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GENERALIZATIONS OF GCD-DOMAINS AND RELATED TOPICS

by

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ABSTRACT

GCD–domains are an important class of integral domains from classical ideal theory. In a GCD–domain, the intersection of any two principal ideals is principal. This property can be generalized in several different ways. A domain for which the intersection of any two invertible ideals is invertible is called a generalized GCD–domain (GGCD–domain). If for elements $a, b \in R - \{0\}$, there is an $n = n(a, b)$ with $a^n R \cap b^n R$ principal, we say $R$ is an almost GCD–domain (AGCD–domain). Combining these two definitions, we get an almost generalized GCD–domain (AGGCD–domain) — for $a, b \in R - \{0\}$, there is an $n = n(a, b)$ with $a^n R \cap b^n R$ invertible.

Anderson and Zafrullah began the study of the first two of these generalizations. They showed that the integral closure of an AGCD–domain is also an AGCD–domain. We show that, in general, an overring of an AGCD–domain need not be an AGCD–domain. Certain special types of overrings do, however, inherit the property. Among these, it is shown, are localizations and LCM–stable overrings. A similar result holds for the AGGCD–domains. Relationships between these classes of domains and the classical domains of ring theory are investigated.

We also investigate how adding the property that $R$ is Noetherian affects an AGCD– or AGGCD–domain. It is shown that a Noetherian AGCD–domain
is almost weakly factorial, that is, $R = \cap R_P$, where $P$ ranges over all rank one primes of $R$, has finite character and $R$ has torsion $t$–class group. Similarly, it is shown that a Noetherian AGGCD–domain is weakly Krull, that is, $R = \cap R_P$, where $P$ ranges over all rank one primes of $R$, has finite character.

Finally, we consider two additional generalizations defined using the ideal $I_n$, where $I_n = \{i^n \mid i \in I\}$. A domain $R$ is called a nearly GCD–domain or NGCD–domain (respectively nearly generalized GCD–domain or NGGCD–domain) if for $a, b \in R - \{0\}$, there is an $n = n(a, b)$ with $[(a, b)_n]_t$ principal (respectively invertible).
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CHAPTER I
PRELIMINARIES

We will restrict ourselves to rings which are commutative with identity. Throughout this thesis, \( n \) will refer to a natural number unless otherwise indicated. The general references for basic results from algebra and commutative ring theory will be [13] and [15], respectively.

Let \( R \) be an integral domain with quotient field \( K \). The set of all nonzero fractional ideals of \( R \) is denoted by \( F(R) \) and \( f(R) \) denotes the subset of \( F(R) \) consisting of finitely generated fractional ideals. The inverse of a fractional ideal \( I \), written \( I^{-1} \), is \( I^{-1} = \{ y \in K \mid yI \subseteq R \} \). \( I^{-1} \) is also a fractional ideal of \( R \) and \( I \) is said to be invertible if \( II^{-1} = R \). The set of invertible fractional ideals of \( R \) is a subgroup of \( F(R) \) which is denoted \( \text{Inv}(R) \). Invertible ideals have some nice properties which will be used extensively. Any invertible ideal is finitely generated and in a semi-quasilocal domain, that is, a domain with a finite number of maximal ideals, any invertible ideal is actually principal. For \( R \) local, the generator can be chosen to be one of the original generators of the ideal. Also, for \( I \) finitely generated, if \( I_M \) is principal for each maximal ideal \( M \), then \( I \) is invertible. Another useful property is that if \( S \) is a multiplicatively closed subset of \( R \) and \( I \in \text{Inv}(R) \), then \( IS \in \text{Inv}(R_S) \). Also, if \( n \) is any natural number and \((a_1, \ldots, a_k)\) is invertible, then \((a_1, \ldots, a_k)^n = (a_1^n, \ldots, a_k^n)\). The
principal fractional ideals of $R$, denoted $\text{Prin}(R)$, are a subgroup of $\text{Inv}(R)$. The quotient group $\frac{\text{Inv}(R)}{\text{Prin}(R)} = C(R)$ is called the class group of $R$.

A $*$-operation is a function $*: F(R) \rightarrow F(R)$ which satisfies (1) $(a)^* = (a)$, $(aA)^* = aA^*$; (2) $A \subseteq A^*$, if $A \subseteq B$, then $A^* \subseteq B^*$; and (3) $(A^*)^* = A$. A fractional ideal $I$ is called a $*$-ideal if $I = I^*$, and a $*$-ideal $I$ is said to be of finite type if $I = J^*$ for some $J \in \mathfrak{f}(R)$. One important property of $*$-operations is for $A, B \in F(R)$, $(AB)^* = (A^*B)^* = (A^*B^*)^*$. For other properties of $*$-operations, see [11]. The trivial $*$-operation, called the $d$-operation, is the $*$-operation for which $I_d = I$ for any $I \in F(R)$. There are two other examples of $*$-operations which are of special interest. If $I \in F(R)$, $(I^{-1})^{-1}$ is denoted $I_v$. The operation on $F(R)$ which sends $I$ to $I_v$ is a $*$-operation. This operation is called the $v$-operation. Another important example, called the $t$-operation, is defined by sending $I \in F(R)$ to $I_t = \cup\{J_v \mid J \subseteq I \text{ and } J \in \mathfrak{f}(R)\}$. Thus if $I$ is finitely generated, $I_t = I_v$. A fractional ideal $I$ is $t$-invertible if there is a fractional ideal $J$ with $(IJ)_t = R$. In this case, we can take $J = I^{-1}$. Clearly, if $I$ is invertible, then $I$ is also $t$-invertible.

Let $f_t(R)$ be the set of finite type $t$-ideals of $R$. Then $f_t(R)$ is a semi-group under the "$t$-product" $I \cdot J = (IJ)_t$. Now $f_t(R)$ need not form a group. An integral domain $R$ in which $f_t(R)$ does form a group is called a Prüfer $v$-multiplication domain (PVMD). The set of $t$-invertible $t$-ideals, denoted $I_t(R)$, is a subgroup of $f_t(R)$. The set of principal ideals, $P(R)$, is a subgroup of $I_t(R)$ and the quotient group $\text{Cl}_t(R) = \frac{I_t(R)}{P(R)}$ is called the $t$-class group of $R$. 
Let \( R \subseteq S \) be an extension of commutative rings. The integral closure of \( R \) in \( S \) is denoted \( \overline{R} \). If no \( S \) is indicated, \( \overline{R} \) is the integral closure of \( R \) in its quotient field. If \( R \subseteq S \) has the property that for each \( s \in S \), there exists an \( n = n(s) \) with \( s^n \in R \), then the extension is called a root extension. It is easily seen that a root extension is an integral extension. A domain \( R \) is said to be integrally closed in \( S \) if \( \overline{R} = R \). Similarly, \( R \) is said to be root closed in \( S \) if for any \( s \in S \), \( s^n \in R \) implies \( s \in R \).

An integral domain \( R \) is a GCD-domain if any two elements of \( R - \{0\} \) have a greatest common divisor. UFD's are well-known examples of GCD-domains. In the definition of a GCD-domain, it is not assumed that the greatest common divisor is a linear combination of the two elements. This is a stronger assumption and can be restated by saying that for any \( a, b \in R \), \( (a, b) \) is principal, or equivalently, every finitely generated ideal is principal. Domains with this property are called Bézout domains. It is well-known that if two elements, \( a \) and \( b \), have a least common multiple, \( \text{lcm}(a, b) \), then they also have a greatest common divisor — namely, \( \text{gcd}(a, b) = \frac{ab}{\text{lcm}(a, b)} \). An integral domain in which every pair of nonzero elements has a least common multiple is called an LCM-domain. Thus an LCM-domain is always a GCD-domain. However, two elements can have a greatest common divisor without having a least common multiple.

**Example 1.1.** Let \( R \) be a ring and consider \( R[x^2, x^3] \), that is, polynomials over \( R \) with no linear term. Now it is easily seen that \( \text{gcd}(x^2, x^3) = 1 \). But \( x^2 \) and \( x^3 \) have no least common multiple in \( R[x^2, x^3] \). For if they did, since
\[ \text{gcd}(x^2, x^3) = 1, \text{ we would have } 1 = \frac{x^2 \cdot x^3}{\text{lcm}(x^2, x^3)} \text{ and hence lcm}(x^2, x^3) = x^5. \] But \[ x^6 = x^2 \cdot x^4 = x^3 \cdot x^3 \text{ is a common multiple and } x^5 \mid x^6. \]

This cannot happen, however, if every pair of nonzero elements has a greatest common divisor, that is, in a GCD-domain. So, in fact, \( R \) is a GCD-domain if and only if \( R \) is an LCM-domain.

The assumption that each pair of nonzero elements, \( a \) and \( b \), in \( R \) have a greatest common divisor can be strengthened still further by insisting that the greatest common divisor be either \( a \) or \( b \) itself. Equivalently, either \( a \mid b \) or \( b \mid a \).

A domain where this condition holds for any pair of nonzero elements, \( a \) and \( b \), is called a valuation domain. As stated earlier, a Bézout domain is an integral domain in which every finitely generated ideal is principal. This condition can be weakened by only requiring that every finitely generated ideal be invertible.

A domain with this property is called a Prüfer domain. It is well–known that an integral domain is a Prüfer domain if and only if for every maximal ideal \( M \) of \( R \), \( R_M \) is a valuation domain.

In [17], Zafrullah first introduced a fundamental definition which is a further generalization of a GCD-domain, called an almost GCD-domain (AGCD–domain). An integral domain \( R \) is called an AGCD–domain if for \( a, b \in R - \{0\} \), there exists an \( n = n(a, b) \) with \( a^nR \cap b^nR \) principal. Since \[ a^nR \cap b^nR = a^n b^n \left( \frac{1}{a^n} R \cap \frac{1}{b^n} R \right) \text{ and } \frac{1}{a^n} R \cap \frac{1}{b^n} R = (a^n, b^n)^{-1}, a^nR \cap b^nR \text{ is principal if and only if } (a^n, b^n)^{-1} \text{ is principal. But, } (a^n, b^n)^{-1} \text{ is principal if and only if } [(a^n, b^n)^{-1}]^{-1} = (a^n, b^n)_v \text{ is principal. Therefore, } a^nR \cap b^nR \text{ is principal if and only if } (a^n, b^n)_v \text{ is principal and we have an equivalent definition} \]
for an AGCD-domain. Similarly, if \( aR : bR = \{ r \in R \mid rb \in aR \} \), then \\
aR \cap bR = (aR : bR)bR. Thus \( aR \cap bR \) is principal if and only if \( aR : bR \) is principal. This gives us another equivalent definition of an AGCD-domain. It is well-known that \( aR \cap bR \) is principal (equivalently \( (a, b)_v \) is principal) if and only if \( a \) and \( b \) have a least common multiple and in this case \( aR \cap bR = cR \) where \( c = \text{lcm}(a, b) \) and \( (a, b)_v = dR \) where \( d = \text{gcd}(a, b) \). It is easily seen that \( R \) is an AGCD-domain if and only if for \( a_1, a_2, \ldots, a_k \in R - \{0\} \), there exists an \( n = n(a_1, \ldots, a_k) \) with \( a_1^nR \cap \ldots \cap a_k^nR \) principal. Some of the important results on AGCD-domains from [17] were summarized in the following theorem from [5] which appeared as Theorem 3.1.

**Theorem 1.2.** (1) Let \( R \) be an AGCD-domain. Then \( \overline{R} \) is an AGCD-domain and \( R \subset \overline{R} \) is a root extension.

(2) Let \( R \) be an integrally closed integral domain. Then \( R \) is an AGCD-domain if and only if \( R \) is a PVMD with torsion \( t \)-class group.

The proof of (2) actually only requires that \( R \) is a root closed domain. So we actually have the following result.

**Theorem 1.3.** Let \( R \) be an integral domain. Then the following statements are equivalent.

(1) \( R \) is an integrally closed AGCD-domain.

(2) \( R \) is a root closed AGCD-domain.

(3) \( R \) is a PVMD with \( Cl_t(R) \) torsion.
Thus a root closed AGCD-domain has a torsion $t$-class group. In [5], it was shown that, in fact, any AGCD-domain has torsion $t$-class group.

Anderson and Zafrullah continued the study of AGCD-domains in [5], and introduced the definitions of some related classes of domains. Now a GCD-domain is also an LCM-domain, and so every pair of nonzero elements has a least common multiple. Therefore, a GCD-domain is characterized by the property that for $a, b \in R \setminus \{0\}$, $(a, b)_v$ is principal. A Bézout domain is defined by the stronger property that for $a, b \in R \setminus \{0\}$, $(a, b)$ is principal. Since an AGCD-domain is defined by the property that for $a, b \in R \setminus \{0\}$, $(a^n, b^n)_v$ is principal for some $n$, Anderson and Zafrullah defined a domain $R$ to be an almost Bézout domain (AB-domain) if for $a, b \in R \setminus \{0\}$, there exists an $n = n(a, b)$ with $(a^n, b^n)$ principal. Now a Prüfer domain can be characterized by the property that $(a, b)$ is invertible for any $a, b \in R \setminus \{0\}$. Thus, they defined an almost Prüfer domain (AP-domain) by the property that for $a, b \in R \setminus \{0\}$, $(a^n, b^n)$ is invertible for some $n$. In the same spirit, they made the following additional definitions.

**Definition 1.4.** Let $R$ be an integral domain.

1. $R$ is an almost principal ideal domain (API-domain) if for any nonempty subset $\{a_\alpha\} \subseteq R \setminus \{0\}$, there exists an $n = n(\{a_\alpha\})$ with $(\{a_\alpha^n\})$ principal.

2. $R$ is an AD-domain if for any nonempty subset $\{a_\alpha\} \subseteq R \setminus \{0\}$, there exists an $n = n(\{a_\alpha\})$ with $(\{a_\alpha^n\})$ invertible.
(3) \( R \) is an almost valuation domain (AV-domain) if for \( a, b \in R - \{0\} \), there exists an \( n = n(a, b) \) with \( a^n | b^n \) or \( b^n | a^n \).

We will be concerned primarily with the AGCD-, AB-, and AP-domains. The following theorem summarizes some of the results from [5] concerning AB- and AP-domains. By an overring of \( R \), we mean a ring \( S \) with \( R \subseteq S \subseteq K \) where \( K \) is the quotient field of \( R \).

**Theorem 1.5.**

(1) Let \( R \) be an AB-domain (respectively, AP-domain) and let \( S \) be an overring of \( R \). Then \( S \) is an AB-domain (respectively, AP-domain).

(2) Let \( R \) be an integral domain and \( S \) an overring with \( R \subseteq S \subseteq R \). Then \( R \) is an AB-domain (respectively, AP-domain) if and only if \( S \) is an AB-domain (respectively, AP-domain) and for each \( s \in S \), there exists an \( n = n(s) \) with \( s^n \in R \).

(3) Let \( R \) be an integral domain. Then the following statements are equivalent:

   a) \( R \) is an integrally closed AB-domain (respectively, AP-domain).

   b) \( R \) is a root closed AB-domain (respectively AP-domain).

   c) \( R \) is a Prüfer domain with torsion class group (respectively, Prüfer domain).

Another generalization of a GCD-domain was first defined in [1]. Here an integral domain \( R \) is called a generalized GCD-domain (G-GCD domain) if the intersection of any two (integral) invertible ideals of \( R \) is invertible. Several
equivalent conditions for an integral domain to be a G-GCD domain were given in [2]. The following theorem summarizes a few of these equivalences.

Theorem 1.6. For an integral domain $R$, the following statements are equivalent.

1. $R$ is a G-GCD domain.
2. Any finite intersection of invertible ideals of $R$ is invertible.
3. For $a, b \in R - \{0\}$, $aR \cap bR$ is invertible.
4. For $a, b \in R - \{0\}$, $aR : bR$ is invertible.

It is well known that if we replace "invertible" by "principal" then the above conditions are all equivalent to $R$ being a GCD-domain. As previously noted, $aR \cap bR$ is invertible if and only if $(a, b)_v$ is invertible. Thus we may add:

5. For $a, b \in R - \{0\}$, $(a, b)_v$ is invertible.

In the spirit of the other definitions in [5], an integral domain $R$ was defined to be an almost generalized GCD-domain (AGGCD-domain) if for each $a, b \in R - \{0\}$, there exists an $n = n(a, b)$ with $a^nR \cap b^nR$ invertible (equivalently, $(a^n, b^n)_v$ invertible). It will be shown that many of the results from [5] concerning AGCD-domains and AP-domains can be extended to AGGCD-domains.
A fundamental definition introduced by Zafrullah in [17] is that of an almost GCD-domain (AGCD-domain). Recall that an integral domain $R$ is called an AGCD-domain if for $a, b \in R - \{0\}$, there exists an $n = n(a, b)$ with $a^n R \cap b^n R$ principal (or equivalently, $(a^n, b^n)_v$ principal). In [5], Anderson and Zafrullah showed that $R$ is an AGCD-domain if and only if for $a_1, \ldots, a_k \in R - \{0\}$, there exists an $n = n(a_1, \ldots, a_k)$ with $a_1^n R \cap \cdots \cap a_k^n R$ principal. It can be seen that this is equivalent to the existence of an $n = n(a_1, \ldots, a_k)$ with $(a_1^n, \ldots, a_k^n)_v$ principal. Unlike in the $k = 2$ case, however, this relies on the property holding for any set of $k$ elements from $R - \{0\}$ rather than just for the elements $a_1, \ldots, a_k$. To see this, let $a_1, \ldots, a_k \in R - \{0\}$. Then $(a_1, \ldots, a_k)^{-1} = \prod_{i=1}^{k} a_i^{-1} R \cap \cdots \cap \prod_{i=1}^{k} a_i^{-1} R$ and hence $a_1 a_2 \cdots a_k (a_1, \ldots, a_k)^{-1} = \hat{a}_1 R \cap \cdots \cap \hat{a}_k R$ where $\hat{a}_i = \frac{a_1 \cdots a_k}{a_i}$. Now if there exists an $n = n(\hat{a}_1, \ldots, \hat{a}_k)$ with $(\hat{a}_1)^n R \cap \cdots \cap (\hat{a}_k)^n R$ principal, then $a_1^n a_2^n \cdots a_k^n (a_1^n, \ldots, a_k^n) = (\hat{a}_1)^n R \cap \cdots \cap (\hat{a}_k)^n R$ is principal. So $(a_1^n, \ldots, a_k^n)^{-1}$ is principal, and hence so is $(a_1^n, \ldots, a_k^n)_v = [(a_1^n, \ldots, a_k^n)^{-1}]^{-1}$.

A similar argument shows that if for any $a_1, \ldots, a_k \in R - \{0\}$ there is an $n = n(a_1, \ldots, a_k)$ with $(a_1^n, \ldots, a_k^n)_v$ principal, then for any $a_1, \ldots, a_k \in R - \{0\}$, there is an $n = n(a_1, \ldots, a_k)$ with $a_1^n R \cap \cdots \cap a_k^n R$ principal. So there are
several properties for a domain $R$ which are equivalent to $R$ being an AGCD-domain. Some common examples of AGCD-domains include LCM-domains or GCD-domains and thus UFD’s.

In [17], Zafrullah proved that if $R$ is an AGCD-domain, then $\overline{R}$ is also an AGCD-domain and $R \subseteq \overline{R}$ is a root extension. In a later paper, [5], Anderson and Zafrullah raised the question of whether the converse to this statement held. That is, if $\overline{R}$ is an AGCD-domain and $R \subseteq \overline{R}$ is a root extension, must $R$ also be an AGCD-domain? Although it at first seemed plausible, the following example shows that, in fact, the converse is false. This example makes use of what is known as a $D + M$ construction. We will use the notation $\gcd_R(a, b)$ (respectively, $\lcm_R(a, b)$) for the greatest common divisor (respectively, least common multiple) of $a$ and $b$ in a ring $R$.

Example 2.1. Let $K \subseteq L$ be a root extension where $K$ and $L$ are both fields (for instance, take $K \subseteq L$ to be a purely inseparable field extension). Let $R = K + (x_1, \ldots, x_n)L[x_1, \ldots, x_n]$ where the $x_i$’s are indeterminates. Then

1) $\overline{R}$ is an AGCD-domain; 2) $R \subseteq \overline{R}$ is a root extension; and 3) $R$ is not an AGCD-domain if $n > 2$.

1) If $R = K + (x_1, \ldots, x_n)L[x_1, \ldots, x_n]$, then $\overline{R} = L[x_1, \ldots, x_n]$ which is a UFD since $L$ is a field. But a UFD is a GCD-domain and hence an AGCD-domain. Thus $\overline{R}$ is an AGCD-domain.

2) Let $f = f(x_1, \ldots, x_n) \in \overline{R} = L[x_1, \ldots, x_n]$. Then $f$ can be written in the form $f = l + g$ where $l \in L$ and $g \in (x_1, \ldots, x_n)L[x_1, \ldots, x_n]$. Since $K \subseteq L$ is a root extension, there exists an $n$ such that $l^n \in K$. 
Now the constant term of \( f^n = (l + g)^n \) is \( l^n \in K \), and, therefore, 
\[ f^n \in K + (x_1, \ldots, x_n)\mathbb{L}[x_1, \ldots, x_n] = R. \] So \( R \subseteq \overline{R} \) is a root extension.

3) Consider the elements \( x_1, x_2 \in R - \{0\} \). We first note that for any \( k \),
\( x_1^k \) and \( x_2^k \) are relatively prime in \( R \). For in \( \overline{R} = \mathbb{L}[x_1, \ldots, x_n] \), a GCD-domain, the only common divisors of \( x_1^k \) and \( x_2^k \) are units — that is, any nonzero constant polynomial. But any common divisor of \( x_1^k \) and \( x_2^k \) in \( R \) is also a common divisor in \( \overline{R} \) and hence must be a nonzero constant polynomial from \( R \). Such a polynomial is a unit in \( R \). Therefore, \( \gcd_R(x_1^k, x_2^k) = 1 \) for any \( k \). So \( x_1^k \) and \( x_2^k \) are relatively prime in \( R \) for any \( k \).

Assume \( R \) is an AGCD-domain. Then there exists an \( n = n(x_1, x_2) \) with \( (x_1^n, x_2^n) \) principal. This is equivalent to saying that \( x_1^n \) and \( x_2^n \) have a least common multiple in \( R \). Then \( x_1^n \) and \( x_2^n \) also have a greatest common divisor, namely \( \gcd_R(x_1^n, x_2^n) = \frac{x_1^n x_2^n}{\lcm_R(x_1^n, x_2^n)} \). From above, we have that \( \gcd_R(x_1^n, x_2^n) = 1 \) and hence \( \lcm_R(x_1^n, x_2^n) = x_1^n x_2^n \).

Finally, let \( l \in L - K \) and consider the element \( lx_1^n x_2^n \) of \( R \). Since \( lx_1^n, lx_2^n, x_1^n \) and \( x_2^n \) are all elements of \( R \), clearly \( lx_1^n x_2^n \) is a common multiple of \( x_1^n \) and \( x_2^n \) in \( R \). So since \( x_1^n x_2^n \) is the least common multiple, we must have \( x_1^n x_2^n \mid lx_1^n x_2^n \). That is, for some \( f \in R \), \( lx_1^n x_2^n = f x_1^n x_2^n \). But this gives that \( l = f \in R \), contradicting that \( l \in L - K \). This contradiction shows that \( R \) is not an AGCD-domain.

It was shown in [5] that by adding one additional hypothesis, the desired conclusion can be reached. The following appeared in [5] as Theorem 5.9.
Theorem 2.2. An integral domain $R$ is an AGCD-domain if and only if

1) $\overline{R}$ is an AGCD-domain,

2) $R \subseteq \overline{R}$ is a root extension, and

3) if $a_1, \ldots, a_k \in R - \{0\}$ satisfy $((a_1, \ldots, a_k)\overline{R})_v = \overline{R}$, then $((a_1, \ldots, a_k)R)_v = R$.

Condition 3) above is a property which Anderson and Zafrullah have called $t$-linked under. An integral domain $R$ is said to be $t$-linked under an overring $S$ if whenever $a_1, \ldots, a_k \in R - \{0\}$ satisfy $((a_1, \ldots, a_k)S)_v = S$, then $((a_1, \ldots, a_k)R)_v = R$. This is the converse of the notion of $S$ being a $t$-linked overring of $R$ which was introduced in [8]. Another case in which $\overline{R}$ an AGCD-domain and $R \subseteq \overline{R}$ a root extension gives that $R$ must be an AGCD-domain was also discussed in [5]. The additional condition needed involved the $t$-dimension of the domain $R$, that is, the length of the longest chain of $t$-ideals. The result showed that if $R$ is of $t$-dimension one, then the conclusion also holds.

Although having the overring $\overline{R}$ of $R$ an AGCD-domain need not give that $R$ is an AGCD-domain, we know that $R$ an AGCD-domain always implies that $\overline{R}$ is an AGCD-domain. This brings up the question of whether the same is true for other overrings of an AGCD-domain. Recall that an integral domain $R$ is called an AB-domain (respectively, AP-domain) if for any $a, b \in R - \{0\}$, there exists an $n = n(a, b)$ with $(a^n, b^n)$ principal (respectively, invertible). It was shown in [5] that every overring of an AB-domain (respectively, AP-domain) is an AB-domain (respectively, AP-domain). Unfortunately, the analogous
statement for AGCD-domains is not true. The next example shows that an overring of an AGCD-domain need not be an AGCD-domain.

Example 2.3. Let \( R = \mathbb{Z}[x] \). Since \( \mathbb{Z} \) is a UFD, so is \( R \). Hence \( R \) is a GCD-domain and, therefore, an AGCD-domain. Consider the overring \( S = \mathbb{Z}[x][\{ \frac{z^{m+1}}{2m} \mid m \geq 1 \}] \) of \( R = \mathbb{Z}[x] \). We first note the following:

1) \( \frac{z}{2} \notin S \): Suppose \( \frac{z}{2} \in S \). Then there exist \( f_0, f_1, \ldots, f_k \in \mathbb{Z}[x] \) with \( \frac{z}{2} = f_0 + f_1 \cdot \frac{z^2}{2} + f_2 \cdot \frac{z^3}{2^2} + \cdots + f_k \cdot \frac{z^{k+1}}{2^k} \). Equivalently, \( x = 2f_0 + f_1x^2 + \cdots + f_k \cdot \frac{z^{k+1}}{2^k} \).

Now the coefficient of the \( x \) term on the right is \( 2f_{0x} \), where \( f_{0x} \) is the coefficient of the linear term of \( f_0 \), and hence is in \( \mathbb{Z} \). Equating coefficients, \( 1 = 2f_{0x} \) and hence \( \frac{1}{2} = f_{0x} \in \mathbb{Z} \), a contradiction. Therefore \( \frac{z}{2} \notin S \).

2) \( \gcd_s(x^k, 2^k) = 2^{k-1} \) for any \( k \): In \( \mathbb{Z}[x] \), a UFD, the only divisors of \( 2^k \), for any \( k \), are \( \{2^j \mid 0 \leq j \leq k\} \) since 2 is irreducible. Clearly, 2 is still irreducible in \( S \) and hence for any \( k \), the divisors of \( 2^k \) in \( S \) are \( \{2^j \mid 0 \leq j \leq k\} \). But in light of 1), the powers of 2 which divide \( x^k \) in \( S \) are those less than \( k \). Therefore, the only common divisors of \( x^k \) and \( 2^k \) in \( S \) are \( \{2^j \mid 0 \leq j \leq k - 1\} \). Since \( 2^i \mid 2^j \) for \( i \leq j \), we have that \( \gcd_s(x^k, 2^k) = 2^{k-1} \) for any \( k \).

Now assume that \( S \) is an AGCD-domain. Then for some \( n = n(x, 2) \), we have \( (x^n, 2^n)_v \) principal and hence \( x^n \) and \( 2^n \) have a least common multiple, \( \text{lcm}_S(x^n, 2^n) \). But then \( x^n \) and \( 2^n \) also have a greatest common divisor, and \( \gcd_s(x^n, 2^n) = \frac{2^n x^n}{\text{lcm}_s(x^n, 2^n)} \). So by 2) above, \( 2^{n-1} = \frac{2^n x^n}{\text{lcm}_s(x^n, 2^n)} \) and hence \( \text{lcm}_S(x^n, 2^n) = 2^n x^n \). Now consider the element \( x^{n+1} \in S \). Since \( x^{n+1} = x \cdot x^n = \)}
\[ \frac{x^{n+1}}{2^n} \cdot 2^n, \ x^{n+1} \] is a common multiple of \( x^n \) and \( 2^n \) in \( S \). So there is some \( s \in S \) with \( x^{n+1} = 2x^n s \). But this says that \( \frac{x}{2} = \frac{x^{n+1}}{2x^n} = s \in S \), contradicting 1) above. Therefore, the overring \( S \) is not an AGCD-domain. \( \square \)

We now know that an overring of an AGCD-domain need not be an AGCD-domain. This leads to the question of whether there are certain types of overrings outside of the integral closure which are also AGCD-domains. The answer to this question is "yes." In particular, any LCM-stable overring of an AGCD-domain is also an AGCD-domain. Recall that an overring \( S \) of a domain \( R \) is said to be LCM-stable over \( R \) if for all \( a, b \in R \), \( (aR \cap bR)S = aS \cap bS \). This property was first studied in [10] and [16].

**Theorem 2.4.** Let \( R \) be an AGCD-domain and \( S \) an LCM stable overring of \( R \). Then \( S \) is also an AGCD-domain.

**Proof:** Let \( \frac{a}{b}, \frac{c}{d} \in S \) where \( a, b, c, d \in R \). Then \( ad \) and \( bc \) are in \( R \), so since \( R \) is an AGCD-domain, there is an \( n = n(ad, bc) \) such that \((ad^n) \cap (bc)^n = R \) is principal. Now since \( S \) is LCM-stable over \( R \), \((ad^n)S \cap (bc)^nS = [ (ad)^n \cap (bc)^n S] S \) and hence is also principal. Therefore, \((\frac{a}{b})^n S \cap (\frac{c}{d})^n S = \frac{a^n}{b^n} S \cap \frac{c^n}{d^n} S = \frac{1}{b^n d^n} [(ad^n S \cap (bc)^n S] \) is principal and \( S \) is an AGCD-domain. \( \square \)

Notice that in the above proof, all that was needed was that \((a \cap b)S = aS \cap bS \) where \( a \cap b \) is a principal ideal. Therefore, LCM-stability is actually a stronger hypothesis than is required. We could call an overring \( S \) of a domain \( R \) principally LCM-stable over \( R \) if for all \( a, b \in R \) with \( aR \cap bR \)
principal, \((aR \cap bR)S = aS \cap bS\). Then any principally stable overring of an AGCD-domain is an AGCD-domain.

Recall that an \(R\)-module \(T\) is flat if the functor \(T \otimes_R \) is exact. (That is, if \(0 \to A \to B \to C \to 0\) is an exact sequence of \(R\)-modules, then \(0 \to T \otimes_R A \to T \otimes_R B \to T \otimes_R C \to 0\) is also exact.) It is well known that flatness implies LCM-stability, but the converse is false, see [16]. This gives us an immediate corollary.

**Corollary 2.5.** Let \(R\) be an AGCD-domain and \(S\) a flat overring of \(R\). Then \(S\) is also an AGCD-domain.

In particular, since every localization of a domain is a flat overring, this gives that every localization of an AGCD-domain is also an AGCD-domain.

It was shown in [5] that, in fact, every flat overring of an AGCD-domain \(R\) is actually a localization \(R\).

One of the most common extensions of an integral domain \(R\) is the polynomial ring \(R[x]\). An interesting question is when a polynomial ring inherits a property from the base ring and conversely. It is well known, for example, that if \(R\) is Noetherian, then \(R[x]\) is also Noetherian. This is what is known as the Hilbert Basis Theorem. If \(R\) is integrally closed, then \(R[x]\) is integrally closed. Also, if \(R\) is a UFD (respectively, GCD-domain) then \(R[x]\) is a UFD (respectively, GCD-domain) [15]. We are interested in whether a similar relationship holds for an AGCD-domain \(R\). We first concern ourselves with the reverse relationship. That is, if \(R[x]\) is an AGCD-domain, does \(R\) necessarily
have the same property? The answer to this question is "yes," as is shown below in Proposition 2.7. We also show below another property which $R$ has whenever the polynomial ring does — the property that the integral closure is a root extension of the ring itself.

**Proposition 2.6.** If $R[x] \subseteq \overline{R[x]}$ is a root extension, then $R \subseteq \overline{R}$ is a root extension.

**Proof:** First note that since $\overline{R}$ is integrally closed, so is $\overline{R[x]}$. Thus since $R[x] \subseteq \overline{R[x]}$, we have $\overline{R[x]} \subseteq \overline{R[x]} = \overline{R[x]}$. But clearly $\overline{R} \subseteq \overline{R[x]}$ and $x \in \overline{R[x]}$ and hence $\overline{R[x]} \subseteq \overline{R[x]}$. Therefore, $\overline{R[x]} = \overline{R[x]}$. Let $a \in \overline{R}$. Then $a \in \overline{R[x]} = \overline{R[x]}$. Since $R[x] \subseteq \overline{R[x]}$ is a root extension, there exists an $n = n(a)$ with $a^n \in R[x]$. But the degree of $a^n$ is 0 and, therefore, $a^n \in R$. Thus $R \subseteq \overline{R}$ is a root extension. \qed

**Proposition 2.7.** Let $R$ be an integral domain. If $R[x]$ is an AGCD-domain, then $R$ is an AGCD-domain.

**Proof:** Let $a, b \in R \subseteq R[x]$. Then since $R[x]$ is an AGCD-domain, there exists an $n = n(a, b)$ with $a^n R[x] \cap b^n R[x] = f R[x]$ for some $f \in R[x]$. Now since $a^n b^n \in a^n R[x] \cap b^n R[x] = f R[x]$, there is a $g \in R[x]$ with $a^n b^n = f g$. The degree of $a^n b^n$ is 0 and hence $0 = \deg(f g) = \deg(f) + \deg(g)$. Therefore, we must have $\deg(f) = \deg(g) = 0$ and so $f \in R$. Then we have $f \in f R[x] \cap R$ and so $f R \subseteq f R[x] \cap R$. But for any $h \in f R[x] \cap R$, there is a $k \in R[x]$ with $fk = h \in R$. Thus as above, $0 = \deg(h) = \deg(f) + \deg(k) = 0 + \deg(k) = \deg(k)$, and so
Therefore, \( h = fk \in fR \) and \( fR[x] \cap R \subseteq fR \). So we actually have that \( fR[x] \cap R = fR \).

Then \((a^nR \cap b^nR)[x] = a^nR[x] \cap b^nR[x] = fR[x], and a^nR \cap b^nR = (a^nR[x] \cap b^nR[x]) \cap R = fR[x] \cap R = fR \) is principal. Thus \( R \) is an AGCD-domain.

\( \square \)

We now turn our attention to the converse of the above result. Is the property of being an AGCD-domain inherited by the polynomial ring? Zafrullah began the study of this question in [17]. He considered the question in the case where \( R \) is a PVMD with torsion \( t \)-class group. Recall that PVMD is an integral domain in which \( ft(R) \) forms a group, that is, where every finite type \( t \)-ideal is \( t \)-invertible. To say that \( R \) has torsion \( t \)-class group says that for each \( t \)-invertible \( t \)-ideal \( I \), there is some \( n \) with \((I^n)_t \) principal. Zafrullah showed that an integrally closed domain \( R \) satisfies these properties if and only if the same holds for \( R[x] \). The following appeared in [17] as Theorem 5.6.

**Theorem 2.8.** Let \( R \) be an integrally closed integral domain and let \( x \) be an indeterminate over \( R \). Then \( R \) is a PVMD with torsion \( t \)-class group if and only if \( R[x] \) has the same property.

This does not immediately seem to have any bearing on the question at hand: if \( R \) is an AGCD-domain, is \( R[x] \) an AGCD-domain? However, it was shown in [12], that to show an integral domain \( R \) is a PVMD, one need only prove that every two generated ideal of \( R \) is \( t \)-invertible. Using this fact, Zafrullah goes on to show that an integrally closed AGCD-domain is a PVMD.
with torsion \( t \)-class group. Thus in the integrally closed case, we have an answer to our question.

**Corollary 2.9.** Let \( R \) be an integrally closed integral domain and let \( x \) be an indeterminate over \( R \). Then \( R \) is an AGCD-domain if and only if \( R[x] \) is an AGCD-domain.

The hypotheses of the above result may actually be weakened slightly. The proof from [17] which shows that an integrally closed AGCD-domain \( R \) is a PVMD with torsion \( t \)-class group relies only on the fact that \( R \) is a root closed AGCD-domain. In order to be integrally closed, a domain \( R \) must contain the roots of all monic polynomials. To be root closed, \( R \) need only contain the roots of polynomials of the form \( x^n - r \) where \( r \in R \) and \( n \) is any natural number. Thus we have a slightly improved version of the above result.

**Corollary 2.10.** Let \( R \) be a root closed integral domain and let \( x \) be an indeterminate over \( R \). Then \( R \) is an AGCD-domain if and only if \( R[x] \) is an AGCD-domain.

In the general case, where \( R \) is any AGCD-domain, the relationship between \( R \) and \( R[x] \) is not as straightforward. We have shown that if \( R[x] \) is an AGCD-domain, the \( R \) must also be an AGCD-domain. The converse, however, does not hold in general. If \( R \) is an AGCD-domain, then \( R[x] \) may (as in the case above) or may not be an AGCD-domain. Furthermore, \( R \) need not be a root closed AGCD-domain in order for the converse to hold. The following
examples illustrate these two points. The first gives a non-root closed AGCD-domain for which the polynomial ring is also an AGCD-domain. The second shows that the property of \( R \) being an AGCD-domain need not be inherited by the polynomial ring \( R[x] \).

For the first example, we will need to recall the definition of the content of a polynomial. Using the notation of [13], if \( R \) is a GCD-domain and \( f = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) is a nonzero polynomial over \( R \), then the greatest common divisor of \( a_0, a_1, \ldots, a_n \) is called the content of \( f \) and is denoted \( C(f) \). In a GCD-domain \( R \), Gauss' Lemma gives us that for \( f, g \in R[x] \), \( C(fg) = C(f)C(g) \). We are now ready for our example.

**Example 2.11.** Let \( R = \mathbb{Z}[2i] \). Then 1) \( R \) is a non-root closed AGCD-domain; 2) if \( f \in \overline{R}[x] - R[x] \) (that is, \( \mathbb{Z}[i][x] - \mathbb{Z}[2i][x] \)) and \( g \in R[x] \) with \( fg \in R[x] \), then \( 2 \mid C(g) \) in \( \mathbb{Z} \); and 3) for any \( h, k \in R[x] \), \( \text{lcm}_{R[x]}(h^2, k^2) \) exists, and thus \( R[x] \) is an AGCD-domain.

1) \( R \) is clearly not root closed, since \( i \) has the property that \( i^2 = -1 \in R \), while \( i \notin R \). In fact, for any \( a + bi \in \mathbb{Z}[i] \), \( (a + bi)^2 \in R \) and \( \overline{R} = \mathbb{Z}[i] \), a PID. Thus \( \overline{R} \) is an AB-domain, for if \( a, b \in \overline{R} - \{0\} \), \( (a, b) \) is principal. But since for any \( s \in \overline{R} \), \( s^2 \in R \) we have that \( R \) is also an AB-domain and hence an AGCD-domain.

2) Let \( f \in \overline{R}[x] - R[x] \), \( g \in R[x] \) with \( fg \in R[x] \). Then there exist \( f_0, f_1, g_0 \) and \( g_1 \in \mathbb{Z}[x] \) with \( f = f_0 + if_1 \) and \( g = g_0 + 2ig_1 \). Now \( fg = (f_0 + if_1)(g_0 + 2ig_1) = (f_0g_0 - 2f_1g_1) + i(2f_0g_1 + f_1g_0) \). Thus if \( fg \in R[x] \), we must have that \( 2 \mid C(2f_0g_1 + f_1g_0) \) in \( \mathbb{Z} \) and hence
2 \mid C(f_1 g_0). But f_1, g_0 \in \mathbb{Z}[x] and \mathbb{Z} is a UFD and hence a GCD-domain. Therefore, \( C(f_1 g_0) = C(f_1) C(g_0) \). So 2 \mid C(f_1) C(g_0) in \mathbb{Z}. But 2 is prime in \mathbb{Z} and so either 2 \mid C(f_1) or 2 \mid C(g_0). By assumption, \( f \notin R[x] \) and so 2 \mid C(f_1). Hence we must have 2 \mid C(g_0), that is, there exists a \( g_0' \in \mathbb{Z}[x] \) with \( g_0 = 2g_0' \). Then \( g = g_0 + 2i g_1 = 2g_0' + 2i g_1 = 2(g_0' + i g_1) \) and 2 \mid C(g) in \mathbb{Z} as desired.

3) Let \( h, k \in R[x] \). Now \( \overline{R[x]} = \mathbb{Z}[i][x] \) is a UFD and hence we can factor \( h \) and \( k \) uniquely into irreducible factors in \( \overline{R[x]} \). Say we have

\[
h = c_1 \cdots c_n h_1 \cdots h_s \quad \text{and} \quad k = c_1 \cdots c_n k_1 \cdots k_t
\]

where each of the \( c_i, 1 \leq i \leq n \), the \( h_i, 1 \leq i \leq s \), and the \( k_i, 1 \leq i \leq t \) are irreducible in \( \overline{R[x]} \). Then we have that

\[
h^2 = c_1^2 \cdots c_n^2 h_1^2 \cdots h_s^2 \quad \text{and} \quad k^2 = c_1^2 \cdots c_n^2 k_1^2 \cdots k_t^2
\]

and thus having \( \overline{R[x]} \) a UFD gives us that \( \gcd_{\overline{R[x]}}(h^2, k^2) = c_1^2 \cdots c_n^2 \). Now since \( \overline{R[x]} \) is a UFD and hence a GCD-domain, it is also a LCM-domain. So \( \text{lcm}_{\overline{R[x]}}(h^2, k^2) \) exists and we have the relationship

\[
c_1^2 \cdots c_n^2 = \gcd_{\overline{R[x]}}(h^2, k^2) = \frac{h^2 k^2}{\text{lcm}_{\overline{R[x]}}(h^2, k^2)}.
\]

Thus \( \text{lcm}_{\overline{R[x]}}(h^2, k^2) = \frac{h^2 k^2}{c_1^2 \cdots c_n^2} = h^2(k_1^2 \cdots k_t^2) = k^2(h_1^2 \cdots h_s^2) \). Now for any \( f \in \overline{R[x]}, f^2 \in R[x] \). For if \( f \in \overline{R[x]} = \mathbb{Z}[i][x] \), \( f \) can be written in the form \( f = f_0 + i f_1 \) where \( f_0, f_1 \in \mathbb{Z}[i] \). Thus \( f^2 = (f_0 + f_1)^2 = (f_0 + i f_1)^2 + 2i f_0 f_1 \in R[x] \). Therefore, \( h^2, k^2, h_1^2 \cdots h_s^2, \) and \( k_1^2 \cdots k_t^2 \) are all
elements of $R[x]$. This says that $\text{lcm}_{R[x]}(h^2, k^2)$ is also a common multiple of $h^2$ and $k^2$ down in $R[x]$. Suppose $f$ is any common multiple of $h^2$ and $k^2$ in $R[x]$. Then we must have that $f$, $\frac{f}{h^2}$ and $\frac{f}{k^2}$ are all elements of $R[x]$. Now clearly, $f$ is also a common multiple of $h^2$ and $k^2$ in $\overline{R[x]}$, and hence we must have that $\text{lcm}_{\overline{R[x]}}(h^2, k^2) | f$ in $\overline{R[x]}$. Thus for some $g \in \overline{R[x]}$, $f = \text{lcm}_{\overline{R[x]}}(h^2, k^2) \cdot g = h^2(k^2_1 \cdots k^2_{t_1})g = k^2(h^2_1 \cdots h^2_{t_2})g$. But from above, we have $f, \frac{f}{h^2} = k^2_1 \cdots k^2_{t_1}g$ and $\frac{f}{k^2} = h^2_1 \cdots h^2_{t_2}g \in R[x]$. So, in particular, if $g \notin R[x]$, then by 2) we must have that $2 | C(k^2_1 \cdots k^2_{t_1})$ and $2 | C(h^2_1 \cdots h^2_{t_2})$ in $\mathbb{Z}$. But this contradicts that $\gcd_{\overline{R[x]}}(h^2, k^2) = c^2_1 \cdots c^2_{t_2}$ as earlier determined. Therefore, $g \in R[x]$ and so $f = \text{lcm}_{\overline{R[x]}}(h^2, k^2) \cdot g = h^2(k^2_1 \cdots k^2_{t_1}) \cdot g$ shows that $\text{lcm}_{\overline{R[x]}}(h^2, k^2) | f$ in $R[x]$. Thus $\text{lcm}_{\overline{R[x]}}(h^2, k^2)$ is a common multiple of $h^2$ and $k^2$ in $R[x]$ which divides any other common multiple of $h^2$ and $k^2$ in $R[x]$. Therefore $\text{lcm}_{\overline{R[x]}}(h^2, k^2) = \text{lcm}_{R[x]}(h^2, k^2)$ and so $h^2$ and $k^2$ have a least common multiple in $R[x]$.

Then for any two elements $h$ and $k \in R[x]$, $\text{lcm}_{R[x]}(h^2, k^2)$ exists. But $h^2$ and $k^2$ have a least common multiple in $R[x]$ if and only if $h^2 R[x] \cap k^2 R[x]$ is principal. Hence $R[x]$ is an AGCD–domain with $n(h, k) = 2$ for any $h, k \in R[x]$.

We now turn to an example of an AGCD–domain $R$ for which $R[x]$ is not an AGCD–domain.

**Example 2.12.** Let $K = GF(3)$, the Galois field of order 3, and $F = K(\sqrt{2}) = K[\sqrt{2}]$ (since $\sqrt{2}$ is algebraic over $K$). Let $R = K + F[[y]]$ so that $\overline{R} = F[[y]]$. 
Then 1) $R$ is an AB-domain and hence an AGCD-domain; and 2) $R[x] \subseteq \overline{R[x]}$ is not a root extension and hence $R[x]$ is not an AGCD-domain.

1) $F$ is a finite field of order $3^2$ and hence $a^{3^2-1} = 1 \in K$ for every $0 \neq a$ in $F$. So $K \subseteq F$ is clearly a root extension. But then $R \subseteq \overline{R}$ is also a root extension. For if $f \in \overline{R} = F[[y]]$, then $f = f_0 + f_1y + f_2y^2 + \cdots$ where each $f_i \in F$. In particular, $f_0 \in F$ and so there is an $n$ with $f_0^n \in K$. Then $f^n = (f_0 + f_1y + f_2y^2 + \cdots)^n = f_0^n + f' \in F[[y]]$. So since $f_0^n \in K$, $f^n = f_0^n + f' \in K + F[[y]] = R$. Now since $F$ is a field, $\overline{R} = F[[y]]$ is a PID and hence an AB-domain. Therefore $R$ is also an AB-domain and hence an AGCD-domain.

2) Let $g \in \overline{R}[x]$. Then $g = g_0 + g_1x + \cdots + g_nx^n$ where each $g_i \in F[[y]]$. So we can write the $g_i$'s, $1 \leq i \leq n$, as follows:

\[
\begin{align*}
g_0 &= a_0 + a_1y + a_2y^2 + \cdots \\
g_1 &= b_0 + b_1y + b_2y^2 + \cdots \\
g_2 &= c_0 + c_1y + c_2y^2 + \cdots \\
&\vdots \\
g_n &= d_0 + d_1y + d_2y^2 + \cdots
\end{align*}
\]

where all the coefficients of the $g_i$'s, $0 \leq i \leq n$, are from $F$. Then $g$ can be rewritten in the form

\[
g = (a_0 + b_0x + c_0x^2 + \cdots + d_0x^n) y^n + \cdots
\]
and can thus be considered as an element of $F[[y]]$. Now in order to have $g^k \in R[x]$, for some $k$, the coefficients of the terms in $g^k$ which involve no $y$ must come from $K$. But the terms not involving $y$ in $g^k$ come from $(a_0 + b_0x + c_0x^2 + \cdots + d_0x^n)^k$. Therefore, in order to have $g^k \in R[x]$, we need to have $(a_0 + b_0x + c_0x^2 + \cdots + d_0x^n)^k \in K[x]$. Thus in order to show that $R[x] \subseteq R[x]$ (that is, $K[x] + F[x][[y]] \subseteq F[x][[y]]$) is not a root extension, it suffices to show that $K[x] \subseteq F[x]$ is not a root extension.

Consider the element $1 + \sqrt{2}x \in F[x]$. Suppose there is a natural number $k$ with $(1 + \sqrt{2}x)^k = 1 + (\sqrt{2}x)^k \in K[x]$. Then, in particular, we must have $(\sqrt{2})^k \in K$. Clearly, $(\sqrt{2})^{2j-1} \not\in K$ for any natural number $j$. Thus $k$ must be a multiple of 2, say $k = 2m$. So we have

$$(1 + \sqrt{2}x)^{2m} = 1 + \binom{2m}{1}\sqrt{2}x + \binom{2m}{2}2x^2 + \binom{2m}{3}2\sqrt{2}x^3 + \cdots + 2^m x^{2m}.$$ 

Now since $\sqrt{2} \not\in K$, the only way for a coefficient involving $\sqrt{2}$ to be in $K$ is if it is zero in $K$. Since $K$ has characteristic 3, this says that $(1 + \sqrt{2}x)^{2m} \in K[x]$ if and only if $3 \not| (\binom{2m}{j})$ for all $j$ odd with $1 \leq j \leq 2m - 1$. Therefore, we must show that for every $m$, there is some odd $j$, $1 \leq j \leq 2m - 1$, with $3 \not| (\binom{2m}{j})$. There are three cases:

a) If $m \equiv 0 \pmod{3}$, say $m = 3^nr$ where $n \geq 1$ and $3 \not| r$. Then $3 \not| (\binom{2m}{3n})$.

b) If $m \equiv 1 \pmod{3}$, then $2m \equiv 2 \pmod{3}$ and thus $3 \not| 2m = (\binom{2m}{1})$.

c) If $m \equiv 2 \pmod{3}$, then $2m \equiv 4 \pmod{3}$ and again $3 \not| 2m = (\binom{2m}{1})$.

We now prove a).
Consider \( \binom{2m}{3n} = \binom{2 \cdot 3^n r}{3^n} = \frac{2 \cdot 3^n r (2 \cdot 3^n r - 1) (2 \cdot 3^n r - 2) \cdots (2 \cdot 3^n r - 3^n + 1)}{3^n \cdot 1 \cdot 2 \cdots (3^n - 1)} \). Notice that the numerator and denominator are each a product of \( 3^n \) consecutive integers and thus have 3 as a factor the same number of times. We can prove this inductively. Clearly \( 3^n \mid 3^n \) and \( 3^n \mid 2 \cdot 3^n r \), but \( 3^{n+1} \nmid 3^n \) and \( 3^{n+1} \nmid 2 \cdot 3^n r \) since 3 \( \nmid r \). Therefore 3 \( \nmid \frac{2 \cdot 3^n r}{3^n} \). Assume that

\[
3 \mid \frac{(2 \cdot 3^n r) (2 \cdot 3^n r - 1) \cdots (2 \cdot 3^n r - i)}{3^n \cdot 1 \cdot 2 \cdots i}, \quad 1 \leq i \leq 3^n - 1. \quad \text{We will show that}
\]

\[
3 \mid \frac{(2 \cdot 3^n r) (2 \cdot 3^n r - 1) \cdots (2 \cdot 3^n r - i)}{3^n \cdot 1 \cdot 2 \cdots (i + 1)}. \quad \text{Now if 3 \mid i, then also 3 \mid (2 \cdot 3^n r - i)}
\]

and so 3 \( \mid (i + 1) \) and 3 \( \nmid (2 \cdot 3^n r - i - 1) \). Therefore 3 \( \mid \frac{(2 \cdot 3^n r - i - 1)}{i + 1} \).

So by the inductive hypothesis, 3 \( \mid \frac{(2 \cdot 3^n r) (2 \cdot 3^n r - 1) \cdots (2 \cdot 3^n r - i)}{3^n \cdot 1 \cdot 2 \cdots i} \cdot \frac{(2 \cdot 3^n r - i - 1)}{i + 1} \) and we have the desired result. Thus we consider the case where 3 \( \nmid i \).

In this case, either 3 \( \mid (i - 1) \) and an argument similar to the one above holds, or 3 \( \nmid (i - 2) \) and hence 3 \( \mid (i + 1) \). So assume \( i + 1 = 3^s i' \) where \( s < n \) and 3 \( \mid i' \). Then 3 \( \mid (i + 1) \) while 3 \( \mid (i + 1) \). Thus we have

\[
2 \cdot 3^n r - (i + 1) = 2 \cdot 3^n r - 3^s i' = 3^s (2 \cdot 3^s r - i') \quad \text{where 3 \mid (2 \cdot 3^s r - i')}
\]

since 3 \( \mid i' \). Therefore, 3 \( \mid (2 \cdot 3^n r - i - 1) \) but 3 \( \mid (2 \cdot 3^n r - i - 1) \). Hence

3 \( \mid \frac{(2 \cdot 3^n r - i - 1)}{i + 1} \). Again by the inductive hypothesis,

\[
3 \mid \frac{(2 \cdot 3^n r) (2 \cdot 3^n r - 1) \cdots (2 \cdot 3^n r - i)}{3^n \cdot 1 \cdot 2 \cdots i} \cdot \frac{(2 \cdot 3^n r - i - 1)}{i + 1}.
\]

So by induction, 3 \( \mid \frac{(2 \cdot 3^n r) (2 \cdot 3^n r - 1) \cdots (2 \cdot 3^n r - i)}{3^n \cdot 1 \cdot 2 \cdots i} \) for any \( i, 1 \leq i \leq 3^n - 1 \). In particular, when \( i = 3^n - 1 \), we have

\[
3 \mid \frac{(2 \cdot 3^n r) (2 \cdot 3^n r - 1) \cdots (2 \cdot 3^n r - 3^n + 1)}{3^n \cdot 1 \cdot 2 \cdots (3^n - 1)} = \binom{2 \cdot 3^n r}{3^n} = \binom{2m}{3^n}
\]

as desired.
Thus for each $m$, there is some odd $j$, $1 \leq j \leq 2m - 1$, with $3 \nmid \binom{2m}{j}$. Therefore, $(1 + \sqrt{2}x)^{2m} \notin K[x]$ for any $m$ and so $K[x] \subseteq F[x]$ is not a root extension. By our previous remarks, this says that $R[x] \subseteq \overline{R[x]}$ is also not a root extension. Hence $R[x]$ is not an AGCD–domain.

We now investigate the consequences of adding the property that $R$ is Noetherian to an AGCD–domain. Such a domain $R$ is an example of a type of domain which was introduced in [7], called an almost weakly factorial domain. Recall that a UFD is also called a factorial domain. Replacing “prime” by “primary” in the definition of a factorial domain, we get the notion of a weakly factorial domain, which was introduced in [4]. Thus an integral domain $R$ is called a weakly factorial domain if every nonzero nonunit element of $R$ may be written as a product of primary elements (that is, elements which generate principal primary ideals). This condition was then weakened slightly in [7] to define an almost weakly factorial domain. In [7], the authors define an integral domain $R$ to be almost weakly factorial if for each nonzero nonunit $a \in R$, there is an $n = n(a)$ so that $a^n$ is a product of primary elements. They included some conditions equivalent to $R$ being almost weakly factorial. One of these conditions can be used to show that a Noetherian AGCD–domain is almost weakly factorial. The equivalence from [7] which we will use is the following.

**Theorem 2.13.** An integral domain $R$ is almost weakly factorial if and only if $R = \cap R_P$, where the intersection is over all rank one primes of $R$, has finite character and $Cl_t(R)$ is torsion.
Here an intersection \( R = \cap R_{P_a} \) is said to have finite character if each nonzero element of \( R \) is in only a finite number of the \( P_a \). In the proof, we will need the notion of a grade one ideal. An ideal \( I \) of a ring \( R \) is said to have grade one if it contains a non-zero-divisor \( x \) with \( I \subseteq Z(R/xR) = \{ r \in R \mid \exists 0 \neq s \in R/xR \text{ with } rs \in xR \} \). With this we prove the following theorem.

**Theorem 2.14.** If \( R \) is a Noetherian AGCD-domain, then \( R \) is almost weakly factorial.

**Proof:** Recall that for any AGCD-domain \( R \), \( Cl_t(R) \) is torsion. It is well-known that if \( R \) is Noetherian, then \( R = \cap R_Q \) where \( Q \) ranges over the maximal primes of principal ideals and this intersection has finite character. (See [15, Thm. 123]). We show that this intersection is equal to \( \cap R_{P} \), where the \( P \) ranges over the rank one primes of \( R \), which still has finite character.

\( \supseteq \): Suppose that \( Q \) is a maximal prime of \( aR \) for \( a \in R \). Since \( R \) is a Noetherian commutative ring, \( R/aR \) is a finitely generated \( R \)-module with \( Q \subseteq Z(R/aR) \) where \( Z(R/aR) = \{ r \in R \mid \exists 0 \neq b \in R/aR \text{ with } rb \in aR \} \). Then there is a \( 0 \neq b \in R/aR \) with \( Qb = 0 \) in \( R/aR \). That is, \( Qb \subseteq aR \). [15, Thm. 8.2].

Recall that \( \{ r \in R \mid rb \subseteq aR \} \) is denoted \( aR : bR \). Thus we have \( Q \subseteq aR : bR \).

Let \( P \) be the prime ideal minimal over \( aR : bR \). So \( Q \subseteq aR : bR \subseteq P \). Since \( R \) is an AGCD-domain, there exists an \( n = u(a,b) \) with \( a^nR : b^nR \) principal.

Then \( P \) is minimal over \( a^nR : b^nR \). For suppose there is a prime \( P_0 \) with \( a^nR : b^nR \subseteq P_0 \subseteq P \). Then for any \( r \in aR : bR \), we have \( rb \in aR \). Hence \( r^n b^n \in a^nR \) and \( r^n \in a^nR : b^nR \subseteq P_0 \). Thus \( r \in P_0 \) since \( P_0 \) is prime.
Therefore, \( aR : bR \subseteq P_0 \subseteq P \). But \( P \) is minimal over \( aR : bR \), so we must have \( P = P_0 \). So \( P \) is minimal over the principal ideal \( a^nR : b^nR \) and hence rank \( P \leq 1 \) [15, Thm. 142]. Since \( R \) is a domain, \( 0 \) is a prime ideal and \( 0 \not\subseteq P \). Thus rank \( P \geq 1 \). Combining these two statements gives that the rank \( P = 1 \). But \( 0 \neq Q \subseteq aR : bR \subseteq P \) and hence \( P = Q \). Therefore rank \( Q = 1 \), so every maximal prime of a principal ideal in \( R \) has rank one. Thus \( \cap R_P \), where \( P \) ranges over all rank one primes, is contained in \( \cap R_Q \) where \( Q \) ranges over all maximal primes of principal ideals.

\( \subseteq \): Let \( x \in \cap R_Q \) where \( Q \) ranges over the maximal primes of principal ideals.

Suppose \( P \) is a rank one prime of \( R \). If \( P \) is a maximal prime of a principal ideal, then \( x \in R_P \) and we are done. So suppose not. Now every rank one prime has grade one. So for some non-zero-divisor \( p \in P \), we have \( P \subseteq Z(R/pR) \) and \( P \) is not maximal in \( Z(R/pR) \). For otherwise \( P \) would be the maximal prime of \( pR \). So there exists a prime ideal \( P_0 \) with \( P \subseteq P_0 \) and \( P_0 \) maximal in \( Z(R/pR) \). This says that \( P_0 \) is the maximal prime ideal of \( pR \). Hence by assumption, \( x \in R_{P_0} \). But since \( P \not\subseteq P_0 \), we have \( R - P_0 \not\subseteq R - P \) and thus \( R_{P_0} \subseteq R_P \). Therefore, \( x \in R_{P_0} \subseteq R_P \) and so \( x \in \cap R_P \) where \( P \) ranges over all rank one primes of \( R \).

Putting the containments together, we have that the two intersections are equal. Therefore, by our earlier remarks, \( R = \cap R_P \) where \( P \) ranges over the rank one primes of \( R \).

Finally, we show that the intersection has finite character. So assume that there is a nonzero nonunit \( r \in R \) such that \( r \) is contained in an infinite number
of the rank one primes of $R$. In the argument above for $\subseteq$, we have shown that each rank one prime is contained in a maximal prime of a principal ideal. Thus $r$ is contained in an infinite number of maximal primes of principal ideals. But this contradicts the fact that $\cap R_Q$, where $Q$ ranges over these maximal primes of principals, has finite character. So every nonzero nonunit of $R$ is contained in only finitely many rank one primes of $R$. Therefore $\cap R_P$, where $P$ ranges over the rank one primes of $R$, also has finite character.

By Theorem 2.13 stated earlier, $R$ is an almost weakly factorial domain. □
CHAPTER III
ALMOST GENERALIZED GCD–DOMAINS

In [1], Anderson introduced the notion of a generalized GCD–domain (G-GCD domain). An integral domain $R$ is said to be a G-GCD domain if the intersection of two integral invertible ideals of $R$ is invertible. It is clear that this extends to the intersection of any finite number of integral invertible ideals. Examples of G-GCD domains were shown to include (1) GCD–domains, (2) Prüfer domains, and (3) $\pi$–domains, that is, domains in which every principal ideal is a product of prime ideals. Several equivalent conditions for an integral domain to be a G-GCD domain were given. The following theorem summarizes a few of these equivalences from [2] along with the additional statement mentioned above. It is well-known that if we replace “invertible” by “principal” then these conditions are all equivalent to $R$ being a GCD–domain.

**Theorem 3.1.** For an integral domain $R$, the following statements are equivalent:

1. $R$ is a G-GCD domain.
2. Any finite intersection of integral invertible ideals of $R$ is invertible.
3. For $a, b \in R - \{0\}$, $aR \cap bR$ is invertible.
4. For $a_1, \ldots, a_k \in R - \{0\}$, $a_1R \cap a_2R \cap \ldots \cap a_kR$ is invertible.
5. For $a, b \in R - \{0\}$, $aR : bR = \{r \in R \mid rb \in aR\}$ is invertible.
A further generalization of a G-GCD domain was introduced in [5]. Anderson and Zafrullah called an integral domain $R$ an almost generalized GCD-domain (AGGCD-domain) if for each $a, b \in R - \{0\}$, there is an $n = n(a, b)$ with $a^n R \cap b^n R$ invertible. It is easily seen that examples of AGGCD-domains include AGCD-domains and AP-domains. Recall that $R$ is an AP-domain if for each $a, b \in R - \{0\}$, there is an $n = n(a, b)$ with $(a^n, b^n)$ invertible. Many of the results from [5] concerning AGCD-domains and AP-domains can be extended to AGGCD-domains.

We begin by giving some equivalent conditions for an integral domain to be an AGGCD-domain. Each of the statements in Theorem 3.1 for G-GCD domains has an analogous statement for AGGCD-domains. Two other equivalences are also included. It is easily seen that similar statements could also be added to Theorem 3.1 for G-GCD domains.

**Theorem 3.2.** For an integral domain $R$, the following statements are equivalent:

1. $R$ is an AGGCD-domain.
2. For $a_1, \ldots, a_k \in R - \{0\}$, there is an $n = n(a_1, \ldots, a_k)$ with $a_1^n R \cap a_2^n R \cap \ldots \cap a_k^n R$ invertible.
3. For integral invertible ideals $A, B$ of $R$, there is an $n = n(A, B)$ with $A^n \cap B^n$ invertible.
4. For integral invertible ideals $A_1, \ldots, A_k$ of $R$, there is an $n = n(A_1, \ldots, A_k)$ with $A_1^n \cap \ldots \cap A_k^n$ invertible.
5. For $a, b \in R - \{0\}$, there is an $n = n(a, b)$ with $(a^n, b^n)_v$ invertible.
(6) For \(a_1, \ldots, a_k \in R - \{0\}\), there is an \(n = n(a_1, \ldots, a_k)\) with 
\((a_1^n, \ldots, a_k^n)_v\) invertible.

(7) For \(a, b \in R - \{0\}\), there is an \(n = n(a, b)\) with \(a^n R : b^n R\) invertible.

For simplicity, we first prove several lemmas which will be used in the proof of the theorem. A very useful result from [5] generalizes the well-known fact from commutative ring theory that for an invertible ideal \(A = (\{a_\alpha\})\), we have \(A^n = (\{a^n_\alpha\})\) for any \(n\). Anderson and Zafrullah showed that for \(\{a_\alpha\} \subseteq R - \{0\}\), \(R\) an integral domain, if \((\{a_\alpha\})_t\) is \(t\)-invertible, then \((\{a^n_\alpha\})_t = ((\{a_\alpha\})^n)_t\) for any \(n\). Another property of \(t\)-invertible ideals which will be of use is the following.

**Lemma 3.3.** Let \(R\) be an integral domain. If \(A_1, \ldots, A_k\) are \(t\)-invertible ideals of \(R\), then \((A_1 A_2 \cdots A_k)^{-1} = (A_1^{-1} A_2^{-1} \cdots A_k^{-1})_t\).

**Proof:** The proof will be by induction on \(k\).

\(k = 2:\) Suppose \(A_1\) and \(A_2\) are \(t\)-invertible ideals of \(R\). Then there are ideals \(I_1\) and \(I_2\) with \((A_1 I_1)_t = (A_2 I_2)_t = R\); furthermore, we may take \(I_1 = A_1^{-1}\) and \(I_2 = A_2^{-1}\). Thus using properties of \(*\)-operations, we have 
\((A_1 A_2 (A_1^{-1} A_2^{-1}))_t = (A_1 A_2 A_1^{-1} A_2^{-1})_t = [(A_1 A_1^{-1})_t (A_2 A_2^{-1})_t]_t = R_t = R\). So \(A_1 A_2\) is \(t\)-invertible with \(t\)-inverse \((A_1^{-1} A_2^{-1})_t\), and hence we have \((A_1^{-1} A_2^{-1})_t = (A_1 A_2)^{-1}\).

Assume the result holds for \(k - 1 \ t\)-invertible ideals. Suppose \(A_1, \ldots, A_k\) are \(t\)-invertible ideals of \(R\). Then clearly \(A_1 A_2 \cdots A_{k-1}\) is also \(t\)-invertible. So by the \(k = 2\) case, we have 
\(((A_1 A_2 \cdots A_{k-1}) A_k)^{-1} = ((A_1 A_2 \cdots A_{k-1})^{-1} A_k^{-1})_t\).
Then using the inductive hypothesis and a property of \(*\)-operations, we have
\[(A_1A_2\cdots A_k)^{-1} = ((A_1A_2\cdots A_{k-1})A_k)^{-1} = ((A_1A_2\cdots A_{k-1})^{-1}A_k^{-1})t =
\[(A_1^{-1}A_2^{-1}\cdots A_{k-1}^{-1})tA_k^{-1}] = ((A_1^{-1}A_2^{-1}\cdots A_{k-1}^{-1})A_k^{-1})t = (A_1^{-1}A_2^{-1}\cdots A_k^{-1})t \] as desired.

In the process of showing that the integral closure of an AGCD-domain is also an AGCD-domain in [17], Zafrullah proved the following result.

Lemma 3.4. Let \( R \) be an AGCD-domain and \( a, b \in R - \{0\} \). If \( aR \cap bR \) is principal, then \( (aR \cap bR)^n = a^nR \cap b^nR \) for any \( n \).

This result is actually true in a more general setting. In the next lemma, we show that, in fact, the above result holds in any integral domain \( R \) and for any finite intersection of principal ideals of \( R \).

Lemma 3.5. Let \( R \) be an integral domain and \( a_1, \ldots, a_k \in R - \{0\} \). If \( a_1R \cap \ldots \cap a_kR \) is principal, then \( (a_1R \cap \ldots \cap a_kR)^n = a_1^nR \cap \ldots \cap a_k^nR \) for any \( n \).

Proof: Suppose that \( a_1R \cap \ldots \cap a_kR = bR \) for some \( b \in R \). We first note that in this case, \( bR = a_1R \cap \ldots \cap a_kR = (a_1^{-1}, \ldots, a_k^{-1})^{-1} \). For since \( b \in a_1R \cap \ldots \cap a_kR \), \( b = a_1r_1 = \ldots = a_kr_k \) for some \( r_i \in R \). Hence for each \( i \), \( ba_i^{-1} = a_ir_i^{-1}a_i^{-1} = r_i^{-1}a_i^{-1} = r_i \in R \) and \( b \in (a_1^{-1}, \ldots, a_k^{-1})^{-1} \).

Also, if \( c \in (a_1^{-1}, \ldots, a_k^{-1})^{-1} \), then for each \( i \), \( ca_i^{-1} = r_i \in R \) and hence \( c = r_ia_i \in a_iR \). Therefore, \( c \in a_1R \cap \ldots \cap a_kR \). From this, we see that \( b^{-1}R = [(a_1^{-1}, \ldots, a_k^{-1})^{-1}]^{-1} = (a_1^{-1}, \ldots, a_k^{-1})t = (a_1^{-1}, \ldots, a_k^{-1})t \) since the ideal is finitely generated. Thus \( (a_1^{-1}, \ldots, a_k^{-1})t = b^{-1}R \) is \( t \)-invertible and
hence so is \((a_1^{-1}, \ldots, a_k^{-1})\). (For \(\left[(a_1^{-1}, \ldots, a_k^{-1})_t\right]_t = R\) for some \(I\) says that 
\(\left[(a_1^{-1}, \ldots, a_k^{-1})_t\right]_t = R\) by a property of \(*\)-operations.) Therefore, for any \(n\), we have

\[
b^n R = (a_1 R \cap \ldots \cap a_k R)^n = \left[(a_1^{-1}, \ldots, a_k^{-1})^{-1}\right]^n
\]
\[
= \left[\left[(a_1^{-1}, \ldots, a_k^{-1})_v\right]^{-1}\right]^n
\]
\[
= \left[\left[(a_1^{-1}, \ldots, a_k^{-1})_v\right]^{-1}\right]^n_t
\]
\[
= \left[\left[(a_1^{-1}, \ldots, a_k^{-1})_v\right]^{-1} \ldots \ldots \left[(a_1^{-1}, \ldots, a_k^{-1})_v\right]^{-1}\right]_t
\]
\[
= \left[(a_1^{-1}, \ldots, a_k^{-1})_v \ldots \ldots (a_1^{-1}, \ldots, a_k^{-1})_v\right]^{-1}
\]
\[
= \left[(a_1^{-1}, \ldots, a_k^{-1})_v\right]^{-1}
\]
\[
= \left[\left[(a_1^{-1}, \ldots, a_k^{-1})_v\right]^{-1}\right]_v
\]
\[
= \left[(a_1^{-1}, \ldots, a_k^{-1})_v\right]^{-1}
\]
\[
= \left[(a_1^{-n}, \ldots, a_k^{-n})\right]^{-1}
\]
\[
= (a_1^{-n}, \ldots, a_k^{-n})^{-1}
\]
\[
= a_1^n R \cap \ldots \cap a_k^n R.
\]

The following was shown in the process of proving the equivalent conditions for G-GCD domains in [2]. We isolate the result and include the proof for completeness. Having done this, we will be able to extend Lemma 3.5 to invertible ideals.

**Lemma 3.6.** Let \(R\) be an integral domain. If \(A = (a_1, \ldots, a_m)\) and \(B = (b_1, \ldots, b_n)\) are invertible ideals of \(R\), then \(A_M \cap B_M = [\sum_{i,j} (a_i R \cap b_j R)]_M = (A \cap B)_M\) for any maximal ideal \(M\) of \(R\).
Proof: Let \( A = (a_1, \ldots, a_m) \) and \( B = (b_1, \ldots, b_n) \) be invertible ideals of \( R \) and \( M \) a maximal ideal of \( R \). Then in \( R_M \), a quasilocal domain, we have \( A_M = a_{i_0}R_M \) and \( B_M = b_{j_0}R_M \) for some \( a_{i_0} \) and \( b_{j_0} \). Clearly, \((A \cap B)_M \subseteq A_M \cap B_M\). Thus we have that \( A_M \cap B_M = a_{i_0}R_M \cap b_{j_0}R_M = (a_{i_0}R \cap b_{j_0}R)_M \subseteq \left[ \sum_{i,j} (a_iR \cap b_jR) \right]_M \subseteq (A \cap B)_M \subseteq A_M \cap B_M\). Therefore, \( A_M \cap B_M = \left[ \sum_{i,j} (a_iR \cap b_jR) \right]_M = (A \cap B)_M \) as desired.

We can now improve upon Lemma 3.5 by using Lemma 3.6 and two well-known results from commutative ring theory. First, if \( A \) is an invertible ideal of a domain \( R \), then \( A_M \) is principal for all maximal ideals \( M \) of \( R \). Secondly, if \( A \) and \( B \) are invertible ideals of a domain \( R \) with \( A_M = B_M \) for all maximal ideals \( M \) of \( R \), then \( A = B \). Using these, we can extend Lemma 3.5 to the case of invertible rather than just principal ideals.

Lemma 3.7. Let \( R \) be an integral domain. If \( A_1, A_2, \ldots, A_k \) are invertible ideals of \( R \) with \( A_1 \cap A_2 \cap \ldots \cap A_k \) also invertible, then \( (A_1 \cap \ldots \cap A_k)^n = A_1^n \cap \ldots \cap A_k^n \) for any \( n \).

Proof: Recall that two ideals \( I \) and \( J \) are equal if and only if \( I_M = J_M \) for any maximal ideal \( M \) of \( R \). So let \( M \) be a maximal ideal of \( R \). It suffices to show that \( [(A_1 \cap \ldots \cap A_k)^n]_M = [A_1^n \cap \ldots \cap A_k^n]_M \). Now we have \( [A_1^n \cap \ldots \cap A_k^n]_M = (A_1^n)_M \cap \ldots \cap (A_k^n)_M \) and for any ideal \( I \), \( (I^n)_M = (I_M)^n \). By assumption, \( A_1 \cap \ldots \cap A_k \) is invertible and so \( (A_1 \cap \ldots \cap A_k)_M \) is a principal ideal. Similarly each \( A_i_M \) is also principal. Therefore, using Lemma 3.5 and the above remarks, we have \( [(A_1 \cap \ldots \cap A_k)^n]_M = [(A_1 \cap \ldots \cap A_k)_M]^n = \ldots \).
\((A_{1M})^n \cap \ldots \cap (A_{kM})^n = (A^n_{1})_M \cap \ldots \cap (A^n_{k})_M = [A^n_1 \cap \ldots \cap A^n_k]_M\) as desired.

Thus \((A_1 \cap \ldots \cap A_k)^n = A^n_1 \cap \ldots \cap A^n_k\) for any \(n\).

We are now ready to prove Theorem 3.2. We will show \((1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (2) \Rightarrow (1)\), \((2) \Leftrightarrow (6)\) and \((5) \Leftrightarrow (1) \Leftrightarrow (7)\).

Proof of Theorem 3.2: \((1) \Rightarrow (3)\) Suppose \(R\) is an AGGCD-domain. Let \(A\) and \(B\) be invertible ideals of \(R\). Then \(A\) and \(B\) are finitely generated, say \(A = (a_1, \ldots, a_s)\) and \(B = (b_1, \ldots, b_t)\). Then for any \(m\), \(A^m = (a_1, \ldots, a_s)^m = (a_1^m, \ldots, a_s^m)\) and \(B^m = (b_1, \ldots, b_t)^m = (b_1^m, \ldots, b_t^m)\) are also invertible. So for any maximal ideal \(M\) of \(R\), \((A^m)_M \cap (B^m)_M = \left[ \sum_{i,j} (a_i^m R \cap b_j^m R) \right]_M = (A^m \cap B^m)_M\) by Lemma 3.6. Thus \(A^m \cap B^m = \sum_{i,j} (a_i^m R \cap b_j^m R)\) for any \(m\). Note that the proof of Lemma 3.6 actually shows that \((A^m \cap B^m)_M = (a_{i_0}^m R \cap b_{j_0}^m R)_M\) for some \(a_{i_0}\) and \(b_{j_0}\). Now for each \(i, j\), there is an \(n_{ij}\) with \(a_i^{n_{ij}} R \cap b_j^{n_{ij}} R\) invertible since \(R\) is an AGGCD-domain. But there are only a finite number of these \(n_{ij}\), namely \(i \cdot j\), so let \(n = \prod_{i,j} n_{ij}\). Then for each \(i, j\), \(a_i^n R \cap b_j^n R = a_i^{n_{ij}} \hat{n}_{ij} R \cap b_j^{n_{ij}} \hat{n}_{ij} R\) where \(\hat{n}_{ij} = \frac{n}{n_{ij}}\). But since \(a_i^{n_{ij}} R \cap b_j^{n_{ij}} R\) is invertible, \(a_i^{n_{ij}} \hat{n}_{ij} R \cap b_j^{n_{ij}} \hat{n}_{ij} R = (a_i^{n_{ij}} R \cap b_j^{n_{ij}} R)^{\hat{n}_{ij}}\) by Lemma 3.7 and hence \(a_i^n R \cap b_j^n R = a_i^{n_{ij}} \hat{n}_{ij} R \cap b_j^{n_{ij}} \hat{n}_{ij} R\) is also invertible.

Now \(A^n \cap B^n = \sum_{i,j} (a_i^n R \cap b_j^n R)\) as above. Hence \(A^n \cap B^n\) is a finite sum of invertible ideals and thus is finitely generated. Also, as noted earlier, for each maximal ideal \(M\) of \(R\), \((A^n \cap B^n)_M = (a_{i_0}^n R \cap b_{j_0}^n R)_M\) for some \(a_{i_0}, b_{j_0}\). But each \(a_{i_0}^n R \cap b_{j_0}^n R\) is invertible and hence \((a_{i_0}^n R \cap b_{j_0}^n R)_M\) is principal. Therefore, \(A^n \cap B^n\) is finitely generated and locally principal. Hence \(A^n \cap B^n\) is invertible.
(3) ⇒ (4) The proof is by induction on \( k \). The case for \( k = 2 \) is (3).

Assume the result holds for \( k - 1 \) invertible ideals and let \( A_1, \ldots, A_k \) be invertible ideals of \( R \). By the inductive hypothesis, there is an \( n = n(A_1, \ldots, A_{k-1}) \) with \( A_1^n \cap \ldots \cap A_{k-1}^n \) invertible. Then consider the two invertible ideals \( A_1^n \cap \ldots \cap A_{k-1}^n \) and \( A_k^n \). By (3), there is an \( m \) depending on these ideals with

\[
(A_1^n \cap \ldots \cap A_{k-1}^n)^m \cap A_k^{nm} \text{ invertible.}
\]

But by Lemma 3.7, \((A_1^n \cap \ldots \cap A_{k-1}^n)^m = A_1^{nm} \cap \ldots \cap A_{k-1}^{nm}\) and thus \((A_1^n \cap \ldots \cap A_{k-1}^n)^m \cap A_k^{nm} = A_1^{nm} \cap \ldots \cap A_k^{nm}\) is invertible.

(4) ⇒ (2) ⇒ (1) Clear.

(2) ⇒ (6) Let \( a_1, \ldots, a_k \in R - \{0\} \). Now \((a_1, \ldots, a_k)^{-1} = a_1^{-1}R \cap \ldots \cap a_k^{-1}R\) and thus \(a_1a_2 \cdots a_k(a_1, \ldots, a_k)^{-1} = (a_1^{-1}R \cap \ldots \cap a_k^{-1}R) = \hat{a}_1 \cap \ldots \cap \hat{a}_k \cap R \) where \( \hat{a}_i = \frac{a_1 \cdots a_k}{a_i} \in R \). By (2), there is an \( n = n(\hat{a}_1, \ldots, \hat{a}_k) \) with \((\hat{a}_1)^nR \cap \ldots \cap (\hat{a}_k)^nR\) invertible. Thus \( a_1^n a_2^n \cdots a_k^n(a_1^n, \ldots, a_k^n)^{-1} = \]

\[
a_1^n a_2^n \cdots a_k^n ((a_1^n)^{-1}R \cap \ldots \cap (a_k^n)^{-1}R) = (\hat{a}_1)^nR \cap \ldots \cap (\hat{a}_k)^nR \text{ is invertible. But this is true if and only if } (a_1^n, \ldots, a_k^n)^{-1} \text{ is invertible and hence } (a_1^n, \ldots, a_k^n)_v = [(a_1^n, \ldots, a_k^n)^{-1}]^{-1} \text{ is invertible.}
\]

(6) ⇒ (2) Let \( a_1, \ldots, a_k \in R - \{0\} \) and again let \( \hat{a}_i = \frac{a_1 \cdots a_k}{a_i} \) for each \( i \).

Then \(a_1 \cap \ldots \cap a_k \cap R = a_1 \cdots a_s((\hat{a}_1)^{-1}R \cap \ldots \cap (\hat{a}_k)^{-1}R) = a_1 \cdots a_k(\hat{a}_1, \ldots, \hat{a}_k)^{-1} \) as before. So again, \( a_1 \cap \ldots \cap a_k \cap R \) is invertible if and only if \((\hat{a}_1, \ldots, \hat{a}_k)^{-1} \) and hence \((\hat{a}_1, \ldots, \hat{a}_k)_v \) is invertible. But by (6), there is an \( n = n(\hat{a}_1, \ldots, \hat{a}_k) \) with \(((\hat{a}_1)^n, \ldots, (\hat{a}_k)^n)_v \) invertible and hence \(a_1^n \cap \ldots \cap a_k^n \cap R \) is also invertible.

(5) ⇒ (1) Let \( a, b \in R - \{0\} \). Then for any \( n \), \((a^n, b^n)^{-1} = \frac{1}{a^n} R \cap \frac{1}{b^n} R \) and hence \(a^n \cap b^n \cap R = a^n b^n((\frac{1}{a^n} R \cap \frac{1}{b^n} R) = a^n b^n(a^n, b^n)^{-1} \). Therefore,
\(a^nR \cap b^nR\) is invertible if and only if \((a^n, b^n)^{-1}\) is invertible. But since 
\[(a^n, b^n)_v = [(a^n, b^n)^{-1}]^{-1}, (a^n, b^n)_v\] is invertible if and only if \((a^n, b^n)^{-1}\) is invertible. Thus for any \(n\), \(a^nR \cap b^nR\) is invertible if and only if \((a^n, b^n)_v\) is invertible.

\((1) \Rightarrow (7)\) Let \(a, b \in R - \{0\}\). Then for any \(n\), \(a^nR : b^nR\) satisfies 
\[a^nR \cap b^nR = (a^nR : b^nR)b^nR.\] For if \(x \in a^nR \cap b^nR\), then \(x = ra^n = sb^n\) 
for some \(r, s \in R\). Then clearly, \(sb^n = ra^n \in a^nR\) and so \(s \in a^nR : b^nR\). So 
\[x = sb^n \in (a^nR : b^nR)b^nR.\] Conversely, if \(y \in (a^nR : b^nR)b^nR\), then \(y = rsb^n\) 
where \(rb^n \in a^nR\). Thus \(y = (rs)b^n = s(rb^n) \in a^nR \cap b^nR\). Therefore, for any \(n\), 
\(a^nR \cap b^nR\) is invertible if and only if \(a^nR : b^nR\) is invertible. \(\square\)

As was true in the case of AGCD–domains, there are several types of 
overrings of an AGGCD–domain which are also AGGCD–domains. Recall 
that any LCM–stable overring of an AGCD–domain is also an AGCD–domain. 
Since flatness implies LCM–stability, any flat overring, in particular any 
localization, of an AGCD–domain is an AGCD–domain. Analogous results hold 
for AGGCD–domains.

**Theorem 3.8.** Let \(R\) be an AGGCD–domain and \(S\) an LCM–stable overring of 
\(R\). Then \(S\) is also an AGGCD–domain.

**Proof:** Let \(\frac{a}{b}, \frac{c}{d} \in S\) where \(a, b, c, d \in R\). Then \(ad\) and \(bc\) are in \(R\), so since \(R\) is 
an AGGCD–domain, there is an \(n = n(ad, bc)\) with \((ad)^nR \cap (bc)^nR\) invertible. 
Now since \(S\) is LCM–stable over \(R\), \((ad)^nS \cap (bc)^nS = [(ad)^nR \cap (bc)^nR]S\)
and hence is also invertible. Therefore, \((\frac{a}{b})^nS \cap (\frac{c}{d})^nS = \frac{a^n}{b^n}S \cap \frac{c^n}{d^n}S = \frac{1}{b^n d^n}[(ad)^nS \cap (bc)^nS]\) is invertible and \(S\) is an AGGCD-domain.

**Corollary 3.9.** Let \(R\) be an AGGCD-domain and \(S\) a flat overring of \(R\). Then \(S\) is also an AGGCD-domain.

So since every localization of a domain is a flat overring, every localization of an AGGCD-domain is also an AGGCD-domain. If we localize an AGGCD-domain at a prime ideal \(P\), we get a stronger result.

**Corollary 3.10.** Let \(R\) be an AGGCD-domain and \(P\) a prime ideal of \(R\). Then \(R_P\) is an AGCD-domain.

**Proof:** By the previous corollary, \(R_P\) is an AGGCD-domain. So for any \(a, b \in R_P\), there is an \(n = n(a, b)\) with \(a^nR_P \cap b^nR_P\) invertible. But \(R_P\) is a quasilocal domain and hence any invertible ideal is principal. Thus \(a^nR_P \cap b^nR_P\) is principal and so \(R_P\) is an AGCD-domain.

For the proof of the above corollary, we needed only the fact that in a quasilocal domain, any invertible ideal is principal. According to [15], in order to conclude that any invertible ideal is principal, it suffices to have a semi-quasilocal domain, that is, a domain with only a finite number of maximal ideals. In this case, we have the following result.

**Theorem 3.11.** Let \(R\) be a semi-quasilocal domain. Then \(R\) is an AGGCD-domain if and only if \(R\) is an AGCD-domain.
Proof: ($\Rightarrow$) In a semi-quasilocal domain, any invertible ideal is principal. Thus if $R$ is an AGGCD-domain, for $a, b \in R - \{0\}$ there is an $n = n(a, b)$ with $(a^n, b^n)_{v}$ invertible and hence principal. So $R$ is an AGCD-domain.

($\Leftarrow$) Clearly if $(a^n, b^n)_{v}$ is principal for some $n$, then it is also invertible. So any AGCD-domain is an AGGCD-domain.

For $R$ an AGCD-domain, we know from [17] that $\overline{R}$ is also an AGCD-domain. A similar result is true for AGGCD domains. In order to prove this result, we will need two lemmas. The first of these lemmas is an extension of a lemma proven in [17]. Zafrullah showed the following in his Lemma 3.5.

Lemma 3.12. If $R$ is an AGCD-domain and $a, b \in R$ with $aR \cap bR = abR$, then $aR \cap bR = abR$.

This result can be extended to a more general situation.

Lemma 3.13. Let $R$ be an AGCD-domain. If $a, b \in R$ with $aR \cap bR$ principal, then $(aR \cap bR)_{R} = aR \cap bR$.

Proof: Recall that $aR \cap bR$ is principal (equivalently $[(a, b)_{R}]_{v}$ is principal) if and only if $a$ and $b$ have a least common multiple in $R$ and in this case $aR \cap bR = cR$ where $c = \text{lcm}(a, b)$ and $[(a, b)_{R}]_{v} = dR$ where $d = \text{gcd}(a, b)$.

Now $\left(\frac{a}{d}, \frac{b}{d}\right)_{R} = \left(\frac{1}{d}(a, b)_{R}\right)_{v} = \frac{1}{d}[(a, b)_{R}]_{v} = \frac{1}{d}(dR) = R$. Thus $\frac{a}{d}$ and $\frac{b}{d}$ have a least common multiple in $R$, and since $\left(\frac{a}{d}, \frac{b}{d}\right)_{R} = R$, $1 = \text{gcd}[\frac{a}{d}, \frac{b}{d}] = \frac{ab}{\text{lcm}[\frac{a}{d}, \frac{b}{d}]}$, we have $\text{lcm}[\frac{a}{d}, \frac{b}{d}] = \frac{ab}{d}$. Therefore $\frac{a}{d}R \cap \frac{b}{d}R = \frac{ab}{d^2}R$ and by Zafrullah's result stated above, $\frac{a}{d}R \cap \frac{b}{d}R = \frac{ab}{d^2}R$. Hence $aR \cap bR = d\left(\frac{a}{d}R \cap \frac{b}{d}R\right) = \frac{ab}{d}R$. 

□
But since \( \text{lcm}(a, b) = c \), \( d = \gcd(a, b) = \frac{ab}{c} \) and hence \( c = \frac{ab}{d} \). So we have
\[
(aR \cap bR)R = cR = \frac{ab}{d}R = aR \cap bR
\]
as desired. \( \square \)

The second lemma we will need extends the above result to invertible ideals rather than just principal ideals when \( R \) is an AGGCD-domain. In order to do this, we will need a well-known fact about localizations from commutative ring theory.

**Fact:** Let \( S, T \) be multiplicatively closed sets in \( R \); write \( T^* \) for the image of \( T \) in \( R_S \). Then \( (R_S)_{T^*} = R_{ST} \). [15]

Using this fact, we can prove the following.

**Lemma 3.14.** Let \( R \) be an AGGCD-domain. If \( I \) and \( J \) are invertible ideals of \( R \) with \( I \cap J \) invertible, then \( (I \cap J)R = IR \cap JR \).

**Proof:** Let \( M \) be a maximal ideal of \( R \). Then \( R_M \) is a quasilocal AGGCD-domain and hence is an AGCD-domain. Since \( I, J \) and \( I \cap J \) are invertible ideals of \( R, I_M, J_M, \) and \( (I \cap J)_M \) are principal ideals of \( R_M \). Therefore, by Lemma 3.13, \( (I_M \cap J_M)R_M = I_M R_M \cap J_M R_M \). But \( R_M = \overline{R_M} \) and so we have
\[
(I \cap J)R_M = (I \cap J)_M R_M = (I_M \cap J_M)R_M = I_M R_M \cap J_M R_M = \overline{I_M R_M} \cap \overline{J_M R_M}.
\]
Thus for any multiplicatively closed subset \( S \) of \( R \) with the form \( S = R - M \), \( M \) a maximal ideal of \( R \),
\[
(I \cap J)R_S = \overline{I_M R_M} \cap \overline{J_M R_M}.
\]

\( (*) \)
Now let $\mathcal{M}$ be a maximal ideal of $\overline{R}$. Then $M = \mathcal{M} \cap R$ is a maximal ideal of $R$.

Let $S = R - M$. Then by the fact above,

$$(\overline{R}_S)_M = (\overline{R}_S)_S$$

where $S^* = R_S - M_S$

$$= \overline{R}_S S$$

$$= \overline{R}_S$$ since $S$ is multiplicatively closed and $1 \in S = R - M$.

Clearly $\overline{R}_S \subseteq \overline{R}_\mathcal{M}$ since $S = R - M = R - (\mathcal{M} \cap R) \subseteq R - \mathcal{M}$. Let $x \in \overline{R}_\mathcal{M}$, so $r, s \in \overline{R}$ with $s \notin \mathcal{M}$. Since $\overline{R}$ is an AGGCD-domain, $R \subseteq \overline{R}$ is a root extension. So there exist natural numbers $n$ and $m$ with $r^n, s^m \in R$. But then $r^{nm}, s^{nm} \in \overline{R}$. Since $s \notin \mathcal{M}$ and $\mathcal{M}$ is a maximal (and hence prime) ideal of $\overline{R}$, $s^{nm} \notin \overline{R} \cap \mathcal{M} = M$. So $\frac{r^{nm}}{s^{nm}} \in R_S \Rightarrow \frac{r}{s} \in \overline{R}_S = \overline{R}_S$.

Therefore $\overline{R}_\mathcal{M} \subseteq \overline{R}_S$ and we have that $\overline{R}_\mathcal{M} = \overline{R}_S = (\overline{R}_S)_M$. Then for any maximal ideal $\mathcal{M}$ of $\overline{R}$,

$$I\overline{R}_\mathcal{M} \cap J\overline{R}_\mathcal{M} = I(\overline{R}_S)_M \cap J(\overline{R}_S)_M$$

where $M = \mathcal{M} \cap R$ and $S = R - M$.

$$= [I\overline{R}_S \cap J\overline{R}_S]_M$$

$$= [(I \cap J)\overline{R}_S]_M \quad \text{by (*)}$$

$$= (I \cap J)(\overline{R}_S)_M$$

$$= (I \cap J)\overline{R}_\mathcal{M}.$$ 

Thus $I\overline{R}_\mathcal{M} \cap J\overline{R}_\mathcal{M} = (I\overline{R} \cap J\overline{R})_\mathcal{M} = (I \cap J)\overline{R}_\mathcal{M}$ for any maximal ideal $\mathcal{M}$ of $\overline{R}$.

So $(I \cap J)\overline{R} = I\overline{R} \cap J\overline{R}$. 

\[ \square \]

We are now ready to prove that the integral closure of an AGGCD-domain is also an AGGCD-domain.

**Theorem 3.15.** If $R$ is an AGGCD-domain, then $\overline{R}$ is an AGGCD-domain.
Proof: Let \( a, b \in \overline{R} \). Since \( R \) is an AGGCD-domain, \( R \subseteq \overline{R} \) is a root extension. So as in the proof of the previous lemma, there is a single natural number \( n \) with \( a^n, b^n \in R \). Then there exists an \( m \) with \( (a^n)^m R \cap (b^n)^m R \) invertible, since \( R \) is an AGGCD-domain. By Lemma 3.14, since principal ideals are invertible, we have \( (a^m R \cap b^m R)\overline{R} = a^m \overline{R} \cap b^m \overline{R} \). But since \( a^n R \cap b^n R = I \) is invertible, \( II^{-1} = R \). Hence \( (I\overline{R})(I^{-1}\overline{R}) = (II^{-1})\overline{R} \cdot \overline{R} = R \cdot \overline{R} = \overline{R} \) and so \( I\overline{R} = (a^m R \cap b^m R)\overline{R} = a^m \overline{R} \cap b^m \overline{R} \) is also invertible. Thus \( \overline{R} \) is an AGGCD-domain. \( \square \)

We have seen that every AGCD-domain is an AGGCD-domain and in the semi–quasilocal case, they are equivalent. There are also other situations in which \( R \) being an AGGCD-domain is equivalent to \( R \) being an AGCD-domain. Recall that the \( t \)-class group of \( R \), denoted \( Cl_t(R) \), is the quotient group of the \( t \)-invertible \( t \)-ideals modulo the principal ideals. Similarly, the class group of \( R \), denoted \( C(R) \), is the quotient of the invertible ideals modulo the principal ideals. Also, according to [5], if \( (a, b)_t \) is \( t \)-invertible, then \( [(a, b)^n]_t = (a^n, b^n)_t \). Using these, we have the following equivalences.

Theorem 3.16. For an integral domain \( R \), the following statements are equivalent:

1) \( R \) is an AGCD-domain.

2) \( R \) is an AGGCD-domain with \( Cl_t(R) \) torsion.

3) \( R \) is an AGGCD-domain with \( C(R) \) torsion.
Proof: 1) ⇒ 2) It was shown in [5], that any AGCD-domain has torsion \( t \)-class group. Also, as remarked above, any AGCD-domain is an AGGCD-domain.

2) ⇒ 3) It suffices to show that \( Cl(R) \) is torsion. Now if \( I \) is an invertible ideal of \( R \), then \( I \) is finitely generated and \( I_t = I_v = (I^{-1})^{-1} = I \). Thus \( I \) is \( t \)-ideal. Also, if \( I \) is invertible, then \( II^{-1} = R \) and hence \( (II^{-1})_t = R \).

Therefore, \( I \) is a \( t \)-invertible \( t \)-ideal. Since \( Cl_t(R) \) is torsion by assumption, there is an \( n \) such that \( (I^n)_t \) is principal. But if \( I \) is invertible, then \( I^n \) is also invertible. Thus \( I^n = ((I^n)^{-1})^{-1} = (I^n)_v = (I^n)_t \) is principal. So there is an \( n \) with \( I^n \) principal and hence \( C(R) \) is torsion.

3) ⇒ 1) Let \( a, b \in R - \{0\} \). Since \( R \) is an AGGCD-domain, there is an \( n = n(a, b) \) with \( (a^n, b^n)_v \) invertible. Thus since \( C(R) \) is torsion, there is an \( m \) with \( [(a^n, b^n)_v]^m \) principal. Therefore, \( ([(a^n, b^n)_v]^m)_v \) is also principal. Using properties of **-operations, we have \( ([(a^n, b^n)_v]^m)_v = ((a^n, b^n)_v \cdots (a^n, b^n)_v)_v = ((a^n, b^n) \cdots (a^n, b^n))_v = [(a^n, b^n)^m)_v \). So \( [(a^n, b^n)^m)_v = [(a^n, b^n)^m]_t \) is principal and hence \( t \)-invertible. Thus we have that \( [(a^n, b^n)^m)_v = [(a^n, b^n)^m]_t = (a^{nm}, b^{nm})_t = (a^{nm}, b^{nm})_v \). Therefore, \( (a^{nm}, b^{nm})_v \) is principal and hence \( R \) is an AGCD-domain.

Another class of examples of AGGCD-domains are the AP-domains first introduced in [5]. Recall that an integral domain \( R \) is called an AP-domain if for \( a, b \in R - \{0\} \), there is an \( n = n(a, b) \) with \( (a^n, b^n) \) invertible. But if \( (a^n, b^n) \) is invertible, then so is \( (a^n, b^n)_v \) and hence \( R \) is an AGGCD-domain.

It was shown in [5] that an integral domain \( R \) is an AP-domain if and only if \( \overline{R} \) is a Prüfer domain (for \( a, b \in R - \{0\} \), \( (a, b) \) is invertible) and \( R \subseteq \overline{R} \) is a
root extension. In particular, \( R \) is an integrally closed AP–domain if and only if \( R \) is Prüfer. This gives us a way to find an example of an AP–domain which is not an AGCD–domain. Recall that if \( R \) is an AGCD–domain, then \( Cl_t(R) \) is torsion. The following result was given by Eakin and Heinzer in [9].

**Theorem 3.17.** Let \( G \) be a finitely generated abelian group, \( \mathbb{Q} \) the rationals, \( \mathbb{Z} \) the integers and \( x \) an indeterminate. Then there is a Dedekind domain \( D \) with class group \( G \) such that \( \mathbb{Z}[x] \subset D \subset \mathbb{Q}[x] \).

Using this result, we can find a Dedekind domain \( D \) with a class group which is not torsion such that \( \mathbb{Z}[x] \subset D \subset \mathbb{Q}[x] \). Thus \( D \) is Prüfer and hence an AP–domain. But, \( D \) is not an AGCD–domain. For if \( D \) were an AGCD–domain, then \( Cl_t(D) \) would be torsion. Then for any invertible ideal \( A \) of \( D \), \( A \) is also \( t \)-invertible and hence \( (A^n)_t \) is principal for some \( n \). But \( A^n \) is invertible since \( A \) is and thus \( A^n \) is a \( t \)-ideal. Therefore, \( A^n \) is principal and \( C(D) \) would be torsion. This contradiction shows that \( D \) is an AP–domain which is not an AGCD–domain.

Many of the results in [5] concerning AGCD–domains and AP–domains can be extended to AGGCD–domains. The first of these is an immediate consequence of the following result which appeared in [5].

**Proposition 3.18.** Let \( R \) be an integral domain and let \( a, b \in R - \{0\} \). Suppose that there exists an \( n = n(a, b) \) with \( a^nR \cap b^nR \) locally principal (e.g. invertible). Then \( \frac{a}{b} \) is integral over \( R \) if and only if \( \left( \frac{a}{b} \right)^n \in R \). In particular, if \( R \) is an integral domain with the property that for each \( a, b \in R - \{0\} \), there exists an
\[ n = n(a, b) \text{ with } a^n R \cap b^n R \text{ locally principal, then } x \in \overline{R} \text{ if and only if } \exists x^m \in R. \]

As an immediate corollary, we get the following.

**Corollary 3.19.** If \( R \) is an AGGCD-domain, then \( R \subseteq \overline{R} \) is a root extension.

**Proof:** If \( R \) is an AGGCD-domain, then for \( a, b \in R - \{0\} \), there is an \( n = n(a, b) \) with \( a^n R \cap b^n R \) invertible and hence locally principal. So by the above proposition, for any \( x \in \overline{R} \), there is an \( m \) such that \( x^m \in R \). Hence \( R \subseteq \overline{R} \) is a root extension. \( \square \)

Given this result, we can extend many of the other results for AGCD-domains from [5] to analogous results for AGGCD-domains. The proofs are similar to the AGCD-domain case, but we include them for completeness. The first result which we will extend appeared in [5] as Theorem 5.3. The statement is given below.

**Theorem 3.20.** Let \( R \) be an AGCD-domain. Let \( P \) be a nonzero prime ideal of \( R \). Then \( P \) is a \( t \)-ideal if and only if \( R_P \) is an AB-domain (for any \( a, b \in R - \{0\} \), there is an \( n = n(a, b) \) with \( (a^n, b^n) \) principal).

To prove the analogous result for AGGCD-domains, we will need some additional information.

In [5], it was shown that if \( R \) is an AGCD-domain and \( P \) is a prime \( t \)-ideal of \( R \), then \( P_P \) is a prime \( t \)-ideal of \( R_P \). This result is also true in the
AGGCD-domain case. It relies on the fact that for \( \{a_\alpha\} \subseteq R - \{0\} \), \( R \) an integral domain, if \( (\{a_\alpha\}_t) \) is \( t \)-invertible, then \( (\{a_\alpha^n\}_t) = ((\{a_\alpha\})^n)_t \). This result was shown in [5].

**Lemma 3.21.** Let \( R \) be an AGGCD-domain. Let \( P \) be a prime \( t \)-ideal of \( R \). Then \( P_P \) is a prime \( t \)-ideal of \( R_P \).

**Proof:** Suppose that \( P_P \) is not a \( t \)-ideal of \( R_P \). Then since \( P_P \) is the unique maximal ideal of \( R_P \), we have that \( (P_P)_t = R_P \). So \( 1 \in (P_P)_t \) and hence there exist \( x_1, \ldots, x_n \in P - \{0\} \) with \( 1 \in [(x_1, \ldots, x_n)_P]_t \). Therefore, \( [(x_1, \ldots, x_n)_P]_t = R_P \). This gives us that \( [(x_1, \ldots, x_n)_P]_t \) is invertible and hence \( t \)-invertible. But if \( [(x_1, \ldots, x_n)_P]_t \) is \( t \)-invertible, then for any \( m \geq 1 \), \( [(x_1^m, \ldots, x_n^m)_P]_t = [(x_1, \ldots, x_n)_P^m]_t = R_P \) by our earlier remarks. Since \( R \) is an AGGCD-domain, there is an \( m \) with \( (x_1^m, \ldots, x_n^m)_t \) invertible and hence \( [(x_1^m, \ldots, x_n^m)_t]_P \) is principal. Now it is well-known that for any nonzero ideal \( I \), \( (I_P)_t = [(I_t)_P]_t \) (see [14]). Thus we have \( R_P = [(x_1^m, \ldots, x_n^m)_P]_t = \left( [(x_1^m, \ldots, x_n^m)_t]_P \right)_t \). But as noted above, \( [(x_1^m, \ldots, x_n^m)_t]_P \) is a principal ideal and hence a \( t \)-ideal. Therefore \( [(x_1^m, \ldots, x_n^m)_t]_P = [(x_1^m, \ldots, x_n^m)_P]_t \). But since \( x_1^m, \ldots, x_n^m \in P \) and \( P \) is a \( t \)-ideal, we have \( [(x_1^m, \ldots, x_n^m)_P] \subseteq (P_t)_P = P_P \). Hence \( R_P = [(x_1^m, \ldots, x_n^m)_P]_t = [(x_1^m, \ldots, x_n^m)_P]_t \subseteq P_P \), a contradiction. \( \Box \)

For the next lemma, we will need to recall the definition of a \( t \)-local domain as defined in [5]. An integral domain \( R \) is called a \( t \)-local domain if \( R \) has a unique maximal \( t \)-ideal. It is easily seen that \( R \) is \( t \)-local if and only if \( R \) has a unique maximal ideal \( M \) and \( M \) is a \( t \)-ideal. For if \( M \) is the unique
maximal ideal of \( R \) and \( M_t = M \), then any ideal \( I \) of \( R \) must have \( I \subseteq M \) and hence \( I_t \subseteq M_t = M \). So \( M \) is the unique maximal \( t \)-ideal. Conversely, suppose \( M = M_t \) is the unique maximal \( t \)-ideal. Then for any ideal \( I \) of \( R \), \( I \subseteq I_t \subseteq M_t = M \). Thus \( M \) is the unique maximal ideal of \( R \) and \( M \) is a \( t \)-ideal.

Now it was shown in [5], that a \( t \)-local AGCD-domain is an AB-domain. We can actually extend this result to the AGGCD-domain case as well.

**Lemma 3.22.** Let \( R \) be a \( t \)-local AGGCD-domain. Then \( R \) is an AB-domain and hence also an AP-domain.

**Proof.** If \( R \) is a \( t \)-local AGGCD-domain, then by the above remarks, \( R \) is a quasilocal AGGCD-domain. But in the quasilocal case, \( R \) is an AGGCD-domain if and only if \( R \) is an AGCD-domain by Theorem 3.15. Hence \( R \) is a \( t \)-local AGCD-domain and thus an AB-domain by the result from [5]. But clearly every AB-domain is an AP-domain. \( \square \)

Recall that every overring of an AB-domain (respectively AP-domain) is also an AB-domain (respectively AP-domain) according to [5]. In addition, Anderson and Zafrullah showed that an integrally closed AP-domain or AB-domain is a Prüfer domain. Now a Prüfer domain is a domain \( R \) in which every finitely generated ideal is invertible, or equivalently, \( R_M \) is a valuation domain for each maximal ideal \( M \) of \( R \). To say that \( R_M \) is a valuation domain says that for any \( r, s \in R_M - \{0\} \), either \( r \mid s \) or \( s \mid r \). This is also equivalent to saying that the ideals of \( R_M \) are totally ordered.
Finally, the set of all prime ideals of a domain \( R \) is denoted by \( \text{Spec}(R) \). Using the terminology from [5], we say that \( \text{Spec}(R) \) is treed if \( \text{Spec}(R) \), as a partially ordered set, is a tree. Equivalently, \( \text{Spec}(R) \) is treed if \( \text{Spec}(R_M) \) is totally ordered for each maximal ideal \( M \) of \( R \). It was shown in [5] that if \( R \subseteq S \) is a root extension of commutative rings, then \( \text{Spec}(R) \) is treed if and only if \( \text{Spec}(S) \) is treed.

We are now ready to prove the following result.

**Theorem 3.23.** Let \( R \) be an AGGCD-domain. Let \( P \) be a nonzero prime ideal of \( R \). Then \( P \) is a \( t \)-ideal if and only if \( R_P \) is an AB-domain.

**Proof.** (\( \Rightarrow \)) Suppose that \( P \) is a nonzero prime \( t \)-ideal of \( R \). Since \( R \) is an AGGCD-domain, \( P_P \) is a prime \( t \)-ideal of \( R_P \) by Lemma 3.21. Now \( R_P \) is also an AGGCD-domain by Corollary 3.13. Thus \( R_P \) is a \( t \)-local AGGCD-domain and hence by Lemma 3.22, an AB-domain.

(\( \Leftarrow \)) Suppose that \( R_P \) is an AB-domain. Then by the previous remarks, \( \overline{R}_P \) is an integrally closed AB-domain and hence a Prüfer domain. Now \( \overline{R}_P \) Prüfer implies that for every maximal ideal \( M \) of \( \overline{R}_P \), \( (\overline{R}_P)_M \) is a valuation domain and hence the ideals are totally ordered. Therefore, \( \text{Spec}((\overline{R}_P)_M) \) is totally ordered for each maximal ideal \( M \) of \( \overline{R}_P \). Equivalently, this says that \( \text{Spec}(\overline{R}_P) \) is treed as noted earlier. Also, since \( R_P \) is an AB-domain and hence an AGGCD-domain, \( R_P \subseteq \overline{R}_P \) is a root extension by Corollary 3.19. So \( \text{Spec}(R_P) \) is also treed, as remarked earlier. Now let \( 0 \neq x \in P \). Shrink \( P \) is a prime ideal \( P_x \) minimal over \( (x) \). It is well-known that any prime minimal over
a principal ideal is a $t$-ideal, so $P_x$ is a $t$-ideal. But $P = \cup \{P_x \mid 0 \neq x \in P\}$ and $P_t = \cup \{I_v \mid I \subseteq P \text{ with } I \in f(R)\}$ by definition. So let $I$ be a finitely generated ideal of $R$ with $I \subseteq P$, say $I = (a_1, \ldots, a_k)$. Then $(a_1, \ldots, a_k) \subseteq P = \cup \{P_x \mid 0 \neq x \in P\}$ and so for each $a_i$, $a_i \in P_{x_i}$ for some $x_i$. But since $\text{Spec}(R_P)$ is treed, the primes contained in $P$ are totally ordered. Therefore $I = (a_1, \ldots, a_k)$ is contained in one of the $P_{x_i}$. So $I_t \subseteq (P_{x_i})_t = P_{x_i}$ since each $P_x$ is a $t$-ideal. Therefore, $I_t \subseteq P$. Since $I$ is finitely generated, we have $I_v = I_t \subseteq P$. Hence for any finitely generated ideal $I$ with $I \subseteq P$, we have $I_v \subseteq P$. Therefore, $P_t = \cup \{I_v \mid I \subseteq P \text{ with } I \in f(R)\} \subseteq P$. But we always have $P \subseteq P_t$, and hence $P = P_t$. So $P$ is a $t$-ideal.

Theorem 3.23 and the analogous result for AGCD-domains from [5], Theorem 3.20, generalize the well-known result that a prime ideal $P$ of a GCD-domain $R$ is a $t$-ideal if and only if $R_P$ is a valuation domain. Given the above result, we can prove the following corollary which extends Corollary 5.4 from [5] to the AGGCD-domain case. Replacing "AGCD-domain" by "AGGCD-domain" and "AB-domain" by "AP-domain" in Anderson and Zafrullah's Corollary 5.4, we get the following result.

Corollary 3.24. Let $R$ be an AGGCD-domain. Then the following statements are equivalent:

1. $R$ is an AP-domain.
2. Every prime ideal of $R$ is a $t$-ideal.
3. Every maximal ideal of $R$ is a $t$-ideal.
(4) Spec(R) is treed.

Proof: (1) ⇒ (2) We know that every overring of an AP-domain is also an AP-domain. Hence for any prime ideal P of an AP-domain R, Rp is an AP-domain. Then by the previous theorem, P is a t-ideal.

(2) ⇒ (3) Clear since every maximal ideal of R is a prime ideal of R.

(3) ⇒ (4) Let M be a maximal ideal of R. Then M is a t-ideal and hence RM is an AP-domain by the previous theorem. Then R̅M = R̅M is an integrally closed AP-domain, being an overring of the AP-domain RM. So, as remarked earlier, R̅M is a Prüfer domain and hence Spec(R̅M) is treed as shown in the proof of Theorem 3.22(⇐). But then Spec(RM) is also treed since RM is an AP-domain implies that RM is an AGGCD-domain and hence RM ⊆ R̅M is a root extension. So Spec(RM) is treed for each maximal ideal M of R and hence Spec(R) must also be treed. For if not, then for some primes P and Q, we have 0 ⊆ P, Q ⊆ N for some maximal ideal N of R where P and Q are not comparable. But then, because of the correspondence of primes between R and RN, we would have 0 ⊆ PN, QN ⊆ NN, where PN and QN are primes of RN which are not comparable. This is impossible since Spec(RN) is treed for any maximal ideal N of R. Thus Spec(R) is treed.

(4) ⇒ (1) Suppose Spec(R) is treed. Since R is an AGGCD-domain, R ⊆ R̅ is a root extension by Corollary 3.19. Thus by earlier remarks from [5], Spec(R̅) is also treed. So for each maximal ideal M of R̅, Spec(R̅M) is totally ordered, and hence each R̅M is a valuation domain. This says that R̅ is a Prüfer domain. Now it was shown in [5], that R is an AP-domain if and only if R̅ is a
Prüfer domain and \( R \subseteq \overline{R} \) is a root extension. Therefore, \( R \) is an AP-domain as required.

Recall that a domain \( R \) is called an almost valuation domain (AV-domain) if for \( a, b \in R - \{0\} \), there exists an \( n = n(a, b) \) with either \( a^n \mid b^n \) or \( b^n \mid a^n \). This definition, given in [5], is a generalization of the definition of a valuation domain. A domain \( R \) is a valuation domain if for \( a, b \in R - \{0\} \), either \( a \mid b \) or \( b \mid a \). It is easily seen that \( R \) is an AV-domain if and only if for each nonzero \( x \) in the quotient field of \( R \), there exists an \( n = n(x) \geq 1 \) with either \( x^n \) or \( x^{-n} \) in \( R \). Now since in the quasilocal case, \( R \) is an AGGCD-domain if and only if \( R \) is an AGCD-domain, Anderson and Zafrullah's Theorem 5.6 from [5] immediately carries over to the AGGCD-domain case. The proof of the theorem stated below is identical to the AGCD-domain case.

**Theorem 3.25.** For an integral domain \( R \), the following statements are equivalent:

1. \( R \) is an AV-domain.
2. \( \overline{R} \) is a valuation domain and \( R \subseteq \overline{R} \) is a root extension.
3. \( R \) is a \( t \)-local AGGCD-domain.
4. \( R \) is a quasilocal AB-domain.

In his paper on almost factoriality [17], Zafrullah showed that an integrally closed AGCD-domain is a PVMD with torsion \( t \)-class group. In fact, the converse is also true. Recall that an integral domain \( R \) is said to be a PVMD (Prüfer \( v \)-multiplication domain) if every finite type \( t \)-ideal is \( t \)-invertible.
It seems reasonable to ask whether integrally closed AGGCD-domains have any special properties. According to Theorem 3.16, unlike an AGCD-domain, an AGGCD-domain need not have a torsion $t$-class group. While any AP-domain is an AGGCD-domain, an AP-domain need not be an AGCD-domain. However, if $R$ is an integrally closed AGGCD-domain, we can still conclude that $R$ is a PVMD. In fact, we have the following result.

**Theorem 3.26.** Let $R$ be an AGGCD-domain. Then the following statements are equivalent:

1. $R$ is integrally closed.
2. $R$ is root closed.
3. $R$ is a PVMD.

**Proof:**

(1) $\Rightarrow$ (2) Always holds.

(2) $\Rightarrow$ (3) According to [12], it suffices to show that every ideal generated by two elements is $t$-invertible. Let $a, b \in R$ and consider the ideal $(a, b)$. Since $R$ is an AGGCD-domain, there exists an $n = n(a, b)$ such that $(a^n, b^n)_v$ is invertible. Now since $(a^n, b^n)$ is finitely generated, we have that $(a^n, b^n)_v = (a^n, b^n)_t$. Thus $(a^n, b^n)_t$ is invertible. So there is some ideal $I$ of $R$ with $(a^n, b^n)_t I = R$. By assumption, $R$ is root closed, and hence $(a^n, b^n)_t = [(a, b)^n]_t$. So we have $[(a, b)^n]_t I = R$, and hence $([(a, b)^n]_t I)_t = R_t = R$. By a property of $*$-operations, this gives us that $[(a, b)^n I]_t = ([(a, b)^n]_t I)_t = R$. Thus we have $[(a, b)((a, b)^{n-1} I)]_t = R$ and so $(a, b)$ is $t$-invertible. Therefore, $R$ is a PVMD.

(3) $\Rightarrow$ (1) It is well-known that any PVMD is integrally closed. See [12].
The above result can be improved slightly by recalling that if \( R \) is a root closed integral domain, then \((\{a^n_\alpha\}_t) = ((\{a_\alpha\})^n)_t\) for any nonempty collection \( \{a_\alpha\} \subseteq R - \{0\} \). Using this, we have the following improved version of the previous theorem.

**Theorem 3.27.** Let \( R \) be an integral domain. Then the following statements are equivalent:

1. \( R \) is an integrally closed AGGCD-domain.
2. \( R \) is a root closed AGGCD-domain.
3. \( R \) is a PVMD with \( Cl_t(R) = C(R) \), that is, every \( t \)-invertible \( t \)-ideal is invertible.

**Proof.** (1) \( \Rightarrow \) (2) \( R \) integrally closed always implies \( R \) root closed.

(2) \( \Rightarrow \) (3) By Theorem 3.26, a root closed AGGCD-domain is a PVMD.

Let \( I \) be a \( t \)-invertible \( t \)-ideal. It is well-known that any \( t \)-invertible \( t \)-ideal has finite type. Thus there are \( a_1, \ldots, a_k \in R \) with \( I = (a_1, \ldots, a_k)_t \). Now since \( R \) is an AGGCD-domain, there is an \( n = n(a_1, \ldots, a_k) \) with \( (a^n_1, \ldots, a^n_k)_t \) invertible.

But \( R \) is root closed, and so by our previous remarks, \( [(a_1, \ldots, a_k)^n]_t = (a^n_1, \ldots, a^n_k)_t \) is invertible. Now notice that if \( I \) is \( t \)-invertible, then \((II^{-1})_t = R\) and hence \([I^n(I^{-1})^n]_t = R\). Thus we have that \((I^{-1})^n = (I^n)^{-1}\). But

\[
(II^{-1})_t = [(a_1, \ldots, a_k)_tI^{-1}]_t = [(a_1, \ldots, a_k)_t]_t = R.
\]

So \((a_1, \ldots, a_k)_t\) is also \( t \)-invertible. Thus we have \( I^n = [(a_1, \ldots, a_k)_t]^n = [(a_1, \ldots, a_k)_v]^n = ([(a_1, \ldots, a_k)^{-1}]^{-1})^{-1} = ([a_1, \ldots, a_k]^n)_{t} \), which is invertible from above. Therefore, \( I^n \)
is invertible and hence so is $I$. Thus every $t$-invertible $t$-ideal is invertible, that is, $Cl_t(R) = C(R)$.

(3) $\Rightarrow$ (1) As noted in the proof of the previous theorem, a PVMD is integrally closed. Suppose $a, b \in R - \{0\}$. Since $R$ is a PVMD, every finite type $t$-ideal is $t$-invertible. Now $(a, b)_t$ is a finite type $t$-ideal and hence is $t$-invertible. So since $Cl_t(R) = C(R)$, we have that $(a, b)_t$ is actually invertible. Thus using $n(a, b) = 1$, we see that $R$ is an AGCD-domain.

Recall that Zafrullah showed in [17] that the integral closure of an AGCD-domain is also an AGCD-domain. Also, as mentioned in the remarks before Theorem 3.26, an integrally closed AGCD-domain is a PVMD with $Cl_t(R)$ torsion. Thus if $R$ is an AGCD-domain, then $\overline{R}$ is a PVMD with $Cl_t(\overline{R})$ torsion. Similarly, we have the following result for AGGCD-domains.

Corollary 3.28. If $R$ is an AGGCD-domain, then $\overline{R}$ is a PVMD with $Cl_t(\overline{R}) = C(\overline{R})$.

Proof. If $R$ is an AGGCD-domain, then $\overline{R}$ is an AGGCD-domain by Theorem 3.11. So $\overline{R}$ is an integrally closed AGCD-domain and hence a PVMD with $Cl_t(\overline{R}) = C(\overline{R})$ by the previous theorem.

The integral closure of an AGGCD-domain has another interesting property. In [8], the authors introduced the notion of a $t$-linked overring of a domain $R$. They defined an overring $S$ of $R$ to be $t$-linked if for each $A \in f(R)$ with $A_v = R$, you have $(AS)_v = S$. It was shown that $S$ is a $t$-linked overring
of $R$ if and only if for each prime $t$-ideal $P$ of $S$, $(P \cap R)_t \neq R$. It is not known whether in general the integral closure of an integral domain is $t$-linked over the domain. However, in [5], it was shown that if $R \subseteq \overline{R}$ is a root extension and \( \overline{R} \) is an AGCD-domain, then \( \overline{R} \) is $t$-linked over $R$. A similar result holds if we replace “AGCD-domain” by “AGGCD-domain.”

**Theorem 3.29.** Let $R$ be an integral domain. If $R \subseteq \overline{R}$ is a root extension and \( \overline{R} \) is an AGGCD-domain, then \( \overline{R} \) is $t$-linked over $R$.

**Proof:** Let $P$ be a prime $t$-ideal of \( \overline{R} \) and $P' = P \cap R$. From the remarks above, it suffices to show that $(P')_t \neq R$. Now $P'$ is a prime ideal in $R$. Since \( \overline{R} \) is an AGGCD-domain, \( \overline{R}_P \) is an AP-domain by Theorem 3.23. Now if $\frac{r}{s} \in \overline{R}_P$, then $r \in \overline{R}$ and $s \in \overline{R} - P$. But $R \subseteq \overline{R}$ is a root extension, so there is an $n = n(r,s)$ with $r^n, s^n \in R$. Also, since $s \notin P$, $s^n \notin P$ and hence $s^n \notin P \cap R = P'$. Thus $(\frac{r}{s})^n \in R_{P'}$ and so $R_{P'} \subseteq \overline{R}_P$ is a root extension. Therefore, as shown in [5], $R_{P'}$ is also an AP-domain. But then every prime ideal of $R_{P'}$ is a $t$-ideal by Corollary 3.24. In particular, $(P'_{P'})_t = P'_{P'}$.

Now assume that $(P')_t = R$. Then there is a finitely generated ideal $I \subseteq P'$ with $I_v = I_t = R$. Thus $R_{P'} = (I_v)_{P'} \subseteq (I_{P'})_v = (I_{P'})_t \subseteq (P'_{P'})_t = P'_{P'}$, a contradiction. Therefore $(P')_t = (P \cap R)_t \neq R$ for each prime $t$-ideal $P$ of \( \overline{R} \). So \( \overline{R} \) is $t$-linked over $R$.

Combining this with previous results gives us the following theorem.

**Theorem 3.30.** If $R$ is an AGGCD-domain, then
1) $\overline{R}$ is a PVMD,

2) $R \subseteq \overline{R}$ is a root extension, and

3) $\overline{R}$ is $t$–linked over $R$.

Proof. Follows immediately from Theorem 3.11, Corollary 3.19, Corollary 3.28 and Theorem 3.29. □

As with AGCD–domains, we would now like to investigate the consequences of adding the property that $R$ is Noetherian to an AGGCD–domain. We showed that a Noetherian AGCD–domain is an example of an almost weakly factorial domain — for each nonunit $a \in R - \{0\}$, there is an $n = n(a)$ so that $a^n$ is a product of primary elements. In doing this, we used the equivalent condition from [7] that an integral domain $R$ is almost weakly factorial if and only if $R = \cap R_P$, where the intersection is over all rank one primes of $R$, has finite character and $Cl_t(R)$ is torsion (see Theorem 2.14). If we leave off the condition that $Cl_t(R)$ is torsion, we have the definition of a weakly Krull domain as introduced in [6]. We will show that a Noetherian AGGCD–domain is an example of such a domain.

Clearly, any almost weakly factorial domain is weakly Krull. So, in particular, a Noetherian AGCD–domain is weakly Krull. In the proof of Theorem 2.14, we showed that if $R$ is a Noetherian AGCD–domain, then $R = \cap R_P$, where the intersection is over all rank one primes of $R$, and this intersection has finite character, i.e., $R$ is weakly Krull. With a few minor changes, this proof carries over to the AGGCD–domain case to show that a
Noetherian AGGCD–domain is also weakly Krull. Recall that an intersection
\[ R = \cap R_{P_a} \] is said to have finite character if each nonzero element of \( R \) is in only
a finite number of the \( P_a \).

**Theorem 3.31.** If \( R \) is a Noetherian AGGCD–domain, then \( R \) is weakly Krull.

**Proof:** As in the proof of Theorem 2.14, since \( R \) is Noetherian, \( R = \cap R_Q \), where
\( Q \) ranges over the maximal primes of principal ideals, and this intersection has
finite character. So it suffices to show that this intersection is equal to \( \cap R_P \),
where \( P \) ranges over the rank one primes of \( R \), which still has finite character.

\( \subseteq \): Suppose that \( x \in \cap R_P \), where \( P \) ranges over all rank one primes of
\( R \), and let \( Q \) be a maximal prime of \( aR \) for some \( a \in R \). Then as in the proof
of Theorem 2.14, there is some \( b \notin aR \) with \( Q \subseteq aR : bR \). Let \( P \) be the prime
minimal over \( aB : bR \), so \( Q \subseteq aR : bR \subseteq P \). Since \( R \) is an AGGCD–domain,
there is an \( n = n(a, b) \) with \( a^nR : b^nR \) invertible. Thus \( a^nR : b^nR \) is locally
principal. In particular, \( (a^nR : b^nR)_P = a^nR_P : b^nR_P \) is principal. So \( P_P \) is
minimal over the principal ideal \( a^nR_P : b^nR_P \) and hence rank \( P_P \leq 1 \) [15, Thm.
142]. Since \( R_P \) is a domain, 0 is a prime ideal and 0 \( \subseteq P_P \). Thus rank \( P_P \geq 1 \).
Together, this tells us that rank \( P_P = 1 \). So by the correspondence of primes
between \( R \) and \( R_P \), we have that rank \( P = 1 \). But 0 \( \subseteq Q \subseteq aR : bR \subseteq P \)
and hence we must have \( Q = P \). Therefore rank \( Q = 1 \), so every maximal ideal
of a principal ideal in \( R \) has rank one. Thus \( x \in \cap R_Q \) where \( Q \) ranges over all
maximal primes of principal ideals.

\( \supseteq \): Identical to the same containment in the proof of Theorem 2.14.
Putting the containments together, we have that the two intersections are equal. Therefore, by our earlier remarks, \( R = \cap R_P \) where \( P \) ranges over the rank one primes of \( R \). The proof that this intersection still has finite character is given in the proof of Theorem 2.14.

Therefore, \( R = \cap R_P \), where \( P \) ranges over the rank one primes of \( R \), and this intersection has finite character. So \( R \) is weakly Krull. \( \square \)
CHAPTER IV
FURTHER GENERALIZATIONS

Most of the definitions of types of domains made by Anderson and Zafrullah in [5] refer to properties of ideals generated by powers of elements of a ring $R$. For example, $R$ is an AB-domain (respectively, AP-domain) if for $a, b \in R - \{0\}$ there is an $n = n(a, b)$ with $(a^n, b^n)$ principal (respectively, invertible). A related topic which is also studied in [5] concerns the ideal generated by all $n$th powers of elements of an ideal $I$, rather than just a set of elements.

In [5], Anderson and Zafrullah defined the ideal $I_n = \{i^n | i \in I\}$. It is easily seen that $I_n$ is an ideal of $R$ satisfying $I_n \subseteq I^n$. In fact, if $I = \{a_\alpha\}$, then $\{a_\alpha^n\} \subseteq I_n \subseteq I^n$. Of special interest is the case where one of these containments is actually an equality. Clearly there is equality for $n = 1$ and any ideal $I$. Their main result concerning $I_n$ is that if $R$ is a commutative ring with identity containing a field of characteristic zero, then $I_n = I^n$ for any ideal $I$ of $R$ and all natural numbers $n$. For more on the ideal $I_n$, see [5].

Using the definition of $I_n$, some classes of integral domains which are closely related to the AB-domains and AP-domains were defined. For example, an integral domain $R$ is said to be a nearly Bézout or NB-domain (respectively, nearly Prüfer or NP-domain) if for each finitely generated nonzero ideal $I$ of
$R$, $I_n$ is principal (respectively, invertible) for some $n = n(I)$. In particular, if $R$ is a NB-domain (respectively, NP-domain), then for $a, b \in R - \{0\}$, there is an $n = n(a, b)$ with $(a, b)_n$ principal (respectively, invertible). We make the following definitions.

**Definition 4.1.** Let $R$ be an integral domain.

- $R$ is a **nearly GCD-domain** (NGCD-domain) if for $a, b \in R - \{0\}$, there is an $n = n(a, b)$ with $[(a, b)_n]_t$ principal.

- $R$ is a **nearly generalized GCD-domain** (NGGCD-domain) if for $a, b \in R - \{0\}$, there is an $n = n(a, b)$ with $[(a, b)_n]_t$ invertible.

- $R$ is a **nearly Prüfer $v$-multiplication domain** (NPVMD) if for each finitely generated nonzero ideal $I$ of $R$, there is an $n = n(I)$ with $I_n$ $t$-invertible.

While the exact relationship between the AB-domains and NB-domains, or the AP-domains and NP-domains was not given in [5], it was shown that the two notions coincide for integrally closed domains. In fact, the following result was proved.

**Theorem 4.2.** Let $R$ be an integral domain. Then the following statements are equivalent.

1. $R$ is an AB-domain (respectively AP-domain) and $R$ is root closed.
2. $R$ is an AB-domain (respectively AP-domain) and $R$ is integrally closed.
3. $R$ is an NB-domain (respectively NP-domain) and $R$ is root closed.
(4) $R$ is an NB-domain (respectively NP-domain) and $R$ is integrally closed.

(5) $R$ is a Prüfer domain with $C(R)$ torsion (respectively, $R$ is a Prüfer domain).

The main ingredient in the proof of this theorem is the relationship between $\left[(\{a_\alpha\})^n\right]_t$, $(\{a^n\})_t$ and $\left[(\{a_\alpha\})_n\right]_t$ for elements $\{a_\alpha\} \in R - \{0\}$, when $R$ is a root closed domain. It was shown in [5], that in the root closed case, $\left[(\{a_\alpha\})^n\right]_t = (\{a^n\})_t = \left[(\{a_\alpha\})_n\right]_t$. With this result, we can show that a result similar to Theorem 4.2 also holds for NGCD-domains.

Theorem 4.3. Let $R$ be an integral domain. Then the following statements are equivalent.

(1) $R$ is an AGCD-domain and $R$ is root closed.

(2) $R$ is an AGCD-domain and $R$ is integrally closed.

(3) $R$ is an NGCD-domain and $R$ is root closed.

(4) $R$ is an NGCD-domain and $R$ is integrally closed.

(5) $R$ is a PVMD with $Cl_t(R)$ torsion.

Proof. (5) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1) See [17].

(1) $\Leftrightarrow$ (3) If $R$ is root closed, then for any $a, b \in R - \{0\}$, $[(a, b)_n]_t = (a^n, b^n)_t$ for any $n$. But $(a^n, b^n)$ is finitely generated and hence $(a^n, b^n)_t = (a^n, b^n)_v$. Thus $[(a, b)_n]_t$ is principal if and only if $(a^n, b^n)_v$ is principal. So $R$ is a NGCD-domain if and only if $R$ is an AGCD-domain.

(2) $\Leftrightarrow$ (4) as in (1) $\Leftrightarrow$ (3).
Along the lines of the definitions of AGCD-domains, AB-domains and AP-domains, we can also define another generalization of a Prüfer \( v \)-multiplication domain.

**Definition 4.4.** Let \( R \) be an integral domain. \( R \) is an almost Prüfer \( v \)-multiplication domain (APVMD) if for \( a, b \in R - \{0\} \), there is an \( n = n(a, b) \) with \((a^n, b^n)\) \( t \)-invertible.

We actually could have defined an APVMD by a formally stronger property. The proof of this fact relies on a result mentioned in [7]. This result will be used again later in this chapter, so we include it for reference.

**Lemma 4.5.** Let \( R \) be an integral domain. An ideal \( A \) of \( R \) is \( t \)-invertible if and only if \( A \) has finite type and \( A_P \) is principal for each maximal \( t \)-ideal \( P \) of \( R \).

This result is a generalization of the fact that an ideal \( A \) is invertible if and only if \( A \) is finitely generated and \( A_M \) is principal for each maximal ideal \( M \) of \( R \). Using Lemma 4.5, we can now prove the following.

**Lemma 4.6.** Let \( R \) be an integral domain. \( R \) is an APVMD if and only if for \( a_1, \ldots, a_k \in R - \{0\} \), there is an \( n = n(a_1, \ldots, a_k) \) with \((a_1^n, \ldots, a_k^n)\) \( t \)-invertible.

**Proof.** \((\Leftarrow)\) Clear.

\((\Rightarrow)\) (By induction on \( k \)) Suppose \( R \) is an APVMD and \( a_1, \ldots, a_k \in R - \{0\} \). The case \( k = 2 \) is the definition. Assume the result holds for any \( k - 1 \) elements of \( R \). Then there is an \( n = n(a_1, \ldots, a_{k-1}) \) with \((a_1^n, \ldots, a_{k-1}^n)\)
t-invertible. So by Lemma 4.5, if \( P \) is a maximal \( t \)-ideal of \( R \), \((a^n_1, \ldots, a^n_{k-1})_P\) is principal and hence equals \((a^n_i)_P\) for some \( i \in \{1, \ldots, k - 1\} \). Now by the \( k = 2 \) case, there is an \( l = l(a^n_1, a^n_i) \) such that \((a^n_l, a^n_k)\) is \( t \)-invertible. But since \((a^n_1, \ldots, a^n_{k-1})_P\) is principal, so is \((a^n_l, a^n_{k-1})_P = [(a^n_1, \ldots, a^n_{k-1})_P]^l\). Then \((a^n_l, \ldots, a^n_{k-1})_P = ((a^n_l, \ldots, a^n_{k-1})_P, a^n_k)_P = ((a^n_l)_P, a^n_k)_P = (a^n_l, a^n_k)_P\) which is principal since \((a^n_l, a^n_k)\) is \( t \)-invertible. Thus \((a^n_l, \ldots, a^n_k)\) is \( t \)-invertible by Lemma 4.5.

In the integrally closed case, the notion of an APVMD and an NPVMD coincide. Thus these classes of domains follow the pattern first established in [5] concerning AB- and NB-domains (respectively AP- and NP-domains), as stated in Theorem 4.2. The pattern continued to hold for AGCD- and NGCD-domains as shown in Theorem 4.3. The following theorem shows that a similar result holds for APVMD’s and NPVMD’s.

**Theorem 4.7.** Let \( R \) be an integral domain. Then the following statements are equivalent.

1. \( R \) is an APVMD and \( R \) is root closed.
2. \( R \) is an APVMD and \( R \) is integrally closed.
3. \( R \) is an NPVMD and \( R \) is root closed.
4. \( R \) is an NPVMD and \( R \) is integrally closed.
5. \( R \) is a PVMD.

**Proof:** We show (3) \( \iff \) (1) \( \iff \) (5) \( \iff \) (2) \( \iff \) (4).
(1) ⇔ (3) If \( R \) is root closed, then for any ideal \( A = \{a_\alpha\}, \{a_\alpha^n\} = (A_n)_t = (A^n)_t \) for any \( n \). Thus if \( I \) is a finitely generated ideal of \( R \), say \( I = (i_1, \ldots, i_k) \), then \( (I_n)_t = (i_1^n, \ldots, i_k^n)_t = (I^n)_t \) for any \( n \). So \( (I_n)_t \) is \( t \)-invertible if and only if \( (i_1^n, \ldots, i_k^n)_t \) is \( t \)-invertible. But if an ideal \( A_t \) is \( t \)-invertible, then there is an ideal \( B \) such that \( (A_tB)_t = R \). Then \( R = (A_tB)_t = (AB)_t \) by a property of \( * \)-operations. Thus \( A \) is also \( t \)-invertible. Therefore \( I_n \) is \( t \)-invertible if and only if \( (i_1^n, \ldots, i_k^n)_t \) is \( t \)-invertible. Using Lemma 4.6, \( R \) is APVMD if and only if \( R \) is NPVMD.

(2) ⇔ (4) Same as (1) ⇔ (3).

(1) ⇒ (5) According to [12] it suffices to show that every two-generated ideal of \( R \) is \( t \)-invertible. So let \( a, b \in R - \{0\} \). Since \( R \) is APVMD, there is an \( n = n(a,b) \) with \( (a^n, b^n) \) \( t \)-invertible. Now \( R \) is root closed, and hence \( [(a,b)^n]_t = (a^n, b^n)_t \) is also \( t \)-invertible. So there is an ideal \( A \) of \( R \) with \( ([(a,b)^n]_tA)_t = R \). Thus \( R = ([(a,b)^n]_tA)_t = [(a,b)^nA]_t = [(a,b)((a,b)^{n-1}A)]_t \). So \( (a,b) \) is also \( t \)-invertible and \( R \) is a PVMD.

(5) ⇒ (1), (2) Clear. Take \( n(a,b) = 1 \) for any \( a, b \in R - \{0\} \) and a PVMD is integrally closed.

(2) ⇒ (5) Same as (1) ⇒ (5).

Beyond the integrally closed and root closed cases, the relationships between the classes of domains defined using the ideal \( I_n \) and the corresponding class defined using powers of generators, was not given in [5]. We have found that the definitions involving \( I_n \) are formally stronger that those involving powers of generators in all the cases we have discussed. To prove this fact, we
will need to use Lemma 4.5 and two additional results from [7]. We state these for reference.

**Lemma 4.8.** Let $R$ be an integral domain. For any fractional ideal $A$ of $R$, 
\[ A_t = \cap \{ (A_t)_P \mid P \text{ a maximal } t\text{-ideal of } R \} \].

**Lemma 4.9.** Let $R$ be an integral domain. If $A$ is a $t$-invertible ideal of $R$, then 
\[ (A_t)_P = A_P \] for each maximal $t$-ideal $P$ of $R$.

Two additional lemmas are needed to lead us to the desired result. As an immediate consequence of the second, we will see that any NB-domain (respectively, NP-domain) is also an AB-domain (respectively, AP-domain).

**Lemma 4.10.** Let $R$ be an integral domain. If $A$ is any ideal of $R$ and $S$ is a multiplicatively closed subset of $R$, then $(A^n)_S = (A_S^n)$ for any $n$.

**Proof:** $\subseteq$: A typical element of $(A^n)_S$ has the form \[ \sum_{i=1}^{m} r_i a^n_s \] where the $a_i \in A$ and the $s \in S$. But 
\[ \sum_{i=1}^{m} r_i a^n_s = \sum_{i=1}^{m} r_i (\frac{a_i}{1})^n \in (A_S^n) \].

$\supseteq$: A typical element of $(A_S^n)$ has the form \[ \sum_{i=1}^{m} r_i (\frac{a_i}{t_i})^n \] where the $a_i \in A$, $t_i \in S$ and \[ \frac{r_i}{t_i} \in R_S \]. Let 
\[ u = \prod_{i=1}^{k} s_i t_i^n \] and 
\[ u_i = \frac{u}{s_i t_i^n} \]. Then $u$ and each $u_i$ are elements of $S$ and 
\[ \sum_{i=1}^{m} u_i r_i a^n_s \in (A^n)_S \].

**Lemma 4.11.** Let $R$ be an integral domain. For $a, b \in R - \{0\}$, if $(a, b)_n$ is invertible, then $(a, b)_n = (a^n, b^n)$.

**Proof:** First assume that $R$ is quasilocal. Then if $(a, b)_n$ is invertible for $a, b \in R - \{0\}$, it is principal. Thus we must have that $(a, b)_n = ((ra + sb)^n)$ for
some $r, s \in R$. Now clearly, $a^n \in (a, b)_n$ and hence $a^n = u(ra + sb)^n$ for some $u \in R$.

i) If $u$ is a unit, then $(a^n) = ((ra + sb)^n)$. But then $(a^n) \subseteq (a^n, b^n) \subseteq (a, b)_n = ((ra + sb)^n) = (a^n)$ and hence $(a, b)_n = (a^n, b^n)$.

ii) If $u$ is not a unit, then $u \in M$, the unique maximal ideal of $R$. Thus $1 - ut$ is a unit for any $t \in R$. Now $a^n = u(ra + sb)^n = u(r^n a^n + \binom{n}{1} r^{n-1} sa^{n-1} b + \cdots + \binom{n}{n-1} r s^{n-1} ab^{n-1} + s^n b^n)$. So $a^n (1 - ur^n) = (\binom{n}{1} r^{n-1} sa^{n-1} b + \binom{n}{2} r^{n-2} s^2 a^{n-2} b^2 + \cdots + \binom{n}{n-1} r s^{n-1} ab^{n-1} + s^n b^n)$. Now $1 - ur^n$ is a unit and hence $a^n = \beta_{n-1} a^{n-1} b + \beta_{n-2} a^{n-2} b^2 + \cdots + \beta_1 a b^{n-1} + \beta_0 b^n$ with each $\beta_i \in R$. Thus $\frac{a^n}{b^n} = \beta_{n-1} \frac{a^{n-1}}{b^{n-1}} + \beta_{n-2} \frac{a^{n-2}}{b^{n-2}} + \cdots + \beta_1 \frac{a}{b} + \beta_0$ and $0 = \left(\frac{a}{b}\right)^n - \beta_{n-1} \left(\frac{a}{b}\right)^{n-1} - \cdots - \beta_1 \left(\frac{a}{b}\right) - \beta_0$. This says that $\frac{a}{b}$ is integral over $R$ and so $\frac{a}{b} \in \overline{R}$. Hence $(a, b)\overline{R} = b\overline{R}$, and $(a, b)\overline{R}$ is invertible. But then $(a^n, b^n)\overline{R} = (a, b)^n\overline{R} = b^n\overline{R}$. Therefore $b^n\overline{R} = (a^n, b^n)\overline{R} \subseteq (a, b)_n\overline{R} \subseteq (a, b)^n\overline{R} = b^n\overline{R}$ and $(a, b)_n\overline{R} = b^n\overline{R}$. But since $(a^n, b^n) \subseteq (a, b)_n = ((ra + sb)^n)$, there is an ideal $A$ of $R$ with $(a^n, b^n) = A(a, b)_n$. Then $b^n\overline{R} = (a^n, b^n)\overline{R} = A\overline{R}(a, b)_n\overline{R} = A\overline{R}b^n\overline{R}$.

So since $b^n\overline{R}$ is principal and hence invertible, $A\overline{R} = \overline{R}$.

Now suppose $A \neq R$. Then $0 \subseteq A \subseteq M$, the unique maximal ideal of $R$.

So since $R \subseteq \overline{R}$ is an integral extension, by Going Up (see [15]), there is a prime ideal $Q$ in $\overline{R}$ with $0 \subseteq A\overline{R} \subseteq Q$ and $Q \cap R = M$. But this says that $0 \subseteq \overline{R} \subseteq Q$ and hence $Q = \overline{R}$. So $M = Q \cap R = \overline{R} \cap R = R$, a contradiction. Hence $A = R$ and so $(a^n, b^n) = A(a, b)_n = R(a, b)_n = (a, b)_n$.

Finally, if $R$ is not quasilocal, let $M$ be any maximal ideal $M$ of $R$. If $(a, b)_n$ is invertible, then $[(a, b)_n]_M$ is principal and hence also invertible. But
\[(a, b)_n M = [(a, b)_M]_n\] by Lemma 4.10. So by the quasilocal case, we have
\[((a, b)_n)_M = [(a, b)_M]_n = (a^n, b^n)_M.\) Thus \([(a, b)_n)_M = (a^n, b^n)_M\) for any maximal ideal \(M\) of \(R\) and hence \((a, b)_n = (a^n, b^n)\) as desired. 

As an immediate consequence, we have the following.

**Corollary 4.12.** Let \(R\) be an integral domain.

1. If \(R\) is a NB–domain, then \(R\) is an AB–domain.

2. If \(R\) is a NP–domain, then \(R\) is an AP–domain.

**Proof:** If \(R\) is a NB–domain (respectively, NP–domain), then for any \(a, b \in R - \{0\}\), there is an \(n = n(a, b)\) such that \((a, b)_n\) is principal (respectively invertible). Then by Lemma 4.11, \((a, b)_n = (a^n, b^n)\) and hence \((a^n, b^n)\) is principal (respectively invertible) and \(R\) is an AB–domain (respectively AP–domain).

A similar statement holds for NGCD– and AGCD–domains, NPVMD’s and APVMD’s, and NGGCD– and AGGCD–domains. In order to prove this, we need the following result.

**Theorem 4.13.** Let \(R\) be an integral domain. For \(a, b \in R - \{0\}\), if \((a, b)_n\) is \(t\)–invertible, then \([(a, b)_n)_t = (a^n, b^n)_t.\)

**Proof.** Suppose \((a, b)_n\) is \(t\)–invertible. Then \([(a, b)_n)_t\) is also \(t\)–invertible. Let \(P\) be a maximal \(t\)–ideal of \(R\). Then by Lemmas 4.5 and 4.9, \([(a, b)_n)_P = \)
is principal. But then \([ (a, b)_n ]_P = (a^n, b^n)_P\) by Lemma 4.11. Finally, combining these facts with Lemma 4.8 gives

\[
[(a, b)_n]_t = \cap \left\{ ([(a, b)_n]_P | P \text{ a maximal } t\text{-ideal of } R \right\} \\
= \cap \left\{ [(a, b)_n]_P | P \text{ a maximal } t\text{-ideal of } R \right\} \\
= \cap \left\{ (a^n, b^n)_P | P \text{ a maximal } t\text{-ideal of } R \right\} \\
\subseteq \cap \left\{ [(a^n, b^n)_t]_P | P \text{ a maximal } t\text{-ideal of } R \right\} \\
= (a^n, b^n)_t.
\]

But we always have \((a^n, b^n) \subseteq (a, b)_n\) and hence \((a^n, b^n)_t \subseteq [(a, b)_n]_t\). Therefore, \([(a, b)_n]_t = (a^n, b^n)_t\).

It now follows that in each of the generalized classes of domains that we have discussed, the definitions involving the ideal \(I_n\) are formally stronger than those involving only powers of generators.

**Corollary 4.14.** Let \(R\) be an integral domain.

1. If \(R\) is a NGCD-domain, then \(R\) is an AGCD-domain.
2. If \(R\) is a NGGCD-domain, then \(R\) is an AGGCD-domain.
3. If \(R\) is a NPVMD, then \(R\) is an APVMD.

**Proof.** Let \(a, b \in R - \{0\}\).

1. If \(R\) is a NGCD-domain, there is an \(n = n(a, b)\) with \(((a, b)_n)_t\) principal and hence \(t\)-invertible. But then \((a, b)_n\) is also \(t\)-invertible. For there is an ideal \(I\) of \(R\) with \(((a, b)_n)_t I = ((a, b)_n)_t = R). So by Theorem
4.13, \((a^n, b^n)_t = [(a, b)_n]_t\) is principal. Since \((a^n, b^n)\) is finitely generated, 
\((a^n, b^n)_v = (a^n, b^n)_t\) is principal and \(R\) is an AGCD–domain.

(2) If \(R\) is a NGGCD–domain, there is an \(n = n(a, b)\) with \([(a, b)_n]_t\)
invertible and hence \(t\)–invertible. As in proof of (1), \((a^n, b^n)_v = (a^n, b^n)_t =
\[(a, b)_n]\_t\) and hence is invertible. So \(R\) is an AGGCD–domain.

(3) If \(R\) is a NPVMD, then there is an \(n = n(a, b)\) with \((a, b)_n\) \(t\)–invertible.
Again as in (1), \((a^n, b^n)_t = [(a, b)_n]_t\) is \(t\)–invertible and hence so is \((a^n, b^n)\).
Therefore \(R\) is an APVMD. □

We have already mentioned the pattern which was first established in [5] concerning the relationship between the almost–( ) domains and the
nearly–( ) domains. In each class of domain we have discussed, the “almost”
and “nearly” notions have coincided in the integrally closed case. The only class
we have left to consider are the AGGCD– and NGGCD–domains. With the
help of Corollary 4.14, we can now show that these domains do not break the
pattern. They also coincide in the integrally closed case. We have the following
relationships.

Theorem 4.15. Let \(R\) be an integral domain. The following statements are
equivalent.

(1) \(R\) is an AGGCD–domain and \(R\) is root closed.

(2) \(R\) is an AGGCD–domain and \(R\) is integrally closed.

(3) \(R\) is a NGGCD–domain and \(R\) is root closed.

(4) \(R\) is a NGGCD–domain and \(R\) is integrally closed.
(5) $R$ is a PVMD and $Cl_t(R) = C(R)$, that is, every $t$–invertible $t$–ideal is invertible.

**Proof.** (1) $\iff$ (2) $\iff$ (5) Theorem 3.27.

(4) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (2) Corollary 4.14.

(5) $\Rightarrow$ (4) Let $a, b \in R - \{0\}$. Since $R$ is a PVMD, every finite type $t$–ideal is $t$–invertible. Hence $(a, b)_t$ is a $t$–invertible $t$–ideal. But by assumption, $Cl_t(R) = C(R)$, and hence $(a, b)_t$ is invertible. So $[(a, b)_1)_t$ is invertible and, therefore, $R$ is a NGGCD–domain. Also, any PVMD is integrally closed. \qed

We have seen some interesting properties of and relationships between the various "almost" and "nearly" classes of domains. In many ways they are similar to the classical domains which they generalize, but there are also many differences. Some questions have been answered, while others will require further investigation.

Of the generalizations of GCD–domains discussed, the most time was spent on the AGCD–domains and the AGGCD–domains. For instance, recall that an AGCD–domain can be characterized by the property that for any $a$ and $b$, there is an $n = n(a, b)$ with $a^nR \cap b^nR$ principal. This statement is equivalent to saying that $a^n$ and $b^n$ have a least common multiple. Now while having a least common multiple guarantees the existence of a greatest common divisor, the converse is not true. It seems reasonable to want to call the type of domain defined above an almost LCM–domain (ALCM–domain). We know that being
a GCD-domain is equivalent to being a LCM-domain. Is the same true in the “almost” case? In other words, if for any \( a, b \in R - \{0\} \), there is an \( n = n(a, b) \) such that \( a^n \) and \( b^n \) have a greatest common divisor, is there an \( m = m(a, b) \) such that \( a^m \) and \( b^m \) have a least common multiple? It is not immediately clear that this should be the case, and yet, one would hope it were true.

Several results concerning GCD-domains and related classes of domains have been generalized to the “almost” and “nearly” cases. One particular topic which was studied was the question of overrings of a given type of domain. We were able to show, for instance, that the integral closure of an AGCD-domain (respectively AGGCD-domain) is also an AGCD-domain (respectively AGGCD-domain). These properties also extend to localizations and other LCM-stable overrings.

Now it is well-known that if every overring of an integral domain \( R \) is a GCD-domain, then \( R \) is Prüfer [15]. This would lead one to hypothesize that if every overring of a domain \( R \) is an AGCD-domain, then \( R \) is an AP-domain. However, since \( R \) itself is an overring of \( R \), this would say that \( R \) is an AGCD-domain and hence \( C(\overline{R}) \) is torsion. In [5], it was shown that if \( R \) is an AP-domain with \( C(\overline{R}) \) torsion, then \( R \) is actually an AB-domain. Therefore, we would like to say the following: If every overring of an integral domain \( R \) is an AGCD-domain, then \( R \) is an AB-domain. In fact, we know that every overring of an AB-domain is also an AB-domain and hence an AGCD-domain. Thus the converse easily holds and the result we would like is the following: Let \( R \) be an integral domain. Then every overring of \( R \) is an AGCD-domain if and
only if $R$ is an AB-domain. Whether this statement is true or false remains a mystery.

A lot of questions have been raised in the study of these new classes of domains. We have been able to answer a number of them, but many others remain for further study and investigation.
REFERENCES


