Front-form calculations of exchange currents in elastic electron-deuteron scattering

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FRONT-FORM CALCULATIONS OF EXCHANGE CURRENTS IN ELASTIC ELECTRON-DEUTERON SCATTERING

by

Yunfei Huang

An Abstract

Of a thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Physics in the Graduate College of The University of Iowa

December 2008

Thesis Supervisor: Professor Wayne N. Polyzou
ABSTRACT

Electron-deuteron scattering is an ideal tool for studying nuclear structure and nuclear forces. The elastic deuteron current matrix elements have been calculated using a Poincaré invariant quantum model with a light-front kinematic symmetry. The impulse approximation violates the Poincaré covariance and current conservation for the interacting system and two-body currents are required to satisfy these constraints. A model two-body current that has a structure motivated by “pair” currents was constructed in this work. The Argonne V18 potential was used as the model nucleon-nucleon interaction and empirical nucleon form factors were used as input in the calculation of the deuteron form factors and structure functions. The sensitivity of the results to different nucleon-nucleon interactions, different nucleon form factors, and different choices of independent current matrix elements was examined. The calculations of elastic electron-deuteron scattering observables in this work are consistent with experiment to within uncertainties in the input.

Abstract Approved: __________________________________________________________

Thesis Supervisor

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Date
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Thesis Supervisor: Professor Wayne N. Polyzou
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To Baishu Sheng and CeeCee Sheng
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7.38 The sensitivity of the deuteron structure function $T_{20}(Q^2)$ to different NN-interactions.
The only two-nucleon bound state, the deuteron, is composed of a proton and a neutron. The study of this simplest bound system of nucleons provides useful information about nuclear structure, nuclear forces and electro-weak currents, which are important input for understanding more complex systems. Among the various deuteron probes, electrons have some advantages and play an important role in studying the structure of the deuteron. Electrons interact with the nucleus through well-understood electromagnetic forces. The electromagnetic fine structure constant, \( \alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \), is approximately equal to \( \frac{1}{137} \). This relatively small coupling constant makes the one-photon-exchange approximation possible \[59\]. Because the higher order diagrams are suppressed compared to the lowest order diagrams, elastic electron-deuteron scattering can be described to high-precision in the one-photon-exchange approximation and it provides reliable information about the electromagnetic structure of the deuteron and the interaction between the nucleons.

Three independent observables can be extracted from elastic electron-deuteron scattering. They can be taken as the Lorentz invariant structure functions \( A(Q^2) \), \( B(Q^2) \) and \( T_{20}(Q^2) \), where \( Q^2 \) is the square of the four-momentum transferred by the electron. The differential cross section for unpolarized elastic electron-deuteron scattering in the
The one-photon-exchange approximation is given by the Rosenbluth formula [61]:

\[
\frac{d\sigma}{d\Omega}(Q^2, \theta) = (\frac{d\sigma}{d\Omega})_{\text{Mott}} \{ A(Q^2) + B(Q^2) \tan^2(\frac{\theta}{2}) \}, \tag{1.1}
\]

where

\[
(\frac{d\sigma}{d\Omega})_{\text{Mott}} = \frac{\alpha^2E'E\cos^2(\frac{\theta}{2})}{4E^3\sin^4(\frac{\theta}{2})} \tag{1.2}
\]

is the Mott cross section for scattering from a point-particle. \( E, E' \) and \( \theta \) are the incident and scattered electron energies and the electron scattering angle respectively. Thus, the structure functions \( A(Q^2) \) and \( B(Q^2) \) can be separated by measuring the unpolarized differential cross section. The measurement of \( T_{20}(Q^2) \) requires polarized targets. It can be extracted from the difference in the cross sections with target deuterons having the z-component of spin 1 and 0:

\[
T_{20}(Q^2, \theta) = \sqrt{2} \frac{\frac{d\sigma}{d\Omega}_1(Q^2, \theta) - \frac{d\sigma}{d\Omega}_0(Q^2, \theta)}{\frac{d\sigma}{d\Omega}(Q^2, \theta)}. \tag{1.3}
\]

The structure functions can be expressed in terms of the three conventional electromagnetic deuteron form factors \( G_0(Q^2), G_1(Q^2) \) and \( G_2(Q^2) \). According to Coester and Ostebee [26], the structure functions \( A(Q^2) \) and \( B(Q^2) \) can be written as

\[
A(Q^2) = G_0^2(Q^2) + G_2^2(Q^2) + \frac{2}{3}\eta G_1^2(Q^2), \tag{1.4}
\]

\[
B(Q^2) = \frac{4}{3}\eta(1+\eta)G_1^2(Q^2), \tag{1.5}
\]
where $\eta \equiv Q^2/4M_d^2$ with $M_d$ the mass of the deuteron. The structure function $T_{20}(Q^2)$ has the following form \[41\]:

$$T_{20}(Q^2) = \frac{\left[G_2^2(Q^2) + \sqrt{8}G_0(Q^2)G_2(Q^2) + \frac{2}{3}\eta G_1^2(Q^2)(\frac{1}{2} + (1 + \eta)\tan^2 \frac{\eta}{2})\right]}{\sqrt{2}(A(Q^2) + B(Q^2)\tan^2 \frac{\eta}{2})}. \quad (1.6)$$

Gourdin \[40\] and Elias et al. \[33\] gave a different set of deuteron form factors:

$$G_C(Q^2) = G_0(Q^2), \quad G_M(Q^2) = G_1(Q^2), \quad G_Q(Q^2) = G_2(Q^2)(\frac{\sqrt{8}\eta}{3})^{-1}. \quad (1.7)$$

These form factors give the charge, magnetic and quadrupole moments of the deuteron in the limit that $Q^2 = 0$:

$$G_C(0) = 1, \quad G_M(0) = \mu_d, \quad G_Q(0) = Q_d. \quad (1.8)$$

The deuteron form factors can also be expressed in terms of the deuteron current matrix elements in the Breit frame, $\langle q/2, \mu'|I^\nu| - q/2, \mu \rangle_c$, as follows:

$$G_0(Q^2) = \frac{1}{3}(\langle 0|I^0|0 \rangle_c + 2\langle 1|I^0|1 \rangle_c), \quad (1.9)$$

$$G_1(Q^2) = -\sqrt{2}\eta(\langle 1|I^1|0 \rangle_c, \quad (1.10)$$

$$G_2(Q^2) = \frac{1}{3}\sqrt{2}(\langle 0|I^0|0 \rangle_c - \langle 1|I^0|1 \rangle_c). \quad (1.11)$$

where $\langle q/2, \mu'|I^\nu| - q/2, \mu \rangle$ is written as $\langle \mu'|I^\nu|\mu \rangle_c$ since the current matrix elements
only depend on the momentum transfer $Q^2 = -q^2$.

The experiment of elastic electron-deuteron scattering was first performed by McIntyre and Hofstadter [54]. Many laboratories have investigated the measurements of the deuteron structure functions since then. The structure function $A(Q^2)$ has been measured out to momentum transfer $Q^2 = 6\text{GeV}^2$ in Hall A in JLab [3]. The measurements for structure function $B(Q^2)$ with the highest $Q^2 = 2.77\text{GeV}^2$ is from SLAC [19] [7]. Polarization experiments to measure the deuteron structure function $T_{20}(Q^2)$ are much more difficult and the first measurement was by Schulze et al [67]. Jlab Hall C have measured $T_{20}(Q^2)$ for $Q^2$ up to $1.72\text{GeV}^2$ [2].

Theoretically, relativity needs to be considered when the momentum transfer and the rest energy of the constituent nucleons are comparable. Relativistic quantum mechanical models that are used to study elastic electron-deuteron scattering can differ widely depending on the assumptions involved. There are two basic approaches. Approaches motivated by quantum field theory utilize models based on quasipotential reductions of the Bethe-Salpeter equation [63]. Approaches motivated by non-relativistic calculations, which is called relativistic Hamiltonian dynamics [31] [10] [42], are based on quantum mechanical models that replace the Galilean symmetry of non-relativistic quantum theory with an exact Poincaré symmetry. While both approaches can be used to successfully model elastic electron-deuteron scattering, the relation between the two approaches is not simple and each approach emphasizes different aspects of the physics.

The second method has been used in this work. An exact Poincaré invariant dynam-
ical model on a two-nucleon Hilbert space was constructed along with a model of a covariant current operator. Precise nucleon-nucleon interactions [75] fit to scattering data were used to construct dynamical representations of the Poincaré group. Current conservation, current covariance and discrete symmetries were used to identify functionally independent current matrix elements. The model independent one-body contributions to the independent current matrix elements were defined as “impulse approximations”. The dynamical constraints of current conservation and current covariance generate the remaining current matrix elements, which necessarily have dynamical contributions. Additional two-body contributions may be added to the independent current matrix elements. These are generally required to fit experimental data; and may be required by the underlying physics. The form of the additional two-body currents depends on both the representation of the dynamics and the choice of independent current matrix elements.

For interacting systems, at least three generators of the Poincaré group require interactions [31] and the remaining generators, which do not involve interactions, generate the Lie algebra of a kinematic subgroup. Different choices of the kinematic subgroup define different forms of dynamics. Initiated by Dirac [31], the three largest kinematic subgroups are the instant-form, point-form and front-form kinematic subgroups. The Euclidean group and Lorentz group are the kinematic subgroups of the instant-form and the point-form respectively. The kinematic subgroup of the front-form is the subgroup of the Poincaré group that leaves the light-front, $x^+ = x^0 + x^3 = 0$, invariant. Currents are representation dependent. Different forms of dynamics have different representa-
tions of the current. A model with a front-form kinematic subgroup has been used in this work because of its notable properties in the treatment of the current matrix elements. For electron scattering the momentum transfer is always space-like, so the orientation of the light front can always be chosen so the + component of the momentum transfer, \( Q^+ \), vanishes. Light-front boosts form a kinematic subgroup so matrix elements of \( I^+(0) \) with \( Q^+ = 0 \) and a non-covariant normalization are independent of frame: All matrix elements of the current can be constructed from the matrix elements of the + component of the current.

The model potential used in this work was the realistic Argonne V18 nucleon-nucleon interaction (NN-interaction) \cite{75}. Interactions need to be introduced in a manner so that the commutation relations of the Poincaré Lie algebra are preserved. The interaction was added to the square of the mass operator in this work following the method of Coester, Pieper and Serduke \cite{27}. Using this method, the deuteron mass eigenvalue problem is the same as the Schrödinger equation and the phase shifts remain the same as in the non-relativistic model. Empirical nucleon form factors were used as input in the calculation of the deuteron current matrix elements. To examine the sensitivity of this model to the empirical nucleon form factors, eight different parameterizations have been used in this work, which are given in detail in Appendix C.

The current operator \( I^\mu(x) \) must satisfy Poincaré covariance and current conservation. The impulse approximation for a composite system assumes that the current is a one-body operator. But the impulse approximation violates the Poincaré covariance and current conservation in interacting theories. Two-body currents, called exchange
currents, are required to restore these constraints. In addition, there are currents that arise from explicit charge exchanges as well as from eliminated degrees of freedom. In this work, the effect of “pair” currents on deuteron elastic electromagnetic form factors was examined. The origin of these currents is the elimination of degrees of freedom in quantum field theory. The model “pair” currents are constructed by coupling the photon to the $\nu$ spinor contribution to the Feynman propagator in a one-pion-exchange diagram. The pion-exchange part of the diagram is replaced by the appropriate contribution of the $AV18$ NN-interaction.

The independent deuteron current matrix elements have been calculated in the impulse approximation and with the “pair” current contribution included. The remaining current matrix elements were generated from the independent current matrix elements using current covariance, current conservation and discrete symmetries as well as obtained by direct calculations. These two sets of results have been compared to examine the size of difference between the directly calculated and the dynamically generated exchange currents in the model. The deuteron structure functions and form factors were calculated using the independent current matrix elements. Different combinations of the independent current matrix elements correspond to different conserved covariant current operators. To examine the sensitivity to different choices, three different combinations were used to calculate the deuteron form factors. The same “pair” current operator constructed using the $AV18$ NN-interaction was combined with deuteron wave functions generated using the CDB [52] and N3LO [34] NN-interactions to test the sensitivity of the “pair” current contribution to different model NN-interactions. The
effects of the “pair” current contribution on the deuteron quadrupole and magnetic moments have also been evaluated. The results were compared to experimental data and to the work done by other groups using different models. The deuteron form factors can be calculated using various methods. The elastic deuteron structure functions have been calculated by Van Orden, Devine and Gross [71] using the spectator equation. Phillips, Wallace and Devine [56] have calculated these structure functions using a three-dimensional formalism. Instant-form calculations have been done by Forest, Schiavilla and Riska [66], by Wiringa, Stoks and Schiavilla [75], and by Arenhövel, Ritz and Wilbois [5]. The deuteron tensor polarization component $T_{20}(Q^2)$ has been calculated recently by Krutov and Troitsky [45]. Allen, Klink and Polyzou calculated the deuteron form factors in the impulse approximation [4] using point-form. Front-form calculations have been investigated by Chung et al. [24], by Frankfurt, Frederico and Strikman [36], by Carbonell and Karmanov [23] and by Lev et al. [48].

This work evaluated the effects of “pair” current contribution to the deuteron form factors using the front-form model. It extended the previous work by Chung et al. [24]. They used a light-front impulse approximation to evaluate a set of independent current matrix elements with various choices of interactions and nucleon form factors. The results were consistent with existing experimental data to within uncertainties in the neutron electric form factors and the choices of NN-interactions. Since then there have been additional measurements, improvements in both the interactions and form factors. This work shows that the improvements in both the interactions and nucleon form factors in the impulse calculations used in [24] are unable to explain the newer
higher precision data. Within uncertainties in the nucleon form factors, the deuteron structure functions computed with the “pair” currents included are consistent with the existing measurements of all three deuteron structure functions.

Properties of relativistic quantum mechanics and the various kinematic subgroups as well as the method of introducing interactions into the Poincaré operators are described in Chapter 2; Chapter 3 explains the constraints on the current operators and the choice of the maximal set of the independent current matrix elements. The relationship between the form factors and the current matrix elements is described and the required “pair” currents for interacting systems are derived in Chapter 3. Chapter 4 describes the method used to calculate the impulse current matrix elements. The calculation of the “pair” current contribution is discussed in Chapter 5. Numerical results are discussed in Chapter 6 and a summary of this work is given in Chapter 7.
CHAPTER 2
RELATIVISTIC QUANTUM MECHANICS

Relativistic quantum mechanics requires that quantum probabilities are invariant under the inhomogeneous Lorentz transformations, the Poincaré transformations. Wigner [73] proved mathematically in 1939 that a necessary and sufficient condition for quantum mechanical probabilities to be independent of the choice of inertial coordinate system is the existence of a unitary representation of the Poincaré group on the quantum mechanical Hilbert space.

Because phase ambiguities occur when one rotates half-integer spin particles through an angle of $2\pi$, the representation of the Poincaré group is a projective representation. Bargmann [11] showed that this can always be replaced by a single-valued representation of the covering group of the Poincaré group.

The covering group of the Poincaré group is inhomogeneous $\text{SL}(2,\mathbb{C})$, $\text{ISL}(2,\mathbb{C})$. To define $\text{ISL}(2,\mathbb{C})$ note that the space-time coordinate $x^\mu$ can be represented by a $2 \times 2$ Hermitian matrix:

$$X \equiv x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \quad x^\mu = \frac{1}{2} \text{Tr}(\sigma_\mu X). \quad (2.1)$$

In this representation, the proper time can be obtained from the determinant of the matrix $X$:

$$\tau^2 = \det |X| = (x^0)^2 - x^2. \quad (2.2)$$
In order to preserve the proper time between events, we need

\[ \tau_{AB}^2 = \tau_{A'B'}^2, \]  

\hspace{1cm} (2.3)

where

\[ \tau_{AB}^2 = \det|X_A - X_B|, \quad \tau_{A'B'}^2 = \det|X_{A'} - X_{B'}|. \]  

\hspace{1cm} (2.4)

The most general linear transformation that satisfies this requirement and preserves the hermitian property of \( X \) has the following form:

\[ X \rightarrow X' = \Lambda X \Lambda^\dagger + a, \]  

\hspace{1cm} (2.5)

with \( \det|\Lambda| = 1 \) and \( a = a^\dagger \). The ordered pairs of matrices \( (\Lambda, a) \) and \( (-\Lambda, a) \) correspond to the same transformation; the Poincaré group provides a label for the corresponding inertial coordinate systems. Note that this does not include time reversal and space reflection. The covering group of the Poincaré group is denoted by \( \mathcal{P} \). The pairs \( (\Lambda, a) \) form a group under composition

\[ (\Lambda_2, a_2) \cdot (\Lambda_1, a_1) \equiv (\Lambda_2 \Lambda_1, \Lambda_2 a_1 (\Lambda_2)^\dagger + a_2), \]  

\hspace{1cm} (2.6)

with inverse

\[ (\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} a (\Lambda^\dagger)^{-1}) \]  

\hspace{1cm} (2.7)

and identity \((I,0)\).
A unitary representation, $U(\Lambda, a)$, of $\mathcal{P}$ on the model Hilbert space $\mathcal{H}$ must satisfy the group representation property:

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U[(\Lambda_2, a_2) \cdot (\Lambda_1, a_1)] = U(\Lambda_2 \Lambda_1, \Lambda_2 a_1 (\Lambda_2)^\dagger + a_2),$$  

with the adjoint and inverse related as follows

$$U(\Lambda, a)\dagger = U(\Lambda, a)^{-1} = U[(\Lambda, a)^{-1}].$$  

The most general $2 \times 2$ matrix $\Lambda$ with $\det|\Lambda| = 1$ and Hermitian matrix $a$ can be represented by

$$\Lambda = \Lambda(\theta, \rho) = \pm \exp(-\frac{i}{2}(\theta + i\rho) \cdot \sigma), \quad a = a^\mu \sigma_\mu,$$  

where $\theta$ represents the angle and axis of a rotation, $\rho$ represents the direction and rapidity of a rotationless Lorentz transformation and $a^\mu$ are coordinates of a space-time translation. The representation $U(\Lambda, a)$ has the following form:

$$U(\Lambda(\theta, \rho), a) \equiv e^{iP \cdot a}e^{-i(J \cdot \theta + K \cdot \rho)}.$$  

Differentiation in a neighborhood of the identity defines abstract generators:

$$P^\mu \equiv -ig^{\mu\nu} \frac{\partial}{\partial a^\nu} U(\Lambda(\theta, \rho), a)|_{\rho=0, a^\mu=0},$$  

(2.12)
They are generators of the ten-parameter Poincaré group. Generators $P^0, P$ and $J$ have the physical meaning of the Hamiltonian, $H$, the total linear momentum and the total angular momentum respectively. $K$ is the generator of rotationless Lorentz boosts. The commutation relations, which follow from the group representation property, are given in Appendix A. The generators of Lorentz transformations are components of the angular momentum tensor $J^{\alpha\beta}$:

$$J^{0j} \equiv K^j, \quad J^{jk} \equiv \epsilon^{jkl} J^l.$$  \hspace{1cm} (2.15)

The commutation relations can be expressed in a covariant form using $J^{\alpha\beta}$ and $P^\mu$. They are also given in Appendix A.

Out of these generators we can construct two Casimir operators, $M^2$ and $W^2$, which are the only independent polynomial functions of the generators that commute with all of the Poincaré generators:

$$M^2 \equiv -P^\mu P_\mu = (P^0)^2 - (P)^2,$$  \hspace{1cm} (2.16)

$$W^2 \equiv W^\mu W_\mu,$$  \hspace{1cm} (2.17)
where $W^\mu$ is the Pauli-Lubanski vector [51] [12]:

$$W^\mu \equiv \frac{1}{2} \epsilon^{\mu\alpha\beta\gamma} P_\alpha J_{\beta\gamma}. \quad (2.18)$$

The above definition combined with Eq. (2.15) gives the components of $W$:

$$W^0 = P \cdot J, \quad W = HJ - P \times K. \quad (2.19)$$

From the definition (2.18) and the commutation relations for the generators, it follows that

$$[P^\mu, W^\nu] = 0, \quad [W^\mu, W^\nu] = i\epsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma. \quad (2.20)$$

$$[J, W^0] = 0, \quad [J^j, W^k] = i\epsilon^{jkl} W^l. \quad (2.21)$$

$$[K, W^0] = -iW, \quad [K^j, W^k] = -i\delta^{jk} W^0. \quad (2.22)$$

Eq. (2.21) and (2.22) indicate that $W^\mu$ transforms like a four-vector under Lorentz transformations.

A physical particle is an eigenstate of the mass operator $M$ and the spin $S$, which is the intrinsic angular momentum of the particle. The mass operator $M$ is defined in (2.16). The construction of the spin operator will be described below. From the definition (2.18) we can see that the Pauli-Lubanski vector is orthogonal to the four-momentum $P^\mu$:

$$W^\mu P_\mu = 0. \quad (2.23)$$
This implies that $W^\mu$ is spacelike for a timelike $P^\mu$ and indicates a way to construct the spin vector. We construct a boost operator which maps the rest momentum $(M, 0, 0, 0)$ to $P^\mu$:

$$L_g(P)^\mu_\nu(M, 0, 0, 0)^\nu = P^\mu, \quad (2.24)$$

and the inverse boost $L_g^{-1}(P)$ maps $P^\mu$ back to $(M, 0, 0, 0)$:

$$L_g^{-1}(P)^\mu_\nu P^\nu = (M, 0, 0, 0)^\nu. \quad (2.25)$$

In what follows $L_g^{-1}(P)$ is considered to be a matrix of operators. Here the subscript $g$ denotes different types of boost operators such as the canonical boost, front-form boost or helicity boost. The canonical boost $L_c(P)$ is the rotationless Lorentz transformation. Front-form boosts $L_f(P)$ are transformations transforming the rest momentum to four momentum $P$ and leaving the light front invariant. The helicity boost $L_h(P)$ is composed of a canonical boost in the $z$ direction to obtain the desired magnitude of the momentum and a rotation from $z$ direction to the desired direction.

If the inverse boost is performed on the Pauli Lubanski operator, Eq. (2.23) indicates that the time component of $L_g^{-1}(P)W$ vanishes and the three space components define the spin vector of the system:

$$(0, S_g) \equiv \frac{1}{M} L_g^{-1}(P)^\mu_\nu W^\nu. \quad (2.26)$$

From Eq. (2.20), we can see that the spin vectors obtained above have the following
properties:

\[ [S^i, S^j] = i\epsilon^{ijk} S^k, \quad S^2 = \frac{W^2}{M^2}, \]  

(2.27)

which indicates that \( S^2 \) commutes with all the Poincaré generators and is Lorentz invariant. Using different types of boosts in Eq. (2.26) defines different spin vectors. The canonical spin is one of the commonly used spins in physics and is obtained using canonical boosts. The canonical boosts have the following \( 2 \times 2 \) matrix form:

\[ L_c(P) = \frac{1}{\sqrt{2M(P^0 + M)}} ((P^0 + M)\sigma_0 + P \cdot \sigma). \]  

(2.28)

They transform any four vector \( A^\mu \) to \( A'^\mu \) in the following way:

\[
\begin{pmatrix}
A^0 \\
A'
\end{pmatrix} = L_c(P) \begin{pmatrix}
A^0 \\
A
\end{pmatrix} = \begin{pmatrix}
\frac{1}{M}(HA^0 + P \cdot A) \\
\frac{P}{M} A^0 + P \frac{P \cdot A}{M(M + H)} + A
\end{pmatrix}.
\]  

(2.29)

The canonical spin \( S_c \) can be obtained using the inverse canonical boost \( L_c^{-1}(P) \) in the definition (2.26):

\[ (0, S_c) \equiv \frac{1}{M} L_c^{-1}(P)^\mu_{\nu} W^\nu. \]  

(2.30)

It has the following operator structure:

\[ S_c = \frac{1}{M} [(HJ - P \times K) - \frac{P(P \cdot J)}{M + H}]. \]  

(2.31)

The mass operator \( M \), the spin \( S^2 \), the momentum \( P \) and the third component of the canonical spin \( S^3_c \) can be chosen to form the complete set of commuting Hermi-
tian operators. The Hilbert space is the space of square integrable functions of the eigenvalues of these operators:

\[ \langle m, s; p, \mu | \Psi \rangle. \]  

\[ (2.32) \]

The normalization condition is chosen to be:

\[ c \langle p', \mu' | p, \mu \rangle_c = \delta_{\mu' \mu} \delta(p' - p), \]

\[ (2.33) \]

\[ \|\Psi\|^2 = \sum_{\mu = -s}^s \int d|\Psi(p, \mu)|^2 = 1, \]

\[ (2.34) \]

where the labels \( m \) and \( s \) are suppressed since there is no sum over the mass and spin.

To obtain the transformation properties of the state vectors on the model Hilbert space, we start from the transformation properties of the observables that define the model Hilbert space. From the previous definition, the four-momentum operator \( P^\mu \) and the spin operator \( S \) have the following transformation properties:

\[ U(\Lambda, a)^\dagger P^\mu U(\Lambda, a) = \Lambda^\mu_\nu P^\nu, \]

\[ (2.35) \]

\[ U(\Lambda, a)^\dagger S_c U(\Lambda, a) = R_c(\Lambda, P) S_c, \]

\[ (2.36) \]

where

\[ R_c(\Lambda, P) = L_c^{-1}(\Lambda P) \Lambda L_c(P) \]

\[ (2.37) \]

is a canonical Wigner rotation.
We start with zero momentum eigenstates, \( p = (m, 0) \equiv p_0 \), to determine the transformation properties of the Hilbert space vectors. If the Lorentz transformation \( \Lambda \) is a pure rotation \( R \), then Eq. (2.37) shows that the Wigner rotation is the rotation itself and \( U(R) \) acts on the rest frame eigenstates like an ordinary rotation:

\[
U(R)|0, \mu\rangle = \sum_{\bar{\mu}=-\mu}^{\mu} |0, \bar{\mu}\rangle D^{s}_{\mu\bar{\mu}}[R]. \tag{2.38}
\]

This property is true for all types of spins so the subscript \( c \) is dropped. If the Lorentz transformation \( \Lambda \) is the boost \( L_c(p) \), then the Wigner rotation becomes the identity so that \( U(L_c(p)) \) boosts the rest momentum to momentum \( p \) without changing the spin:

\[
U(L_c(p))|0, \mu\rangle_c = \sqrt{\frac{\omega_m(p)}{m}} |p, \mu\rangle_c, \tag{2.39}
\]

where \( \omega_m(p) = \sqrt{m^2 + p^2} \) and the constant \( \sqrt{\omega_m(p)/m} \) is determined by unitarity and choice of the normalization condition (2.33). The space-time translation operator \( T(a) \) has the form

\[
U(T(a))|p, \mu\rangle_c = e^{ip \cdot a} |p, \mu\rangle_c \tag{2.40}
\]

since the four-momentum commutes with the spin. Finally, for any Poincaré transformation, \( U(\Lambda, a) \) can be obtained by following transformations in sequence:

\[
U(\Lambda, a) = T(a)U(L_c(\Lambda p))U(R_c(\Lambda, p))U(L_c^{-1}(p)). \tag{2.41}
\]
This equation combined with the transformation properties (2.38) (2.39) (2.40) gives the following expression for any Poincaré transformation on arbitrary Hilbert space basis vectors:

\[ U(\Lambda, a)|p, \mu\rangle_c = e^{ip_{\Lambda} \cdot a} \sqrt{\frac{\omega_m(p_{\Lambda})}{\omega_m(p)}} \sum_{\vec{\mu} = -s}^s [p_{\Lambda}, \vec{\mu}]_c D_{\vec{\mu}\mu}^s[R_c(\Lambda, p)], \quad (2.42) \]

where \( p_{\Lambda} \equiv \Lambda p \).

2.1 Front-form Dynamics

For systems consisting of interacting particles, Dirac [31] pointed out that the Lie algebra of the Poincaré group must be satisfied when interactions are introduced. But if interactions are included in the Hamiltonian only, the commutation relations will not be preserved. Dirac [31] suggested three ways to introduce the interactions in a minimal number of generators. Dirac named dynamical theories according to the three largest kinematic subgroups: the instant-form, point-form and front-form kinematic subgroups. Different choices of the kinematic subgroup define equivalent formulations of the dynamical problems.

In the instant-form, the kinematic subgroup is the three-dimensional Euclidean group. It is generated by space translations and rotations and leaves the instant plane \( t = t_c \) invariant. The Hamiltonian and Lorentz boosts are dynamical transformations.

The point-form kinematic subgroup is the Lorentz group and it leaves the Lorentz surface, \( t^2 - x^2 = a^2 \), invariant. The Hamiltonian and space translations are dynamical
transformations.

In the front-form, the kinematic subgroup leaves the light front, which is a hyper-plane tangent to the light cone, invariant. It is the largest possible kinematic subgroup with seven kinematic generators. Front-form representations of the dynamics have important advantages \[28\] \[42\] \[69\] \[15\] \[16\] \[47\] in the calculation of electromagnetic current matrix elements for space-like momentum transfers.

Let \(\hat{n}\) be a fixed unit vector. The light-front coordinates of a four-vector are defined by:

\[
x^\pm = x^0 \pm \hat{n} \cdot x, \quad (2.43)
\]

\[
x_\perp = \hat{n} \times x. \quad (2.44)
\]

The light-front is the hyperplane of space-time points satisfying \(x^+ = 0\). If we choose \(\hat{n} = \{0, 0, 1\}\), then \(x^+ = x^0 + x^3\).

The light-front components of the four-momentum are defined by

\[
\tilde{P} \equiv (P^+ = P^0 + P^3, P_\perp = (P^1, P^2)), \quad P^- \equiv P^0 - P^3. \quad (2.45)
\]

Here \(P^-\) plays the role of the Hamiltonian in front-form dynamics.

The front-form kinematic subgroup, which leaves the light front invariant, contains seven kinematic generators:

\[
\{P^1, P^2, P^+, J^3, K^3, E_\perp \equiv K_\perp - \hat{n} \times J_\perp\}. \quad (2.46)
\]
The other three generators

\[ \{ P^-, F_\perp \equiv K_\perp + \hat{n} \times J_\perp \} \quad (2.47) \]

are dynamical generators. They generate the transformations that take points off of the light front. Sometimes the operator \( F_\perp \) is replaced by the transverse components of the total angular momentum \( J_\perp \). The commutation relations of these generators are given in Appendix A.2.

The mass operator \( M \) in a front-form model can expressed as:

\[ M = \sqrt{P^+ P^- - P^2_\perp}. \quad (2.48) \]

The front-form boost is generated by \( K^3 \) and \( E_\perp \). These three generators form a closed subalgebra so that front-form boosts form a group. The front-form boosts leave the light front invariant and map the light-front four momentum at rest to \( P^\mu \):

\[ L_f(P)(M, M, 0, 0) = (P^-, P^+, P_\perp). \quad (2.49) \]

The \( SL(2,c) \) matrix form of the front-form boost is:

\[ L_f(P) = \frac{1}{\sqrt{MP^+}} \begin{pmatrix} P^+ & 0 \\ P_\perp & M \end{pmatrix}, \quad (2.50) \]

where \( P_\perp = P^1 + iP^2 \). The front-form components of any four vector \( A \) are transformed.
in the following way by a front-form boost:

\[
\begin{pmatrix}
A^{+'} \\
A_{\perp}' \\
A^{-'}
\end{pmatrix} = L_f(P)
\begin{pmatrix}
A^+ \\
A_{\perp} \\
A^-
\end{pmatrix} =
\begin{pmatrix}
\frac{P^+}{M}A^+ \\
A_{\perp} + \frac{P^\perp}{P^+}A^+ \\
\frac{P^2}{MP^+}A^+ + 2\frac{P^\perp A_{\perp}}{P^+} + \frac{M}{P^+}A^-
\end{pmatrix}.
\tag{2.51}
\]

The front-form spin, \( S_f \), is obtained by using the front-form boost \( L_f^{-1}(P) \) in Eq. (2.26):

\[
(0, S_f) \equiv \frac{1}{M} L_f^{-1}(P)^\mu_{\nu} W^\nu.
\tag{2.52}
\]

It is related to the Poincaré generators by

\[
S_f^3 = \frac{W^+}{P^+} = \frac{1}{P^3 + \frac{\hat{z} \cdot (E_{\perp} \times P_{\perp})}{P^+}},
\tag{2.53}
\]

\[
S_{f\perp} = \frac{1}{M} (W_{\perp} - \frac{W^+}{P^+} P_{\perp})
\tag{2.54}
\]

\[
\begin{align*}
&= \frac{1}{M} (\hat{z} \times \frac{1}{2} (P^- E_{\perp} - P^+ F_{\perp}) \\
&+ P_{\perp} K^3 - \frac{P_{\perp}}{P^+} [P^+ J^3 + \hat{z} \cdot (E_{\perp} \times P_{\perp})]),
\end{align*}
\]

where \( W^\pm = W^+ \pm W^- \). The front-form spin thus defined is invariant under front-form boosts since they form a closed group.

Two different types of spins can be related as follows:

\[
S_a^j = L_a^{-1}(P)^j_\nu L_b(P)^\nu_k S_b^k \equiv R_{ab}(P)^j_k S_b^k.
\tag{2.55}
\]
The operation

\[ R_{ab}(P) = L_{a}^{-1}(P)L_{b}(P), \]  

(2.56)

is a transformation that maps a rest four-momentum to momentum \( P \) using one type of boost followed by the inverse boost of another type, which maps the four-momentum \( P \) back to its rest momentum. So the net effect of this Lorentz transformation is a rotation. The rotation \( R_{ef}(P) \) that transforms the front-form spin to canonical spin is called a Melosh rotation \([55]\).

The mass operator \( M \), the total front-form momentum \( \tilde{P} \), the front-form spin \( S_{f}^{2} \) and the third component of the spin \( S_{f}^{3} \) can be chosen to be a complete set of commuting operators for a particle. The third component of spin takes integer or half-integer values from \(-s\) to \( s\). The spectrum of \( P_{\perp} \) ranges over the possible physical values on \((-\infty, \infty)\) and \( P^{+} \) over \((0, \infty)\). Their eigenvalues are used to label the Hilbert space basis states which are normalized as follows:

\[ \Psi(\tilde{p}, \mu) \equiv \langle m, s; \tilde{p}, \mu | \Psi \rangle, \]  

(2.57)

\[ f(\tilde{p}, \mu | \tilde{p}, \mu) = \delta_{\mu, \mu} \delta(\mu - \mu^{\prime}) \delta^{2}(p_{\perp}^{\prime} - p_{\perp}). \]  

(2.58)

In the representation (2.57) and (2.58) the transformation properties of the state vector can be determined by replacing Eq. (2.38) (2.39) (2.40) (2.42) by:

\[ U(R_{f})|\tilde{0}, \mu\rangle = \sum_{\mu = -s}^{s} |\tilde{0}, \mu\rangle D_{\mu \mu}^{s}[R_{f}], \]  

(2.59)
\[ U(L_f(p))|\vec{0}, \mu\rangle_f = \sqrt{\frac{p^+}{m}}|\vec{p}, \mu\rangle_f, \quad (2.60) \]

\[ U(T(a))|\vec{p}, \mu\rangle_f = e^{ip\cdot a}|\vec{p}, \mu\rangle_f, \quad (2.61) \]

\[ U(\Lambda, a)|\vec{p}, \mu\rangle_f = e^{ip\cdot \Lambda} \sqrt{\frac{p^+}{p^+}} \sum_{\mu=-\Lambda}^{\Lambda} |\vec{p}, \mu\rangle_f D^j_\mu\bar{\mu}[R_f(\Lambda, p)]. \quad (2.62) \]

### 2.2 Two-body System

The Hilbert space for a two-particle system of mass \( m_i \) and spin \( s_i \) \((i = 1, 2)\) is the tensor product of the single-particle Hilbert spaces:

\[ \mathcal{H}(2) = \mathcal{H}_{m_1}^{s_1} \otimes \mathcal{H}_{m_2}^{s_2} \quad (2.63) \]

A basis on the two-particle space can be constructed from single-particle bases:

\[ |\vec{p}_1, \mu_1, \vec{p}_2, \mu_2\rangle \equiv |m_1, s_1; \vec{p}_1, \mu_1\rangle \otimes |m_2, s_2; \vec{p}_2, \mu_2\rangle. \quad (2.64) \]

States on this space are represented by square integrable functions

\[ \Psi(\vec{p}_1, \mu_1, \vec{p}_2, \mu_2) \equiv \langle \vec{p}_1, \mu_1, \vec{p}_2, \mu_2|\Psi\rangle, \quad (2.65) \]

with normalization condition

\[ f^\dagger(\vec{p}_1', \mu_1', \vec{p}_2', \mu_2')|\vec{p}_1, \mu_1, \vec{p}_2, \mu_2\rangle_f = \delta_{\mu_1', \mu_1} \delta_{\mu_2', \mu_2} \delta(\vec{p}_1' - \vec{p}_1) \delta(\vec{p}_2' - \vec{p}_2). \quad (2.66) \]
If there is no interaction between the two particles, then the unitary representation for the two free particles is the tensor product of the two single-particle representations:

$$U_0(\Lambda, a) = U_1(\Lambda, a) \otimes U_2(\Lambda, a).$$  \hspace{1cm} (2.67)

The infinitesimal generators are sums of the one-body generators:

$$G_0^i = G_1^i + G_2^i,$$  \hspace{1cm} (2.68)

where $i$ represents one of the ten generators given in (2.46) and (2.47).

It is also convenient to choose a complete set of commuting variables so that the overall system motion and the internal motion can be separated. The internal variables are the relative momentum, $k \equiv k_1 = -k_2$, which is defined by

$$k_i \equiv L_f^{-1}(P_0) p_i, \quad i = 1, 2.$$  \hspace{1cm} (2.69)

Note that $P_0$ is the total momentum of the non-interacting two-body system. The invariant mass and spin operator can be constructed using the definitions in the previous section. The free mass operator of the non-interacting two-body system can be expressed in terms of the relative momentum by using equation (2.48):

$$M_0 = \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2}.$$  \hspace{1cm} (2.70)
Notice that the invariant mass has continuous eigenvalues that depend on the relative momentum \( k \). We define the longitudinal momentum fraction \( \xi \) as

\[
\xi \equiv \frac{p_1^+}{P_0^+} = 1 - \frac{p_2^+}{P_0^+}. \tag{2.71}
\]

The perpendicular components of the relative momentum can be expressed as

\[
k_\perp = p_{1\perp} - \xi P_\perp, \tag{2.72}
\]

and the invariant mass is

\[
M_0 = \sqrt{\frac{m_1^2 + k_1^2}{\xi}} + \sqrt{\frac{m_2^2 + k_1^2}{1 - \xi}}. \tag{2.73}
\]

The composite front-form spin can be obtained by applying a Melosh rotation on the canonical spin:

\[
s_0 = R_{fc}(k)s_{c1} + R_{fc}(-k)s_{c2}. \tag{2.74}
\]

The internal angular momentum of the two-particle system is

\[
L = i \nabla \times k. \tag{2.75}
\]

Since \( s \) and \( L \) transform identically under rotations, the total angular momentum can be expressed as the sum of the composite spin of the system and the orbital angular
momentum $\mathbf{L}$:

$$j_0 = \mathbf{L} + s_0. \quad (2.76)$$

The invariant mass $M_0$, the total front-form momentum $\tilde{P}_0$, the relative momentum $\mathbf{k}^2$, the total spin $\mathbf{s}^2$, the orbital angular momentum, which is $l^2$, and the total angular momentum $j^2$ and its third component can be chosen to be the complete set of commuting operators for the two-particle system. The basis vectors in this Poincaré irreducible representation can be expressed in terms of the eigenvalues of these operators:

$$|\tilde{P}, j, \mu, (k, l, s)\rangle \quad (2.77)$$

and the state vectors can be expressed as

$$\Psi(\tilde{P}, j, \mu, (k, l, s)) \equiv \langle \tilde{P}, j, \mu, (k, l, s)|\Psi\rangle \quad (2.78)$$

with the normalization condition:

$$f \langle \tilde{P}', j', \mu', (k'l's) | \tilde{P}, j, \mu, (kls) \rangle_f = \delta_{jj'} \delta_{\mu\mu'} \delta_{ss'} \delta(P^+ - P'^+)\delta^2(\mathbf{P}_\perp - \mathbf{P}'_\perp) \frac{1}{k'^2} \delta(k - k'). \quad (2.79)$$

In this representation, the overall rest states transform like free particle states described by Eq. (2.59), except that the mass and spin are replaced by the invariant rest mass and total spin. Poincaré transformations of states with any momentum can be obtained by
the method used in section 2:

\[ U_0(\Lambda, a)\tilde{P}_0, j, \mu, (kls)\rangle_f = e^{i\Lambda a} \left( \frac{P^{+}_0}{P^{+}} \right)^{\frac{1}{2}} \sum_{\tilde{\mu} = -j}^j D^j_{\tilde{\mu} \mu} [R_f(\Lambda, p_0)]\tilde{P}_0, j, \tilde{\mu}, (kls)\rangle_f. \]  

(2.80)

The tensor product basis vectors (2.64) and the irreducible basis vectors (2.77) are related by Clebsh-Gordan coefficients for the Poincaré group in a light-front basis

\[ \langle \tilde{p}'_1, \mu'_1, \tilde{p}'_2, \mu'_2 | \tilde{P}, j, \mu, (k, l, s)\rangle. \]  

(2.81)

Notice that only the canonical spins can be coupled with ordinary SU(2) Clebsh-Gordan coefficients because the canonical Wigner rotation of a rotation is the rotation itself. Thus, in the calculation of the front-form Poincaré Clebsh-Gordan coefficients we first need to transform the front-form spins to canonical spins in the rest frame, couple them to SU(2) irreducible representations, and then use the front-form boosts to boost the rest vectors to states of the corresponding momentum.

From Eq. (2.38) (2.39) and (2.60), the front-form spin states can be related to canonical spin states by the following equation:

\[ |\tilde{p}, \mu\rangle_f = \sqrt{\frac{\omega_{m}(p)}{p^+}} |p, \bar{\mu}\rangle_c D^j_{\tilde{\mu} \mu} [R_{cf}(k)]. \]  

(2.82)

In the overall rest frame, \( k \equiv p_1 = -p_2 \), so the tensor product vector can be expressed
in the following form:

\[
\begin{align*}
\langle \tilde{k}_1, \mu_1, \tilde{k}_2, \mu_2 \rangle_f &= \sqrt{\frac{\omega_{m_1}(k)}{k^+}} \sqrt{\frac{\omega_{m_2}(-k)}{-k^+}} D^{s_1}_{\tilde{\mu}_1 \mu_1} [R_m(k)] D^{s_2}_{\tilde{\mu}_2 \mu_2} [R_m(-k)] |k, \tilde{\mu}_1, -k, \tilde{\mu}_2\rangle_c.
\end{align*}
\]  

(2.83)

We then express this tensor product state in terms of vectors in a Poincaré irreducible representation with the angle portion of the relative momentum \( \tilde{k} \) being expanded with spherical harmonics:

\[
\begin{align*}
|k\rangle &= |k, l, \mu_l\rangle \langle l, \mu_l| \tilde{k} = |k, l, \mu_l\rangle Y_l^m(\tilde{k}).
\end{align*}
\]  

(2.84)

The canonical spins can be coupled with ordinary Clebsch-Gordan coefficients so Eq. (2.83) becomes:

\[
\begin{align*}
\langle \tilde{k}_1, \mu_1, \tilde{k}_2, \mu_2 \rangle_f &= \sqrt{\frac{\omega_{m_1}(k)}{k^+}} \sqrt{\frac{\omega_{m_2}(-k)}{-k^+}} D^{s_1}_{\tilde{\mu}_1 \mu_1} [R_m(k)] D^{s_2}_{\tilde{\mu}_2 \mu_2} [R_m(-k)] \\
&\times |0, j, \mu, (kls)\rangle \langle j\mu|l, \mu_l, s, \mu_s\rangle \langle s, \mu_s|s_1, \tilde{\mu}_1, s_2, \tilde{\mu}_2\rangle Y_j^{s_\mu}(\tilde{k}).
\end{align*}
\]  

Then we boost the state vectors on both sides of the equation above:

\[
\begin{align*}
U(L_f(p_1))|\tilde{k}_1, \mu_1\rangle \otimes U(L_f(p_2))|\tilde{k}_2, \mu_2\rangle &= \sqrt{\frac{p^+_1 p^+_2}{k^+_1 k^+_2}} |\tilde{p}_1, \mu_1, \tilde{p}_2, \mu_2\rangle, \\
U(L_f(P))|0, j, \mu(kls)\rangle &= \sqrt{\frac{P^+}{M_0}} |\tilde{P}, j, \mu(kls)\rangle.
\end{align*}
\]  

(2.86)
The combination of Eq. (2.85) (2.86) and normalization condition (2.79) gives the following front-form Clebsch-Gordan coefficients:

\[
\langle \tilde{p}_1', \mu_1', \tilde{p}_2', \mu_2' | \tilde{P}, j, \mu, (k, l, s) \rangle = \delta(\tilde{P} - \tilde{p}_1' - \tilde{p}_2') \frac{1}{|k|^2} \delta(|k(\tilde{p}_1', \tilde{p}_2')| - |k|) \\
\times \sqrt{\frac{\omega_{m_1}(k)\omega_{m_2}(-k)P^+}{p_1^+ p_2^+ M_0}} \sum D_{\mu_1 \mu_1}^{s_1} [R_{fc}(k)] D_{\mu_2 \mu_2}^{s_2} [R_{fc}(-k)] \\
\times Y_{\mu_1}^{l}(\hat{k}(\tilde{p}_1, \tilde{p}_2)) \langle s_1, \mu_1, s_2, \mu_2 | s, \mu_s \rangle \langle l, \mu_1, s, \mu_s | j, \mu \rangle. \tag{2.87}
\]

### 2.3 Introduction of Interactions

For a composite system, Dirac identified three basic ways simplifying the addition of interactions. But the problem of introducing the interactions is not trivial and was not provided by Dirac. The key requirement of this task is that the commutation relations need to be preserved when interactions are introduced. In an irreducible representation the Poincaré generators in a light-front basis have the form:

\[
P^+ = p^+, \quad P^i = p^i (i = 1, 2), \tag{2.88}
\]

\[
P^- = M_0^2 + p_+^2 = p^-, \tag{2.89}
\]

\[
E^i = -ip^+ \frac{\partial}{\partial p^i}, (i = 1, 2), \tag{2.90}
\]

\[
K^3 = -ip^+ \frac{\partial}{\partial p^+}. \tag{2.91}
\]
\[ J^3 = j^3_f - \frac{\hat{z} \cdot (P \times E)}{P^+}, \quad (2.92) \]

\[ J_\perp = \frac{1}{P^+} \left[ \frac{1}{2}(P^+ - P^-)(\hat{z} \times E_\perp) - (\hat{z} \times P_\perp)K^3 + P_\perp j^3_f + M_0 j^3_f \right]. \quad (2.93) \]

We notice that the Hamiltonian \( P^- \) and mass operator also appear in the operator \( J_\perp \). If we add interactions to the Hamiltonian \( P^- \), \( J_\perp \) must also be modified in order to preserve the commutation relations. On the other hand, the mass operator is a Casimir operator and commutes with all of the generators. Following Coester, Pieper and Serduke [27], Leutwyler and Stern [47] and Coester and Polyzou [28], it is convenient to introduce the interaction to the free mass operator.

\[ M \equiv M_0 + V \quad \text{or} \quad M^2 \equiv M_0^2 + U, \quad (2.94) \]

where \( U \) and \( V \) are related by

\[ U \equiv M^2 - M_0^2 = V^2 + \{M_0, V\}_+. \quad (2.95) \]

In front-form model, the interaction \( V \) must commute with the kinematic generators and the front-form spin:

\[ [E_\perp, V] = [K^3, V] = [j_0, V] = [P_\perp, V] = [P^+, V] = 0, \quad (2.96) \]

where the front-form spin vector is defined in Eq. (2.53) and (2.54). If we construct the Lie algebra using these operators, the commutation relations are preserved after
the interaction is introduced to the mass operator. The operator $P^-$ and $J$ can be obtained by using Eq. (2.89), (2.92) and (2.93) with the invariant mass $M_0$ replaced by $M$:

$$P^- = \frac{M^2 + P_-^2}{P^+},$$  \hfill (2.97)

$$J^3 = j^3_f - \frac{\hat{z}}{P^+} \cdot (P_\perp \times E_\perp),$$  \hfill (2.98)

$$J_\perp = \frac{1}{P^+} \left[ \frac{1}{2} (P^+ - P^-) (\hat{z} \times E_\perp) - (\hat{z} \times P_\perp) K^3 + P_\perp j^3_f + M j_\perp \right].$$  \hfill (2.99)

Sometimes the operator $F_\perp$ is used instead of $J_\perp$, which can be expressed as the following using definition (2.47):

$$F_\perp = 2 \frac{K^3}{P^+} P_\perp + \frac{M^2 + P_-^2}{(P^+)^2} E_\perp + \frac{2}{P^+} (j^3_f P_\perp + M j_\perp).$$  \hfill (2.100)

The remaining generators do not include interactions and are simply sums of the one-body generators. The mass operator does not appear in the generator $J^3$ either so that $J^3$ is also a kinematic operator. Thus the front-form model has the largest kinematic subgroup which contains seven kinematic generators.

The mass and front-form spin operator act only on the inner Hilbert space vector $|k, l, s\rangle$. Thus all the dynamics can be built on the inner Hilbert space. For each fixed mass eigenvalue $M$, the eigenstates $|\vec{P}, j, \mu, M\rangle_f$ have the same transformation properties as (2.80), with the non-interacting two-body mass replaced by the interacting
mass eigenvalue:

\[ U(\Lambda, a)|\tilde{P}, j, \mu, M\rangle_f = e^{i\Lambda P \cdot a} \sqrt{\frac{P^+}{P^+}} \sum_{\bar{\mu}=-j}^j D^{ij}_{\bar{\mu}\mu}[R_f(\Lambda, P)]|\tilde{P}, j, \bar{\mu}M\rangle_f. \] (2.101)

This defines the dynamical representation of the Poincaré group.
CHAPTER 3
CURRENT MATRIX ELEMENTS

In a relativistic model, the current operator $I^\mu(x)$ must transform covariantly under Poincaré transformations:

$$U(\Lambda, a)I^\mu(x)U^\dagger(\Lambda, a) = (\Lambda^{-1})^\mu_\nu I^\nu(\Lambda x + a), \quad (3.1)$$

which means that it must be separately Lorentz covariant and space-time translational covariant:

$$U(\Lambda)I^\mu(x)U^\dagger(\Lambda) = (\Lambda^{-1})^\mu_\nu I^\nu(\Lambda x), \quad (3.2)$$

$$[P^\mu, I^\nu(x)] = i \frac{\partial I^\nu(x)}{\partial x^\mu}. \quad (3.3)$$

The current operator must also satisfy current conservation:

$$\frac{\partial I^\mu(x)}{\partial x^\mu} = 0. \quad (3.4)$$

Eq. (3.3) and (3.4) imply that the current operator satisfies:

$$[P^\mu, I^\mu(x)] = 0. \quad (3.5)$$

Besides the Poincaré covariance and current conservation, the discrete symmetries
of time reversal and space inversion also impose constraints on current operators:

\[
TI^0(x)T^{-1} = I^0(-x^0, \mathbf{x}), \quad TI(x)T^{-1} = -I(-x^0, \mathbf{x}),
\]

\[
\mathcal{P}I^0(x)\mathcal{P}^{-1} = I^0(x^0, -\mathbf{x}), \quad \mathcal{P}I(x)\mathcal{P}^{-1} = -I(x^0, -\mathbf{x}).
\]

These constraints relate different current matrix elements. Thus, not all current matrix elements are independent [58]. All of the current matrix elements can be derived from a maximum set of independent current matrix elements, which are usually expressed in terms of invariant form factors. One task of this chapter is to use these constraints to find out the maximum set of the independent current matrix elements and define the Lorentz invariant form factors.

In general, for electron-hadron scattering, the current operator \( I^\mu(x) \) includes an electron component \( I^\mu_e(x) \), a strong component \( I^\mu_s(x) \) and a mixed remainder. In this work, the one-photon-exchange approximation is assumed. In this approximation, the representation \( U(\Lambda, a) \) in equation (3.1) becomes a tensor product of the representation associated with the strong interaction, \( U_s(\Lambda, a) \), and the representation associated with free electrons, \( U_e(\Lambda, a) \):

\[
U(\Lambda, a) \rightarrow U_s(\Lambda, a) \otimes U_e(\Lambda, a).
\]

The current becomes the sum of a strong current component and an electron current

\[
I^\mu(x) \rightarrow I^\mu_s(x) + I^\mu_e(x).
\]
They transform covariantly under the corresponding Poincaré representations $U_s(\Lambda, a)$ and $U_e(\Lambda, a)$.

The canonical spin electron current matrix elements are known:

$$\sqrt{\omega_e(p)/m_e} \langle p', \mu' | I_\alpha^e(x) | p, \mu \rangle \sqrt{\omega_e(p)/m_e} = \bar{u}(p', \mu') \gamma^\alpha u(p, \mu),$$

where the canonical spin Dirac $u$ spinors are described in Appendix $B$.

This work focuses on the strong interaction components, $I^\mu_s(x)$. The subscript $s$ will be dropped and the operator $I^\mu(x)$ will always stand for the hadron current operator unless otherwise indicated.

Impulse approximations use the one-body part of the current operator to compute the independent current matrix elements. For a system composed of interacting particles, the impulse approximation violates the requirements of Poincaré covariance (3.1) and current conservation (3.5). Two-body currents must be added to satisfy these requirements.

### 3.1 Independent Current Matrix Elements and Form Factors

Electromagnetic form factors [39] [76] [40] [29] are Lorentz invariant observables. They can be expressed in terms of a maximal set of independent current matrix elements. These matrix elements are functions of the single variable, the square of the four-momentum transfer $Q^2$. The task of defining form factors is the same as the task of determining the maximum set of independent current matrix elements.
After applying the constraints of the Poincaré covariance (3.1) and current conservation (3.5) as well as discrete Poincaré symmetries (3.6) and (3.7), there are $2s + 1$ independent matrix elements left, where $s$ is the spin of the particle. This result is independent of basis. We derive this conclusion using canonical spin.

Poincaré covariance (3.1) and the transformation properties of the Hilbert space state vectors under Poincaré transformations (2.42) lead to the following relations among different current matrix elements:

\[
\langle p_f, \nu_f | I_\mu(x) | p_i, \nu_i \rangle_c = e^{i(p_i - p_f)\cdot a}(\Lambda^{-1})^\mu_{\mu'}
\]

\[
\times D^{ij*}_{\nu_f' \nu_f}[L_c^{-1}(p_f')\Lambda L_c(p_f)]D^j_{\nu_i' \nu_i}[L_c^{-1}(p_i')\Lambda L_c(p_i)]
\]

\[
\times \sqrt{\frac{\omega_f'}{\omega_f}}\langle p_f', \nu_f' | I_{\mu'}(\Lambda a + a) | p_i', \nu_i' \rangle_c \sqrt{\frac{\omega_i'}{\omega_i}}.
\]

Here $p_f, p_i$ and $\nu_f, \nu_i$ are the final and initial momenta and canonical spin indices respectively. This equation and the current conservation as well as the discrete Poincaré symmetries can be used to determine the independent current matrix elements. First we use the ten independent Poincaré transformations to relate the general matrix elements to a specific set of matrix elements. From Eq. (3.11) we see that any current matrix element at space-time point $x$ can be expressed in terms of the current matrix elements at $x = 0$ by a space-time translation:

\[
\langle p_f, \nu_f | I_\mu(x) | p_i, \nu_i \rangle_c = e^{-i(p_i - p_f)\cdot x}\langle p_f, \nu_f | I_\mu(0) | p_i, \nu_i \rangle_c.
\]
This uses four of the ten transformations and now we only need to discuss the current matrix elements at the space-time point \( 0 \). The current matrix elements have a simpler form in the Breit frame, where the spatial components of \( p_f - p_i \) vanish. Various form factors are traditionally interpreted \([40]\) as the electric and magnetic moments in this frame. So the next step is to use the three Lorentz boosts to transform the system to a Breit frame with

\[
p_f' = \left( \sqrt{m_f^2 + \frac{q^2}{4}}, \frac{1}{2} q \right), \quad p_i' = \left( \sqrt{m_i^2 + \frac{q^2}{4}}, -\frac{1}{2} q \right).
\]  

(3.13)

Then all current matrix elements can be expressed in terms of the current matrix element in this frame:

\[
\langle p_f, \nu_f | I^\mu(0) | p_i, \nu_i \rangle
\]

(3.14)

\[
= \left( \Lambda^{-1}\right)^\mu_\nu D^{ij}_{\nu_f' \nu_f} \left[ L_c^{-1}(p_f') \Lambda L_c(p_f) \right] D^i_{\nu_i' \nu_i} \left[ L_c^{-1}(p_i') \Lambda L_c(p_i) \right]
\]

\[
\times \sqrt{\frac{\omega_f'}{\omega_f}} \langle p_f', \nu_f' | I^{\mu'}(0) | p_i', \nu_i' \rangle \sqrt{\frac{\omega_i'}{\omega_i}}.
\]

The current matrix elements in the Breit frame can be labeled by the spin indices as follows:

\[
\langle \nu_f | I^\mu(0) | \nu_i \rangle \equiv \langle \frac{q}{2}, \nu_f | I^\mu(0) | -\frac{q}{2}, \nu_i \rangle.
\]  

(3.15)

It is always possible to orient a space-like momentum transfer, \( q = p_f' - p_i' \), along a specific axis. We use two rotations to orient it along the \( \hat{z} \) direction, which is the axis of spin quantization. The last transformation is rotations about the \( \hat{z} \) direction, which
will not change the spin. These rotations yield the following relations:

\[ \langle \nu_f | I^0(0) | \nu_i \rangle = \delta_{\nu_f, \nu_i} \langle \nu_f | I^0(0) | \nu_i \rangle, \tag{3.16} \]

\[ \langle \nu_f | I^z(0) | \nu_i \rangle = \delta_{\nu_f, \nu_i} \langle \nu_f | I^z(0) | \nu_i \rangle, \tag{3.17} \]

\[ \langle \nu_f | I^x(0) | \nu_i \rangle = \frac{1}{2} \left( \delta_{\nu_f, \nu_i+1} + \delta_{\nu_f, \nu_i-1} \right) \langle \nu_f | I^x(0) | \nu_i \rangle \]
\[ + \frac{1}{2i} \left( \delta_{\nu_f, \nu_i+1} - \delta_{\nu_f, \nu_i-1} \right) \langle \nu_f | I^y(0) | \nu_i \rangle, \tag{3.18} \]

\[ \langle \nu_f | I^y(0) | \nu_i \rangle = \frac{1}{2} \left( \delta_{\nu_f, \nu_i+1} + \delta_{\nu_f, \nu_i-1} \right) \langle \nu_f | I^y(0) | \nu_i \rangle \]
\[ - \frac{1}{2i} \left( \delta_{\nu_f, \nu_i+1} - \delta_{\nu_f, \nu_i-1} \right) \langle \nu_f | I^x(0) | \nu_i \rangle. \tag{3.19} \]

Eq. (3.16) and (3.17) require \( \nu_f = \nu_i \) and Eq. (3.18) and (3.19) require \( |\nu_f - \nu_i| = 1 \).

From Eq. (3.18) and (3.19), we can also relate the non-zero matrix elements of \( I^y(0) \) to those of \( I^x(0) \):

\[ \langle \nu_i \pm 1 | I^y(0) | \nu_i \rangle = \pm i \langle \nu_i \pm 1 | I^x(0) | \nu_i \rangle. \tag{3.20} \]

Thus by using the continuous Poincaré symmetries, there are \( 2s + 1 \) non-zero independent matrix elements of \( I^0(0) \) and \( I^z(0) \) and \( 4s + 1 \) independent matrix elements of \( I^x(0) \). These matrix elements are not constrained by Poincaré invariance.
The number of independent matrix elements can be reduced by current conservation and discrete symmetries. Current conservation requires

\[ q_\mu \langle \nu_f | I^\mu(0) | \nu_i \rangle = 0. \] (3.21)

Since \( q^0 = q_0 = 0 \) and \( \mathbf{q} = q^z \), the third component of the current matrix elements has to be zero:

\[ \langle \nu_f | I^z(0) | \nu_i \rangle = 0. \] (3.22)

So the remaining independent matrix elements are:

\[ \langle \nu | I^0(0) | \nu \rangle, \] (3.23)
\[ \langle \nu | I^x(0) | \nu \pm 1 \rangle. \] (3.24)

Parity can be used to further reduce the number of independent matrix elements. The combination of parity and a rotation by \( \pi \) about the \( x \) axis yields:

\[ \langle \nu | I^0(0) | \nu \rangle = (-1)^{P_f + P_i} \langle -\nu | I^0(0) | -\nu \rangle, \] (3.25)
\[ \langle \nu \pm 1 | I^x(0) | \nu \rangle = -(-1)^{P_f + P_i} \langle -\nu \pm 1 | I^x(0) | -\nu \rangle, \] (3.26)

where \( P_f \) and \( P_i \) are the intrinsic parities of the initial and final states. Eq. (3.25) reduces the number of independent matrix elements of \( I^0(0) \) to \( s + 1 \) and \( s + \frac{1}{2} \) for integer and half-integer spin particles. Eq. (3.26) gives \( s \) independent matrix elements if \( s \) is
an integer and \( s + 1/2 \) if \( s \) is a half-integer. Thus totally there are \( 2s + 1 \) independent current matrix elements left. We can choose the ones with \( \nu \geq 0 \) to form the maximal set of independent matrix elements. In addition, time reversal invariance requires that all elastic form factors are real [25] which can be seen from the combination of time reversal, the Hermitian property of the current operator and a rotation by \( \pi \) about the \( \hat{x} \)-axis:

\[
\langle \nu | I^0(0) | \nu \rangle = (-1)^{s_f+s_i-2\nu} \langle \nu | I^0(0) | \nu \rangle^*, \tag{3.27}
\]

\[
\langle \nu \pm 1 | I^x(0) | \nu \rangle = -(-1)^{s_f+s_i-2\nu} \langle \nu \pm 1 | I^x(0) | \nu \rangle^*. \tag{3.28}
\]

As mentioned in Chapter 1, observables measured in the laboratory can be related to Lorentz invariant form factors. They are a conveniently chosen maximal set of independent functions of the independent matrix elements with canonical spin. Form factors so defined are normally interpreted [40] as the electric and magnetic moments in the Breit frame. The number of invariant independent form factors depends on the spin and is the same as the number of independent current matrix elements.

For spin-0 particles, there is only one form factor and it is defined as:

\[
F(Q^2) = \langle 0 | I^0(0) | 0 \rangle. \tag{3.29}
\]

For spin-\( \frac{1}{2} \) particles, like the neutron and the proton, there are two independent form factors, the Dirac and Pauli form factors, \( F_1(Q^2) \) and \( F_2(Q^2) \). Ernst, Sachs and Wali [35] also defined the so-called Sachs’ form factors, the Sachs’ electric and magnetic
form factors $G_E(Q^2)$ and $G_M(Q^2)$. Non-relativistically, the Sachs’ form factors can be interpreted \cite{62} as Fourier transforms of spatial distributions of nucleon charge and magnetization in Breit frame. They are defined in terms of the independent current matrix elements as the following:

$$G_E(Q^2) = \sqrt{1 + \tau} \langle \frac{1}{2} | I^0(0) | \frac{1}{2} \rangle_c,$$

$$G_M(Q^2) = \sqrt{1 + \tau} \langle \frac{1}{2} | I^1(0) | - \frac{1}{2} \rangle_c.$$  

Here $\tau \equiv Q^2 / 4m^2$ with $m$ being the nucleon mass. The Sachs’ form factors are related to the Dirac and Pauli form factors by:

$$F_1(Q^2) = \frac{G_E(Q^2) + \tau G_M(Q^2)}{1 + \tau},$$

$$F_2(Q^2) = \frac{G_M(Q^2) - G_E(Q^2)}{1 + \tau}.$$  

A spin-1 particle, such as the deuteron, has three independent form factors. The conventional form factors are defined by Eq. (1.9) (1.10) (1.11) following Coester and Ostebee \cite{26} in this work.

3.2 Form Factors in Light-Front Quantum Kinematics

This calculation is carried out using a light-front model because of its convenience for calculating current matrix elements. In light-front quantum mechanics, all of the current matrix elements can be derived from the matrix elements of the plus component
of the current operator, $I^+(0)$, in a Breit frame with $Q^+ = 0$. The covariant single nucleon current matrix elements have the structure:

$$
\langle \tilde{p}', \mu' | I^+(0) | \tilde{p}, \mu \rangle_{\text{cov}} = \bar{u}_f(\tilde{p}', \mu') K^+(q) u_f(\tilde{p}, \mu),
$$

(3.34)

where the front-form spinors $u_f$ and $\bar{u}_f$ are given in Appendix B. $K^\alpha(q)$ is the single nucleon current kernel:

$$
K^\alpha(q) = \gamma^\alpha F_1(q^2) + \frac{1}{2} [\gamma^\alpha, \gamma^\beta \frac{q_\beta}{2m}] F_2(q^2).
$$

(3.35)

Here $F_1(q^2)$ and $F_2(q^2)$ are the Dirac and Pauli nucleon form factors respectively. The non-covariant current matrix elements can be expressed in terms of these covariant matrix elements by:

$$
f(\tilde{p}', \mu' | I^+(0) | \tilde{p}, \mu)_f = \sqrt{\frac{m}{p^+}} \langle \tilde{p}', \mu' | I^+(0) | \tilde{p}, \mu \rangle_{\text{cov}} \sqrt{\frac{m}{p^+}}.
$$

(3.36)

The matrix elements of $I^+(0)$ with a non-covariant normalization are invariant under front-form boosts. This leads to a factorization theorem, an useful property for computing impulse current matrix elements. To see this, we express the transformation properties of the current operator $I^+(0)$ and the front-form state vectors under front-form boosts in the follow form:

$$
U(L_f(P)) I^+(0) U^+(L_f(P)) = \frac{1}{v^+} I^+(0),
$$

(3.37)
\[ U(L_f(P))|p^+, p_\perp, \mu\rangle = \sqrt{v^+}|v^+ p^+, p_\perp + v_\perp p^+, \mu\rangle. \] (3.38)

Note that the factors \( v^+ = P^+/m \) are parameters of the kinematic boost rather than momentum transfers. They cancel in the current matrix elements, which gives the following identity under the front-form boosts:

\[ \langle p^+, p_\perp, \mu|I^+(0)|p^+, p_\perp, \mu\rangle = \langle v^+ p^+, p_\perp + v_\perp p^+, \mu|I^+(0)|v^+ p^+, p_\perp + v_\perp p^+, \mu\rangle. \]

In electron scattering, the four momentum transfer \( q \) is space-like. So it is always possible to Lorentz transform the system to a frame where the plus component of the momentum transfer, \( q^+ = p_\perp^+ - p^+ \), vanishes. In this frame the current matrix elements have the following form

\[ \langle p^+, p_\perp + q_\perp, \mu|I^+(0)|p^+, p_\perp, \mu\rangle. \] (3.40)

Thus, under a light-front boost, Eq. (3.39) can be written as:

\[ \langle p^+, p_\perp + q_\perp, \mu|I^+(0)|p^+, p_\perp, \mu\rangle = \langle v^+ p^+, p_\perp + v_\perp p^+, \mu|I^+(0)|p^+, p_\perp, \mu\rangle. \] (3.41)

Eq. (3.41) shows that the current matrix elements of \( I^+(0) \) only depend on the initial
and final light-front spins and the momentum transfer. They are invariant under front-form boosts.

For a system composed of $N$ particles, assume that the momentum is transferred to particle 1. The subscript $F$ and $I$ represent the total final and initial variables of the system. Then the one-body current matrix elements can be exactly factored out of the $N$-body current matrix elements of $I^+(0)$ using Eq. (3.41):

$$
\langle p_1^+, p_{I\perp} + q_\perp, \mu_F | I^+(0) | p_1^+, p_{I\perp}, \mu_I \rangle
= \int \langle p_1^+, p_{I\perp} + q_\perp, \mu_F | p_1^+, p_{1\perp} + q_\perp, \mu_1', \cdots, \tilde{p}_N, \mu_N \rangle
\times \langle p_1^+, p_{1\perp} + q_\perp, \mu_1' | I^+(0) | p_1^+, p_{1\perp}, \mu_1 \rangle
\times d\tilde{p}_1, \cdots, d\tilde{p}_N \langle \tilde{p}_1, \mu_1, \cdots, \tilde{p}_N, \mu_N | \tilde{p}_I, \mu_I \rangle
$$

$$
= \int I_{\mu_1' \mu_1}(q_\perp) \langle p_1^+, p_{I\perp} + q_\perp, \mu_F | p_1^+, p_{1\perp} + q_\perp, \mu_1', \cdots, \tilde{p}_N, \mu_N \rangle
\times d\tilde{p}_1, \cdots, d\tilde{p}_N \langle \tilde{p}_1, \mu_1, \cdots, \tilde{p}_N, \mu_N | \tilde{p}_I, \mu_I \rangle.
$$

Here $I_{\mu_1' \mu_1}(q_\perp)$ is the single particle current matrix elements of particle 1. This property is called the Factorization theorem and will simplify the calculation of the one-body deuteron current matrix elements.

The independent current matrix elements in a light-front model can be determined by the transformation properties of the current operator and the front-form state vectors under front-form boosts as well as current conservation and discrete symmetries. We orient the space-like momentum transfer along the $x$-axis. Then the continuity equation
requires that matrix elements of the \( x \) component of the current has to vanish:

\[
\langle \mu' | I^1(0) | \mu \rangle = 0. \tag{3.43}
\]

The matrix elements of \( I^-(0) \) can be obtained from the plus components by rotations by \( \pi \) about the \( \hat{x} \) axis:

\[
\langle \mu' | I^-(0) | \mu \rangle \tag{3.44}
= \langle \mu' | U^\dagger [R_x(\pi)] I^+(0) U[R_x(\pi)] | \mu \rangle.
\]

Rotations by \( \pi/2 \) about the \( \hat{x} \) axis performed on the matrix elements of \( I^3 \), which equals to \( [I^+(0) - I^-(0)]/2 \), give the matrix elements of \( I^2(0) \):

\[
\langle \mu' | I^2(0) | \mu \rangle \tag{3.45}
= \frac{1}{2} \langle \mu' | U^\dagger [R_x(\frac{1}{2} \pi)] [I^+(0) - I^-(0)] U[R_x(\frac{1}{2} \pi)] | \mu \rangle.
\]

Thus, for \( Q^+ = 0 \), all current matrix elements can be obtained from the plus components of the current matrix elements in Breit frame. Note that the charge operator,

\[
I^0(0) = \frac{1}{2} [I^+(0) + I^-(0)], \tag{3.46}
\]

is invariant with respect to arbitrary rotations. A rotation by \( \pi/2 \) about the \( \hat{x} \) axis on
the charge operator gives:

\[ \langle m', j'; \mathbf{p}', \mu' | [I^+ + I^-(0)] | m, j; \mathbf{p}, \mu \rangle \]  
\[ = \langle m', j'; \mathbf{p}', \mu' | U[R_x(\frac{1}{2}\pi)] | I^+(0) \rangle U[R_x(\frac{1}{2}\pi)] | m, j; \mathbf{p}, \mu \rangle. \]  

Eq. (3.47) together with Eq. (3.44) puts extra constraints on the matrix elements of \( I^+(0) \):

\[ (1 + 2\eta) I_{11}^+ + I_{1-1}^+ - I_{00}^+ - 2\sqrt{2\eta} I_{10}^+ = 0. \]  

This constraint is called angular condition. It is a rotational covariance condition that is used to generate required two-body current matrix elements in interacting systems.

Using these independent current matrix elements of \( I^+(0) \), we can define the Lorentz invariant independent form factors. For spin-0 particles, the form factor in front-from is also defined by (3.29).

For spin-\( \frac{1}{2} \) particles, the Dirac and Pauli form factors are defined in terms of the plus components of the current matrix elements in the following form:

\[ F_1(Q^2) = \langle \frac{1}{2} I^+(0) | \frac{1}{2} \rangle, \]  
\[ F_2(Q^2) = -\frac{1}{\sqrt{\tau}} \langle \frac{1}{2} I^+(0) | - \frac{1}{2} \rangle. \]  

Relations (3.32) and (3.33) can be used to determine the form of the electric and magnetic form factors, \( G_M(Q^2) \) and \( G_E(Q^2) \), in terms of the matrix elements of \( I^+(0) \).
For the deuteron, while the form factors can be calculated using the independent current matrix elements, there are many possible combinations of the independent current matrix elements. Following the work in [24], labeled the CCKP method, the deuteron form factors are defined in terms of the independent current matrix elements, $I_{11}^+, I_{00}^+, I_{10}^+$ and $I_{1-1}^+$, as follows:

\[ F_{0d}(Q^2) = \frac{1}{2(1 + \eta)} \left( \langle 0 | I^+ | 0 \rangle + \langle 1 | I^+ | 1 \rangle \right), \]  
\[ F_{1d}(Q^2) = -\sqrt{2} \frac{1}{\eta (1 + \eta)} \langle 1 | I^+ | 0 \rangle, \]  
\[ F_{2d}(Q^2) = \frac{-1}{(1 + \eta)} (1 | I^+ | -1). \]

They are related to conventional deuteron form factors $G_0(Q^2)$, $G_1(Q^2)$ and $G_2(Q^2)$ by:

\[ G_0(Q^2) = F_{0d}(Q^2) + \frac{1}{6} F_{2d}(Q^2) - \frac{2}{3} \eta (F_{0d}(Q^2) + F_{2d}(Q^2) + \frac{5}{2} F_{1d}(Q^2)), \]  
\[ G_1(Q^2) = 2F_{0d}(Q^2) + F_{2d}(Q^2) + (1 - \eta) F_{1d}(Q^2), \]  
\[ G_2(Q^2) = \sqrt{\frac{8}{3}} (F_{2d}(Q^2) + \eta (\frac{1}{2} F_{2d}(Q^2) - F_{0d} - F_{1d}(Q^2))). \]

The relation of the deuteron form factors $G_0(Q^2)$, $G_1(Q^2)$, $G_2(Q^2)$ to the independent deuteron current matrix elements is:

\[ G_0(Q^2) = \frac{1}{6(1 + \eta)} [(3 - 2\eta) (I_{11}^+ + I_{00}^+) + 10\sqrt{2} I_{10}^+ + (4\eta - 1) I_{1-1}^+], \]
\[ G_1(Q^2) = \frac{1}{1 + \eta} [(I_{11}^+ + I_{00}^\mu) - \frac{2}{\sqrt{2\eta}}(1 - \eta)I_{10}^+ - I_{i-1}^+], \quad (3.58) \]

\[ G_2(Q^2) = \frac{\sqrt{2}}{3(1 + \eta)} [-\eta(I_{11}^+ + I_{00}^\mu) + \sqrt{8\eta}I_{10}^+ - (2 + \eta)I_{i-1}^+]. \quad (3.59) \]

Here \( I_{\mu\mu}^+ \) is a short hand for the current matrix element \( \langle \mu' | I^+(0) | \mu \rangle \). The remaining current matrix elements, \( I_{11}^+ - I_{00}^+ \), can be generated from the independent ones using the angular condition (3.48):

\[ I_{11}^+ - I_{00}^+ = 2\eta \left[ - \frac{1}{2(1 + \eta)}(I_{11}^+ + I_{00}^\mu) + \frac{\sqrt{(2/\eta)}}{1 + \eta} I_{10}^+ \right] - \frac{1}{1 + \eta} I_{i-1}^+. \quad (3.60) \]

The deuteron form factors can also be calculated using other combinations of the independent current matrix elements. Following Frankfurt, Frederico and Strikman, the deuteron form factors \( G_1(Q^2) \) and \( G_2(Q^2) \) are defined in the same way as in Eq. (3.58) and (3.59) and \( G_0(Q^2) \) is defined as:

\[ G_0(Q^2) = \frac{1}{3(1 + \eta)} [(2\eta + 3)I_{11}^+ + 2\sqrt{2\eta}I_{10}^+ - \eta I_{00}^+ + (2\eta + 1)I_{i-1}^+]. \quad (3.61) \]

When the deuteron form factor \( G_1(Q^2) \) is defined in by Eq. (3.58) and only \( I_{11}^+ \) and \( I_{i-1}^+ \) are used in addition, \( G_0(Q^2) \) and \( G_2(Q^2) \) are defined as follows:

\[ G_2(Q^2) = \frac{2\sqrt{2}}{3}(\eta I_{11}^+ - I_{i-1}^+ - \eta G_1(Q^2)), \quad (3.62) \]

\[ G_0(Q^2) = \eta I_{11}^+ + I_{i-1}^+ + \frac{G_2(Q^2)}{\sqrt{2}}. \quad (3.63) \]
Then the deuteron structure functions $A(Q^2)$, $B(Q^2)$ and $T_{20}(Q^2)$ can be calculated using Eq. (1.4)- (1.6).

3.3 Exchange Currents

The current operator for a composite system has the general form:

$$I^\mu(x) = \sum_i I^\mu_i(x) + \sum_{i<j} I^\mu_{ij}(x) + \cdots. \quad (3.64)$$

The first term corresponds to a one-body current, which acts on each constituent in turn. The second term is a two-body current, which vanishes if particles $i$ and $j$ are widely separated. The two-body current operators and the many-body current operators are also called exchange currents. For the deuteron, there are only one-body and two-body terms.

As discussed in the previous section, Poincaré covariance and current conservation impose dynamical constraints on current operators. Poincaré covariance (3.1) requires that the current operator transforms like a four-vector density under Lorentz transformations [58]. Thus, the commutation relations of current operators at space-time point 0 with the Lorentz generators are identical to the commutators of the four-momentum with the Lorentz generators. The commutators of the current operators with the rotation generators are:

$$[\mathbf{J}, I^0(0)] = 0, \quad [\mathbf{J}^i, I^j(0)] = i\epsilon_{ijk}I^k(0). \quad (3.65)$$
The commutators of $J_\perp$ with the plus and minus components of the current operator can be obtained:

$$[J^j, I^+(0)] = i\epsilon_{j3k}I^k(0), \quad [J^j, I^-(0)] = -i\epsilon_{j3k}I^k(0).$$ (3.66)

If the two-body parts of $J_\perp$ do not commute with the one-body current, the current must include two-body terms to satisfy rotational covariance. The exchange current contributions that are required to satisfy current covariance and current conservation are called required exchange currents. In addition, it is also possible, and often necessary to include exchange currents generated by additional dynamical considerations. In general, current matrix elements are related to observables. Unitary transformations that are applied to the states in these matrix elements must be canceled by transformations that are applied to the current in order to keep the observable representation independent. While it is possible to treat exchange currents like unknowns, that can be fixed by experiment, ideally it is desirable to have a framework where the interaction and exchange currents can be derived together.

Dynamical two-body currents are representation dependent. The interactions used in Poincaré invariant quantum mechanics are normally realistic interactions that include some long- and intermediate-range meson exchange physics along with some fitting and regularization. The corresponding current operators should also include some long- and intermediate-range physics motivated by meson exchange. In this spirit we construct a model two-body current that includes the long-range part of the physics associated
with the so-called “pair” currents, and investigate its impact on electromagnetic observables. These currents are normally associated with degrees of freedom in perturbative quantum field theory. To construct model currents we calculate S-matrix elements in the Born approximation using tree-level Feynman diagrams for a one-pion-exchange interaction, and for one-pion-exchange in the presence of an external electromagnetic field.

The kernel of the interaction term is extracted. It is replaced with a phenomenological interaction fit to scattering data that has the kinematic symmetries associated with a light-front dynamics. The phenomenological interaction is not completely arbitrary; it is assumed to have a long-range part associated with one-pion-exchange. Next the same tree-level diagram is calculated in the presence of an external electromagnetic field. The nucleon propagators connecting the external photon to the pion vertex are replaced by the $v$ spinor contribution to the nucleon propagator. This is what is normally called the “pair” term. A current can be extracted by taking the functional derivative of this expression with respect to the external field. This current has the general structure of an interaction multiplied by an effective one-body current. The effective interaction looks like the one-pion-exchange interaction with the $\gamma_5$ at the vertex that couples to the “pair” part of the propagator replaced by a $\beta$. The effective current also gets modified due to the presence of the $v$ spinor. The model “pair” current is obtained by replacing the model interaction with a similarly modified version of the model interaction. The modification is dictated by the change in the form of the one-pion-exchange interaction when one factor of $\gamma_5$ is replaced by a $\beta$. 
Finally, in the phenomenological interaction we only keep the long-range contribution due to one-pion-exchange, and we make the same type of truncations that are used in the construction of one-pion-exchange interactions. The effective current is defined in a preferred frame, and in other frames by kinematic covariance; current conservation and current covariance are enforced by evaluating independent current matrix elements and generating the remaining matrix elements using the constraints imposed by covariance and current conservation. The steps outlined above give an expression for a model current, however it should be considered in the same spirit that is normally used to construct phenomenological two and three-body interactions. It provides operator structures that contain the desired physics (the long-range contribution to the “pair” current in this case) with intermediate-range cutoffs that may have to be adjusted. The details of the construction is outlined below.

We begin by considering a model of nucleons interacting via a pseudoscalar one-pion-exchange interaction:

\[
\mathcal{L}(x) = -ig_\pi : \bar{\Psi}(x)\gamma_5 \Psi(x) \mathbf{T} \cdot \phi(x) :. \tag{3.67}
\]

We use this interaction to compute tree-level scattering matrix elements for NN scattering and NN scattering in an external electromagnetic field. The S-matrix and transition operator are related by

\[
S_{fi} = \langle f | i \rangle - 2\pi i \delta^4(p_f - p_i) \langle f | T | i \rangle, \tag{3.68}
\]
which in the Born (tree level) approximation becomes

\[ S_{fi} \to \langle f|i \rangle - 2\pi i\delta^4(p_f - p_i)\langle f|V|i \rangle. \]  

(3.69)

The current operator is extracted from

\[ \frac{\delta S_{fi}}{\delta A_\mu(q)} = -2\pi i\delta^4(p_f - p_i - q)e\langle f_+|I^\mu(0)|i_\rangle \]  

(3.70)

using a tree level truncation, where

\[ A_\mu(q) \equiv \frac{1}{(2\pi)^4} \int e^{-iq\cdot y}A_\mu(y)d^4y \]  

(3.71)

is the Fourier transform of the external field.

We assume a delta function normalization for the nucleon creation and annihilation operators:

\[ [a(p, s), a^\dagger(p', s')]_+ = \delta_{ss'}\delta(p - p'). \]  

(3.72)

These operators create particles with momentum \( p \) and z-component of canonical spin \( s \). With the normalization (3.72) the covariant nucleon fields have the form:

\[ \Psi(x) \equiv \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega(p)}} [u_c(p, s)e^{ip\cdot x}a(p) + v_c(p, s)e^{-ip\cdot x}b^\dagger(p)], \]  

(3.73)

\[ \bar{\Psi}(x) \equiv \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega(p)}} [\bar{u}_c(p, s)e^{-ip\cdot x}a^\dagger(p) + \bar{v}_c(p, s)e^{ip\cdot x}b^\dagger(p)], \]  

(3.74)
and the pion fields are defined as:
\[
\phi(x) \equiv \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{1}{2\omega(p)}} [e^{ip \cdot x}a(p) + e^{-ip \cdot x}b^\dagger(p)], \tag{3.75}
\]
\[
\phi(x)\dagger \equiv \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{1}{2\omega(p)}} [e^{-ip \cdot x}a^\dagger(p) + e^{ip \cdot x}b(p)], \tag{3.76}
\]
where \( p^0 = \sqrt{p^2 + m^2} \) and the canonical spinors \( u_c(p, s) \) and \( v_c(p, s) \) are defined in Appendix B. The pion field \( \phi(x) \) is a three-component isovector field corresponding to \( \pi_0 \) and \( \pi_{\pm} \). The initial and final momentum and spin indices for particle 1 and 2 are labeled by \( p_1, s_1, p_2, s_2, p_1', s_1', p_2', s_2' \) respectively. Internal variables are labeled with a double prime and the momentum transfer is labeled by \( q \).

The \( S \) matrix for NP scattering in the Born approximation gives
\[
-2\pi i\delta^4(p_1' + p_2' - p_1 - p_2)\langle p_1', p_2' | V | p_2, p_1 \rangle \tag{3.77}
\]
\[
eq \frac{(-i)^2}{2!} \int d^4xd^4y(-i)^2 g^2 \langle 0 | a(p_1', s_1') a(p_2', s_2')
\times T[j : \bar{\Psi}(x) \tau_1 \cdot \phi(x) \gamma^5 \Psi(x) :: \bar{\Psi}(y) \tau_2 \cdot \phi(y) \gamma^5 \Psi(y) ]
\times a^\dagger(p_2, s_2) a^\dagger(p_1, s_1) | 0 \rangle,
\]
where \( \tau \) are Pauli matrices for the isospin \( I = \tau/2 \). There are two identical terms associated with the two contractions of the above equation, which cancel the factor \( 1/2! \) and lead to the following expression:
\[
-2\pi i\delta^4(p_1' + p_2' - p_1 - p_2)\langle p_1', p_2' | V | p_2, p_1 \rangle \tag{3.78}
\]
\[ g_\pi^2 \int d^4x d^4y \langle 0 | a(p_1', s_1') \bar{\Psi}(y) | 0 \rangle \gamma_5 \langle 0 | \Psi(y) a^\dagger(p_1, s_1) | 0 \rangle \times \langle 0 | [ \tau_1 \cdot \phi(x) \tau_2 \cdot \phi(y) ] | 0 \rangle \times \langle 0 | a(p_2', s_2') \bar{\Psi}(x) | 0 \rangle \gamma_5 \langle 0 | \Psi(x) a^\dagger(p_2, s_2) | 0 \rangle. \]

The quantities in Eq. (3.78) are:

\[ \langle 0 | a(p, s) \bar{\Psi}(x) | 0 \rangle = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega(p)}} \bar{u}_c(p, s) e^{-ip_x}, \quad (3.79) \]

\[ \langle 0 | \Psi(x) a^\dagger(p, s) | 0 \rangle = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega(p)}} u_c(p, s) e^{ip_x}, \quad (3.80) \]

\[ \langle 0 | [ \tau_1 \cdot \phi(x) \tau_2 \cdot \phi(y) ] | 0 \rangle = -\frac{i}{(2\pi)^4} \tau_1 \cdot \tau_2 \int d^4k \frac{e^{ik \cdot (x-z)}}{k^2 + m_\pi^2 - i0^+}. \quad (3.81) \]

Combining all the above equations gives the following:

\[ -2\pi i \delta^4(p_1' + p_2' - p_1 - p_2) \langle p_1', p_2' | V | p_2, p_1 \rangle \]

\[ = -i \frac{g_\pi^2}{(2\pi)^4} \int d^4x d^4y d^4k e^{i(-p_1' + p_1 + k) \cdot x} e^{i(-p_2' + p_2 - k) \cdot y} \times \sqrt{\frac{m}{\omega(p_1')}} \sqrt{\frac{m}{\omega(p_1)}} \sqrt{\frac{m}{\omega(p_2')}} \sqrt{\frac{m}{\omega(p_2)}} \times \bar{u}(p_1') \gamma_5 u(p_1) \frac{\tau_1 \cdot \tau_2}{m_\pi^2 + k^2 - i0^+} \bar{u}(p_2') \gamma_5 u(p_2). \]

The integrals over \( x \) and \( y \) in Eq. (3.82) give four-momentum conserving delta functions at each vertex:

\[ \frac{1}{(2\pi)^4} \int d^4x e^{i(-p_1' + p_1 + k) \cdot x} = \delta^4(-p_1' + p_1 + k), \quad (3.83) \]
\begin{equation}
\frac{1}{(2\pi)^4} \int d^4y e^{i(-p_2' + p_2 - k)y} = \delta^4(-p_2' + p_2 - k). \quad (3.84)
\end{equation}

Then Eq. (3.82) becomes

\begin{equation}
-2\pi i \delta^4(p_1' + p_2' - p_1 - p_2)(p_1', p_2'|V|p_2, p_1) \quad (3.85)
\end{equation}

\begin{align*}
&= -i \delta^4(p_1' + p_2' - p_1 - p_2) \frac{g_\pi^2}{(2\pi)^2} \\
&\times \left[ \frac{m}{\omega(p_1')} \right] \left[ \frac{m}{\omega(p_1)} \right] \left[ \frac{m}{\omega(p_2')} \right] \left[ \frac{m}{\omega(p_2)} \right] \\
&\times \bar{u}(p_1') \gamma_5 u(p_1) \left( \frac{\tau_1 \cdot \tau_2}{m_\pi^2 + (p_1' - p_1)^2 - i0^+} \right) \bar{u}(p_2') \gamma_5 u(p_2).
\end{align*}

The interaction is the coefficient of

\begin{equation}
-2\pi i \delta(p_1' + p_2' - p_1 - p_2). \quad (3.86)
\end{equation}

The interaction to be used in the mass operator is obtained by evaluating this in the rest frame of the two-body system:

\begin{align*}
p_1' \rightarrow k', & \quad p_2' \rightarrow -k', & \quad p_1 \rightarrow k, & \quad p_2 \rightarrow -k. \quad (3.87)
\end{align*}

In order to relate this to a model interaction we evaluate the interaction in the rest frame of the two nucleon system:

\begin{align*}
&\delta(p_1' + p_2' - p_1 - p_2) \\
&\times \frac{g_\pi^2}{(2\pi)^3 \omega(k')} \bar{u}(k') \gamma_5 u(k) \frac{\tau_1 \cdot \tau_2}{m_\pi^2 + (k' - k)^2 - i0^+} \bar{u}(-k') \gamma_5 u(-k) \frac{m}{\omega(k)}.
\end{align*}
The rotationally invariant kernel

\[
\frac{g_\pi^2 m}{(2\pi)^3 \omega(k')} \bar{u}(k') \gamma_5 u(k) \frac{\tau_1 \cdot \tau_2}{m_\pi^2 + (k' - k)^2 - i0^+} \bar{u}(-k') \gamma_5 u(-k) \frac{m}{\omega(k)}
\]  

is identified with the rotationally invariant kernel of the phenomenological two-body interaction:

\[
\frac{g_\pi^2 m}{(2\pi)^3 \omega(k')} \bar{u}(k') \gamma_5 u(k) \frac{\tau_1 \cdot \tau_2}{m_\pi^2 + (k' - k)^2 - i0^+} \bar{u}(-k') \gamma_5 u(-k) \frac{m}{\omega(k)} \rightarrow \langle k', \mu_1', \mu_2'|V|k, \mu_1, \mu_2 \rangle.
\]  

(3.90)

In phenomenological interactions a static truncation of the term

\[
\sqrt{\frac{m}{\omega(k')}} \bar{u}(k') \gamma_5 u(k) \sqrt{\frac{m}{\omega(k)}}
\]

leads to the replacement

\[
\sqrt{\frac{m}{\omega(k')}} \bar{u}(k') \gamma_5 u(k) \sqrt{\frac{m}{\omega(k)}} \rightarrow \frac{(k - k') \cdot \sigma}{2m}.
\]  

(3.92)

The kernel (3.89) has the non-relativistic limit:

\[
- \frac{g_\pi^2}{(2\pi)^3} \frac{(k - k') \cdot \sigma_1}{2m} \frac{\tau_1 \cdot \tau_2}{m_\pi^2 + (k' - k)^2} \frac{(k - k') \cdot \sigma_2}{2m} = - \frac{1}{(2\pi)^3 m_\pi^2} \frac{f_\pi^2}{m_\pi^2} (k - k') \cdot \sigma_1 \frac{\tau_1 \cdot \tau_2}{m_\pi^2 + (k - k')^2} (k - k') \cdot \sigma_2,
\]

(3.93)
which contributes to the tensor force in

\[ \langle \mathbf{k}', \mu_1', \mu_2'| V | \mathbf{k}, \mu_1, \mu_2 \rangle. \]  

(3.94)

In Eq. (3.93) we have used the identity

\[ \frac{g_\pi^2}{4m^2} = \frac{f_\pi^2}{m_\pi^2}. \]  

(3.95)

The desired interaction to add to the mass operator in a theory with a light-front kinematic subgroup is:

\[ \langle \tilde{p}_1', \mu_1', \tilde{p}_2', \mu_2' | V | \tilde{p}_1, \mu_1, \tilde{p}_2, \mu_2 \rangle \]  

(3.96)

\[ = \langle \tilde{p}_1', \mu_1', \tilde{p}_2', \mu_2' | \tilde{P}', \mu', (k', l', s', j') \rangle \]

\[ \times \delta(\tilde{P} - \tilde{P}')\delta_{\mu' \mu} \langle k', l', s'| V^j | k, l, s \rangle \]

\[ \times \langle \tilde{P}, \mu, (k, l, s, j) | \tilde{p}_1, \mu_1, \tilde{p}_2, \mu_2 \rangle \]

where

\[ \langle \tilde{p}_1, \mu_1, \tilde{p}_2, \mu_2 | \tilde{P}, \mu, (k, l, s, j) \rangle \]  

(3.97)

are the Poincaré Clebsch-Gordan coefficients in a light-front basis (2.87) and

\[ \langle k', l', s'| V^j | k, l, s \rangle \]  

(3.98)
is the standard partial wave expansion of the rotationally invariant kernel

$$\langle k', \mu_1', \mu_2' | V | k, \mu_1, \mu_2 \rangle.$$ (3.99)

We identify the coefficient of the isospin dependent operator in (3.93)

$$\tau_1 \cdot \tau_2 (k - k') \cdot \sigma_1 (k - k') \cdot \sigma_2$$ (3.100)

as the one-pion-exchange potential:

$$V_{OPE}(|k - k'|) \equiv \frac{1}{(2\pi)^3} \frac{f_\pi^2}{m_\pi^2} \frac{1}{m_\pi^2 + (k - k')^2}.$$ (3.101)

The same type of truncation can be used to identify the “pair” currents. Eq. (3.70) shows that the currents can be obtained from the S matrix in Born approximation for the system in an external electromagnetic field. In this case, the S matrix in the Born approximation gives:

$$-2\pi i \delta^4(p'_1 + p'_2 - p_1 - p_2 - q) \langle p'_1, p'_2 | J^\mu(0) A_\mu(q) | p_1, p_2 \rangle$$ (3.102)

$$= -i e g_\pi^2 \int d^4 x d^4 y d^4 z \langle 0 | a_1(p'_1, s'_1) a_2(p'_2, s'_2) T[ : \bar{\Psi}(x) \tau_1 \cdot \phi(x) \gamma_5 \Psi(x) : ] : \bar{\Psi}(y) K^\mu A_\mu(y) \Psi(y) : | a_1^\dagger(p_2, s_2) a_2^\dagger(p_1, s_1) | 0 \rangle.$$

The terms contributing to the “pair” currents are associated with the four different
contractions of Eq. (3.102):

\[ (1) = -i e g_\pi^2 \int d^4 x d^4 y d^4 z \langle 0 | a(p'_1, s'_1) \bar{\Psi}(y) | 0 \rangle \]
\[ \times \ K_1^\mu A_\mu(y) \langle 0 | T[\bar{\Psi}(y) \bar{\Psi}(z)] | 0 \rangle \gamma^5 \langle 0 | \Psi(z) a^\dagger(p_1, s_1) | 0 \rangle \]
\[ \times \ \langle 0 | T[\tau \cdot \phi(z)] | 0 \rangle \langle 0 | a(p'_2, s'_2) \bar{\Psi}(x) | 0 \rangle \gamma^5 \langle 0 | \Psi(x) a^\dagger(p_2, s_2) | 0 \rangle; \]

\[ (2) = -i e g_\pi^2 \int d^4 x d^4 y d^4 z \langle 0 | a(p'_2, s'_2) \bar{\Psi}(y) | 0 \rangle \]
\[ \times \ K_1^\mu A_\mu(y) \langle 0 | T[\bar{\Psi}(y) \bar{\Psi}(z)] | 0 \rangle \gamma^5 \langle 0 | \Psi(z) a^\dagger(p_2, s_2) | 0 \rangle \]
\[ \times \ \langle 0 | T[\tau \cdot \phi(z)] | 0 \rangle \langle 0 | a(p'_1, s'_1) \bar{\Psi}(x) | 0 \rangle \gamma^5 \langle 0 | \Psi(x) a^\dagger(p_1, s_1) | 0 \rangle; \]

\[ (3) = -i e g_\pi^2 \int d^4 x d^4 y d^4 z \langle 0 | a(p'_1, s'_1) \bar{\Psi}(z) | 0 \rangle \]
\[ \times \ \gamma^5 \langle 0 | T[\bar{\Psi}(y) \bar{\Psi}(z)] | 0 \rangle K_2^\mu A_\mu(y) \langle 0 | \Psi(y) a^\dagger(p_1, s_1) | 0 \rangle \]
\[ \times \ \langle 0 | T[\tau \cdot \phi(z)] | 0 \rangle \langle 0 | a(p'_2, s'_2) \bar{\Psi}(x) | 0 \rangle \gamma^5 \langle 0 | \Psi(x) a^\dagger(p_2, s_2) | 0 \rangle; \]

\[ (4) = -i e g_\pi^2 \int d^4 x d^4 y d^4 z \langle 0 | a(p'_2, s'_2) \bar{\Psi}(z) | 0 \rangle \]
\[ \times \ \gamma^5 \langle 0 | T(\bar{\Psi}(y) \bar{\Psi}(z)) | 0 \rangle K_2^\mu A_\mu(y) \langle 0 | \Psi(y) a^\dagger(p_2, s_2) | 0 \rangle \]
\[ \times \ \langle 0 | T[\tau \cdot \phi(z)] | 0 \rangle \langle 0 | a(p'_1, s'_1) \bar{\Psi}(x) | 0 \rangle \gamma^5 \langle 0 | \Psi(x) a^\dagger(p_1, s_1) | 0 \rangle. \]
The nucleon propagator can be decomposed into \(u\) and \(v\) spinor contributions:

\[
\langle 0| T(\Psi(x) \bar{\Psi}(y)) |0 \rangle = -\frac{i}{(2\pi)^4} \int d^4p'' \frac{m}{\omega(p'')} \left[ \frac{u_c(p'', s'') u_c(p'', s'')}{\omega(p'')} - \frac{u_c(-p'', s'') \bar{v_c}(-p'', s'')}{\omega(p'')} \right] e^{ip'' \cdot (x-y)}. \tag{3.107}
\]

The "pair" current contributions are identified with the \(v\) spinor contributions to Eq. (3.107). Eq. (3.79)- (3.81) and the above equation can be used to simplify the four expressions (3.103)- (3.106). To construct the "pair" current we define

\[
\langle 0| T[\Psi(x) \bar{\Psi}(y)] |0 \rangle_{\text{pair}} = \frac{i}{(2\pi)^4} \int d^4p'' m \frac{v_c(-p'', s'') \bar{v_c}(-p'', s'')}{\omega(p'')} e^{ip'' \cdot (x-y)} \tag{3.108}
\]

and make the following replacement in Eq. (3.103)- (3.106):

\[
\langle 0| T[\Psi(x) \bar{\Psi}(y)] |0 \rangle \rightarrow \langle 0| T[\Psi(x) \bar{\Psi}(y)] |0 \rangle_{\text{pair}}. \tag{3.109}
\]

Evaluating (3.103) with the nucleon propagator replaced by Eq. (3.108) gives

\[
(1) = -ie g^2_{\pi} \frac{1}{(2\pi)^{14}} \sqrt{\frac{m}{\omega(p_1)}} \sqrt{\frac{m}{\omega(p_2)}} \sqrt{\frac{m}{\omega(p_1')}} \sqrt{\frac{m}{\omega(p_2')}} \int d^4k d^4p'' d^4x d^4y d^4z \bar{u_c}(p_1', s_{1}') e^{-ip'' \cdot y} K^\mu_1 A_\mu(y) \times \frac{m}{\omega(p_1')} \frac{v_c(-p'', s'') \bar{v_c}(-p'', s'')}{\omega(p'')} e^{ip'' \cdot (y-z)} \gamma_5 u_c(p_1, s_1) \times e^{ip_1 \cdot z} e^{ik \cdot (x-z)} \frac{\tau_1 \cdot \tau_2}{k^2 + m^2_{\pi}} e^{i(p_2 - x) \cdot \tau_2} \gamma_5 u_c(p_2, s_2) e^{ip_2 \cdot x}. \tag{3.110}
\]
We simplify this expression using the following identities:

\[ v_c(p, s) = \gamma^5 \beta u_c(-p, s), \]  
(3.111)

\[ \bar{v}_c(p, s) \gamma^5 = -\bar{u}_c(-p, s) \beta. \]  
(3.112)

Eq. (3.110) then becomes

\[
(1) = i e g^2 \frac{1}{(2\pi)^4} \frac{1}{\omega(p_1)} \frac{1}{\omega(p_2)} \frac{1}{\omega(p_1')} \frac{1}{\omega(p_2')} \times \int d^4k d^4p''_1 d^4x d^4y d^4z e^{-ip'_1 y} A_\mu(y) e^{ip_1''(y-z)} e^{ip_1' z} e^{ik(x-z)} e^{-ip_2' x} e^{ip_2 x} 
\times \frac{m}{\omega(p_1') \omega(p_2')} \epsilon^{\nu \rho} \beta^\nu \gamma^5 \beta u_c(p_1'', s_1'') \]

\[
\times \bar{u}_c(p_1', s_1') \beta u_c(p_1, s_1) \frac{\tau_1 \cdot \tau_2}{k^2 + m^2 - i\epsilon} \bar{u}_c(p_2', s_2') \gamma^5 u_c(p_2, s_2). \]  
(3.113)

The integrals over \( x \) and \( z \) in Eq. (3.113) give four-momentum delta functions at each vertex:

\[
\frac{1}{(2\pi)^4} \int d^4x e^{ikx} e^{-ip_2' x} e^{ip_2 x} = \delta^4(k - p_2' + p_2), \]  
(3.114)

\[
\frac{1}{(2\pi)^4} \int d^4z e^{-ip_1'' z} e^{ip_1 z} e^{-ikz} = \delta^4(p_1'' - p_1 - k). \]  
(3.115)

The integration over \( y \) gives the Fourier transformation of the external field defined by Eq. (3.71)

\[ A^\mu(q) = \frac{1}{(2\pi)^4} \int d^4y e^{-ip_1 y} A_\mu(y) e^{ip_1'' y} \]  
(3.116)
with a delta function

\[ q \equiv p'_1 - p''_1 = p'_1 + p'_2 - p_1 - p_2 \]  \hspace{1cm} (3.117)

The “pair” current contribution to (3.103) then becomes:

\[
\begin{align*}
(1) &= \frac{i e}{(2\pi)^2} g_\pi^2 \sqrt{\frac{m}{\omega(p'_1)}} \sqrt{\frac{m}{\omega(p'_1)}} \sqrt{\frac{m}{\omega(p'_2)}} \sqrt{\frac{m}{\omega(p'_2)}} \\
&\times \int d^4 k d^4 p'_1 \delta^4(k - p'_2 + p_2) \delta^4(p''_1 + k - p_1) \delta^4(q - p'_1 + p''_1) \\
&\times A_\mu(q) \frac{m}{\omega(p''_1) (\omega(p''_1) + p''_1^0 - i\epsilon)} \bar{u}(p''_1, s''_1) K_1^\mu \gamma^5 \beta u(p''_1, s''_1) \\
&\times \frac{\tau_1 \cdot \tau_2}{k^2 + m^2 - i0^+} \bar{u}(p'_2, s'_2) \gamma^5 u(p_2, s_2).
\end{align*}
\]  \hspace{1cm} (3.118)

Comparing this with Eq. (3.102) shows that the current matrix element is the coefficient of

\[-i(2\pi) \int d^4 q \delta(q - p'_1 + p''_1) A_\mu(q) e,\]  \hspace{1cm} (3.119)

which is

\[
\begin{align*}
\langle p'_1, p'_2 | I^\mu | p_2, p_1 \rangle &= \frac{-1}{(2\pi)^3 g_\pi^2} \sqrt{\frac{m}{\omega(p'_1)}} \sqrt{\frac{m}{\omega(p'_1)}} \sqrt{\frac{m}{\omega(p'_2)}} \sqrt{\frac{m}{\omega(p'_2)}} \\
&\times \frac{m}{\omega(p''_1) (\omega(p''_1) + p''_1^0 - i\epsilon)} \bar{u}(p''_1, s''_1) K_1^\mu \gamma^5 \beta u(p''_1, s''_1) \\
&\times \bar{u}(p'_2, s'_2) \gamma^5 u(p_2, s_2).
\end{align*}
\]  \hspace{1cm} (3.120)

It contains exactly the same factor as in Eq.(3.85) except the \( \gamma_5 \) on one vertex is replaced by a \( \beta \). The other term looks like a modified current operator evaluated
in a non-covariant basis multiplied by the energy denominator from the “pair” term. Similar expression can be evaluated for each of the other three terms. To make the identification with the phenomenological interaction, (3.120) was evaluated in the rest frame of the initial two nucleon system:

\[ p_1 = -p_2 = k, \quad p''_1 = -p'_2 = k''. \] (3.121)

Then (3.120) becomes:

\[
-\frac{g_\pi^2}{(2\pi)^3 2m} \frac{1}{\sqrt{m \omega(p'')} \bar{u}(p'_1, s'_1) K''_{1} \gamma^5 \beta u(p''_1, s''_1) \sqrt{m \omega(k'')}} \\
\times \frac{\sqrt{m}}{\omega(k'')} \bar{u}(k'', s''_1) \beta u(k, s_1) \frac{\tau_1 \cdot \tau_2}{\sqrt{m \omega(k')(k - k'')^2 + m^2_\pi}} \\
\times \frac{\sqrt{m}}{\omega(k'')} \bar{u}(-k'', s'_2) \gamma^5 u(-k, s_2) \sqrt{m \omega(k)},
\]

where

\[
\frac{1}{(\omega(p''_1) + p''_1^0 - i\epsilon)} \rightarrow \frac{1}{2\omega(k)} \rightarrow \frac{1}{2m}
\] (3.122)

using the delta functions in the low energy limit.

As in Eq. (3.92), in the low energy limit,

\[
\frac{\sqrt{m}}{\omega(k'')} \bar{u}(-k'', s'_2) \gamma^5 u(-k, s_2) \sqrt{m \omega(k)} \rightarrow \frac{(k'' - k) \cdot \sigma_2}{2m}
\] (3.123)

Applying the same expansion to the term \( \bar{u}(k'') \beta u(k_1) \), which couples to the “pair"
propagator by replacing $\gamma^5$ by $\beta$, we have

$$\sqrt{\frac{m}{\omega(k'')}} \bar{u}_e(k'', s''_1) \beta u_e(k, s_1) \sqrt{\frac{m}{\omega(k)}} \to 1 + O\left(\frac{k''^2}{m^2}, \frac{k^2}{m^2}\right)$$  (3.124)

where the corrections are of $(k/m)^2$. For small values of $k$ and $k'$, the above term becomes 1. Then Eq.(3.120) becomes:

$$\frac{-1}{(2\pi)^3} \frac{g^2}{2m \cdot 4m^2} \sqrt{\frac{m}{\omega(p'_1)}} \bar{u}(p'_1, s'_1) K_{1''} \gamma^5 \beta u(p''_1, s''_1) \sqrt{\frac{m}{\omega(k'')}}$$

$$\times \frac{\tau_1 \cdot \tau_2}{m^2 + (k - k'')^2} 2m (k - k'') \cdot \sigma_2,$$  (3.125)

where $2m k \cdot \sigma_2$ replaces $k \cdot \sigma_1 k \cdot \sigma_2$ in Eq. (3.93).

According to Eq. (3.93) and (3.101), the following factors are identified as the one-pion-exchange potential and is replaced by

$$\frac{1}{(2\pi)^3} \frac{g^2}{4m^2} \frac{1}{m^2 + (k - k'')^2} \to V_{OPE}(|k - k''|).$$  (3.126)

Now Eq. (3.125) can be written as:

$$\langle p'_1, p'_2 | I^\mu | p_1, p_2 \rangle$$  (3.127)

$$= - \sqrt{\frac{m}{\omega(p'_1)}} \bar{u}(p'_1, s'_1) K_{1''} \gamma^5 \beta u(p''_1, s''_1) \sqrt{\frac{m}{\omega(k'')}}$$

$$\times V_{OPE}(|k - k''|) \tau_1 \cdot \tau_2 (k - k'') \cdot \sigma_2,$$

After all of the truncations, we obtain the operator structure of the first “pair” current
\[ \langle p'_1, \mu'_1, p'_2, \mu'_2 | I_{\text{ex}1}^\mu | p_1, \mu_1, p_2, \mu_2 \rangle \]
\[ = \sqrt{\frac{m}{\omega(p_1)}} \bar{u}(p'_1, s'_1) K_1^{\mu} \gamma_5 \beta u(p'', s''_1) \sqrt{\frac{m}{\omega(k'')}} \times V_{\text{OPE}}(|k'' - k|) \tau_1 \cdot \tau_2 \cdot (k'' - k) \cdot \sigma_2. \]

Similarly, the other three terms give:

\[ \langle p'_1, \mu'_1, p'_2, \mu'_2 | I_{\text{ex}2}^\mu | p_1, \mu_1, p_2, \mu_2 \rangle \]
\[ = V_{\text{OPE}}(|k' - k''|) \tau_1 \cdot \tau_2 \cdot (k' - k'') \cdot \sigma_2 \times \sqrt{\frac{m}{\omega(k'')}} \bar{u}(p'', s''_1) \beta \gamma_5 K_2^\mu (q) u(p_1, s_1) \sqrt{\frac{m}{\omega(p_1)}}, \]

\[ \langle p'_1, \mu'_1, p'_2, \mu'_2 | I_{\text{ex}3}^\mu | p_1, \mu_1, p_2, \mu_2 \rangle \]
\[ = \sqrt{\frac{m}{\omega(p_2)}} \bar{u}(p'_2, s'_2) K_2^{\mu} (q) \gamma_5 \beta u(p'', s''_2) \sqrt{\frac{m}{\omega(k'')}} \times (k'' - k) \cdot \sigma_1 V_{\text{OPE}}(|k'' - k|) \tau_1 \cdot \tau_2, \]

\[ \langle p'_1, \mu'_1, p'_2, \mu'_2 | I_{\text{ex}4}^\mu | p_1, \mu_1, p_2, \mu_2 \rangle \]
\[ = V_{\text{OPE}}(|k' - k''|) \tau_1 \cdot \tau_2 \cdot (k' - k'') \cdot \sigma_1 \times \sqrt{\frac{m}{\omega(k'')}} \bar{u}(p'', s''_2) \beta \gamma_5 K_2^\mu (q) u(p_2, s_2) \sqrt{\frac{m}{\omega(p_2)}}. \]
Note that each term is computed in a frame where the interaction is at rest. We use kinematic covariance to define these matrix elements in the other frame. The exchange current operator depends on the model NN-interaction. The long-range part of this current is obtained by replacing the coefficient of the operator $\tau_1 \cdot \tau_2 k \cdot \sigma_1 k \cdot \sigma_2$ in the phenomenological interaction by the one-pion-exchange contribution. Thus, this operator structure can be used to calculate exchange current matrix elements for any suitable realistic potential model. In this work, the Argonne V18 NN-interaction is used and the calculation is given in detail in Chapter (5).
CHAPTER 4

IMPULSE CURRENTS

As discussed previously, all the other current matrix elements can be expressed in terms of a maximal set of independent matrix elements of $I^+(0)$ in a front-form dynamical model with $Q^+ = 0$. The deuteron form factors can be expressed as linear combinations of these matrix elements using Eq. (3.51) - (3.53). Thus, the deuteron structure functions can be calculated from these current matrix elements using Eq. (1.4) - (1.6) and Eq. (3.54) - (3.56). The main task of this chapter is to calculate the independent deuteron current matrix elements in the impulse approximation.

Suppose the proton is particle 1 and the neutron is particle 2. The plus component of the deuteron current matrix elements can be expressed in the following form:

$$
\langle \tilde{P}_d', \mu_d' | I^+(0) | \tilde{P}_d, \mu_d \rangle
$$

$$
= \sum \int \langle \tilde{P}_d', \mu_d'| \tilde{P}_p', \mu_p', \tilde{P}_n', \mu_n' \rangle d\tilde{P}_p'd\tilde{P}_n'
\times \langle \tilde{P}_p', \mu_p', \tilde{P}_n', \mu_n' | I^+(0) | \tilde{P}_p, \mu_p, \tilde{P}_n, \mu_n \rangle
\times d\tilde{P}_p d\tilde{P}_n \langle \tilde{P}_p, \mu_p, \tilde{P}_n, \mu_n | \tilde{P}_d, \mu_d \rangle.
$$

It is an integral of the product of the full final and initial deuteron wave functions,

$$
\langle \tilde{P}_d', \mu_d' | \tilde{P}_p', \mu_p', \tilde{P}_n', \mu_n' \rangle, \quad \langle \tilde{P}_p, \mu_p, \tilde{P}_n, \mu_n | \tilde{P}_d, \mu_d \rangle
$$

and the matrix elements of the current operator in the two particle tensor product
representation space

\[ \langle \tilde{p}_p^\prime, \mu_p^\prime, \tilde{p}_n^\prime, \mu_n^\prime | I^+(0) | \tilde{p}_p, \mu_p, \tilde{p}_n, \mu_n \rangle. \]  \hspace{1cm} (4.3)

The full deuteron wave function can be expressed in terms of the conventional deuteron wave functions \( u_i(k) \) in the non-interacting Poincaré irreducible representation space using the light-front Clebsch-Gordan coefficients (2.87):

\[
\langle \tilde{p}_p, \mu_p, \tilde{p}_n, \mu_n | \tilde{P}_d, \mu_d \rangle \equiv \Psi_{\tilde{P}_d, \mu_d}(\tilde{p}_p, \tilde{p}_n, \mu_p, \mu_n) = \int \langle \tilde{p}_p, \mu_p, \tilde{p}_n, \mu_n | \tilde{P}, j, \mu_j, (k, l, s) | \tilde{P}_d, \mu_d \rangle \langle \tilde{P}, j, \mu_j, (k, l, s) | \tilde{P}_d, \mu_d \rangle \delta(\tilde{P} - \tilde{p}_p - \tilde{p}_n) \frac{1}{|k|^2} \delta(|k(\tilde{p}_p, \tilde{p}_n)| - |k|) \times \sqrt{\frac{\omega_{m_\mu}(k) \omega_{m_\nu}( -k) P^+_{\mu_p, \mu_n}}{p^+_{\mu_p, \mu_n} M_0}} D^*_{\mu_p, \mu_n}[R_{fc}(k)] D^*_{\mu_n, \mu_n}[R_{fc}(-k)] \times \langle s_p, \mu_p, s_n, \mu_n | l, \mu_l, s, \mu_s | j, \mu \rangle Y^l_{\mu_l}(\hat{k}(\tilde{p}_p, \tilde{p}_n)) u_l(k) \equiv \Psi_{\tilde{P}_d, \mu_d}(\tilde{p}_p, \tilde{p}_n, \mu_p, \mu_n)
\]

The normalization condition of Eq. (4.4) is

\[
\int d\mathbf{k} \Psi_{\tilde{P}_d, \mu_d}(\mathbf{k}, \mu_1, \mu_2) \Psi_{\tilde{P}_d, \mu_d}^*(\mathbf{k}, \mu_1, \mu_2) = \delta(\tilde{P}_d - \tilde{P}_d) \delta_{\mu_1, \mu_1} \delta_{\mu_2, \mu_2}. \]  \hspace{1cm} (4.5)

The deuteron is a spin-1 particle with \( s = 1 \). In the case of \( j = 1 \), the \( l \) values are limited only to be 0 or 2 by parity. They correspond to the conventional deuteron S-state wave function \( u_0(k) \) and D-state wave function \( u_2(k) \) respectively with the
normalization condition
\[ \int_0^\infty dk k^2 (u_0^2(k) + u_2^2(k)) = 1. \] (4.6)

They can be obtained by solving the mass eigenvalue equation:

\[ M^2 |\Psi\rangle \equiv M_d^2 |\Psi\rangle, \] (4.7)

where \( M_d \) is the mass eigenvalue.

As discussed in Chapter 2, the interaction can be added into the mass operator as in Eq. (2.94) without changing the commutation relations. The deuteron is a bound state of two nucleons, the proton and the neutron, whose masses, \( m \), are assumed to be the same. We introduce the interaction in the following way [27]:

\[ M^2 \equiv 4(m^2 + k^2) + 4mV, \] (4.8)

where the \( m \) in \( 4mV \) term is included so that \( V \) has the same dimensions as a non-relativistic interaction and the factor of four makes it possible to identify \( V \) with the non-relativistic interaction. The above equation can be rewritten in the form:

\[ \frac{k^2}{m} + V = \frac{M^2 - 4m^2}{4m}. \] (4.9)

We recognize that if the right side of the above equation is identified as the energy \( E \):

\[ \frac{M^2 - 4m^2}{4m} \rightarrow E, \] (4.10)
the deuteron bound state mass eigenvalue equation has exactly the same form as of the non-relativistic Schrödinger equation

\[
\left( \frac{k^2}{m} + V \right) |\Psi\rangle = E |\Psi\rangle. \tag{4.11}
\]

The only difference is that the eigenvalue in non-relativistic case, \( M_d - 2m \), is replaced by \( (M_d^2 - 4m^2)/4m \) in the relativistic model. In the case of deuteron, \( M_d = 1.87561339 \text{GeV} \) and \( m = 0.938919 \text{GeV} \), the difference is negligible:

\[
\frac{[M_d - 2m] - [M_d^2 - 4m^2]/4m}{[M_d^2 - 4m^2]/4m} \approx \frac{1}{2000}. \tag{4.12}
\]

The relativistic deuteron wave functions, which are the eigenfunctions of the mass operator, are identical to the non-relativistic deuteron wave functions and the phase shifts are unchanged.

The model of nucleon-nucleon interaction used in this calculation is the realistic Argonne V18 potential [75]. The interaction is a sum of product of spin-isospin dependent operators with radial coefficient functions:

\[
v = \sum_{p=1}^{18} v_p(r) O^p. \tag{4.13}
\]

where \( O^p \) are the 18 spin-isospin projection operators and \( v_p(r) \) are their corresponding radial coefficient functions. The first 14 operators are the same as in the Argonne V14
potential [74] which are charge independent:

$$O^{p=1,14} =$$

$$1, \tau_i \cdot \tau_j, \sigma_i \cdot \sigma_j, (\sigma_i \cdot \sigma_j)(\tau_i \cdot \tau_j), S_{ij}, S_{ij}(\tau_i \cdot \tau_j),$$

$$L \cdot S, L \cdot S(\tau_i \cdot \tau_j), L^2, L^2(\tau_i \cdot \tau_j), L^2(\sigma_i \cdot \sigma_j),$$

$$L^2(\sigma_i \cdot \sigma_j)(\tau_i \cdot \tau_j), (L \cdot S)^2, (L \cdot S)^2(\tau_i \cdot \tau_j),$$

(4.14)

where $\sigma_i$ and $\tau_i$ are the spin and isospin vector of each nucleon respectively and $L$ and $S$ are the relative orbital angular momentum and the total spin.

$$S_{12} = 3(\sigma_1 \cdot \hat{r}_{12})(\sigma_2 \cdot \hat{r}_{12}) - \sigma_1 \cdot \sigma_2$$

(4.15)

is the tensor operator. The four additional operators are charge dependent:

$$O^{p=15,18} = T_{ij}, (\sigma_i \cdot \sigma_j)T_{ij}, S_{ij}T_{ij}, (\tau_{zi} + \tau_{zj}),$$

(4.16)

where $T_{12} = 3\tau_{z1}\tau_{z2} - \tau_1 \cdot \tau_2$ is the isotensor operator.

In solving for the deuteron wave function, we used the partial wave method. The S- and D-state wave functions are expressed as a superposition of the partial wave basis functions. Each operator $O_i$ in the Argonne V18 potential model is decomposed into partial waves. Since the operators are spin-isospin dependent, we need to express the matrix coefficients of the operator $O_i$ in a basis of total internal angular momentum
states:

\[ |j', \mu', l', s', \tau_1', \tau_2'| O_i j, \mu, l, s, \tau_1, \tau_2 \rangle. \quad (4.17) \]

The basis functions developed by Keister and Polyzou [43] are used in this work. The configuration-space basis functions are expanded in the complete set of \( n \) radial basis functions:

\[ \phi_{ln}(r) \equiv \frac{1}{\sqrt{N_{ln}}} (x)^l L_n^{2l+2}(2x)e^{-x}, \quad x \equiv sr, \quad (4.18) \]

where \( s \) is a scale parameter and the normalization coefficient is

\[ N_{ln} \equiv s^{-3} \left( \frac{1}{2} \right)^{2l+3} \frac{\Gamma(n + 2l + 2 + 1)}{n!}. \quad (4.19) \]

The associated Laguerre polynomials have the following form:

\[ L_n^\alpha(x) = \sum_{m=0}^{n} (-)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!}, \quad (4.20) \]

with \( \alpha = 2l + 2 \) in this case. The recurrence relation for the Laguerre polynomials is

\[ (n + 1)L_{n+1}^\alpha(x) = (2n + 1 + \alpha - x)L_n^\alpha(x) - (n + \alpha)L_{n-1}^\alpha(x) \quad (4.21) \]

with the first and second term being

\[ L_0^\alpha(x) = 1, \quad L_1^\alpha(x) = \alpha + 1 - x. \quad (4.22) \]
Functions defined in this way satisfy the orthogonality

\[
\int_0^\infty r^2 \phi_{ln}(r) \phi_{lm}(r) = \delta_{mn}.
\]  

(4.23)

The momentum-space basis functions are obtained by the Fourier transformation of the basis functions in the configuration space:

\[
\phi_{ln}(k) \equiv \sqrt{\frac{2}{\pi}} \int_0^\infty dr r^2 j_l(kr) \phi_{ln}(r),
\]  

(4.24)

where \( j_l(kr) \) is the spherical Bessel functions:

\[
j_n(x) = (-x)^n \left( \frac{1}{x} \frac{d}{dx} \right)^n \sin x \frac{x}{x}.
\]  

(4.25)

These basis functions can be expressed in terms of the Jacobi polynomials as

\[
\tilde{\phi}_{ln}(k) = \frac{1}{\sqrt{\tilde{N}_{ln}}} \frac{(k/s)^l}{((k/s)^2 + 1)^{l+\frac{1}{2}}} P_n^{l+\frac{3}{2}, l+\frac{1}{2}} \left[ \frac{k^2 - s^2}{k^2 + s^2} \right].
\]  

(4.26)

The Jacobi polynomials and the normalization coefficients are:

\[
P_n^{l+\frac{3}{2}, l+\frac{1}{2}}(x) = \frac{\Gamma(l + \frac{3}{2} + n + 1)}{n!(2l + n + 2)!} \sum_{m=1}^n \frac{n!}{m!(n-m)!} \frac{(2l + n + 2 + m)!}{2^m \Gamma(l + \frac{3}{2} + m + 1)} (x - 1)^m,
\]  

(4.27)

\[
\tilde{N}_{ln} \equiv \frac{s^3}{2(2n + 2l + 3)} \frac{\Gamma(n + l + \frac{3}{2}) \Gamma(n + l + \frac{3}{2})}{n!(n + 2l + 2)!}.
\]  

(4.28)

With the help of these basis functions, we can use Eq. (4.11) to obtain the deuteron
wave function. The most general interaction $V$ satisfying Eq. (2.96) has the following form:

$$f\langle \tilde{P}', j', \mu', k', l', s' | V | \tilde{P}, j, \mu, k, l, s \rangle_f = \delta_{jj'} \delta_{\mu\mu'} \delta(\tilde{P}' - \tilde{P}) \langle k'l's' | V | kl's \rangle. \quad (4.29)$$

For the deuteron, $s = j = 1$ and the interaction depends only on the momentum $k$ and the $l$ value:

$$\langle j, \mu', k', l', s | V | j, \mu, k, l, s \rangle = \delta_{\mu\mu'} \langle k', l' | v^1,s=1 | k, l \rangle. \quad (4.30)$$

Suppose the S- and D-state wave functions in the configuration space are written as:

$$u_l(r) = \sum_n c_{ln} \phi_{ln}(r), \quad (4.31)$$

then the advantage of these basis functions is that we can express the corresponding wave functions in the momentum space using the same set of coefficients $c_{ln}$ as follows:

$$u_l(k) = (-i)^l \sum_n c_{ln} \phi_{ln}(k). \quad (4.32)$$

The normalization condition is:

$$\sum_n (c_{0n}^2 + c_{2n}^2) = 1. \quad (4.33)$$
These coefficients can be obtained by solving the mass eigenvalue equation:

\[
\left( \frac{k^2}{m} - E_{0n} \right) u_0(k) = - \int_{0}^{\infty} \langle k, 0 | v | k' \rangle k'^2 dk' u_0(k') - \int_{0}^{\infty} \langle k, 0 | v | k' \rangle k'^2 dk' u_2(k'),
\]

\[
\left( \frac{k^2}{m} - E_{2n} \right) u_2(k) = - \int_{0}^{\infty} \langle k, 2 | v | k' \rangle k'^2 dk' u_0(k') - \int_{0}^{\infty} \langle k, 2 | v | k' \rangle k'^2 dk' u_2(k').
\] (4.34)

The full deuteron wave function can be obtained by using Eq. (4.4).

The current operator \( I^+(0) \) is the sum of a one-body part, \( I^+_p(0) \) and \( I^+_n(0) \), and a two-body exchange current contribution \( I^+_{ex}(0) \):

\[
I^+(0) = I^+_p(0) + I^+_n(0) + I^+_{ex}(0).
\] (4.35)

Expressions for the single-nucleon current matrix elements are given by Eq. (3.34) and the exchange current matrix elements are the sum of the four exchange current terms (3.128)- (3.132).

In the impulse approximation, the currents are the sum of the proton and neutron currents:

\[
I^+_{im}(0) = I^+_p(0) + I^+_n(0).
\] (4.36)

The form of the non-covariant single nucleon current matrix elements are given in Eq. (3.34). So the deuteron current matrix element (4.1) in the impulse approximation
can be expressed in the following form in terms of the full deuteron wave function:

\[
\langle \tilde{P}_d', \mu_d' | I_{im}^+(0) | \tilde{P}_d, \mu_d \rangle = \sum \int d\tilde{P}'_d d\tilde{k} d\tilde{p}_p d\tilde{p}_p' d\tilde{p}_n d\tilde{p}_n' \Psi_{\tilde{P}'_d, \mu_d'}(\tilde{p}_p', \tilde{p}_n', \mu_p', \mu_n') (4.37)
\]

\[
\times \langle \tilde{p}_p', \mu_p', \tilde{p}_n', \mu_n' | I_{im}^+(0) | \tilde{p}_p, \mu_p, \tilde{p}_n, \mu_n \rangle \delta^2(\tilde{P}' - \tilde{P}_d) \delta^2(\tilde{P} - \tilde{P}_d)
\]

For elastic scattering, the momentum is transferred to one of the nucleons with the other being a spectator. For the case of the deuteron, there is a symmetry under interchanging of the neutron and the proton. Thus it is enough to calculate the current matrix elements for one case and obtain the total impulse deuteron current matrix elements by doubling the results using the isoscalar nucleon form factor in the single nucleon current matrix elements. Assume the neutron is the spectator, the deuteron impulse current matrix elements can be written in the following form by using the combination of Eq. (3.34) (3.35) (3.36) (4.1) (4.4):

\[
\langle \tilde{P}_d', \mu_d' | I_{im}^+(0) | \tilde{P}_d, \mu_d \rangle = \sum \int d\tilde{P}'_d d\tilde{k} d\tilde{p}_p d\tilde{p}_p' d\tilde{p}_n d\tilde{p}_n' \Psi_{\tilde{P}'_d, \mu_d'}(\tilde{p}_p', \tilde{p}_n', \mu_p', \mu_n') (4.38)
\]

\[
\times \delta^2(\tilde{P}' - \tilde{P}_d) \delta^2(\tilde{P} - \tilde{P}_d)
\]

\[
\times \sqrt{m_p} \tilde{f}_p(\tilde{p}_p', \mu_p') (\gamma^\alpha F_1(Q^2) + \frac{1}{2} \gamma^\beta \frac{q_\beta}{2m} F_2(Q^2)) u_f(\tilde{p}_p', \mu_p') \sqrt{m_p} + \tilde{p}_p' \tilde{p}_n' M_0
\]

\[
\times \sum \frac{\omega_{m_p}(\tilde{k'}) \omega_{m_n}(\tilde{k}) \tilde{P}_{P_p} \tilde{P}_{P_n} M_0}{\tilde{P}_{P_p} \tilde{P}_{P_n} M_0} \sqrt{\omega_{m_p}(\tilde{k}) \omega_{m_n}(\tilde{k}) \tilde{P}_{P_p} \tilde{P}_{P_n} M_0}
\]

\[
\times D_{\mu_p, \mu_p'}^{\tilde{P}_{P_p}, \tilde{P}_{P_n}}[R_f(\tilde{k'})] D_{\mu_n, \mu_n'}^{\tilde{P}_{P_p}, \tilde{P}_{P_n}}[R_{f_c}(\tilde{k})] D_{\mu_n, \mu_n'}^{\tilde{P}_{P_p}, \tilde{P}_{P_n}}[R_{f_c}(\tilde{k})]
\]

\[
\times \langle s_p', \mu_p', s_n', \mu_n' | l', \mu_l, s', \mu_s | j', \mu_j \rangle \langle s_p, \mu_p, s_n, \mu_n | l, \mu_l, s, \mu_s \rangle u_l(k) u_l(k)
\]

\[
\times Y_{\mu_l}^{l'}(\tilde{k}(\tilde{p}_p, \tilde{p}_n)) \tilde{Y}_{\mu_i}^{l'}(\tilde{k}(\tilde{p}_p, \tilde{p}_n)) u_l(k) u_l(k).
\]
Here \( u_f(\bar{p}, s) \) and \( \bar{u}_f(\bar{p}, s) \) are the front-form spinors given in Appendix B. The empirical nucleon form factors, the Dirac form factor \( F_1(Q^2) \) and the Pauli form factor \( F_2(Q^2) \), depend only on the four-momentum transfer. So they can be factored out in calculation of the deuteron current matrix elements and can be included as an overall factor when we calculate the deuteron form factors and structure functions.

The integration in Eq. (4.38) can be carried out on either the initial or the final variables. Front-form kinematics relates the initial state variables to the final state variables. The Delta functions in Eq. (4.38) reduce the integration to a single three-dimensional momentum variable, which can be expressed in terms of the relative momentum \( k \) or \( k' \). If the final relative momentum \( k' \) is chosen to be the integration variable, Eq. (4.38) becomes the following:

\[
\langle \bar{P}_d', \mu_d' | I^+_{p} (0) | \bar{P}_d, \mu_d \rangle = \sum \int d\mathbf{k}' \frac{1}{|\partial(\bar{P}_d', \mathbf{k}')/\partial(\bar{P}_d, \mathbf{k})|^{\frac{1}{2}} |\partial(\bar{P}_p', \mathbf{p}')/\partial(\bar{P}_p, \mathbf{p})|^{\frac{1}{2}}} \\
\times \sqrt{\frac{m}{p_p'}} u_f(\bar{p}_p', \mu_p') (\gamma^\alpha F_1(q^2) + \frac{1}{2} [\gamma^\alpha, \gamma^\beta] q^\beta/2m) F_2(q^2) u_f(\bar{p}_p, \mu_p) \sqrt{\frac{m}{p_p}} \\
\times D^s_{\mu_p' \mu_p} [R_{ef}(\mathbf{k})] D^{s*}_{\mu_n' \mu_n} [R_{ef}(-\mathbf{k})] D^s_{\mu_p' \mu_p} [R_{f c}(\mathbf{k})] D^{s*}_{\mu_n' \mu_n} [R_{f c}(-\mathbf{k})] \\
\times \langle s'_p, \bar{s}_n', \bar{s}_n' | s', \mu_s' \rangle \langle \bar{t}', \mu_t', \bar{t}_s' | j', \mu_j' \rangle \langle s_p, \bar{s}_p, s_n | s, \mu_s \rangle \langle \bar{t}_p, \mu_t, \bar{t}_n \rangle u_l(k') u_l(k),
\]

where the overall jacobian is:

\[
\left|\frac{\partial(\bar{p}_p', \mathbf{p}_n')}{\partial(\bar{P}_d', \mathbf{k}')}\right|^{-\frac{1}{2}} \left|\frac{\partial(\bar{P}_d', \mathbf{k}')}{\partial(\bar{P}_d, \mathbf{k})}\right|^{-\frac{1}{2}} = \sqrt{\frac{\omega_m(\mathbf{k})\omega_m(-\mathbf{k}) p_p'^+ p_n'^+ P + M_0^*}{\omega_m(\mathbf{k})\omega_m(-\mathbf{k}) p_p'^+ p_n'^+ P + M_0}} = \sqrt{\frac{M_0'}{M_0^*}}.
\]
Note that the square root factors occurring in the Clebsh-Gordan coefficients are already included in the overall jacobian. We could also integrate on the integration relative momentum $k$ with the jacobian replaced by

$$
\frac{\partial (\vec{p}_p, \vec{p}_n)}{\partial (\vec{P}, \vec{k})} \frac{1}{2} \frac{\partial (\vec{P}', \vec{k}')}{\partial (\vec{p}_p', \vec{p}_n')} = \sqrt{\frac{M_0'}{M_0}}.
$$

(4.41)

In this work, I carry out the integration on the final relative momentum $k'$ which is decomposed into integrals on the magnitude of $k$ and the angle part $\theta$ and $\phi$:

$$
\int dk \rightarrow \int_0^\infty k^2 dk \int_0^\pi \sin(\theta) d\theta \int_0^{2\pi} d\phi
$$

(4.42)

The integral on the magnitude $k$ on the range of $(0, \infty)$ was mapped to $(0, 1)$ using a special quadrature. I make the following transformation for the integral variable:

$$
k \rightarrow x = \frac{\alpha^2}{k^2 + \alpha^2}, \quad dk \rightarrow dx = -\frac{2k}{(\alpha + k^2)^2} dk,
$$

(4.43)

$$
k = \sqrt{\frac{\alpha - x}{x}}, \quad dk = \frac{1}{2x^2 \sqrt{\alpha - x/x}} dx.
$$

(4.44)

The integration on the new variable $x$ can be used to carry out the integration on $k$:

$$
\int_0^1 x \frac{\alpha^2 dx}{2x^2} f(k(x)) = \int_0^\infty kdkf(k).
$$

(4.45)

We do the integral using Gauss-Legendre points $x_i$ and the corresponding weights $dx_i$ on the interval $(0, 1)$ and then use relation (4.44) to obtain the abscissa and weights,
$k_i$ and $dk_i$. The integral on $k$ then becomes

$$\int_0^\infty k^2 dk f(k) \approx \sum k_i^2 dk_i f(k_i). \quad (4.46)$$

Here $\alpha$ is a scaling constant which controls the density of grid points. $\alpha = 1.0$ gives good convergence. The integral on the angle part is carried out by using evenly distributed integral points and weights on the domain of integration.

The term with the proton as the spectator is the same as in the case where the neutron is the spectator. The total impulse deuteron current matrix elements can be obtained from the term with the proton as the spectator by multiplying the nucleon isoscalar form factor $F_1(Q^2)$ and $F_2(Q^2)$. 
CHAPTER 5

“PAIR” CURRENTS

The “pair” current contribution to the full current matrix element (4.1) has the following form:

\[
\langle \tilde{P}_d', \mu'_d | I_{ex}^+(0) | \tilde{P}_d, \mu_d \rangle
\]

\[
= \sum \int d\tilde{p}_p' d\tilde{p}_n ' \Psi_{\tilde{P}_d', \mu_d'} (\tilde{p}_p', \tilde{p}_n', \mu'_p, \mu'_n)
\times \langle \tilde{p}_p', \mu_p, \tilde{p}_n', \mu_n' | I_{ex}^+(0) | \tilde{p}_p, \mu_p, \tilde{p}_n, \mu_n \rangle
\times d\tilde{p}_p d\tilde{p}_n \Psi_{\tilde{P}_d, \mu_d} (\tilde{p}_p, \tilde{p}_n, \mu_p, \mu_n).
\]

It has the same structure as the impulse contributions (4.37) except that the single-nucleon current kernel is replaced by the “pair” current kernel, which is the sum of Eq. (3.128)-Eq. (3.132). The contribution from the first and the third term in (5.1) are identical up to the nucleon form factors. Similarly, the contribution from the second and fourth term in (5.1) are identical and the adjoint of the first and third terms.

The calculation of the exchange current matrix elements requires the specific form of \( V_{OPE}(|k|) \), which is the isospin dependent tensor component of the pion-exchange interaction. Schiavilla, Pandharipande and Riska [60] outlined a method to construct current operators satisfying the continuity equation while we satisfy both covariance and current conservation by using independent matrix elements. According to their convention, the spin amplitudes of the tensor and spin-spin components of the model
nucleon-nucleon potential in momentum space are defined as:

\[ \tilde{\Omega}_T = \sigma_1 \cdot \sigma_2 k^2 - 3 \sigma_1 \cdot k \sigma_2 \cdot k, \quad (5.2) \]

\[ \tilde{\Omega}_{SS} = k^2 \sigma_1 \cdot \sigma_2. \quad (5.3) \]

They correspond to the operator \((\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2)\) and \(S_{12}(\tau_1 \cdot \tau_2)\) in Argonne V18 NN-interaction \((4.14)\) respectively, except for the factor \(\tau_1 \cdot \tau_2\), which is not included in Eq. \((5.2)\) \((5.3)\). The operators associated with the pseudoscalar and pseudovector potentials are \(\sigma_1 \cdot k \sigma_2 \cdot k\) and \((\sigma_1 \times k) \cdot (\sigma_2 \times k)\), corresponding to the pion and \(\rho\)-meson exchange respectively. They can be written in terms of the tensor and spin-spin operators:

\[ \sigma_1 \cdot k \sigma_2 \cdot k = -\frac{1}{3}(\tilde{\Omega}_T - \tilde{\Omega}_{SS}), \quad (5.4) \]

\[ (\sigma_1 \times k) \cdot (\sigma_2 \times k) = \frac{1}{3}[2\tilde{\Omega}_{SS} + \tilde{\Omega}_T]. \quad (5.5) \]

In order to extract the tensor component of the pion-exchange potential, we separate the coefficients of \(\Omega_T\) and \(\Omega_{SS}\), \(v_T(|k|)\) and \(v_{SS}(|k|)\), into contributions from pion and \(\rho\)-meson exchange interactions:

\[ v_T(|k|) = v_T^\pi(|k|) + v_T^\rho(|k|), \quad (5.6) \]

\[ v_{SS}(|k|) = v_{SS}^\pi(|k|) + v_{SS}^\rho(|k|). \quad (5.7) \]
Eq. (5.4) and (5.5) indicate:

\[ v_{SS}^\pi(|k|) = -v_{T}^\pi(|k|), \quad (5.8) \]

\[ v_{SS}^\rho(|k|) = 2v_{T}^\rho(|k|). \quad (5.9) \]

We first use these relations to eliminate the spin-spin contributions and get:

\[ v_{SS}(|k|) = -v_{T}^\pi(|k|) + 2v_{T}^\rho(|k|). \quad (5.10) \]

Then we use Eq. (5.6)-(5.10) to eliminate the \( \rho \)-meson contribution and obtain the tensor component of the pion-exchange potential:

\[ v_{T}^\pi(|k|) = \frac{1}{3}(2v_{T}(|k|) - v_{SS}(|k|)). \quad (5.11) \]

This is exactly the same as Eq. (2.10) obtained by Schiavilla, Pandharipande and Riska in reference [64]. Then in [65] Schiavilla, Pandharipande and Riska use this relation to make the following replacement:

\[ \frac{f_{\pi}^2}{3m_{\pi}^2} \frac{1}{m_{\pi}^2 + k^2} \rightarrow \frac{1}{3}(2v_{T}(|k|) - v_{SS}(|k|)). \quad (5.12) \]

This potential, extracted from the Argonne V18 interaction, is also used in calculation of the “pair” current matrix elements. The one-pion-exchange potential defined in
Eq. (3.101) now becomes:

\[ V_{OPE}(|k|) \rightarrow \frac{1}{(2\pi)^3} (2v^{\sigma\tau}(|k|) - v^{\sigma\tau}(|k|)). \] (5.13)

In this procedure, the isospin-dependent factor \( \tau_1 \cdot \tau_2 \), which is 3 for deuteron, has been factored out. Here \( v_{SS}(|k|) \) and \( v_T(|k|) \) are identified as the isospin dependent spin-spin and tensor components, \( v^{\sigma\tau}(|k|) \) and \( v^{t\tau}(|k|) \) respectively, in the Argonne V18 NN-interaction. They are the Fourier transformation of the corresponding radial function of the potential in r-space \[64] \[75], \( v^{\sigma\tau}(r) \) and \( v^{t\tau}(r) \). Note that \( k \) is used in Eq. (5.2) and (5.3) while in configuration space, unit vector \( \hat{r} \) is used in Eq. (4.14) and (4.15). Thus a factor of \( 1/k^2 \) is required when we Fourier transform the r-space potential \[64]:

\[ v^{\sigma\tau}(|k|) = \frac{4\pi}{|k|^2} \int_0^\infty dr r^2 v^{\sigma\tau}(r)(j_0(|k|r) - 1), \] (5.14)

\[ v^{t\tau}(|k|) = \frac{4\pi}{|k|^2} \int_0^\infty dr r^2 v^{t\tau}(r)j_2(|k|r). \] (5.15)

The subtraction in Eq. (5.14) ensures that \( |k|^2 v(|k|) \) vanishes when \( |k| = 0 \). In the Argonne V18 potential, the radial function in configuration space is defined as follows:

\[ v_4(r) \equiv \frac{f_\pi^2}{3} \left\{ \left( \frac{m_0}{m_\pm} \right)^2 m_0 Y(\mu_0, r) + 2m_\pm Y(\mu_\pm, r) \right\} / 3 + v^c(r), \] (5.16)

\[ v_6(r) \equiv \frac{f_\pi^2}{3} \left\{ \left( \frac{m_0}{m_\pm} \right)^2 m_0 T(\mu_0, r) + 2m_\pm T(\mu_\pm, r) \right\} / 3 + v^t(r), \] (5.17)

where the coupling constant \( f_\pi^2/4\pi = 0.075 \) and \( m_0 \) and \( m_\pm \) are the masses of the \( \pi_0 \)
and \( \pi \pm \). The corresponding \( \mu' \)'s are obtained by dividing by the factor \( \hbar c \). \( Y(\mu, r) \) and \( T(\mu, r) \) are the Yukawa and tensor functions with the exponential cutoff parameter \( c = 2.1 fm \):

\[
Y(\mu, r) = \frac{e^{-\mu r}}{\mu r} \left( 1 - e^{-cr^2} \right),
\]

(5.18)

\[
T(\mu, r) = \left( 1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2} \right) Y(\mu, r)(1 - e^{-cr^2}).
\]

(5.19)

\( v^e(r) \) and \( v^t(r) \) in Eq. (5.16) and (5.17) are the short- and intermediate-range potential which are supposed to be eliminated by using (5.13) to obtain the one-pion-exchange potential. To examine the effects of the short- and intermediate-range potential in the “pair” current contribution, the effective one-pion-exchange potential has also been calculated with \( v^e(r) \) and \( v^t(r) \) removed from Eq. (5.16) and (5.17).

The strategy used to calculate the “pair” current matrix elements is to factor the “pair” current kernel into a product of a one-body and two-body operator. The one-body part is treated as a modified impulse current and the two-body part modifies the deuteron wave function to generate a “pseudo wave function”. Then the “pair” current matrix elements will have the same form as the impulse approximation (4.38) when the single nucleon current is replaced by the modified current and one of the deuteron wave functions, either initial or final, is replaced by the pseudo wave function.

Note that the four exchange current matrix elements, Eq. (3.128)- (3.132), have similar operator structures. They can be factored into two parts:

\[
\sqrt{\frac{m}{\omega(p')}} \bar{u}_c(p', \mu') \gamma^5 \beta u_c(p'', \mu'') \sqrt{\frac{m}{\omega(k'')}} \quad \text{(5.20)}
\]
or
\[ \sqrt{\frac{m}{\omega(k'')}} \bar{u}_c(p'', \mu') \beta \gamma^5 K^\mu(q) u_c(p, \mu) \sqrt{\frac{m}{\omega(p)}} \] (5.21)

and
\[ (k'' - k) \cdot \sigma V_{OPE}(|k'' - k|), \] (5.22)
\[ V_{OPE}(|k' - k''|)(k' - k'') \cdot \sigma. \]

The first part, Eq. (5.20) and (5.21), is similar to the single nucleon current matrix elements, Eq. (3.34) in the impulse approximation, except for the factors of \( \gamma^5 \) and \( \beta \). We call this part the “equivalent impulse current”. The second part, Eq. (5.22), is used to modify the initial or final deuteron wave function in the rest frame to generate the pseudo wave function. The variables and spins need to be transformed into front-form variables. We do this by defining the operator in the initial or final state in laboratory frame, imposing a light front symmetry, and then defining it in other frames using light-front boosts. Take the first term (3.128) as an example. We apply the operator \((k'' - k) \cdot \sigma V_{OPE}(|k'' - k|)\) in (5.22) to the initial deuteron wave function (4.4) and call it the pseudo wave function:

\[ \chi_{F_d,\mu_d}^{(c)}(k'', \mu_{p''}, \mu_{n''}) \] (5.23)
\[ \equiv \int d\mathbf{k}(k'' - k) \cdot \langle \mu_{n'} | \sigma | \mu_{n} \rangle_c V_{OPE}(|k'' - k|) \]
\[ \times \langle s_p, \mu_{p''}, s_n, \mu_{n} | s, \mu_s \rangle \langle s, \mu_s, l, \mu_l | j, \mu \rangle Y_{\mu l}^j(\hat{k}) u_l(k). \]
Note that the interaction kernel is a scalar so that the pseudo wave function still has spin-1. We change the variables to front-from variables and replace the canonical spinors by front-form spinors, inserting a Melosh rotation and its inverse between the spinors. The pseudo wave function (5.23) becomes the following in terms of light-front spins:

\[
\chi^{(f)}_{\tilde{P}_d, \mu_d} (k'', \mu''_p, \mu'_n) \\
\equiv D^{s_p}_{\mu'_p \tilde{\mu}'_p} [R_{fc}(k'')] D^{s_n}_{\mu'_n \tilde{\mu}'_n} [R_{fc}(-k'')] \chi^{(c)}_{\tilde{P}_d, \mu_d} (k'', \tilde{\mu}'_p, \tilde{\mu}'_n) \\
= D^{s_p}_{\mu'_p \tilde{\mu}'_p} [R_{fc}(k'')] D^{s_n}_{\mu'_n \tilde{\mu}'_n} [R_{fc}(-k'')] \\
\times \int d\vec{k}(k'' - \vec{k}) \cdot \langle \tilde{\mu}'_n | \sigma | \tilde{\mu}''_n \rangle V_{OP}(|k'' - \vec{k}|) \\
\times \langle s_p, \tilde{\mu}'_p, s_n, \tilde{\mu}'_n | s, \mu_s \rangle \langle s, \mu_s, l, \mu_l | j, \mu \rangle Y^l_{\mu_l}(\hat{k}) u_l(k),
\]

where the canonical nucleon spins are labeled by a bar on top.

Since we assumed that the initial frame was a rest frame, we can use kinematic covariance to define the current matrix elements in any other frame. We use a front-form boost to transform the system to the Breit frame. Note that the effective current, Eq. (5.21), which looks like a one-body current is the usual one-body current multiplied by \( \gamma_5 \beta \). The factor \( \beta \) is not covariant. Kinematic covariance requires that it is replaced by \(-P \cdot \gamma/M_d\) in the Breit frame. This reduces to \( \beta \) in the rest frame of the deuteron. In the Breit frame the total initial and final momenta are:

\[
\tilde{P} = (\sqrt{M_d^2 + \frac{q^2}{4}}, -\frac{q}{2}, 0)
\]
and

\[ \tilde{P}' = \left( \sqrt{M_d^2 + \frac{q^2}{4}}, \frac{q}{2}, 0 \right). \]  

(5.26)

The related front-form boost is:

\[ L_f(q) = \begin{pmatrix} \sqrt{1 + \frac{q^2}{4M_d^2}} & 0 \\ -\frac{q}{2M_d}/\sqrt{1 + \frac{q^2}{4M_d^2}} & 1/\sqrt{1 + \frac{q^2}{4M_d^2}} \end{pmatrix}. \]  

(5.27)

If we define

\[ \hat{P} \equiv \frac{P}{M_d} = \left( \sqrt{1 + \frac{q^2}{4M_d^2}}, -\frac{q}{2M_d}, 0, 0 \right) \equiv \eta^\mu, \]  

(5.28)

then in the Breit frame

\[ -\frac{P \cdot \gamma}{M_d} = -\hat{P} \cdot \gamma = \eta^0 \gamma^0 - \eta^1 \gamma^1. \]  

(5.29)

The equivalent impulse current (5.20) now becomes:

\[ \langle \vec{p}_p', \vec{\mu}_p' | I_{1E_p}^\mu | \vec{p}_p'', \vec{\mu}_p'' \rangle \]  

\[ \equiv \sqrt{\frac{m}{p_p'}} \bar{u}_f(\vec{p}_p', \vec{\mu}_p') K_p^\mu(q) \gamma^5 (\eta^0 \gamma^0 - \eta^1 \gamma^1) u_f(\vec{p}_p'', \vec{\mu}_p'') \sqrt{\frac{m}{p_p''}}. \]  

(5.30)

When the neutron is the spectator, the final and initial neutron spins are the same, thus the first term of the exchange current contribution can be written in the following form:
\[
\langle \hat{P}_d', \mu_d'| I_{ex1}^+(0) | \hat{P}_d, \mu_d \rangle 
= \sum \int dk'dk'' \chi_{P,\mu}^*(k', \mu'_p, \mu_n) \langle \hat{P}_p', \mu_p' | I_{E_p}^+(0) | \hat{P}_p'' \rangle \chi^{(f)}_{\hat{E}_{d,\mu_d}}(k''', \mu''_p, \mu'_n) 
\]
\[
= \sum \int dk'dk'' dk \frac{\partial (\hat{P}_p', k')}{\partial (\hat{P}_p, k''')} \left| \frac{1}{2} \frac{\partial (\hat{P}_p', k''')}{\partial (\hat{P}_p'' \hat{P}_n''')} \right| \frac{1}{2} \left| \frac{\partial (\hat{P}_p, k'')}{\partial (\hat{P}_p' \hat{P}_n'')} \right| \frac{1}{2} \sqrt{\frac{m}{p_p'}} \sqrt{\frac{m}{p_p''}} \times \\
\times \bar{u}_f(\hat{P}_p', \mu_p') (\eta_0^0 \gamma - \eta_1^1 \gamma) \gamma_5 (\gamma \alpha_1 F_1(q^2) + \frac{1}{2} [\gamma \alpha, \gamma \beta \frac{|q_\beta|}{2m} F_2(q^2))] u_f(\hat{P}_p', \mu_p') 
\times D_{\mu_p' \mu_n'}^{s_{k'}}(R_{cf}(k')) D_{\mu_p \mu_p}^{s_{k''}}(R_{cf}(-k'')) D_{\mu_n' \mu_n}^{s_{k'''}}(R_{cf}(-k''')) 
\times \langle s_p', \mu_p', s_n', \mu_n | s, \mu_s | \langle l, \mu_l | j, \mu_l \rangle 
\times Y_{\mu_l^*}(\hat{k}(\hat{P}_p', \hat{P}_n')) Y_{\mu_l}(\hat{k}(\hat{P}_p, \hat{P}_n)) u_l(k') u_l(k) 
\]

The adjoint term can be derived by the same method:

\[
\langle \hat{P}_d', \mu_d'| I_{ex2}^+(0) | \hat{P}_d, \mu_d \rangle 
= \sum \int dk'' dk \chi_{P,\mu}^*(k'', \mu'_p, \mu_n) \langle \hat{P}_p', \mu_p' | I_{E_p}^+(0) \rangle \chi^{(f)}_{\hat{E}_{d,\mu_d}}(k'', \mu''_p, \mu'_n) 
\]
\[
= \sum \int dk'dk'' dk \frac{\partial (\hat{P}_p', k'')}{\partial (\hat{P}_p, k'')} \left| \frac{1}{2} \frac{\partial (\hat{P}_p', k'')}{\partial (\hat{P}_p'' \hat{P}_n'')} \right| \frac{1}{2} \left| \frac{\partial (\hat{P}_p, k'')}{\partial (\hat{P}_p' \hat{P}_n'')} \right| \frac{1}{2} \sqrt{\frac{m}{p_p'}} \sqrt{\frac{m}{p_p''}} \times \\
\times \bar{u}_f(\hat{P}_p', \mu_p') (\eta_0^0 \gamma - \eta_1^1 \gamma) \gamma_5 (\gamma \alpha_1 F_1(q^2) + \frac{1}{2} [\gamma \alpha, \gamma \beta \frac{|q_\beta|}{2m} F_2(q^2))] u_f(\hat{P}_p', \mu_p') 
\times D_{\mu_p' \mu_n'}^{s_{k}}(R_{cf}(k'')) D_{\mu_p \mu_p}^{s_{k''}}(R_{cf}(k'')) D_{\mu_n' \mu_n}^{s_{k'''}}(R_{cf}(-k''')) 
\times \langle s_p', \mu_p', s_n', \mu_n | s, \mu_s | \langle l, \mu_l | j, \mu_l \rangle 
\times Y_{\mu_l^*}(\hat{k}(\hat{P}_p', \hat{P}_n')) Y_{\mu_l}(\hat{k}(\hat{P}_p, \hat{P}_n)) u_l(k') u_l(k). 
\]
The momentum transfer is $q = p_p' - p_p''$ and $q = p_p'' - p_p$ for term (5.31) and (5.32) respectively.

Terms with the neutron as the active particle can also be obtained by the same method and the whole exchange current contribution is the sum of the four terms. By taking advantage of the symmetry, results are obtained by doubling the sum of term (5.31) and term (5.32), and replacing $F_1(q^2)$ and $F_2(q^2)$ by isoscalar nucleon form factors.

This is a seven-dimension integration, which is difficult to perform directly. We use a partial wave expansion to carry out the angular part of the integral, which results in a significant saving of computer time. We expand the pseudoscalar potential, $V_{OPE}(|k - k''|)$, in terms of partial waves as follows:

$$V_{OPE}(|k - k''|) = \sum_{lm} V_l(k, k'')Y^*_{lm}(\hat{k})Y_{lm}(\hat{k}''), \quad \text{(5.33)}$$

where

$$V_l(k, k'') = 2\pi \int_{-1}^{1} P_l(u)V_{OPE}(\sqrt{k^2 + k''^2 - 2kk''u})du. \quad \text{(5.34)}$$

Note that the quantity $V_l(k, k'')$ is a scalar depending only on the magnitude of $k$ and $k''$. The vectors $k$, $k''$ and $k \cdot k''$ can be expanded in spherical harmonics $Y_{1m}$.

$$k = k(\sqrt{\frac{2\pi}{3}}(Y_{1-1}(\hat{k}) - Y_{11}(\hat{k})), i\sqrt{\frac{2\pi}{3}}(Y_{1-1}(\hat{k}) + Y_{11}(\hat{k})), \sqrt{\frac{4\pi}{3}}Y_{10}(\hat{k})). \quad \text{(5.35)}$$
Eq. (5.23) becomes:

\[
\chi^{(c)}_{\tilde{P}_d,\mu_d} (\mathbf{k}''', \mu_p', \mu_n') \equiv \int \hat{k}^2 dk \sum_{l=0,2} \sum_{\mu_l=-l}^{l} \sum_{l'=0}^{\infty} \sum_{\mu_l'=-l'} V_{l'}(k,k''') Y_{l'}(\hat{k}'') Y_{l'}^{*}(\hat{k}) u_{l}(k) \]

\[
\times \left( k'' \left( \sqrt{\frac{2\pi}{3}} (Y_{l-1}(\hat{k}'') - Y_{l+1}(\hat{k}'')) , i \sqrt{\frac{2\pi}{3}} (Y_{l-1}(\hat{k}) + Y_{l+1}(\hat{k})), \sqrt{\frac{4\pi}{3}} Y_{10}(\hat{k}) \right) \right)
\]

\[
- k \left( \sqrt{\frac{2\pi}{3}} (Y_{l-1}(\hat{k}) - Y_{l+1}(\hat{k})), i \sqrt{\frac{2\pi}{3}} (Y_{l-1}(\hat{k}) + Y_{l+1}(\hat{k})), \sqrt{\frac{4\pi}{3}} Y_{10}(\hat{k}) \right) \right]
\]

\[
\cdot \langle \mu'_n | \sigma | \mu_n \rangle \langle s_p, \mu'_p, s_n, \mu_n | s, \mu_s \rangle \langle s, \mu_s, l, \mu_l | j, \mu \rangle Y_{l}^{*}(\hat{k}) u_{l}(k).
\]

The following identities are used to perform the integral over angles:

\[
\int \hat{k}^2 dk \sum_{l=0}^{2} \sum_{\mu_l=-l}^{l} \sum_{l'=0}^{\infty} \sum_{\mu_l'=-l'} Y_{l'}(\hat{k}) Y_{l'}^{*}(\hat{k}) = \delta_{l,l'} \delta_{\mu_l,\mu_l'} \tag{5.37}
\]

\[
\int \hat{k}^2 dk \sum_{l=0}^{2} \sum_{\mu_l=-l}^{l} \sum_{l'=0}^{\infty} \sum_{\mu_l'=-l'} Y_{l'}(\hat{k}) Y_{l'}^{*}(\hat{k})
\]

\[
= \sqrt{\frac{(2 \cdot 1 + 1)(2 \cdot l + 1)}{4\pi(2 \cdot l' + 1)}} \langle 1, \mu''_l, l, \mu_l | l', \mu_l' \rangle \langle 1, 0, l | 0, l' \rangle. \tag{5.38}
\]

The expression for the pseudo wave function is the following after carrying out the integral on \( \hat{k} \):

\[
\chi^{(c)}_{\tilde{P}_d,\mu_d} (\mathbf{k}''', \mu_p', \mu_n') \equiv \int k^2 dk \sum_{l=0,2} \sum_{\mu_l=-l}^{l} \sum_{l'=0}^{\infty} \sum_{\mu_l'=-l'} V_{l'}(k,k''') Y_{l'}(\hat{k}'') Y_{l'}^{*}(\hat{k}) u_{l}(k)
\]

\[
\times \langle s_p, \mu'_p, s_n, \mu_n | s, \mu_s \rangle \langle s, \mu_s, l, \mu_l | j, \mu \rangle.
\]

\[
\tag{5.39}
\]
\begin{align*}
&\times \left[ \frac{2\pi}{3} \langle \mu'_n | \sigma_x | \mu_n \rangle [k''(Y_{1-1}(\hat{k}'') - Y_{11}(\hat{k}'')) \delta_{l'l''} \delta_{\mu'_l \mu_l} \\
&+ \ k \sqrt{\frac{(2 \cdot 1 + 1)(2 \cdot l + 1)}{4\pi(2 \cdot l' + 1)}} \langle 1, 0, l, 0| l', 0 \rangle (-\langle -1, -1, l, \mu_l | l', \mu_{l'} \rangle + \langle 1, 1, l, \mu_l | l', \mu_{l'} \rangle) ] \\
&+ \ i \sqrt{\frac{2\pi}{3}} \langle \mu'_n | \sigma_y | \mu_n \rangle [k''(Y_{1-1}(\hat{k}'') + Y_{11}(\hat{k}'')) \delta_{l'l''} \delta_{\mu'_l \mu_l} \\
&+ \ k \sqrt{\frac{(2 \cdot 1 + 1)(2 \cdot l + 1)}{4\pi(2 \cdot l' + 1)}} \langle 1, 0, l, 0| l', 0 \rangle (-\langle -1, -1, l, \mu_l | l', \mu_{l'} \rangle - \langle 1, 1, l, \mu_l | l', \mu_{l'} \rangle) ] \\
&+ \ \sqrt{\frac{4\pi}{3}} \langle \mu'_n | \sigma_z | \mu_n \rangle [k'' Y_{10}(\hat{k}'') \delta_{l'l''} \delta_{\mu'_l \mu_l} \\
&- \ k \sqrt{\frac{(2 \cdot 1 + 1)(2 \cdot l + 1)}{4\pi(2 \cdot l' + 1)}} \langle 1, 0, l, 0| l', 0 \rangle \langle 1, 0, l, \mu_l | l', \mu_{l'} \rangle ] \right].
\end{align*}

The properties of the Clebsch-Gordan coefficients and $\langle 1, 0, 2, 0| 2, 0 \rangle = 0$ limit the $l'$ values that survive after the angular integrals. It can only be 0 or 1 for $l = 0$ and 1, 2 or 3 when $l = 2$. This limits the infinite sum on $l'$ to the first four partial waves, $l' = 0, 1, 2, 3$. Note that the $l = 3$ partial wave appears because we are using $I^+(0)$ rather than $I^0(0)$. The final expression of the pseudo wave function can be written as follows:

$$
\chi_{P_{\theta, \phi \psi}}^{(c)}(k'', \mu''_p, \mu'_n) = \sum_{l''} I_{l''}(k'') f_{l''}(\hat{k}'', \mu''_p, \mu'_n),
$$

(5.40)

where $f_{l''}(\hat{k}'', \mu''_p, \mu'_n)$ are the angle dependent coefficients and $I_{l''}(k'')$ are scalar quantities:

$$
I_{l''}(k'') \equiv \int k^2 dk V_l(k, k'') u_{l''}(k)
$$

(5.41)

or

$$
I_{l''}(k'') \equiv \int k^3 dk V_l(k, k'') u_{l''}(k).
$$

(5.42)
All allowed combinations of $l$ and $l'$ pairs give the following integrals, $I_{ll'}(k'')$:

\begin{align*}
I_{00}(k'') &\equiv \int V_0(k, k'') u_0(k) k^2 dk, \\
I_{10}(k'') &\equiv \int V_1(k, k'') u_0(k) k^3 dk, \\
I_{12}(k'') &\equiv \int V_1(k, k'') u_2(k) k^3 dk, \\
I_{22}(k'') &\equiv \int V_2(k, k'') u_2(k) k^2 dk, \\
I_{32}(k'') &\equiv \int V_3(k, k'') u_2(k) k^3 dk.
\end{align*}

(5.43) \quad (5.44) \quad (5.45) \quad (5.46) \quad (5.47)

These integrals are carried out by using the basis function expansion. For each $ll'$ pair, $I_{ll'}(k)$ can be defined as follows:

\[ I_{ll'}(k'') \equiv \sum_n d_{n0}^{ll'} \phi_{n0}(k''). \]  

(5.48)

The coefficients $d_{n0}^{ll'}$ are

\[ d_{n0}^{ll'} = \int_0^\infty k'^2 dk' \phi_{n0}(k'') \int_0^\infty k^2 dk V_l(k, k') u_{l'}(k) \]  

(5.49)

and

\[ d_{n0}^{ll'} = \int_0^\infty k'^2 dk' \phi_{n0}(k'') \int_0^\infty k^3 dk V_l(k, k') u_{l'}(k) \]  

(5.50)

for 00 and 22 pairs and for 10, 12 and 32 pairs respectively. The basis functions $\phi_{n0}(k)$ are defined in the previous chapter.
By using Eq. (5.24), the pseudo wave function (5.39) can be transformed to front-form representation. The exchange current matrix elements (5.31) and (5.32) can be calculated. As discussed in the previous chapter, the integral on the single nucleon momenta are suppressed and front-form kinematics relates the initial and final state variables. Variables with double primes are integration variables. The jacobians have similar forms to (4.41) and (4.40) except that variables with double primes are replaced by the corresponding initial or final variables in impulse approximation:

\[
J_i = \left| \frac{\partial (\tilde{p}'', \tilde{p}_n''')} {\partial (\tilde{P}'', k'')} \right|^{\frac{1}{2}} \left| \frac{\partial (\tilde{P}', k')} {\partial (\tilde{p}', \tilde{p}_n')} \right|^{\frac{1}{2}} = \sqrt{\frac{M_0'''}{M_0''}},
\]

\[
J_f = \left| \frac{\partial (\tilde{p}_p'', \tilde{p}_n''')} {\partial (\tilde{P}_p'', k''')} \right|^{\frac{1}{2}} \left| \frac{\partial (\tilde{P}, k')} {\partial (\tilde{p}_p', \tilde{p}_n')} \right|^{\frac{1}{2}} = \sqrt{\frac{M_0}{M_0'''}}
\]

corresponding to integrals on the initial and final relative momenta respectively.

Using the final relative momenta to carry out the integral, the total exchange current contribution (5.1) can be written as:

\[
\langle \tilde{P}_d', \mu_d' | I_{\text{ex}}^+(0) | \tilde{P}_d, \mu_d \rangle = \sum \int d\mathbf{k}' J_i
\]

\[
\times \left[ \bar{\Psi}_{\tilde{P}_d, \mu_d}^{*}(\mathbf{k}', \mu_p', \mu_n) [\bar{u}_f(\tilde{p}_p', \mu_p) K_p^+(q) \gamma^5 (\eta^0 \gamma^0 - \eta^1 \gamma^1) u_f(\tilde{p}_p, \mu_p)] \chi^{(f)}_{\tilde{P}_d, \mu_d}(\mathbf{k}, \mu_p, \mu_n) \right]
\]

\[
+ \left[ \bar{\Psi}_{\tilde{P}_d, \mu_d}^{*}(\mathbf{k}, \mu_p', \mu_n) [\bar{u}_f(\tilde{p}_p', \mu_p) (\eta^0 \gamma^0 - \eta^1 \gamma^1) \gamma^5 K_p^+(q) u_f(\tilde{p}_p, \mu_p)] \Psi_{\tilde{P}_d, \mu_d}(\mathbf{k}, \mu_p, \mu_n) \right]
\]

\[
+ [p \leftrightarrow n].
\]

The integral over \( \mathbf{k} \) is decomposed into the magnitude and angle parts and calculated by the same quadrature used in the impulse approximation.
The exchange current contribution is added to the impulse current matrix elements to examine its effect on the deuteron form factors and structure functions.
CHAPTER 6

NUMERICAL RESULTS

The main task of this research is to calculate the deuteron structure functions, $A(Q^2)$, $B(Q^2)$, and $T_{20}(Q^2)$, by calculating independent deuteron current matrix elements, and using the constraints of current conservation and current covariance to generate the remaining current matrix elements. The independent current matrix elements have been calculated in the impulse approximation and with a model “pair” current contribution included, using Poincaré invariant quantum mechanics with a light-front symmetry. The deuteron wave functions are constructed using the realistic AV18 NN-interaction [75] and empirical nucleon form factors [22] [17] [44] [20] [50] [8] are used as input. We used eight nucleon form factor parameterizations to test the sensitivity of the deuteron form factors to the input nucleon form factors. The one-pion-exchange contribution to the Argonne V18 NN-interaction is extracted and used to generate the model “pair” current. The same “pair” current is also used with deuteron wave functions generated using the CDB [52] and N3LO [34] NN-interactions to test the sensitivity of the “pair” current contribution to different model NN-interactions that are fit to the same scattering data. While the deuteron form factors can be calculated consistently using independent current matrix elements, there are many combinations of independent current matrix elements. Different choices correspond to different conserved covariant current operators. To examine the sensitivity to the choice of independent current matrix elements, the current matrix elements generated using angular condition (3.48) are compared to the results obtained by direct calculation. We also examine the sensitivity
to the choice of independent current matrix elements by using different independent linear combinations of current matrix elements to calculate the deuteron form factors. The effect of the “pair” current contribution on the deuteron quadrupole and magnetic moments is examined. Finally, the results are compared to the work done by other groups using different models.

The deuteron S- and D-state wave functions of the Argonne V18 NN-interaction in momentum space are obtained by Fourier transforming the wave functions in configuration space. They are plotted in Figures (7.1) and (7.2) and compared to the wave functions constructed using the CDB [52] and N3LO [34] NN-interaction. While these potentials are all fit to the same scattering data, the corresponding deuteron wave functions are quite different. The differences show up for momenta above $1.5 \text{fm}^{-1}$ and $2.0 \text{fm}^{-1}$ for the S-state and D-state wave functions respectively. Note that the wave functions generated using the N3LO NN-interaction differ greatly from the wave functions generated using the other two NN-interactions in the high momentum range. This is because the high momentum contributions to the N3LO NN-interaction are suppressed. The D-state probability is 5.78% for Argonne V18, 4.82% for CDB and 4.48% for N3LO. These differences are related to different short-range features in the different interactions that do not affect the fit to scattering data.

A one-pion-exchange potential was extracted from the Argonne V18 NN-interaction. While the bare meson propagator is replaced by the corresponding one-pion-exchange potential with the short-range cutoffs used in the AV18 NN-interaction, additional short-range contributions are eliminated by using the method given in Eq. (5.13). We
examined this by removing the additional short-range interactions $v^c(r)$ and $v^l(r)$ in
Eq. (5.16) and (5.17). The effective one-pion-exchange potential constructed including
the short-range potential is labeled WS, and with the short-range potential removed
is labeled NS. They are plotted in Figures (7.3) and (7.4) and compared to the bare
pion propagator. We can see that the differences increase with respect to increasing
momenta. This indicates the presence of some short- and intermediate-range physics
remaining in the effective one-pion-exchange potential.

The five different parameterizations of the empirical nucleon form factors, labeled
the BBA [22], BI [17], Kelly [44], BBBA [20], Lomon [50] and three hybrid parameteri-
zations, labeled the Kelly-AMT4 [8], Kelly-BBBA and Kelly-BI, are described in detail
in Appendix C. To examine the sensitivity to these different parameterizations, the
proton and neutron electric and magnetic form factors of the first six parameterizations
are compared in Figures (7.5)–(7.10). The Kelly-AMT4 uses the same neutron parame-
terization as in Kelly so that the neutron form factors are the same as shown in Figures
(7.7), (7.8) and (7.10). Note that we use the parameters in Table 4 in Ref. [8] for the
proton parameterization in Kelly-AMT4, which are fit to the one-photon-exchange data
while all other parameterizations are fit to the two-photon-exchange data. Figure (7.5)
and (7.6) displays a large difference between the one-photon-exchange and the two-
photon-exchange data fit for the proton electric form factor. The differences in the
other parameterizations are mainly from the neutron electric form factor, which are
shown in Figures (7.7) and (7.8). The isoscalar electric and magnetic nucleon form
factors, $F_{1N}(Q^2)$ and $F_{2N}(Q^2)$, were used in this calculation to generate the deuteron
current matrix elements. Figures (7.11) and (7.12) show the difference of isoscalar nucleon form factors using different parameterizations. We can see that Kelly-AMT4 differs greatly from the other parameterizations in $F_{1N}(Q^2)$ while the difference is less obvious in $F_{2N}(Q^2)$. This indicates that the proton form factors mainly contribute to $F_{1N}(Q^2)$. The differences in the neutron electric form factors also contribute to the differences in $F_{1N}(Q^2)$. The difference between the BBBA and BI parameterizations represents the largest uncertainty. The differences in $F_{2N}(Q^2)$ are caused by the differences in the neutron and proton electric and magnetic form factors. The largest difference for $Q^2 > 3 GeV^2$ is between the BBBA and BI parameterizations in $F_{1N}(Q^2)$. From Figure (7.8), we can see that this difference is caused by the difference in the neutron electric form factors.

The CCKP [24] method was used to calculate the deuteron structure functions, in which the deuteron form factors $G_0(Q^2), G_1(Q^2), G_2(Q^2)$ are defined in terms of the independent deuteron current matrix elements given by Eq. (3.57)–(3.59). Figures (7.13)–(7.15) display the deuteron form factors, $G_0(Q^2), G_1(Q^2)$ and $G_2(Q^2)$. Curves are for the deuteron form factors calculated in the impulse approximation and with the “pair” current contribution included. To test the sensitivity to empirical nucleon form factors, the BBBA, BI and AMT4 parameterizations are used to calculate the deuteron form factors. The effective one-pion-exchange potential was constructed without removing the residual short-range potential. All three figures show that the deuteron form factors exhibit similar patterns. $G_0(Q^2)$ and $G_2(Q^2)$ exhibit sensitivities to the empirical nucleon form factors while $G_1(Q^2)$ is not very sensitive to them. The sensitivity of
$G_0(Q^2)$ and $G_2(Q^2)$ to the nucleon form factors decreases when the “pair” current contributions are included while it increases with increasing momentum transfer. The difference between the results using BBBA and BI parameterizations is mainly caused by the different neutron electric form factors. There is also a difference due to the uncertainty in the proton form factors, which is shown in Figure (7.13) and (7.15). In the impulse approximation, the position of the first zero in $G_0(Q^2)$ and $G_1(Q^2)$ is not sensitive to the nucleon form factors while the position of the second zero in $G_0(Q^2)$ and the first one in $G_2(Q^2)$ exhibit a large sensitivity to the nucleon form factors. The slope of the curves change very little when “pair” currents are included but the position of the first zero in $G_0(Q^2)$ and $G_1(Q^2)$ is sensitive to the nucleon form factors. These three figures also show that $G_2(Q^2)$ is much smaller than $G_0(Q^2)$ and $G_1(Q^2)$. These sensitivities in the nucleon form factors lead to sensitivities in the deuteron structure functions $A(Q^2)$, $B(Q^2)$ and $T_{20}(Q^2)$.

The deuteron structure functions $A(Q^2)$, $B(Q^2)$ and $T_{20}(Q^2)$ can be obtained using Eq. (1.4)- (1.6). Figure (7.16) shows that $A(Q^2)$ is very sensitive to the nucleon form factors in the impulse approximation. The difference between the curve obtained using the Kelly-AMT4 parameterization and the other curves in the structure function $A(Q^2)$ shown in Figure (7.16) is caused by the difference in the proton form factors, exhibiting the experimental differences between the one-photon-exchange and two-photon-exchange data. Figure (7.8) shows that the main sensitivity in the deuteron structure function $A(Q^2)$ is from uncertainties in the neutron electric form factor. To show this we use hybrid nucleon form factor parameterizations that only differ in the neutron
electric form factor to calculate the structure function $A(Q^2)$. We replace the neutron electric form factor in the Kelly parameterization by the corresponding ones in the BI and BBBA parameterizations. The difference between using BI and BBBA parameterizations in Figure (7.16) is almost the same as the difference between using Kelly-BI and Kelly-BBBA parameterizations in Figure (7.17). This means that the sensitivity of the deuteron form factors to the nucleon form factors mainly comes from the neutron electric form factor. Also shown in Figure (7.17), the size of the differences generated in the deuteron structure function $A(Q^2)$ due to differences in the proton form factors and due to differences in the neutron electric form factors are comparable. The Kelly-AMT4 parameterization gives the best fit in the impulse approximation. Figures (7.18) and (7.19) show that the deuteron structure functions $B(Q^2)$ and $T_{20}(Q^2)$ are not sensitive to the nucleon form factors in the impulse approximation. Experimental data in (7.18) indicates a minimum of $B(Q^2)$ near $Q^2 = 2 GeV^2$ which is missed in the impulse approximation. Figure (7.19) shows that the curves of $T_{20}(Q^2)$ are much lower than the experimental data in the range of momentum transfer $Q^2 > 0.5 GeV^2$.

All three deuteron structure functions require additional current contributions beyond the impulse currents to fit the experimental data.

The deuteron structure functions $A(Q^2)$, $B(Q^2)$ and $T_{20}(Q^2)$ calculated with the “pair” current contribution included are shown in Figures (7.20)–(7.22), where the short- and intermediate-range potential is included in the construction of the effective one-pion-exchange potential. We can see that all three structure functions are fit better to the experimental data with the “pair” current contribution included. We also can see
that the sensitivity of $A(Q^2)$ to the nucleon form factors becomes smaller than in the impulse approximation and $T_{20}(Q^2)$ exhibits some sensitivities to the nucleon form factors with the “pair” current contribution included. It appears that the different nucleon form factors have little effect on $B(Q^2)$. Within these small uncertainties, the BBBA parameterization gives the best fit. The other parameterizations give results for $A(Q^2)$ above the experimental data. The Kelly-AMT4 parameterization gives the worst fit. The “pair” current contribution not only moves the minimum of the structure function $B(Q^2)$ closer to the right position but it also fits the experimental data much better. The structure function $T_{20}(Q^2)$ moves closer to the experimental data but it is still in disagreement with the data. The best fit is given by using the BI parameterization. This indicates that some additional exchange current corrections are needed.

To test the effect of the residual short-range potential, the effective one-pion-exchange potential was also constructed with the short-range potential turned off. The corresponding structure functions are shown in Figures (7.23)- (7.26). Figure (7.23) shows good agreement of the structure function $A(Q^2)$ with the experimental data, with and without the residual short- and intermediate-range potential contribution to the exchange current for momentum transfer up to $2GeV^2$. With the residual short-range potential removed, the deuteron structure function $A(Q^2)$ is also consistent with the experimental data in the high-momentum transfer range to within uncertainties in the nucleon form factors, which is shown in Figure (7.24). $B(Q^2)$ and $T_{20}(Q^2)$ are not affected by the short- and intermediate-range potential contribution to the exchange current. Note that the deuteron structure function $A(Q^2)$ calculated using the Kelly-AMT4
parameterization, where the proton form factors are fit to the one-photon-exchange data, also fits the experimental data. This indicates that the one-photon-exchange approximation used in this work gives reliable information about the deuteron in the elastic-electron scattering.

To examine the sensitivity to the choice of independent current matrix elements, the fourth plus component of the deuteron current matrix element, $I_{11}^+ - I_{00}^+$, is computed by direct calculation and by using Eq. (3.60). The calculations are compared in Figures (7.27) and (7.28). We see that differences exist in both cases, with and without “pair” current contribution, for both the BBBA and BI nucleon form factor parameterizations. However, the sensitivity to the choice of independent matrix elements is much larger when the “pair” current is included. We also calculated the deuteron structure functions using the choice of independent current matrix elements suggested by Frankfurt, Frederico and Strikman [36] and by Coester as described in Chapter 3. In the FFS method, the deuteron form factors $G_1(Q^2)$ and $G_2(Q^2)$ are defined by Eq. (3.58) and (3.59), as in the CCKP method, and $G_0(Q^2)$ is defined by Eq. (3.61). $G_1(Q^2)$ is also defined by Eq. (3.58) in the Coester method and $G_0(Q^2)$ and $G_2(Q^2)$ are defined by Eq. (3.63) and (3.62). The structure function $B(Q^2)$ is the same for all these three methods since it only depends on $G_1(Q^2)$, which is identical in all three methods. Results of $A(Q^2)$ and $T_{20}(Q^2)$ calculated using the FFS and the Coester method are shown in Figures (7.29)-(7.30) and Figures (7.31)-(7.32) respectively. We see that the FFS method and the Coester method give similar results. They give the best fit for $A(Q^2)$ and $T_{20}(Q^2)$ using the BBBA nucleon form factor.
parameterization. While $A(Q^2)$ agrees with the experimental data using any of these three methods combined with a specific nucleon form factor parameterization, the FFS and Coester methods give a much better fit of $T_{20}(Q^2)$ with the BBBA nucleon form factor parameterization. Comparing Figures (7.24) (7.25) (7.26) (7.29) (7.30), we see that the deuteron structure functions $A(Q^2)$, $B(Q^2)$ and $T_{20}(Q^2)$ calculated using the FFS and Coester methods for the choice of independent current matrix elements and the BBBA parameterization happen to give the best fit to the experimental data. Results are shown in Figures (7.33)- (7.35) and compared to the results in the impulse approximation.

Deuteron wave functions constructed using different nucleon-nucleon interaction are used to test the importance of the consistency between the “pair” current operator and the nucleon-nucleon interaction used to generate the deuteron wave functions. We used the same “pair” current operator combined with deuteron wave functions constructed using the CDB and N3LO potential to estimate the “pair” current contribution. Since these interactions are fit to the same data, but have different treatments of the short range physics, we expect that the “pair” current contribution used in AV18 would have similar effects on the deuteron structure functions in the low-momentum transfer range while different in the high-momentum transfer range. Figures (7.36)- (7.38) show that with the “pair” current contribution calculated using the CDB wave functions give deuteron structure functions consistent with experiment for $Q^2 < 1GeV^2$. We notice that the large difference between the deuteron wave functions generated using the N3LO and the other two model NN-interactions produces a very large difference in the
deuteron structure functions when the same “pair” current operator is used. These plots indicate that the “pair” current contribution needs to be calculated in a manner that is consistent with the model potential used to generate the deuteron wave functions. This is particularly true for the N3LO wave functions which have a significantly different high momentum behavior than the $AV_{18}$ and CDB wave functions. This suggests that even the long-range part of the exchange current used in $AV_{18}$ would require modification before it could be used in N3LO.

The static deuteron quadrupole and magnetic moments are related to the deuteron form factors, $G_1(Q^2)$ and $G_2(Q^2)$, in the limit of zero momentum transfer:

$$Q_d = \lim_{Q^2 \to 0} 3 \sqrt{2} \frac{G_2(Q^2)}{Q^2}, \quad \mu_d = \frac{m}{M_d} G_1(0).$$

(6.1)

The units of $Q_d$ and $\mu_d$ are $fm^2$ and $\mu_N$ respectively. They have been calculated in the impulse approximation and with the “pair” current contribution included. They are not sensitive to the nucleon form factors and the choice of the independent current matrix elements. Table (6.1) lists the value of magnetic moments and quadrupole moments under each condition. The deuteron quadrupole moment is 0.2698 in the impulse approximation. It is 0.2748 and 0.2752 including the “pair” current contribution constructed with and without removing the residual short-range potential. The magnetic moment is 0.8535 in the impulse approximation and 0.8573 and 0.8596 including the “pair” current contribution constructed with and without removing the residual short-range potential. Without removing the residual short-range potential,
Table 6.1: Deuteron magnetic and quadrupole moments evaluated in the impulse approximation and including the “pair” current contribution constructed with(WS) and without(NS) the residual short-range potential removed. The values are the same using all six different nucleon form factor parameterizations and three combinations of independent current matrix elements. Argonne V18 potential is used in the calculation. The values labeled with WSS are from [75]. The experimental values are $0.2860 \pm 0.0015 \text{fm}^2$ [18] and $0.857406 \pm 0.000001 \mu_N$ [49].

<table>
<thead>
<tr>
<th></th>
<th>IM</th>
<th>IM+Pair(WS)</th>
<th>IM+Pair(NS)</th>
<th>IM(WSS)</th>
<th>IM+MEC(WSS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_d$</td>
<td>0.2698</td>
<td>0.2752</td>
<td>0.2748</td>
<td>0.270</td>
<td>0.275</td>
</tr>
<tr>
<td>$\mu_d$</td>
<td>0.8535</td>
<td>0.8596</td>
<td>0.8573</td>
<td>0.847</td>
<td>0.871</td>
</tr>
</tbody>
</table>

The deuteron quadrupole moment is improved by 2% by the “pair” current contribution but is still 3.78% below the experimental data and the magnetic moment is improved by 0.71% by the “pair” current but is 2.57% above the experimental data. With the residual short-range potential removed, the “pair” current contribution improves the quadrupole moment by 1.85% but is still 3.92% below the experimental data and the magnetic moment is improved by 0.45% by the “pair” current and is only 0.01% below the experimental data. The values Wiringa, Stoks and Schiavilla [75] obtained in their calculation are also listed in Table (6.1) and labeled by WSS.

The deuteron structure functions have also been calculated by other groups using different models. The calculation carried out by Van Orden, Devine and Gross [71] using the spectator equation includes the “pair” currents and the $\rho\pi\gamma$ exchange currents. Their results fit the old experimental data but need to be examined with the new data. Phillips, Wallace and Devine [56] also calculated the deuteron structure functions with the “pair” currents and the $\rho\pi\gamma$ meson-exchange current contribution. $T_{20}(Q^2)$ had a good agreement with the experimental data but $A(Q^2)$ and $B(Q^2)$ still need to be
improved. Wiringa, Stoks and Schiavilla [75] used the same AV18 NN-potential as we did in their calculation with exchange current included in instant-form. In their results, the deuteron structure function $A(Q^2)$, $B(Q^2)$ fit the experimental data but $T_{20}(Q^2)$ was above the data with the exchange currents included. The moments calculated in this work are also compared with their results in Table 6.1. Arenhövel, Ritz and Wilbois [5] had calculated the deuteron structure functions with various exchange current contribution with the momentum transfer up to $Q^2 \leq 1.16 GeV^2$. The behavior in the high-momentum transfer range still needs to be examined. The front-form calculations carried out by Chung et al. [24] in the impulse approximation gave results that were consistent with the existing experimental data. But their results do not agree with the newer higher precision data. Carbonell and Karmanov [23] calculated the deuteron structure functions with the exchange current contribution in the front-form dynamic. There is a good agreement of $T_{20}(Q^2)$ with the experimental data in their results. But $A(Q^2)$ and $B(Q^2)$ are not consistent with experiments for momentum transfer larger than $2 GeV^2$. We also notice that the minimum of $B(Q^2)$ moves in the opposite direction with respect to the position indicated by the experimental data when the exchange current contribution is included.

This work shows that the with improved NN-interaction and nucleon form factors, the impulse calculation no longer agrees with experiment. We improved the calculation by adding the “pair” current contribution to the one-body currents. All three deuteron structure functions, $A(Q^2)$, $B(Q^2)$ and $T_{20}(Q^2)$, are improved with the “pair” current contribution included. Within the uncertainty of the empirical nucleon form factors,
the results happen to fit the data perfectly using the BBBA nucleon form factor parameterization and the FFS and Coester methods of combinations of the independent current matrix elements with the residual short-range potential removed. The magnetic moment is improved by 0.45% by the “pair” current contribution and is only 0.01% below the experimental data. The quadrupole moment was improved by 1.85% but is still 3.92% below the experimental data.
CHAPTER 7

CONCLUSIONS

In this work, we have examined the impact of “pair” currents in the evaluation of elastic electron-deuteron scattering using an exactly Poincaré invariant model of the nuclear dynamics with a light-front kinematic subgroup. The dynamics is given by adding a realistic nucleon-nucleon interaction multiplied by $4m$ to the square of the invariant mass operator of a two nucleon system. The interaction is designed to commute with and be independent of the light-front components of the total four momentum and the spins of the non-interacting two-body system. Simultaneous eigenstates of this mass operator, the light-front components of the momentum and the non-interacting light-front spin are complete and transfer irreducibly with respect to the Poincaré group. This representation defines the strong interaction dynamics. This method of including realistic interactions leads to a relativistic model that fits the same experimental data as the corresponding non-relativistic model. In this calculation the Argonne V18 NN-interaction was used.

In the one-photon-exchange approximation, electromagnetic observables can be expressed as quadratic functions of conserved covariant current matrix elements. The dynamical constraints on the current operator are difficult to realize, however they imply linear relations among different current matrix elements. A set of independent current matrix elements were calculated and the linear constraint was used to generate all matrix elements of the conserved covariant current operator that agrees with the calculated independent matrix elements.
We orient the light front so the plus component of the momentum transfer vanishes. This is always possible for electron scattering. Light-front dynamics has the following advantages:

- All matrix elements can be determined from matrix elements of the plus component of the current.

- Matrix elements of the plus component of the current transformation are invariant under light-front boosts.

- The light-front boosts form a group so the spins are invariant with respect to light-front boosts.

- Momentum transferred to the nucleons in the impulse current matrix elements is identical to the momentum transferred to the deuteron.

This work extends previous work by Chung et al. [24]. They used a light-front impulse approximation to evaluate a set of independent current matrix elements with various choices of interactions and nucleon form factors. The results were consistent with existing experimental data to within uncertainties in the neutron electric form factors and the choices of nucleon-nucleon interactions. Since then there have been additional measurements, improvements in both the interactions and form factors. This work shows that using the improved interactions and nucleon form factors in the impulse calculations used in [24] are inconsistent with the newer higher precision data.

There has been some success in resolving this discrepancy between theory and experiments by including additional two-body current operators. This has been done in
a variety of frameworks. Van Orden, Devine and Gross [71] had included the “pair” currents and $\rho\pi\gamma$ exchange currents in their calculation using the spectator equation. Phillips, Wallace and Devine [56] also calculated the deuteron structure functions with the “pair” currents and $\rho\pi\gamma$ meson-exchange current contribution. In the work of Wiringa, Stoks and Schiavilla [75], the same AV18 NN-potential was used to calculate the structure functions with exchange current included in the instant-form. Arenhövel, Ritz and Wilbois [5] had calculated the deuteron structure functions with various exchange current contribution included in the front-form.

In this work we investigated the impact of “pair” currents contributions in elastic electron-deuteron scattering in Poincaré invariant quantum mechanics with a light-front symmetry. In general, both the current and dynamics are representation independent, however most model interactions have a long range contributions due to pion-exchange. We used the long range part of the model interactions to construct a model current. We extract a one-pion-exchange interaction by considering a tree-level calculation of the S-matrix for NN-scattering, and then impose a light front kinematic symmetry on the interaction kernel. We introduced an external electromagnetic field and use the same steps to construct a “pair” current related to the interaction kernel. We replaced the one-pion-exchange interaction by a realistic interaction; and we use the long-range part of this operator to construct the corresponding model “pair” current.

The deuteron structure functions, $A(Q^2)$, $B(Q^2)$ and $T_{20}(Q^2)$, were calculated using independent deuteron current matrix elements. The independent current matrix elements were calculated in the impulse approximation and with the model “pair” current
contribution included. The model pair current was constructed consistently with the
Argonne V18 NN-interaction, which is used to generate the deuteron wave function.
Deuteron wave functions generated using the CDB [52] and N3LO [34] NN-interactions
were used to test the sensitivity of the model “pair” current to different model NN-
interactions that are fit to the same scattering data. The residual short-range potential
was removed to examine its effect on the model “pair” current. Empirical nucleon form
factors were used as input and various nucleon form factor parameterizations were used
to test the sensitivity of the deuteron form factors to the input nucleon form factors.
The angular condition was examined by comparing the current matrix elements calcu-
lated directly to the results generated using dynamical constraints. We also tested the
sensitivity to different choices of independent current matrix elements by calculating the
deuteron structure functions with three different choices of independent current matrix
elements . The effect of the “pair” current contribution on the deuteron quadrupole
and magnetic moments has also been investigated.

The deuteron structure functions $A(Q^2)$, $B(Q^2)$ and $T_{20}(Q^2)$ calculated in the im-
pulse approximation using the realistic AV18 NN-interaction do not agree with experi-
ment. They were improved by the “pair” current contributions constructed consistently
with the AV18 NN-interaction. The “pair” current constructed with the residual short-
and intermediate-range contribution removed gives a better fit to the experimental data.
The deuteron structure functions are sensitive to the choice of the independent current
matrix elements. The results calculated using the CCKP [24] choice of independent
current matrix elements give $A(Q^2)$ and $B(Q^2)$ in good agreement with experiment for
all of the empirical nucleon form factors used in this work. But $T_{20}(Q^2)$ is still below the experimental data. The FFS [36] and Coester choices of independent current matrix elements give good agreement of $T_{20}(Q^2)$ to experimental data. All three deuteron structure functions happen to fit the data perfectly using the FFS and Coester choices independent current matrix elements and the BBBA nucleon form factor parameterization. The magnetic moment is improved by 0.45% by the “pair” current contribution and is only 0.01% below the experimental data. The quadrupole moment was improved by 1.85% but is still 3.92% below the experimental data. This work suggest that the “Pair” current contributions are sufficient to resolve the difference between theory and experiments in all deuteron structure functions, except for the quadruple moments. The origin of the quadrupole moment is the non-zero D-state probability in the deuteron wave functions. Different realistic NN-interactions give equivalent wave functions with different D-state probabilities. This calculation indicates a sensitivity to different phase equivalent interactions. But we did not compute the “pair” current associated with each interaction. It may be that the resolution of the discrepancy between the calculated and observed quadrupole moment can be attributed to this sensitivity. Future work is needed on the calculation of electromagnetic observables using different NN-interaction and consistently constructed exchange currents.
Figure 7.1: Deuteron S-state wave functions in momentum space. The realistic NN-interactions used are the AV18, CDB and N3LO NN-interactions.
Figure 7.2: Deuteron D-state wave functions in momentum space. The realistic NN-interactions used are the AV18, CDB and N3LO NN-interactions.
One-Pion Exchange Potential

Figure 7.3: Linear plot of the one-pion-exchange potential extracted from the Argonne V18 NN-interaction with (WS) and without (NS) the short- and intermediate-range contribution. Also shown is the bare pion propagator.
One-Pion Exchange Potential

Figure 7.4: Log plot of the one-pion-exchange potential extracted from the Argonne V18 NN-interaction with (WS) and without (NS) the short- and intermediate-range contribution. Also shown is the bare pion propagator.
Nucleon Form Factors

Figure 7.5: Proton electric form factors $\mu_p G_{E_p}/G_{M_p}$. The parameterizations used are BBA [22], BI [17], Kelly [44], BBBA [20], Lomon [50] and the hybrid parameterization Kelly-AMT4 [44] [8].
Figure 7.6: Proton electric form factors $G_{Ep}/G_d$. The nucleon form factor parameterizations used are the same as in Figure (7.5).
Nucleon Form Factors

Figure 7.7: Neutron electric form factors $\mu_n G_{E_n}/G_{M_n}$. The nucleon form factor parameterizations used are the same as in Figure (7.5).
Figure 7.8: Neutron electric form factors $G_{E_n}/G_d$. The nucleon form factor parameterizations used are the same as in Figure (7.5).
Figure 7.9: Proton magnetic form factors $G_{M_p}/\mu_p G_d$. The nucleon form factor parameterizations used are the same as in Figure (7.5).
Figure 7.10: Neutron magnetic form factors $G_{M_n}/\mu_n G_d$. The nucleon form factor parameterizations used are the same as in Figure 7.5.
Figure 7.11: Isoscalar nucleon form factors $F_{1N}$. The nucleon form factor parameterizations used are the same as in Figure (7.5).
Figure 7.12: Isoscalar nucleon form factors $F_{2N}$. The nucleon form factor parameterizations used are the same as in Figure (7.5).
Figure 7.13: The deuteron form factor $G_0(Q^2)$ calculated with(IM+Pair) and without(IM) the “pair” current contribution. The “pair” current contribution is constructed without removing the residual short- and intermediate-range potential from the AV18 NN-interaction. The CCKP choice of the independent current matrix elements is used. The curves with the BBBA and BI parameterizations represent the sensitivity to the nucleon electric form factors. The curve with the Kelly-AMT4 parameterization represents the uncertainty in the proton form factors, the one-photon-exchange data fit for Kelly-AMT4 and the two-photon-exchange data fit for the BBBA and BI parameterizations.
Deuteron Form Factors

Figure 7.14: Deuteron form factor $G_1(Q^2)$ calculated with (IM+Pair) and without (IM) the “pair” current contribution. The “pair” current contribution is constructed without removing the residual short- and intermediate-range potential from the AV18 NN-interaction. The CCKP choice of the independent current matrix elements is used. The curves with the BBBA and BI parameterizations represent the sensitivity to the nucleon electric form factors. The curve with the Kelly-AMT4 parameterization represents the uncertainty in the proton form factors, the one-photon-exchange data fit for Kelly-AMT4 and the two-photon-exchange data fit for the BBBA and BI parameterizations.
Figure 7.15: Deuteron form factor $G_2(Q^2)$ calculated with(IM+Pair) and without(IM) the “pair” current contribution. The “pair” current contribution is constructed without removing the residual short- and intermediate-range potential from the AV18 NN-interaction. The CCKP choice of the independent current matrix elements is used. The curves with the BBBA and BI parameterizations represent the sensitivity to the nucleon electric form factors. The curve with the Kelly-AMT4 parameterization represents the uncertainty in the proton form factors, the one-photon-exchange data fit for Kelly-AMT4 and the two-photon-exchange data fit for the BBBA and BI parameterizations.
Figure 7.16: Deuteron structure function $A(Q^2)$ calculated in the impulse approximation. The deuteron wave functions are generated using the realistic AV18 NN-interaction. Empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are from: Stanford Mark III [21], CEA [33], Orsay [13], SLAC E101 [6], Saclay ALS [57], DESY [37], Bonn [30], Mainz [68], JLab Hall C [1], JLab Hall A [3] and Monterey [14].
Figure 7.17: Sensitivity of the deuteron structure function $A(Q^2)$ to the empirical nucleon form factors in the impulse approximation. The difference between the solid line and the dotted line represents the uncertainty in the proton form factors fit to the one-photon-exchange(Kelly-AMT4) or two-pion-exchange(Kelly) data. The deferences among the solid, dashed and dash-dotted lines are from the different neutron electric form factors. Experimental data are the same as in Figure 7.16.
Figure 7.18: Deuteron structure function $B(Q^2)$ calculated in the impulse approximation. The deuteron wave functions are generated using the realistic $AV18\;NN$-interaction. Empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are from: SLAC NPSA NE4 [19], Martin [53], Bonn [30], Saclay ALS [9], Mainz [68], Stanford Mark III [21].
Deuteron Structure Function $T_{20}$

Figure 7.19: Deuteron structure function $T_{20}(Q^2)$ calculated in the impulse approximation. The deuteron wave functions are generated using the realistic AV18 NN-interaction. Empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are taken from: Novosibirsk-85 [32] [72], Novosibirsk-90 [38], Bates-84 [67], Bates-91 [70] and JLab Hall C [2]
Figure 7.20: The deuteron structure function $A(Q^2)$ calculated with the “pair” current contribution constructed without removing the residual short- and intermediate-range potential. The empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are the same as in Figure (7.16).
Figure 7.21: The deuteron structure function $B(Q^2)$ calculated with the “pair” current contribution constructed without removing the residual short- and intermediate-range potential. The empirical nucleon form factors used are the same as in Figure (7.5) Experimental data are the same as in Figure (7.18).
Figure 7.22: The deuteron structure function $T_{20}(Q^2)$ calculated with the “pair” current contribution constructed without removing the residual short- and intermediate-range potential. The empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are the same as in Figure (7.19).
Figure 7.23: The effect of the residual short- and intermediate-range contribution on the deuteron structure function $A(Q^2)$ in the range of momentum transfer up to $1 GeV^2$. The “Pair” current contribution are constructed with (WS) and without (NS) removing the residual short- and intermediate-range potential. The empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are the same as in Figure (7.16).
Figure 7.24: The deuteron structure function $A(Q^2)$ calculated with the “pair” current contribution constructed with the residual short- and intermediate-range potential removed. The CCKP method is used to choose the independent current matrix elements. The empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are the same as in Figure (7.16).
Figure 7.25: The deuteron structure function $B(Q^2)$ calculated with the “pair” current contribution constructed with the residual short- and intermediate-range potential removed. The CCKP method is used to choose the independent current matrix elements. The empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are the same as in Figure (7.18).
Figure 7.26: The deuteron structure function $T_{20}(Q^2)$ calculated with the “pair” current contribution constructed with the residual short- and intermediate-range potential removed. The CCKP method is used to choose the independent current matrix elements. The empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are the same as in Figure (7.19).
Deuteron Current Matrix Elements

Figure 7.27: Deuteron current matrix elements obtained by direct calculation (Cal) and derived using dynamics (Drv) with BBBA nucleon form factor parameterization. They are calculated in the impulse approximation (IM) and with the “pair” current contribution included (IM+Pair). The “pair” current contribution are constructed with (WS) and without (NS) removing the residual short- and intermediate-range potential.
Deuteron Current Matrix Elements

Figure 7.28: Deuteron current matrix elements obtained by direct calculation (Cal) and derived using dynamics (Drv) with BI nucleon form factor parameterization. They are calculated in the impulse approximation (IM) and with the “pair” current contribution included (IM+Pair). The “pair” current contribution are constructed with (WS) and without (NS) removing the residual short- and intermediate-range potential.
Figure 7.29: The deuteron structure function $A(Q^2)$ calculated using the FFS choice of the independent current matrix elements with the “pair” current contribution included. The residual short- and intermediate-range potential in the $AV18$ NN-interaction is removed. The empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are the same as in Figure (7.16).
Figure 7.30: The deuteron structure function $T_{20}(Q^2)$ calculated using the FFS choice of the independent current matrix elements with the “pair” current contribution included. The residual short- and intermediate-range potential in the $AV18$ NN-interaction is removed. The empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are the same as in Figure (7.19).
Figure 7.31: The deuteron structure function $A(Q^2)$ calculated using the Coester choice of the independent current matrix elements with the “pair” current contribution included. The residual short- and intermediate-range potential in the AV18 NN-interaction is removed. The empirical nucleon form factors used are the same as in Figure 7.5. Experimental data are the same as in Figure 7.16.
Figure 7.32: The deuteron structure function $T_{20}(Q^2)$ calculated using the Coester choice of the independent current matrix elements with the “pair” current contribution included. The residual short- and intermediate-range potential in the $AV18$ NN-interaction is removed. The empirical nucleon form factors used are the same as in Figure (7.5). Experimental data are the same as in Figure (7.19).
Deuteron Structure Function $A$}

Figure 7.33: The best fit of the deuteron structure function $A(Q^2)$ compared to the impulse approximation. The residual short- and intermediate-range potential in the $AV18$ NN-interaction is removed. The FFS and Coester choices of the independent current matrix elements are used. The empirical nucleon form factors used is the BBBA parameterization. Experimental data are the same as in Figure (7.16).
Figure 7.34: The best fit of the deuteron structure function $B(Q^2)$ compared to the impulse approximation. The residual short- and intermediate-range potential in the AV18 NN-interaction is removed. The FFS and Coester choices of the independent current matrix elements are used. The empirical nucleon form factors used is the BBBA parameterization. Experimental data are the same as in Figure (7.18).
Figure 7.35: The best fit of the deuteron structure function $T_{20}(Q^2)$ compared to the impulse approximation. The residual short- and intermediate-range potential in the AV18 NN-interaction is removed. The FFS and Coester choices of the independent current matrix elements are used. The empirical nucleon form factors used is the BBBA parameterization. Experimental data are the same as in Figure (7.19).
Figure 7.36: The sensitivity of the deuteron structure function $A(Q^2)$ to different NN-interactions. The CDB and N3LO deuteron wave functions are used in calculating $A(Q^2)$ in the impulse approximation(IM) and with the pair current contribution included(IM+Pair). The “pair” current contribution is constructed with the residual short- and intermediate-range potential removed. The BBBA nucleon form factor parameterization is used. The CCKP choice of the independent current matrix elements is used. Experimental data are the same as in Figure (7.16).
Figure 7.37: The sensitivity of the deuteron structure function $B(Q^2)$ to different NN-interactions. The CDB and N3LO deuteron wave functions are used in calculating $B(Q^2)$ in the impulse approximation(IM) and with the pair current contribution included(IM+Pair). The “pair” current contribution is constructed with the residual short- and intermediate-range potential removed. The BBBA nucleon form factor parameterization is used. The CCKP choice of the independent current matrix elements is used. Experimental data are the same as in Figure (7.18).
Figure 7.38: The sensitivity of the deuteron structure function $T_{20}(Q^2)$ to different NN-interactions. The CDB and N3LO deuteron wave functions are used in calculating $T_{20}(Q^2)$ in the impulse approximation (IM) and with the pair current contribution included (IM+Pair). The “pair” current contribution is constructed with the residual short- and intermediate-range potential removed from the AV18 NN-interaction. The BBBA nucleon form factor parameterization is used. The CCKP choice of the independent current matrix elements is used. Experimental data are the same as in Figure (7.19).
APPENDIX A

COMMUTATION RELATIONS OF POINCARÉ GROUP

A.1 Canonical-form

\[ [P^j, P^0] = 0, \quad (A.1) \]
\[ [P^j, P^k] = 0, \quad (A.2) \]
\[ [J^j, J^k] = i \epsilon^{jkl} J^l, \quad (A.3) \]
\[ [K^j, K^k] = -i \epsilon^{jkl} J^l, \quad (A.4) \]
\[ [J^j, K^k] = i \epsilon^{jkl} K^l \quad (A.5) \]
\[ [J^j, P^0] = 0, \quad (A.6) \]
\[ [K^j, P^0] = -i P^j, \quad (A.7) \]
\[ [J^j, P^k] = i \epsilon^{jkl} P^l, \quad (A.8) \]
\[ [K^j, P^k] = -i \delta^{jk} P^0 \quad (A.9) \]

where Eq. (A.1) and (A.2) gives the Lie algebra of space-time translations; Eq. (A.3)-(A.5) are the commutation relations for the Lorentz group, which includes the group of rotation and rotationless boosts; The remaining commutation relations constrain \( P^\mu \) to transform as a four-vector under Lorentz transformations. If the generators \( J^\mu \) and
$K^\mu$ are written in form of the angular momentum tensor $J^{\alpha\beta}$:

$$J^{0j} \equiv K^j, \quad J^{jk} \equiv \epsilon^{jkl} J^l,$$  \hspace{1cm} (A.10)

then the commutation relations can be expressed in the covariant form as follows:

$$[P^\mu, P^\nu] = 0,$$  \hspace{1cm} (A.11)

$$[J^{\rho\sigma}, P^\nu] = i(g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho),$$  \hspace{1cm} (A.12)

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\sigma} J^{\mu\rho} - g^{\nu\rho} J^{\mu\sigma} + g^{\mu\rho} J^{\nu\sigma} - g^{\mu\sigma} J^{\nu\rho}).$$  \hspace{1cm} (A.13)

### A.2 Front-form

Front-form generators are consist of seven kinematic generators and three dynamic generators. The commutation relations among the seven kinematic generators, $P^1, P^2, P^+, J^3, K^3, E^1, E^2$, have the following form:

$$[E^j, E^k] = 0,$$  \hspace{1cm} (A.14)

$$[K^3, \vec{E}] = -i \vec{E},$$  \hspace{1cm} (A.15)

$$[P^+, \vec{E}] = 0,$$  \hspace{1cm} (A.16)

$$[J^3, P^+] = 0.$$  \hspace{1cm} (A.17)
\[ [P^+, P^j] = 0, \quad (A.18) \]
\[ [J^3, E_i] = -i\hat{z} \times \vec{E}, \quad (A.19) \]
\[ [P^j, E^k] = i\delta^{jk}P^+, \quad (A.20) \]
\[ [K^3, P^+] = -iP^+. \quad (A.21) \]

These generators form a closed Lie algebra and generate transformations that leave the light front invariant. Notice that Eq. (A.14) and (A.14) imply that the generators \( E_1, E_2 \) and \( K^3 \) form a closed subalgebra so that the corresponding transformations, the front-form boosts, also form a group.

The three dynamic generators which take points off of the light front are \( P^-, F^1 \) and \( F^2 \). They have the following commutation relations:

\[ [F^j, F^k] = 0, \quad (A.22) \]
\[ [\vec{F}, P^-] = 0. \quad (A.23) \]

The mixed commutators, commutators among the kinematic and dynamic generators, are listed below:

\[ [P^-, J^3] = 0, \quad (A.24) \]
\[ [P^+, P^-] = 0, \quad (A.25) \]
\[ [F^j, E^j] = 2iK^3, \quad (A.26) \]
\[ [F^j, E^k] = -i \epsilon^{jk3} J^3, \quad (A.27) \]

\[ [\vec{F}, K^3] = -i \vec{F}, \quad (A.28) \]

\[ [F^j, J^3] = i \epsilon^{jk3} F^k, \quad (A.29) \]

\[ [F^j, P^k] = -i \delta^{jk} P^-, \quad (A.30) \]

\[ [\vec{F}, P^+] = -2i \vec{P}, \quad (A.31) \]

\[ [P^-, \vec{E}] = 2i \vec{P}, \quad (A.32) \]

\[ [P^-, K^3] = i P^+. \quad (A.33) \]
APPENDIX B

DIRAC SPINORS

The following convention are used to construct the Dirac spinors used in our work:

(1). Minkowski metric:

\[
g_{\mu\nu} = g_{\mu\nu} \equiv \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}. \tag{B.1}
\]

(2). Pauli matrices and the identity:

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.2}
\]

(3). Dirac gamma matricies:

If we define

\[
\bar{\sigma}_\mu \equiv \sigma_2 \sigma_\mu^* \sigma_2 = (\sigma_0, -\bar{\sigma}), \tag{B.3}
\]

the Dirac gamma matricies can be expressed as

\[
\gamma_\mu \equiv \begin{pmatrix} 0 & -\sigma_\mu \\ -\bar{\sigma}_\mu & 0 \end{pmatrix}, \quad \gamma^\mu \equiv \begin{pmatrix} 0 & \bar{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}. \tag{B.4}
\]
Gamma matrices thus defined are:

\[ \beta = \gamma^0 = -\gamma_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma^1 = \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (B.5) \]

\[ \gamma^2 = \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (B.6) \]

\[ \gamma^5 = \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (B.7) \]

They satisfy the following relations:

\[ [\gamma_\mu, \gamma_\nu]^+ = -2g_{\mu\nu}I, \quad \frac{1}{2}[\gamma_0, \gamma_j] = \begin{pmatrix} -\sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}, \quad \frac{i}{2}[\gamma_i, \gamma_j] = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}. \quad (B.8) \]

In order to construct the four component Dirac spinors, we need an independent basis
spinors which are chosen to be:

\[
u(0, \sigma) \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 \\ \sigma_0 \end{pmatrix}, \quad \nu(0, \sigma) \equiv \gamma^5 u(0, \sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 \\ -\sigma_0 \end{pmatrix},
\]

where \(\sigma = \frac{1}{2}\) is the first column and \(\sigma = -\frac{1}{2}\) is the second column. For any \(\Lambda \in SL(2, C)\), we define the representation \(S(\Lambda)\)

\[
S(\Lambda) \equiv \begin{pmatrix} \Lambda & 0 \\ 0 & \bar{\Lambda} \end{pmatrix},
\]

where

\[
\bar{\Lambda} = (\Lambda^\dagger)^{-1} = \sigma_2 \Lambda^* \sigma_2.
\]

The canonical and front-form Dirac spinors can be obtained by applying the corresponding representation \(S(\Lambda)\) on those basis spinors:

\[
u_c(p, \sigma) \equiv S(L_c(p)) u(0, \sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} L_c(p) \\ \tilde{L}_c(p) \end{pmatrix} u(0, \sigma),
\]

\[
\bar{\nu}_c(p, \sigma) = u_c(p, \sigma) \beta = \frac{1}{\sqrt{2}} (L_c^{-1}(p), L_c^\dagger(p)) u(0, \sigma),
\]

\[
v_c(p, \sigma) \equiv S(L_c(p)) v(0, \sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} L_c(p) \\ -\tilde{L}_c(p) \end{pmatrix} v(0, \sigma),
\]

\[
\bar{v}_c(p, \sigma) = v_c(p, \sigma) \beta = \frac{1}{\sqrt{2}} (-L_c^{-1}(p), L_c^\dagger(p)) v(0, \sigma),
\]
\[
\begin{align*}
\bar{u}_f(p, \sigma) &= u_f(p, \sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} L_f(p) \\ \tilde{L}_f(p) \end{pmatrix} u(0, \sigma), \quad (B.16) \\
\bar{v}_f(p, \sigma) &= v_f(p, \sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} L_f(p) \\ -\tilde{L}_f(p) \end{pmatrix} v(0, \sigma), \quad (B.18)
\end{align*}
\]

Spinors thus defined satisfy

\[
\begin{align*}
\bar{u}_c(p, \sigma) u_c(p, \sigma') &= \sigma_0, \quad \bar{u}_f(p, \sigma) u_f(p, \sigma') = \sigma_0, \quad (B.20) \\
\bar{v}_c(p, \sigma) v_c(p, \sigma') &= -\sigma_0, \quad \bar{v}_f(p, \sigma) v_f(p, \sigma') = -\sigma_0. \quad (B.21)
\end{align*}
\]
APPENDIX C

NUCLEON FORM FACTORS

Nucleon electromagnetic form factors are fundamental quantities describing the intrinsic charge and magnetization distributions inside nucleons. The empirical nucleon form factors are used as input in calculation of the deuteron current matrix elements in our work. Just as the electron-deuteron scattering used to study deuteron structure, electrons are also useful tools to probe the nucleon structure. The differential cross-section for electron-nucleon scattering has the same form as Eq. (1.1) except the deuteron mass $M_d$ is replaced by the nucleon mass $m$ in $\tau$. Here the nucleon structure functions $A(Q^2)$ and $B(Q^2)$ are related to the conventional nucleon electric and magnetic form factors $G_E(Q^2)$ and $G_M(Q^2)$ as follows:

$$A(Q^2) = \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1 + \tau}, \quad (C.1)$$

$$B(Q^2) = 2\tau G_M^2(Q^2). \quad (C.2)$$

The electric and magnetic form factors can be extracted by using the Rosenbluth separation technique [61], where the cross section for a fixed $Q^2$ value are measured versus varied electron scattering angle. Experimental data gives the following approximate expression in the dipole approximation:

$$G_E^p(Q^2) = G_D(Q^2), \quad G_E^n(Q^2) = 0, \quad (C.3)$$

$$G_M^p(Q^2) = \mu_p G_D(Q^2), \quad G_M^n(Q^2) = \mu_n G_D(Q^2).$$
The standard dipole parameterization is

\[ G_D(Q^2) = \frac{1}{(1 + \frac{Q^2}{M_V^2})^2}, \tag{C.4} \]

where \( M_V^2 = 0.71\text{GeV}^2 \) and the proton and neutron magnetic moments \( \mu_p \) and \( \mu_n \) are 2.793 and -1.913 respectively in the units of the nucleon magneton \( \mu_N = e/2m \). The Dirac and Pauli form factors are related to these form factors by Eq. (3.32) and (3.33). Form factors are also obtained from a variety of theoretical models by describing these experimental data consistently. Two parametrizations of the nucleon form factors are used in our calculation.

### C.1 BBA Parameterization

The so-called BBA form factor used in our calculation was a nucleon parameterization presented by Budd, Bodek and Arrington ([22]). They use the following inverse polynomial fit to account for the deviations from the dipole form (C.3) for proton electric and proton and neutron magnetic form factors:

\[ G_{E,M}^N(Q^2) = \frac{G_{E,M}^N(Q^2 = 0)}{1 + a_2 Q^2 + a_4 Q^4 + a_6 Q^6 + \cdots}, \tag{C.5} \]

where \( G_{E,M}^N(Q^2 = 0) \) are calculated using the dipole form (C.3). The coefficients \( a_i \) are given in table (C.1). They use the same parameterization as Krutov et. al. [46] for
neutron electric form factor:

\[ G_E^n(Q^2) = -\mu_n \frac{a\tau}{1 + b\tau} G_D(Q^2), \quad (C.6) \]

where \( a = 0.942 \) and \( b = 4.61 \).

### C.2 BI Parameterization

Bijker and Iachello [17] gave a parameterization of the spacelike and timelike nucleon electromagnetic form factors. The spacelike form factors have been used in our work. In their definition, the Sachs form factors can be obtained from the isoscalar and isovector form factors as follows:

\[ G_{M,p}(Q^2) = (F_1^S(Q^2) + F_1^V(Q^2)) + \tau(F_2^S(Q^2) + F_2^V(Q^2)), \quad (C.7) \]

\[ G_{E,p}(Q^2) = (F_1^S(Q^2) + F_1^V(Q^2)) - \tau(F_2^S(Q^2) + F_2^V(Q^2)), \quad (C.8) \]

\[ G_{M,n}(Q^2) = (F_1^S(Q^2) - F_1^V(Q^2)) + \tau(F_2^S(Q^2) - F_2^V(Q^2)), \quad (C.9) \]

\[ G_{E,n}(Q^2) = (F_1^S(Q^2) - F_1^V(Q^2)) - \tau(F_2^S(Q^2) - F_2^V(Q^2)). \quad (C.10) \]
The isoscalar and isovector form factors are defined as:

\[ F_S^1(Q^2) = \frac{1}{2} g(Q^2) \left[ 1 - \beta_\omega - \beta_\phi + \beta_\omega \frac{m_\omega^2}{m_\omega^2 + Q^2} + \beta_\phi \frac{m_\phi^2}{m_\phi^2 + Q^2} \right], \quad (C.11) \]

\[ F_V^1(Q^2) = \frac{1}{2} g(Q^2) \left[ 1 - \beta_\rho + \beta_\rho \frac{m_\rho^2}{m_\rho^2 + Q^2} \right], \quad (C.12) \]

\[ F_S^2(Q^2) = \frac{1}{2} g(Q^2) \left[ (\mu_p + \mu_n - 1 - \alpha_\phi) \frac{m_\omega^2}{m_\omega^2 + Q^2} + \alpha_\phi \frac{m_\phi^2}{m_\phi^2 + Q^2} \right], \quad (C.13) \]

\[ F_V^2(Q^2) = \frac{1}{2} g(Q^2) \left[ \frac{(\mu_p + \mu_n - 1 - \alpha_\phi)}{1 + \gamma Q^2} + \alpha_\phi \frac{m_\rho^2}{m_\rho^2 + Q^2} \right], \quad (C.14) \]

where \( g(Q^2) = (1 + \gamma Q^2)^{-2} \). The factor \( \frac{m_\rho^2}{m_\rho^2 + Q^2} \) is replaced as:

\[ \frac{m_\rho^2}{m_\rho^2 + Q^2} \rightarrow \frac{m_\rho^2 + 8\Gamma_\rho m_\pi/\pi}{m_\rho^2 + Q^2 + (4m_\pi^2 + Q^2)\Gamma_\rho \alpha(Q^2)/m_\pi}, \quad (C.15) \]

where

\[ \alpha(Q^2) = \frac{2}{\pi} \left[ \frac{4m_\pi^2 + Q^2}{Q^2} \right]^{1/2} \ln \left( \frac{\sqrt{4m_\pi^2 + Q^2} + \sqrt{Q^2}}{2m_\pi} \right). \quad (C.16) \]

The values of the parameters used in the above equations are listed in table (C.2).

<table>
<thead>
<tr>
<th>( \mu_p )</th>
<th>( m_\phi )</th>
<th>( 1.019\text{GeV} )</th>
<th>( \alpha_\rho )</th>
<th>( 2.675 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_n )</td>
<td>( \beta_\rho )</td>
<td>( 0.512 )</td>
<td>( \alpha_\phi )</td>
<td>( -0.200 )</td>
</tr>
<tr>
<td>( m_\rho )</td>
<td>( \beta_\omega )</td>
<td>( 1.129 )</td>
<td>( \gamma )</td>
<td>( 0.515(\text{GeV})^{-2} )</td>
</tr>
<tr>
<td>( m_\omega )</td>
<td>( \beta_\phi )</td>
<td>( -0.263 )</td>
<td>( \Gamma_\rho )</td>
<td>( 0.112 )</td>
</tr>
</tbody>
</table>

Table C.2: Parameters used in the BI nucleon form factor parameterization.
Table C.3: Parameters used in the Kelly nucleon form factor parameterization.

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_E^p$</td>
<td>-0.24</td>
<td>10.98</td>
<td>12.82</td>
<td>21.97</td>
</tr>
<tr>
<td>$G_M^p/\mu_p$</td>
<td>0.12</td>
<td>10.97</td>
<td>18.86</td>
<td>6.55</td>
</tr>
<tr>
<td>$G_M^n/\mu_n$</td>
<td>2.33</td>
<td>14.72</td>
<td>24.20</td>
<td>84.1</td>
</tr>
</tbody>
</table>

C.3 Kelly Parameterization

Kelly ([44]) presented a simple parametrization for the proton electric and proton and neutron magnetic form factors using the following form:

$$G(Q^2) = \frac{\sum_{i=0}^{n} a_i \tau_i}{1 + \sum_{i=1}^{n+2} b_i \tau_i},$$

(C.17)

where $a_0 = 1$ and $n = 1$. The coefficients $a_i$ and $b_i$ are given in table (C.3). The neutron electric form factor is defined as:

$$G_E^n = \frac{A\tau}{1 + B\tau} G_D.$$

(C.18)

where $G_D$ is the dipole form factor (C.3) and $A = 1.70, B = 3.30$.

C.4 BBBA Parameterization

Bradford, Bodek, Budd and Arrington ([20]) presented the following form to parameterize nucleon form factors:

$$G(Q^2) = \frac{\sum_{k=0}^{n} a_k \tau^k}{1 + \sum_{i=1}^{n+2} b_k \tau^k}.$$

(C.19)
It was the first time for this particular form being employed for all four form factors. The coefficients $a_k$ and $b_k$ are given in table (C.4).

### C.5 Lomon Parameterization

Lomon presented a parameterization of the nucleon form factors based on the extended Gari-Krumpelmann model in reference [50]. They decompose the Dirac and Pauli form factors into isoscalar and isovector parts:

\[
2F_i^p = (F_i^{is} + F_i^{iv}), \quad 2F_i^n = (F_i^{is} - F_i^{iv}), \quad (i = 1, 2) \tag{C.20}
\]

In Lomon’s definition the Sachs form factors are:

\[
G_{En}(Q^2) = F_1^N(Q^2) - \tau F_2^N(Q^2) \tag{C.21}
\]
\[
G_{Mn}(Q^2) = F_1^N(Q^2) + \tau F_2^N(Q^2).
\]
The four isotopic electromagnetic form factors in the GKex model has the following form:

\[ F_{iv}^{1}(Q^2) = \frac{N}{2} \frac{1.0317 + 0.0875(1 + Q^2/0.3176)^{-2}}{1 + Q^2/0.5496} F_{1}^{\rho}(Q^2) \]  
\[ + \frac{g_{\rho}'}{f_{\rho}'} \frac{m_{\rho}^2}{m_{\rho}^2 + Q^2} F_{1}^{\rho}(Q^2) + (1 - 1.1192N/2 - \frac{g_{\rho}'}{f_{\rho}'} F_{1}^{D}(Q^2), \]  
\[ F_{iv}^{2}(Q^2) = \frac{N}{2} \frac{5.7824 + 0.3907(1 + Q^2/0.1422)^{-1}}{1 + Q^2/0.5362} F_{2}^{\rho}(Q^2) \]  
\[ + \kappa_{\rho} \frac{g_{\rho}'}{f_{\rho}'} \frac{m_{\rho}^2}{m_{\rho}^2 + Q^2} F_{2}^{\rho}(Q^2) + (\kappa_{\rho} - 6.1731N/2 - \kappa_{\rho} \frac{g_{\rho}'}{f_{\rho}'} F_{2}^{D}(Q^2), \]  
\[ F_{is}^{1}(Q^2) = \frac{g_{\omega}}{f_{\omega}} \frac{m_{\omega}^2}{m_{\omega}^2 + Q^2} F_{1}^{\omega}(Q^2) + \frac{g_{\omega}'}{f_{\omega}'} \frac{m_{\omega}^2}{m_{\omega}^2 + Q^2} F_{1}^{\omega}(Q^2) \]  
\[ + \frac{g_{\phi}}{f_{\phi}} \frac{m_{\phi}^2}{m_{\phi}^2 + Q^2} F_{1}^{\phi}(Q^2) + (1 - \frac{g_{\omega}}{f_{\omega}} - \frac{g_{\omega}'}{f_{\omega}'} F_{1}^{D}(Q^2), \]  
\[ F_{is}^{2}(Q^2) = \frac{\kappa_{\omega}}{f_{\omega}} \frac{m_{\omega}^2}{m_{\omega}^2 + Q^2} F_{2}^{\omega}(Q^2) + \frac{\kappa_{\omega}}{f_{\omega}'} \frac{m_{\omega}^2}{m_{\omega}^2 + Q^2} F_{2}^{\omega}(Q^2) \]  
\[ + \frac{\kappa_{\phi}}{f_{\phi}} \frac{m_{\phi}^2}{m_{\phi}^2 + Q^2} F_{2}^{\phi}(Q^2) + (\kappa_{s} - \frac{g_{\omega}}{f_{\omega}} - \frac{g_{\phi}}{f_{\phi}} F_{2}^{D}(Q^2). \]  

The hadronic form factors are parameterized as follows:

\[ F_{1}^{\omega, D}(Q^2) = \frac{\Lambda_{1,D}^2}{\Lambda_{1,D}^2 + Q^2}, \]  
\[ F_{2}^{\omega, D}(Q^2) = \frac{\Lambda_{2,D}^2}{\Lambda_{2,D}^2 + Q^2}, \]  
\[ F_{1}^{\phi, D}(Q^2) = \frac{\Lambda_{1,D}^2}{\Lambda_{1,D}^2 + Q^2}, \]  
\[ F_{2}^{\phi, D}(Q^2) = \frac{\Lambda_{2,D}^2}{\Lambda_{2,D}^2 + Q^2}. \]
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\(\kappa_v\) & 3.076 & \(g(\rho')/f(\rho')\) & 0.007208 & \(g(\omega')/f(\omega')\) & 0.164 \\
\hline
\(\kappa_s\) & -0.12 & \(\kappa(\rho')\) & 12.0 & \(\kappa(\omega')\) & -2.973 \\
\hline
\(m_\rho\) & 0.776GeV & \(g(\omega)/f(\omega)\) & 0.7021 & \(\Lambda_1\) & 0.93088 \\
\hline
\(m_\omega\) & 0.784Gev & \(\kappa_\omega\) & 0.4027 & \(\Lambda_D\) & 1.181 \\
\hline
\(m_\phi\) & 1.019GeV & \(g(\phi)/f(\phi)\) & -0.1711 & \(\Lambda_2\) & 2.6115 \\
\hline
\(m'_\rho\) & 1.45GeV & \(\kappa_\phi\) & 0.01 & \(\Lambda_{QCD}\) & 0.015 \\
\hline
\(m'_\omega\) & 1.419GeV & \(\mu_\phi\) & 0.2 & \(N\) & 1.0 \\
\hline
\end{tabular}
\caption{Parameters used in the Lomon nucleon form factor parameterization.}
\end{table}

\[ F_{2,1,2}^\alpha(Q^2) = \frac{\Lambda_{1,2}^2}{\Lambda_{1,2}^2 + Q^2} (\frac{\Lambda_{1,2}^2}{\Lambda_{1,2}^2 + Q^2})^{1.5}, \]

where \(\alpha = \rho, \omega\). \(\Lambda_{1,2}\) is \(\Lambda_1\) for \(F_1^\alpha\) and \(\Lambda_D\) for \(F_i^D\) respectively and

\[ F_1^\phi(Q^2) = F_1^\alpha(Q^2)(\frac{Q^2}{\Lambda_1^2 + Q^2})^{1.5}, \quad F_1^\phi(0) = 0, \]

\[ F_2^\phi(Q^2) = F_2^\alpha(Q^2)(\frac{\Lambda_1^2 Q^2 + \mu_\phi^2}{\mu_\phi^2 \Lambda_1^2 + Q^2})^{1.5}, \]

\[ \tilde{Q}^2 = Q^2 \frac{\ln[(\Lambda_D^2 + Q^2)/\Lambda_{QCD}^2]}{\ln(\Lambda_D^2/\Lambda_{QCD}^2)}. \]

The values of the parameters in the above equations are listed in table (C.5).

### C.6 Hybrid Parameterizations

The hybrid nucleon form factor parameterizations are used in our calculation to test the origin of the difference in the deuteron structure functions. The Kelly-AMT4 parameterization uses the neutron form factor parameterization as in the Kelly parameterization \[44\] and the proton form factor parameterization presented by Arrington,
Table C.6: Parameters used in the hybrid nucleon form factor parameterizations.

<table>
<thead>
<tr>
<th>AMT4</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
<th>$b_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G^p_E$</td>
<td>-1.651</td>
<td>1.287</td>
<td>-0.185</td>
<td>9.531</td>
<td>0.591</td>
<td>0.000</td>
<td>0.000</td>
<td>4.994</td>
</tr>
<tr>
<td>$G^p_M$</td>
<td>-2.151</td>
<td>4.261</td>
<td>0.159</td>
<td>8.647</td>
<td>0.001</td>
<td>5.245</td>
<td>82.817</td>
<td>14.191</td>
</tr>
</tbody>
</table>

Melnitchouk and Tjon [8]:

\[ G^p_E, \quad G^p_M/\mu_p = \frac{1 + \sum_{i=1}^n a_i \tau^i}{1 + \sum_{i=1}^{n+2} b_i \tau^i}, \]  

where $n$ is taken to be 3. We use the parameters $a_i$ and $b_i$ in Table 4 in Ref. [8], which are fit to the one-pion-exchange data and listed listed in Table (C.6). Refer to C.3 for the neutron form factor parameterization. Similarly, in the Kelly-BBBA and Kelly-BI parameterizations, the proton form factor parameterization are the same as in the Kelly parameterization [44] and the neutron parameterization are the ones used in BBBA and BI parameterizations respectively. Parameters are listed in Tables C.3, C.4 and C.2.
BIBLIOGRAPHY


