{k=0}^{\infty} \|z\|^{2k} \right) \\ &\le \left(\frac{1}{1 - \|z\|^2} \right) < \infty. \end{aligned}$$

□

Proposition 4.2. *Let $\mathfrak{u}$ be a unitary representation on $U(d)$, such that for $g \in U(d)$*

$$\mathfrak{u}_g : H_d^2 \longrightarrow H_d^2 \text{ defined by } \mathfrak{u}_g f(z) = f(g^{-1} z^\top).$$

Then $\|\mathfrak{u}_g f\|^2_{H_d^2} = \|f\|^2_{H_d^2}$ and $\mathfrak{u}_g$ is a unitary operator on the Drury-Arveson space.

Proof.

$$\mathfrak{u}_g f(z) = \sum_{n=0}^{\infty} < \xi_n, (g^{-1} z^\top)^{\otimes n} >_{\mathbf{E}^n} \tag{4.2}$$

$$= \langle (\otimes^n g) \xi_n, z^n \rangle_{\mathbf{E}^n} . \tag{4.3}$$

Since $\otimes^n \xi_n$ is a symmetric tensor, we have that $\|(\otimes^n g)\xi_n\| = \|\xi_n\|$. Therefore, it follows that $\|\mathfrak{u}_g f\|^2_{H^2_d} = \|f\|^2_{H^2_d}$. □

The following lemma provides an orthogonality condition useful for our results. For $z \in \mathbb{B}_d$, we already have that $\{z^\alpha\}_{\alpha \in \mathbb{N}_0^d}$ is an orthogonal system. Let $\widehat{\mathfrak{u}}$ be another unitary representation on $U(d)$, such that given $g \in U(d)$,

$$\widehat{\mathfrak{u}}_g : L^2(\mu) \longrightarrow L^2(\mu) \text{ is defined by } (\widehat{\mathfrak{u}}_g \widehat{f})(b) = \widehat{f}(g^{-1} b^\top).$$

Lemma 4.3 (Main Lemma). *Denote* $\mathbf{E} = \mathbb{C}^d$*, as before, let* $z \in \mathbb{B}_d$*, and let* $b \in \partial\mathbb{B}_d$*. Then* $z^{\otimes n}, b^{\otimes n} \in \mathbf{E}^n$ *and we have the following two orthogonality conditions:*

H^2_d: *With respect to the inner product on* H^2_d*, we have that* $\{< \xi_n, z^{\otimes n} >\}_{\xi_n \in \mathbf{E}^n}$ *is an orthogonal system, and*

$L^2(\mu)$: *With respect to the inner product on* $L^2(\mu)$*, we have that* $\{< \xi_n, b^{\otimes n} >\}_{\xi_n \in \mathbf{E}^n}$ *is an orthogonal system,*

where (ξ_n) *ranges over the symmetric Fock space* $\mathfrak{F}_+(\mathbf{E})$*.*

Proof. Let $W_\mu : H^2_d \longrightarrow L^2(\mu)$ be the mapping that sends a function f from the Drury-Arveson space to $\widehat{f} \in L^2(\mu)$, where

$$\widehat{f}(b) \underset{a.e}{:=} \lim_{r \to 1^-} f(rb) = \lim_{r \to 1^-} < f(z), K_{DA}(z, rb) >_{H^2_d} .$$

H^2_d: The orthogonality of $\{\varphi_n\}$, where $\varphi_n(z) =< \xi_n, z^{\otimes n} >_{\mathbf{E}^n}$, follows from the isomorphism between $H^2_d \cong \mathfrak{F}_+(\mathbf{E})$. That is,

$$< \varphi_n, \varphi_m >_{H^2_d} =< (\xi_n), (\xi_m) >_{\mathfrak{F}_+(\mathbf{E})} .$$

It follows from proposition 3.3 (and properties of the tensor product) that

$$< (\xi_n), (\xi_m) >_{\mathfrak{F}_+(\mathbf{E})} = 0 \text{ when } n \neq m.$$

$L^2(\mu)$: Now consider the $L^2(\mu)$-inner product of $< \xi_n, b^{\otimes n} >$ and $< \xi_m, b^{\otimes m} >$,

$$\int_{U(d)/U(d-1)} < \xi_n, b^{\otimes n} > \overline{< \xi_m, b^{\otimes m} >} d\mu(b).$$

Let $g = \begin{pmatrix} e^{i\theta} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta} \end{pmatrix} \in U(d)$ and define $\widehat{\mathfrak{u}}_g :< \xi_n, b^{\otimes n} > \longmapsto < \xi_n, (g^{-1}b^{\top})^{\otimes n} >$, where $b = (b_1, b_2, \ldots, b_d)$. Then it follows that

$$\widehat{\mathfrak{u}}_g W_\mu = W_\mu \mathfrak{u}_g. \qquad \text{(Intertwining Property)}$$

For $n \neq m$, it follows that $< \xi_n, b^{\otimes n} >$ and $< \xi_m, b^{\otimes m} >$ span distinct eigenspaces; thus are orthogonal. □

Note that the Arveson kernel is translation invariant with respect to the Lie group $U(d)$. In this chapter the Arveson kernel will be denoted by $K_{DA}(z, w)$, that is

$$K_{DA}(z, w) = \frac{1}{1- < z, w >}, z, w \in \mathbb{B}_d.$$

The following theorem is the main theorem. The theorem provides a response to the issue that arose in chapter 3.2. By relaxing the isometric condition, for the case of $d = 1$, we have a generalization (of what we observe in the Hardy space) to the Drury-Arveson space. Arveson showed that we cannot have an isometry to an $L^2(\mathbb{C}^d)$-space; thus, we show that we have contractivity instead. This leads to other results and open questions.

Theorem 4.4 (Main Theorem)**.** *Let* $W_\mu : H^2_d \longrightarrow L^2(\mu)$ *be the mapping that sends a function* f *from the Drury-Arveson space to* $\widehat{f} \in L^2(\mu)$*, where*

$$\widehat{f}(b) \underset{a.e}{:=} \lim_{r\to 1^-} f(rb) = \lim_{r\to 1^-} < f(z), K_{DA}(z, rb) >_{H^2_d} .$$

Then the following conditions hold

(i) $(W^*_\mu \varphi)(z) = \int_{U(d)/U(d-1)} k^{\partial\mathbb{B}_d}_z(b)\varphi(b)d\mu(b)$

(ii) $\|\widehat{f}\|^2_{L^2(\mu)} \leq \|f\|^2_{H^2_d}$ *(equality only for* $d = 1$*)*

Proof. Let $W_\mu : H^2_d \longrightarrow L^2(\mu)$ be as above.

(i) Then we get

$$(W^*_\mu \varphi)(z) =< W^*_\mu \varphi(\cdot), K_{DA}(z, \cdot) >_{H^2_d} =< \varphi(\cdot), W_\mu(K_{DA}(z, \cdot)) >_{L^2(\mu)} .$$

(ii) Futhermore, it follows that

$$\begin{aligned}
\|\widehat{f}\|^2_{L^2(\mu)} &= \int_{U(d)/U(d-1)} \left| \sum_{n=0}^{\infty} < \xi_n, b^{\otimes n} >_{\mathbf{E}^n} \right|^2 d\mu(b) \\
&= \int_{U(d)/U(d-1)} \sum_{n,m=0}^{\infty} < \xi_n, b^{\otimes n} >_{\mathbf{E}^n} \overline{< \xi_m, b^{\otimes m} >}_{\mathbf{E}^m} d\mu(b) \\
&= \int_{U(d)/U(d-1)} \sum_{n=0}^{\infty} \left| < \xi_n, b^{\otimes n} >_{\mathbf{E}^n} \right|^2 d\mu(b) \qquad \text{(Lemma 4.3)} \\
&\leq \sum_{n=0}^{\infty} \|\xi_n\|^2_{\mathbf{E}^n} \\
&= \|f\|^2_{H^2_d}.
\end{aligned}$$

□

The above theorem yields the following contraction

$$\|W^f_\mu\|_{L^2(U(d)/U(d-1),\mu)} \leq \|f\|_{H^2_d}.$$

Remark. See the Appendix B for open questions stemming from the main theorem.

Following definition 2.5, the map $\Phi : \mathbb{B}_d \longrightarrow L^2({}^{U(d)}/_{U(d-1)}, \mu)$ defined by $\Phi(z) = K^{\partial \mathbb{B}_d}(z, b)$ is a feature map. Thus, K^{new} is a positive definite kernel induced by the feature map,

$$\begin{aligned} K^{new}(z,w) &= \left\langle K^{\partial \mathbb{B}_d}(z,b), K^{\partial \mathbb{B}_d}(w,b) \right\rangle_{L^2(\mu)} \\ &= \int_{U(d)/U(d-1)} K^{\partial \mathbb{B}_d}(z,b) \overline{K^{\partial \mathbb{B}_d}(w,b)} d\mu(b) \\ &= \int_{U(d)/U(d-1)} \frac{1}{1- < z, b >} \frac{1}{1- < b, w >} d\mu(b). \end{aligned}$$

We denote $\mathcal{H}_d = \mathcal{H}_d(\mathbb{B}_d)$ to be the RKHS for K^{new}.

Corollary 4.5. *The map $T_\mu : \mathcal{H}_d \longrightarrow L^2({}^{U(d)}/_{U(d-1)}, \mu)$ defined by*

$$(T_\mu K^{new}(z, \cdot))(b) := K^{\partial \mathbb{B}_d}(z, b) \tag{4.4}$$

is an isometric embedding of $\mathcal{H}_d(\mathbb{B}_d)$ into $L^2({}^{U(d)}/_{U(d-1)}, \mu)$. That is,

$$\|\tilde{f}\|^2_{\mathcal{H}_d} = \int_{U(d)/U(d-1)} |\widehat{f}(b)|^2 d\mu(b),$$

where $\hat{f} := T_\mu(\tilde{f})$.

Proof. Define the map

$$T_\mu : \mathcal{H}_d \longrightarrow L^2({}^{U(d)}/_{U(d-1)}, \mu)$$

$$(T_\mu K^{new}(z, \cdot))(b) := K^{\partial \mathbb{B}_d}(z, b).$$

Case 1: Assume that $\tilde{f} \in \operatorname{span}\{K^{new}(z_j, \cdot) | z_j \in \mathbb{B}_d\}$. Then we have that

$$\begin{aligned}
\|\tilde{f}\|^2_{\mathcal{H}_d} &= \sum_{i,j \in \mathbb{N}_0^d} \alpha_i \overline{\alpha_j} \left\langle K^{new}(z_i, \cdot), K^{new}(z_j, \cdot) \right\rangle_{\mathcal{H}_d} \\
&= \sum_{i,j \in \mathbb{N}_0^d} \alpha_i \overline{\alpha_j} \left\langle K^{\partial \mathbb{B}_d}(z_i, \cdot), K^{\partial \mathbb{B}_d}(z_j, \cdot) \right\rangle_{L^2(\mu)} \\
&= \|T_\mu(\tilde{f})\|^2_{L^2 \mu}.
\end{aligned}$$

Case 2: Assume that $\tilde{f} \notin \operatorname{span}\{K^{new}(z_j, \cdot) | z_j \in \mathbb{B}_d\}$. Then there exists a sequence of functions $(\tilde{f}_m) \subseteq \operatorname{span}\{K^{new}(z_j, \cdot) | z_j \in \mathbb{B}_d\}$ with $\|\tilde{f} - \tilde{f}_m\|_{\mathcal{H}_d} \longrightarrow 0$. We have that $\tilde{f}(z) = \lim_m \tilde{f}_m(z_j)$, for every $z_j \in \mathbb{B}_d$.

$$\begin{aligned}
\|\tilde{f}\|^2_{\mathcal{H}_d} &= \lim_{m \to \infty} \|\tilde{f}_m\|^2_{\mathcal{H}_d} \\
&= \lim_{m \to \infty} \left\| \sum_{j \in \mathbb{N}_0^d} \alpha_j K_m^{new}(z_j, \cdot) \right\|^2_{\mathcal{H}_d} \\
&= \lim_{m \to \infty} \left\| \sum_{j \in \mathbb{N}_0^d} \alpha_j K_m^{\partial \mathbb{B}_d}(z_j, \cdot) \right\|^2_{L^2(\mu)} \\
&= \lim_{m \to \infty} \int_{L^2(\mu)} \left| \sum_{j \in \mathbb{N}_0^d} \alpha_j K_m^{\partial \mathbb{B}_d}(z_j, \cdot) \right|^2 d\mu(b) \\
&= \lim_{m \to \infty} \|T(\tilde{f}_m)\|^2_{L^2 \mu} \\
&= \|T_\mu(\tilde{f})\|^2_{L^2 \mu}.
\end{aligned}$$

□

Remark. We also have that

$$\sup_{r \in [0,1)} \int_{U(d)/U(d-1)} |f(rb)|^2 d\mu(b) = \|\tilde{f}\|^2_{\mathcal{H}_d}.$$

Proposition 4.6. *Given $z, w \in \mathbb{B}_d$, it follows from the main lemma that K^{new} simplifies as follows,*

$$K^{new}(z,w) = \int_{U(d)/U(d-1)} \frac{1}{1- < z, b >< b, w >} d\mu(b). \tag{4.5}$$

It is clear that the RKHS $\mathcal{H}_d$ is not equivalent to the Drury-Arveson space H_d^2 for $d > 1$. For $d = 1$, we have that K_{DA} and K^{new} both are just the Szego kernel for the Hardy space $H^2(\mathbb{D})$. Consider the below calculation to see that, for $(d = 1)$ the new reproducing kernel Hilbert space, $\mathcal{H}_1$, coincides with the Hardy space;

$$\begin{aligned} K^{new}(z,w) &= \int_{\partial\mathbb{B}_1} \frac{1}{1- < z, e^{i\theta} > \overline{< w, e^{i\theta} >}} d\theta \\ &= \frac{1}{1 - z\overline{w}}. \end{aligned}$$

By construction, we have that (4.5) is a boundary factorization for K^{new}. Also, we see next that the inclusion map $i : \mathcal{H}_d \hookrightarrow H_d^2$ is well-defined. Recall that in order to show that $K^{new} \in H_d^2$ we must show that there is a finite constanf $C = C_{K^{new}}$, depending on K^{new}, such that for all $d \in \mathbb{N}$, $\{z_1, \cdots, z_d\} \subset \mathbb{B}_d$, and $\{c_1, \cdots, c_d\} \subset \mathbb{C}$, we have:

$$\sum_{i,j=1}^{d} c_i \overline{c_j} K^{new}(z_i, z_j) \leq C \sum_{i,j=1}^{d} a_i \overline{a_j} K_{DA}(z_i, z_j);$$

in the literature this is denoted as $K^{new} << K_{DA}$.

Corollary 4.7. *Let H_d^2 and $\mathcal{H}_d$ be as before. Then $K^{new} << K_{DA}$; hence, $\mathcal{H}_d$ is a proper subset of H_d^2 when $d > 1$.*

Proof. Let $\{z_1, \cdots, z_d\} \subset \mathbb{B}_d$, and $\{c_1, \cdots, c_d\} \subset \mathbb{C}$. Then

$$\begin{aligned}
\sum_{i,j=1}^{d} c_i\overline{c_j}K^{new}(z_i, z_j) &= \sum_{i,j=1}^{d} c_i\overline{c_j}\int_{U(d)/U(d-1)} \frac{1}{1-<z_i, b>}\frac{1}{1-<b, z_j>}d\mu(b)\\
&= \int_{U(d)/U(d-1)} \sum_{i,j=1}^{d} \frac{c_i}{1-<z_i, b>}\frac{\overline{c_j}}{1-<b, z_j>}d\mu(b)\\
&= \int_{U(d)/U(d-1)} \left|\sum_{i=1}^{d} \frac{c_i}{1-<z_i, b>}\right|^2 d\mu(b)\\
&\leq \sum_{i=1}^{d} |c_i|^2\|K(z_i, w)\|^2_{H^2_d} \qquad \text{(Main Theorem)}\\
&= \sum_{i,j=1}^{d} c_i\overline{c_j}K_{DA}(z_i, z_j).
\end{aligned}$$

□

It is important to note that this is an unbounded inclusion. A key comparison of the Drury-Arveson space and the new RKHS is seen in view of an analogue to Fatou's theorem.

$\mathbf{d = 1}:$ For $f \in H^2(\mathbb{D}), \theta \in [0, 2\pi)$, and $r \in [0, 1)$ we have that

$$\lim_{r\to 1^-} f(re^{i\theta}) = \lim_{r\to 1^-} f_r(\theta) = f^*(\theta),$$

where $f^* \in L^2(\mathbb{T})$. Moreover, $\|f\|_{H^2(\mathbb{D})} = \|f^*\|_{L^2(\mathbb{T})}$.

$\mathbf{d > 1}:$ For $f \in H^2_d, b \in \partial\mathbb{B}_d$, and $r \in [0, 1)$ we have that

$$\lim_{r\to 1^-} f(rb) = \lim_{r\to 1^-} f_r(b) = \widehat{f}(b),$$

where $\widehat{f} \in L^2(\mu)$. Moreover, $\|\widetilde{f}\|_{\mathcal{H}_d} = \|\widehat{f}\|_{L^2(\mu}$.

That is, when $d > 1$, we have to pass to the new RKHS to keep the isomoetric properties of the norms. The following diagram visualizes the relationship between the functions

$$f \in H_d^2, \widehat{f} \in L^2(\mu), \widetilde{f} \in \mathcal{H}_d;$$

$$\begin{array}{ccc} \mathcal{H}_d & \overset{i}{\hookrightarrow} & H_d^2 \\ & \underset{\text{(isometry)}}{\overset{T_\mu}{\searrow}} & \Big\downarrow W_\mu \\ & & L^2\big(U(d)/U(d-1), \mu\big) \end{array}$$

It follows that $\widehat{f}(b) := (W_\mu f)(b) = (T_\mu \tilde{f})(b)$.

We close this chapter with one final conclusion about the new RKHS $\mathcal{H}_d$ relative to the Drury-Arveson RKHS. That is, the Schur product of the two kernels K_{DA} and K^{new} is K^{new}:

$$< K_{DA}(z, \cdot), K^{new}(w, \cdot) >_{H_d^2} = K^{new}(z, w).$$

Moreover, extending this to $f \in H_d^2$ yields that

$$< f(\cdot), K^{new}(z, \cdot) >_{H_d^2} = \widetilde{f}(z), \widetilde{f} \in \mathcal{H}_d,$$

where $\widetilde{f} \in \mathcal{H}_d$.

Chapter 5
OPEN QUESTIONS

5.1 Reproducing Kernel Hilbert Spaces

1. What is a minimal boundary factorization for the Arveson Kernel? *(The motivating question.)*

2. For the Drury-Arveson Space, what are the important boundaries and what are their properties?

3. What are the kernels that yield interesting determinantal point processes? For those, what applications result?

4. Are there multivariable versions of the theorem of Peres et al, [11]? (Gaussian analytic functions.)

5. Characterize the determinantal point processes which yield point distributions of random analytic functions, RAF?

6. So far we only know the case from Peres et al: Gaussian RAF, the disk, and Bergmann kernel. Compare to the multivariable Bergmann. Compare to the case of the Drury-Arveson kernel.

7. Does the Drury-Arveson kernel yield interesting determinantal point processes?

5.2 Relation between Szego Kernel and Bergman Kernel on $\mathbb{D}$

As discussed in chapter two, the Szego kernel on $\mathbb{D} \times \mathbb{D}$ is defined as

$$K(z, w) = \frac{1}{1 - z\overline{w}}.$$

This is the reproducing kernel for the Hardy space. The Hardy space $H^2(\mathbb{D})$ is the space of analytic functions on the disk determined from an L^2-condition via a measure on the boundary of the disk. While the Bergman space $B^2(\mathbb{D})$ is the space of analytic functions on the disk determined from an L^2-condition via an area measure on the disk. The Bergman kernel on $\mathbb{D} \times \mathbb{D}$ is the square of the Szego kernel, that is,

$$K(z, w) = \left(\frac{1}{1 - z\overline{w}} \right)^2.$$

When $d > 1$, the Hardy space has been generalized to two RKHS:

1. The Multivariable Hardy space,
2. The Drury-Arveson space.

The reproducing kernel for the Multivariable Hardy space $H^2(\mathbb{D}^d)$ is, $z, w \in \mathbb{D}^d$,

$$K(z, w) = \prod_{i=1}^{d} \frac{1}{1 - z_i w_i}.$$

It turns out that the relationship between the reproducing kernels of $H^2(\mathbb{D})$ and $B^2(\mathbb{D})$ generalizes to the polydisc, where the reproducing kernel for $B^2(\mathbb{D}^d)$ is, $z, w \in \mathbb{D}^d$,

$$K(z, w) = \prod_{i=1}^{d} \left(\frac{1}{1 - z_i w_i} \right)^2.$$

Viewing our new RKHS $\mathcal{H}_d$ as a generalization of $H^2(\mathbb{D})$, we can ask the following question:

1. Does the relation between the Szego kernel and Bergman kernel generalize to the Bergman space on $\mathbb{B}_d$ with a generalization of the Hardy space? If so, then is it one of the generalizations mentioned above?

In the above question the Bergman space on $\mathbb{B}_d$, $B^2(\mathbb{B}_d)$ is understood to mean the following,

$$B^2(\mathbb{B}_d) = L^2(\mathbb{B}_d) \cap \mathrm{Hol}(\mathbb{B}_d).$$

5.3 The Sequence $\{\alpha_n\}$

Let φ_n be a function $\varphi_n \in L^2({}^{U(d)}/{}_{U(d-1)})$ defined by $\varphi_n(b) =< \xi_n, b^{\otimes n} >_{\mathbf{E}^n}$. Then it makes sense to consider the following quadratic form

$$\int_{U(d)/U(d-1)} \left|< \xi_n, b^{\otimes n} >\right|^2 d\mu(b) := Q_n(\xi).$$

Then we get that $Q_n(\xi) =< \xi, M_n\xi >_{\mathbf{E}^n} =< g\xi, M_n g\xi >_{\mathbf{E}^n}$, which says that the quadratic form is invariant under the action of $U(d)$. Therefore, it follows that $g^{-1}M_n g = M$; and by the Schur-Weyl Lemma, we have that $M_n = \alpha_n I$.

When $n = 0$, we get that $\alpha_0 = 1$; and when $n = 1$ we know that $\alpha_1 \in (0, 1)$.

1. What do we get for α_1?
2. What can be determined as a relationship between α_1 and α_n?

REFERENCES

[1] Agler, J., McCarthy, J. E. (2002). Pick Interpolation and Hilbert Function Spaces.

[2] Alpay, D., Jørgensen, P. (2012). *Spectral theory for Gaussian processes: Reproducing kernels, random functions, boundaries, and* $\mathbf{L}^2$*-wavelet generators with fractional scales.* ArXiv:1208.2918 [Math]. Retrieved from http://arxiv.org/abs/1208.2918

[3] Arcozzi, N., Rochberg, R., Sawyer, E. (2002). *Carleson measures for analytic Besov spaces.* Revista Matemática Iberoamericana, 443–510. https://doi.org/10.4171/RMI/326

[4] Arcozzi, N., Rochberg, R., Sawyer, E. (2008). *Carleson measures for the Drury–Arveson Hardy space and other Besov–Sobolev spaces on complex balls.* Advances in Mathematics, 218(4), 1107–1180.

[5] Aronszajn, N. (1950). *Theory of reproducing kernels.* Transactions of the American Mathematical Society, 68(3), 337–337. https://doi.org/10.1090/S0002-9947-1950-0051437-7

[6] Arveson, Whilliam. *Subalgebras of* C^**-algebras III: Multivariable operator theory.* Acta Math. 181 (1998), no. 2, 159–228. doi:10.1007/BF02392585. https://projecteuclid.org/euclid.acta/1485891178

[7] Jørgensen, Palle E. T., Steen Pedersen, and Feng Tian. *Extensions of Positive Definite Functions : Applications and Their Harmonic Analysis.* Lecture Notes in Mathematics (Springer-Verlag); 2160. Cham] Switzerland: Springer, 2016.

[8] Jørgensen, Palle, Steen Pedersen, and Feng Tian. *Extensions of Positive Definite Functions: Applications and Their Harmonic Analysis.* 2015.

[9] Parzen, Emanuel. (1970). *Statistical inference on time series by RKHS methods.* 73.

[10] Paulsen, V., Raghupathi, M. (2016). *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces* (Cambridge Studies in Advanced Mathematics). Cambridge: Cambridge University Press. doi:10.1017/CBO9781316219232

[11] Peres, Y., Virag, B. (2005). *Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process.* Acta Mathematica, 194(1), 1–35. https://doi.org/10.1007/BF02392515

[12] Rudin, W. (1991). *Functional Analysis.* Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York

[13] Stegenga, D. A. (1980). *Multipliers of the Dirichlet space. Illinois Journal of Mathematics,* 24(1), 113–139. https://doi.org/10.1215/ijm/1256047800